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Probabilistic Choice Induced by Strength of Preference

Daniel R. Cavagnaro[1] · Michel Regenwetter[2]

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Abstract

Just as we formulate detailed theories of utility or preference, so too should we theorize carefully about strength of preference. Likewise, because behavior is inherently uncertain, we need a theoretical framework for understanding choice probabilities. This paper fleshes out the simple premise that more strongly preferred options are more likely to be chosen. The resulting *distribution-free Fechnerian models* (DFMs) eschew convenience assumptions underlying popular models like the logit and probit, revealing which aspects of a core decision theory do or do not remain invariant across different ways of constructing strengths of preference, as well as across different monotonic links between those strengths of preference and choice probabilities. We formulate DFMs in a unifying polyhedral geometric space that allows for direct comparisons of theories that can be as categorically different as, say, regret theory, expected utility theory, and lexicographic semiorders. The geometric representation also provides a nuanced perspective on theoretical parsimony beyond parameter counting. Through a series of examples, we demonstrate the derivation and mathematical characterization of DFMs for decision theories with and without utilities and the inferences one can draw from data. We show how DFMs provide a multi-layered quantitative approach to the identifiability of hypothetical constructs. We highlight specific cases where DFMs protect the researcher against mistaken conclusions caused by overspecified models.

Keywords Decision-making · Fechnerian models · Nonparametric models · Probabilistic choice · Strength of preference

Introduction

Across a broad range of tasks, human choice behavior is often inherently uncertain. A decision maker may be unsure about what to choose. A decision maker may carry out mental randomizations to make a choice. Different neural pathways may compete in triggering a response to a stimulus. There are many possible reasons for choice to be probabilistic.

While probabilistic choice deserves thorough theoretical attention in its own right, much decision research puts its emphasis elsewhere. Entire research programs debate how various attributes of decision stimuli are perceived, compared, and/or combined into preferences or utilities: Some of these paradigms dissect the fine details of discounting functions in certain models of intertemporal choice, others propose and study competing mathematical formulations for probability weighting in choice under risk or uncertainty. More broadly, the specifics of utility functions are of great concern across a variety of domains. The bulk of that research stops short of theorizing in equal detail about how those preferences or utilities manifest in observable behavior, such as choices or judgments. Instead, applications of such theories to data typically account for choice uncertainty with an off-the-shelf probabilistic response model like a logit (softmax) or probit, with little or no discussion of the theoretical underpinnings for this parametric form. Essentially, this approach treats choice variability as a nuisance to be accounted for in the statistical analysis, rather than treating the underlying choice uncertainty as a theoretical primitive.

Because probabilistic response models provide the mechanisms through which scholars infer preferences, risk attitudes, levels of impatience, attitudes towards ambiguity, etc., from observed behavior, probabilistic response models are both heavily used, and can be pivotal in inferences and conclusions that scholars draw from their data. Indeed, many findings and conclusions in research on functional forms for discounting, subjective value, probability weighting, or other hypothetical constructs could hinge on the validity of

✉ Daniel R. Cavagnaro
dcavagnaro@fullerton.edu

[1] California State University, Fullerton, Fullerton, CA, USA

[2] University of Illinois at Urbana-Champaign, Champaign, IL, USA

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their models for response mechanisms. Whenever we use a misspecified response model, we may misclassify a patient decision maker as impatient, misdiagnose a risk seeking decision maker as risk averse, or mislabel an inadequate policy as effective. In sum, probabilistic response mechanisms provide a bridge between theory and data. They deserve thorough theoretical attention both in their own right and in the role they play for much decision-making research across basic and applied paradigms. Understanding that link better can elucidate consumer behavior, intertemporal or risky choice, and multi-attribute choice such as medical, health, and legal decision-making. It can also help connect policy and human behavior.

This paper considers ways to relax or even remove certain 'convenience' assumptions[1] in a prominent class of probabilistic response models for binary choice data. Convenience assumptions can be characterized as any assumptions that serve the role of enabling statistical inference with standard parametric methods, but which are not grounded in psychological or decision theories. For probabilistic models of binary choice, such assumptions are rarely, if ever, tested for empirical accuracy, in part because the field has lacked suitable alternatives for modeling uncertainty in choice behavior. However, auxiliary assumptions should be treated with great caution because, as Hey (2005, p. 325) warns, "if one makes the wrong assumptions about the stochastic structure of the noise, then one usually makes wrong inferences from the data." Indeed, a growing body of research documents the profound impact of such assumptions on the estimated parameters of decision theories, and ultimately about the drawn conclusions regarding their descriptive and predictive performances (Bhatia & Loomes, 2017; Loomes & Sugden, 1995; Buschena & Zilberman, 2000; Stott, 2006; Wilcox, 2008; Blavatskyy & Pogrebna, 2010; Drichoutis & Lusk, 2014; Broomell & Bhatia, 2014; Regenwetter et al., 2017). In other words, quoting Alós-Ferrer et al. (2021, p. 1), "the dependence on possibly unwarranted assumptions is not just an abstract theoretical problem but is known to plague empirical research." We believe that this comment applies beyond just empirical research. While there was active work on the theory of probabilistic choice in the 1950s and 1960s (e.g., Block & Marschak, 1960; Becker et al., 1963; Debreu, 1958; Luce, 1959; Luce & Suppes, 1965), probabilistic response models are still widely used today as tools of convenience rather than being given full consideration as critical theoretical components of a fully specified model of choice behavior. Rather than add more empirical evidence to the existing literature about misspecification and misleading inference, we take these problems as a given. We provide a mathematical structure for understanding these problems and emphasize possible solutions.

The core conceptual premise in this paper is that all overly specific characterizations of choice uncertainty will be "wrong" assumptions in that they lack sufficient theoretical grounding. In particular, they do not give enough attention to the interplay between hypothetical constructs, psychological theory, and empirical measures. These are examples of "coordination problems" (Kellen et al., 2021) and "construct-behavior gaps" (Regenwetter & Robinson, 2017). As we explore better theoretical grounding, we pay careful attention to the relationship between theoretical parsimony, diagnosticity of stimuli, identifiability of hypothetical constructs, and invariance of theoretical claims across different assumptions. Instead of entertaining overly stylized parametric specifications, in what follows, we formalize the qualitative nature of choice uncertainty.

Our core theoretical primitive is *strength of preference*. We begin with the basic and intuitive theoretical premise that the more strongly a decision maker prefers one option over another, the more likely they are to choose it (Alós-Ferrer & Garagnani, 2022b, a). Much prior research on decision-making has utilized models based on this premise, most notably various types of so-called Fechnerian models, such as the well-known logit and probit models that predominate in econometric analyses. One of our major mathematical findings is that, in determining a decision theory's predictions about behavior, the details of how we construct strengths of preference can be just as important as the details of how we construct preferences or utilities. Another major mathematical finding is that overly specific probabilistic specifications are prone to artifacts.

There are two main features of commonly used Fechnerian models that we eschew here. One is the limitation to theories that assign utilities to options and define strength of preference as the arithmetic difference between utilities. For utility theories, we do not wish to ignore alternative mechanisms for generating strengths of preference. Beyond utility theories, we do not want to leave out theories that quantify directly how much one choice alternative is preferred to another without evoking utilities. These include theories like regret theory (Loomes & Sugden, 1982; Quiggin, 1994) in risky choice and the tradeoff model (Scholten & Read, 2010) in intertemporal choice. Our theoretical approach can interface with any theory that derives or predicts strengths of preference. The other feature that we eschew is the reliance on parametric distributional assumptions to link strengths of preference to binary choice probabilities. For example, the probit model utilizes a normal cumulative distribution

[1] Throughout, we use "double quotes" to indicate verbatim quotations or technical terms that are later defined formally, and we use 'single quotes' to acknowledge vague or ill-defined concepts.

Table 1 Risk sensitivities, utilities, and strengths of preference for three hypothetical decision makers

Decision maker	Risk sensitivity r	Utility $u_r(x)$	$u_r(y)$	$u_r(z)$	Strength of preference $S_r(zx)$	$S_r(yx)$	$S_r(zy)$
A	−0.4	45.0	62.1	64.0	19.0	17.1	1.9
B	1.2	0.3	1.3	2.4	2.1	1.0	1.1
C	0.8	0.8	2.7	4.5	3.7	1.9	1.8

Note. In this example, strength of preference is the arithmetic difference between utilities

function (CDF), while the logit model utilizes a logistic CDF. By abstracting away from these features, we create a 'distribution-free' modeling framework that identifies what inferences about a decision maker follow from the Fechnerian premise alone, without the added structure imposed by technical convenience assumptions.

Other papers have studied distribution-free cases of error models, random preference models, and distribution-free random utility models (e.g., Becker et al., 1963; Block & Marschak, 1960; Marschak, 1960; Loomes & Sugden, 1995; Heck & Davis-Stober, 2019; Regenwetter et al., 2018, 2014; Zwilling et al., 2019, and many others). In this paper, we provide a similarly general perspective on models that translate strengths of preference into choice probabilities. We aim to disentangle the core principles of the Fechnerian approach from the distributional and auxiliary assumptions that are not grounded in psychological theory. In doing so, we expand a broad class of classical models to a general, polyhedral-geometry representation that is characterized by linear equality and inequality constraints on binary choice probabilities. These general models are subject to modern order-constrained frequentist and Bayesian statistical inference methods, which are available in public-domain software (Zwilling et al., 2019; Heck & Davis-Stober, 2019; Regenwetter et al., 2014). We also show later how these general Fechnerian models are mathematically distinguishable from each other, from other distribution-free models, and from classical models.

To set the stage, we begin with a toy example that illustrates more concretely what we have described so far. It is also the starting point for a sequence of examples that build on each other throughout the rest of the paper.

Example 1 Suppose that three different retailers have the same computer on sale, and that each of them offers a different promotion to attract potential buyers. The first retailer will enter the buyer into a drawing for a \$100 cash prize, in which they will have a 1/10 chance of winning. The second retailer will let the buyer spin a prize wheel on which half of the wheel offers a prize of \$40, and the other half nothing. The third retailer offersa \$25 rebate on the purchase price. Consider two hypothetical buyers, A and B, each considering two of these retailers at a time. We represent this scenario as pairwise choices on three gambles, x, y, and z, that are defined as follows:

x is a 10% chance of winning \$100, otherwise nothing,

y is a 50% chance of winning \$40, otherwise nothing,

z is a sure win of \$25.

Without loss of generality, and for concreteness, suppose that both decision makers have preferences consistent with expected utility and constant relative risk aversion (henceforth CRRA-EU), but they may differ in their values of the risk sensitivity parameter, r. Under CRRA-EU, for an observer with risk sensitivity r, the utility of a gamble g that pays m with probability q is $u_r(g) = q \times \frac{m^{1-r}-1}{1-r}$. Values of r greater than zero imply risk aversion, and values of r less than zero imply risk seeking.[2] For now, we calculate strength of preference between options i and j via the very commonly used arithmetic difference among utilities, $S_r(ij) = u_r(i) - u_r(j)$. Subsequent examples will consider other cases. Table 1 shows the risk sensitivities, utilities, and strengths of preference for these two decision makers (ignore decision maker C for now).

Both decision makers like z the most and x the least, but they have different orders of the strengths of preference between gambles. In particular, A's preference for y over x [with $S_{-0.4}(yx) = 17.1$] is stronger than her preference for z over y [with $S_{-0.4}(zy) = 1.9$], while B's preference for z over y [with $S_{1.2}(zy) = 1.1$] is stronger than her preference for y over x [with $S_{1.2}(yx) = 1.0$].

What does it mean if, according to our premise, more strongly preferred options are more likely to be chosen? It means that the order of choice probabilities matches the order of strengths of preference. Here, writing $P(ij)$ for the probability of choosing i from the unordered pair $\{i, j\}$, and noting

[2] Division by $1 - r$ is necessary for increasing utility when $r > 1$. When $r = 1$, $u_r(g) = q \times \ln(m)$.

that $S(ji) = -S(ij)$, this principle states that the choice probabilities of decision maker A satisfy

$$P(zx) > P(yx) > P(zy) > P(yz) > P(xy) > P(xz),$$

whereas those of decision maker B satisfy

$$P(zx) > P(zy) > P(yx) > P(xy) > P(yz) > P(xz).$$

Intuitively, with enough data, one may be able to infer that B has a different risk attitude than A based on the order of the choice probabilities alone.

Now, consider decision maker C, given in the last row of Table 1. Although this decision maker is risk averse like B, the order of strengths of preference matches that of A, who is risk seeking. Therefore, for these lotteries, the order of strengths of preference alone (according to the arithmetic difference between utility values in CRRA-EU) is not sufficient to tell apart a decision maker like C from one like A. For example, imagine that a scientist is able to infer unambiguously from data that a fourth decision maker, say, D, satisfies

$$P(zx) > P(yx) > P(zy) > P(yz) > P(xy) > P(xz).$$

This order matches that of A (who is risk seeking) as well as that of C (who is risk averse). Therefore, using only the assumption that choice probabilities are monotonically related to strengths of preference, the scientist cannot ascertain whether D is risk seeking or risk averse. In our view, it is crucially important that the analyst be able to notice this ambiguity.

Suppose that, instead, the researcher proceeds to estimate decision maker D's risk sensitivity parameter r by plugging the arithmetic difference of CRRA-EU utility values into a Thurstonian specification of choice probabilities (i.e., a binary probit model). Then, thanks to the probit, with a large enough sample size, the researcher may obtain a very tight confidence interval for the inferred value of r. From that, they may also be able to infer whether D is risk seeking or risk averse. This confidence interval captures statistical uncertainty related to sampling variability. However, it leaves out theoretical ambiguity about how exactly strengths of preference translate into choice probabilities. Instead, it presumes that, in addition to being monotonic, this relationship follows the cumulative normal distribution of the probit model. Short of using tools like those we reviewed for decision makers A, B, and C to determine what inferences are robust, one should consider conclusions or predictions drawn through the lens of a probit or logit model to be only as valid as the distributional convenience assumptions in that probit or logit model.

As the preceding example illustrates, for one and the same core algebraic decision theory, different distributional assumptions may generate different empirical predictions. These predictions interact with inference from data in that different assumptions may yield different parameter point estimates and support different substantive conclusions from the same empirical data. As Blavatskyy and Pogrebna (2010, p. 981) state, "when a researcher exogenously picks a model of stochastic choice, this has a profound effect on the estimated parameters of decision theories, which ultimately affects the drawn conclusions about their descriptive validity." More generally, when fitting a theory of decision-making to empirical data using a specific probabilistic model such as a logit or probit, identifiability issues can arise at many levels. These range from the type of core theory (say expected utility vs. regret), to functional form (say power vs. exponential utility), to parameter ranges (say risk seeking or risk averse), to specific parameter point estimates (say $r = 0.5$ vs. $r = 0.6$). This hazard also affects out-of-sample prediction. As Wilcox (2008, abstract) states, "Econometric comparisons suggest that for the purpose of prediction (as opposed to explanation), choices of stochastic models may be far more consequential than choices of structures such as expected utility or rank-dependent utility." Even with a fixed probabilistic model, such as a logit, identifiability issues may arise from tradeoffs between the parameters of the core theory and the parameter of the logit (Krefeld-Schwalb et al., 2021; Olschewski et al., 2022). In this paper, rather than adding another voice to the chorus of warnings, we aim to address these problems by providing a framework for identifying what predictions or inferences (e.g., risk aversion within CRRA-EU) remain invariant across a universe of probabilistic choice models that derive choice probabilities from strengths of preference. More generally, we provide new perspectives on building a theoretical bridge across the construct-behavior gap (Regenwetter & Robinson, 2017). To accomplish that goal, in the following two sections, we first review and generalize a class of probabilistic models, then change perspective to ask what theoretical constructs and properties are or are not identifiable through the lens of these models. In the process and in later sections, we also offer new nuance to questions about theoretical parsimony (see, e.g., Regenwetter et al., 2022) and the coordination problem in decision research (see, e.g., Kellen et al., 2021).

The rest of the paper is organized as follows. In the next section, we give an overview of existing Fechnerian models of binary choice. The "Distribution-Free Fechnerian Models" section introduces distribution-free Fechnerian models (DFMs) and discusses their mathematical and geometric properties. The "DFMs for Utility Theories" section walks the reader through the development of DFMs for utility-based decision theories in combination with different theories about strength of preference. It also contrasts distribution-free models with logit models of the same core utility theory. The

"Comparing DFMs with Other Probabilistic Choice Models" section distinguishes DFMs from "random utility," "supermajority," and "constant error" models. In "Inference from Data," we highlight specific examples where DFMs guard against mistaken conclusions that would result from auxiliary distributional assumptions. Following our "Conclusions and Discussion" section, Appendix 1 defines DFMs for more general decision theories. It moves beyond utility theories to demonstrate applications of the framework more generally, including to theories that violate transitivity. It works through examples ranging from well-known core theories to novel axiomatizations of strengths of preference. Two additional appendices deal with mathematics.

Background: Fechnerian Models of Binary Choice

Fechnerian models of binary choice are named after Gustav Theodor Fechner (1801–1887), who was an early pioneer in experimental psychology and is widely regarded as the founder of psychophysics. Fechner studied how one can relate the magnitude of a physical stimulus to the magnitude of psychological sensation. Seminal papers, such as Fechner (1860) and later Thurstone (1927), develop probability models for psychophysics and psychoacoustics that correspond to what are now known in economics as strong utility models and random utility models. Many authors in decision theory use the terms "strong utility model" and "Fechnerian model" interchangeably. However, as we describe in this section, various generalizations of the strong utility model, as well as other seemingly unrelated models, may also be regarded as Fechnerian models. We now review some of these models and their relationships. For more, see, e.g., Becker et al. Becker et al. (1963), Luce and Suppes Luce & Suppes (1965), and Wilcox Wilcox (2008).

At their core, Fechnerian models of preferential choice are based on the theoretical primitive that the more strongly a person prefers one option to another, the more likely they are to choose it. In the strong utility model (Debreu, 1958; Luce & Suppes, 1965), the "strength of preference" for one alternative over another takes the form of the arithmetic difference between utility values, and this strength of preference is mapped into a choice probability via a (nondecreasing) distribution function.

Formally, throughout the paper, let $\mathcal{C}$ be a finite set of unordered pairs of choice alternatives. We think of $\mathcal{C}$ as the collection of pairwise choice stimuli presented to a participant in a study. Let $\mathcal{D} = \{ij \mid \{i, j\} \in \mathcal{C}\}$. Notice that $\mathcal{D}$ contains both ordered pairs ij and ji, for each $\{i, j\} \in \mathcal{C}$. We refer to $\mathcal{D}$ as a *domain* of choice pairs. When a decision maker must choose either i or j, i.e., when $ij \in \mathcal{D}$, let $P(ij)$ be the probability that the person chooses i.

Definition 1 (Luce & Suppes, 1965) Writing F for a cumulative distribution function with the property that $F(0) = 1/2$, and u for a mapping from choice alternatives into utilities (i.e., a utility function), the collection of binary choice probabilities defined by

$$P(ij) = F[u(i) - u(j)], \qquad \forall ij \in \mathcal{D}. \tag{1}$$

is called the *strong utility model for F and u*. We say that a collection of binary choice probabilities is *a* strong utility model if it is *the* strong utility model for some such F and u.

In applications of strong utility models, researchers typically assume that F belongs to a parametric family that makes statistical inference, such as parameter estimation, tractable. The well-known (binary) logit and probit models are two such special cases that have become standard tools for modeling individual choice. The *logit*, also known as the *softmax choice rule*, follows by assuming that F in Eq. 1 is a logistic CDF. The *probit* follows by assuming that F is a normal CDF. Such distributional assumptions are not required by the theory behind strong utility, but they facilitate parameter estimation of decision theories such as expected utility, rank-dependent utility, and hyperbolic discounting.

The *strict utility* model (Marschak, 1960), also known as *Luce's choice rule* (Luce, 1959), is another prominent probabilistic choice model that follows by adding additional assumptions to Eq. 1. It states that

$$P(ij) = \frac{u(i)}{u(i) + u(j)}, \qquad \forall ij \in \mathcal{D}. \tag{2}$$

By setting $F(x) = (1 - e^{-x})^{-1}$ and $u'(k) = \ln[u(k)]$, Eq. 2 is equivalent to the strong utility formulation

$$P(ij) = F[u'(i) - u'(j)], \qquad \forall ij \in \mathcal{D}.$$

Many papers add a scaling parameter γ and consider choice probabilities of the form $P(ij) = \frac{u(i)^\gamma}{u(i)^\gamma + u(j)^\gamma}$, which is equivalent to the strong utility formulation above with $u'(k) = \gamma \ln(u(k))$. This model offers the technical convenience that one can sometimes estimate the parameters of the underlying theory using logistic regression (Stewart et al., 2014; Alempaki et al., 2019). Both the original Luce choice rule and some recent extensions have an axiomatic grounding (see, e.g., Tserenjigmid, 2021). For a related early result linking strong utility and strict utility, see Yellott (1977).

Strong utility models can also be conceptualized naturally as 'noise' models in which the decision maker's perception of the difference between the alternatives is distorted. To see this, notice that, if we let $\boldsymbol{\epsilon}$ be a continuous, symmetric, mean-zero random variable whose CDF is F, and if we write p for the associated probability measure, then Eq. 1 becomes

$$P(ij) = p\big(u(i) - u(j) - \boldsymbol{\epsilon} > 0\big), \qquad \forall ij \in \mathcal{D}. \tag{3}$$

It is common to refer to $\boldsymbol{\epsilon}$ as the "error term." This definition has led to several extensions of strong utility models that vary in their assumptions about the error term. For instance, Blavatskyy and Pogrebna (2010) coin the term "Fechner models of heteroscedastic random errors" to refer to the class of models taking the form of Eq. 3 in which $\boldsymbol{\epsilon}$ can be heteroscedastic. In these models, the standard deviation of the errors is assumed to depend on certain properties of the decision problem, such as the number of possible outcomes in the lotteries (Hey, 1995), the arithmetic difference between the utilities of the lotteries (Buschena & Zilberman, 2000), or the range of possible outcomes in the lotteries (e.g., in contextual utility, see Wilcox, 2008). These variations do not discuss the distributional form of $\boldsymbol{\epsilon}$ but they typically assume that it is either normal, yielding a *heteroscedastic probit* model, or extreme value, yielding a *heteroscedastic logit* model.

Let $\sigma_{ij} = d(i, j)$ denote the standard deviation of $\boldsymbol{\epsilon}$ in Eq. 3. Then, we can also write Fechner models of heteroscedastic random errors as

$$P(ij) = F\left[\frac{u(i) - u(j)}{d(i, j)}\right], \qquad \forall ij \in \mathcal{D}, \tag{4}$$

where F is a CDF.

Definition 2 (Halff, 1976; He & Natenzon, 2019) A model of the form given in Eq. 4, in which $d(i, j)$ is a distance metric, is called the *moderate utility model* for F, u, and d. We say that a collection of binary choice probabilities is *a* moderate utility model if it is *the* moderate utility model for some such F, u, and d.

The close connection between the representations in Eqs. 3 and 4 means that one can also formalize assumptions regarding the error term in Eq. 3 as assumptions about the strength of preference through the lens of Eq. 4. In other words, we can view Fechner models of heteroscedastic random errors alternatively as Fechnerian models with homoscedastic random errors in which the strength of preference is a stimulus-dependent function of the utility values.

Moving away from strong and moderate utility models, a *constant error* model (Harless & Camerer, 1994; Wakker et al., 1994) assigns a constant probability e to choices of the alternative with lower utility. Formally, in the constant error model for u,

$$P(ij) = \begin{cases} 1 - e & \text{if } u(i) > u(j), \\ e & \text{if } u(j) > u(i), \\ \frac{1}{2} & \text{if } u(i) = u(j), \end{cases} \tag{5}$$

where $0 < e < \frac{1}{2}$ is a constant. With an appropriate formulation of strength of preference, the constant error model satisfies the theoretical primitive for a Fechnerian model. Namely, if the strength of preference is

$$S(ij) = \begin{cases} c & \text{if } u(i) > u(j), \\ -c & \text{if } u(j) > u(i), \\ 0 & \text{if } u(i) = u(j), \end{cases}$$

for some positive constant c, then the constant error model has the property that the stronger the preference the higher the choice probability. The fact that there are only three specific strength-of-preference values may not be theoretically appealing. However, this example highlights the fact that the theoretical primitive for Fechnerian models is strength of preference, and that strength of preference can be very different from the arithmetic difference between utilities. Geometrically, the constant error model represents an extreme boundary case of strong utility. This is apparent from Fig. 5 in Example 11.

Some, but not all, Fechnerian models can also be characterized as binary random utility models (Block & Marschak, 1960). Formally, a binary random utility model on $\mathcal{C}$ is a collection of binary choice probabilities for which there is a random vector $\boldsymbol{U}$ on $\mathcal{C}$ and a probability measure p, such that $P(ij) = p(\boldsymbol{U}_i > \boldsymbol{U}_j)$. The binary probit model, which we formulated earlier as a strong utility model, is also a binary random utility model in which the utilities are independent normal random variables. The latter formulation is also known as a Thurstone Case V model (Thurstone, 1927). Similarly, the binary logit model is a random utility model in which the utilities are independent extreme-value random variables. Many papers have explored the nuanced relationship between strong utility and random utility models (e.g., Becker et al., 1963; Luce & Suppes, 1965). In general, neither family is a subset of the other. That is, there are binary random utility models that are not strong utility models and there are strong utility models that are not random utility models. Other papers have explored the polyhedral geometry of distribution-free random utility models (e.g., Doignon et al., 2006; Doignon & Regenwetter, 1997; Fiorini, 2001, 2004; Regenwetter et al., 2011; Regenwetter & Davis-Stober, 2012; Regenwetter et al., 2014). This paper focuses on distribution-free extensions of Fechnerian models, including strong, moderate, and strict utility models. We explore the relationship between these families in a later section. We also later illustrate these visually in Figs. 4 and 5.

Many published papers utilize variations of the Fechnerian models described above. As we have shown, each variation entails specific functional and distributional assumptions beyond just the notion that more strongly preferred options should be more likely to be chosen. While the vast majority of these papers pay relatively little attention to such assumptions, a few notable exceptions document specific consequences of the shape of the error term. With the models and concepts we have reviewed in this section, we can now provide more specifics on some of those exceptions. Buschena and Zilberman (2000) use probit models to select among generalized expected utility models and find that, for one and the same data set, one would select different models depending on whether one assumed homoscedastic or heteroscedastic errors. In the same vein, Drichoutis and Lusk (2014) find that contextual utility can produce different characterizations of risk preferences depending on whether one assumes the CDF to be normal or logistic (i.e., a heteroscedastic probit or heteroscedastic logit). Furthermore, Blavatskyy (2007, 2014) show that truncating the error term in a strong utility model can force preferences to satisfy first-order stochastic dominance. These papers provide empirical evidence for our premise that auxiliary assumptions beyond the general Fechnerian model can be problematic.

We have reviewed existing models that are "Fechnerian" in the sense of requiring that the more strongly a decision maker prefers an alternative, the more likely they are to choose it. We decomposed these models into assumptions about the relationship between utilities and strengths of preference and, in turn, about the relationship between strengths of preference and binary choice probabilities. In the next section, we introduce formal concepts for developing a more general definition of Fechnerian models — one that begins directly with strength of preference as a theoretical primitive, and which also does not require the assumption of a parametric family of CDFs for mapping strengths of preference into binary choice probabilities.

Distribution-Free Fechnerian Models

We now consider Fechnerian models in a more general sense than those we specified in Eqs. 1, 3, or 4. At their full level of generality, Fechnerian models use strengths of preference, not necessarily derived from utilities, and they permit any strictly monotonic relationship between the strengths of preference and the choice probabilities. Therefore, beginning with the basic premise that the more strongly a person prefers one option to another, the more likely they are to choose it, we define what we call "distribution-free Fechnerian models." These models bridge the gap between core (algebraic) decision theories and probabilistic choice data, without explicit reference to utilities or error distributions.

Definitions

Throughout the rest of the paper, we utilize the following notational conventions. We use i, j, k, ℓ to denote generic choice options and w, x, y, z for specific ones. We are concerned with the predictions of a given decision theory about observable choice behavior on a given domain $\mathcal{D}$ of choice pairs. For a given decision theory and probabilistic response model, these predictions take the form of mappings from choice pairs into probabilities. We call these mappings binary choice (probability) vectors. By and large, we treat binary choice probabilities as different from zero or one. These premises motivate the following formal definition.

Definition 3 Let $]0,1[$ denote the open unit interval and let $]0,1[^{\mathcal{D}}$ denote the collection of all mappings from $\mathcal{D}$ into $]0,1[$. A mapping $P \in\ ]0,1[^{\mathcal{D}}$ is a *binary choice (probability) vector* on $\mathcal{D}$ if $P(ij) = 1 - P(ji), \forall ij \in \mathcal{D}$.

For a fixed domain, the binary choice vectors produced by a given decision theory and probabilistic response model may depend on free parameters. For example, the predictions of CRRA-EU, with a logit response model, depend on both the risk sensitivity parameter of CRRA-EU and the scaling parameter of the logit. More generally, for a given theory and probabilistic response model, as we vary parameter values, we obtain different binary choice vectors. We formalize such collections with a definition.

Definition 4 A *probabilistic model of binary choices* on $\mathcal{D}$ is a collection of binary choice vectors on $\mathcal{D}$.

Probabilistic choice models in the literature, such as a logit specification of CRRA-EU on a fixed domain, comprise all of the binary choice vectors that can be derived from the decision theory (here, CRRA-EU) through the response model (here, logit). In this way, the response model links the decision theory to the binary choice probabilities. The next definition introduces another such link that is more general than a logit or probit model. It characterizes probabilistic models in which the binary choice probabilities are monotonically related to the strengths of preference implied by an underlying decision theory.

Definition 5 Let S be an odd real-valued function on the domain $\mathcal{D}$, i.e., suppose that $S(ij) = -S(ji),\ \ \forall ij \in \mathcal{D}$. We also refer to $S(ij)$ as the *strength of preference* for i over

j. A binary choice vector P satisfies the *Fechnerian property* for S if and only if

$$S(ij) > S(k\ell) \quad \Leftrightarrow \quad P(ij) > P(k\ell), \qquad \forall ij, k\ell \in \mathcal{D}. \quad (6)$$

A *Fechnerian model* (for S) is a collection of binary choice vectors that satisfy the Fechnerian property.

A special case of the Fechnerian property applies when $i = k$: If i is preferred more strongly over j than over k, then the probability of choosing i from $\{i, j\}$ is greater than the probability of choosing i from $\{i, k\}$. For the choice options in Example 1, because decision maker A prefers the sure thing, z, more strongly to gamble x than to gamble y, the Fechnerian property says that decision maker A is more likely to choose the sure thing when it is paired against gamble x than when it is paired against gamble y.

Having laid out the above preliminaries and definitions, we are ready to consider some core properties of Fechnerian models. We prove the following insight in Appendix 2.

Proposition 1 *Let S be an odd real-valued function on $\mathcal{D}$ and let P be a binary choice vector on $\mathcal{D}$. The Fechnerian property implies that*

$$S(ij) > 0 \quad \Leftrightarrow \quad P(ij) > \tfrac{1}{2}, \qquad \forall ij \in \mathcal{D}.$$

The next proposition, which we state without proof, follows directly from the definitions.

Proposition 2 *Let S be an odd real-valued function on $\mathcal{D}$, let P be a binary choice vector on $\mathcal{D}$, and let F be the function on the range of S given by*

$$F(S(ij)) = P(ij), \qquad \forall ij \in \mathcal{D}. \quad (7)$$

P satisfies the Fechnerian property (6) for S if and only if F is strictly increasing. Furthermore, F satisfies $F(S(ij)) = 1 - F(-S(ij))$, for all $ij \in \mathcal{D}$.

Proposition 2 implies that strong utility and moderate utility specifications of a core utility theory (Eqs. 1 and 4, respectively), with strictly increasing F, are Fechnerian models in the sense of Definition 5. They generate binary choice vectors from Eq. 7 with various assumptions about S and F. Some of these, most notably logit and probit models, restrict F in Eq. 7 to belong to a parametric family of CDFs.

The Fechnerian property (6) and the resulting relationship in Eq. 7 satisfy Kellen et al.'s (2021) prescriptive criterion of a "monotonic coordination function." Kellen et al. (2021) advocate broadly that, in order to avoid overspecified relationships between hypothetical constructs and observable measures, scholars in all areas of psychology should only assume that observable variables are monotonic functions of latent variables, without specifying those monotonic relationships in too much detail. Equation 7 implements their recommendation for the relationship between latent strengths of preference and probabilities of observable binary choices by dropping the additional auxiliary structure that a parametric CDF would impose. The next definition defines a Fechnerian model for a generic collection of strength-of-preference functions, without reference to utilities, and without distributional assumptions about F, by relying only on the Fechnerian property (6).

Definition 6 Let $\mathcal{S}$ be a collection of odd real-valued functions, each on the domain $\mathcal{D}$. For a given $S \in \mathcal{S}$, the *distribution-free Fechnerian model*, or *DFM*, for S is the collection $\mathcal{M}_S = \{P \in\ ]0,1[^{\mathcal{D}}$ such that P satisfies the Fechnerian property for $S\}$, i.e., the union of all Fechnerian models for S. Likewise, the *DFM* for $\mathcal{S}$ is the collection of binary choice vectors given by $\mathcal{M}_{\mathcal{S}} = \cup_{S\in\mathcal{S}}\mathcal{M}_S$.

When considering Definition 6, note that a core decision theory with its parameters held fixed may yield a single strength-of-preference function S. As the parameters of the core theory vary, it may yield a collection $\mathcal{S}$ of strength-of-preference functions. For an example involving a utility theory, consider CRRA-EU with risk sensitivity parameter r. When strength of preference is the arithmetic difference of utilities, a fixed value of r yields a fixed mapping S_r from pairs of choice alternatives into strengths of preference, such as those illustrated in the rows of Table 1. The collection of such mappings that arise by considering each value of r in a continuum constitutes a collection $\mathcal{S}$ of strength-of-preference functions.

Geometric Characterization

We now show that a DFM is a well-defined geometric object whose mathematical properties are surprisingly tractable. The geometric framework provides tools from polyhedral combinatorics for understanding these models in ways that can be difficult to achieve with algebraic formulations alone. In particular, as we will see in more detail later, the geometric approach helps to generate powerful substantive theoretical insights, especially related to theoretical parsimony and construct identifiability.

The first geometric concept we need is that of a *hypercube*, a generalization of the 2D square and of the 3D cube to higher dimensions (we will consider examples of these concepts when we discuss Fig. 1). The hypercube is an example of a more general object called a *convex polytope*, which we now define semi-formally. This is an object with 'flat surfaces' and no 'holes' or 'gaps.' One way of characterizing a convex polytope is through its *vertex description:* Take a finite collection of points, so-called *vertices*, and consider everything 'between' any of them, i.e., their *convex hull* (intuitively, the space defined by 'tightly shrink-wrapping' the vertices). Being *convex* means that, for any two points in the polytope, the line segment connecting those two points is completely contained in the polytope. A 3D cube is convex, and it forms the convex hull of its eight 'corners.' The same cube is also the space that is 'sandwiched' between the cube's six (square-shaped) 2D faces. In general, a d-dimensional convex polytope is also the space 'sandwiched' between a collection of (sometimes very many) $(d-1)$-dimensional *facets*. For instance, the facets of a 4D hypercube are, themselves, 3D cubes. Finding either the *vertex description* (i.e., a complete list of all its vertices and their coordinates) or the *facet description* (i.e., a comprehensive list of facets and their descriptions) of a convex polytope can range from trivial to computationally intractable. For one and the same polytope, it is not unusual for one of these, either the vertex description or the facet description, to be easy to obtain and the other to be extremely difficult to find.

A probability distribution over a finite set can be thought of as a convex combination of elements of that set, i.e., a weighted average with nonnegative weights that sum to one. This is equivalent to saying that the elements of that set form the *vertex description* of a polytope. The convex polytope is the collection of all such probability distributions: Every probability distribution over that set forms a point in the polytope. The power of convex geometry concepts like these lies in the potential for characterizing certain probabilistic models in useful ways. Here, we accomplish that goal using the *facet descriptions* of probabilistic models that form convex polytopes. Such a description provides a smallest collection of nonredundant affine inequalities that jointly characterize the model completely. In other words, they form a smallest possible set of necessary and sufficient conditions for a point (here a probability vector) to belong to the polytope (here a model). Suppose that a scholar has derived affine inequalities describing a probabilistic model and wants to determine whether the inequalities characterize the model completely and efficiently. If that model forms a convex polytope then a collection of affine inequalities is necessary and sufficient for that model as soon as these inequalities form a facet description for that polytope, in which case they are also nonredundant. Facet descriptions typically take the form of *facet-defining inequalities*, each of which defines a half-space. The intersection of an open or closed d-dimensional probability hypercube $]0,1[^d$ or $[0,1]^d$ with one or more d-dimensional half-spaces forms a convex polytope (or is empty).[3] In addition to mathematical and theoretical insights, the geometric approach leads to powerful inferential statistics capabilities: Contemporary order-constrained statistical inference methods connect naturally to facet descriptions.

We begin our discussion of geometric properties of Fechnerian models with an observation that specifies a common space for representing probabilistic models of binary choices. We define a pairwise preference in favor of i over j, denoted as $i \succ j$, via $i \succ j \Leftrightarrow P(ij) > 1/2$. The next proposition, which we illustrate in Example 2 and Fig. 1, follows naturally from the concepts we have discussed.

Proposition 3 *The set* $]0,1[^{\mathcal{D}}$ *forms a unit hypercube of dimension* $|\mathcal{D}|$. *The set of all binary choice vectors on* $\mathcal{D}$, *which is a proper subset of* $]0,1[^{\mathcal{D}}$, *also forms a unit hypercube. It has dimension* $|\mathcal{D}|/2$ *because* $P(ij) = 1 - P(ji)$ *for each distinct pair* ij. *A minimal coordinate system for representing this hypercube can be obtained by arbitrarily selecting, for each unordered pair* $\{i, j\}$, *either the probability* $P(ij)$ *or the probability* $P(ji)$. *Consider any point in the unit hypercube* $]0,1[^{|\mathcal{D}|/2}$ *such that, for each coordinate* $P(k\ell)$ *in the minimal coordinate system,* $P(k\ell) \neq 1/2$. *This point lies in one of* $2^{|\mathcal{D}|/2}$ *distinct half-unit hypercubes, each of which defines a binary preference relation on choice options.*

Example 2 To illustrate Proposition 3, let $\mathcal{D}_0$ be the collection of all ordered pairs of choice alternatives among the gambles x, y, and z of Example 1. Hence, $\mathcal{D}_0 = \{xy, yx, xz, zx, yz, zy\}$. Using $P(xy)$, $P(xz)$, $P(yz)$ as a coordinate system, Fig. 1 shows two different half-unit cubes embedded in the unit cube of binary choice probabilities. These are two out of $2^3 = 8$ such half-unit cubes that make up the 3D unit cube. The half-unit cube on the left is defined by the constraints that $P(xy) < 1/2$, $P(xz) < 1/2$, and $P(yz) < 1/2$. It corresponds to the binary preference relation $y \succ x$, $z \succ x$, and $z \succ y$. The half-unit cube on the right (located in the space above the cube shown on the left) is the space where $P(xy) < 1/2$, $P(xz) > 1/2$, and $P(yz) < 1/2$ corresponding to the (intransitive) binary preference relation $y \succ x$, $x \succ z$, and $z \succ y$. We later review how the left-hand side cube can be partitioned into 6 distinct tetrahedra (together with additional lower-dimensional polytopes) corresponding to the six different orders of strengths of preference that are consistent with the preference pattern $y \succ x$, $z \succ x$, and $z \succ y$.

The unit hypercube defined in Proposition 3 provides a common space to test and compare theories of binary choice (e.g., Cavagnaro & Davis-Stober, 2014; Dai, 2017; Morrison, 1963; Regenwetter et al., 2011, 2014, 2018, Zwilling et al., 2019). However, decision theorists have not typically worked in this space because the most commonly used (parametric) probabilistic specifications do not form convex polytopes and hence are not amenable to analysis methods from polyhedral combinatorics. We will see a 3D illustration later, in Example

[3] In many cases, in this paper, we consider open hypercubes $]0,1[^d$. For ease of reading, we do not explicitly distinguish verbally between open and closed hypercubes.

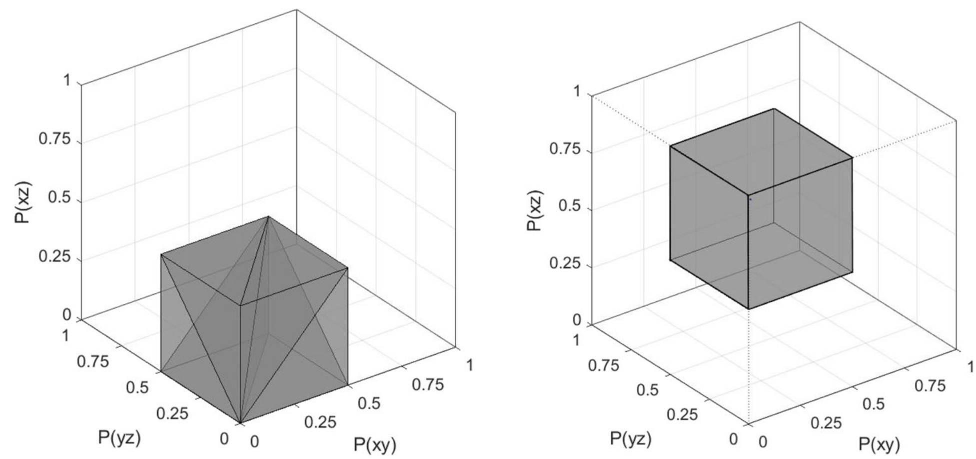

Fig. 1 Two half-unit cubes in $]0, 1[^3$. The left panel represents the binary preference pattern $y \succ x$, $z \succ x$, and $z \succ y$ via $P(xy) < 1/2$, $P(xz) < 1/2$, and $P(yz) < 1/2$. The right panel represents the (intransitive) binary preference pattern $y \succ x$, $x \succ z$, and $z \succ y$ via $P(xy) < 1/2$, $P(xz) > 1/2$, and $P(yz) < 1/2$

6 and Fig. 3. As we unpack further below, the distribution-free Fechnerian specification of any core theory forms a finite set of convex polytopes in the unit-hypercube, each fully characterized by an easy-to-derive facet description. As such, these models are subject to modern and fast order-constrained methods for drawing inferences from data.

Next, we show an example to illustrate how the facet-defining inequalities of a DFM follow directly from the strengths of preference. Following the example, we will state the general result formally.

Example 3 Consider again the gambles x, y, z of Example 1 and $\mathcal{D}_0 = \{xy, yx, xz, zx, yz, zy\}$ as in Example 2. Consider decision maker A in Table 1, whose strengths of preference are

$$0 < S_{-0.4}(zy) = 1.9 < S_{-0.4}(yx) = 17.1 < S_{-0.4}(zx) = 19.0.$$

Abstracting away from the specific value of $r = -0.4$, suppose more generally that

$$S(xz) < S(xy) < S(yz) < 0 < S(zy) < S(yx) < S(zx).$$

By the Fechnerian property (6), the DFM for S on $\mathcal{D}_0$ comprises the binary choice vectors in $]0, 1[^{\mathcal{D}_0}$ that satisfy

$$0 < P(xz) < P(xy) < P(yz) < \frac{1}{2} < P(zy) < P(yx) < P(zx) < 1. \tag{8}$$

Taking into account that $P(xy) = 1 - P(yx)$; $P(xz) = 1 - P(zx)$; $P(yz) = 1 - P(zy)$, like in Example 2 we can consider just the three probabilities $P(xy), P(xz), P(yz)$ as a coordinate system. The model is fully characterized by an irreducible system of four nonredundant affine inequalities,

$$0 < P(xz) \tag{9}$$

$$P(xz) < P(xy) \tag{10}$$

$$P(xy) < P(yz) \tag{11}$$

$$P(yz) < \frac{1}{2}. \tag{12}$$

These form the four facet-defining inequalities of the polytope displayed in the left-most panel of Fig. 2. Inequality 9 states that the polytope is 'above' the horizontal plane defined by $P(xz) = 0$. Inequality 10 states that the polytope is 'below' the 'diagonally leaning' plane defined by

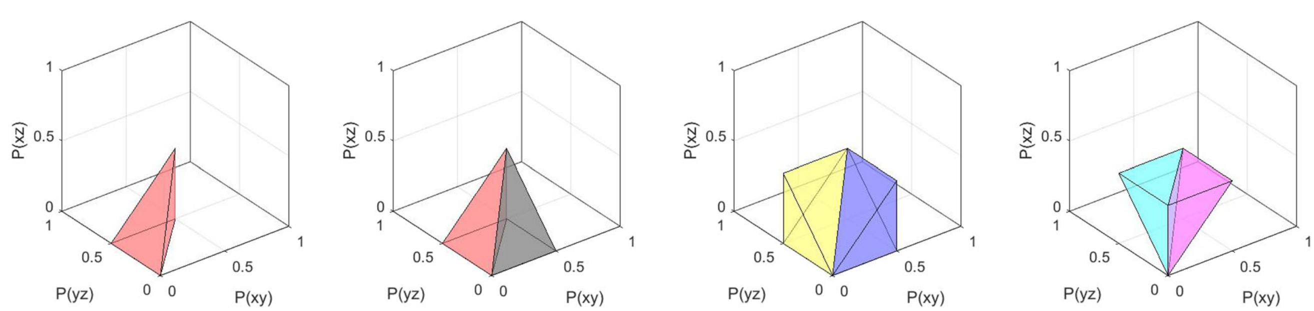

Fig. 2 Tetrahedra corresponding to the orders of strengths of preference compatible with the binary preference relation $y \succ x$, $z \succ x$, and $z \succ y$

$P(xz) = P(xy)$. Inequality 11 states that the polytope is 'on the left of' the vertical plane defined by $P(xy) = P(yz)$. Inequality 12 states that the polytope is in the 'southeast half' of the unit cube. The polytope is a tetrahedron and the four facets of that polytope are the four triangular faces of the tetrahedron.

The following two propositions, both of which are proven in Appendix 2, state these ideas and properties generally and formally.

Proposition 4 *Let S be an odd, real-valued function on a domain $\mathcal{D}$. Then, the collection of all binary choice probability vectors that satisfy the Fechnerian property for S, i.e., $\mathcal{M}_S$, forms a convex polytope of dimension less than or equal to $|\mathcal{D}|/2$, whose facet-defining inequalities follow directly from Condition 6. If S is one-to-one then the corresponding polytope has dimension $|\mathcal{D}|/2$.*

Proposition 5 *If $\mathcal{S}$ is a collection of odd, real-valued functions, each on the domain $\mathcal{D}$, then $\mathcal{M}_\mathcal{S}$ forms a finite disjoint union of convex polytopes of the forms discussed in Proposition 4.*

As we explain throughout the rest of the paper, Proposition 5 can be useful for understanding the nuances and parsimony of theoretical predictions, for mapping out the identifiability of various theoretical constructs, and even for designing diagnostic stimuli. The next proposition wraps up our general geometric results by spelling out that one can partition the entire empirical sample space into polytopes that represent different orders of strengths of preference (including lower dimensional polytopes corresponding to orders of strengths of preference with ties).

Proposition 6 *For a given domain $\mathcal{D}$, and for any binary choice vector P, there exists an odd, real-valued function S on $\mathcal{D}$ such that $P \in \mathcal{M}_S$. Therefore, there are collections $\mathcal{S}$ such that $\mathcal{M}_\mathcal{S}$ partitions the $|\mathcal{D}|/2$-dimensional hypercube of binary choice vectors.*

Example 4 Building on the stimuli of Example 1 and the domain $\mathcal{D}_0 = \{xy, yx, xz, zx, yz, zy\}$ of Examples 2 and 3, Fig. 2 illustrates Propositions 4–6. These illustrations also connect with later results in the paper.

First off, Proposition 4 means that the polytopes in question must have dimension less than or equal to $|\mathcal{D}_0|/2 = 3$. Each panel of the figure visualizes one or two 3-dimensional DFMs, shown in the subspace of $]0, 1[^6$ that is spanned by the coordinates $P(xy)$, $P(xz)$, and $P(yz)$. Most notably, the figure shows various 3-dimensional polytopes that form tetrahedra. Each one of these polytopes has the point $(1/2, 1/2, 1/2)$, which corresponds to perfectly calibrated and unbiased guessing on all paired comparisons, as one of its vertices.

Returning to the model whose facet-description we already considered in Example 3, the tetrahedron in the left-most panel of Fig. 2 is formed by the (open) convex hull of the vertex $(1/2, 1/2, 1/2)$ and three vertices that are all located in the plane $P(xz) = 0$. In the second panel from the left, we have 'added' another such tetrahedron, which is defined by

$$0 < P(xz) < P(yz) < P(xy) < \frac{1}{2}. \tag{13}$$

This tetrahedron is the DFM for S' on $\mathcal{D}$ satisfying

$$S'(xz) < S'(yz) < S'(xy) < 0 < S'(yx) < S'(zy) < S'(zx).$$

These two tetrahedra are on opposite sides of a shared facet, the dark-shaded triangle defined by

$$0 < P(xz) < P(xy) = P(yz) < \frac{1}{2}, \tag{14}$$

i.e., the DFM for S'' on $\mathcal{D}$ satisfying

$$S''(xz) < S''(xy) = S''(yz) < 0$$
$$< S''(zy) = S''(yx) < S''(zx).$$

This triangle is the convex hull of the three vertices shared by the two tetrahedra. Together, these two tetrahedra and their shared face form an open pyramid, which is also the convex hull[4] of the vertex $(1/2, 1/2, 1/2)$ and four vertices located in the plane $P(xz) = 0$, two of which are shared by the two tetrahedra. We provide the mathematical details of the remaining two panels of Fig. 2 in Appendix 3.

Figure 2 also partially illustrates Proposition 6 in that it outlines the partition in the left of Fig. 1. The six tetrahedra shown across the panels of Fig. 2 match the six possible orders of strengths of preference among $S(yx) > 0$, $S(zx) > 0$, and $S(zy) > 0$. Together with various 2-dimensional and 1-dimensional polytopes, as well as the point $(1/2, 1/2, 1/2)$, the polytopes form a partition of the (open) half-unit cube in the left panel of Fig. 1. To fully illustrate the partition of the entire unit cube in Proposition 6, one needs to consider also the other seven half-unit cubes that make up $]0, 1[^3$. These correspond to the seven ways in which one can switch the sign of one or more of these strengths of preference, and thereby make one or more of these binary choice probabilities greater than $1/2$. We later revisit some of the polytopes in

[4] From here on, we omit the technical detail that when we refer to a convex hull, we mean the corresponding open set that excludes the vertices and faces.

Fig. 2, as they represent specific situations that we see in later examples.

To illustrate the theoretical significance of Proposition 6, imagine an extreme case of a decision theory with a free parameter r, a collection of stimuli, and an associated collection of strength-of-preference functions, for which the resulting DFM forms the entire unit hypercube. This means that, as one varies r in the decision theory, one arrives at every possible order of strengths of preference over the collection of stimuli. This has two major implications: Loosely speaking, if $|\mathcal{D}|$ is 'large' then the partition in Proposition 6 will contain 'numerous' polytopes, each of which is associated with a specific range of values of r (or even a unique value). Partitioning the domain for r into 'many' separate ranges means that the Fechnerian property alone provides a 'high level of resolution' into the 'identifiability' of r. On the other hand, if the DFM indeed forms the entire probability hypercube, for a given set of stimuli, that also makes it vacuous in that it cannot be rejected by data when using those stimuli. A researcher would want to notice this, because it means that a parametric model such as a logit specification would make the theory testable exclusively through the extra constraints imposed by the functional form of that logit and not through features of the core theory.

The above insights suggest new perspectives on theoretical parsimony and identifiability as we consider the correspondence between polytopes and associated components of the core theory. For each component polytope of a DFM, we can look up the values of the theory's parameters that map into that polytope, to see whether individual values of the parameters can be identified through that DFM or what ranges of parameter values map into one and the same polytope. For instance, as we later unpack in Table 2, for CRRA-EU on the stimuli of Examples 1–4, the polytope characterized by Condition 8 corresponds to $-0.465 < r < 1.017$. The polytope characterized by Condition 13 corresponds to $r > 1.017$. The facet shared by these two polytopes, which is a two-dimensional polytope characterized by Condition 14, corresponds to $r = 1.017$. Thus, the parameter value of $r = 1.017$ is identifiable through the DFM, while point parameter values for r greater than 1.017, even though they are distinguishable from those less than or equal to 1.017, are nevertheless not identifiable without additional assumptions. More generally, for a given set of stimuli that define $\mathcal{D}$, the correspondence between parameter values of a core theory and the associated polytopes constraining choice probabilities is 'uneven' across the parameter space: For some polytopes, the underlying parameter values that give rise to the associated order of strengths of preference may be a unique combination of values, i.e., fully identifiable, while for others, there may be many combinations of parameter values that give rise to the same order of strengths of preference, hence to the same polytope constraining choice probabilities.

Beyond identifiability issues, it is also important to consider the parsimony and falsifiability of models. According to an algebraic heuristic, the number of parameters in a model is a proxy for its parsimony. This is not literally true. For example, the constraint $0.5 < p < 0.6$ is far more parsimonious than $0.1 < p < 0.9$, even though both constraints involve a single parameter p. According to a related algebraic heuristic, a model is only falsifiable if it has strictly fewer free parameters than there are degrees of freedom in the data. This is also not literally true. Both constraints $0.5 < p < 0.6$ and $0.1 < p < 0.9$ are falsifiable with a single proportion. Returning to DFMs, a DFM on a domain $\mathcal{D}$ can have as many as $|\mathcal{D}|/2$ free parameters, and there are $|\mathcal{D}|/2$-many degrees of freedom in the data, so a DFM can have a free parameter for each stimulus. Nevertheless, as we have seen through the lens of polyhedral geometry, even full-dimensional DFMs

Table 2 Correspondence between values of the risk sensitivity parameter in CRRA-EU and the facet-defining inequalities of polytopes in the DFM, when the strength of preference is $S(ij) = u_r(i) - u_r(j)$, i.e., the strong utility assumption

Parameter of core theory	Utility ranking	Order of positive strengths of preference	Facet-defining inequalities
$r > 1.017$	$u(z) > u(y) > u(x)$	$S(zx) > S(zy) > S(yx) > 0$	$1 > 1 - P(xz) > 1 - P(yz) > 1 - P(xy) > \frac{1}{2}$
$1.017 > r > -0.465$		$S(zx) > S(yx) > S(zy) > 0$	$1 > 1 - P(xz) > 1 - P(xy) > 1 - P(yz) > \frac{1}{2}$
$-0.465 > r > -0.607$	$u(y) > u(z) > u(x)$	$S(yx) > S(zx) > S(yz) > 0$	$1 > 1 - P(xy) > 1 - P(xz) > P(yz) > \frac{1}{2}$
$-0.607 > r > -0.658$		$S(yx) > S(yz) > S(zx) > 0$	$1 > 1 - P(xy) > P(yz) > 1 - P(xz) > \frac{1}{2}$
$-0.658 > r > -0.698$	$u(y) > u(x) > u(z)$	$S(yz) > S(yx) > S(xz) > 0$	$1 > P(yz) > 1 - P(xy) > P(xz) > \frac{1}{2}$
$-0.698 > r > -0.755$		$S(yz) > S(xz) > S(yx) > 0$	$1 > P(yz) > P(xz) > 1 - P(xy) > \frac{1}{2}$
$-0.755 > r > -0.959$	$u(x) > u(y) > u(z)$	$S(xz) > S(yz) > S(xy) > 0$	$1 > P(xz) > P(yz) > P(xy) > \frac{1}{2}$
$-0.959 > r$		$S(xz) > S(xy) > S(yz) > 0$	$1 > P(xz) > P(xy) > P(yz) > \frac{1}{2}$

Note. We omit the subscript r for brevity are falsifiable because they imply inequality constraints on probabilities. Furthermore, the smaller the volume[5] of the corresponding union of convex polytopes, the more parsimonious the DFM.

In sum, the geometric approach offers new insights into the identifiability of hypothetical constructs, as well as a novel perspective on theoretical parsimony, based on the number of polytopes that form a DFM, their dimensions, as well as a DFM's volume. Because a DFM is directly connected to the stimuli under consideration, our approach also better accommodates that, ultimately, identifiability and parsimony should be considered in the context of the stimuli being used. These themes will accompany our development of DFMs in the next section and beyond.

DFMs for a Utility Theory

Motivated by the classical Fechnerian models that we discussed in **Background**, we now consider DFMs for utility theories and how they relate to some other probabilistic choice models. We rely on CRRA-EU to serve as illustrative case studies of how DFMs connect to a range of research programs. Since utility theories do not generate strengths of preference directly, one must theorize about how utility relates to strength of preference. We translate different assumptions about this relationship into the resulting DFMs. These considerations open up insights into a broad spectrum of theoretical questions: Under a given fixed core utility theory, what predictions are invariant across distributional assumptions within a fixed strength of preference theory? Equivalently, which core constructs of the utility theory are identifiable without distributional assumptions, through the mere assumption of the Fechnerian property? What predictions remain invariant across different strength of preference theories and to what extent can one distinguish different strength of preference theories from each other? In the example of CRRA-EU, these questions translate into questions about risk attitudes, the degree to which they constrain behavior without reliance on additional assumptions, the degree to which they are identifiable without such assumptions, and the ways in which risk attitudes and a strength of preference theory interact. Likewise, we could ask, under a given fixed strength of preference theory, to what extent different core utility theories are identifiable without distributional assumptions.

A major goal along the way is to better understand the polyhedral geometry of DFMs. We explore the geometric relationships between different DFMs, as well as other model classes such as random utility, supermajority, and constant error. We discuss the role of these geometric representations for testing a utility theory, estimating its parameters, and comparing it to other theories (both empirically and theoretically). Those insights are, in turn, closely intertwined with theoretical parsimony. In particular, DFMs permit the statistical complexity of a model to grow hand-in-hand with the complexity of the data generating process as one adds more and more stimuli to an experimental design, a feature not shared by parametric Fechnerian models.

Strong Utility DFM

Perhaps the most prominent way to introduce strength of preference induced by a utility function is to define it via the arithmetic difference between utility values, as in the strong utility model of Definition 1.

Definition 7 Let u be a utility function on a domain on $\mathcal{D}$. An assignment S of strengths of preference to elements of $\mathcal{D}$ satisfies the *strong utility* assumption with respect to u if

$$S(ij) = u(i) - u(j), \qquad \forall ij \in \mathcal{D}. \tag{15}$$

The DFM for S in Eq. 15 is called the *strong utility DFM* on $\mathcal{D}$ for u. We say that a DFM on $\mathcal{D}$ is *a* strong utility DFM if it is *the* strong utility DFM for some such u.

Note that Definition 7 extends naturally to an entire utility theory by considering all utility functions associated with that theory (e.g., by varying free parameters in the theory), all S associated with those utility functions through Eq. 15, and taking the union of those DFMs. Through the lens of Definition 6, one may view a strong utility DFM as the collection of binary choice vectors satisfying the Fechnerian property under the strong utility assumption. Equivalently, through the lens of Proposition 2 and Definition 1, one may view a strong utility DFM as the union of all strong utility models that can arise by varying F in Eq. 1 and requiring that F to be strictly increasing.

Example 5 We develop a strong utility DFM for CRRA-EU, building on Examples 1–4. As we vary r in CRRA-EU, the utilities of x, y, and z vary. The strong utility assumption therefore assigns different strengths of preference to elements of $\mathcal{D}_0$, depending on r. Writing $S_r(ij) = u_r(i) - u_r(j)$ for the strength of preference induced by CRRA-EU with fixed r, $\mathcal{S} = \{S_r\}_{r\in\mathbb{R}}$ comprises the strength-of-preference assignments under all possible values of r on the real number line $\mathbb{R}$.

[5] For a probability model that forms a convex polytope, the reciprocal of the volume of that polytope is the upper bound on the Bayes factor that one can obtain in comparing the model against an unconstrained reference (Klugkist and Hoijtink, 2007; Zwilling et al., 2019). In particular, Constraint Set 14 characterizes a 2-dimensional polytope in 3-dimensional space, for $r = 1.017$. This polytope has volume zero in $]0, 1[^3$. This means that there is no upper limit as to how much evidence a very large, very well-fitting data set can provide in favor of the DFM for CRRA-EU restricted to $r = 1.017$.

The resulting $\mathcal{M}_{\mathcal{S}}$ is the strong utility DFM for CRRA-EU on this domain.

Since r takes values in a continuum, understanding $\mathcal{M}_{\mathcal{S}}$ would seem to be a formidable task. However, by Proposition 5, $\mathcal{M}_{\mathcal{S}}$ is a finite disjoint union of convex polytopes. Moreover, by Proposition 4, each polytope corresponds to an order of strengths of preference whose facet-defining inequalities follow directly from the Fechnerian property. This means that, to write down the constraints on choice probabilities that characterize $\mathcal{M}_{\mathcal{S}}$, we only need to find the (finitely many) orders of strengths of preference, for the stimuli in $\mathcal{D}_0$, that can be generated from CRRA-EU via the strong utility assumption (15) as we vary r. There are a variety of approaches for doing this, ranging from analytical derivations to computational approximations (see, e.g., Pitt et al., 2006).

For simplicity, in this example, we followed the approximation route: We generated the orders of strengths of preference for the DFM by running a grid search of r between -3 and $+3$, with a precision of 0.001. Table 2 lists these orders along with the corresponding values of r. To keep the table concise, we only show the order among positive strengths of preference. In each case, the full order of strengths of preference follows from the fact that $S(ij) = -S(ji)$. For example, the order of strengths of preference for $r > 1.017$ is

$$S(zx) > S(zy) > S(yx) > 0 > S(xy) > S(yz) > S(xz).$$

The table is organized by decreasing values of r from top to bottom. As r varies, we get different orders of strengths of preference in different rows of the table. In addition, we group together orders of strengths of preference that are compatible with the same ranking of utility values, shown in the second column of the table. Recall from Example 2 and Fig. 1 that such groupings have a geometric interpretation in terms of half-unit cubes. Polytopes corresponding to the same ranking of utility values are nested in the same half-unit cube. They have a common vertex (not included in the open polytope) at a corner of the probability cube, in which $P(ij) \in \{0, 1\}$ for all ij. This would correspond to deterministic choice in accordance with the ranking of utility values. For example, the first two rows in Table 2 characterize the two tetrahedra in the second panel from the left in Fig. 2. These are nested in the half-unit cube in the left panel of Fig. 1, and they share the vertex $P(xy) = P(xz) = P(yz) = 0$. That vertex, in turn, corresponds to deterministic choice of y over x, of z over x, and of z over y.

For each order of strengths of preference, Table 2 also shows the facet-defining inequalities of the corresponding polytope in the coordinate system we selected earlier. The facet-defining inequalities follow from the orders of positive strengths of preference as follows (see also Proposition 4 and Example 3). First, for $ij \in \{xy, xz, yz\}$, replace $S(ij)$ with $P(ij)$. Then, for $k\ell \notin \{xy, xz, yz\}$, replace $S(k\ell)$ with $1 - P(\ell k)$. Finally, add a strict upper bound of 1 and a strict lower bound of $1/2$ to the system of inequalities. The lower bound follows from the fact that the strengths of preference in the table are strictly positive.

Note that the table omits orders of strengths of preference that contain ties. These occur at point values of r that join the open intervals shown in the table. Geometrically, such orders also correspond to lower-dimensional polytopes that join two or more higher dimensional polytopes, so that their union is convex. For example, $r = 1.017$ joins the open intervals shown in the first two rows of the table, and yields $S(zx) > S(zy) = S(yx) > 0$. The corresponding constraints on binary choice probabilities, in the coordinate system we selected earlier, are $1 > 1-P(xz) > 1-P(yz) = 1 - P(xy) > 1/2$. These constraints characterize a triangular face that joins the two tetrahedra corresponding to the first two rows of the table, which appear in the second panel from the left in Fig. 2. Notice how the geometry pinpoints $r = 1.017$ as an identifiable knife-edge parameter value that generates a far more restrictive, lower-dimensional polytope, using nothing more than the strong utility assumption (15) applied to CRRA-EU.

In all, on this particular set of choice stimuli, the strong utility DFM for CRRA-EU comprises eight open, 3-dimensional, convex polytopes: one for each order of strengths of preference (without ties) shown in Table 2. It also includes seven lower-dimensional polytopes corresponding to orders of strengths of preference with ties. Four of these correspond to cases with equality between two nonzero strengths of preference, and therefore equality between two choice probabilities that are also different from $1/2$. These polytopes are (open) 2-dimensional faces (triangles) that join two tetrahedra in the DFM. The other three lower dimensional polytopes correspond to the cases where one of the choice probabilities equals $1/2$. This happens at very specific values of r where the utilities of two alternatives are equal to each other, so that the corresponding strength of preference is zero. In those cases, the other two strengths of preference are equal to each other. Consequently, the corresponding choice probabilities are also equal to each other but may vary freely between zero and $1/2$. These polytopes are (open) 1-dimensional edges that join two neighboring tetrahedra (i.e., line segments). The left panels of Fig. 3 depict the union of all 15 polytopes from two different angles of view.

We can review these insights through the lens of the coordination problem (e.g., Kellen et al., 2021) and Bhatia and Loomes' (2017) distinction of "preference noise" (e.g., fluctuating utility values) and "response noise" (e.g., choice 'errors'). There are specific values of r in which we obtain equality among strengths of preference (r values of 1.017,

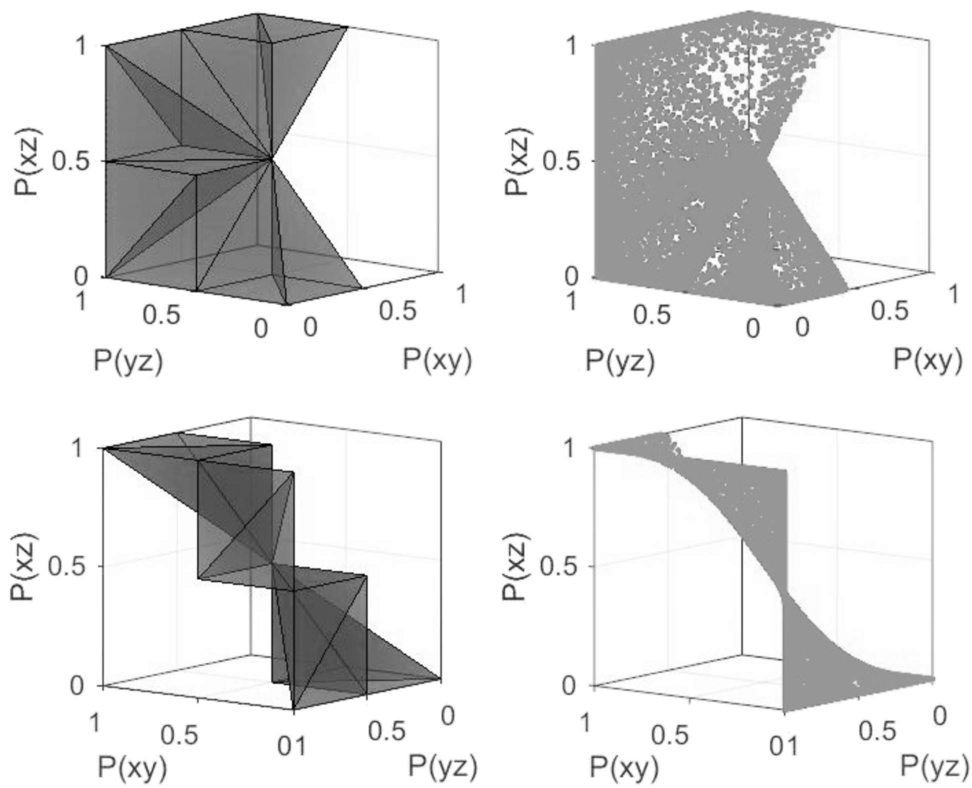

Fig. 3 Probabilistic specifications of CRRA-EU: strong utility DFM (left panels) and strong utility logit (right panels), from two different angles of view (upper and lower panels)

−0.607, −0.698, −0.959) while having strict inequalities among utilities. There are other values of r at which one or more strengths of preference are zero and where utilities are equal to each other (r values of −0.465, −0.658, −0.755). This shows the 'unevenness' of the relationship between r values and choice probabilities when viewed through the lens of a strong utility DFM, and hence the 'uneven' coordination between constructs and behavior without additional auxiliary constraints imposed by parametric distributional assumptions (see also Broomell & Bhatia, 2014, for a related discussion of parameter identifiability). Regarding Bhatia and Loomes' "preference noise," as long as that noise concentrates all its probability mass in a range that stays within one line in Table 2, the order of strengths of preference in a strong utility DFM is not affected, and hence neither are the order-constraints on the choice probabilities.

Before we proceed to the next model class, we compare the strong utility DFM with a typical logit specification of the same utility theory, defined as $P(ij) = [1 + e^{-\gamma(u(i)-u(j))}]^{-1}$, where $\gamma > 0$ is a scaling parameter. We refer to this model as the *strong utility logit* specification of the utility theory.[6] It is a Fechnerian model because it can be written in the form of Eq. 7, where F is a logistic CDF with scaling parameter γ, and S is the strength of preference under the strong utility assumption (i.e., the arithmetic difference of utility values). This restriction on F is what distinguishes the strong utility logit from the strong utility DFM.

Example 6 The right panels of Fig. 3 depict[7] the binary choice probability vectors that can be obtained from the strong utility logit specification in CRRA-EU for $-3 < r < 3$ and $0 < \gamma < 10$ for the same stimuli we have considered in earlier examples. Comparing the panels on the left (DFM) with those on the right (logit) shows how restrictive the strong utility logit is relative to the strong utility DFM. The DFM, having three free parameters, is full-dimensional within the cube, while the logit, having just two free parameters, defines a two dimensional surface (looking like a twisted, wavy piece of foil) that is nested within the DFM.

More generally, by virtue of having a fixed number of free parameters regardless of the number of choice pairs in the domain, parametric models like this logit become increasingly restrictive relative to the DFM as the number of choice pairs increases. The contrast between the strong utility logit and the strong utility DFM for CRRA-EU becomes particularly striking when we think about external validity of

[6] Other models taking the form of Eq. 7, with different assumptions about how strengths of preference are related to utilities, may also be regarded as logit specifications whenever F is a logistic CDF. Similarly, models of the form of Eq. 7 may be regarded as probit specifications whenever F is a normal CDF.

[7] The pictures approximate a continuous shape with a discrete set of points.

such models: What does it mean for the model to apply to a hundred different types of choices, not just those examined in the laboratory? If one were to consider these models in, say, a hundred-dimensional space spanned by a coordinate system of 100 distinct gamble pairs, the strong utility logit of CRRA-EU remains a two-dimensional shape in 100-dimensional space, whereas the strong utility DFM forms a union of polytopes of many different dimensions, ranging from 1-dimensional to 100-dimensional. It is a mathematical fact that, with enough data and enough stimuli, the strong utility logit will eventually be rejected on virtually any conceivable data. This feature is not shared by the DFM. As we have already alluded to, the arbitrary restrictiveness of the strong utility logit makes the process of inference from data simpler computationally, but it can also lead to false conclusions such as biased parameter estimates, incorrect model selection, or misclassification of qualitative features like risk seeking/aversion. We unpack this issue further in **Inference from Data**. We also show how a DFM allows us to avoid various inherent downsides of parametric specifications.

Moderate Utility DFMs

Different DFMs for the same core utility theory arise from different assumptions about the relationship between utility and strength of preference. For example, for the case where choice alternatives are gambles, Wilcox (2011) defines a *contextual utility model* of the form

$$P(ij) = F\left[\frac{u(i) - u(j)}{a_{ij} - b_{ij}}\right], \tag{16}$$

where i and j are gambles, a_{ij} is the utility of (a degenerate gamble with) the largest possible reward in either i or j, b_{ij} is the utility of (a degenerate gamble with) the smallest possible reward in either i or j, and F is a CDF with $F(x) = 1 - F(-x)$, hence $F(0) = 1/2$.

This model motivates the following definition.

Definition 8 Let u be a utility function on gambles. An assignment S of strengths of preference to elements of a domain $\mathcal{D}$ satisfies the *contextual utility* assumption with respect to u if

$$S(ij) = \frac{u(i) - u(j)}{a_{ij} - b_{ij}}, \qquad \forall ij \in \mathcal{D}, \tag{17}$$

where a_{ij} is the utility of (a degenerate gamble with) the largest possible reward in either i or j, and b_{ij} is the utility of (a degenerate gamble with) the smallest possible reward in either i or j. The DFM for S satisfying Eq. 17 is the *contextual utility DFM* on $\mathcal{D}$ for u. We say that a DFM on $\mathcal{D}$ is *a* contextual utility DFM if it is *the* contextual utility DFM for some such u.

Like Definition 7, so does Definition 8 extend naturally to an entire utility theory. When researchers fit contextual utility models to data, they typically treat F in Eq. 16 as a parametric CDF, and consider all permissible utility functions u in that theory. For example, Wilcox (2011) assumes a logistic CDF, resulting in a specification that could be called a *contextual utility logit*. Drichoutis and Lusk Drichoutis & Lusk (2014) assumes a normal CDF, yielding what we would call a *contextual utility probit* specification. These contextual utility models are also Fechnerian models in the sense of Definition 5. They are special cases of the contextual utility DFM for the function u, which, in turn, comprises all contextual utility models for u obtained by varying F in Eq. 16, and requiring that F to be strictly increasing.

Example 7 To illustrate the contextual utility DFM on $\mathcal{D}_0$ for CRRA-EU, we first generated the orders of strengths of preference that arise as we vary r from -3 to $+3$ in increments of 0.001. Table 3 lists the resulting utility rankings and orders of strengths of preference, together with their corresponding ranges of values of r. For brevity, as with Table 2, we only show the orders among positive strengths of preference, and we omit orders with ties. The second column of the table shows all six orders of strengths of preference that are possible within each utility ranking in the first column. Recall that these groupings correspond to partitions of a half-unit cube into six tetrahedra (see Figs. 1 and 2, for example). Columns 3 and 4 show the values of r, if any, that yield the corresponding order of strengths of preference under strong and contextual utility, respectively.

The table also provides insight into the polyhedral geometry of these two models. Each model is a union of tetrahedra — one for each permissible order of strengths of preference — along with the lower-dimensional polytopes that 'glue' the tetrahedra together. Columns 3 and 4 of the table show that, as r varies, each model generates 8 different orders of strengths of preference — hence 8 tetrahedra. Six tetrahedra are shared by the two DFMs, although in some cases for different values of r (e.g., the first two rows in the second grouping). Each DFM also includes two tetrahedra that the other does not.

The contextual utility assumption represents one very specific way of infusing 'context' into strengths of preference derived from utilities. It is also limited to cases where choice alternatives are gambles. For generic choice alternatives and a much broader notion of stimulus-dependent contexts, we can build on the general moderate utility model of Eq. 4.

Definition 9 Let u be a utility function. An assignment S of strengths of preference to elements in a domain $\mathcal{D}$ satisfies the *moderate utility* assumption with respect to u if

$$S(ij) = \frac{u(i) - u(j)}{d(i, j)}, \qquad \forall ij \in \mathcal{D}, \tag{18}$$

Table 3 Risk sensitivity parameter values that generate each possible order of strengths of preference, under weak, strong, and contextual utility in Examples 5, 7, and 9. Angled brackets indicate orders that are not compatible with moderate utility (Example 8)

Utility ranking and associated risk sensitivity parameter values	Positive strengths of preference under weak utility,	and compatible orders of strengths of preference	Risk sensitivity parameter values under strong utility	Risk sensitivity parameter values under contextual utility
$u_r(z) > u_r(y) > u_r(x)$, $r > -0.465$	$S(yx), S(zx), S(zy)$	$S(zx) > S(zy) > S(yx)$	$r > 1.017$	$r > -0.186$
		$S(zx) > S(yx) > S(zy)$	$1.017 > r > -0.465$	$-0.186 > r > -0.465$
		$S(zy) > S(zx) > S(yx)$	–	–
		$S(yx) > S(zx) > S(zy)$	–	–
		$\langle S(yx) > S(zy) > S(zx)\rangle$	–	–
		$\langle S(zy) > S(yx) > S(zx)\rangle$	–	–
$u_r(y) > u_r(z) > u_r(x)$, $-0.465 > r > -0.658$	$S(yx), S(zx), S(yz)$	$S(yx) > S(zx) > S(yz)$	$-0.465 > r > -0.607$	$-0.465 > r > -0.542$
		$S(yx) > S(yz) > S(zx)$	$-0.607 > r > -0.658$	$-0.542 > r > -0.554$
		$S(yz) > S(yx) > S(zx)$	–	$-0.554 > r > -0.658$
		$S(zx) > S(yx) > S(yz)$	–	–
		$\langle S(zx) > S(yz) > S(yx)\rangle$	–	–
		$\langle S(yz) > S(zx) > S(yx)\rangle$	–	–
$u_r(y) > u_r(x) > u_r(z)$, $-0.658 > r > -0.755$	$S(yx), S(xz), S(yz)$	$S(yz) > S(yx) > S(xz)$	$-0.658 > r > -0.698$	$-0.658 > r > -0.698$
		$S(yz) > S(xz) > S(yx)$	$-0.698 > r > -0.755$	$-0.698 > r > -0.755$
		$S(yx) > S(yz) > S(xz)$	–	–
		$S(xz) > S(yz) > S(yx)$	–	–
		$\langle S(yx) > S(xz) > S(yz)\rangle$	–	–
		$\langle S(xz) > S(yx) > S(yz)\rangle$	–	–
$u_r(x) > u_r(y) > u_r(z)$, $-0.755 > r$	$S(xy), S(xz), S(yz)$	$S(xz) > S(yz) > S(xy)$	$-0.755 > r > -0.959$	–
		$S(xz) > S(xy) > S(yz)$	$-0.959 > r$	–
		$S(yz) > S(xz) > S(xy)$	–	$-0.755 > r$
		$S(xy) > S(xz) > S(yz)$	–	–
		$\langle S(yz) > S(xy) > S(xz)\rangle$	–	–
		$\langle S(xy) > S(yz) > S(xz)\rangle$	–	–

Note. We omit the subscript r from strengths of preference for brevity

where $d(i, j)$ is a distance metric. If S satisfies the moderate utility assumption with respect to u, for some d, then the DFM for S is the *moderate utility DFM* on $\mathcal{D}$ for u and d. We say that a DFM on $\mathcal{D}$ is *a* moderate utility DFM if it is *the* moderate utility DFM for some such u and d.

The contextual utility DFM is one example of a moderate utility DFM. By considering other distance metrics in Eq. 18 one can construct many other moderate utility DFMs either for one utility function or, more broadly, for an entire core theory. We refer to the DFM for the family $\mathcal{S}_u$ that one obtains by considering all possible distance metrics d in Eq. 18 as the *comprehensive moderate utility DFM* for u. Likewise, the comprehensive moderate utility DFM for a utility theory is the union of the comprehensive moderate utility DFMs over utility functions in that theory. Through the lens of Proposition 2 and Definition 2, the comprehensive moderate utility DFM for a given u (or a given utility theory) can also be viewed as the union of all moderate utility models for that u (or that theory), as one considers all F and d in Eq. 4, and as one requires F to be strictly increasing.

A necessary set of inequality constraints for moderate utility DFMs follows from the fact that Eq. 4 is equivalent to a strengthened version of *moderate stochastic transitivity*,[8] given by

$$P(ij) \geq \frac{1}{2} \quad \wedge \quad P(jk) \geq \frac{1}{2}$$
$$\Rightarrow \begin{cases} P(ik) > min\{P(ij), P(jk)\} \\ \quad\text{or} \\ P(ik) = P(ij) = P(jk) \end{cases}, \qquad \forall ij, jk, ik \in \mathcal{D}. \quad (19)$$

This equivalence, proved by He & Natenzon (2019), implies that any moderate utility DFM must satisfy moderate stochastic transitivity (in both its strengthened and conventional form). However, most moderate utility DFMs are far more restrictive than moderate stochastic transitivity (19). This is because they typically do not accommodate all conceivable preference rankings. Rather, they are constrained by those rankings of utility values that the core utility theory permits. We now illustrate this and other features of moderate utility DFMs on the example of CRRA-EU.

[8] Technically, moderate stochastic transitivity treats $\mathcal{D}$ as the set of all ordered pairs of choice options from a master set of choice alternatives, even if some of those pairs were never used as stimuli.

Example 8 As we have seen in earlier examples, CRRA-EU permits four utility rankings among the choice options x, y, z. These are listed in the left-most column of Table 3. In general, for three choice options, any such utility ranking permits six different orders of strengths of preference. Within each group of six orders, two are incompatible with moderate stochastic transitivity. In the example of $\mathcal{D}_0$, the second column of Table 3 marks the incompatible cases with angled brackets, two of which are also illustrated in the right-most panel of Fig. 2. To see why they are incompatible, note that, for any $\mathcal{D}$, the strengthened moderate stochastic transitivity in Eq. 19, together with the Fechnerian property (6), implies that

$$S(ij) \geq 0 \quad \wedge \quad S(jk) \geq 0$$
$$\Rightarrow \begin{cases} S(ik) > min\{S(ij), S(jk)\} \\ \quad\quad \text{or} \\ S(ik) = S(ij) = S(jk) \end{cases}, \qquad \forall ij, jk, ik \in \mathcal{D}. \tag{20}$$

Returning to $\mathcal{D}_0$, we have seen in column 3 of Table 3 that 24 orders of strengths of preference are compatible with CRRA-EU on $\mathcal{D}_0$, when allowing any conceivable strength-of-preference function. Eliminating those that violate Implication 20 leaves 16 that are also compatible with moderate stochastic transitivity. Thus, the moderate utility DFM for CRRA-EU on $\mathcal{D}_0$ comprises 16 tetrahedra, together with some of the lower dimensional polytopes that join them[9].

A moderate utility DFM embodies weaker assumptions than a strong or contextual utility DFM regarding how utilities relate to strengths of preference. It specifies a functional form, but not a specific function. The tradeoff for this generality is that, under a moderate utility DFM, there is no direct correspondence between strengths of preference and parameter values in the core theory: A given parameter value in the core theory is compatible with multiple orders of strengths of preference.

Next, we consider a DFM for utility theories based on an even broader class of strength-of-preference assignments. These may violate the moderate utility assumption.

Weak Utility DFM

We can also derive a DFM for a utility theory from the simple assumption that the strength of preference for the option with the higher utility is greater than zero.

Definition 10 Let u be a utility function. An assignment S of strengths of preference to elements in a domain $\mathcal{D}$ satisfies the *weak utility* assumption with respect to u if

$$S(ij) > 0 \quad \Leftrightarrow \quad u(i) > u(j), \qquad \forall ij \in \mathcal{D}. \tag{21}$$

If S satisfies the weak utility assumption with respect to u then the DFM for S is a *weak utility DFM* for u on $\mathcal{D}$.

Every strong, contextual, or moderate utility DFM is also a weak utility DFM because the strong, contextual, and moderate utility assumptions each imply the weak utility assumption. The union of all weak utility DFMs, for a given utility function u and domain $\mathcal{D}$, is itself a DFM. We call this the comprehensive weak utility DFM for u. It is the DFM for the family $\mathcal{S}_u$ of all S satisfying the weak utility assumption for u, and it comprises all binary choice vectors $P \in [0, 1]^{\mathcal{D}}$ with the property that

$$P(ij) > \frac{1}{2} \quad \Leftrightarrow \quad u(i) > u(j), \qquad \forall ij \in \mathcal{D}. \tag{22}$$

A collection of probabilities satisfying Condition 22 is known as a *weak utility model* (Block & Marschak, 1960; Luce & Suppes, 1965). Thus, the comprehensive weak utility DFM for a function u can be viewed as the union of all weak utility models for that function. We can extend the above more broadly to a weak utility DFM (or the comprehensive weak utility DFM) for an entire theory by considering all utility functions associated with the theory.

For a collection of binary choice probabilities, the existence of a function u that satisfies Condition 22 is equivalent to *weak stochastic transitivity*[10] (Block & Marschak, 1960), henceforth $\mathcal{WST}$, given by

$$P(ij) \geq \tfrac{1}{2} \quad \wedge \quad P(jk) \geq \tfrac{1}{2}$$
$$\Rightarrow \quad P(ik) \geq \tfrac{1}{2} \qquad \forall ij, ik, jk \in \mathcal{D}. \tag{23}$$

Therefore, any weak utility DFM must satisfy weak stochastic transitivity. Geometrically, a weak utility DFM is a union of half-unit-hypercubes. We illustrate this by deriving the comprehensive weak utility DFM for CRRA-EU on $\mathcal{D}_0$ in the next example.

Example 9 Consider CRRA-EU on the domain $\mathcal{D}_0$ from Example 1. On this domain, as r varies, CRRA-EU can generate four different utility rankings. These are shown in the first column of Table 3. Under the weak utility assumption (21), each utility ranking determines a set of positive strengths of preference. These are shown in the second column of Table 3. In turn, by the Fechnerian property, each set

[9] Eq. 20 also rules out some orders of strengths of preference with ties, which are not shown in Table 3. The corresponding lower dimensional polytopes are not included in the DFM.

[10] Technically, weak stochastic transitivity treats $\mathcal{D}$ as the set of all ordered pairs of choice options from a master set of choice alternatives, even if some of those pairs were never used as stimuli.

of positive strengths of preference determines a half-unit-cube in $]0, 1[^{\mathcal{D}_0}$. For example, for $r > -0.465$ the utility ranking under CRRA-EU is $u_r(z) > u_r(y) > u_r(x)$. Under weak utility, this is equivalent to $S(zy), S(zx), S(yx) > 0$, which corresponds to $P(zy), P(zx), P(yx) > 1/2$. Recall that the half-unit cube given by these constraints is shown in the left panel of Fig. 1 (but in a different coordinate system). Likewise, as r is allowed to vary freely, CRRA-EU generates three other utility rankings corresponding to three other half-unit cubes. In all, the comprehensive weak utility DFM for CRRA-EU on this domain comprises the four half-unit cubes corresponding to the four rankings of utility values in the first column of Table 3.

Comparing DFMs with Other Probabilistic Choice Models

Other families of probabilistic models for utility theories have been proposed in the literature, some with mathematical and conceptual formulations that may be hard to differentiate from DFMs. We now compare these models with DFMs in a common geometric framework. The polyhedral geometry representation helps to disentangle these families in ways that may not be apparent when viewing them algebraically.

Distribution-Free Random Utility and Random Preference Models

As we have seen earlier, a binary random utility model on $\mathcal{C}$ is a collection of binary choice probabilities for which there is a random vector $\boldsymbol{U}$ on $\mathcal{C}$ and a probability measure p, such that $P(ij) = p(\boldsymbol{U}_i > \boldsymbol{U}_j)$. We have seen the special case of independent normal random utilities as a random utility formulation of a probit model. We can derive a general *distribution-free random utility model* for a utility theory by permitting any (joint) probability distribution over the parameter(s) of the core theory and defining $P(ij)$ to be the probability of those parameter values for which $u(i) > u(j)$ (see, e.g., Marley & Regenwetter 2016; Regenwetter et al., 2014; Zwilling et al., 2019). The corresponding *random preference model* allows for any probability distribution over permissible preference states and defines $P(ij)$ to be the probability of those preference states $\succ$ in which $i \succ j$ (for more details, including more precise mathematical definitions, see the references above). For simplicity, we state the next example also semi-formally.

Example 10 Like in earlier examples, consider again CRRA-EU on the same stimuli. According to the general distribution-free random utility model for CRRA-EU, there exists a probability distribution Pr over r such that the binary choice probabilities are marginals of that distribution, namely

$$P(ij) = Pr(\{r \mid u_r(i) > u_r(j)\}).$$

In words, the probability of choosing i over j is the marginal probability that r takes a value for which $u_r(i) > u_r(j)$. The equivalent random preference model for CRRA-EU considers the preference patterns $\succ_r$ associated with the values of r and defines

$$P(ij) = Pr(\{r \mid i \succ_r j\}).$$

In words, the probability of choosing i over j is the marginal probability that r takes a value for which i is preferred to j.

As we have seen in Table 2, there are four different utility rankings associated with CRRA-EU, on these stimuli. Accordingly, the random preferences (and distribution-free random utility) model forms the convex hull of four vertices. In the coordinate system given by $P(xy), P(xz), P(yz)$, these four vertices have coordinates

(0, 0, 0), which corresponds to the preference pattern
$y \succ x,\ z \succ x,\ z \succ y$ having probability 1,
(0, 0, 1), which corresponds to the preference pattern
$y \succ x,\ z \succ x,\ y \succ z$ having probability 1,
(0, 1, 1), which corresponds to the preference pattern
$y \succ x,\ x \succ z,\ y \succ z$ having probability 1,
(1, 1, 1), which corresponds to the preference pattern
$x \succ y,\ x \succ z,\ y \succ z$ having probability 1.

Figure 4 shows the strong utility DFM for CRRA-EU and the random preference model for the same stimuli from two different angles of view. The convex hull of the above four vertices is outlined with thick lines. As we have seen before, the strong utility DFM consists of eight tetrahedra (as well as some lower dimensional polytopes). Four of its tetrahedra are nested inside the random preference model. These are lightly shaded. The other four tetrahedra, which are dark shaded, lie outside the convex polytope that forms the random preference model. Notice that there are unshaded regions in the random preference model, visible from the angle of view in the right side panel of the figure. As the figure illustrates, the strong utility DFM and the random preference model overlap, but neither is nested in the other.

Modal Choice, Supermajority, and Constant Error Models

Some DFMs are closely related to other model classes in the literature. Most notably, the weak utility model (22) is also known as the *modal choice specification* or the *majority*

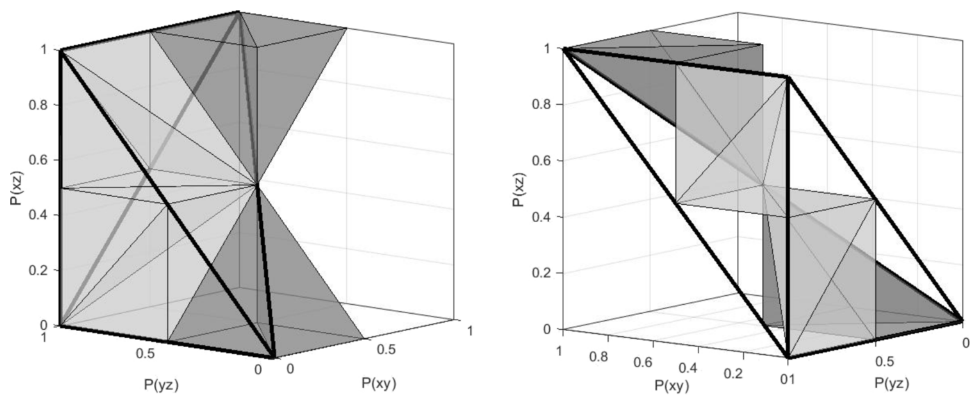

Fig. 4 Strong utility DFM and random preference model of CRRA-EU from two angles of view. The random preference model is the tetrahedron whose edges are the thick lines in the figure. The intersection of the strong utility DFM and the random preference model is shaded light gray. The parts of the DFM that are not in the random preference model are shaded dark gray. Unshaded regions in the random preference model are choice probabilities that are consistent with the random preference model but ruled out by the strong utility DFM

specification of u. It is a special case of a broader class of "supermajority" specifications and "distance-based" probabilistic specifications of u (Regenwetter et al., 2014; Zwilling et al., 2019), all of which also form geometric objects, in most cases, collections of convex polytopes. These models assume the existence of a fixed deterministic core preference pattern or utility function and derive constraints on choice probabilities through various assumptions about probabilistic errors in responses. We illustrate these, and their relationship with DFMs, with another example.

Example 11 Once again, consider CRRA-EU on the domain from Examples 1–10, but with the additional constraint that $r > -0.465$, where $u_r(z) > u_r(y) > u_r(x)$. Figure 5 shows six different probabilistic models for CRRA-EU with this constraint and these stimuli. For better visibility of these shapes, the figure offers a different angle of view than earlier figures. Each panel of the figure shows one gray-shaded object to denote the model of interest. The gray-shaded pyramid in the upper left panel is the strong utility DFM. We have discussed it already in Examples 3 and 4, including the fact that it is the disjoint union of the two tetrahedra in the left two panels of Fig. 2 together with a triangle that forms a shared facet. The upper right panel's gray shape (in Fig. 5) is the comprehensive moderate utility DFM. It also forms a disjoint union of polytopes we have seen before, namely the four tetrahedra in the left three panels of Fig. 2 and their shared facets. The gray shape in the center left panel is the comprehensive weak utility DFM. Its contours are also shown with dashed outlines in the other panels of the figure to facilitate the visual comparison between models. Like the strong utility DFM and the comprehensive moderate utility DFM, the comprehensive weak utility DFM can be partitioned into a disjoint union of polytopes, in this case according to all possible orders of strengths of preference that are compatible with $u(z) > u(y) > u(x)$. The center right panel shows the 0.75-supermajority specification of CRRA-EU with the same constraint on r and the same stimuli. Notice that the dashed outline of the comprehensive weak utility DFM is also an outline of the 0.50-majority specification since those models are identical. The bottom left panel shows the constant error model (Harless & Camerer, 1994; Wakker et al., 1994) for $u(z) > u(y) > u(x)$, with error rate e in Eq. 5 constrained to be $0 \leq e \leq 0.5$. That constant error model forms a 1-dimensional line segment connecting the points with coordinates (0, 0, 0) and $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ in the coordinate system given by $P(xy)$, $P(xz)$, and $P(yz)$. This line segment also forms a 1-dimensional edge shared by the two (respectively four or six) tetrahedra composing the strong (respectively moderate or weak) utility DFM in the upper left (respectively upper right or center left) panel. The bottom right panel shows the random preference model of CRRA-EU, with the same constraint on r. Since $u_r(z) > u_r(y) > u_r(x)$ for all $r > -0.465$, it follows that the random preference model places all probability mass on a single preference pattern $\succ$ given by $z \succ y$, $y \succ x$, $z \succ x$, no matter what probability distribution we consider over values of $r > -0.465$. That means that this particular random preference model forms a single point at coordinates (0, 0, 0) in the probability cube with this coordinate system. In other words, it predicts 'deterministic' behavior on these stimuli: choose y over x with probability 1, z over x with probability 1, and z over y with probability 1.

Inference from Data

DFMs recast inference about a core decision theory in terms of inference about orders of binary choice probabilities. They allow the scientist to test various hypotheses about the core theory by evaluating equivalent hypotheses regarding orders

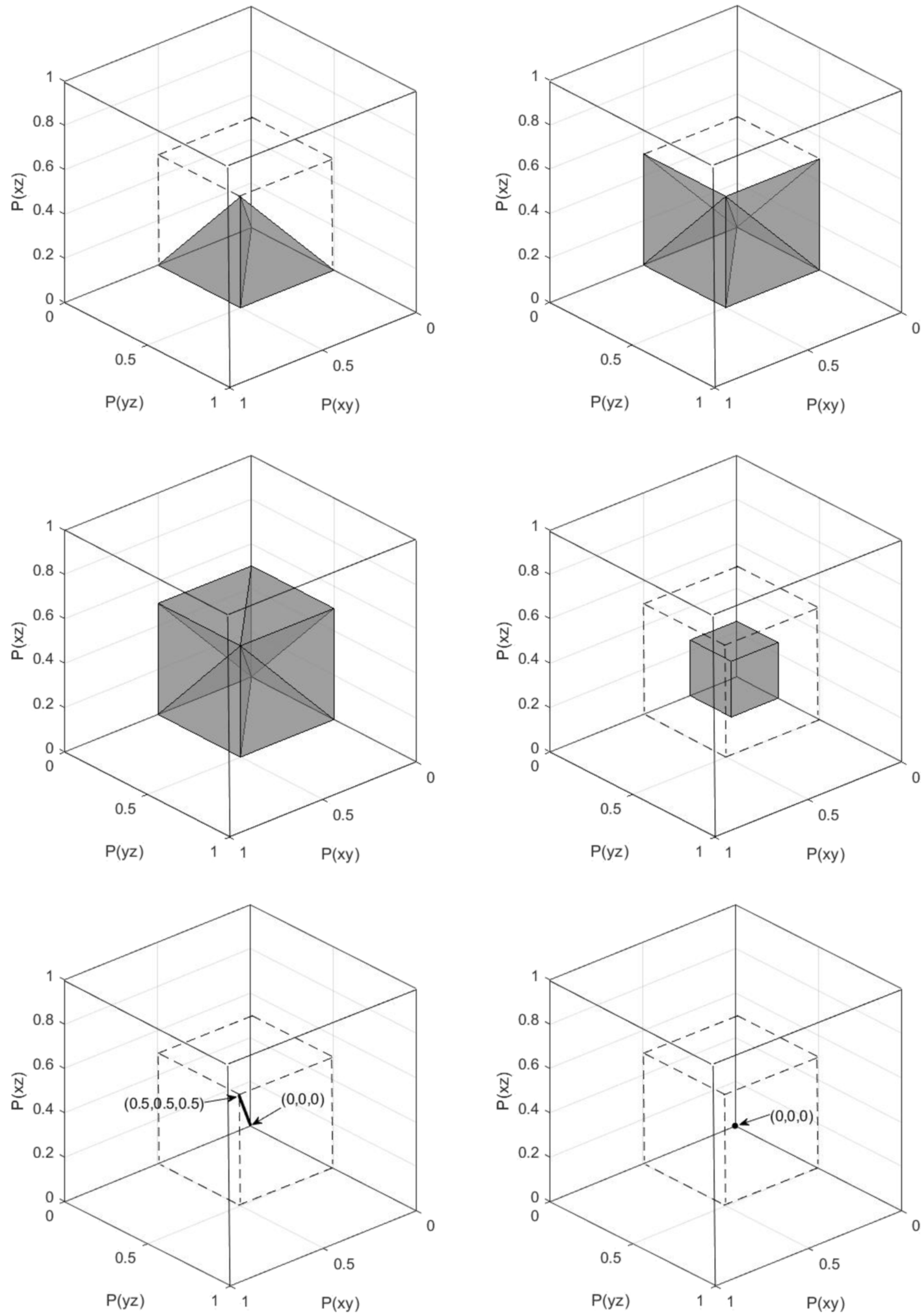

Fig. 5 Six probabilistic specifications for CRRA-EU with $r > -0.465$; left to right, top to bottom: strong utility DFM and comprehensive moderate utility DFM; comprehensive weak utility DFM and 0.75-supermajority specification; constant error model and random preference

of binary choice probabilities. These hypotheses can pertain to parameter values or qualitative features of the core theory, or even the viability of the theory itself. Put differently, as long as the analyst can infer the order of binary choice probabilities, they can draw equivalent conclusions about the core theory under investigation.

The concept map in Fig. 6 decomposes the relationship between data and theory into a sequence of steps. Earlier, we discussed how to move from a theory to order constraints on choice probabilities, and back. We now consider the fact that choice probabilities are not observable directly. To draw inferences from data, the first step is to infer the order of binary choice probabilities, subject to any restrictions imposed by the theory. These restrictions are encoded in the polytope(s) that characterize(s) the distribution-free model. The second step uses the Fechnerian property to identify the order of strengths of preference from the order of binary choice probabilities. The third and final step uses a

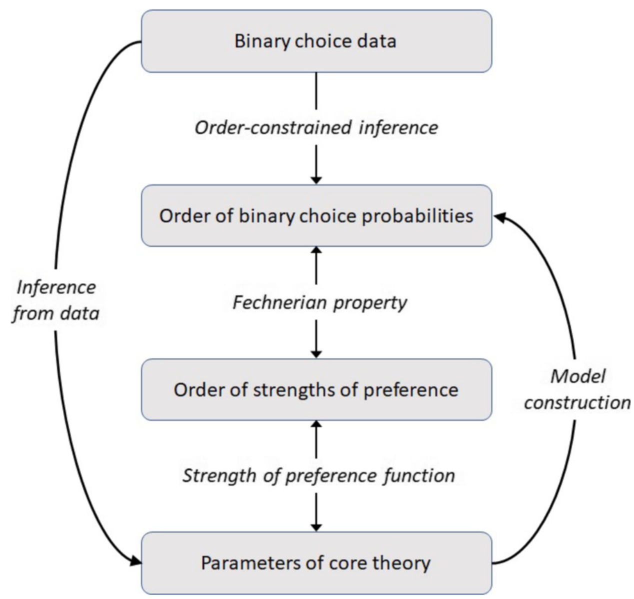

Fig. 6 Concept map relating data to parameters of the core theory

strength-of-preference function to link the order of strengths of preference with parameters of the core theory. Although the strength-of-preference function typically maps parameters of the core theory into strengths of preference, one can also use the pre-image of this function to infer the compatible parameter values of the core theory from an order of strengths of preference.

The next example demonstrates these steps in a simple, exaggerated, hypothetical case where sample sizes are so large that choice probabilities can be inferred from choice proportions without a statistical analysis. For the sake of simplicity and continuity, we keep the domain of choice pairs from our running examples, and the corresponding DFMs for CRRA-EU under strong and strict utility, respectively. Later, we discuss how this type of inference extends naturally to experiments utilizing larger pools of stimuli, and with fewer repetitions on each choice pair.

Example 12 Suppose that a decision maker has made repeated, independent choices from each pairwise combination of the gambles x, y, and z of Example 1 Let $pr(ij)$ denote the proportion of times that i was chosen from $\{i, j\}$. (Thus, $1 - pr(ij)$ is the proportion of times that j was chosen from $\{i, j\}$.) For concreteness, we consider the case when $pr(yz) = .90$, $pr(yx) = .75$, and $pr(xz) = .60$ and thus the order $pr(yz) > pr(yx) > pr(xz) > 1/2$ among choice proportions. If, for the sake of argument, the sample size is large enough that $pr(ij)$ is almost indistinguishable from $P(ij)$, then the order of choice probabilities must be $P(yz) > P(yx) > P(xz) > 1/2$. By the Fechnerian property, this is equivalent to $S(yz) > S(yx) > S(xz) > 0$. Through the lens of CRRA-EU, from Table 2, this order of strengths of preference corresponds to a coefficient of relative risk aversion between −0.658 and −0.698. What's more, by Table 3, this inference holds irrespective of whether one assumes strong utility or contextual utility. However, notice that this invariance also makes the functional form for strength of preference nonidentifiable from these data.

As this example shows, when the data perfectly align with an order of choice probabilities predicted by a theory, and the sample size is sufficiently large, we can look up the ranges of parameter values in the core theory, such as by consulting Tables 2 and 3 for CRRA-EU. Typically, that order of strengths of preference will correspond to a best-fitting range of parameter values in the core theory, rather than a best-fitting unique parameter value. In the example, while we do not infer a unique value of the parameter r, we obtain a precise interval estimate of $-0.658 < r < -0.698$. In the same example, if the data were to support $S(zx) > S(yx) > S(zy) > 0$ (matching decision makers A and C from Example 1) then, under strong utility, we would infer that the coefficient of risk aversion is between −0.465 and +1.017 (see row 2 of Table 2). Such cases fail to generate a precise estimate of the risk sensitivity parameter. They also cannot disambiguate whether the decision maker is risk seeking or risk averse. Those insights are valuable because they protect scholars from drawing overly precise conclusions that are imposed by distributional assumptions not grounded in the theory.

On the other hand, when data mismatch all orders permitted by a theory, they refute the entire theory, provided that the sample size is sufficiently large. In the example above, any set of data that strongly favors $S(zy) > S(zx) > S(xy) > 0$ will challenge CRRA-EU under any DFM on the same stimuli, since this order of strengths of preference is impossible in that theory even under the weak utility assumption (21). This feature of inference under DFMs can facilitate model selection. When multiple theories are under consideration, the data may refute one theory but not another. For example, the order $S(zx) > S(yz) > S(xy)$ (matching decision maker B from Example 1) refutes contextual utility CRRA-EU, but is compatible with strong utility CRRA-EU for $-0.959 < r < -0.755$.

This type of inference also leads to testable, out-of-sample predictions about new choice problems for which data are not yet available. Consider, for example, two additional pairwise choice stimuli $\{s, t\}$ and $\{w, v\}$, where

s is a 75%chance of winning \$30, otherwise nothing,
t is a 65%chance of winning \$32, otherwise nothing,
w is a 40%chance of winning \$50, otherwise nothing,
v is a 25%chance of winning \$60, otherwise nothing.

Without any sample data, using CRRA-EU with $-2 < r < 2$, the strong utility DFM allows five different order constraints on the choice probabilities P(st) and P(wv). These constraints, and their associated values of r, are as follows:

$$
\begin{aligned}
0.5 > \quad & P(st) > P(wv) && \text{for } -2.000 < r < -1.652,\\
0.5 > \quad & P(wv) > P(st) && \text{for } -1.652 < r < -1.578,\\
1 - P(st) > P(wv) \quad & > 0.5 && \text{for } -1.578 < r < -1.521,\\
P(wv) > 1 - P(st) \quad & > 0.5 && \text{for } -1.521 < r < -1.216,\\
P(wv) > P(st) \quad & > 0.5 && \text{for } -1.216 < r < +2.000.
\end{aligned}
$$

However, if, as in Example 12, we have inferred from a large sample using stimuli $\{x, y\}$, $\{x, z\}$ and $\{y, z\}$ that r is between -0.698 and -0.658, then we can eliminate the first four cases. This is because the strong utility DFM for CRRA-EU predicts $P(wv) > P(st) > 0.5$ throughout the range $-0.658 > r > -0.698$.

We can refine the model's predictions further by considering what orders of strengths of preference are possible jointly for all five stimuli, when $-0.658 > r > -0.698$, namely,

$$
\begin{aligned}
&S(wv) > S(yz) > S(yx) > S(xz) > S(st) > 0,\\
&\text{for } -0.698 < r < -0.682,\\
&S(wv) > S(yz) > S(yx) > S(st) > S(xz) > 0,\\
&\text{for } -0.682 < r < -0.658.
\end{aligned}
$$

Replacing strengths of preference with choice probabilities and incorporating point estimates from Example 12, we get

$$
\begin{aligned}
&1 > P(wv) > 0.9 > 0.75 > 0.6 > P(st) > 0.5,\\
&\text{for } -0.698 < r < -0.682,\\
&1 > P(wv) > 0.9 > 0.75 > P(st) > 0.6 > 0.5,\\
&\text{for } -0.682 < r < -0.658.
\end{aligned}
$$

From the above, it follows that the model predicts $P(wv) > 0.9$ across the entire inferred range of r. For the other new stimulus, $\{s, t\}$, the model predicts different constraints for different subranges. In the subrange $-0.698 < r < -0.682$, it predicts $0.6 > P(st) < 0.5$. For the remaining subrange $-0.682 < r < -0.658$, it predicts $0.75 > P(st) > 0.6$ (and for $r = -0.682$, up to rounding, it yields the point prediction $P(st) = 0.6$).

In sum, for the new stimulus $\{w, v\}$, the strong utility DFM for CRRA-EU makes a single, rejectable prediction, $P(wv) > 0.9$. For the other new stimulus, $\{s, t\}$, it makes the rejectable prediction $0.75 > P(st)$, while also providing the opportunity to further refine the range estimate of r.

In the examples we have considered so far, the large sample size contributed to the ease and precision of inference. Notice that models with distributional assumptions, like a logit or probit, do not benefit from large sample sizes the way that DFMs do. For example, there is no combination of parameters of CRRA-EU and the logit model that yields exactly $P(xz) = .90$, $P(yz) = .75$, and $P(xy) = .60$. Therefore, choice proportions $pr(xz) = .90$, $pr(yz) = .75$, and $pr(xy) = .60$, with enough data, will reject the logit specification of CRRA-EU. More generally, even choice proportions that perfectly match an order of strengths of preference permitted by a core theory, because they mismatch the specific probabilities permitted under a logit, will reject the theory, under that logit, in a large enough sample.

In practice, inference about the underlying order of strengths of preference typically requires a statistical analysis because there are not enough observations to infer choice probabilities unambiguously from the data. This applies even when the choice proportions align perfectly with an order of choice probabilities predicted by the theory, and hence with an order of strengths of preference. Here, a statistical analysis may determine which other orders of strengths of preference can or cannot account for the same data as well, in addition to the one that matches perfectly. Similarly, when the choice proportions mismatch all orders of choice probabilities permitted by the theory, a statistical analysis can identify any permissible orders that provide an acceptable explanation of the data when taking into account sampling variability. Such analyses require specialized, order-constrained statistical methods (Silvapulle & Sen, 2011). For example, public domain software such as QTest (Regenwetter et al., 2014; Zwilling et al., 2019) or *multinomineq* (Heck & Davis-Stober, 2019) allows the user to identify the best order of choice probabilities for a given set of data, compute frequentist or Bayesian p values for specific orders of choice probabilities or collections thereof, and compute Bayes factors for selecting between specific orders or collections of orders of strengths of preference. Regenwetter and Cavagnaro (2019) provide a related tutorial. See also Dunn and Rao (2019) for a related approach based on state trace and signed difference analyses.

With the statistical methods described above, one can identify the best order of strengths of preference to explain a given set of binary choice data. That best order may or may not be strongly supported. If the data support multiple orders of strengths of preference, a Bayesian analysis can assign appropriate weights to the corresponding parameter ranges in a way that reflects their relative support.

This type of analysis also extends naturally to model selection. DFMs for different models on the same set of stimuli differ in the collections of orders of strengths of preference that they permit. The methods and software described above also allow the user to compute Bayes factors for selecting between such collections (Heck & Davis-Stober, 2019; Silvapulle & Sen, 2011; Zwilling et al., 2019), regardless of whether they are distinct, overlapping, or nested among each other, and irrespective of sample size.

Just as larger sample sizes can improve precision of inference under a DFM, so too can larger domains of choice stimuli. The DFM for a given theory on a larger domain may include more possible orders of strengths of preference, which may also partition the core theory's parameter space into smaller parts. This, in turn, brings potential for far greater precision in parameter estimation, as long as one can determine all relevant orders of strengths of preference. However, this precision can only be unlocked with precise estimation of the order of strengths of preference, which, in turn, is facilitated by a larger sample size for each stimulus. At the same time, scholars often wish to limit the total number of trials in the experiment. Therefore, as a compromise, we recommend experimental designs with at least several repetitions on each of a manageable number of choice pairs.

More specifically, for frequentist analyses, since one needs to estimate each binomial, a rule of thumb would be to have at least 20 observations per choice pair. For parametric models like logit and probit, this is not the case because one does not need to estimate each binary choice probability directly. Instead, such analyses often have only a single observation per choice pair. While the Bayesian analyses for DFMs permits a single observation per choice pair as well, more research is needed to understand the conditions under which such an analysis would clearly distinguish between competing orders of strengths of preference. Following up on the above compromise, and based on our own experience with order-constrained inference, if using Bayesian methods, we recommend at least a handful of observations per choice pair.

A Note of Caution

We briefly explore the viability of a potential heuristic method for identifying the best order of strengths of preference to explain a given set of binary choice data. An analyst could first directly estimate the parameters of a core theory using a logit. They might then look up the order of strengths

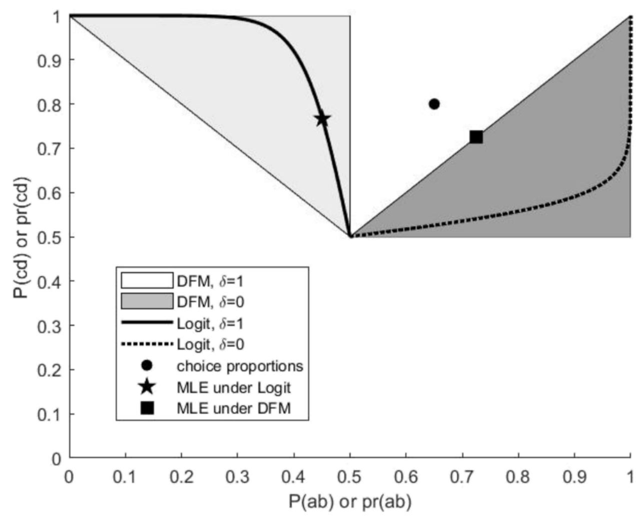

Fig. 7 Maximum likelihood estimates under DFM and logit specifications of a hypothetical decision theory. *Note.* To superimpose the data on the same plot with the models, axes represent probabilities in the case of the models and proportions in the case of the data

of preference consistent with that parameter estimate and relax the distributional assumption of the logit by taking the full range of parameter values that yield the same order. This approach may seem computationally appealing because it involves fitting fewer parameters, and it circumnavigates various other computations associated with the distribution-free approach. However, that approach need not generate the truly best fitting order of strengths of preference. We illustrate this insight with a simple example.

Example 13 Let $\mathcal{C}$ consist of just two pairs of choice alternatives, labeled $\{a, b\}$ and $\{c, d\}$, and consider a core theory with a single binary parameter $\delta \in \{0, 1\}$. Suppose this theory predicts $S(ab) = 100$, $S(cd) = 10$ for $\delta = 0$, and $S(ab) = -10$, $S(cd) = 100$ for $\delta = 1$. Table 4 lists the orders of strengths of preference for this theory, along with the constraints on choice probabilities given by the corresponding facet-defining inequalities. It also lists the associated parameter values. The polytopes characterizing the DFM for this theory are 2-dimensional and shown in Fig. 7. The triangle in the upper-right represents $S(ab) > S(cd) > 0$ and $\delta = 0$ through $P(ab) > P(cd) > 1/2$. The triangle in the upper-left represents $S(cd) > S(ba) > 0$ and $\delta = 1$ through $P(cd) > P(ba) > 1/2$.

While the logit specification of this theory satisfies the same inequality constraints on the choice probabilities, it is far more restrictive in that it also assumes $P(ij) = [1 + e^{-\gamma S(ij)}]^{-1}$ for all (i, j), with $\gamma > 0$. As γ varies, the logit probabilities form the curves within the triangles in Fig. 7. Fixing $\delta = 0$, as γ ranges from 0 to infinity, the logit specification traces a curve beginning at the midpoint of the square, where $P(ab) = P(cd) = 1/2$, and converging to the upper-left vertex, where $P(ab) = 0$ and

Table 4 DFM for a hypothetical decision theory with binary parameter δ

Order of strengths of preference	Facet-defining inequalities	Parameter of core theory
$S(ab) > S(cd) > 0$	$1 > P(ab) > P(cd) > \frac{1}{2}$	$\delta = 0$
$S(cd) > S(ba) > 0$	$1 > P(cd) > 1 - P(ab) > \frac{1}{2}$	$\delta = 1$

$P(cd) = 1$. Fixing $\delta = 1$, as γ ranges from 0 to infinity, the logit specification traces a curve beginning at the midpoint of the square and converging to the upper-right vertex, where $P(ab) = P(cd) = 1$.

Now, suppose that, in an experiment with 20 choices per pair of alternatives, a was chosen 13 times from $\{a, b\}$, while c was chosen 16 times from $\{c, d\}$. These choice proportions, $pr(ab) = .65$ and $pr(cd) = .8$, are represented by the black dot in Fig. 7. Under the logit specification, the maximum likelihood estimates, shown with a $\star$ in Fig. 7, are $\delta = 1$ and $\gamma = 1.7978$. Therefore, through the lens of a logit, the most likely order of strengths of preference appears to be $S(cd) > S(ba) > 0$, as this is the order associated with $\delta = 1$. Yet, under the DFM, the maximum likelihood estimate of the binary choice probabilities is $P(ab) = P(cd) = 0.725$, shown as a ■ in Fig. 7, which is located on a facet of the triangle corresponding to $\delta = 0$. In all, this example shows that one cannot in general use a parametric approach to infer the order of strengths of preference in a DFM.

Conclusions and Discussion

Thanks to decades of research, scholars' understanding of preferences, utilities, or strengths of preference can be mathematically grounded in a decision theory. Likewise, statistical inference may be mathematically grounded in a formal probability model of the data generating process. Yet, a gap between constructs and behavior persists. When decision theory guides decision analysis, such as assessing a stake holder's risk tolerance, loss aversion, or patience, there often remains at least one weak link between data and theory. The deductive path from theory to data, and the inferential path from data back to theory, commonly lead through at least one a-theoretical link that is primarily driven by computational convenience, intuition, or even just disciplinary tradition. This is the link between hypothetical constructs such as the strength of preference on the one hand, and data generating choice probabilities on the other hand. In this paper, we have fleshed out a compelling theoretical characterization of that link: the Fechnerian property. It states that, the stronger the preference for i over j, the more likely that i is chosen over j. The Fechnerian property enables the researcher to either eliminate or at least better understand some a-theoretical auxiliary convenience assumptions that would otherwise infuse an ad hoc element into their analysis. It thereby enables the analyst to extract more robust information, e.g., about risk tolerance, loss aversion, or patience. The Fechnerian property is an example of a monotonic coordination function between strength of preference and choice probability in the spirit of Kellen et al.'s (2021) prescriptive recommendation for improving psychological science broadly. While there is a long tradition for looking at both theoretical and empirical underpinnings of the Fechnerian property in decision-making (e.g., Luce, 1959), this is not a large active area of research (with Alós-Ferrer & Garagnani, 2021, a notable exception).

For some decision theories, such as regret theory, the Fechnerian property fully closes the loop between decision theory and choice probabilities. This is because regret theory provides uniquely defined strengths of preference. In such a case, the researcher only requires auxiliary assumptions at the statistical inference stage, such as iid sampling, for instance. When one analyzes data without the bottleneck of ad hoc models like logits or probits, one can better understand which inferences are robust across a spectrum of Fechnerian response mechanisms.

However, in many domains, unlike regret theory or the tradeoff model, strength of preference is not yet an inherent part of decision theory. Many utility theories, including those for risky or intertemporal choice, only provide mathematically grounded pairwise preferences among options, or mathematically grounded utilities of individual choice alternatives. Through the lens of our paper, the coordination function linking those primitives to observable behavior depends on the actual definition of strengths of preference. We have considered several very specific strength of preference theories, such as, most prominently, strong and contextual utility. However, deeply understanding the theoretical primitive of strength of preference appears to us a wide-open question. Existing research in all areas of decision theory can provide precedents and guidance: Cumulative prospect theory uses functional forms with free parameters to accommodate either individual differences or within-person fluctuations within a theory. The same applies to many other core theories across many decision theoretic paradigms and tasks. Future work on strength of preference theories could allow for within- and between-person heterogeneity as well, most naturally by considering functional forms of strength of preference that also feature their own free parameters. The DFM framework makes it possible to track the testability, identifiability, and parsimony of such models theoretically via their geometric structure, as well as empirically and statistically via order-constrained inference methods. In an effort to bridge the construct-behavior gap in existing research, studying strength of preference seems to us key to understanding decision-making in a very broad spectrum of decision tasks and research paradigms.

While the Fechnerian model class builds on orders of strengths of preference as a theoretical primitive, other models have provided more direct links from utilities or binary preferences to choice probabilities, most notably distribution-free random utility and random preference models. Interestingly, some Fechnerian models, such as logit and probit models, besides being nested in the DFMs, are also nested in certain distribution-free random utility models. This leads us to another important feature of ad hoc models, which we have touched upon earlier. With every additional stimulus used in a study, the complexity of the data generating process increases. However, for standard Fechnerian models like logit and probit, when combined with commonly used theories like Expected Utility Theory, or, say, Cumulative Prospect Theory, the parameter space has a fixed, low dimension. For instance, Tversky and Kahneman's (1992) original version of Cumulative Prospect Theory evokes five parameters. In conjunction with a logistic specification (Eq. 2), and depending on the types of stimuli being used, this creates at most a five-dimensional model in the space of binary choice probabilities. In contrast, if the model is aimed at explaining the choice among hundreds or even thousands of pairs of potential choice alternatives, the data generating model is a potentially thousand-dimensional unit-hypercube. It is conceptually very difficult to imagine how a six-dimensional model can accommodate thousand-dimensional behavioral lab and/or field data. With a rich enough set of stimuli and enough data, rejecting such a model is almost a forgone conclusion. This is, perhaps, one of the rationales for the common saying that "all models are wrong, but some are useful," often attributed to statistician George Box. We would advocate that more useful models and models with higher face-validity are those whose complexity grows hand-in-hand with the complexity of the data generating process. Indeed, DFMs, distribution-free random utility models, random preference models, as well as "aggregation-" and "distance-based" probabilistic specifications (Regenwetter et al., 2014), all have the property that their number of parameters often equals the number of degrees of freedom in the empirical choice data. Furthermore, all of those models form collections of convex polytopes in a shared geometric space and all are amenable to order-constrained statistical inference.

While the parametric flexibility of DFMs is an asset when it comes to face-validity, it can come at a cost in terms of computational complexity. For a set of n choice pairs, where $|\mathcal{D}| = 2n$, there are 2^n possible preference relations, and for each preference relation there are $n!$ many orders of strengths of preference. For example, there are more than 3 billion possible orders of strengths of preference for 10 choice pairs. This can make the process of precomputing and enumerating all orders of strengths of preference for a given theory extremely laborious. On the other hand, each predicted order of strengths of preference corresponds to a range of parameter values in the core theory. For example, in Tables 2 and 3, we have provided the ranges of values of r corresponding to orders of strengths of preference generated by CRRA-EU. Even for just 10 choice pairs, the benefit is that the Fechnerian property can potentially partition the core theory's parameter space into as many as 3 billion distinct parts. In all, the flip side of the computational cost of DFMs is that even a modest number of well-designed stimuli can give very high resolution to the actual combinations of parameter values that may have generated a set of binary choice data.

In contrast to parameter estimation, theory testing can benefit from an experimental design with a large number of choice pairs that drastically restrict the permissible orders of strengths of preference under a core theory. The higher the dimension of the space, and the smaller the number of orders of strengths of preference, the smaller the volume of the resulting union of convex polytopes. This means that, even though the resulting union of polytopes may be n-dimensional in n-dimensional space, it may be extremely parsimonious. Furthermore, studies aimed at discriminating between two or more theories, say discounting vs. tradeoff in intertemporal choice, benefit from carefully designed stimuli that imply not just very restrictive but also distinct unions of convex polytopes. In other words, a good experimental design for model selection minimizes model mimicry by minimizing the number of orders of strengths of preference shared by the theories. While we used an enumeration of orders of strengths of preference in our examples, other methods, such as axiomatic approaches, may help to derive such diagnostic stimuli more directly. These approaches may also help circumnavigate the computational cost of precomputing orders of strengths of preference and the associated facet-defining inequalities. Relatedly, future work may incorporate adaptive experimental designs to minimize the number of trials needed in an experiment (see, e.g., Cavagnaro et al., 2010). That work must also navigate various issues of computational complexity.

Standard Fechnerian models like logit and probit do not require computing constraints on permissible binary choice probabilities explicitly. The constraints are imposed directly by the distributional assumptions, which essentially re-parambreaketerize the space of permissible binary choice probabilities according to one or more scaling parameters, in addition to the parameters of the core theory. This makes statistical inference under such models computationally more efficient than under a DFM, but also arbitrary and potentially misleading, as we have illustrated. Modern methods in operations research have started to look at computing Bayes factors for convex polytopes whose fact-defining inequalities are unknown (Smeulders et al., 2017). This would alleviate many of the computational bottlenecks in computing Bayes factors for high-dimensional DFMs. However, that work is so far custom-designed for specific polytopes. Future work may find innovative ways to expand such approaches to more general classes of models like the DFMs we have considered here.

All things considered, we do not contend that DFMs should replace parametric models in all circumstances. While we consider most parametric models as over-specified, some applications may not be concerned with worries about over-specification. Furthermore, good experimental designs featuring many choice stimuli that are informative for the research question under investigation can potentially alleviate some of the identification problems of parametric models that we have described here. However, the surest way to evaluate whether a given design permits robust inference under a parametric Fechnerian model is to cross-check it against the distribution-free model for the same design. Another strength of parametric models such as the logit and probit is that they extend naturally from binary choice to multiple choice. Just as scholars have considered distribution-free random utility and random preference models for multiple choice and best-worst choice (see Marley & Regenwetter, 2016, for a review), future work could consider extending DFMs to such paradigms.

Appendix 1. DFMs for more general decision theories

In this section, we illustrate how DFMs work for theories without utility functions. We walk the reader through examples ranging from DFMs for well-known core theories to ideas for novel axiomatizations of strengths of preference. We start with a case where a decision theory generates strengths of preference directly.

Example 14 We briefly consider regret theory, as spelled out by Loomes et al. (1991) with choice options that follow the blue-print of Loomes and Sugden's Table I (p. 430). The bottom three rows of our Table 5 indicate three choice alternatives,[11] x, y, z. The right three columns denote three different states of the world, SW_1, SW_2, SW_3, and their probabilities. A decision maker who chooses x will receive \$5 if the state of the world is SW_1, which has probability 0.2 of occurring, and \$2 otherwise. A person who chooses y receives \$4 if states of the world SW_1 or SW_2 happen, or \$1 if SW_3 happens, which has a probability of 0.3 of occurring. Anyone who chooses z will receive \$3 irrespective of the state of the world.

Let i_m and j_m denote the outcomes of options i and j in SW_m, and let p_m denote the probability of the state SW_m. According to this version of regret theory, letting Ψ be an odd function that is increasing in its first argument and that satisfies $\Psi(a, c) > \Psi(a, b) + \Psi(b, c)$, when $a > b > c$ are monetary amounts, the strength of preference for i over j is $S(ij) = \sum_{m=1}^{3} p_m \Psi(i_m, j_m)$. Consider $\Psi(i_m, j_m) = (i_m - j_m)^3$. This yields $S(xy) = -3.5$, $S(xz) = 0.8$, and $S(yz) = -1.7$. The resulting DFM forms a tetrahedron, which, using the analogous coordinate system $P(xy)$, $P(xz)$, $P(yz)$ as before, is characterized by the constraints

$$1 \geq 1 - P(xy) > 1 - P(yz) > P(xz) > \tfrac{1}{2}. \qquad (24)$$

These inequalities document how regret theory, on this domain, produces strengths of preference that are not compatible with the weak utility assumption. Namely, they imply $P(xz) > 1/2$, $P(zy) > 1/2$, and $P(yx) > 1/2$, which violates $\mathcal{WST}$.

The inequalities in Condition 24 are the facet-defining inequalities of the tetrahedron in the upper-left panel of Fig. 8. That tetrahedron is nested in the half-unit cube shown in the upper-right panel of Fig. 8 (the same as the right panel of Fig. 1, in a different angle of view, which is also the majority specification of an intransitive preference pattern). It also has no overlap with $\mathcal{WST}$, shown on the lower panels of Fig. 8, which we explain in Examples 15 and 16. The insight that this DFM is not contained in $\mathcal{WST}$, together with the earlier insight that DFMs need not contain $\mathcal{WST}$ (see, e.g., Fig. 1), implies the following formal result.

Proposition 7 *Let $\mathcal{M}_{\mathcal{S}}$ be a DFM as defined in Definition 6. In general, $\mathcal{WST} \not\subseteq \mathcal{M}_{\mathcal{S}} \not\subseteq \mathcal{WST}$.*

We proceed to DFMs for axiomatic theories that likewise need not evoke the concept of utility. We start with the axioms of transitivity and asymmetry. For the rest of this section, we assume that $\mathcal{D}$ contains all pairs of options of a given finite choice set.

Let $\mathcal{WST}^*$ denote $\mathcal{WST}$ (Condition 23) with the added constraint that $P(ij) \neq 1/2$ for all $ij \in \mathcal{D}$. Defining, as before,

$$i \succ j \quad \Leftrightarrow \quad S(ij) > 0 \quad \Leftrightarrow \quad P(ij) > \frac{1}{2}, \qquad \forall ij \in \mathcal{D}, \qquad (25)$$

Table 5 Stimuli for regret theory

	State of the world and probability		
Option	SW_1 $p_1 = 0.2$	SW_2 $p_2 = 0.5$	SW_3 $p_3 = 0.3$
x	\$5	\$2	\$2
y	\$4	\$4	\$1
z	\$3	\$3	\$3

[11] Retooling the same labels x, y, z for new options allows for a shared coordinate system across figures to compare the geometric structure.

it is well known that $\mathcal{WST}^*$ is equivalent to *transitivity* of the preference relation $\succ$, namely,

$$i \succ j \quad \wedge \quad j \succ k \quad \Rightarrow \quad i \succ k, \qquad \forall ij, jk, ik \in \mathcal{D}.$$

Writing $i \not\succ j$ to denote that $i \succ j$ does not hold, since $S(ij) > 0 \Rightarrow S(ji) \not> 0$, the preference relation $\succ$ defined in Condition 25 is *asymmetric*, i.e., $i \succ j \Rightarrow j \not\succ i$. This gives us the tools to better understand the geometry associated with $\mathcal{WST}$ and $\mathcal{WST}^*$, and axiomatic theories more broadly (for early related work, including geometric visualizations of $\mathcal{WST}$ and other types of stochastic transitivity, see Morrison, 1963).

Example 15 If $S(ij) \neq 0$, $\forall ij \in \mathcal{D}$ then the preference relation $\succ$ defined in Condition 25 is *complete*, i.e., for all distinct choice options i, j it holds that either $i \succ j$ or $j \succ i$. A complete, asymmetric, transitive binary relation is a *strict linear order* (ranking, permutation) of the choice options. $\mathcal{WST}^*$ rules out $P(ij) = 1/2$, and therefore implies $S(ij) \neq 0$ through Condition 25. As a consequence, $\mathcal{WST}^*$ could be labeled the DFM for the collection of all strict linear orders through the lens of Condition 25. For three choice alternatives, this DFM forms a disjoint union of six open half-unit cubes (lower-left panel of Fig. 8). The figure uses the coordinate system $P(xy)$, $P(xz)$, $P(yz)$. Characterizing the six component cubes of $\mathcal{WST}^*$ involves facet-defining inequalities of the form $P(ij) > 1/2$, or $P(ij) > 0$, or $P(ij) < 1$ (each with suitably selected ij).

While Example 15 constructed the DFM for complete, asymmetric, transitive preference relations, we now consider the DFM for a slightly different combination of axioms.

Example 16 When $S(ij)$ is also allowed to be zero, the relation $\succ$ need not be complete. A binary relation $\succ$ is *negatively transitive* if and only if $i \not\succ j \wedge j \not\succ k \Rightarrow i \not\succ k$, for all i, j, k. If $\succ$ is defined as in Condition 25 then $\mathcal{WST}$ holds if and only if $\succ$ is negatively transitive. Asymmetric negatively transitive binary relations are *strict weak orders*. $\mathcal{WST}$ allows $P(ij) = 1/2$ for some pairs and therefore allows $S(ij) = 0$ for some pairs. Consequently, $\mathcal{WST}$ could be labeled the DFM for the collection of all strict weak orders through the lens of Condition 25. In the case of three choice alternatives, there are 13 strict weak orders. Here, the DFM consists of the disjoint union of the six half-unit open cubes we just reviewed for the six strict linear orders, together with six open 2-dimensional half-unit squares and the point $(1/2, 1/2, 1/2)$. Figure 8 shows the 3-dimensional polytopes of this model in the lower-left panel, and the lower-dimensional polytopes in the lower-right panel, all in the coordinate system $P(xy)$, $P(xz)$, $P(yz)$. Line segments of the form $P(ij) = P(jk) = 1/2 < P(ik) < 1$ (written in a suitable coordinate system) are not included in this model because the corresponding preferences violate negative transitivity (i.e., $i \not\succ j$ and $j \not\succ k$ but $i \succ k$). These line segments represent "semiorders" with transitive preference and intransitive indifference (Luce, 1956), which we sketch in the next example. That example moves beyond all examples we have seen so far in that it explores how to convert axioms, here those for interval orders and semiorders, into novel axioms about strengths of preference.

Example 17 A binary relation $\succ$ is an *interval order* if and only[12] if

$$[i \succ j \quad \wedge \quad k \succ \ell] \quad \Rightarrow \quad [i \succ \ell \quad \vee \quad k \succ j].$$

An interval order $\succ$ is a *semiorder* if and only if

$$[i \succ j \quad \wedge \quad j \succ k] \quad \Rightarrow \quad [i \succ \ell \quad \vee \quad \ell \succ k].$$

These two axioms could inspire novel axioms about strengths of preference, such as

$$[S(ij) > 0 \quad \wedge \quad S(k\ell) > 0] \quad \Rightarrow \quad \max(S(i\ell), S(kj)) \geq \min(S(ij), S(k\ell)),$$

respectively

$$[S(ij) > 0 \quad \wedge \quad S(jk) > 0] \quad \Rightarrow \quad \max(S(i\ell), S(\ell k)) \geq \min(S(ij), S(jk)).$$

We finish this section with a generalization of hierarchical decision strategies, such as "lexicographic semiorders" (see, e.g., Tversky, 1969).

Example 18 One can readily imagine DFMs for various sorts of lexicographic decision processes, say in consumer choice. Consider a shopper comparing two computers or televisions who follows a hierarchy of attributes, say, price, then screen resolution, followed by screen size, etc. When considering an attribute, they stop the decision process when that attribute meets a stopping criterion, otherwise proceed to the next attribute. For instance, writing $S_n(ij)$ for the strength of preference for i over j according to attribute n, suppose that

$$S_1(xy) = 2; \quad S_2(xy) = 0; \quad S_3(xy) = 1; \quad S_4(xy) = 50; \quad S_5(xy) = -1; \quad S_6(xy) = -20;$$

and

$$S_1(wz) = 1; \quad S_2(wz) = 60; \quad S_3(wz) = 1; \quad S_4(wz) = -30; \quad S_5(wz) = 15; \quad S_6(wz) = 10.$$

[12] Throughout this example, for notational and conceptual simplicity, we omit universal quantifiers.

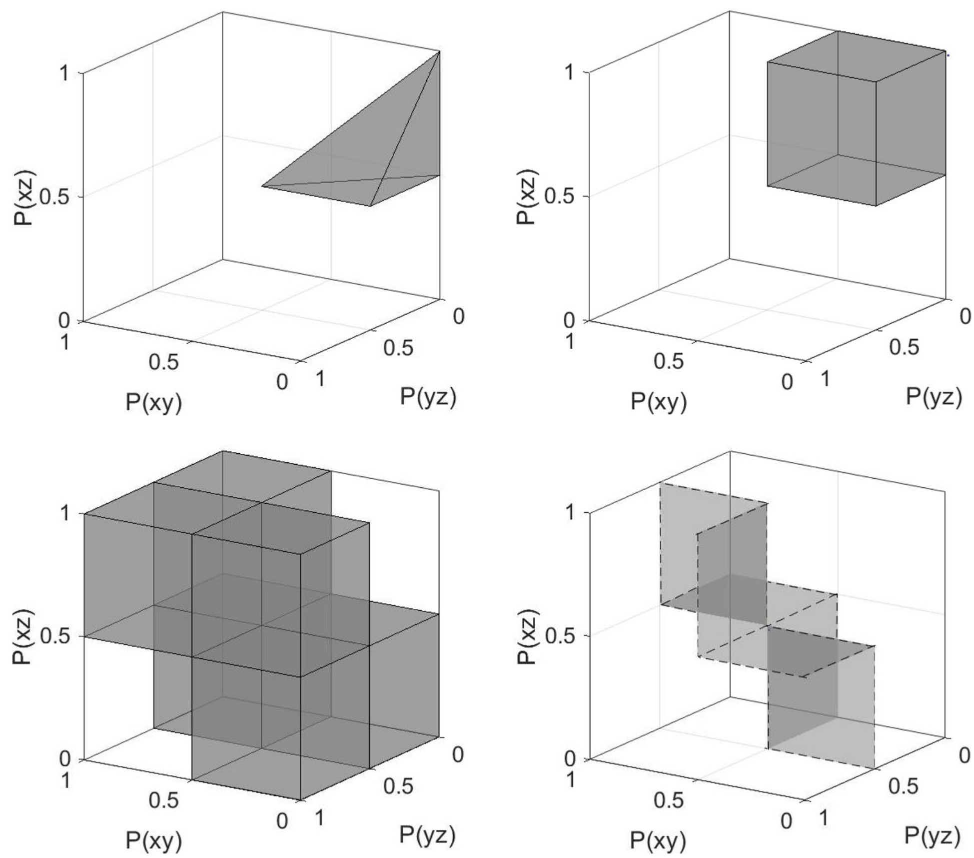

Fig. 8 Regret DFM of Example 14 (upper-left panel), half-unit cube for intransitive preference $y \succ x, x \succ z, z \succ y$ (upper-right panel), and geometric representations of $\mathcal{WST}^*$ (lower-left panel) and $\mathcal{WST}$ (union of lower panels). *Note.* Dashed lines in the lower-right panel are for enhancing 3D visualization and have no special meaning

Suppose the decision maker has some decision threshold δ and sequentially, i.e., lexicographically, considers attributes until $|S_m(ij)| > \delta$ for some m, then adopts the strength of preference according to that attribute. A decision maker with $\delta = 0$ stops the search process at the first attribute for both $\{x, y\}$ and $\{w, z\}$, with overall strengths of preference $S(xy) = S_1(xy) = 2$ and $S(wz) = S_1(wz) = 1$ and, hence $P(xy) > P(wz)$. A decision maker with $\delta = 10$ has strengths of preference $S(xy) = S_4(xy) = 50$ and $S(wz) = S_2(wz) = 60$ and, hence $P(xy) < P(wz)$. One can also design novel DFMs in which attribute-wise strengths of preference are somehow accumulated to weigh reasons in favor or against one option over another, similar to the "perceived relative argument" model of Loomes (2010).

Appendix 2. Proofs

Proof of Proposition 1

$$\begin{aligned} S(ij) > 0 &\Leftrightarrow S(ji) < 0 \\ &\Leftrightarrow S(ji) < S(ij) \\ &\Leftrightarrow P(ji) < P(ij) \\ &\Leftrightarrow 1 - P(ij) < P(ij) \\ &\Leftrightarrow P(ij) > \frac{1}{2}. \end{aligned}$$

□

Proof of Proposition 2 Let S be an odd, real-valued function on the domain $\mathcal{D}$ and let $\mathcal{D}'$ be any maximal subset of $\mathcal{D}$, such that S is one-to-one on $\mathcal{D}'$, i.e., for any distinct ij and $k\ell$ in $\mathcal{D}'$, $S(ij) \neq S(k\ell)$. In particular, if S is a *one-to-one* function on $\mathcal{D}$ then $\mathcal{D}' = \mathcal{D}$. Let $\mathcal{D}'_{S+} = \{ij \in \mathcal{D}' \mid S(ij) > 0\}$. If $ij \in \mathcal{D}'$ then $S(ij) \neq 0$ and therefore $ji \in \mathcal{D}'$, since $S(ji) = -S(ij)$. Therefore, for each $ij \in \mathcal{D}'$, exactly one of ij or ji is in $\mathcal{D}'_{S+}$. As a consequence, $|\mathcal{D}'_{S+}| = \frac{|\mathcal{D}'|}{2}$. We denote $|\mathcal{D}'_{S+}|$ as n. Since S is one-to-one on $\mathcal{D}'$, there is a unique one-to-one function $f : X = \{1, 2, \ldots, n\} \rightarrowtail \mathcal{D}'_{S+}$, which orders the elements of $\mathcal{D}'_{S+}$ by decreasing strength of preference, i.e., such that $S(f(x)) > S(f(x + 1))$ for $1 \leq x \leq n - 1$. According to the Fechnerian property (6) for S, a binary choice vector P is in $\mathcal{M}_S$ if and only if

$$1 > P(f(1)), \tag{26}$$

$$P(f(x)) > P(f(x + 1)) \text{ for } 1 \leq x \leq n - 1, \tag{27}$$

$$P(f(n)) > \frac{1}{2}, \tag{28}$$

by construction. Each of these $n + 1$ inequality constraints defines a half-space in $\mathbb{R}^n$. The intersection of these half-spaces is contained in $]0, 1[^n$ and, therefore, forms a convex polytope of dimension n. Because they form $n + 1$ nonredundant constraints describing a convex polytope, Inequalities 26–28 are facet-defining for that polytope. This construction of a polytope and of facets does not depend on the initial choice of $\mathcal{D}'$. As $\mathcal{D}'$ varies, the facet-defining inequalities (26–28) are stated in terms of different coordinates in $\mathbb{R}^n$, but in conjunction with the equivalence $S(ij) = S(k\ell) \Leftrightarrow P(ij) = P(k\ell)$, they form the same convex polytope in $]0, 1[^{\mathcal{D}}$. □

Proof of Proposition 3 Let $S_1, S_2 \in \mathcal{S}$ be strength-of-preference functions on $\mathcal{D}$, and suppose $P \in \mathcal{M}_{S_1} \cap \mathcal{M}_{S_2}$. Then, by definition

$$S_1(ij) > S_1(k\ell) \Leftrightarrow P(ij) > P(k\ell) \Leftrightarrow S_2(ij) > S_2(k\ell).$$

Therefore, by the construction in the proof of Proposition 4, $\mathcal{M}_{S_1}$ and $\mathcal{M}_{S_2}$ form identical polytopes nested in $]0, 1[^{\mathcal{D}}$. Since there are only finitely many rankings (with possible ties) of $\frac{|\mathcal{D}|}{2}$ many choice probabilities from largest to smallest, there are also only finitely many such convex polytopes possible. □

Proof of Proposition 4 Now, suppose that $\mathcal{S}$ is maximal. Since we have already shown that the convex polytopes under consideration are disjoint, we only need to show that they form a complete partition of $]0, 1[^{\mathcal{D}}$. Let $P \in]0, 1[^{\mathcal{D}}$ be a binary choice vector on $\mathcal{D}$. To show that there exists a Fechnerian model family $\mathcal{M}_S$ containing P, define a strength-of-preference function S on $\mathcal{D}$ by $S(ij) = P(ij) - 1/2$, for all $ij \in \mathcal{D}$. Then P satisfies Condition 6, hence P is a Fechnerian model for S. □

Appendix 3. Facet-defining inequalities for panels 3 and 4 of Fig. 2

In the third panel from the left, the left tetrahedron is defined by

$$0 < P(xy) < P(xz) < P(yz) < \frac{1}{2} < P(zy) < P(zx) < P(yx) < 1,$$

i.e., $S(xy) < S(xz) < S(yz) < 0 < S(zy) < S(zx) < S(yx)$. In that panel, the right tetrahedron is defined by

$$0 < P(yz) < P(xz) < P(xy) < \frac{1}{2} < P(yx) < P(zx) < P(zy) < 1,$$

i.e., $S(yz) < S(xz) < S(xy) < 0 < S(yx) < S(zx) < S(zy)$. These are on opposite sides of the line segment defined by $0 < P(yz) = P(xz) = P(xy) < \frac{1}{2}$, which is also the convex hull of $\left(0, 0, 0\right)$ and $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.

In the right-most panel of Fig. 2, the left tetrahedron is defined by

$$0 < P(xy) < P(yz) < P(xz) < \frac{1}{2} < P(zx) < P(zy) < P(yx) < 1,$$

i.e., $S(xy) < S(yz) < S(xz) < 0 < S(zx) < S(zy) < S(yx)$. In that panel, the left tetrahedron is defined by

$$0 < P(yz) < P(xy) < P(xz) < \frac{1}{2} < P(zx) < P(yx) < P(zy) < 1,$$

i.e., $S(yz) < S(xy) < S(xz) < 0 < S(zx) < S(yx) < S(zy)$. These are on opposite sides of the triangle defined by $0 < P(yz) = P(xy) < P(xz) < \frac{1}{2}$.

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