Bifurcation of finger-like structures in traveling waves of epithelial tissues spreading

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Abstract

We consider a continuum active polar fluid model for the spreading of epithelial monolayers introduced by R. Alert, C. Blanch-Mercader, and J. Casademunt, 2019. The corresponding free boundary problem possesses flat front traveling wave solutions. Linear stability of these solutions under periodic perturbations is considered. It is shown that the solutions are stable for short-wave perturbations while exhibiting long-wave instability under certain conditions on the model parameters (if the traction force is sufficiently strong). Then, considering the prescribed period as the bifurcation parameter, we establish the emergence of nontrivial traveling wave solutions with a finger-like periodic structure (pattern). We also construct asymptotic expansions of the solutions in the vicinity of the bifurcation point and study their stability. We show that, depending on the value of the contractility coefficient, the bifurcation can be a subcritical or a supercritical pitchfork.

Keywords: tissue spreading, free boundary problem, traveling waves, pitchfork bifurcation, stability analysis.

1 Introduction

The spreading of epithelial tissues plays an important role in the physiology of living organisms. For instance, epithelial cells heal wounds by the collective migration of large sheets of cells bound together by intercellular connections [15]. Other examples include tissue morphogenesis and tumor invasion. It is observed in experiments both in vivo and in vitro that the tissue front experiences instabilities similar to the celebrated Saffman-Taylor instabilities [22], leading via multicellular protrusions to the formation of finger-like patterns, see, e.g., [19, 24, 18].

In this work, we study this phenomenon in the framework of a free-boundary model for epithelial monolayers spreading introduced in [2], that is based on the theory of active polar fluids [20]. The epithelial monolayer is regarded as a compressible fluid flowing subject to hydrodynamic viscous forces, cell-substrate friction, surface tension, and active traction and contractile forces. The active forces are described by the polarity field.

We establish nontrivial traveling wave solutions describing the onset of finger-like patterns. These patterns emerge for a critical scale as the result of competition of destabilizing traction forces with stabilizing contractile stresses and surface tension. It was observed in [2] by means of linear stability analysis that solutions with flat interfaces are unstable under long-wavelength perturbations via the following kinematic mechanism. A small perturbation of the monolayer edge results in a velocity gradient that makes peaks move faster than troughs. This leads, as shown in [23] via numerical simulations, to the formation of finger-like patterns.

In the present work we *analytically* establish traveling wave solutions with finger-like patterns. Specifically, we find flat-front traveling wave solutions, study their stability under periodic perturbations with a prescribed period, and, considering the period as a bifurcation parameter show that at a critical value of the period, a pitchfork bifurcation occurs and a new branch of nontrivial traveling wave solutions emerges. Next, we study the linear stability of these new traveling wave solutions and identify whether the bifurcation is subcritical or supercritical. This stability issue has important biophysical implications. Namely, a subcritical bifurcation corresponds to an abrupt onset of finger-like patterns while a supercritical bifurcation implies a gradual transition. We show that the type of bifurcation depends on the mechanical properties of tissue and both subcritical and supercritical pitchforks can happen. Specifically, varying the contractility parameter we observe that a subcritical pitchfork corresponds to large or sufficiently small values of the contractility, while in another range of rather small values of the contractility, there occurs a supercritical pitchfork.

Many non-equilibrium physics systems, in particular hydrodynamic systems manifest pattern formation phenomenon [12]. An important example of interfacial patterns in fluids confined in a quasi-two-dimensional geometry, the Hele-Shaw cell, was first addressed by P. Saffman and G. I. Taylor in the seminal work [22]. The corresponding free boundary model has been attracting a lot of attention in both physical and mathematical communities, see, e.g., [5], [13], [17]. Moreover, free boundary problems of this type (with additional scalar field) appear in recent biological models of tumor growth [10], [11] or cell motility [4], [3], [14], [8], [21]. In [9] A. Friedman and F. Reitich discovered symmetry-breaking steady states bifurcating from radial solutions of the tumor growth free boundary problem, thus revealing pattern formation in this model. Another example of symmetry breaking bifurcation is studied in work [21] dealing with a cell motility model, where stability issue is also addressed.

The paper is organized as follows. Section 2 is devoted to the description of the model. In Section 3 we study the linearized operator, in particular, we show that it has a discrete spectrum. Next, we consider the flat front traveling wave solution and calculate its spectral representation via the Fourier analysis. The explicit formula for eigenvalues is given and analyzed in Section 4, while its derivation is presented in Appendix A. Then we study the case of the critical period such that the kernel of the linearized operator (around the flat traveling wave) has nonconstant eigenfunction. We show that a new branch of traveling waves with a finger-like structure bifurcates and study their stability. The theory of M. Crandall and P. Rabinowitz [6], [7] is used to study both bifurcation and stability questions. Namely, bifurcation of nonflat traveling waves is established in Section 5, where we exploit symmetries of the problem to adjust functional setting for applying Theorem 1.7 from [6] (Theorem 3 below). Addressing stability of traveling waves, we use results of Section 3 and Theorem 1.16 from [7] (Theorem 5 below) to conclude that stability is determined by the fact whether the period (bifurcation parameter in the problem) increases or decreases when departing from the bifurcation point. This makes us construct several terms in the asymptotic expansion of the traveling wave solutions. Section 6 deals with these constructions, while many technical calculations are transferred to Appendices B–E. Finally, Section 7 contains some numerical results and conclusions. There, in particular, we describe how the stability/instability of bifurcating traveling waves depends on the value of the contractility parameter.

2 Model

Following [2], we employ a continuum active polar fluid model of tissue spreading, described by a polarity field $\mathbf{p}(x, y, t)$ and a velocity field $\mathbf{v}(x, y, t)$. A tissue monolayer spreads by extending its edge towards free space. The phenomenon is mainly caused by traction forces generated by cells close to the monolayer edge. These cells polarize perpendicular to the edge, where we prescribe $\mathbf{p} = \mathbf{n}$ (the unit outward normal). The field $\mathbf{p}$ is assumed to follow purely relaxational dynamics and equilibrate fast (compared to the spreading dynamics) to the minimum of the energy with density $L_c^2|\nabla \mathbf{p}|^2 + |\mathbf{p}|^2$, where L_c is the characteristic length describing the decay rate of $\mathbf{p}$ in the bulk. For simplicity, we set $L_c = 1$ that can always be achieved by an appropriate scaling of spatial variables. Thus, $\mathbf{p}$ solves

$$\Delta \mathbf{p} = \mathbf{p} \quad \text{in } \Omega(t), \qquad \mathbf{p} = \mathbf{n} \quad \text{on } \partial\Omega(t), \tag{1}$$

where $\Omega(t)$ denotes the domain occupied by the tissue and $\mathbf{n}$ is the unit outward normal vector to the boundary.

The force balance equation reads

$$\text{div}\sigma + \mathbf{f} = 0 \quad \text{in } \Omega(t), \tag{2}$$

where σ is the stress tensor and $\mathbf{f}$ is the stress field given by the following constitutive equations for a compressible active polar fluid:

$$\sigma = \mu(\nabla \mathbf{v} + (\nabla \mathbf{v})^T) - \zeta \mathbf{p} \otimes \mathbf{p}, \qquad \mathbf{f} = -\xi \mathbf{v} + \zeta_i \mathbf{p} \qquad \text{in } \Omega(t), \tag{3}$$

where $\mathbf{v}$ is the velocity field, $\mu > 0$ is the constant effective viscosity, $\zeta < 0$ is the constant contractility coefficient, $\xi > 0$ is the constant friction coefficient, and ζ_i is the constant contact active force coefficient. On the free boundary σ satisfies

$$\sigma \cdot \mathbf{n} = -\gamma \kappa \mathbf{n} \quad \text{on } \partial\Omega(t), \tag{4}$$

where κ denotes the curvature of the boundary and $\gamma > 0$ is the constant surface tension of the monolayer edge.

While the model in [2] deals with small (linear) perturbations of a rectangular monolayer of epithelial tissue, in this work we consider half-plane type domains. This corresponds to modeling of the local behavior near the boundary of sufficiently large tissue specimens. Mathematically, considering half-plane type domains allows us to go beyond linear stability analysis and describe the formation of finger patterns via bifurcation of traveling wave solutions.

The evolution of the boundary $\partial\Omega(t)$ is described by equation $y = \rho(x,t)$, assuming that $\Omega(t) = \{(x,y) \,|\, y < \rho(x,t)\}$. Then the normal vector is given by

$$\mathbf{n} = \frac{1}{\sqrt{1+\rho'^2}} \begin{pmatrix} -\rho' \\ 1 \end{pmatrix}, \tag{5}$$

where ρ' denotes the partial derivative of ρ in x.

Assuming the continuity of velocities up to the boundary we have the following kinematic boundary condition relating the normal velocity of the boundary and the normal component of the tissue velocity field:

$$\mathbf{v} \cdot \mathbf{n} = V_{\mathbf{n}} = \frac{1}{\sqrt{1+\rho'^2}} \frac{\partial \rho}{\partial t}. \tag{6}$$

Taking (1)–(6) together, we have the equation

$$\frac{\partial \rho}{\partial t} = \mathcal{A}(\rho), \quad \mathcal{A}(\rho) = (v_y - v_x \rho')\big|_{y=\rho(x,t)}, \tag{7}$$

with $\mathbf{v} = (v_x, v_y),\ \mathbf{p} = (p_x, p_y)$ solving

$$\Delta \mathbf{p} = \mathbf{p} \qquad \text{for } y < \rho(x,t) \tag{8}$$

$$\mu(\Delta \mathbf{v} + \nabla \mathrm{div} \mathbf{v}) - \zeta \mathrm{div}(\mathbf{p} \otimes \mathbf{p}) - \xi \mathbf{v} + \zeta_i \mathbf{p} = 0 \qquad \text{for } y < \rho(x,t) \tag{9}$$

$$\mathbf{p} = \mathbf{n} \qquad \text{for } y = \rho(x,t) \tag{10}$$

$$\big(\mu(\nabla \mathbf{v} + (\nabla \mathbf{v})^T) - \zeta \mathbf{p} \otimes \mathbf{p}\big)\, \mathbf{n} = -\gamma \kappa \mathbf{n} \qquad \text{for } y = \rho(x,t). \tag{11}$$

We also assume that $\mathbf{v}$ and $\mathbf{p}$ vanish as $y \to -\infty$.

3 Linearized operator and its spectrum

Let $\rho(x)$ be an arbitrary function from the space $C^{k,\delta}_{\#}(0,\Pi)$ of k ($k \in \{3, 4, \dots\}$) times differentiable Π-periodic functions whose k-th derivatives are Hölder continuous with the exponent $0 < \delta < 1$. Problem (8)–(11) has a unique Π-periodic in x and vanishing as $y \to -\infty$ solution in the subgraph domain $y < \rho(x)$. Therefore the operator $\mathcal{A}(\rho)$ is well-defined by (7). By applying elliptic estimates from [1] to problem (8)-(11) we get that the operator $\mathcal{A}$ maps $\rho \in C^{k,\delta}_{\#}(0,\Pi)$ to $\mathcal{A}(\rho) \in C^{k-1,\delta}_{\#}(0,\Pi)$.

In this section, we consider the linearized operator $\partial_\rho \mathcal{A}(\rho)$ and show that it has a discrete spectrum and high magnitude eigenvalues are stable (have negative real parts). The following lemma establishes differentiability of $\mathcal{A}(\rho)$ and provides a formula for the first derivative.

Lemma 1. *The operator $\mathcal{A}(\rho)$ is of the class $C^\infty\left(C^{k,\delta}_{\#}(0,\Pi), C^{(k-1),\delta}_{\#}(0,\Pi)\right)$, $k \in \{3, 4 \dots\}$. Its first derivative is given by*

$$\partial_\rho \mathcal{A}(\rho)\tilde{\rho} = (\tilde{w}_y - \rho' \tilde{w}_x - \tilde{\rho}' v_x)\,\big|_{y=\rho(x)}$$

where $\mathbf{p}$, $\mathbf{v}$ solve (8)-(11), $\tilde{\mathbf{w}}$ is the solution to the system

$$\begin{aligned} \mu\left(\Delta \tilde{w}_x - \partial_x \mathrm{div} \tilde{\mathbf{w}} - 2\tilde{\rho}'' \partial_y v_x - \tilde{\rho}'(\partial^2_{yy} v_y + 4\partial^2_{xy} v_x)\right) - \xi \tilde{w}_x \\ + \zeta_i \tilde{q}_x + \zeta \tilde{\rho}' \partial_y p_x^2 = \zeta \big(\mathrm{div}\big(\mathbf{p} \otimes \tilde{\mathbf{q}} + \tilde{\mathbf{q}} \otimes \mathbf{p}\big)\big)_x \quad \textit{for } y < \rho(x), \end{aligned} \tag{12}$$

$$\begin{aligned} \mu\left(\Delta \tilde{w}_y - \partial_y \mathrm{div} \tilde{\mathbf{w}} - \tilde{\rho}'' \partial_y v_y - \tilde{\rho}'(\partial^2_{yy} v_x + 2\partial^2_{xy} v_y)\right) - \xi \tilde{w}_y \\ + \zeta_i \tilde{q}_y + \zeta \tilde{\rho}' \partial_y (p_x p_y) = \zeta \big(\mathrm{div}\big(\mathbf{p} \otimes \tilde{\mathbf{q}} + \tilde{\mathbf{q}} \otimes \mathbf{p}\big)\big)_y \quad \textit{for } y < \rho(x), \end{aligned} \tag{13}$$

with boundary conditions

$$\begin{aligned} -2\mu\rho' \partial_x \tilde{w}_x + \mu\left(\partial_y \tilde{w}_x + \partial_x \tilde{w}_y\right) - 2\mu\big(\tilde{\rho}' \partial_x v_x - \rho' \tilde{\rho}' \partial_y v_x\big) - \mu \tilde{\rho}' \partial_y v_y \\ = -\tilde{\rho}'(\zeta - \gamma\kappa) - \gamma \rho' \Big(\big(1 + \rho'^2\big)^{-3/2} \tilde{\rho}'\Big)' \quad \textit{for } y = \rho(x), \end{aligned} \tag{14}$$

$$\begin{aligned} -\mu\rho' \big(\partial_y \tilde{w}_x + \partial_x \tilde{w}_y\big) + 2\mu \partial_y \tilde{w}_y - \mu \tilde{\rho}' \left(\partial_y v_x - \rho' \partial_y v_y + \partial_x v_y\right) \\ = \gamma \Big(\big(1 + \rho'^2\big)^{-3/2} \tilde{\rho}'\Big)' \quad \textit{for } y = \rho(x), \end{aligned} \tag{15}$$

and $\tilde{\mathbf{q}}$ *satisfies*

$$\Delta\tilde{\mathbf{q}} - \tilde{\rho}''\partial_y\mathbf{p} - 2\tilde{\rho}'\partial^2_{xy}\mathbf{p} = \tilde{\mathbf{q}} \qquad \text{for } y < \rho(x), \tag{16}$$

$$\tilde{q}_x = -\frac{\tilde{\rho}'}{\left(1+(\rho')^2\right)^{3/2}} \qquad \text{for } y = \rho(x), \tag{17}$$

$$\tilde{q}_y = -\frac{\rho'\tilde{\rho}'}{\left(1+(\rho')^2\right)^{3/2}} \qquad \text{for } y = \rho(x). \tag{18}$$

Proof. To find $\partial_\rho\mathcal{A}(\rho)$ consider the perturbation $\rho^{(\varepsilon)} = \rho + \varepsilon\tilde{\rho}$ of the domain, where ε is a small parameter. Let $\mathbf{p}$, $\mathbf{p}^{(\varepsilon)}$ be the solutions of the following problems in domains with boundaries $y = \rho(x)$ and $y = \rho(x) + \varepsilon\tilde{\rho}(x)$ respectively:

$$\Delta\mathbf{p} = \mathbf{p} \quad \text{for } y < \rho(x), \qquad \mathbf{p} = \mathbf{n} \quad \text{for } y = \rho(x), \tag{19}$$

$$\Delta\mathbf{p}^{(\varepsilon)} = \mathbf{p}^{(\varepsilon)} \quad \text{for } y < \rho(x) + \varepsilon\tilde{\rho}(x), \qquad \mathbf{p}^{(\varepsilon)} = \mathbf{n} \quad \text{for } y = \rho(x) + \varepsilon\tilde{\rho}(x). \tag{20}$$

Represent $\mathbf{p}^{(\varepsilon)}$ in the form

$$\mathbf{p}^{(\varepsilon)}(x,y) = \mathbf{p}(x, y - \varepsilon\tilde{\rho}(x)) + \varepsilon\tilde{\mathbf{p}}^{(\varepsilon)}(x, y - \varepsilon\tilde{\rho}(x)), \tag{21}$$

substitute in the equation (20) to find (after changing variables) that

$$\begin{aligned}
&\Delta\tilde{\mathbf{p}}^{(\varepsilon)} - \tilde{\rho}''\partial_y\big(\mathbf{p} + \varepsilon\tilde{\mathbf{p}}^{(\varepsilon)}\big) - \tilde{\rho}'\big(2\partial_x - \varepsilon\tilde{\rho}'\partial_y\big)\partial_y\big(\mathbf{p} + \varepsilon\tilde{\mathbf{p}}^{(\varepsilon)}\big) = \tilde{\mathbf{p}}^{(\varepsilon)} && \text{for } y < \rho(x),\\
&\tilde{p}^{(\varepsilon)}_x = \frac{\rho'}{\varepsilon\sqrt{1+(\rho')^2}} - \frac{\rho'+\varepsilon\tilde{\rho}'}{\varepsilon\sqrt{1+(\rho'+\varepsilon\tilde{\rho}')^2}} && \text{for } y = \rho(x),\\
&\tilde{p}^{(\varepsilon)}_y = \frac{1}{\varepsilon\sqrt{1+(\rho'+\varepsilon\tilde{\rho}')^2}} - \frac{1}{\varepsilon\sqrt{1+(\rho')^2}} && \text{for } y = \rho(x)
\end{aligned}$$

Passing to the limit as $\varepsilon \to 0$ in this problem using elliptic estimates we see that $\tilde{\mathbf{p}}^{(\varepsilon)}$ converges in $C^{k,\delta}(K)$ on every compact $K \subset \{(x,y) |\, y \le \rho(x)\}$ to the solution $\tilde{\mathbf{p}}$ of (16)–(18).

Similarly one can show that if $\mathbf{v}^{(\varepsilon)}$ is represented as

$$\mathbf{v}^{(\varepsilon)}(x,y) = \mathbf{v}(x, y - \varepsilon\tilde{\rho}(x)) + \varepsilon\tilde{\mathbf{v}}^{(\varepsilon)}(x, y - \varepsilon\tilde{\rho}(x)), \tag{22}$$

then $\tilde{\mathbf{v}}^{(\varepsilon)}$ converges in $C^{k,\delta}(K)$ on every compact $K \subset \{(x,y) |\, y \le \rho(x)\}$ to the solution $\tilde{\mathbf{v}}$ of (12)-(15). Here the limit transition can be justified by using elliptic estimates from [1].

Reasoning analogously one establishes the existence of higher order derivatives. $\square$

Notice that setting

$$\tilde{\mathbf{w}}(x,y) = \tilde{\rho}(x)\partial_y\mathbf{v}(x,y) + \mathbf{w}(x,y), \quad \tilde{\mathbf{q}}(x,y) = \tilde{\rho}(x)\partial_y\mathbf{p}(x,y) + \mathbf{q}(x,y), \tag{23}$$

we can simplify the boundary value problem (12)–(15) as follows. The operator $\partial_\rho\mathcal{A}(\rho)$ can be written as

$$\partial_\rho\mathcal{A}(\rho)\tilde{\rho} = (\mathbf{w}\cdot\mathbf{n} + \tilde{\rho}\partial_y\mathbf{v}\cdot\mathbf{n})\big|_{y=\rho(x)}\sqrt{1+\rho'^2} - \tilde{\rho}'v_x\big|_{y=\rho(x)} \tag{24}$$

with $\mathbf{w}, \mathbf{q}$ solving

$$\begin{aligned}
&\Delta \mathbf{q} = \mathbf{q} && \text{for } y < \rho(x), && (25)\\
&\mu(\Delta \mathbf{w} + \nabla \mathrm{div} \mathbf{w}) - \zeta \mathrm{div}(\mathbf{p} \otimes \mathbf{q} + \mathbf{q} \otimes \mathbf{p}) - \xi \mathbf{w} + \zeta_i \mathbf{q} = 0 && \text{for } y < \rho(x), && (26)\\
&\mathbf{q} = \tfrac{\tilde{\rho}'}{1+(\rho')^2} \mathbf{t} - \tilde{\rho} \partial_y \mathbf{p} && \text{for } y = \rho(x), && (27)\\
&\mu(\nabla \mathbf{w} + (\nabla \mathbf{w})^T)\mathbf{n} = \gamma \Big(\big(1 + \rho'^2\big)^{-3/2} \tilde{\rho}' \Big)' \mathbf{n} + \mathbf{G} && \text{for } y = \rho(x), && (28)
\end{aligned}$$

where

$$\begin{aligned}
G_x =& \tfrac{\mu}{\sqrt{1+(\rho')^2}} \big(\tilde{\rho}' \left(2\partial_x v_x + \rho' \left(\partial_x v_y + \partial_y v_x \right) - 2\partial_y v_y \right) + 2\rho' \tilde{\rho} \partial^2_{xy} v_x \\
& \qquad - \tilde{\rho} \left(\partial^2_{xy} v_y + \partial^2_{yy} v_x \right) \big), \\
G_y =& \tfrac{\mu}{\sqrt{1+(\rho')^2}} \left(\tilde{\rho}' \left(\partial_y v_x + \partial_x v_y \right) + \rho' \tilde{\rho} \left(\partial^2_{xy} v_y + \partial^2_{yy} v_x \right) - 2\tilde{\rho} \partial_{yy} v_y \right),
\end{aligned}$$

and $\mathbf{t}$ is the unit tangent vector, $t_x = -\frac{1}{\sqrt{1+(\rho')^2}}$, $t_y = -\frac{\rho'}{\sqrt{1+(\rho')^2}}$.

For every $\tilde{\rho} \in H^{3/2}_{\#}(0, \Pi)$ formula (24) defines $\partial_\rho \mathcal{A}(\rho)\tilde{\rho} \in H^{1/2}_{\#}(0, \Pi)$ since problem (25)–(28) has a unique solution pair (vanishing as $y \to -\infty$) and $\|\mathbf{w}\|_{H^1(\Omega_{\#,\rho})} \leq C \|\tilde{\rho}\|_{H^{3/2}_{\#}(0,\Pi)}$, where $\Omega_{\#,\rho} = \{(x, y) \,|\, 0 \leq x < \Pi,\ y < \rho(x)\}$ (the period of the domain). Hereafter C denotes a generic finite constant whose value may change from line to line.

Theorem 2. *Assume that $\rho \in C^{3,\delta}_{\#}(0, \Pi)$, $\delta > 0$. Then $\partial_\rho \mathcal{A}(\rho)$ is a closed operator in $H^{1/2}_{\#}(0, \Pi)$ and the domain of $\partial_\rho \mathcal{A}(\rho)$ is $H^{3/2}_{\#}(0, \Pi)$. Its spectrum comprises at most countable set of eigenvalues (of finite multiplicities) without finite accumulation points. Moreover, for every eigenvalue λ it holds that*

$$\mathrm{Re}(\lambda) \leq C - \eta_* |\lambda|, \tag{29}$$

where $\eta_* > 0$ is independent of λ.

Proof. Notice that by elliptic estimates $\partial_\rho \mathcal{A}(\rho)\tilde{\rho} \in H^{1/2}_{\#}(0, \Pi)$, $\forall \tilde{\rho} \in H^{3/2}_{\#}(0, \Pi)$, and the operator $\partial_\rho \mathcal{A}(\rho)$ annihilates constant functions, therefore it is well defined on $H^{3/2}_{\#}(0, \Pi)/\mathbb{R}$. Consider for $\Lambda > 0$ the equation

$$\partial_\rho \mathcal{A}(\rho)\tilde{\rho} - \Lambda \tilde{\rho} = f \quad \text{in } H^{1/2}_{\#}(0, \Pi)/\mathbb{R} \tag{30}$$

and show that for sufficiently large $\Lambda > 0$ there is a unique solution $\tilde{\rho} \in H^{3/2}_{\#}(0, \Pi)/\mathbb{R}$ for every $f \in H^{1/2}_{\#}(0, \Pi)/\mathbb{R}$. To this end, we write down a weak formulation of (30) multiplying by $\Big(\big(1 + \rho'^2\big)^{-3/2} \phi'\Big)'$ and integrating over the period,

$$\mathcal{E}(\tilde{\rho}, \phi) = -\int_0^\Pi f' \phi' \big(1 + \rho'^2\big)^{-3/2}\, dx \quad \forall \phi \in H^{3/2}_{\#}(0, \Pi), \tag{31}$$

where the form $\mathcal{E}(\tilde\rho,\phi)$ is given by

$$\mathcal{E}(\tilde\rho,\phi)=\int_0^{\Pi}\left(\left(\left(1+\rho'^2\right)^{-3/2}\phi'\right)'\partial_\rho\mathcal{A}(\rho)\tilde\rho+\Lambda\left(1+\rho'^2\right)^{-3/2}\tilde\rho'\phi'\right)dx. \tag{32}$$

Since $\forall\tilde\phi\in H_{\#}^{-1/2}(0,\Pi)$ with zero mean value the equation $\left(\left(1+\rho'^2\right)^{-3/2}\phi'\right)'=\tilde\phi$ has a solution $\phi\in H_{\#}^{3/2}(0,\Pi)$, the variational problem (31) and the equation (30) are equivalent. Next we show that for sufficiently large Λ the form (32) satisfies conditions of the Lax-Milgram theorem. The continuity of $\mathcal{E}(\tilde\rho,\phi)$ on $H_{\#}^{3/2}(0,\Pi)/\mathbb{R}$ follows from the boundness of the operator (24), and we proceed with its coercivity. Consider an arbitrary function $\tilde\rho\in H_{\#}^{3/2}(0,\Pi)$ with zero mean value. Take the dot product of (26) with $\mathbf{w}$ and integrate over $\Omega_{\#,\rho}$. Using (26), (28) we obtain via integration by parts,

$$\begin{aligned}\int_{\Omega_{\#,\rho}}&\left(\tfrac{\mu}{2}\left|\nabla\mathbf{w}+\nabla\mathbf{w}^T\right|^2-\zeta\left(\nabla\mathbf{w}+\nabla\mathbf{w}^T\right)\mathbf{p}\cdot\mathbf{q}+\xi|\mathbf{w}|^2-\zeta_i\mathbf{q}\cdot\mathbf{w}\right)dxdy\\&\qquad+\zeta\int_0^{\Pi}\left(\mathbf{n}\cdot\mathbf{q}\,\mathbf{p}\cdot\mathbf{w}+\mathbf{n}\cdot\mathbf{p}\,\mathbf{q}\cdot\mathbf{w}\right)\big|_{y=\rho(x)}\sqrt{1+\rho'^2}dx\\&\qquad=\mu\int_0^{\Pi}(\nabla\mathbf{w}+(\nabla\mathbf{w})^T)\mathbf{n}\cdot\mathbf{w}\big|_{y=\rho(x)}\sqrt{1+\rho'^2}dx,\end{aligned}\tag{33}$$

Furthermore, by (28) and (24) we have

$$\begin{aligned}\mu\int_0^{\Pi}&(\nabla\mathbf{w}+(\nabla\mathbf{w})^T)\mathbf{n}\cdot\mathbf{w}\big|_{y=\rho(x)}\sqrt{1+\rho'^2}dx=\gamma\mathcal{E}(\tilde\rho,\tilde\rho)\\&-\frac{\gamma}{2}\int_0^{\Pi}\left(\left(1+\rho'^2\right)^{3/2}v_x\right)'\frac{\tilde\rho'^{\,2}}{\left(1+\rho'^2\right)^3}dx-\gamma\Lambda\int_0^{\Pi}\left(1+\rho'^2\right)^{-3/2}\tilde\rho'^{\,2}\,dx\\&+\gamma\int_0^{\Pi}\left(\tilde\rho\partial_y\mathbf{v}\cdot\mathbf{n}\sqrt{1+\rho'^2}\right)'\frac{\tilde\rho'dx}{\left(1+\rho'^2\right)^{3/2}}+\int_0^{\Pi}\mathbf{G}\cdot\mathbf{w}\sqrt{1+\rho'^2}dx.\end{aligned}\tag{34}$$

We combine (33) and (34), then applying the Korn inequality and the Cauchy-Schwarz inequality we find

$$\begin{aligned}\mathcal{E}(\tilde\rho,\tilde\rho)\geq&\eta_1\|\mathbf{w}\|^2_{H^1(\Omega_{\#,\rho})}-C_1\|\mathbf{w}\|_{H^1(\Omega_{\#,\rho})}\|\mathbf{q}\|_{L^2(\Omega_{\#,\rho})}\\&-C_2\|\tilde\rho\|_{H^1(0,\Pi)}\|\mathbf{w}(x,\rho(x))\|_{H^{1/2}_{\#}(0,\Pi)}+(\eta_2\Lambda-C_3)\|\tilde\rho\|^2_{H^1(0,\Pi)},\end{aligned}\tag{35}$$

where $\eta_{1,2}>0$ are independent of $\tilde\rho$ and Λ. Next, we use the inequality for traces $\|\mathbf{w}(x,\rho(x))\|_{H^{1/2}_{\#}(0,\Pi)}\leq C\|\mathbf{w}\|_{H^1(\Omega_{\#,\rho})}$ and the following bound

$$\int_{\Omega_{\#,\rho}}|\mathbf{q}|^2dxdy\leq C\int_0^{\Pi}|\tilde\rho'|^2dx,$$

which follows from (25), (27). Thus, for sufficiently large $\Lambda>0$ (35) yields the following bound

$$\mathcal{E}(\tilde\rho,\tilde\rho)\geq\eta_3\|\mathbf{w}\|^2_{H^1(\Omega_{\#,\rho})}+\|\tilde\rho\|^2_{H^1(0,\Pi)},\tag{36}$$

with $\eta_3 > 0$. This inequality in turn implies that $\mathcal{E}(\tilde\rho,\tilde\rho) \geq \eta_4\|\tilde\rho\|^2_{H^{3/2}_\#(0,\Pi)}$ for some $\eta_4 > 0$. Otherwise, for a sequence $\tilde\rho^{(n)}$ with $\|\tilde\rho^{(n)}\|_{H^{3/2}_\#(0,\Pi)} = 1$ it holds that $\|\tilde\rho^{(n)}\|_{H^1(0,\Pi)} \to 0$ and the corresponding solutions $\mathbf{w}^{(n)}$ converge to 0 strongly in $H^1(\Omega_{\#,\rho})$. Then in view of (26) and (28) we have that $(\tilde\rho^{(n)})'' \to 0$ strongly in $H^{-1/2}_\#(0,\Pi)$. Therefore, by compactness of the embedding of $H^{3/2}_\#(0,\Pi)$ into $H^{1/2}_\#(0,\Pi)$ the spectrum of $\partial_\rho\mathcal{A}(\rho)$ is discrete.

Assume now that λ is an eigenvalue and $\tilde\rho$ is a corresponding eigenfunction. Subtract from $\tilde\rho$ its mean value then the resulting function, still denoted by $\tilde\rho$, satisfies $\partial_\rho\mathcal{A}(\rho)\tilde\rho - \Lambda\tilde\rho = (\lambda-\Lambda)\tilde\rho$ up to a constant. Therefore arguing as above (but working with real and imaginary parts of $\tilde\rho$) one can show that if $\tilde\rho$ is normalized by $\|\tilde\rho\|_{H^1(0,\Pi)} = 1$ then

$$\mathrm{Re}(\lambda) \leq C - \eta_4\|\tilde\rho\|^2_{H^{3/2}_\#(0,\Pi)}. \tag{37}$$

On the other hand

$$\begin{aligned}|\lambda| &\leq C|\lambda|\int_0^\Pi \left(1+\rho'^2\right)^{-3/2}|\tilde\rho'|^2dx \\ &\leq C_1\|\partial_\rho\mathcal{A}(\rho)\tilde\rho\|_{H^{1/2}_\#(0,\Pi)}\left(\|\tilde\rho''\|_{H^{-1/2}_\#(0,\Pi)} + \|\tilde\rho'\|_{H^{-1/2}_\#(0,\Pi)}\right) \leq C_2\|\tilde\rho\|^2_{H^{3/2}_\#(0,\Pi)}.\end{aligned}$$

Combining this bound with (37) we obtain (29). ☐

4 Flat front solutions and their stability analysis

We are interested in a particular form of solutions of problem (8)–(11) that evolve translationally, traveling waves. The system (8)–(11) has, inter alia, a flat front traveling wave solution whose boundary is a straight line moving with constant velocity $V^{(0)}$ along y-axis. This solution does not depend on the x-variable and is stationary in the moving frame: $\mathbf{v}(x,y,t) = \mathbf{V}(y - V^{(0)}t)$, $\mathbf{p}(x,y,t) = \mathbf{P}(y - V^{(0)}t)$. Moreover, it is defined up to a translation in the direction of y-axis and we stick to the one with $\rho = 0$. Then we have $P_x = 0$, $V_x = 0$ and

$$\begin{cases}\partial^2_{yy}P_y = P_y & \text{for } y<0,\\ P_y = 1 & \text{for } y=0,\end{cases} \qquad \text{i.e. } P_y = e^y,$$

$$\begin{cases}2\mu\partial^2_{yy}V_y - \xi V_y - 2\zeta e^{2y} + \zeta_i e^y = 0 & \text{for } y<0,\\ 2\mu\partial_y V_y = \zeta & \text{for } y=0.\end{cases} \tag{38}$$

The unique (vanishing as $y\to-\infty$) solution of (38) is given by

$$V_y(y) = \tfrac{\zeta}{8\mu-\xi}\left(2e^{2y} - \tfrac{\sqrt{\xi}}{\sqrt{2\mu}}e^{\sqrt{\xi}y/\sqrt{2\mu}}\right) + \tfrac{\zeta_i}{2\mu-\xi}\left(\tfrac{\sqrt{2\mu}}{\sqrt{\xi}}e^{\sqrt{\xi}y/\sqrt{2\mu}} - e^y\right), \tag{39}$$

so that it travels with constant velocity

$$V^{(0)} = V_y\big|_{y=0} = \tfrac{\zeta}{4\mu+\sqrt{2\mu\xi}} + \tfrac{\zeta_i}{\sqrt{2\mu\xi}+\xi}. \tag{40}$$

A flat front traveling wave solution can be considered as a periodic one with an arbitrary period Π. Our interest however is in finding non-flat Π-periodic in x traveling wave solutions, that is we seek a pair of Π-periodic function $\rho(x)$ and a constant C_ρ (velocity of this wave) such that

$$\mathcal{A}(\rho) = C_\rho, \tag{41}$$

where the operator $\mathcal{A}(\rho)$ is given by (7) via Π-periodic in x and vanishing as $y \to -\infty$ solution of (8)–(11).

In order to perform the bifurcation analysis of the flat front traveling wave solution ($\rho = 0$) consider the linearized operator

$$\mathcal{L}\tilde{\rho} = \partial_\rho \mathcal{A}(0)\tilde{\rho}. \tag{42}$$

By (25)–(28),

$$\mathcal{L}\rho = (\partial_y V_y \rho + v_y)\big|_{y=0} = \frac{\zeta}{2\mu}\rho + v_y\big|_{y=0}, \tag{43}$$

where the (linearized) velocity $\mathbf{v} = (v_x, v_y)$ and the (linearized) polarity $\mathbf{p} = (p_x, p_y)$ fields solve the system

$$\begin{cases} \mu(\Delta\mathbf{v} + \nabla \mathrm{div}\mathbf{v}) - \zeta \mathrm{div}(\mathbf{p}\otimes\mathbf{P} + \mathbf{P}\otimes\mathbf{p}) - \xi\mathbf{v} + \zeta_i\mathbf{p} = 0 & \text{for } y < 0, \\ \Delta\mathbf{p} = \mathbf{p} & \text{for } y < 0, \\ \mu(\partial_x v_y + \partial_y v_x) = -\zeta\rho', \quad 2\mu\partial_y v_y = -2\mu\partial^2_{yy} V_y\big|_{y=0}\rho + \gamma\rho'' & \text{for } y = 0, \\ p_x = -\rho',\ p_y = -\partial_y P_y\big|_{y=0}\rho = -\rho & \text{for } y = 0. \end{cases} \tag{44}$$

Next, we study the spectral properties of the operator $\mathcal{L}$ that amounts to finding solutions $\mathbf{v} = e^{iqx}\hat{\mathbf{v}}(y)$, $\mathbf{p} = e^{iqx}\hat{\mathbf{p}}(y)$ (Fourier modes) for $\rho(x) = e^{iqx}$, so that

$$\mathcal{L}e^{iqx} = \big(\tfrac{\zeta}{2\mu} + \hat{v}_y(0)\big)e^{iqx}. \tag{45}$$

We have

$$p_x = -iqe^{iqx+\sqrt{q^2+1}y}, \quad p_y = -e^{iqx+\sqrt{q^2+1}y}, \tag{46}$$

while components of $\mathbf{v}$ satisfy for $y < 0$ the equations

$$\begin{aligned} \mu(\Delta v_x + \partial_x \mathrm{div}\mathbf{v}) - \xi v_x &= iq\zeta_i e^{iqx+\sqrt{q^2+1}y} \\ &\quad - iq\zeta(\sqrt{q^2+1}+1)e^{iqx+(\sqrt{q^2+1}+1)y}, \end{aligned} \tag{47}$$

$$\begin{aligned} \mu(\Delta v_y + \partial_y \mathrm{div}\mathbf{v}) - \xi v_y &= \zeta_i e^{iqx+\sqrt{q^2+1}y} \\ &\quad + q^2\zeta e^{iqx+(\sqrt{q^2+1}+1)y} - 2\zeta(\sqrt{q^2+1}+1)e^{iqx+(\sqrt{q^2+1}+1)y} \end{aligned} \tag{48}$$

with boundary conditions on the line $y = 0$,

$$\mu(\partial_x v_y + \partial_y v_x) = -iq\zeta e^{iqx}, \tag{49}$$

$$2\mu\partial_y v_y = -\zeta\Big(\tfrac{\xi}{4\mu+\sqrt{2\mu\xi}} + 2\Big)e^{iqx} + \zeta_i \tfrac{\sqrt{2\mu}}{\sqrt{\xi}+\sqrt{2\mu}} e^{iqx} - q^2\gamma e^{iqx}. \tag{50}$$

Here we have used the formula

$$\partial^2_{yy} V_y\big|_{y=0} = \frac{\xi\zeta}{2\mu\big(4\mu + \sqrt{2\mu\xi}\big)} - \frac{\zeta_i}{\sqrt{2\mu\xi} + 2\mu} + \frac{\zeta}{\mu}$$

which follows from (38) and (40).

We represent the solution of (47)–(50) as the sum

$$\mathbf{v} = \zeta_i \mathbf{v}^t + \zeta\mathbf{v}^c + \gamma\mathbf{v}^s \tag{51}$$

of three terms caused by the traction force, the contractile stress and the surface tension. More precisely, $\mathbf{v^t}$ solves (47)–(50) with $\zeta_i = 1$, $\zeta = \gamma = 0$; for $\mathbf{v^c}$ we set $\zeta = 1$, $\zeta_i = \gamma = 0$; for $\mathbf{v^s}$ we set $\gamma = 1$, $\zeta = \zeta_i = 0$. Solutions of the corresponding problems (103), (104), (105) are found explicitly in Appendix A, so that

$$v^t_y\big|_{y=0} = \Lambda^t(q,\mu,\xi)\, e^{iqx},\;\; v^c_y\big|_{y=0} = \Lambda^c(q,\mu,\xi)\, e^{iqx},\;\; v^s_y\big|_{y=0} = \Lambda^s(q,\mu,\xi)\, e^{iqx}, \tag{52}$$

where

$$\begin{aligned}\Lambda^t(q,\mu,\xi) = &\frac{\xi\sqrt{q^2+\frac{\xi}{2\mu}}\Big(2\mu\big(1-\sqrt{q^2+1}\big)+\sqrt{2\mu\xi}\big(1-\sqrt{1+\frac{2\mu q^2}{\xi}}\big)\Big)}{D(q,\mu,\xi)\Big(\sqrt{q^2+1}+\sqrt{q^2+\frac{\xi}{2\mu}}\Big)\big(\sqrt{2\mu\xi}+2\mu\big)}\\ &+\frac{\xi q^2\big(1-\sqrt{q^2+1}\big)\sqrt{q^2+\frac{\xi}{2\mu}}}{D(q,\mu,\xi)\Big(\sqrt{q^2+\frac{\xi}{\mu}}+\sqrt{q^2+\frac{\xi}{2\mu}}\Big)\Big(\sqrt{q^2+1}+\sqrt{q^2+\frac{\xi}{2\mu}}\Big)\Big(\sqrt{q^2+\frac{\xi}{\mu}}+\sqrt{q^2+1}\Big)}\end{aligned} \tag{53}$$

$$\begin{aligned}\Lambda^c(q,\mu,\xi) = &\frac{\xi\sqrt{q^2+\frac{\xi}{2\mu}}}{D(q,\mu,\xi)\,\Big(\sqrt{q^2+1}+1+\sqrt{q^2+\frac{\xi}{2\mu}}\Big)}\Bigg(q^2-2\sqrt{q^2+1}-2\\ &+\frac{2q^4}{\big(\sqrt{q^2+1}+1+\sqrt{q^2+\frac{\xi}{\mu}}\big)\big(\sqrt{q^2+\frac{\xi}{2\mu}}+\sqrt{q^2+\frac{\xi}{\mu}}\big)}\Bigg)\\ &+\frac{\xi\sqrt{q^2+\frac{\xi}{2\mu}}}{D(q,\mu,\xi)}\Bigg(\tfrac{\xi}{4\mu+\sqrt{2\mu\xi}}+2-\frac{q^2}{\sqrt{q^2+\frac{\xi}{2\mu}}+\sqrt{q^2+\frac{\xi}{\mu}}}\Bigg),\end{aligned} \tag{54}$$

$$\Lambda^s(q,\mu,\xi) = \frac{\xi\sqrt{q^2+\frac{\xi}{2\mu}}}{D(q,\mu,\xi)}q^2, \tag{55}$$

$$D(q,\mu,\xi) = 4\mu^2 q^2 \sqrt{q^2+\tfrac{\xi}{2\mu}}\sqrt{q^2+\tfrac{\xi}{\mu}} - (2\mu q^2+\xi)^2, \tag{56}$$

Thus, according to (45) we have the following formula for the eigenvalue of the linearized operator corresponding to the mode e^{iqx}:

$$\Lambda(q,\mu,\zeta,\zeta_i,\xi,\gamma) = \zeta_i \Lambda^t(q,\mu,\xi) + \tfrac{\zeta}{2\mu}\left(1+2\mu\Lambda^c(q,\mu,\xi)\right) + \gamma\Lambda^s(q,\mu,\xi). \tag{57}$$

For simplicity hereafter we omit dependence on the parameters $\mu,\zeta,\zeta_i,\xi,\gamma$ and write $\Lambda(q)$ and $\Lambda^{t,c,s}(q)$. Notice that $\Lambda^c(q) \to -\frac{1}{4\mu}$, $\Lambda^s(q) = -\frac{|q|}{\mu} + O(\frac{1}{|q|})$ and $\Lambda^t(q) = O(\frac{1}{q^2})$ as $|q|\to\infty$, i.e.

$$\Lambda(q) = -\tfrac{\gamma}{\mu}|q| + \tfrac{\zeta}{4\mu} + O(\tfrac{1}{|q|}) \quad \text{as } |q|\to\infty. \tag{58}$$

Expanding Λ in a neighborhood of $q=0$ we get

$$\Lambda(q) = \left(\tfrac{\mu\zeta_i}{\xi(\sqrt{2\mu\xi}+2\mu)} + \tfrac{\zeta}{2\mu\xi}\left(3-2\sqrt{2}-\tfrac{2\sqrt{2\mu}(\sqrt{\xi}+\sqrt{2\mu})}{(2\sqrt{2\mu}+\sqrt{\xi})^2}\right) - \tfrac{\gamma}{\sqrt{2\mu\xi}}\right) q^2 + O(q^4). \tag{59}$$

Thus in a neighborhood of $q=0$, $\Lambda>0$ for large enough ζ_i while $\Lambda\to-\infty$ when $|q|\to\infty$. Then there exists a non-zero root $q=q_0>0$ of the transcendental equation

$$\Lambda(q) = 0. \tag{60}$$

We show below that this q_0 defines a critical period $\Pi_0 = 2\pi/q_0$ for which a bifurcation of nontrivial traveling wave solutions occurs.

5 Bifurcation of traveling wave solutions

Let $q_0>0$ be a solution of (60). We apply the celebrated theory of bifurcation from the simple eigenvalue (see Theorem 3) to show that there emerges a family of nontrivial traveling wave solutions with periods close to the critical period $\Pi_0 = 2\pi/q_0$.

It is convenient to pass from the prescribed period Π to another bifurcation parameter $\theta = \Pi/\Pi_0$ (scaling factor). Introduce new coordinates $x=\theta\tilde{x}$, $y=\theta\tilde{y}$ and change the unknowns $\tilde{\rho} = \frac{1}{\theta}\rho(\theta\tilde{x})$, $\tilde{\mathbf{v}} = \frac{1}{\theta}\mathbf{v}(\theta\tilde{x},\theta\tilde{y})$ and $\tilde{\mathbf{p}} = \mathbf{p}(\theta\tilde{x},\theta\tilde{y})$. This allows us to reduce the analysis to the fixed period Π_0, while the parameter θ appears in the rescaled version of problem (8)-(11) (where we drop ˜ to simplify the notation)

$$\begin{aligned}
&\Delta\mathbf{p} = \theta^2\mathbf{p} && \text{for } y<\rho(x),\\
&\mathbf{p} = \mathbf{n} && \text{for } y=\rho(x),\\
&\mu(\Delta\mathbf{v}+\nabla\mathrm{div}\mathbf{v}) - \zeta\mathrm{div}(\mathbf{p}\otimes\mathbf{p}) - \theta^2\xi\mathbf{v} + \theta\zeta_i\mathbf{p} = 0 && \text{for } y<\rho(x),\\
&\mu(\nabla\mathbf{v}+(\nabla\mathbf{v})^T)\mathbf{n} = (\zeta - \frac{\gamma}{\theta}\kappa)\mathbf{n} && \text{for } y=\rho(x).
\end{aligned} \tag{61}$$

Then the problem (41) of finding traveling wave solutions is equivalent to

$$\mathcal{A}(\rho,\theta) = (v_y - v_x\rho')\big|_{y=\rho(x)} = C_\rho, \tag{62}$$

where C_ρ is an unknown constant, and $\mathbf{v}$, $\mathbf{p}$ solve (61).

Notice that the linearized operator $\mathcal{L}(\theta)$ has the eigenvalue $\Lambda\left(\frac{q_0}{\theta}, \mu, \zeta, \zeta_i, \xi, \gamma\right)$ given by (57) whose corresponding eigenfunction is $\rho = e^{iq_0x}$. This eigenvalue becomes zero for $\theta = 1$. In this case, however, zero is a multiple eigenvalue since $\rho = 1$ and $\rho = e^{-iq_0x}$ are also eigenfunctions. To get rid of this multiplicity issue observe that $\mathcal{A}(\rho + C, \theta) = \mathcal{A}(\rho, \theta)$, $\forall C = \text{const}$. Therefore we can pass to the quotient spaces $C^{k,\delta}_{\#}(\Pi_0)/\mathbb{R}$ and $C^{k-1,\delta}_{\#}(\Pi_0)/\mathbb{R}$ identifying constant functions with zero. This eliminates the eigenfunction $\rho = 1$. Furthermore, the multiplicity can be reduced to one by assuming the natural symmetry $\rho(x) = \rho(-x)$.

In view of above mentioned we can apply the following

Theorem 3 (Crandall-Rabinowitz [6])**.** *Let X, Y be Banach spaces. Let $U \subset X$ be a neighborhood of* 0 *and let*

$$\Phi : U \times (1 - \theta_0, 1 + \theta_0) \to Y \tag{63}$$

have the following properties:

(i) $\Phi(0, \theta) = 0$ *for all* $\theta \in (1 - \theta_0, 1 + \theta_0)$,

(ii) $\Phi \in C^2(U \times (1 - \theta_0, 1 + \theta_0))$,

(iii) $\dim \operatorname{Ker}(\partial_x \Phi(0, 1)) = \operatorname{codim} \operatorname{Ran}(\partial_x \Phi(0, 0)) = 1$,

(iv) $\partial^2_{\theta x} \Phi(0, 1) x_0 \notin \operatorname{Ran}(\partial_x \Phi(0, 1))$ *where* $\operatorname{Ker}(\partial_x \Phi(0, 1)) = \operatorname{Span}\{x_0\}$.

Then if $\tilde{X}$ is any complement of $\operatorname{Span}\{x_0\}$ *in X, there exists $\varepsilon > 0$ and continuously differentiable functions $\psi : (-\varepsilon, \varepsilon) \to \mathbb{R}$ and $\phi : (-\varepsilon, \varepsilon) \to \tilde{X}$ such that $\phi(0) = 0$, $\psi(0) = 0$, and $\Phi(\alpha x_0 + \alpha\phi(\alpha), 1 + \psi(\alpha)) = 0 \quad \forall \alpha \in (-\varepsilon, \varepsilon)$. Moreover, $\Phi^{-1}(\{0\})$ near $(0, 1)$ consists precisely of the curves $x = 0$ and $(\alpha x_0 + \alpha\phi(\alpha), 1 + \psi(\alpha))$, $\alpha \in (-\varepsilon, \varepsilon)$*

Using Theorem 3 we establish bifurcation of nontrivial traveling wave solutions for the problem (8)–(11), (41).

Theorem 4. *Assume that equation* (60) *has a root $q_0 > 0$ and $\Lambda(jq_0) \neq 0$ for $j = 2, 3, \ldots$ Assume also that $\partial_q \Lambda(q_0) \neq 0$. Then there is a family of nontrivial (non-flat) traveling wave solutions $\rho = \rho(x, \alpha)$ of* (41) *with periods $\Pi = 2\pi\theta(\alpha)/q_0$, depending on a small parameter α. Moreover, $\rho(x, \alpha)$ and $\theta(\alpha)$ smoothly depend on the parameter α and $\rho(x, 0) = 0$, $\theta(0) = 1$.*

Proof. Recall that problem (41), where the prescribed period Π is considered as a bifurcation parameter, is reduced via rescaling with the factor $\theta > 0$ to equation (62) with fixed period $\Pi_0 = 2\pi/q_0$. Observe that for every even function $\rho \in C^{k,\delta}_{\#}(0, \Pi_0)$ there is a unique Π_0-periodic in x and vanishing as $y \to -\infty$ solution $\mathbf{v}$, $\mathbf{p}$ of the rescaled problem (61), and the symmetry

$$\begin{aligned} v_y(-x, y) &= v_y(x, y), & p_y(-x, y) &= p_y(x, y), \\ v_x(-x, y) &= -v_x(x, y), & p_x(-x, y) &= -p_x(x, y) \end{aligned}$$

holds. Thus we can apply Theorem 3 to the family of operators $\mathcal{A}(\rho,\theta)$ with X and Y being subspaces of $C^{k,\delta}_{\#}(0,\Pi_0)/\mathbb{R}$ and $C^{k-1,\delta}_{\#}(0,\Pi_0)/\mathbb{R}$ $(0<\delta<1, k=2,3,\dots)$ of even functions.

The flat front traveling wave solution for $\rho=0$ constructed in Section 4 satisfies the condition (i) of Theorem 3. By virtue of Theorem 1, (ii) is also satisfied. Since the linearized operator $\mathcal{L}(\theta)$ has the following spectral representation:

$$\mathcal{L}(\theta) : \cos jq_0x \mapsto \Lambda\left(\tfrac{jq_0}{\theta}\right)\cos jq_0x, \quad j=1,2,\dots, \tag{64}$$

the kernel of $\mathcal{L}=\mathcal{L}(1)$ is one-dimensional and is spanned by $\{\cos q_0x\}$.

We claim that

$$\mathrm{Ran}\,(\mathcal{L}) = \left\{\rho\in C^{k-1,\delta}_{\#}(0,\Pi_0)/\mathbb{R}\,\middle|\,\rho \text{ is even, } \int_{\Pi_0}\rho\cos q_0x dx=0\right\}. \tag{65}$$

Indeed, consider the equation $\mathcal{L}\varrho=\rho$, where ρ belongs to the space given by the right-hand side of (65). Then expanding ρ into Fourier series $\rho=\sum_{j\geq 2}c_j\cos jq_0x$ we have

$$\begin{aligned}\varrho = \sum_{j\geq 2}\frac{c_j}{\Lambda(jq_0)}\cos jq_0x = &-\sum_{j\geq 2}\left(\frac{\mu}{\gamma q_0 j}+\frac{\mu\zeta}{4(\gamma q_0 j)^2}\right)c_j\cos jq_0x\\ &+\sum_{j\geq 2}\left(\frac{1}{\Lambda(jq_0)}+\frac{\mu}{\gamma q_0 j}+\frac{\mu\zeta}{4(\gamma q_0 j)^2}\right)c_j\cos jq_0x.\end{aligned} \tag{66}$$

Let us show that $\varrho\in C^{k,\delta}_{\#}(0,\Pi_0)$. It follows from (58) that the second term in the right-hand side of (66) belongs to $W^{k+2,2}_{\#}(0,\Pi_0)$ and hence to $C^{k,\delta}_{\#}(0,\Pi_0)$. Using the Sokhotski-Plemelj formulas another term can be represented as

$$-\frac{\mu}{\gamma}\mathcal{K}\rho-\frac{\mu\zeta}{4\gamma^2}\mathcal{K}^2\rho, \quad \text{where } \mathcal{K}\rho=\frac{1}{\Pi_0}\text{p.v.}\int_0^{\Pi_0}\cot\tfrac{q_0(x-z)}{2}\int_0^z\rho(s)dsdz. \tag{67}$$

Since the Hilbert transform involved in (67) continuously maps $C^{k,\delta}_{\#}(0,\Pi_0)$ to $C^{k,\delta}_{\#}(0,\Pi_0)$, the first term of right-hand side of (66) also belongs to $C^{k,\delta}_{\#}(0,\Pi_0)$.

And finally, the transversality condition (iv) is satisfied since

$$\partial_\theta\mathcal{L}(1)\cos q_0x = -q_0\partial_q\Lambda(q_0)\cos q_0x\notin \mathrm{Ran}\,(\mathcal{L})\,. \tag{68}$$

Thus all of the conditions of Theorem 3 are fulfilled. Also according to Theorem 1.18 from [6], $\rho(x,\alpha)$ and $\theta(\alpha)$ are infinitely differentiable in α functions. □

Remark 1. *Note that traveling wave problem* (62) *is invariant with respect to shifts in x-axis, moreover, for any even solution ρ of* (62)*, its shift by the half-period is still an even solution. Thus we can assume that $\forall\alpha$ $\rho(x,-\alpha)=\rho(x-\Pi_0/2,\alpha)$, $\theta(-\alpha)=\theta(\alpha)$.*

Now we address the issue of stability of traveling wave solutions. To this end we apply the following result obtained in [7].

Theorem 5. *Assume that conditions of Theorem 3 are fulfilled and $X \subseteq Y$ with continuous embedding. Then for sufficiently small α there exists the smallest in absolute value simple eigenvalue $\lambda(\alpha)$ of the linearized operator $\partial_x \Phi(\alpha x_0 + \alpha\phi(\alpha), 1 + \psi(\alpha))$ and*

$$\lambda(\alpha) = -\alpha\tilde{\lambda}'(1)\theta'(\alpha)(1 + O(\alpha)) \text{ as } \alpha \to 0, \tag{69}$$

where $\tilde{\lambda}(\theta)$ is the smallest in absolute value eigenvalue of $\partial_x \Phi(0, \theta)$.

Consider traveling wave solution and linearized operator $\partial_\rho \mathcal{A}(\rho, \theta(\alpha))$ near the bifurcation point, i.e. when $|\alpha|$ is small. Its spectrum has the following structure. There is a zero eigenvalue of multiplicity two corresponding to infinitesimal shifts with eigenfunctions equal 1 (vertical shifts) and ρ' (horizontal shifts) respectively. By Theorem 5 the smallest in absolute value nonzero eigenvalue of $\partial_\rho \mathcal{A}(\rho, \theta(\alpha))$ is given by the asymptotic formula

$$\lambda(\alpha) = q_0 \partial_q \Lambda(q_0)\alpha\theta'(\alpha)(1 + O(\alpha)) \text{ as } \alpha \to 0, \tag{70}$$

so the sign of $\lambda(\alpha)$ is determined by that of $\alpha\theta'(\alpha)$. Other eigenvalues either remain bounded and converge as $\alpha \to 0$ to those of $\partial_\rho \mathcal{A}(0, 1)$, or have a negative sufficiently large in absolute value imaginary part (and therefore do not affect stability).

6 Asymptotic expansions of traveling wave solutions near the bifurcation point

Let $q_0 > 0$ be a solution of (60), and assume that other conditions of Theorem 4 are satisfied. Then we have a family of Π_0-periodic ($\Pi_0 = 2\pi/q_0$) traveling wave solutions $\rho(x, \alpha)$ of problem (62). The corresponding polarisation and velocity fields, $\mathbf{p}(x, y, \alpha)$, $\mathbf{v}(x, y, \alpha)$ are unique solutions of (61).

For $\alpha = 0$ we have flat front solution $\rho(x, 0) = 0$, $\theta(0) = 1$, and $\mathbf{p}(x, y, 0) = \mathbf{P}(y)$, $\mathbf{v}(x, y, 0) = \mathbf{V}(y)$, where $\mathbf{P}(y) = (0, e^y)$ and $\mathbf{V}(y) = (0, V_y(y))$ with $V_y(y)$ given by (39). Now, for small α we consider asymptotic expansions

$$\theta(\alpha) = 1 + b\alpha^2 + O(\alpha^4) \tag{71}$$

$$C_\rho = V^{(0)} + \alpha^2 V^{(2)} + O(\alpha^4), \tag{72}$$

$$\rho(x, \alpha) = \alpha\rho^{(1)}(x) + \alpha^2\rho^{(2)}(x) + \alpha^3\rho^{(3)}(x) + O(\alpha^4), \tag{73}$$

$$\mathbf{p}(x, y, \alpha) = \mathbf{P}(y) + \alpha\mathbf{p}^{(1)}(x, y) + \alpha^2\mathbf{p}^{(2)}(x, y) + \alpha^3\mathbf{p}^{(3)}(x, y) + O(\alpha^4), \tag{74}$$

$$\mathbf{v}(x, y, \alpha) = \mathbf{V}(y) + \alpha\mathbf{v}^{(1)}(x, y) + \alpha^2\mathbf{v}^{(2)}(x, y) + \alpha^3\mathbf{v}^{(3)}(x, y) + O(\alpha^4), \tag{75}$$

where $V^{(0)}$ is given by (40), $\rho^{(k)}(x)$ are even functions with zero mean value, and in view of Remark 1, the expansions (71)–(72) contain only even powers of α. Our main objective here is to identify the sign of the parameter b that is decisive for the stability of the traveling wave solution.

We substitute the expansions above into (61)–(62) and equate the terms corresponding to the same powers of α. At the order α we arrive at the linearized problem (44) for $\mathbf{p} = \mathbf{p}^{(1)}$ and $\mathbf{v} = \mathbf{v}^{(1)}$ coupled with the equation $\mathcal{L}\rho^{(1)} = 0$ for Π_0-periodic even fuction $\rho^{(1)}$. According to spectral analysis of the operator $\mathcal{L}$ (Section 3) we have, up to a multiplicative constant,

$$\rho^{(1)} = \cos q_0 x, \tag{76}$$

and

$$p_x^{(1)} = q_0 e^{y\sqrt{q_0^2+1}} \sin q_0 x,\ \ p_y^{(1)} = -e^{y\sqrt{q_0^2+1}} \cos q_0 x, \tag{77}$$

while $\mathbf{v}^{(1)}$ is the real part of $\mathbf{v} = \zeta_i \mathbf{v}^t + \zeta \mathbf{v}^c + \gamma \mathbf{v}^s$ explicitely found in Appendix A.

Next, equating terms of the order α^2 in (61) we find the following system

$$\Delta \mathbf{p}^{(2)} = \mathbf{p}^{(2)} + 2b\mathbf{P} \qquad \text{for } y<0, \tag{78}$$

$$p_x^{(2)} = -\partial_y p_x^{(1)} \rho^{(1)} - [\rho^{(2)}]' \qquad \text{for } y=0, \tag{79}$$

$$p_y^{(2)} = -\partial_y P_y \rho^{(2)} - \tfrac{1}{2}\partial^2_{yy} P_y [\rho^{(1)}]^2 - \partial_y p_y^{(1)} \rho^{(1)} - \tfrac{1}{2}{[\rho^{(1)}]'}^2 \qquad \text{for } y=0, \tag{80}$$

$$\begin{aligned}
&\mu(\Delta \mathbf{v}^{(2)} + \nabla \mathrm{div} \mathbf{v}^{(2)}) - \zeta \mathrm{div}\big(\mathbf{p}^{(2)} \otimes \mathbf{P} + \mathbf{P} \otimes \mathbf{p}^{(2)}\big) \\
&\qquad - \zeta \mathrm{div}\big(\mathbf{p}^{(1)} \otimes \mathbf{p}^{(1)}\big) - \xi \mathbf{v}^{(2)} - 2\xi b \mathbf{V} + \zeta_i \mathbf{p}^{(2)} + \zeta_i b \mathbf{P} = 0 \qquad \text{for } y<0, \qquad (81)\\
&\mu\big(\partial_x v_y^{(2)} + \partial_y v_x^{(2)}\big) = 2\mu \partial_x v_x^{(1)} [\rho^{(1)}]' - \mu\big(\partial^2_{xy} v_y^{(1)} + \partial^2_{yy} v_x^{(1)}\big)\rho^{(1)} \\
&\qquad\qquad - \zeta [\rho^{(2)}]' - \gamma [\rho^{(1)}]'[\rho^{(1)}]'' \qquad \text{for } y=0, \qquad (82)\\
&2\mu \partial_y v_y^{(2)} = \mu\big(\partial_x v_y^{(1)} + \partial_y v_x^{(1)}\big)[\rho^{(1)}]' - 2\mu \partial^2_{yy} V_y \rho^{(2)} - \mu \partial^3_{yyy} V_y [\rho^{(1)}]^2 \\
&\qquad - 2\mu \partial^2_{yy} v_y^{(1)} \rho^{(1)} + \gamma [\rho^{(2)}]'' \qquad \text{for } y=0, \qquad (83)
\end{aligned}$$

and from (62) we get

$$v_y^{(2)} + \partial_y V_y \rho^{(2)} = -\tfrac{1}{2}\partial^2_{yy} V_y [\rho^{(1)}]^2 + v_x^{(1)} [\rho^{(1)}]' - \partial_y v_y^{(1)} \rho^{(1)} + V^{(2)} \quad \text{for } y=0. \tag{84}$$

Observe that $\mathbf{p}^{(2)}$ can be represented as

$$\mathbf{p}^{(2)} = \mathbf{p}^{(21)} + \mathbf{p}^{(22)}, \tag{85}$$

where

$$p_x^{(21)} = -\tfrac{q_0\sqrt{q_0^2+1}}{2}\, e^{y\sqrt{4q_0^2+1}}\, \sin 2q_0 x, \tag{86}$$

$$p_y^{(21)} = \tfrac{q_0^2 + 2\sqrt{q_0^2+1} - 1}{4}\, e^{y\sqrt{4q_0^2+1}} \cos 2q_0 x + \Big(by + \tfrac{2\sqrt{q_0^2+1} - q_0^2 - 1}{4}\Big) e^y, \tag{87}$$

and $\mathbf{p}^{(22)}$ solves the problem

$$\begin{cases} \Delta \mathbf{p}^{(22)} = \mathbf{p}^{(22)} & \text{for } y<0, \\ p_x^{(22)} = -[\rho^{(2)}]', \quad p_y^{(22)} = -\rho^{(2)} & \text{for } y=0. \end{cases} \tag{88}$$

Bearing in mind (85) we represent the vector $\mathbf{v}^{(2)}$ as

$$\mathbf{v}^{(2)} = \mathbf{v}^{(21)} + \mathbf{v}^{(22)}, \tag{89}$$

with $\mathbf{v}^{(21)}$ found explicitly in Appendix B and $\mathbf{v}^{(22)}$ satisfying

$$\begin{cases} \mu(\Delta\mathbf{v}^{(22)} + \nabla\mathrm{div}\mathbf{v}^{(22)}) - \xi\mathbf{v}^{(22)} & \\ \qquad -\zeta\mathrm{div}\big(\mathbf{p}^{(22)}\otimes\mathbf{P} + \mathbf{P}\otimes\mathbf{p}^{(22)}\big) + \zeta_i\mathbf{p}^{(22)} = 0 & \text{for } y<0, \\ \mu\big(\partial_x v_y^{(22)} + \partial_y v_x^{(22)}\big) = -\zeta[\rho^{(2)}]', & \text{for } y=0, \\ 2\mu\partial_y v_y^{(22)} = -2\mu\partial^2_{yy}V_y\rho^{(2)} + \gamma[\rho^{(2)}]'' & \text{for } y=0. \end{cases} \tag{90}$$

Then from (88), (90) we have

$$v_y^{(22)} = -\frac{\zeta}{2\mu}\rho^{(2)} + \mathcal{L}\rho^{(2)}. \tag{91}$$

Substituting expressions $\rho^{(1)}$, $\mathbf{v}^{(1)}$, $v_y^{(21)}$ into (84) and taking into account (91) we conclude that $\mathcal{L}\rho^{(2)} \in Span\{1, \cos 2q_0x\}$. This in turn implies that $\rho^{(2)} = \beta\cos 2q_0x$ and we find that constants β and $V^{(2)}$ are given by formulas (130) and (131), see Appendix B. Having established $\rho^{(2)}$ we get the following explicit form of $\mathbf{p}^{(22)}$:

$$p_x^{(22)} = 2\beta q_0 e^{y\sqrt{4q_0^2+1}}\sin 2q_0x, \quad p_y^{(22)} = -\beta e^{y\sqrt{4q_0^2+1}}\cos 2q_0x. \tag{92}$$

Now considering term of the order α^3 in (61) we get

$$\Delta\mathbf{p}^{(3)} = \mathbf{p}^{(3)} + 2b\mathbf{p}^{(1)}, \qquad \text{for } y<0, \tag{93}$$

$$\begin{aligned} p_x^{(3)} &= -\partial_y p_x^{(1)}\rho^{(2)} - \partial_y p_x^{(2)}\rho^{(1)} - \tfrac{1}{2}\partial^2_{yy}p_x^{(1)}[\rho^{(1)}]^2 - [\rho^{(3)}]' + \tfrac{1}{2}{[\rho^{(1)}]'}^3, \\ p_y^{(3)} &= -\partial_y p_y^{(1)}\rho^{(2)} - \partial_y p_y^{(2)}\rho^{(1)} - \tfrac{1}{2}\partial^2_{yy}p_y^{(1)}\,[\rho^{(1)}]^2 - \partial_y P_y\,\rho^{(3)} - \partial^2_{yy}P_y\,\rho^{(1)}\rho^{(2)} \\ &\quad - \tfrac{1}{6}\partial^3_{yyy}P_y\,[\rho^{(1)}]^3 - [\rho^{(1)}]'[\rho^{(2)}]', \qquad \text{for } y=0; \end{aligned} \tag{94}$$

$$\begin{aligned} &\mu(\Delta\mathbf{v}^{(3)} + \nabla\mathrm{div}\mathbf{v}^{(3)}) - \zeta\mathrm{div}\left(\mathbf{p}^{(3)}\otimes\mathbf{P} + \mathbf{P}\otimes\mathbf{p}^{(3)} + \mathbf{p}^{(1)}\otimes\mathbf{p}^{(2)} + \mathbf{p}^{(2)}\otimes\mathbf{p}^{(1)}\right), \\ &\qquad - \xi\mathbf{v}^{(3)} - 2\xi b\mathbf{v}^{(1)} + \zeta_i\mathbf{p}^{(3)} + \zeta_i b\mathbf{p}^{(1)} = 0 \qquad \text{for } y<0, \end{aligned} \tag{95}$$

and for $y=0$ we have

$$\begin{aligned} \mu\big(\partial_x v_y^{(3)} + \partial_y v_x^{(3)}\big) &= 2\mu\partial^2_{xy}v_x^{(1)}\rho^{(1)}[\rho^{(1)}]' + 2\mu\partial_x v_x^{(1)}[\rho^{(2)}]' + 2\mu\partial_x v_x^{(2)}[\rho^{(1)}]' \\ &- \mu\big(\partial^2_{xy}v_y^{(1)} + \partial^2_{yy}v_x^{(1)}\big)\rho^{(2)} - \tfrac{\mu}{2}\big(\partial^3_{xyy}v_y^{(1)} + \partial^3_{yyy}v_x^{(1)}\big)[\rho^{(1)}]^2 \\ &- \mu\big(\partial^2_{xy}v_y^{(2)} + \partial^2_{yy}v_x^{(2)}\big)\rho^{(1)} - \zeta[\rho^{(3)}]' - \gamma[\rho^{(1)}]''[\rho^{(2)}]' - \gamma[\rho^{(2)}]''[\rho^{(1)}]', \end{aligned} \tag{96}$$

$$\begin{aligned} 2\mu\partial_y v_y^{(3)} &= \mu\big(\partial_x v_y^{(1)} + \partial_y v_x^{(1)}\big)[\rho^{(2)}]' + \mu\big(\partial^2_{xy}v_y^{(1)} + \partial^2_{yy}v_x^{(1)}\big)\rho^{(1)}[\rho^{(1)}]' \\ &+ \mu\big(\partial_x v_y^{(2)} + \partial_y v_x^{(2)}\big)[\rho^{(1)}]' - 2\mu\partial^2_{yy}V_y\rho^{(3)} - 2\mu\partial^3_{yyy}V_y\rho^{(1)}\rho^{(2)} \\ &- \tfrac{\mu}{3}\partial^4_{yyyy}V_y[\rho^{(1)}]^3 - 2\mu\partial^2_{yy}v_y^{(1)}\rho^{(2)} - \mu\partial^3_{yyy}v_y^{(1)}[\rho^{(1)}]^2 - 2\mu\partial^2_{yy}v_y^{(2)}\rho^{(1)} \\ &+ \gamma[\rho^{(3)}]'' - \gamma b[\rho^{(1)}]'' - \tfrac{3}{2}\gamma{[\rho^{(1)}]'}^2[\rho^{(1)}]'', \end{aligned} \tag{97}$$

Finally, collecting terms of the order α^3 in (62) we derive that for $y = 0$

$$v_y^{(3)} + \partial_y V_y \rho^{(3)} + \partial_{yy}^2 V_y \rho^{(1)} \rho^{(2)} + \tfrac{1}{6} \partial_{yyy}^3 V_y [\rho^{(1)}]^3 + \partial_y v_y^{(1)} \rho^{(2)} + \tfrac{1}{2} \partial_{yy}^2 v_y^{(1)} [\rho^{(1)}]^2 \\ + \partial_y v_y^{(2)} \rho^{(1)} - \partial_y v_x^{(1)} \rho^{(1)} [\rho^{(1)}]' - v_x^{(2)} [\rho^{(1)}]' - v_x^{(1)} [\rho^{(2)}]' = 0. \tag{98}$$

Similarly to the previous two steps the solution of (93)–(97) can be found in an explicit form. However, this task is rather cumbersome. Instead, we only identify the parameter b appearing in the expansion (71) of θ, whose sign determines the bifurcation type, $b > 0$ ($b < 0$) corresponds to the supercritical (subcritical) bifurcation (see (70) and Figure 2). We notice that calculations of the Fourier coefficient corresponding to $\cos q_0 x$ in (98) give a linear function of b, $k_1 b + k_2$, moreover

$$k_1 \cos q_0 x = \partial_\theta \mathcal{L}(1) \cos q_0 x = \partial_\theta \Lambda(q_0/\theta)\Big|_{\theta=1} \cos q_0 x = -q_0 \partial_q \Lambda(q_0) \cos q_0 x. \tag{99}$$

To find k_2 first we represent $\mathbf{p}^{(3)}$, $\mathbf{v}^{(3)}$ as

$$\mathbf{p}^{(3)} = \mathbf{p}^{(31)} + \mathbf{p}^{(32)}, \quad \mathbf{v}^{(3)} = \mathbf{v}^{(31)} + \mathbf{v}^{(32)}, \tag{100}$$

where $\mathbf{p}^{(32)}$ and $\mathbf{v}^{(32)}$ are solutions of problem (44) with $\rho = \rho^{(3)}$. Observe that $\left(v_y^{(32)} + \partial_y V_y \rho^{(3)}\right)\Big|_{y=0}$ is orthogonal to $\cos q_0 x$ in $L^2(0, \Pi_0)$. Next we write $\mathbf{p}^{(31)}$ and $\mathbf{v}^{(31)}$ in the form

$$\mathbf{p}^{(31)} = \mathbf{p}^{(311)} + \mathbf{p}^{(312)}, \quad \mathbf{v}^{(31)} = \mathbf{v}^{(311)} + \mathbf{v}^{(312)}, \tag{101}$$

where

$$v_x^{(311)} = \hat{v}_x^{(311)}(y) \sin q_0 x, \quad v_y^{(311)} = \hat{v}_y^{(311)}(y) \cos q_0 x,$$

while $p_x^{(312)}$ and $v_x^{(312)}$ ($p_y^{(312)}$ and $v_y^{(312)}$) absorbs all the terms that contain the factor $\sin 3q_0 x$ ($\cos 3q_0 x$) or/and b. An explicit form (147)–(148) of the vector function $\mathbf{v}^{(311)}$ is found in Appendixes C, D. Then considering the Fourier coefficients of functions in (98) we find (see Appendix E) that for $y = 0$

$$v_y^{(311)} + \Bigg(\beta D_1 + \frac{D_2}{8\mu\sqrt{4q_0^2 + \frac{\xi}{2\mu}}} + \frac{1}{4\mu}\Big(-\frac{\zeta}{2} + \zeta\sqrt{q_0^2+1} - \frac{\zeta_i}{8} + \frac{\zeta\xi}{16\mu} - \frac{5\zeta q_0^2}{8} - \beta\gamma q_0^2\Big) \\ + \frac{3\beta}{4}\Big(\frac{\xi\zeta}{2\mu(4\mu + \sqrt{2\mu\xi})} - \frac{\zeta_i}{\sqrt{2\mu\xi} + 2\mu} + \frac{\zeta}{\mu}\Big) - b q_0 \partial_q \Lambda(q_0)\Bigg) \cos q_0 x = 0, \tag{102}$$

where D_1 and D_2 are defined by (122) and (138). Substituting (148) into (102) we get equation (170) for finding b.

7 Conclusions

In this section, we present and discuss several numerical results relevant to the bifurcation of nonflat traveling waves. Computations are done for the typical value of the characteristic length $L_c = 25\,\mu m$ [2] by adjusting to the case of the unit length via a spatial rescaling.

The finger-like pattern in the shape of the traveling wave solution is shown in Figure 1, where the approximate shape corresponding to the two-term expansion $\rho = \alpha\rho^{(1)} + \alpha^2\rho^{(2)}$ ($\alpha = 0.5$) is depicted. The shape is computed by using explicit formulas (111) for $\rho^{(1)}$ and (135) for $\rho^{(2)}$, taking some typical values of parameters [2].

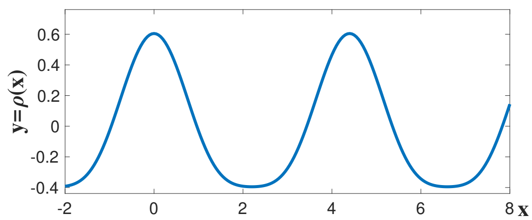

Figure 1: Approximate shape of the traveling wave.

Figure 2 depicts graphs of the eigenvalue $\Lambda(q)$ (growth rate, computed by the formula (57)) for different values of the intercellular contractility $-\zeta$.

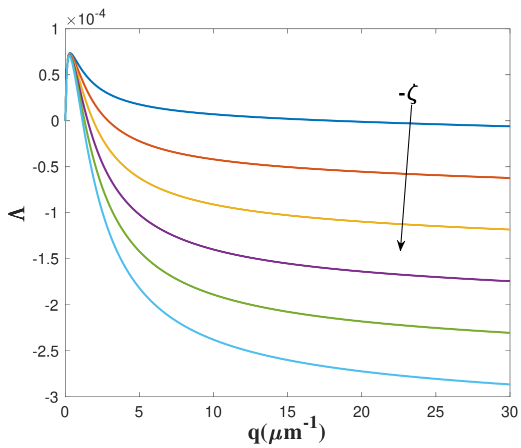

Figure 2: Growth rate $\Lambda(q)$ as a function of wave number q for different contractilities $-\zeta$. For this plot $-\zeta = 0,\ 6,\ 12,\ 18,\ 24,\ 30$ (kPa). Other parameters are $\zeta_i = 0.1\,\mathrm{kPa}/\mu\mathrm{m}$, $\xi = 100\,\mathrm{Pa}\cdot\mathrm{s}/\mu\mathrm{m}^2$, $\gamma = 0.2\,\mathrm{mN/m}$, $\mu = 25\,\mathrm{MPa}\cdot\mathrm{s}$.

Next, we study the dependence of the critical period $\Pi_0 = \frac{2\pi}{q_0}$ on the contractility. It amounts to numerical solving of the equation $\Lambda(q_0) = 0$. The results are depicted in the Figure 3. Note that for contractility $-\zeta \sim 20$ kPa the value of the period is close to one hundred micrometers which is in agreement with the measured finger spacing [24].

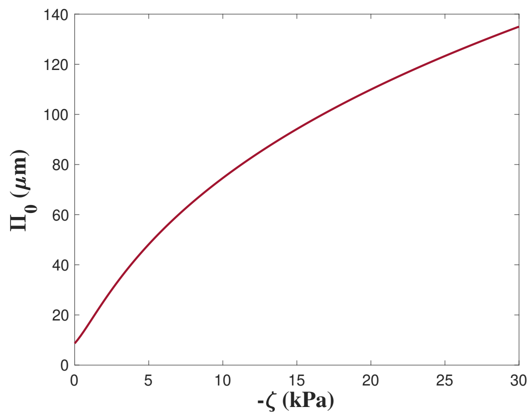

Figure 3: Dependence of the critical period on the contractility coefficient $-\zeta$. Computations carried out for $\zeta_i = 0.1\,\mathrm{kPa}/\mu\mathrm{m}$, $\xi = 100\,\mathrm{Pa}\cdot\mathrm{s}/\mu\mathrm{m}^2$, $\gamma = 0.2\,\mathrm{mN/m}$, $\mu = 25\,\mathrm{MPa}\cdot\mathrm{s}$

Finally Figures 4 and 5 present results of computations of the coefficient b. Recall that b is the coefficient in the asymptotic expansion (71) of $\theta(\alpha) = \Pi/\Pi_0$. It follows from (70) that the sign of the smallest in absolute value eigenvalue of the operator linearized around the traveling wave solution coincides (for sufficiently small α) with the sign of the product $b\partial_q\Lambda(q_0)$, while other nonzero eigenvalues have a negative real part. Since $\partial_q\Lambda(q_0) < 0$ (see Figure 2), $b > 0$ correspond to stable case, while $b < 0$ correspond to unstable case. In other words, for $b > 0$ we have a supercritical bifurcation, while for $b < 0$ we have a subcritical one. Notice that $b < 0$ for large values of the contractility as seen from Figure 4.

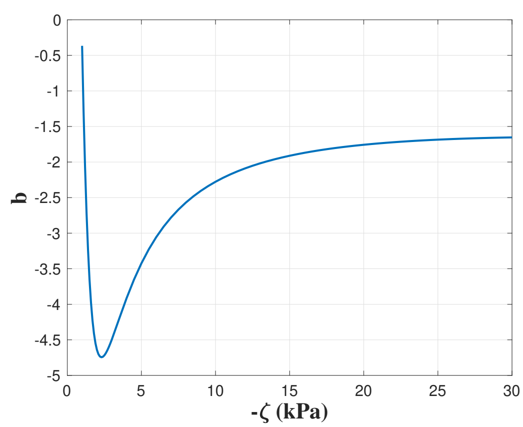

Figure 4: Coefficient b for contractility $-\zeta$ from 1 kPa to 30 kPa. Other parameters are $\zeta_i = 0.1\,\mathrm{kPa}/\mu\mathrm{m}$, $\xi = 100\,\mathrm{Pa}\cdot\mathrm{s}/\mu\mathrm{m}^2$, $\gamma = 0.2\,\mathrm{mN/m}$, $\mu = 25\,\mathrm{MPa}\cdot\mathrm{s}$.

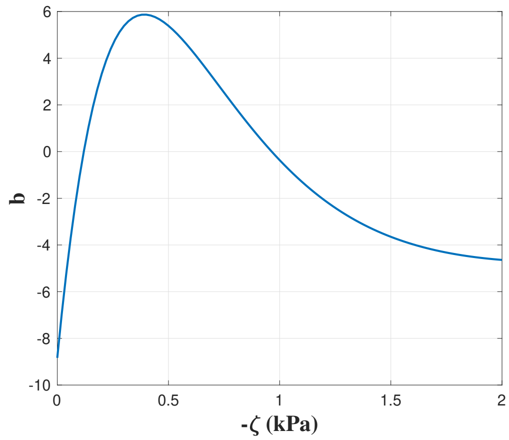

Figure 5: Coefficient b for contractility $-\zeta$ from 0 to 2 kPa. Other parameters are $\zeta_i = 0.1\,\mathrm{kPa}/\mu\mathrm{m}$, $\xi = 100\,\mathrm{Pa}\cdot\mathrm{s}/\mu\mathrm{m}^2$, $\gamma = 0.2\,\mathrm{mN/m}$, $\mu = 25\,\mathrm{MPa}\cdot\mathrm{s}$.

It is interesting to observe in Figure 5 that for smaller $-\zeta$ both cases $b > 0$ and $b < 0$ occur. Thus the model exhibits both subcritical and supercritical bifurcation.

8 Acknowledgments

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Appendix A Fourier analysis of the linearized operator

The solution of (47)-(50) can be represented as $\mathbf{v} = \zeta_i \mathbf{v}^t + \zeta \mathbf{v}^c + \gamma \mathbf{v}^s$ with $\mathbf{v}^t$, $\mathbf{v}^c$ and $\mathbf{v}^s$ solving

$$\begin{cases} \mu(\Delta v_x^t + \partial_x \mathrm{div}\mathbf{v}^t) - \xi v_x^t = iqe^{iqx+\sqrt{q^2+1}y} & \text{for } y < 0, \\ \mu(\Delta v_y^t + \partial_y \mathrm{div}\mathbf{v}^t) - \xi v_y^t = e^{iqx+\sqrt{q^2+1}y} & \text{for } y < 0, \\ \mu(\partial_x v_y^t + \partial_y v_x^t) = 0 & \text{for } y = 0, \\ 2\mu \partial_y v_y^t = \frac{\sqrt{2\mu}}{\sqrt{\xi}+\sqrt{2\mu}} e^{iqx} & \text{for } y = 0, \end{cases} \tag{103}$$

$$\begin{cases} \mu(\Delta v_x^c + \partial_x \mathrm{div}\mathbf{v}^c) - \xi v_x^c = -iq(\sqrt{q^2+1}+1)e^{iqx+y+y\sqrt{q^2+1}} & \text{for } y<0, \\ \mu(\Delta v_y^c + \partial_y \mathrm{div}\mathbf{v}^c) - \xi v_y^c & \\ \quad = q^2 e^{iqx+y+y\sqrt{q^2+1}} - 2(\sqrt{q^2+1}+1)e^{iqx+y+y\sqrt{q^2+1}} & \text{for } y<0, \\ \mu(\partial_x v_y^c + \partial_y v_x^c) = -iqe^{iqx} & \text{for } y=0, \\ 2\mu\partial_y v_y^c = -\left(\frac{\xi}{4\mu+\sqrt{2\mu\xi}}+2\right)e^{iqx} & \text{for } y=0, \end{cases} \tag{104}$$

$$\begin{cases} \mu(\Delta\mathbf{v}^s + \nabla\mathrm{div}\mathbf{v}^s) - \xi\mathbf{v}^s = 0 & \text{for } y<0, \\ \mu(\partial_x v_y^s + \partial_y v_x^s) = 0 & \text{for } y=0, \\ 2\mu\partial_y v_y^s = -q^2 e^{iqx} & \text{for } y=0. \end{cases} \tag{105}$$

We find explicit solutions to these problems, starting with problem (103). Represent the equations in (103) as

$$\begin{aligned} &\mu(\Delta\mathbf{v}^t + \nabla\mathrm{div}\mathbf{v}^t) - \xi\mathbf{v}^t \\ &\quad = \left(\sqrt{q^2+1}-q^2\right)\nabla e^{iqx+\sqrt{q^2+1}y} + iq\left(1-\sqrt{q^2+1}\right)\nabla^\perp e^{iqx+\sqrt{q^2+1}y}, \end{aligned} \tag{106}$$

where $\nabla^\perp = (-\partial_y, \partial_x)$. Then

$$\begin{aligned} \mathbf{v}^t = &\tfrac{-q^2+\sqrt{q^2+1}}{2\mu-\xi}\nabla e^{iqx+\sqrt{q^2+1}y} + iq\tfrac{1-\sqrt{q^2+1}}{\mu-\xi}\nabla^\perp e^{iqx+\sqrt{q^2+1}y} \\ &+ A_1\tilde{\mathbf{v}}^{(1)} + A_2\tilde{\mathbf{v}}^{(2)}, \end{aligned} \tag{107}$$

where

$$\tilde{\mathbf{v}}^{(1)} = \nabla^\perp e^{iqx+y\sqrt{q^2+\xi/\mu}}, \quad \tilde{\mathbf{v}}^{(2)} = \nabla e^{iqx+y\sqrt{q^2+\xi/(2\mu)}}$$

are linearly independent solutions of the corresponding homogenous equations, vanishing as $y \to -\infty$. Substituting (107) into boundary conditions of (103) we get a linear system for constants A_1, A_2, resolving which we obtain

$$\begin{aligned} \mathbf{v}^t = &\tfrac{\sqrt{q^2+1}-q^2}{2\mu-\xi}\left(\nabla e^{iqx+y\sqrt{q^2+1}} + A_{111}\tilde{\mathbf{v}}^{(1)} + A_{112}\tilde{\mathbf{v}}^{(2)}\right) \\ &+ iq\tfrac{\sqrt{q^2+1}-1}{\mu-\xi}\left(\nabla^\perp e^{iqx+y\sqrt{q^2+1}} + A_{121}\tilde{\mathbf{v}}^{(1)} + A_{122}\tilde{\mathbf{v}}^{(2)}\right) \\ &+ A_{131}\tilde{\mathbf{v}}^{(1)} + A_{132}\tilde{\mathbf{v}}^{(2)} \end{aligned} \tag{108}$$

with

$$A_{111} = -\tfrac{2\mu iq\sqrt{q^2+1}}{D(q,\mu,\xi)}\left(2\mu q^2 + \xi - 2\mu\sqrt{q^2+\xi/(2\mu)}\sqrt{q^2+1}\right),$$

$$A_{112} = \tfrac{2\mu\sqrt{q^2+1}}{D(q,\mu,\xi)}\left((2\mu q^2+\xi)\sqrt{q^2+1} - 2\mu q^2\sqrt{q^2+\xi/\mu}\right),$$

$$A_{121} = \tfrac{2\mu^2}{D(q,\mu,\xi)}\sqrt{q^2+\xi/(2\mu)}\left((2q^2+1)\sqrt{q^2+\xi/(2\mu)} - 2q^2\sqrt{q^2+1}\right),$$

$$A_{122} = -\tfrac{iq\mu}{D(q,\mu,\xi)}\left(2\mu(2q^2+1)\sqrt{q^2+\xi/\mu} - 2(2\mu q^2+\xi)\sqrt{q^2+1}\right),$$

$$A_{131} = \frac{-2\mu iq\sqrt{q^2+\xi/(2\mu)}}{D(q,\mu,\xi)}\frac{\sqrt{2\mu}}{\sqrt{\xi}+\sqrt{2\mu}}, \quad A_{132} = \frac{-(2\mu q^2+\xi)}{D(q,\mu,\xi)}\frac{\sqrt{2\mu}}{\sqrt{\xi}+\sqrt{2\mu}}$$

and $D(q,\mu,\xi)$ given by (56). Analogously we find solutions $\mathbf{v}^c$, $\mathbf{v}^s$ to problems (104) and (105). We have

$$\begin{aligned}\mathbf{v}^c = & \frac{q^2-\sqrt{q^2+1}-1}{4\mu(\sqrt{q^2+1}+1)-\xi}\Big(\nabla e^{iqx+y+y\sqrt{q^2+1}} + A_{211}\tilde{\mathbf{v}}^{(1)} + A_{212}\tilde{\mathbf{v}}^{(2)}\Big)\\ & + \frac{q^3}{(\xi-2\mu(\sqrt{q^2+1}+1))(\sqrt{q^2+1}+1)}\Big(\nabla^{\perp} e^{iqx+y+y\sqrt{q^2+1}} + A_{221}\tilde{\mathbf{v}}^{(1)} + A_{222}\tilde{\mathbf{v}}^{(2)}\Big)\\ & + A_{231}\tilde{\mathbf{v}}^{(1)} + A_{232}\tilde{\mathbf{v}}^{(2)},\end{aligned} \tag{109}$$

where

$$A_{211} = -\frac{2\mu iq(\sqrt{q^2+1}+1)}{D(q,\mu,\xi)}\Big(2\mu q^2+\xi-2\mu\sqrt{q^2+\xi/(2\mu)}\big(\sqrt{q^2+1}+1\big)\Big),$$

$$A_{212} = \frac{2\mu(\sqrt{q^2+1}+1)}{D(q,\mu,\xi)}\Big((2\mu q^2+\xi)\big(\sqrt{q^2+1}+1\big)-2\mu q^2\sqrt{q^2+\xi/\mu}\Big),$$

$$A_{221} = \frac{4\mu^2\sqrt{q^2+\xi/(2\mu)}}{D(q,\mu,\xi)}\Big(\sqrt{q^2+\xi/(2\mu)}\big(q^2+\sqrt{q^2+1}+1\big)-q^2\big(\sqrt{q^2+1}+1\big)\Big),$$

$$A_{222} = -\frac{2\mu iq}{D(q,\mu,\xi)}\Big(2\mu\sqrt{q^2+\xi/\mu}\big(q^2+\sqrt{q^2+1}+1\big)-(2\mu q^2+\xi)\big(\sqrt{q^2+1}+1\big)\Big),$$

$$A_{231} = \frac{2\mu iq}{D(q,\mu,\xi)}\sqrt{q^2+\xi/(2\mu)}\Big(\frac{\xi}{4\mu+\sqrt{2\mu\xi}}-\sqrt{q^2+\xi/(2\mu)}+2\Big),$$

$$A_{232} = \frac{1}{D(q,\mu,\xi)}\Big((2\mu q^2+\xi)\Big(\frac{\xi}{4\mu+\sqrt{2\mu\xi}}+2\Big)-2\mu q^2\sqrt{q^2+\xi/\mu}\Big),$$

and

$$\mathbf{v}^s = \frac{2\mu iq^3\sqrt{q^2+\xi/(2\mu)}}{D(q,\mu,\xi)}\tilde{\mathbf{v}}^{(1)} + \frac{q^2(2\mu q^2+\xi)}{D(q,\mu,\xi)}\tilde{\mathbf{v}}^{(2)}. \tag{110}$$

Notice that $v^{(1)}$ appearing in the first order term of the expansion (75) can be obtain by taking the real part of $\mathbf{v} = \zeta_i\mathbf{v}^t + \zeta\mathbf{v}^c + \gamma\mathbf{v}^s$. This yields $v_y^{(1)}\mid_{y=0} = \Big(\zeta_i\Lambda^t(q_0,\mu,\xi)+\zeta\Lambda^c(q_0,\mu,\xi)+\gamma\Lambda^s(q_0,\mu,\xi)\Big)\cos q_0x$ (with $\Lambda^{t,c,s}(q_0,\mu,\xi)$ given by (53)–(55)). For q_0 satisfying (60) this formula is simplified to

$$v_y^{(1)}\mid_{y=0} = -\frac{\zeta}{2\mu}\cos q_0x. \tag{111}$$

To find higher order terms in the expansions (71)-(75) we will also need to compute $v_x^{(1)}\mid_{y=0}$. Although an explicit formula for $v_x^{(1)}$ is available via (108)–(110), we can derive a more compact expression in the case $q=q_0$. Recall that $\mathbf{v}^{(1)}$ solves equations

$$\begin{aligned}\mu\big(\Delta v_x^{(1)} + \partial_x\mathrm{div}\mathbf{v}^{(1)}\big) - \xi v_x^{(1)} = & -q_0\zeta_i e^{y\sqrt{q_0^2+1}}\sin q_0x\\ & + q_0\zeta\Big(\sqrt{q_0^2+1}+1\Big)e^{y+y\sqrt{q_0^2+1}}\sin q_0x,\end{aligned} \tag{112}$$

$$\mu\big(\Delta v_y^{(1)}+\partial_y \mathrm{div}\mathbf{v}^{(1)}\big)-\xi v_y^{(1)}=\zeta_i e^{y\sqrt{q_0^2+1}}\cos q_0x+q_0^2\zeta e^{y+y\sqrt{q_0^2+1}}\cos q_0x$$
$$-2\zeta\Big(\sqrt{q_0^2+1}+1\Big)e^{y+y\sqrt{q_0^2+1}}\cos q_0x \tag{113}$$

with boundary conditions for $y=0$

$$\mu(\partial_x v_y^{(1)}+\partial_y v_x^{(1)})=\zeta q_0\sin q_0x \tag{114}$$
$$2\mu\partial_y v_y^{(1)}=\Big(-\zeta\Big(\tfrac{\xi}{4\mu+\sqrt{2\mu\xi}}+2\Big)+\zeta_i\tfrac{\sqrt{2\mu}}{\sqrt{\xi}+\sqrt{2\mu}}-\gamma q^2\Big)\cos q_0x. \tag{115}$$

It follows from (113) that for $y=0$

$$2\mu\partial^2_{yy}v_y^{(1)}=\big(\mu q_0^2+\xi\big)v_y^{(1)}-\mu\partial^2_{xy}v_x^{(1)}+\Big(\zeta_i+q_0^2\zeta-2\zeta\big(\sqrt{q_0^2+1}+1\big)\Big)\cos q_0x. \tag{116}$$

On the other hand differentiating (114) in x we get $\mu\partial^2_{xy}v_x^{(1)}=\mu q_0^2v_y^{(1)}+q_0^2\zeta\cos q_0x$ for $y=0$, and substituting (111) we obtain

$$\mu\partial^2_{xy}v_x^{(1)}\Big|_{y=0}=\frac{q_0^2\zeta}{2}\cos q_0x. \tag{117}$$

Therefore (116) yields

$$2\mu\partial^2_{yy}v_y^{(1)}\Big|_{y=0}=\Big(\zeta_i-2\zeta\Big(\sqrt{q_0^2+1}+1+\tfrac{\xi}{4\mu}\Big)\Big)\cos q_0x. \tag{118}$$

Now we find $\mathrm{div}\mathbf{v}^{(1)}$. From (112)–(113)

$$2\mu\Delta\mathrm{div}\mathbf{v}^{(1)}-\xi\mathrm{div}\mathbf{v}^{(1)}=\zeta_i\Big(\sqrt{q_0^2+1}-q_0^2\Big)\cos q_0x\,e^{y\sqrt{q_0^2+1}}$$
$$+2\zeta\Big(q_0^2\sqrt{q_0^2+1}-2\sqrt{q_0^2+1}-2\Big)\cos q_0x\,e^{y+y\sqrt{q_0^2+1}}, \tag{119}$$

also by (117)–(118) we have for $y=0$

$$\mu\partial_y\mathrm{div}\mathbf{v}^{(1)}=\Big(\tfrac{\zeta_i}{2}-\zeta\Big(\sqrt{q_0^2+1}+1+\tfrac{\xi}{4\mu}-\tfrac{q_0^2}{2}\Big)\Big)\cos q_0x. \tag{120}$$

One can find an explicit solution to the problem (119)–(120), in particular

$$\mathrm{div}\mathbf{v}^{(1)}\Big|_{y=0}=D_1\cos q_0x, \tag{121}$$

where

$$D_1=\frac{\zeta_i\Big(\sqrt{q_0^2+\frac{\xi}{2\mu}}+q_0^2\Big)}{2\mu\sqrt{q_0^2+\frac{\xi}{2\mu}}\Big(\sqrt{q_0^2+\frac{\xi}{2\mu}}+\sqrt{q_0^2+1}\Big)}$$
$$-\frac{\zeta}{\mu\sqrt{q_0^2+\frac{\xi}{2\mu}}}\left(\sqrt{q_0^2+1}+1+\tfrac{\xi}{4\mu}-\tfrac{q_0^2}{2}+\frac{q_0^2\sqrt{q_0^2+1}-2\sqrt{q_0^2+1}-2}{\sqrt{q_0^2+1}+1+\sqrt{q_0^2+\frac{\xi}{2\mu}}}\right). \tag{122}$$

Then taking into account (115)

$$\partial_x v_x^{(1)}=\Big(D_1+\tfrac{\zeta\xi}{2\mu\big(4\mu+\sqrt{2\mu\xi}\big)}-\tfrac{\zeta_i}{\sqrt{2\mu\xi}+2\mu}+\tfrac{2\zeta+\gamma q_0^2}{2\mu}\Big)\cos q_0x, \tag{123}$$

and

$$v_x^{(1)}=\tfrac{1}{q_0}\Big(D_1+\tfrac{\zeta\xi}{2\mu\big(4\mu+\sqrt{2\mu\xi}\big)}-\tfrac{\zeta_i}{\sqrt{2\mu\xi}+2\mu}+\tfrac{2\zeta+\gamma q_0^2}{2\mu}\Big)\sin q_0x. \tag{124}$$

Appendix B Calculations of $\mathbf{v}^{(2)}$ and $\rho^{(2)}$

From (81) and taking into account explicit formulas (86)–(87) for $\mathbf{p}^{(21)}$ we have that $\mathbf{v}^{(21)}$ satisfies the equations

$$\mu(\Delta v_x^{(21)} + \partial_x \mathrm{div}\mathbf{v}^{(21)}) - \xi v_x^{(21)} \tag{125}$$
$$= -\zeta_i p_x^{(21)} - \zeta \tfrac{q_0\sqrt{q_0^2+1}(1+\sqrt{4q_0^2+1})}{2} e^{y+y\sqrt{4q_0^2+1}} \sin 2q_0 x$$
$$+ \zeta q_0\big(q_0^2 - \sqrt{q_0^2+1}\big) e^{2y\sqrt{q_0^2+1}} \sin 2q_0 x,$$
$$\mu(\Delta v_y^{(21)} + \partial_y \mathrm{div}\mathbf{v}^{(21)}) - \xi v_y^{(21)} \tag{126}$$
$$= 2b\xi V_y - b\zeta_i P_y - \zeta_i p_y^{(21)} + \zeta\big(\sqrt{q_0^2+1} - q_0^2\big) e^{2y\sqrt{q_0^2+1}} \cos 2q_0 x$$
$$+ \zeta\left(\tfrac{(q_0^2+2\sqrt{q_0^2+1}-1)(1+\sqrt{4q_0^2+1})}{2} - q_0^2\sqrt{q_0^2+1}\right) e^{y+y\sqrt{4q_0^2+1}} \cos 2q_0 x$$
$$+ \zeta\sqrt{q_0^2+1} e^{2y\sqrt{q_0^2+1}} + \zeta\big(2b + 4by + 2\sqrt{q_0^2+1} - q_0^2 - 1\big) e^{2y}.$$

Moreover, from (82)–(83) using (38), (112), (114), (118), (123) we find the boundary conditions for $\mathbf{v}^{(21)}$ as $y = 0$:

$$\mu(\partial_x v_y^{(21)} + \partial_y v_x^{(21)}) = T_x \sin 2q_0 x$$

$$2\mu\partial_y v_y^{(21)} = T_y^{(1)} + T_y^{(2)} \cos 2q_0 x,$$

where

$$T_x = \tfrac{q_0}{2}\Big(\zeta_i - \zeta\sqrt{q_0^2+1} - \zeta\Big) - \tfrac{\gamma q_0^3}{2}$$
$$- \tfrac{1}{2q_0}\left(4\mu q_0^2 + \xi\right)\left(D_1 + \tfrac{\zeta\xi}{2\mu\left(4\mu + \sqrt{2\mu\xi}\right)} - \tfrac{\zeta_i}{\sqrt{2\mu\xi}+2\mu} + \tfrac{2\zeta + \gamma q_0^2}{2\mu}\right),$$
$$T_y^{(1)} = \zeta\Big(\sqrt{q_0^2+1} - \tfrac{q_0^2}{2} + \tfrac{\xi}{8\mu}\Big) - \tfrac{1}{4}\zeta_i, \quad T_y^{(2)} = \zeta\left(\sqrt{q_0^2+1} + \tfrac{q_0^2}{2} + \tfrac{\xi}{8\mu}\right) - \tfrac{1}{4}\zeta_i.$$

We represent $\mathbf{v}^{(21)}$ in the following way

$$v_x^{(21)} = \Big(B_x^{(1)} e^{y\sqrt{4q_0^2+1}} + B_x^{(2)} e^{y+y\sqrt{4q_0^2+1}} + B_x^{(3)} e^{2y\sqrt{q_0^2+1}}\Big) \sin 2q_0 x$$
$$- \Big(2Eq_0 e^{y\sqrt{4q_0^2+\xi/(2\mu)}} + F\sqrt{4q_0^2+\xi/\mu}\, e^{y\sqrt{4q_0^2+\xi/\mu}}\Big) \sin 2q_0 x, \tag{127}$$

$$v_y^{(21)} = \Big(B_y^{(1)} e^{y\sqrt{4q_0^2+1}} + B_y^{(2)} e^{y+y\sqrt{4q_0^2+1}} + B_y^{(3)} e^{2y\sqrt{q_0^2+1}}\Big) \cos 2q_0 x$$
$$+ \Big(E\sqrt{4q_0^2+\xi/(2\mu)}\, e^{y\sqrt{4q_0^2+\xi/(2\mu)}} + 2q_0 F e^{y\sqrt{4q_0^2+\xi/\mu}}\Big) \cos 2q_0 x$$
$$+ B_y^{(4)} e^y + B_y^{(5)} y e^y + B_y^{(6)} e^{2y} + B_y^{(7)} y e^{2y} + B_y^{(8)} e^{2y\sqrt{q_0^2+1}}$$
$$+ B_y^{(9)} y e^{y\sqrt{\xi}/\sqrt{2\mu}} + G e^{y\sqrt{\xi}/\sqrt{2\mu}} \tag{128}$$

where

$$B_x^{(1)} = \tfrac{\zeta_i}{4\mu^2-6\mu\xi+2\xi^2}\Big(\big(\mu(4q_0^2+2)-\xi\big)q_0\sqrt{q_0^2+1}-\mu q_0\sqrt{4q_0^2+1}\big(q_0^2+2\sqrt{q_0^2+1}-1\big)\Big),$$

$$\begin{aligned}B_x^{(2)} = &-\frac{\zeta q_0\sqrt{q_0^2+1}\big(1+\sqrt{4q_0^2+1}\big)\big(4\mu(q_0^2+\sqrt{4q_0^2+1}+1)-\xi\big)}{64\mu^2q_0^2+4(8\mu^2-3\mu\xi)(1+\sqrt{4q_0^2+1})+2\xi^2}\\ &+\frac{\zeta q_0\mu(1+\sqrt{4q_0^2+1})\big((q_0^2+2\sqrt{q_0^2+1}-1)(1+\sqrt{4q_0^2+1})-2q_0^2\sqrt{q_0^2+1}\big)}{32\mu^2q_0^2+2(8\mu^2-3\mu\xi)(1+\sqrt{4q_0^2+1})+\xi^2},\end{aligned}$$

$$B_x^{(3)} = \tfrac{\zeta\left(q_0^2-\sqrt{q_0^2+1}\right)}{32\mu^2-12\mu\xi+\xi^2}\Big(4\mu q_0^3+(8\mu-\xi)q_0-4\mu q_0\sqrt{q_0^2+1}\Big),$$

$$B_y^{(1)} = \tfrac{\zeta_i}{2\mu^2-3\mu\xi+\xi^2}\Big(-\mu q_0^2\sqrt{4q_0^2+1}\sqrt{q_0^2+1}-\tfrac{1}{4}\big(\mu(1-4q_0^2)-\xi\big)\big(q_0^2+2\sqrt{q_0^2+1}-1\big)\Big),$$

$$\begin{aligned}B_y^{(2)} = &\frac{\zeta q_0^2\mu\sqrt{q_0^2+1}\big(4q_0^2+2\sqrt{4q_0^2+1}+2\big)}{32\mu^2q_0^2+2(8\mu^2-3\mu\xi)(1+\sqrt{4q_0^2+1})+\xi^2}\\ &+\frac{\zeta\big(\mu(-4q_0^2+2\sqrt{4q_0^2+1}+2)-\xi\big)\big((q_0^2+2\sqrt{q_0^2+1}-1)(1+\sqrt{4q_0^2+1})-2q_0^2\sqrt{q_0^2+1}\big)}{64\mu^2q_0^2+4(8\mu^2-3\mu\xi)(1+\sqrt{4q_0^2+1})+2\xi^2},\end{aligned}$$

$$B_y^{(3)} = \tfrac{\zeta\left(q_0^2-\sqrt{q_0^2+1}\right)}{32\mu^2-12\mu\xi+\xi^2}\Big(-4\mu q_0^2\sqrt{q_0^2+1}+4\mu q_0^2-4\mu+\xi\Big),$$

$$B_y^{(4)} = -\tfrac{\zeta_i}{2\mu-\xi}\Big(\tfrac{2\sqrt{q_0^2+1}-q_0^2-1}{4}-b\Big),\quad B_y^{(5)} = -\tfrac{\zeta_i b}{2\mu-\xi},$$

$$B_y^{(6)} = \tfrac{\zeta}{8\mu-\xi}\Big(2\sqrt{q_0^2+1}-q_0^2-1-2b\Big),\quad B_y^{(7)} = \tfrac{4\zeta b}{8\mu-\xi},$$

$$B_y^{(8)} = \tfrac{\zeta\sqrt{q_0^2+1}}{8\mu(q_0^2+1)-\xi},\qquad B_y^{(9)} = \tfrac{\zeta_i b}{2\mu-\xi}-\tfrac{\zeta b\xi}{2\mu(8\mu-\xi)},$$

$$\begin{aligned}E = -\tfrac{1}{D(2q_0)}\Bigg(&4\mu q_0\sqrt{4q_0^2+\xi/\mu}\,\Big(-\mu\Big(B_x^{(1)}\sqrt{4q_0^2+1}+B_x^{(2)}\big(1+\sqrt{4q_0^2+1}\big)+2B_x^{(3)}\sqrt{q_0^2+1}\Big)\\ &\qquad+2\mu q_0\big(B_y^{(1)}+B_y^{(2)}+B_y^{(3)}\big)+T_x\Big)\\ &+(8\mu q_0^2+\xi)\Big(-2\mu\Big(B_y^{(1)}\sqrt{4q_0^2+1}+B_y^{(2)}\big(1+\sqrt{4q_0^2+1}\big)+2B_y^{(3)}\sqrt{q_0^2+1}\Big)+T_y^{(2)}\Big)\Bigg),\end{aligned}$$

$$F = \tfrac{1}{D(2q_0)}\Bigg((8\mu q_0^2+\xi)\Big(-\mu\left(B_x^{(1)}\sqrt{4q_0^2+1}+B_x^{(2)}\big(1+\sqrt{4q_0^2+1}\big)+2B_x^{(3)}\sqrt{q_0^2+1}\right)$$
$$+2\mu q_0\big(B_y^{(1)}+B_y^{(2)}+B_y^{(3)}\big)+T_x\Big)$$
$$+4\mu q_0\sqrt{4q_0^2+\xi/(2\mu)}\Big(-2\mu\Big(B_y^{(1)}\sqrt{4q_0^2+1}+B_y^{(2)}\big(1+\sqrt{4q_0^2+1}\big)+2B_y^{(3)}\sqrt{q_0^2+1}\Big)+T_y^{(2)}\Big)\Bigg),$$

$$G = \tfrac{1}{\sqrt{2\mu\xi}}\,T_y^{(1)} - \sqrt{\tfrac{2\mu}{\xi}}\left(B_y^{(4)}+B_y^{(5)}+2B_y^{(6)}+B_y^{(7)}+2B_y^{(8)}\sqrt{q_0^2+1}+B_y^{(9)}\right).$$

From (84) we get

$$\begin{aligned} &v_y^{(22)}+\tfrac{\zeta}{2\mu}\rho^{(2)} \\ &\quad= -v_y^{(21)}+\tfrac12\big(\partial^2_{yy}V_y+\tfrac{\gamma q_0^2}{\mu}\big)\cos^2 q_0x - q_0v_x^{(1)}\sin q_0x + V^{(2)} \quad \text{for } y=0. \end{aligned} \tag{129}$$

Observe that the right-hand side of (129) is a linear combination of a constant function and $\cos 2q_0x$. Then it follows from the spectral representation (64) (for $\mathcal{L}=\mathcal{L}(1)$) that $\rho^{(2)}=\beta\cos 2q_0x$. Moreover, using (38), (115), (123), (127)–(128) we obtain

$$\begin{aligned} \beta = &-\tfrac{1}{\Lambda(2q_0)}\left(B_y^{(1)}+B_y^{(2)}+B_y^{(3)}+E\sqrt{4q_0^2+\xi/(2\mu)}+2q_0F\right) \\ &+\tfrac{1}{2\Lambda(2q_0)}\left(D_1+\tfrac{\gamma q_0^2}{\mu}\right)+\tfrac{3}{4\Lambda(2q_0)}\left(\tfrac{\zeta\xi}{2\mu\left(4\mu+\sqrt{2\mu\xi}\right)}-\tfrac{\zeta_i}{\sqrt{2\mu\xi}+2\mu}+\tfrac{\zeta}{\mu}\right), \end{aligned} \tag{130}$$

and

$$\begin{aligned} V^{(2)} &= B_y^{(4)}+B_y^{(6)}+B_y^{(8)}+G+\tfrac12 D_1+\tfrac14\Big(\tfrac{\zeta\xi}{2\mu\left(4\mu+\sqrt{2\mu\xi}\right)}-\tfrac{\zeta_i}{\sqrt{2\mu\xi}+2\mu}+\tfrac{\zeta}{\mu}\Big) \\ &= -bV^{(0)}+\left(2\sqrt{q_0^2+1}-q_0^2-1\right)\Big(\tfrac{\zeta_i}{4\left(\sqrt{2\mu\xi}+\xi\right)}-\tfrac{\zeta}{2\sqrt{2\mu\xi}+\xi}\Big) \\ &\quad-\tfrac{\zeta\sqrt{q_0^2+1}}{2\sqrt{2\mu\xi}\sqrt{q_0^2+1}+\xi}+\tfrac{1}{\sqrt{2\mu\xi}}\,T_y^{(1)}+\tfrac12 D_1+\tfrac14\Big(\tfrac{\zeta\xi}{2\mu\left(4\mu+\sqrt{2\mu\xi}\right)}-\tfrac{\zeta_i}{\sqrt{2\mu\xi}+2\mu}+\tfrac{\zeta}{\mu}\Big) \\ &=: V^{(21)}-bV^{(0)}. \end{aligned} \tag{131}$$

Next, we establish the boundary values of components of $\mathbf{v}^{(2)}$ and some of their derivatives on the line $y=0$, that will be used in (98) and in the calculations of the stress vector for $\mathbf{v}^{(3)}$. To this end substitute the expressions for components of $\mathbf{V}$, $\mathbf{p}^{(1)}$ and $\mathbf{p}^{(2)}$ in (81) (see (39), (77), (85)–(87), (92)):

$$\begin{aligned} \mu(\Delta v_x^{(2)}+\partial_x\mathrm{div}\mathbf{v}^{(2)})-\xi v_x^{(2)} &= \zeta_i q_0\Big(\tfrac{\sqrt{q_0^2+1}}{2}-2\beta\Big)\,e^{y\sqrt{4q_0^2+1}}\,\sin 2q_0x \\ &\quad-\zeta q_0\big(1+\sqrt{4q_0^2+1}\big)\Big(\tfrac{\sqrt{q_0^2+1}}{2}-2\beta\Big)e^{y+y\sqrt{4q_0^2+1}}\sin 2q_0x \\ &\quad+\zeta q_0\big(q_0^2-\sqrt{q_0^2+1}\big)e^{2y\sqrt{q_0^2+1}}\sin 2q_0x, \end{aligned} \tag{132}$$

$$
\begin{aligned}
\mu(\Delta v_y^{(2)} + \partial_y \mathrm{div}\mathbf{v}^{(2)}) - \xi v_y^{(2)} &= \zeta_i\Big(\beta - \tfrac{q_0^2+2\sqrt{q_0^2+1}-1}{4}\Big)\, e^{y\sqrt{4q_0^2+1}}\cos 2q_0x \\
&- \zeta_i\Big(\tfrac{2b\xi}{2\mu-\xi} + b + by + \tfrac{2\sqrt{q_0^2+1}-q_0^2-1}{4}\Big)e^y + 2b\xi\Big(\tfrac{\zeta_i}{2\mu-\xi}\tfrac{\sqrt{2\mu}}{\sqrt{\xi}} - \tfrac{\zeta}{8\mu-\xi}\tfrac{\sqrt{\xi}}{\sqrt{2\mu}}\Big)e^{\sqrt{\xi}y/\sqrt{2\mu}} \\
&+ \zeta\Big(\tfrac{4b\xi}{8\mu-\xi} + 2b + 4by + 2\sqrt{q_0^2+1} - q_0^2 - 1\Big)e^{2y} \\
&+ \zeta\Big(\sqrt{q_0^2+1} - q_0^2\Big)e^{2y\sqrt{q_0^2+1}}\cos 2q_0x + \zeta\sqrt{q_0^2+1}e^{2y\sqrt{q_0^2+1}} \\
&+ \zeta\Big(\tfrac{(q_0^2+2\sqrt{q_0^2+1}-1-4\beta)(1+\sqrt{4q_0^2+1})}{2} - q_0^2\sqrt{q_0^2+1} + 4\beta q_0^2\Big)e^{y+y\sqrt{4q_0^2+1}}\cos 2q_0x.
\end{aligned} \tag{133}
$$

Let $\mathbf{w}^{(2)}(x,y)$ denote the vector function obtained by subtracting from $\mathbf{v}^{(2)}(x,y)$ its average in x over the period. Find the divergence of $\mathbf{w}^{(2)}(x,y)$ for $y=0$. Taking derivative of (82) in x and using (123), (112) we obtain that

$$
\begin{aligned}
&\mu\big(\partial^2_{xx}w_y^{(2)} + \partial^2_{xy}w_x^{(2)}\big) \\
&= \big(\zeta_i q_0^2 - \zeta q_0^2\big(\sqrt{q_0^2+1}+1\big)\big)\cos 2q_0x + (4\zeta q_0^2\beta - \gamma q_0^4)\cos 2q_0x \\
&- (4\mu q_0^2+\xi)\Big(D_1 + \tfrac{\zeta\xi}{2\mu\left(4\mu+\sqrt{2\mu\xi}\right)} - \tfrac{\zeta_i}{\sqrt{2\mu\xi}+2\mu} + \tfrac{2\zeta+\gamma q_0^2}{2\mu}\Big)\cos 2q_0x,
\end{aligned} \tag{134}
$$

for $y=0$, also thanks to (129) we have

$$
\begin{aligned}
v_y^{(2)} &= -\tfrac{\zeta\beta}{2\mu}\cos 2q_0x + \tfrac12\Big(D_1 + \tfrac{3\zeta\xi}{4\mu\left(4\mu+\sqrt{2\mu\xi}\right)} - \tfrac{3\zeta_i}{2\sqrt{2\mu\xi}+4\mu} + \tfrac{6\zeta+4\gamma q_0^2}{4\mu}\Big)\cos 2q_0x \\
&- \tfrac12 D_1 + \tfrac14\Big(\tfrac{\zeta_i}{\sqrt{2\mu\xi}+2\mu} - \tfrac{\zeta\xi}{2\mu\left(4\mu+\sqrt{2\mu\xi}\right)} - \tfrac{\zeta}{\mu}\Big) + V^{(2)}.
\end{aligned} \tag{135}
$$

Since

$$
\mu(\Delta w_y^{(2)} + \partial_y\mathrm{div}\mathbf{w}^{(2)}) - \xi w_y^{(2)} = 2\mu\partial_y\mathrm{div}\mathbf{w}^{(2)} - \mu\big(\partial^2_{xx}w_y^{(2)} + \partial^2_{xy}w_x^{(2)}\big) - \big(8\mu q_0^2+\xi\big)w_y^{(2)},
$$

from (133)–(135) we get the following boundary condition for $y=0$:

$$
\begin{aligned}
2\mu\partial_y\mathrm{div}\mathbf{w}^{(2)} &= \big(2\mu q_0^2 - \tfrac{\xi}{4}\big)\Big(\tfrac{\zeta\xi}{2\mu\left(4\mu+\sqrt{2\mu\xi}\right)} - \tfrac{\zeta_i}{\sqrt{2\mu\xi}+2\mu} + \tfrac{\zeta}{\mu}\Big)\cos 2q_0x \\
&+ \Big(-\tfrac{\xi}{2}D_1 + \gamma q_0^4 + \zeta_i q_0^2 + 2\zeta q_0^2\big(4\beta - \sqrt{q_0^2+1} - 1\big) + \zeta\sqrt{q_0^2+1}\Big)\cos 2q_0x \\
&+ \Big(\zeta\tfrac{1+\sqrt{4q_0^2+1}}{2} - \tfrac{\zeta_i}{4}\Big)\Big(q_0^2 + 2\sqrt{q_0^2+1} - 1 - 4\beta\Big)\cos 2q_0x.
\end{aligned} \tag{136}
$$

Then using (132)–(133) we obtain that $\mathrm{div}\mathbf{w}^{(2)}$ satisfies the equation

$$
\begin{aligned}
&2\mu\Delta\mathrm{div}\mathbf{w}^{(2)} - \xi\mathrm{div}\mathbf{w}^{(2)} = 2\zeta\Big(q_0^2 - \sqrt{q_0^2+1}\Big)^2 e^{2y\sqrt{q_0^2+1}}\cos 2q_0x \\
&+ \zeta_i\Big(q_0^2\big(\sqrt{q_0^2+1} - 4\beta\big) + \tfrac14\sqrt{4q_0^2+1}\big(4\beta - q_0^2 - 2\sqrt{q_0^2+1}+1\big)\Big)\, e^{y\sqrt{4q_0^2+1}}\cos 2q_0x \\
&+ \tfrac{\zeta}{2}\big(1+\sqrt{4q_0^2+1}\big)\Big(\big(q_0^2 + 2\sqrt{q_0^2+1} - 1 - 4\beta\big)\big(1+\sqrt{4q_0^2+1}\big) \\
&\qquad -2q_0^2\sqrt{q_0^2+1} + 8\beta q_0^2\Big)e^{y+y\sqrt{4q_0^2+1}}\cos 2q_0x,
\end{aligned}
$$

solving which with boundary condition (136) we derive

$$\mathrm{div}\mathbf{w}^{(2)}\Big|_{y=0} = \frac{D_2}{2\mu\sqrt{4q_0^2+\frac{\xi}{2\mu}}}\cos 2q_0x, \tag{137}$$

where

$$\begin{aligned}
D_2 = &-\tfrac{\xi}{2}D_1 + \gamma q_0^4 + \zeta_i q_0^2 + 2\zeta q_0^2\big(4\beta - \sqrt{q_0^2+1} - 1\big) + \zeta\sqrt{q_0^2+1} \\
&+ (2\mu q_0^2 - \xi/4)\Big(\tfrac{\zeta\xi}{2\mu\left(4\mu+\sqrt{2\mu\xi}\right)} - \tfrac{\zeta_i}{\sqrt{2\mu\xi}+2\mu} + \tfrac{\zeta}{\mu}\Big) \\
&+ \tfrac{1}{4}\big(2\zeta + 2\zeta\sqrt{4q_0^2+1} - \zeta_i\big)\big(q_0^2 + 2\sqrt{q_0^2+1} - 1 - 4\beta\big) \\
&+ \frac{\zeta_i}{\left(\sqrt{4q_0^2+\frac{\xi}{2\mu}}+\sqrt{4q_0^2+1}\right)}\Big(4\beta q_0^2 - q_0^2\sqrt{q_0^2+1} + \tfrac{1}{4}\sqrt{4q_0^2+1}\big(q_0^2 + 2\sqrt{q_0^2+1} - 1 - 4\beta\big)\Big) \\
&- \frac{2\zeta}{\left(\sqrt{4q_0^2+\frac{\xi}{2\mu}}+2\sqrt{q_0^2+1}\right)}\big(q_0^2 - \sqrt{q_0^2+1}\big)^2 \\
&- \frac{\zeta\left(1+\sqrt{4q_0^2+1}\right)}{\left(\sqrt{4q_0^2+\frac{\xi}{2\mu}}+1+\sqrt{4q_0^2+1}\right)}\Big(4\beta q_0^2 - q_0^2\sqrt{q_0^2+1} \\
&\qquad + \tfrac{1}{2}\big(q_0^2 + 2\sqrt{q_0^2+1} - 1 - 4\beta\big)\big(1 + \sqrt{4q_0^2+1}\big)\Big).
\end{aligned} \tag{138}$$

Now using (83) we get

$$\begin{aligned}
\partial_y v_y^{(2)} = &\tfrac{1}{2\mu}\Big(-\tfrac{3}{4}\zeta q_0^2 - \tfrac{\zeta_i}{4} + \zeta\sqrt{q_0^2+1} + \tfrac{\zeta\xi}{8\mu}\Big) + \tfrac{1}{2\mu}\Big(\tfrac{3\zeta}{4}q_0^2 + \tfrac{\zeta\xi}{8\mu} + \zeta\sqrt{q_0^2+1} \\
&- 2\mu\beta\Big(\tfrac{\xi\zeta}{2\mu\left(4\mu+\sqrt{2\mu\xi}\right)} - \tfrac{\zeta_i}{\sqrt{2\mu\xi}+2\mu} + \tfrac{\zeta}{\mu}\Big) - 4\gamma q_0^2\beta - \tfrac{\zeta_i}{4}\Big)\cos 2q_0x,
\end{aligned} \tag{139}$$

therefore

$$\begin{aligned}
v_x^{(2)} = &\tfrac{\sin 2q_0x}{4q_0\mu}\Big(\frac{D_2}{\sqrt{4q_0^2+\frac{\xi}{2\mu}}} - \tfrac{\zeta}{4}\Big(3q_0^2 + \tfrac{\xi}{2\mu} + 4\sqrt{q_0^2+1} - 4\beta\big(\tfrac{\xi}{4\mu+\sqrt{2\mu\xi}} + 2\big)\Big) \\
&+ 4\gamma q_0^2\beta - \zeta_i\Big(\tfrac{\beta\sqrt{2\mu}}{\sqrt{\xi}+\sqrt{2\mu}} - \tfrac{1}{4}\Big)\Big).
\end{aligned} \tag{140}$$

Appendix C Representations of $\mathbf{p}^{(3)}$ and $\mathbf{v}^{(3)}$

The vector functions $\mathbf{p}^{(3)}, \mathbf{v}^{(3)}$ appearing at the order α^3 in the expansions (74), (75) are represented as

$$\mathbf{p}^{(3)} = \mathbf{p}^{(311)} + \mathbf{p}^{(312)} + \mathbf{p}^{(32)}, \quad \mathbf{v}^{(3)} = \mathbf{v}^{(311)} + \mathbf{v}^{(312)} + \mathbf{v}^{(32)}, \tag{141}$$

and to find the coefficient b in (71) we need to calculate only $\mathbf{p}^{(311)}$ and $\mathbf{v}^{(311)}$, whose x-component (y-component) contains all terms with the factor $\sin q_0x$ ($\cos q_0x$), except for those additionally having the multiplier b. It follows from (94) that on the line $y = 0$

$$p_x^{(311)} = \Big(\beta q_0\tfrac{\sqrt{q_0^2+1}}{2} + \sqrt{4q_0^2+1}\big(\tfrac{q_0}{4}\sqrt{q_0^2+1} - \beta q_0\big) - \tfrac{q_0^3}{2} - \tfrac{q_0}{8}\Big)\sin q_0x \tag{142}$$

$$
\begin{aligned}
p_y^{(311)} = \Big(& \tfrac{\beta}{2}\big(\sqrt{q_0^2+1} - 2q_0^2 - 1\big) - \tfrac{\sqrt{q_0^2+1}}{2} + \tfrac{5}{8}q_0^2 + \tfrac{1}{2} \\
& - \sqrt{4q_0^2+1}\Big(\tfrac{\sqrt{q_0^2+1}}{4} + \tfrac{q_0^2}{8} - \tfrac{1}{8} - \tfrac{\beta}{2}\Big)\Big)\cos q_0 x. \qquad (143)
\end{aligned}
$$

Then since $\Delta \mathbf{p}^{(311)} = \mathbf{p}^{(311)}$ we have $\mathbf{p}^{(311)}(x,y) = \mathbf{p}^{(311)}(x,0)e^{y\sqrt{q_0^2+1}}$. We substitute this expression for $\mathbf{p}^{(311)}$ in place of $\mathbf{p}^{(3)}$ in (95) and use (77) and (85)–(87) with (88) to conclude that components of $\mathbf{v}^{(311)}$ satisfy the following equations for $y<0$

$$
\begin{aligned}
&\mu(\Delta v_x^{(311)} + \partial_x \mathrm{div}\mathbf{v}^{(311)}) - \xi v_x^{(311)} \\
&\quad = \big(-\zeta_i H_x^{(1)} + \zeta H_x^{(2)} e^y + \zeta H_x^{(3)} e^{y\sqrt{4q_0^2+1}}\big) e^{y\sqrt{q_0^2+1}} \sin q_0 x,
\end{aligned}
$$

$$
\begin{aligned}
&\mu(\Delta v_y^{(311)} + \partial_x \mathrm{div}\mathbf{v}^{(311)}) - \xi v_y^{(311)} \\
&\quad = \big(-\zeta_i H_y^{(1)} + \zeta H_y^{(2)} e^y + \zeta H_y^{(3)} e^{y\sqrt{4q_0^2+1}}\big) e^{y\sqrt{q_0^2+1}} \cos q_0 x,
\end{aligned}
$$

where

$$
H_x^{(1)} = \beta q_0 \tfrac{\sqrt{q_0^2+1}}{2} + \sqrt{4q_0^2+1}\Big(\tfrac{q_0}{4}\sqrt{q_0^2+1} - \beta q_0\Big) - \tfrac{q_0^3}{2} - \tfrac{q_0}{8},
$$

$$
\begin{aligned}
H_y^{(1)} = & \tfrac{\beta}{2}\big(\sqrt{q_0^2+1} - 2q_0^2 - 1\big) - \tfrac{\sqrt{q_0^2+1}}{2} + \tfrac{5}{8}q_0^2 + \tfrac{1}{2} \qquad (144) \\
& - \sqrt{4q_0^2+1}\Big(\tfrac{\sqrt{q_0^2+1}}{4} + \tfrac{q_0^2}{8} - \tfrac{1}{8} - \tfrac{\beta}{2}\Big),
\end{aligned}
$$

$$
\begin{aligned}
H_x^{(2)} = & \, q_0\big(1+\sqrt{q_0^2+1}\big)\Big(\tfrac{\sqrt{q_0^2+1}}{2} - \tfrac{q_0^2}{4} - \tfrac{1}{4}\Big) \\
& + \big(1+\sqrt{q_0^2+1}\big)\Big(\beta q_0 \tfrac{\sqrt{q_0^2+1}}{2} + \sqrt{4q_0^2+1}\Big(\tfrac{q_0}{4}\sqrt{q_0^2+1} - \beta q_0\Big) - \tfrac{q_0^3}{2} - \tfrac{q_0}{8}\Big),
\end{aligned}
$$

$$
\begin{aligned}
H_y^{(2)} = & \Big(\tfrac{\sqrt{q_0^2+1}}{2} - \tfrac{q_0^2}{4} - \tfrac{1}{4}\Big)\big(q_0^2 - 2\big(1+\sqrt{q_0^2+1}\big)\big) \\
& + q_0\Big(\beta q_0 \tfrac{\sqrt{q_0^2+1}}{2} + \sqrt{4q_0^2+1}\Big(\tfrac{q_0}{4}\sqrt{q_0^2+1} - \beta q_0\Big) - \tfrac{q_0^3}{2} - \tfrac{q_0}{8}\Big) \\
& + 2\big(1+\sqrt{q_0^2+1}\big)\Big(\tfrac{\beta}{2}\big(\sqrt{q_0^2+1} - 2q_0^2 - 1\big) - \tfrac{\sqrt{q_0^2+1}}{2} + \tfrac{5}{8}q_0^2 + \tfrac{1}{2} \\
& - \sqrt{4q_0^2+1}\Big(\tfrac{\sqrt{q_0^2+1}}{4} + \tfrac{q_0^2}{8} - \tfrac{1}{8} - \tfrac{\beta}{2}\Big)\Big),
\end{aligned}
$$

$$
H_x^{(3)} = \tfrac{q_0^3}{2}\sqrt{q_0^2+1} - 2\beta q_0^3 + \big(\sqrt{q_0^2+1} + \sqrt{4q_0^2+1}\big)\Big(\tfrac{q_0}{8} - \tfrac{q_0^3}{8} - \tfrac{\beta q_0}{2}\Big),
$$

$$
H_y^{(3)} = \tfrac{q_0^2}{8} - \tfrac{q_0^4}{8} - \tfrac{\beta q_0^2}{2} - \big(\sqrt{q_0^2+1} + \sqrt{4q_0^2+1}\big)\Big(\tfrac{q_0^2}{4} + \tfrac{\sqrt{q_0^2+1}}{2} - \tfrac{1}{4} - \beta\Big).
$$

We also have boundary conditions for $y=0$:

$$\mu(\partial_x v_y^{(311)} + \partial_y v_x^{(311)}) = Q \sin q_0 x, \tag{145}$$
$$2\mu \partial_y v_y^{(311)} = R \cos q_0 x, \tag{146}$$

where constants Q, R are obtained from (96)–(97) (see computations in Appendix D). The solution $\mathbf{v}^{311}$ is represented as follows:

$$\begin{aligned} v_x^{(311)} =& \big(I_x^{(1)} e^y + I_x^{(2)} + I_x^{(3)} e^{y\sqrt{4q_0^2+1}}\big) e^{y\sqrt{q_0^2+1}} \sin q_0 x \\ &- \Big(q_0 M e^{y\sqrt{q_0^2+\xi/(2\mu)}} + \sqrt{q_0^2 + \tfrac{\xi}{\mu}} N e^{y\sqrt{q_0^2+\xi/\mu}}\Big) \sin q_0 x, \end{aligned} \tag{147}$$

$$\begin{aligned} v_y^{(311)} =& \big(I_y^{(1)} e^y + I_y^{(2)} + I_y^{(3)} e^{y\sqrt{4q_0^2+1}}\big) e^{y\sqrt{q_0^2+1}} \cos q_0 x \\ &+ \Big(\sqrt{q_0^2 + \tfrac{\xi}{2\mu}} M e^{y\sqrt{q_0^2+\xi/(2\mu)}} + q_0 N e^{y\sqrt{q_0^2+\xi/\mu}}\Big) \cos q_0 x, \end{aligned} \tag{148}$$

where

$$\begin{aligned} I_x^{(1)} &= \tfrac{\zeta}{d_1} \Big(H_x^{(2)} \Big(\mu\big(q_0^2 + 4\sqrt{q_0^2+1} + 4\big) - \xi\Big) + H_y^{(2)} \mu q_0 \big(1 + \sqrt{q_0^2+1}\big)\Big), \\ I_y^{(1)} &= \tfrac{\zeta}{d_1} \Big(H_y^{(2)} \Big(\mu\big(2\sqrt{q_0^2+1} + 2 - q_0^2\big) - \xi\Big) - H_x^{(2)} \mu q_0 \big(1 + \sqrt{q_0^2+1}\big)\Big), \end{aligned}$$

$$d_1 = 8\mu^2 q_0^2 + (16\mu^2 - 6\mu\xi)\big(1 + \sqrt{q_0^2+1}\big) + \xi^2,$$

$$\begin{aligned} I_x^{(2)} &= -\tfrac{\zeta_i}{2\mu^2 - 3\mu\xi + \xi^2} \Big(H_x^{(1)} (\mu q_0^2 + 2\mu - \xi) + H_y^{(1)} \mu q_0 \sqrt{q_0^2+1}\Big), \\ I_y^{(2)} &= -\tfrac{\zeta_i}{2\mu^2 - 3\mu\xi + \xi^2} \Big(H_y^{(1)} (\mu - \mu q_0^2 - \xi) - H_x^{(1)} \mu q_0 \sqrt{q_0^2+1}\Big), \end{aligned}$$

$$\begin{aligned} I_x^{(3)} &= \tfrac{\zeta}{d_2} \Big(\big(9\mu q_0^2 + 4\mu + 4\mu\sqrt{4q_0^4 + 5q_0^2 + 1} - \xi\big) H_x^{(3)} \\ &\qquad + \mu q_0 \big(\sqrt{q_0^2+1} + \sqrt{4q_0^2+1}\big) H_y^{(3)}\Big), \\ I_y^{(3)} &= \tfrac{\zeta}{d_2} \Big(\big(3\mu q_0^2 + 2\mu + 2\mu\sqrt{4q_0^4 + 5q_0^2 + 1} - \xi\big) H_y^{(3)} \\ &\qquad - \mu q_0 \big(\sqrt{q_0^2+1} + \sqrt{4q_0^2+1}\big) H_x^{(3)}\Big), \\ d_2 &= 4\mu q_0^2 \big(16\mu q_0^2 + 18\mu - 3\xi\big) \\ &\quad + 2\mu\big(16\mu q_0^2 + 8\mu - 3\xi\big)\sqrt{4q_0^4 + 5q_0^2 + 1} + 16\mu^2 - 6\mu\xi + \xi^2. \end{aligned} \tag{149}$$

The last two terms in (147) and (148) represent the linear combination

$$M\nabla(e^{y\sqrt{q_0^2+\xi/(2\mu)}} \cos q_0 x) + N\nabla^{\perp}(e^{y\sqrt{q_0^2+\xi/\mu}} \sin q_0 x)$$

of vector functions satisfying the homogeneous equation $\mu(\Delta\ \cdot\ +\nabla\mathrm{div}\cdot) - \xi\cdot\ = 0$, and coefficients M and N are found from the boundary conditions (145)–(146),

$$M = -\tfrac{1}{D(q_0,\mu,\xi)}\Big(2\mu q_0\sqrt{q_0^2+\tfrac{\xi}{\mu}}\,\tilde Q + (2\mu q_0^2+\xi)\,\tilde R\Big), \tag{150}$$

$$N = \tfrac{1}{D(q_0,\mu,\xi)}\Big((2\mu q_0^2+\xi)\,\tilde Q + 2\mu q_0\sqrt{q_0^2+\tfrac{\xi}{2\mu}}\,\tilde R\Big), \tag{151}$$

where

$$\begin{aligned}\tilde Q = Q &+ \mu q_0\big(I_y^{(1)} + I_y^{(2)} + I_y^{(3)}\big)\\ &- \mu\Big(I_x^{(1)} + \sqrt{q_0^2+1}\big(I_x^{(1)} + I_x^{(2)} + I_x^{(3)}\big) + \sqrt{4q_0^2+1}I_x^{(3)}\Big),\end{aligned} \tag{152}$$

$$\tilde R = R - 2\mu\Big(I_y^{(1)} + \sqrt{q_0^2+1}\big(I_y^{(1)} + I_y^{(2)} + I_y^{(3)}\big) + \sqrt{4q_0^2+1}I_y^{(3)}\Big),$$

and Q, R are obtained below.

Appendix D Boundary conditions for $\mathbf{v}^{(311)}$

In order to establish the coefficient Q in (145) we consider each term in the right-hand side of the first boundary condition of (96) and collect coefficients in front of $\sin q_0x$.

Equation (117) yields

$$2\mu\partial^2_{xy}v_x^{(1)}\rho^{(1)}[\rho^{(1)}]' = -\tfrac{\zeta q_0^3}{4}\sin q_0x + C\sin 3q_0x. \tag{153}$$

Hereafter C denotes a generic constant whose value may possibly change from line to line. Next, by (123) we have

$$\begin{aligned}&2\mu\partial_x v_x^{(1)}[\rho^{(2)}]'\\ &= -2\mu\beta q_0\Big(D_1 + \tfrac{\zeta\xi}{2\mu\left(4\mu+\sqrt{2\mu\xi}\right)} - \tfrac{\zeta_i}{\sqrt{2\mu\xi}+2\mu} + \tfrac{2\zeta+\gamma q_0^2}{2\mu}\Big)\sin q_0x + C\sin 3q_0x,\end{aligned} \tag{154}$$

and (140) entails

$$\begin{aligned}2\mu\partial_x v_x^{(2)}[\rho^{(1)}]' = \tfrac{q_0}{2}\Big(&\tfrac{D_2}{\sqrt{4q_0^2+\frac{\xi}{2\mu}}} - \tfrac{\zeta}{4}\Big(3q_0^2 + \tfrac{\xi}{2\mu} + 4\sqrt{q_0^2+1} - 4\beta\big(\tfrac{\xi}{4\mu+\sqrt{2\mu\xi}}+2\big)\Big)\\ &+ 4\gamma q_0^2\beta - \zeta_i\Big(\tfrac{\beta\sqrt{2\mu}}{\sqrt{\xi}+\sqrt{2\mu}} - \tfrac14\Big)\Big)\sin q_0x + C\sin 3q_0x.\end{aligned} \tag{155}$$

Then considering (112) for $y=0$ and using (123) we get

$$\begin{aligned}\mu\big(\partial^2_{xy}v_y^{(1)} + \partial^2_{yy}v_x^{(1)}\big)\rho^{(2)} = -\tfrac{\beta}{2}\sin q_0x\Big(&q_0\zeta\big(1+\sqrt{q_0^2+1}\big) - q_0\zeta_i\\ &+ 2\mu q_0\Big(D_1 + \tfrac{\zeta\xi}{2\mu\left(4\mu+\sqrt{2\mu\xi}\right)} - \tfrac{\zeta_i}{\sqrt{2\mu\xi}+2\mu} + \tfrac{2\zeta+\gamma q_0^2}{2\mu}\Big)\\ &+\tfrac{\xi}{q_0}\Big(D_1 + \tfrac{\zeta\xi}{2\mu\left(4\mu+\sqrt{2\mu\xi}\right)} - \tfrac{\zeta_i}{\sqrt{2\mu\xi}+2\mu} + \tfrac{2\zeta+\gamma q_0^2}{2\mu}\Big)\Big) + C\sin 3q_0x.\end{aligned} \tag{156}$$

Differentiating (112) in y and substituting (118) we obtain, for $y=0$

$$\begin{aligned} &-\tfrac{\mu}{2}\big(\partial^3_{xyy}v_y^{(1)}+\partial^3_{yyy}v_x^{(1)}\big)[\rho^{(1)}]^2 \\ &\quad=\tfrac{q_0}{8}\sin q_0x\Big(\zeta_i\sqrt{q_0^2+1}-\tfrac{\zeta\xi}{2\mu}-2\zeta\big(q_0^2+\sqrt{q_0^2+1}+1\big)\Big)+C\sin 3q_0x. \end{aligned}\tag{157}$$

Similarly to (156) using equation (132) and (140) we get

$$\begin{aligned} -\mu\big(\partial^2_{xy}v_y^{(2)}+\partial^2_{yy}v_x^{(2)}\big)\rho^{(1)} &= -\tfrac{\zeta q_0}{2}\big(q_0^2-\sqrt{q_0^2+1}\big)\sin q_0x \\ &+\tfrac{\zeta q_0}{4}\big(\sqrt{q_0^2+1}-4\beta\big)\big(1+\sqrt{4q_0^2+1}\big)\sin q_0x-\tfrac{\zeta_i q_0}{4}\big(\sqrt{q_0^2+1}-4\beta\big)\sin q_0x \\ &+\tfrac{-8\mu q_0^2-\xi}{8\mu q_0}\Bigg(\tfrac{D_2}{\sqrt{4q_0^2+\frac{\xi}{2\mu}}}-\tfrac{\zeta}{4}\Big(3q_0^2+\tfrac{\xi}{2\mu}+4\sqrt{q_0^2+1}-4\beta\Big(\tfrac{\xi}{4\mu+\sqrt{2\mu\xi}}+2\Big)\Big) \\ &\qquad+4\gamma q_0^2\beta-\zeta_i\Big(\tfrac{\beta\sqrt{2\mu}}{\sqrt{\xi}+\sqrt{2\mu}}-\tfrac{1}{4}\Big)\Bigg)\sin q_0x+C\sin 3q_0x. \end{aligned}\tag{158}$$

Finally,

$$-\gamma[\rho^{(1)}]''[\rho^{(2)}]'-\gamma[\rho^{(2)}]''[\rho^{(1)}]'=\beta\gamma q_0^3\sin q_0x+C\sin 3q_0x.\tag{159}$$

Thus combining (153)–(159) we have

$$\begin{aligned} Q=&(\beta\gamma-\zeta)q_0^3+\frac{\zeta q_0}{4}\big(\sqrt{q_0^2+1}-4\beta\big)\big(1+\sqrt{4q_0^2+1}\big)+\tfrac{\zeta q_0}{4}\sqrt{q_0^2+1} \\ &-\beta\big(\mu q_0-\tfrac{\xi}{2q_0}\big)\Big(D_1+\tfrac{\zeta\xi}{2\mu\left(4\mu+\sqrt{2\mu\xi}\right)}-\tfrac{\zeta_i}{\sqrt{2\mu\xi}+2\mu}+\tfrac{2\zeta+\gamma q_0^2}{2\mu}\Big) \\ &+\tfrac{\beta}{2}\Big(q_0\zeta\big(1+\sqrt{q_0^2+1}\big)+q_0\zeta_i\Big)-\tfrac{q_0}{8}\Big(\zeta_i\sqrt{q_0^2+1}+\tfrac{\zeta\xi}{2\mu}+2\zeta\Big) \\ &-\tfrac{4\mu q_0^2+\xi}{8\mu q_0}\Bigg(\tfrac{D_2}{\sqrt{4q_0^2+\frac{\xi}{2\mu}}}+4\gamma q_0^2\beta-\zeta_i\Big(\tfrac{\beta\sqrt{2\mu}}{\sqrt{\xi}+\sqrt{2\mu}}-\tfrac{1}{4}\Big) \\ &-\tfrac{\zeta}{4}\Big(3q_0^2+\tfrac{\xi}{2\mu}+4\sqrt{q_0^2+1}-4\beta\Big(\tfrac{\xi}{4\mu+\sqrt{2\mu\xi}}+2\Big)\Big)\Bigg). \end{aligned}\tag{160}$$

Next we consider (97) and establish that

$$2\mu\partial_y v_y^{(3)}=b\big(\gamma q_0^2-2\zeta+\zeta_i-\xi V^{(0)}\big)\cos q_0x+R\cos q_0x+C\cos 3q_0x,\tag{161}$$

where

$$\begin{aligned} R=&-2\beta\zeta q_0^2-\tfrac{\zeta_i\beta}{2}-\tfrac{\zeta_i}{4}+\Big(\tfrac{\xi}{8}-\tfrac{3\mu q_0^2}{4}\Big)D_1-\tfrac{3\mu q_0^2}{4}\Big(\tfrac{\zeta\xi}{2\mu\left(4\mu+\sqrt{2\mu\xi}\right)}-\tfrac{\zeta_i}{\sqrt{2\mu\xi}+2\mu}\Big) \\ &+\beta\zeta\big(\sqrt{q_0^2+1}+\sqrt{4q_0^2+1}\big)+\tfrac{3\zeta}{2}-2\zeta\sqrt{q_0^2+1}-\xi\Big(V^{(21)}-\tfrac{\zeta\beta}{4\mu}\Big)-\tfrac{\xi\gamma q_0^2}{8\mu} \\ &-\tfrac{\zeta}{4}\big(q_0^2+2\sqrt{q_0^2+1}-1\big)\big(1+\sqrt{4q_0^2+1}\big)+\tfrac{3\zeta_i}{8}\sqrt{q_0^2+1}-\tfrac{1}{4}\gamma q_0^4+\zeta q_0^2. \end{aligned}\tag{162}$$

For the first term in the right hand side of (97) we have, by (114),

$$\mu\big(\partial_x v_y^{(1)}+\partial_y v_x^{(1)}\big)[\rho^{(2)}]'=-\beta\zeta q_0^2\cos q_0x+C\cos 3q_0x.\tag{163}$$

Next two terms are transformed as follows

$$
\begin{aligned}
&\mu\big(\partial_x v_y^{(2)}+\partial_y v_x^{(2)}\big)[\rho^{(1)}]'+\mu\big(\partial^2_{xy}v_y^{(1)}+\partial^2_{yy}v_x^{(1)}\big)\rho^{(1)}[\rho^{(1)}]'=2\mu\partial_x v_x^{(1)}[\rho^{(1)}]'^2\\
&\quad-\zeta[\rho^{(2)}]'[\rho^{(1)}]'-\gamma[\rho^{(1)}]'^2[\rho^{(1)}]''=\big(\tfrac{\gamma q_0^4}{4}-\zeta\beta q_0^2\big)\cos q_0x\\
&\quad+\tfrac{\mu q_0^2}{2}\Big(D_1+\tfrac{\zeta\xi}{2\mu\left(4\mu+\sqrt{2\mu\xi}\right)}-\tfrac{\zeta_i}{\sqrt{2\mu\xi}+2\mu}+\tfrac{2\zeta+\gamma q_0^2}{2\mu}\Big)\cos q_0x+C\cos 3q_0x,
\end{aligned}
\tag{164}
$$

where we have used (82) and (123). With the help of (38) we find, for $y=0$

$$
\begin{aligned}
&-2\mu\partial^3_{yyy}V_y\rho^{(1)}\rho^{(2)}-\tfrac{\mu}{3}\partial^4_{yyyy}V_y[\rho^{(1)}]^3=\cos q_0x\Big(\tfrac{\beta\zeta_i}{2}-2\beta\zeta-\zeta+\tfrac{\zeta_i}{8}\\
&\qquad-\tfrac{\beta\zeta\xi}{4\mu}-\tfrac{\zeta\xi}{8\mu}-\tfrac{\zeta\xi^2}{16\mu\left(4\mu+\sqrt{2\mu\xi}\right)}+\tfrac{\zeta_i\xi}{8(\sqrt{2\mu\xi}+2\mu)}\Big)+C\cos 3q_0x.
\end{aligned}
\tag{165}
$$

It follows from (118) that

$$
-2\mu\partial^2_{yy}v_y^{(1)}\rho^{(2)}=\cos q_0x\Big(\beta\zeta\big(1+\sqrt{q_0^2+1}+\tfrac{\xi}{4\mu}\big)-\tfrac{\beta\zeta_i}{2}\Big)+C\cos 3q_0x.
\tag{166}
$$

Differentiating (112) in x and subtracting the derivative of (113) in y, setting $y=0$, and using (115), (123) we get

$$
\begin{aligned}
-\mu\partial^3_{yyy}v_y^{(1)}[\rho^{(1)}]^2=&-\tfrac{3}{8}\cos q_0x\Big(\zeta_i\big(q_0^2+\sqrt{q_0^2+1}\big)-(2\mu q_0^2+\xi)D_1\\
&-2(\mu q_0^2+\xi)\Big(\tfrac{\zeta\xi}{2\mu\left(4\mu+\sqrt{2\mu\xi}\right)}-\tfrac{\zeta_i}{\sqrt{2\mu\xi}+2\mu}+\tfrac{2\zeta+\gamma q_0^2}{2\mu}\Big)\\
&-2\zeta\big(1+\sqrt{q_0^2+1}\big)^2\Big)+C\cos 3q_0x.
\end{aligned}
\tag{167}
$$

Since $2\mu\partial^2_{yy}v_y^{(2)}=\mu(\Delta v_y^{(2)}+\partial_y\mathrm{div}\mathbf{v}^{(2)})-\mu\partial_x(\partial_x v_y^{(2)}+\partial_y v_x^{(2)})$ we derive with the help of (133)–(135) that

$$
\begin{aligned}
2\mu\partial^2_{yy}v_y^{(2)}\rho^{(1)}=&\Bigg(\big(2\mu q_0^2+\tfrac{\xi}{4}\big)D_1+\xi\big(V^{(2)}+2bV^{(0)}-\tfrac{\zeta\beta}{4\mu}+\tfrac{5\zeta}{8\mu}\big)\\
&+\big(2\mu q_0^2+\tfrac{5\xi}{8}\big)\Big(\tfrac{\zeta\xi}{2\mu\left(4\mu+\sqrt{2\mu\xi}\right)}-\tfrac{\zeta_i}{\sqrt{2\mu\xi}+2\mu}\Big)+\big(3\mu q_0^2+\xi\big)\tfrac{\gamma q_0^2}{2\mu}+\zeta_i\big(\tfrac{\beta}{2}-b\big)\\
&+\tfrac{\zeta}{4}\Big(\big(q_0^2+2\sqrt{q_0^2+1}-1-4\beta\big)\big(1+\sqrt{4q_0^2+1}\big)-14\sqrt{q_0^2+1}-4q_0^2\Big)\\
&+\zeta(2b-1)-\tfrac{3\zeta_i}{8}\big(q_0^2+2\sqrt{q_0^2+1}-1\big)\Bigg)\cos q_0x+C\cos 3q_0x.
\end{aligned}
\tag{168}
$$

Finally, we have

$$
-\gamma b[\rho^{(1)}]''-\tfrac{3}{2}\gamma[\rho^{(1)}]'^2[\rho^{(1)}]''=\cos q_0x\Big(b\gamma q_0^2+\tfrac{3}{8}\gamma q_0^4\Big)+C\cos 3q_0x.
\tag{169}
$$

Combining (163)–(169) (and taking into account (131)) we obtain (161) with R given by (162) .

Appendix E The equation for the coefficient b

Now we consider the boundary condition (98) and find the coefficient b which determines the bifurcation type. Since $2\mu\partial_y v_y^{(1)} = -2\mu\partial^2_{yy}V_y\rho^{(1)} + \gamma[\rho^{(1)}]''$ for $y=0$, we have

$$\partial^2_{yy}V_y\rho^{(1)}\rho^{(2)} + \partial_y v_y^{(1)}\rho^{(2)} = -\tfrac{\beta\gamma q_0^2}{4\mu}\cos q_0x + C\cos 3q_0x.$$

By (118), (139)–(140) and (124) we get, correspondingly,

$$\tfrac{1}{2}\partial^2_{yy}v_y^{(1)}[\rho^{(1)}]^2 = \tfrac{3}{16\mu}\Big(\zeta_i - 2\zeta\big(\sqrt{q_0^2+1}+1+\tfrac{\xi}{4\mu}\big)\Big)\cos q_0x + C\cos 3q_0x,$$

$$\begin{aligned}\partial_y v_y^{(2)}\rho^{(1)} - v_x^{(2)}\rho^{(1)\prime} &= \tfrac{1}{2\mu}\Big(-\tfrac{3}{4}\zeta q_0^2 - \tfrac{\zeta_i}{4} + \zeta\sqrt{q_0^2+1} + \tfrac{\zeta\xi}{8\mu}\Big)\cos q_0x\\ &+ \tfrac{1}{8\mu}\Big(\tfrac{D_2}{\sqrt{4q_0^2+\frac{\xi}{2\mu}}} + \tfrac{3\zeta}{4}q_0^2 + \tfrac{\zeta\xi}{8\mu} + \zeta\sqrt{q_0^2+1} - 4\gamma q_0^2\beta - \tfrac{\zeta_i}{4}\\ &-2\mu\beta\Big(\tfrac{\xi\zeta}{2\mu(4\mu+\sqrt{2\mu\xi})} - \tfrac{\zeta_i}{\sqrt{2\mu\xi}+2\mu} + \tfrac{\zeta}{\mu}\Big)\Big)\cos q_0x + C\cos 3q_0x,\end{aligned}$$

and

$$-v_x^{(1)}[\rho^{(2)}]' = \beta\Big(D_1 + \tfrac{\zeta\xi}{2\mu(4\mu+\sqrt{2\mu\xi})} - \tfrac{\zeta_i}{\sqrt{2\mu\xi}+2\mu} + \tfrac{2\zeta+\gamma q_0^2}{2\mu}\Big)\cos q_0x + C\cos 3q_0x.$$

Using (38) we obtain

$$\tfrac{1}{6}\partial^3_{yyy}V_y[\rho^{(1)}]^3 = \tfrac{1}{16\mu}\Big(\tfrac{\zeta\xi}{2\mu} + 4\zeta - \zeta_i\Big)\cos q_0x + C\cos 3q_0x,$$

and from (117) we have

$$-\partial_y v_x^{(1)}\rho^{(1)}[\rho^{(1)}]' = \frac{\zeta q_0^2}{8\mu}\cos q_0x + C\cos 3q_0x.$$

Substituting the expressions obtained above into (98), combining all the terms with $\cos q_0x$, and taking into account (99) we get the relation (102). Finally, using (147)–(148) we obtain

$$\begin{aligned}-bq_0\partial_q\Lambda(q_0) &= \tfrac{D(q_0)+2\mu\xi q_0^2+\xi^2}{2\mu q_0 D(q_0)}Q + \tfrac{\xi\sqrt{q_0^2+\frac{\xi}{2\mu}}}{D(q_0)}R - I_y^{(123)}\\ &+ \tfrac{D(q_0)+2\mu\xi q_0^2+\xi^2}{2q_0D(q_0)}\Big(q_0I_y^{(123)} - I_x^{(1)} - \sqrt{q_0^2+1}I_x^{(123)} - \sqrt{4q_0^2+1}I_x^{(3)}\Big)\\ &- \tfrac{1}{4\mu}\Big(\zeta\sqrt{q_0^2+1} - \tfrac{\zeta}{2} - \tfrac{\zeta_i}{8} + \tfrac{\zeta\xi}{16\mu} - \tfrac{5\zeta q_0^2}{8} - \beta\gamma q_0^2\Big)\\ &- \tfrac{2\mu\xi\sqrt{q_0^2+\frac{\xi}{2\mu}}}{D(q_0)}\Big(I_y^{(1)} + \sqrt{q_0^2+1}I_y^{(123)} + \sqrt{4q_0^2+1}I_y^{(3)}\Big)\\ &- \beta D_1 - \tfrac{D_2}{8\mu\sqrt{4q_0^2+\frac{\xi}{2\mu}}} - \tfrac{3\beta}{4}\Big(\tfrac{\xi\zeta}{2\mu(4\mu+\sqrt{2\mu\xi})} - \tfrac{\zeta_i}{\sqrt{2\mu\xi}+2\mu} + \tfrac{\zeta}{\mu}\Big),\end{aligned}\tag{170}$$

where $I_x^{(123)} = I_x^{(1)} + I_x^{(2)} + I_x^{(3)}$ and $I_y^{(123)} = I_y^{(1)} + I_y^{(2)} + I_y^{(3)}$.

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