Minimum degree ensuring that a hypergraph is hamiltonian-connected

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Abstract

A hypergraph H is *hamiltonian-connected* if for any distinct vertices x and y, H contains a hamiltonian Berge path from x to y. We find for all $3 \leq r < n$, exact lower bounds on minimum degree $\delta(n, r)$ of an n-vertex r-uniform hypergraph H guaranteeing that H is hamiltonian-connected. It turns out that for $3 \leq n/2 < r < n$, $\delta(n, r)$ is 1 less than the degree bound guaranteeing the existence of a hamiltonian Berge cycle. Moreover, unlike for graphs, for each $r \geq 3$ there exists an r-uniform hypergraph that is hamiltonian-connected but does not contain a hamiltonian Berge cycle.

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1 Introduction and results

A hypergraph H is a family of subsets of a ground set. We refer to these subsets as the *edges* of H and the elements of the ground set as the *vertices* of H. We use $E(H)$ and $V(H)$ to denote the set of edges and the set of vertices of H respectively. We say that H is r*-uniform* (an r-graph, for short) if every edge of H contains exactly r vertices. A *graph* is a 2-graph. The *degree* $d_H(v)$ of a vertex v in a hypergraph H is the number of edges containing v. When it is clear from the context, we may simply write $d(v)$ to mean $d_H(v)$. The *minimum degree*, $\delta(H)$, is the minimum over degrees of all vertices of H.

A *hamiltonian cycle* (path) in a graph is a cycle (path) that visits every vertex. A graph is *hamiltonian* if it contains a hamiltonian cycle. Furthermore, a graph is *hamiltonian-connected* if there exists a hamiltonian path between every pair of vertices.

It is well known that determining whether a graph is hamiltonian is an NP-complete problem. Sufficient conditions for existence of hamiltonian cycles in graphs have been well-studied. In particular, the famous Dirac's Theorem [5] says that for any $n \geq 3$ each n-vertex graph G with $\delta(G) \geq n/2$ contains a hamiltonian cycle.

Every hamiltonian-connected graph is also hamiltonian, but the converse is not true. For example for even $n \geq 4$, the complete bipartite graph $K_{n/2,n/2}$ is hamiltonian but not hamiltonian-connected. The example of $K_{n/2,n/2}$ also shows that for even n, condition $\delta(G) \geq n/2$ does not provide that G

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is hamiltonian-connected. On the other hand, Ore [10] proved that a slightly stronger restriction on minimum degree of a graph implies hamiltonian-connectedness:

Theorem 1.1 (Ore [10])**.** *Let $n \geq 3$ and G be an n-vertex graph. If $d(u) + d(v) \geq n + 1$ for every $u, v \in V(G)$ with $uv \notin E(G)$, then G is hamiltonian-connected. In particular, if $\delta(G) \geq (n+1)/2$, then G is hamiltonian-connected.*

Note that for odd n, the restriction on minimum degree is the same as in Dirac's Theorem.

Dirac's Theorem and Theorem 1.1 have been generalized and refined in several directions by Posa [11], Lick [7] and many others. Among generalizations, there were different extensions of the theorems to cycles and paths in hypergraphs, in particular, in r-graphs.

Definition 1.2. *A* **Berge cycle of** *length s in a hypergraph is a list of s distinct vertices and s distinct edges $v_1, e_1, v_2, \ldots, e_{s-1}, v_s, e_s, v_1$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ for all $1 \leq i \leq s$ (we always take indices of cycles of length s modulo s). We call vertices $v_1, \ldots, v_s$ the* **defining vertices** *of C and write $V(C) = \{v_1, \ldots, v_s\}$, $E(C) = \{e_1, \ldots, e_s\}$. Similarly, a* **Berge path** *of length ℓ is a list of $\ell + 1$ distinct vertices and ℓ distinct edges $v_1, e_1, v_2, \ldots, e_\ell, v_{\ell+1}$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ for all $1 \leq i \leq \ell$, with* **defining vertices** *$V(P) = \{v_1, \ldots, v_{\ell+1}\}$ and $E(P) = \{e_1, \ldots, e_\ell\}$.*

For simplicity, we will say a hypergraph is *hamiltonian* if it contains a hamiltonian Berge cycle, and is *hamiltonian-connected* if it contains a hamiltonian Berge path between any pair of vertices.

Approximate bounds on the minimum degree of an n-vertex r-graph H that provide that H is hamiltonian were obtained for $r \leq \frac{n-4}{2}$ by Bermond, Germa, Heydemann, and Sotteau [1]; Clemens, Ehrenmüller, and Person [3]; and Ma, Hou, and Gao [8]. Coulson and Perarnau [4] gave exact bounds in the case $r = o(\sqrt{n})$ (and large n). The present authors resolved the problem for all $3 \leq r < n$:

Theorem 1.3 ([6])**.** *Let $n > r \geq 3$. Suppose H is an n-vertex, r-graph such that (1) $r \leq (n-1)/2$ and $\delta(H) \geq \binom{\lfloor (n-1)/2 \rfloor}{r-1} + 1$, or (2) $r \geq n/2$ and $\delta(H) \geq r$. Then H contains a hamiltonian Berge cycle.*

The inequalities in this result are best possible for all $3 \leq r < n$. Very recently, Salia [12] proved sharp results of Pósa type for Berge hamiltonian cycles. He described the sequences $(d_1, \ldots, d_n)$ with $d_1 \leq d_2 \leq \ldots \leq d_n$ of two types: (a) for $r < n/2$ every n-vertex r-graph with degree sequence $(d'_1, \ldots, d'_n)$ such that $d'_i > d_i$ for all i has a hamiltonian Berge cycle and also (b) every n-vertex hypergraph with degree sequence $(d'_1, \ldots, d'_n)$ such that $d'_i > d_i$ for all i has a hamiltonian Berge cycle. The first of these nice results implies Part (a) of Theorem 1.3 for odd n.

Since we consider mostly Berge cycles and paths, from now on, we will drop the word "Berge" and simply use *cycle* and *path* to refer to a Berge cycle and a Berge path, respectively.

Note that while every hamiltonian-connected graph is hamiltonian, this is not true for r-graphs when $3 \leq r < n$. In the next section, for every $3 \leq r < n$ we present a hamiltonian-connected r-graph that has no hamiltonian cycles.

The main result of this paper is the following.

Theorem 1.4. *Let $n \geq r \geq 3$. Suppose H is an n-vertex r-graph such that*
(1) $r \leq n/2$ and $\delta(H) \geq \binom{\lfloor n/2 \rfloor}{r-1} + 1$, or (2) $n - 1 \geq r > n/2 \geq 3$ and $\delta(H) \geq r - 1$,
or (3) $r = 3$, $n = 5$ and $\delta(H) \geq 3$. Then H is hamiltonian-connected.

Note that the conditions in Theorem 1.4 for $3 \leq r \leq n/2$ and even n are stronger than in Theorem 1.3, for $3 \leq r \leq n/2$ and odd n are the same, and for $3 \leq n/2 < r \leq n-1$ are weaker than in Theorem 1.3. These bounds are sharp, and extremal examples will be given in the next section.

Similarly to [6], we elaborate the idea of Dirac [5] of choosing a longest cycle plus a longest path. We also use a series of lemmas on subsets of edges and vertices in graph paths.

The structure of the paper is as follows. In the next section, we show extremal examples for Theorems 1.3 and 1.4 and also examples of hamiltonian-connected r-graphs that have no hamiltonian cycles. In Section 3 we prove lemmas on subsets of graph paths. In Section 4 we set up the main proofs for all cases: we define "best" extremal substructures in possible counter-examples to our theorem and prove some properties of such substructures. In the subsequent three sections, we analyze all possible cases that can arise in counter-examples, and settle these cases. We finish the paper with some concluding remarks.

2 Examples

2.1 Examples for Theorems 1.3 and 1.4

For all $n > 3$ and $3 \leq r \leq (n-1)/2$, let $H_1 = H_1(n,r)$ be the r-graph formed by a clique Q of size $\lceil \frac{n+1}{2} \rceil$ and a clique R of size $\lfloor \frac{n+1}{2} \rfloor$ sharing exactly one vertex. Then $\delta(H_1) = \binom{\lfloor \frac{n-1}{2} \rfloor}{r-1}$, and H_1 is non-hamiltonian because it has a vertex whose deletion disconnects the r-graph.

Another example for $3 \leq r \leq (n-1)/2$, is the r-graph $H_2 = H_2(n,r)$ whose vertex set is $A \cup B$ where $|A| = \lceil \frac{n+1}{2} \rceil$, $|B| = \lfloor \frac{n-1}{2} \rfloor$, $A \cap B = \emptyset$ and whose edges are sets $X \subset A \cup B$ with $|X| = r$ and $|X \cap A| \leq 1$. Again, $\delta(H_2) = \binom{\lfloor \frac{n-1}{2} \rfloor}{r-1}$. Also, each cycle in H_2 has no two consecutive vertices in A. Since $|A| > n/2$, this yields that H_2 is not hamiltonian.

For $n/2 \leq r \leq n-1$, $H_3 = H_3(n,r)$ is obtained by removing a single edge from any r-regular r-graph. Then $\delta(H_3) = r-1$ and H_3 has $n-1$ edges. Hence H_3 cannot have a hamiltonian cycle.

The r-graphs above show sharpness of the bounds in Theorem 1.3. The following slight modifications of them show sharpness of the bounds in Theorem 1.4.

For all $n > 3$ and $3 \leq r \leq n/2$, let $H_1' = H_1'(n,r)$ be the r-graph formed by a clique Q of size $\lceil \frac{n+2}{2} \rceil$ and a clique R of size $\lfloor \frac{n+2}{2} \rfloor$ sharing exactly two vertices, say x and y. Then $\delta(H_1') = \binom{\lfloor \frac{n}{2} \rfloor}{r-1}$, and H_1' has no hamiltonian x,y-path, since any x,y-path should miss either $Q - \{x,y\}$ or $R - \{x,y\}$.

Another example for $3 \leq r \leq n/2$, is the r-graph $H_2' = H_2'(n,r)$ whose vertex set is $A \cup B$ where $|A| = \lceil \frac{n}{2} \rceil$ and $|B| = \lfloor \frac{n}{2} \rfloor$, $A \cap B = \emptyset$ and whose edges are sets $X \subset A \cup B$ with $|X| = r$ and $|X \cap A| \leq 1$. Now $\delta(H_2') = \binom{\lfloor \frac{n}{2} \rfloor}{r-1}$. Also, for distinct $x, y \in B$ each x,y-path in H_2' has no two consecutive vertices in A. Since $|A| \geq n/2$, this yields that H_2' has no hamiltonian x,y-path.

For $r > n/2$, let $H_3' = H_3'(n,r)$ be obtained from $H_3(n,r)$ by removing any edge. Then $\delta(H_3') = r-2$ and H_3' has $n-2$ edges. Hence H_3' cannot have any hamiltonian path.

For $r = 3$, $n = 5$, let $V(H_4) = \{1,2,3,4,5\}$ and $E(H_4) = \{\{1,5,2\},\{1,5,3\},\{1,5,4\},\{2,3,4\}\}$. Then $\delta(H_4) = 2$ but there is no hamiltonian path from 1 to 5.

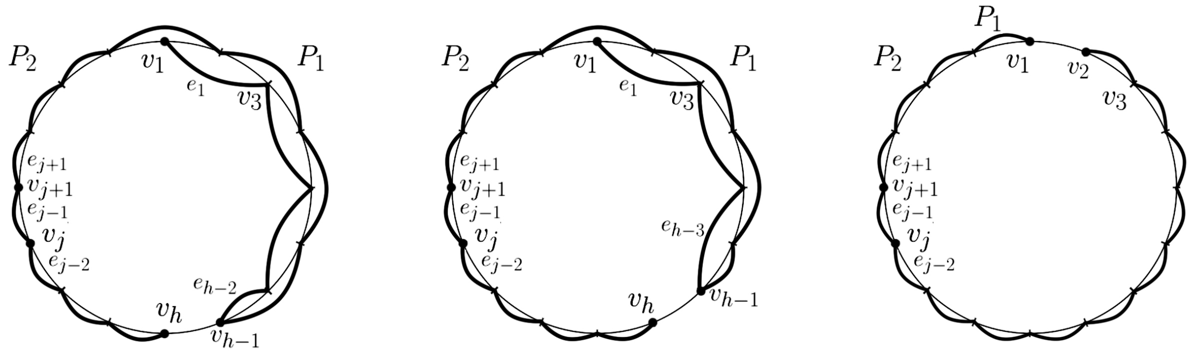

Figure 1: The three cases of a hamiltonian v_1, v_h-path in $C'(n,r)$.

2.2 Hamiltonian-connected r-graphs with no hamiltionian cycles

By the (n,r)-*tight cycle* $C(n,r)$ we denote the r-graph with vertex set $V = \{v_1, \dots, v_n\}$ and edge set $E = \{e_1, \dots, e_n\}$, where $e_i = \{v_i, v_{i+1}, \dots, v_{i+r-1}\}$ for all $i = 1, \dots, n$ and indices count modulo n.

Our example $C'(n,r)$ is obtained from $C(n,r)$ by deleting one edge. Since $C'(n,r)$ has $n-1$ edges, it has no hamiltonian cycles. We claim that for $3 \le r < n$, $C'(n,r)$ is hamiltonian-connected.

Indeed, by symmetry we may assume that we need a hamiltionian v_1, v_h-path and that we have deleted e_j from $C(n,r)$. Also by symmetry, we may assume that $h \le j+1 \le n$. We construct a hamiltionian v_1, v_h-path slightly differently for odd h, for even $h \ge 4$ and for $h = 2$. In all cases, the subpath from v_n to v_h will be

$$P_2 = v_n, e_{n-1}, v_{n-1}, e_{n-2}, \dots, \dots, v_{j+2}, e_{j+1}, v_{j+1}, e_{j-1}, v_j, e_{j-2}, v_{j-1}, \dots, e_h, v_{h+1}, e_{h-1}, v_h.$$

Our final hamiltonian v_1, v_h-path will be of the form $P_1 \cup P_2$ (see Figure 1) where the subpath P_1 is as follows:

If h is odd, then

$$P_1 = v_1, e_1, v_3, e_3, v_5, \dots, v_{h-2}, e_{h-2}, v_{h-1}, e_{h-3}, v_{h-3}, e_{h-5}, \dots, e_2, v_2, e_n, v_n.$$

If h is even and $h > 2$, then

$$P_1 = v_1, e_1, v_3, e_3, v_5, \dots, v_{h-1}, e_{h-2}, v_{h-2}, e_{h-4}, v_{h-4}, e_{h-6}, \dots, e_2, v_2, e_n, v_n.$$

Finally if $h = 2$, then $P_1 = v_1, e_n, v_n$.

3 Lemmas on graph paths

In this section we derive some properties of subsets of graph paths that will be heavily used in our proofs. The reader can skip their proofs at the first reading.

Lemma 3.1. *Let $Q = v_1, e_1, \dots, e_{s-1}, v_s$ be a graph path. Let A and B be nonempty subsets of $V(Q)$ such that A is an independent set, $B - A \ne \emptyset$, and for each $v_i \in A$ and $v_j \in B - A$, $|i-j| \ge q \ge 1$. Then* *(i) If $q \geq 2$, then $s \geq 2|A| + |B - A| + q - 2$ (and therefore $|A| \leq (s - |B - A| - q + 2)/2$).*

(ii) If $q = 1$, then $s \geq 2|A| + |B - A| - 2$.

Moreover, if B is also an independent set, then $s \geq 2|A| + 2|B - A| + q - 3$.

Proof. Let $v_j \in B - A$. Without loss of generality, we may suppose there exists a vertex $v_i \in A$ such that $i < j$ and $v_k \notin A \cup B$ for all $i < k < j$. Then $V_1 := \{v_{i+1}, \ldots, v_{j-1}\}$ is a set of of at least $q - 1$ vertices which does intersect $A \cup B$. Similarly, if there exists $v_{i'} \in A$ such $i < j < i'$ (and $v_k \notin A \cup B$ for all $j < k < i'$), then $V_2 := \{v_{j+1}, \ldots, v_{i'-1}\}$ also contains at least $q - 1$ vertices and does not intersect $A \cup B$. In this case, set $V' = V_1 \cup V_2$. Otherwise, set $V' = V_1$.

For each $v_k \in A - \{v_s\}$, v_{k+1} does not intersect $A \cup B$, and only one v_{k+1}, namely v_{i+1}, is in V'. Therefore

$$s \geq |A| + |B - A| + |\{v_{k+1} : v_k \in A, k \notin \{i, s\}\}| + |V'|.$$

If $V' = V_1 \cup V_2$, then $s \geq 2|A| + |B - A| - 2 + 2q - 2$ which is at least $2|A| + |B - A| + q - 2$ if $q \geq 2$, and at least $2|A| + |B - A| + 2$ if $q = 1$.

If $V' = V_1$, then in this case $v_s \notin A$, so we have $s \geq |A| + |B - A| + (|A| - 1) + q - 1 = 2|A| + |B - A| + q - 2$.

Suppose now that B is also an independent set, and let $v_j \in B - A$. Again we may suppose there exists $v_i \in A$ with $i < j$. Between v_j and v_i there is a set V' of at least $q - 1$ vertices not in $A \cup B$, and for any $v_k \in A \cup B$, $v_{k+1} \notin A \cup B$. Therefore

$$\begin{aligned} s &\geq |A \cup B| + |\{v_{k+1} : v_k \in A \cup B, k \notin \{i, s\}\}| + |V'| \\ &\geq |A| + |B - A| + (|A| + |B - A| - 2) + q - 1 = 2|A| + 2|B - A| + q - 3. \end{aligned}$$

□

Lemma 3.2. *Let $q \geq 2$ and $s > a \geq 1$. Let $Q = v_1, e_1, \ldots, e_{s-1}, v_s$ be a graph path, and I be a non-empty independent subset of $\{v_1, \ldots, v_s\}$. If A' is a set of a edges of Q such that the distance in Q from any edge in A' to any vertex in I is at least q, then $|I| \leq \lfloor \frac{s-a-q+1}{2} \rfloor$ if $q \geq 2$, and $|I| \leq \lfloor \frac{s-a+1}{2} \rfloor = \lceil \frac{s-a}{2} \rceil$ if $q = 1$.*

Proof. Applying Lemma 3.1 with $A = I$ and $B = \bigcup_{\{i : e_i \in A'\}} \{v_i, v_{i+1}\}$ (so $|B| \geq a + 1$) gives the desired bounds. □

Lemma 3.3. *Let $Q = v_1, e_1, \ldots, e_{s-1}, v_s$ be a graph path. Let A and B be nonempty subsets in $V(Q)$ such that*

$$\text{for each } v_i \in A \text{ and } v_j \in B, \text{ either } i = j \text{ or } |i - j| \geq q \geq 2. \tag{1}$$

(i) If $A = B$, then $s \geq 1 + q(|A| - 1)$ with equality only if $A = \{v_1, v_{1+q}, v_{1+2q}, \ldots, v_s\}$.

(ii) If $B \neq A$, then $s \geq |A| + |B| + q - 2$ with equality only if $A \subset B$ or $B \subset A$.

Proof. Part (i) is obvious. We prove (ii) by induction on $|A \cap B|$.

If $A \cap B = \emptyset$, then Q contains $|A| + |B|$ vertices in $A \cup B$ and at least $q - 1$ vertices outside of $A \cup B$ between A and a closest to A vertex in B.

Suppose now that (ii) holds for all A' and B' with $|A' \cap B'| < t$ and that $|A \cap B| = t$, say $v_i \in A \cap B$. By symmetry, we may assume $|A| \leq |B|$. If $A = \{v_i\}$, then Q has $|B| - 1$ vertices in $B - A$ and at least $q - 1$ vertices between v_i and a closest to v_i vertex in $B - A$ (such a vertex exists since $B \neq A$). Thus, $s \geq 1 + |B - A| + q - 1 = |B| + |A| - 2$, as claimed.

Finally, suppose $|A| \geq 2$. By definition, $(A \cup B) \cap \{v_{i-q+1}, v_{i-q+2}, \ldots, v_{i+q-1}\} = \{v_i\}$. So since $|B| \geq |A| \geq 2$, $i \geq q+1$ or $i \leq s-q$ (or both). By symmetry, assume $i \geq q+1$. Define $e'_i = v_{i-q+1}v_{i+1}$ and let $A' = A - v_i$, $B' = B - v_i$, and $Q' = v_1, e_1, \ldots, v_{i-q+1}, e'_i, v_{i+1}, e_{i+1}, \ldots, e_{s-1}, v_s$. By definition, A' and B' satisfy (1). So, by induction, $|V(Q')| \geq |A'| + |B'| + q - 2$ with equality only if $A' \subset B'$. Hence

$$s \geq q + |V(Q')| \geq q + (|A| - 1) + (|B| - 1) + q - 2 = |A| + |B| + 2(q - 2),$$

with equality only if $A \subset B$. Since $q \geq 2$, this proves (ii). □

Lemma 3.4. *Let $Q = v_1, e_1, \ldots, e_{s-1}, v_s$ be a graph path. Let A' and B' be nonempty subsets of $E(Q)$ such that*

$$\text{for each } e_i \in A' \text{ and } e_j \in B', \text{ either } i = j \text{ or } |i - j| \geq q \geq 2. \tag{2}$$

(i) If $A' = B'$, then $s - 1 \geq 1 + q(|A'| - 1)$ with equality only if $A' = \{e_1, e_{1+q}, e_{1+2q}, \ldots, e_{s-1}\}$.

(ii) If $B' \neq A'$, then $s - 1 \geq |A'| + |B'| + q - 2$ with equality only if $A' \subset B'$ or $B' \subset A'$.

Proof. Let $A = \{v_i : e_i \in A'\}$ and $B = \{v_i : e_i \in B'\}$. Since $v_s \notin A \cup B$, the sets A, B are vertex subsets of the path $Q' = v_1, e_1, \ldots, e_{s-2}, v_{s-1}$. So, Lemma 3.3 applied to A, B and Q' yields the desired bounds. □

Lemma 3.5. *Let $Q = v_1, e_1, \ldots, e_{s-1}, v_s$ be a graph path. Suppose $F \subset E(Q)$ and $f = |F|$. Let A and B be subsets of $\{v_1, \ldots, v_s\}$ that are vertex-disjoint from all edges in F and such that*

$$\text{for each } v_i \in A \text{ and } v_j \in B, \text{ either } i = j \text{ or } |i - j| \geq 2. \tag{3}$$

(i) If $A = B$, then $s \geq |A| + |B| + f - 1$.

(ii) If $B \neq A$, then $s \geq |A| + |B| + f$ with equality only if $A \subset B$ or $B \subset A$.

Proof. Let $Q' = v'_1, e'_1, v'_2, \ldots, e'_{s'-1}, v'_{s'}$ be a path obtained from Q by iteratively contracting f edges of F. In particular, $s' = s - f$. Since A and B are both vertex-disjoint from F, each $v_i \in A \cup B$ was unaffected by the edge contractions and hence still exists as some $v'_{i'}$ in Q'. Moreover, (3) still holds for A and B in Q'.

So, Lemma 3.3 for $q = 2$ applied to A, B and Q' yields that if $A = B$, then $s' \geq |A| + |B| - 1$, and if $B \neq A$, then $s' \geq |A| + |B|$ with equality only if $A \subset B$ or $B \subset A$. Since $s' = s - f$, this proves our lemma. □

4 Setup for Theorem 1.4

Bounds of the theorem differ for $r \leq n/2$ and $r > n/2$. Naturally, the proofs also will be different, but they will have similar structure. In both proofs, for given vertices x, y in an r-graph H we attempt to find a hamiltonian x, y-path. Both proofs will have three steps.

In Step 1 we construct an x, y-path Q with at least $\max\{\lceil \frac{n+2}{2} \rceil, r+1\}$ vertices.

Then we consider pairs (Q, P) of vertex-disjoint paths in H such that Q is an x, y-path. We will say that such a pair (Q, P) *is better* than a similar pair (Q', P') if

(i) $|E(Q)| > |E(Q')|$, or

(ii) $|E(Q)| = |E(Q')|$ and $|E(P)| > |E(P')|$, or

(iii) $|E(Q)| = |E(Q')|$, $|E(P)| = |E(P')|$ and the total number of vertices in $V(P)$ in the edges in Q (counted with multiplicities) is greater than the total number of vertices in $V(P')$ in the edges in Q'.

We consider best pairs and study their properties. Some properties will be proven in the next subsection. Using these properties together with the lemmas on graph paths from the previous section in Step 2 we show that the path P in a best pair cannot have exactly one vertex. In the final Step 3 we handle all cases when P has at least two vertices.

Below we assume that (Q, P) is a best pair, $Q = v_1, e_1, \ldots, e_{s-1}, v_s$, and $P = u_1, f_1, \ldots, f_{\ell-1}, u_\ell$.

We consider three subhypergraphs, H_Q, H_P and H' of H with the same vertex set $V(H)$: $E(H_Q) = \{e_1, \ldots, e_{s-1}\}$, $E(H_P) = \{f_1, \ldots, f_{\ell-1}\}$ and $E(H') = E(H) - E(H_P) - E(H_Q)$. By definition, the edge sets of these three subhypergraphs form a partition of the edge set of H. For a hypergraph F and a vertex u, we denote by $N_F(u) = \{v \in V(F) : \{u, v\} \subseteq e \text{ for some } e \in F\}$. For $i \in \{1, \ell\}$, set $B_i = \{e_j \in E(Q) : u_i \in e_j\}$ and $b_i = |B_i|$.

4.1 Claims on best pairs

The claims below apply to all best pairs (Q, P), regardless of the uniformity r.

Claim 4.1. *In a best pair (Q, P), $N_{H'}(u_1)$ cannot contain a pair of vertices that are consecutive in Q.*

Proof. Suppose toward a contradiction that v_i, v_{i+1} are contained in edges of H' with u_1. Let $e, e' \in E(H')$ be such that $u_1, v_i \in e$ and $u_1, v_{i+1} \in e'$. If $e \neq e'$, then replacing e_i with e, u_1, e' gives a longer x, y-path than Q, a contradiction. Thus we may assume $e = e'$.

If there is $1 \leq j \leq \ell$ such that $u_j \in e_i$, then by replacing the path v_i, e_i, v_{i+1} in Q with the longer path $v_i, e, u_1, f_1, u_2, \ldots, f_{j-1}, u_j, e_i, v_{i+1}$, we obtain a longer x, y-path than Q. Thus $e_i \cap V(P) = \emptyset$. Then replacing e_i with e in Q gives a path Q' with (Q', P) better than (Q, P) by criterion (iii). □

Symmetrically, the claim holds for u_ℓ in place of u_1.

Claim 4.2. *For any $u \notin V(Q)$, if $u \in e_i$, then $v_i, v_{i+1} \notin N_{H-H_Q}(u)$.*

Proof. Suppose $v_i \in N_{H-H_Q}(u)$, and let $e \in E(H) - E(H_Q)$ be such that $\{u, v_i\} \subseteq e$. Then we can find a longer cycle by replacing e_i with e, u, e_i, a contradiction to our choice of Q. A similar argument holds for v_{i+1}. □

Claim 4.3. *For every $e_i \in B_1, e_j \in B_\ell$ either $i = j$ or $|i - j| \geq \ell$.*

Proof. Suppose there exists $e_i \in B_1, e_j \in B_\ell$ such that without loss of generality $j > i$ and $j-i \leq \ell-1$. Then the path obtained by replacing $v_i, e_i, \dots, e_j, v_{j+1}$ in Q with $v_i, e_i, u_1, f_1, \dots, f_{\ell-1}, u_\ell, e_j, v_{j+1}$ has $|V(Q)| - (i-j) + \ell > |V(Q)|$ vertices, a contradiction. □

Claim 4.4. *If there exists distinct edges $e, f \in E(H')$ such that $\{u_1, v_i\} \subset e$ and $\{u_\ell, v_j\} \subset f$, then $|i-j| \geq \ell+1$.*

Proof. If $|i-j| \leq \ell$, replace the subpath in Q from v_i to v_j with the path v_i, e, P, f, v_j to get a longer x, y-path. □

Claim 4.5. *For every $v_i \in N_{H'}(u_1)$ and $e_j \in B_\ell$, if $i \leq j$ then $j-i \geq \ell$ and if $i > j$ then $i-j \geq \ell+1$.*

Proof. Let $e \in E(H')$ contain v_i and u_1. If $i \leq j$, let Q' be the path obtained by replacing the segment $v_i, e_i, \dots, e_j, v_{j+1}$ in Q with the path $v_i, e, u_1, P, u_\ell, e_j, v_{j+1}$. If $i > j$, let Q' be obtained from Q by replacing $v_j, e_j, \dots, v_i$ with $v_j, e_j, u_\ell, f_{\ell-1}, \dots, f_1, u_1, e, v_i$. In the first case, $|V(Q')| = |V(Q)| - (j-i) + \ell$, and in the second case $|V(Q')| = |V(Q)| - (i-(j+1)) + \ell$. The Claim follows since $|V(Q)| \geq |V(Q')|$ by the choice of (Q, P). □

Claim 4.6. *For any $e \in E(H')$, if $v_i, v_j \in e$, then at most one of e_i, e_j is in B_1 and at most one of e_{i-1}, e_{j-1} is in B_1.*

Proof. If $e_{i-1}, e_{j-1} \in B_1$, then we get a longer path

$$v_1, e_1, v_2, \dots, v_{i-1}, e_{i-1}, u_1, e_{j-1}, v_{j-1}, e_{j-2}, v_{j-2}, \dots, v_i, e, v_j, e_j, v_{j+1}, \dots, v_s.$$

The argument for e_i, e_j is similar. □

Claim 4.7. *Let $B_1^- = \{v_i : e_i \in B_1\}$ and $B_1^+ = \{v_{i+1} : e_i \in B_1\}$.*
(i) For any edge $e \in E(H')$, $b_1 \leq s - |e \cap V(Q)| + 1$ with equality only if $B_1 = \{e_i, e_{i+1}, \dots, e_j\}$ for some $i < j$ and $e \cap V(Q) = \{v_1, \dots, v_i\} \cup \{v_{j+1}, \dots, v_s\}$.
(ii) $b_1 \leq s - 1 - |N_{H'}(u_1) \cap V(Q)|$ with equality only if B_1 is a set of b_1 consecutive edges in Q and $N_{H'}(u_1) \cap V(Q) = V(Q) - (B_1^- \cup B_1^+)$.

Proof. Set $e' = e \cap V(Q)$. By Claim 4.6, $|B_1^- \cap e| \leq 1$. Hence

$$b_1 - 1 + |e \cap V(Q)| - 1 \leq |B_1| - 1 + |e' - \{v_s\}| \leq |\{v_1, \dots, v_{s-1}\}| = s - 1,$$

i.e., $b_1 \leq s - |e'| + 1$ with equality only if $v_s \in e$, $|B_1^- \cap e| = 1$, and $e \cup B_1^- = V(Q)$. Symmetrically, $v_1 \in e$, $|e \cap B_1^+| = 1$, and $V(Q) \setminus e \subset B_1^+$. So, $V(Q) \setminus e \subseteq B_1^- \cap B_1^+$, and therefore $|B_1^- \cap B_1^+| \geq s - |e'| = b_1 - 1$. This means, the symmetric difference of B_1^- and B_1^+ has only two vertices. For this, the b_1 edges in B_1 must be consecutive on Q, and $e' = V(Q) - (B_1^- \cap B_2^+)$. This proves (i).
For (ii), by Claim 4.2 $(B_1^- \cup B_1^+) \cap (N_{H'}(u_1) \cap V(Q)) = \emptyset$. Therefore $|B_1^- \cup B_1^+| \leq |V(Q)| - |N_{H'}(u_1) \cap V(Q)| = s - |N_{H'}(u_1) \cap V(Q)|$. We have $|B_1^- \cup B_1^+| \geq b_1 + 1$ with equality only if B_1 is a set of consecutive edges in Q. These inequalities together give our result. □

Corollary 4.8. *When $\ell = 1$, $b_1 \leq n - 3$. If in addition, $|N_{H'}(u_1) \cap V(Q)| \geq 2$, then $b_1 \leq n - 4$.*

Proof. Since $|E(Q)| < n-1$, there must be at least one edge $e \in E(H')$, and since $\ell = 1$, e contains at least $r-1 \geq 2$ vertices in Q if $u_1 \in e$, and at least $r \geq 3$ otherwise. By Claim 4.7, $b_1 \leq s-2 \leq n-3$.

The second part follows from Claim 4.7 (ii). □

5 Finding a longish x, y-path

In this section, we will show that there exists an x, y-path of length at least $\max\{n/2+1, r+1\}$.

Lemma 5.1. *Let $3 \leq r \leq n/2$, and let H be an n-vertex r-graph. Let $x, y \in V(H)$. If $\delta(H) \geq \binom{\lfloor n/2 \rfloor}{r-1}$, then H contains an x, y-path with at least $n/2+1$ vertices.*

Proof. The restrictions on $\delta(H)$ in Theorem 1.3 are not stronger than in Theorem 1.4. So, by Theorem 1.3, H contains a hamiltonian cycle C in H. The longer of the two x, y-paths along C has at least $n/2+1$ vertices. □

For $r > n/2$, we need much more effort, see below.

Lemma 5.2. *Let $n \geq r > n/2$, and let H be an n-vertex r-graph. Let $x, y \in V(H)$. If $\delta(H) \geq r-1$, then H contains an x, y-path with at least $r+1$ vertices.*

Proof. We will first show that there exists some x, y-path in H. If there exists an edge $e \in E(H)$ with $\{x, y\} \subseteq e$, then we are done. Otherwise since $r > n/2$, any two edges $e, f \in E(H)$ such that $x \in e, y \in f$ have a common vertex, say $v \in e \cap f$. Then x, e, v, f, y is an x, y-path in H.

Now let $Q = v_1, e_1, \ldots, e_{s-1}, v_s$ be a longest x, y-path in H (so $x = v_1, y = v_s$). Moreover, choose Q so that if $\{v_1, \ldots, v_s\} \in E(H)$, then this edge is used in Q. Suppose $s \leq r$.

Construct a new hypergraph $\hat{H}$ as follows: $V(\hat{H}) = V(H) - V(Q)$, and $E(\hat{H}) = \{e \cap V(\hat{H}) : e \in E(H) - E(Q)\}$. Note that $\hat{H}$ is not necessarily a uniform hypergraph. We have a mapping from the edges of H to the edges of $\hat{H}$ given by $e \mapsto e - V(Q)$ (which is not necessarily one-to-one).

Let $D_1, D_2, \ldots, D_q$ be the vertex sets of the connected components of $\hat{H}$. For $1 \leq j \leq q$, let $d_j = |\{e_i \in E(Q) : e_i \cap D_j \neq \emptyset\}|$. Since $|V(Q)| \leq r$, at most one edge $e_i \in E(Q)$ may be contained in $V(Q)$. It follows that

$$\sum_{i=1}^{q} d_i \geq |E(Q)| - 1 = s - 2. \tag{4}$$

Claim 5.3. *For any $1 \leq j \leq q$, if $e_i \cap D_j \neq \emptyset$, then the edges of $E(H) - E(Q)$ containing v_i or v_{i+1} cannot intersect D_j.*

Proof. Let $v \in D_j \cap e_i$. Suppose $v_i \in h \in E(H) - E(Q)$ and $u \in h \cap D_j$. Then $\hat{H}$ contains a u, v-path which we can lift to a u, v-path P in H that avoids $E(Q)$. If $h \notin E(P)$, then by replacing the segment v_i, e_i, v_{i+1} in Q with the path v_i, h, P, e_i, v_{i+1}, we obtain a longer x, y-path. Otherwise let P' be the subpath of P starting with h. Then we replace v_i, e_i, v_{i+1} with v_i, P', e_i, v_{i+1} to get a longer path. The argument for v_{i+1} is similar. □

Claim 5.4. *For any $1 \le j \le q$ and any $1 \le i \le s-1$, there are no distinct edges $e, f \in E(H)-E(Q)$ such that e and f intersect D_j, $v_i \in e$, and $v_{i+1} \in f$.*

Proof. Let P' be a shortest path in $\hat{H}$ from $e \cap D_j$ to $f \cap D_j$. Lift P' to a path P in H which avoids $E(Q)$. By the minimality of P', $e \notin E(P)$ and $f \notin E(P)$. Then we may replace the segment v_i, e_i, v_{i+1} in Q with v_i, e, P, f, v_{i+1} to get a longer x, y-path. □

Claim 5.5. *For any $1 \le j \le q$, if at least 2 edges in $E(H) - E(Q)$ intersect D_j, then*

$$|D_j| \ge r - \lceil (s - d_j)/2 \rceil + 1.$$

Proof. Suppose $|D_j| \le r - \lceil (s-d_j)/2 \rceil$, and let $e, g \in E(H) - E(Q)$ be distinct edges that intersect D_j. Let $A = e \cap V(Q)$, $B = g \cap V(Q)$, and $F = \{\{v_i, v_{i+1}\} : D_j \cap e_i \neq \emptyset\}$. By definition, $|F| = d_j$, and each of A and B has at least $r - |D_j| \ge \lceil (s - d_j)/2 \rceil$ vertices.

By Claim 5.3, A and B are disjoint from all pairs in F. By Claim 5.4, (3) holds. So Lemma 3.5 together with the lower bounds on $|A|$ and $|B|$ imply that if $A \neq B$, then

$$s \ge |A| + |B| + d_j \ge 2\frac{s - d_j}{2} + d_j = s, \tag{5}$$

with equality only if $A \subset B$ or $B \subset A$. But if $A \subset B$ or $B \subset A$ and $A \neq B$, then $|A|+|B| \ge 1+2\frac{s-d_j}{2}$. Hence, if $A \neq B$, then in the RHS of (5) we get at least $s+1$, a contradiction.

Thus $A = B$. Since e and g are distinct but coincide on Q, $e \cap D_j$ and $g \cap D_j$ are distinct sets each with at least $r - \lceil (s-d_j)/2 \rceil$ vertices. It follows that $|D_j| \ge r - \lceil (s - d_j)/2 \rceil + 1$. □

Claim 5.6. *For any $1 \le j \le q$, if exactly one edge in $E(H) - E(Q)$ intersects D_j, then $|D_j| \ge r$.*

Proof. Suppose $|D_j| \le r - 1$ and e is the unique edge in $E(H) - E(Q)$ that intersects D_j. Then by the definition of $\hat{H}$, $D_j = e - V(Q)$. Let $v \in D_j$. Since $|e| = r$, e contains at least one vertex v_i in Q. By symmetry, we may suppose $i < s$. In order to have $d(v) \ge r - 1$, v must belong to at least $r - 2$ edges of $E(Q)$. By Claim 5.3, none of these at least $r - 2$ edges is e_{i-1} or e_i. This is possible only if $s = r$, $e \cap V(Q) = \{v_1\}$ and $v \in e_2 \cap e_3 \cap \ldots \cap e_{s-1}$. This implies $|D_j| = r - 1$ and each vertex in D_j belongs to e_2 by symmetry. But then $\{v_2, v_3\} \cup D_j \subseteq e_2$, contradicting the fact that $|e_2| = r$. □

Claim 5.7. *For any $1 \le j \le q$, at least one edge in $E(H) - E(Q)$ intersects D_j.*

Proof. Suppose not. By the definition of $\hat{H}$, D_j is a single vertex, say v. Since $d(v) \ge r-1$, v must belong to at least $r - 1$ edges of Q, which is only possible if $|V(Q)| = r$. In this case v is contained in all edges of Q. By the choice of Q, we have $\{v_1, \ldots, v_s\} \notin E(H)$.

Since $|E(H)| \ge n - 1 > r - 1$, there exists an edge $g \in E(H) - E(Q)$. By the choice of Q, g intersects some D_h. If $|D_h| \ge r - 1$, then $|V(H)| \ge |V(Q)| + |D_h| + |D_j| \ge r + r - 1 + 1 > n$, a contradiction. In particular, by Claim 5.6, this implies that at least two edges in $E(H) - E(Q)$ intersect D_h. We claim that for each such edge e,

$$e \cap V(Q) \subseteq \{v_1, v_s\}. \tag{6}$$

Suppose this is not the case. Then since $|D_h| \le r-2$, there exists a pair $\{v_i, v_{i'}\} \neq \{v_1, v_s\}$ and edges $e, f \in E(H) - E(Q)$ such that e and f intersect D_h, $v_i \in e$, and $v_{i'} \in f$. Without loss of generality, we may assume $i < i' < s$. Let P be a $v_i, v_{i'}$-path in H avoiding $E(Q)$ (it could be the case that P contains only one edge). Then

$$v_1, \ldots, v_i, P, v_{i'}, e_{i'-1}, \ldots, v_{i+1}, e_i, v, e_{i'}, v_{i'+1}, \ldots, e_{s-1}, v_s$$

is a longer x, y-path in H. Therefore (6) holds. Since at least two edges in $E(H) - E(Q)$ intersect D_h, $|D_h| \ge r-1$, a contradiction. □

Claim 5.8. *$\hat{H}$ has at least 2 components.*

Proof. Suppose $q = 1$. If $|V(Q)| \le r-1$, then each edge e_i intersects D_1 and each $v_i \in V(Q)$ is contained in an edge $h \in E(H) - E(Q)$, and h must also intersect D_1 since $|h| > |V(Q)|$, contradicting Claim 5.3. So we may assume $|V(Q)| = r$.

By Claim 5.7, some edge $h \in E(H) - E(Q)$ intersects D_1. If $h \subset D_1$, then $|V(H)| \ge |V(D_1)| + |V(Q)| \ge r + r > n$, a contradiction. For each $v_i \in h \cap V(Q)$, each of e_i and e_{i-1} must be contained in $V(Q)$. As $r = V(Q)$, only one such edge in Q can satisfy this. Hence without loss of generality, we may assume $h \cap V(Q) \subseteq \{v_1\}$ and $e_1 = V(Q)$. It follows that $|D_1| \ge r-1$. If $|D_1| \ge r$, then again we get $|V(H)| > n$.

Hence, the last possibility is that $|D_1| = r-1$ and $h \cap V(Q) = \{v_1\}$. In particular, by Claim 5.6, some other edge $h' \in E(H) - E(Q)$ intersects D_1. Since $s = r \ge 3$, $e_1 \neq e_{s-1}$. So by the same argument as for h, we have $h' \cap V(Q) = \{v_1\}$. Since $h' \neq h$ and $D_1 \supseteq h \cup h' - \{v_1\}$, we get $|D_1| > r-1$, a contradiction. □

Now we are ready to finish the proof of the lemma. By Claims 5.7, 5.6 and 5.5, $|V(D_j)| \ge r - \lceil (s-d_j)/2 \rceil + 1$ for all j. Therefore

$$|V(H)| \ge |V(Q)| + \sum_{j=1}^{q} (r - \lceil (s-d_j)/2 \rceil + 1) \ge s + q(r - \frac{s+1}{2} + 1) + \sum_{i=1}^{q} \frac{d_j}{2}.$$

Since $r \ge s$, the quantity $q(r - \frac{s+1}{2} + 1)$ is minimized when $q = 2$. By (4),

$$\begin{aligned} |V(H)| &\ge s + 2(r - \frac{s+1}{2} + 1) + \sum_{i=1}^{q} \frac{d_j}{2} \\ &\ge s + 2r - (s+1) + 2 + (s-2)/2 \\ &= 2r + s/2 \\ &> n, \end{aligned}$$

a contradiction. □

6 Proof of Theorem 1.4 for $r \le n/2$

In the next two sections, we set $t' = \lfloor n/2 \rfloor$ and consider a best pair (Q, P) with $Q = v_1, e_1, \ldots, e_{s-1}, v_s$ and $P = u_1, f_1, \ldots, f_{\ell-1}, u_\ell$. By Lemma 5.1, $s \ge t' + 1$ if s is even and $s \ge t' + 2$ if s is odd. In both cases we get $\ell \leq n-s \leq t'-1$ and $s \geq n/2+1$. Recall that for $i \in \{1, \ell\}$, $B_i = \{e_j : u_i \in e_j\}$, and $b_i = |B_i|$.

6.1 Finding a nontrivial path P

Lemma 6.1. *In a best pair (Q, P), $|V(P)| \geq 2$.*

Proof. Suppose that $|V(P)| = \ell = 1$, i.e., $P = u_1$. Then $s \leq n-1$. By condition (ii) of (Q, P) being a best pair, every edge of H' contains at most one vertex outside Q.
Claims 4.1, 4.2 and Lemma 3.2 imply that $|N_{H'}(u_1)| \leq \lceil (s-b_1)/2 \rceil$. Therefore

$$1 + \binom{t'}{r-1} \leq d_H(u_1) \leq b_1 + \binom{\lceil (s-b_1)/2 \rceil}{r-1} \leq b_1 + \binom{\lceil (n-1-b_1)/2 \rceil}{r-1}. \tag{7}$$

Case 1: $b_1 = 0$. By (7), $1 + \binom{t'}{r-1} \leq \binom{\lceil (n-1)/2 \rceil}{r-1} = \binom{t'}{r-1}$, a contradiction.

Case 2: $b_1 = 1$. Again by (7), $1 + \binom{t'}{r-1} \leq 1 + \binom{\lceil (s-1)/2 \rceil}{r-1} \leq 1 + \binom{\lceil (n-2)/2 \rceil}{r-1}$. If n is even, we immediately obtain a contradiction. If n is odd, then we reach a contradiction when $s < n-1$. So suppose n is odd, $s = n-1$, $|N_{H'}(u_1)| = \lceil (s-b_1)/2 \rceil = s/2 = t'$, and u_1 is contained in all $\binom{t'}{r-1}$ possible edges within $N_{H'}(u_1) \cup \{u_1\}$.
Consider the unique edge e_i of Q containing u_1. Then $|N_{H'}(u_1)| \leq \lceil (i-1)/2 \rceil + \lceil (n-1-(i+1))/2 \rceil$ by Claim 4.1 and Claim 4.2. If i is odd, then this gives $|N_{H'}(u_1)| \leq (i-1)/2 + (n-i-2)/2 = (n-3)/2$, a contradiction. Thus, i is even and $X := N_{H'}(u_1) = \{v_1, v_3, \ldots, v_{i-1}, v_{i+2}, v_{i+4}, \ldots, v_s\}$.
Replacing e_{i-1} in Q with the edge $e \in E(H')$ containing u_1, v_{i-1} and replacing v_i with u_1 creates a new path Q' which only misses v_i. Since (Q, P) is a best pair, by condition (iii) of choosing a best pair, e_i and e_{i-1} can be the only edges of Q which contain v_i and in fact (Q', v_i) is also a best pair. Thus applying the same arguments to v_i and Q' as we did to u_1 and Q, we obtain that $N_{H-Q'}(v_i) = X$. Notice that we can apply a symmetric argument to v_{i+1} and corresponding path Q'' to get $N_{H-Q''}(v_{i+1}) = X$.
We will find an edge $g \neq e_i$ with $|g - X| \geq 2$ and $|g \cap \{v_2, v_4, \ldots, v_{i-2}, v_{i+3}, v_{i+5}, \ldots, v_{s-1}\}| \geq 1$, and then we will use g to find the desired hamiltonian path. Choose $v_j \notin e_i$ with $v_j \notin X$, which exists because $|(X \cup e_i) \cap V(Q)| \leq |X| + r - 1 \leq 2t' - 1 = s - 1$. Since $d_H(v_j) > \binom{t'}{r-1}$ and $|X| = t'$, there is an edge g containing v_j and some vertex outside X. Since $v_j \notin X$ and $v_j \notin e_i$, that vertex cannot be u_1 and must instead be some $v_k \in V(H) - (X \cup \{u_1\}) = V(Q) - X$. Suppose without loss of generality that $j < k$.
Case 2.1: $g \in E(H')$. Since v_j is in neither X nor e_i, $v_{j-1} \in X$. Thus let $f \in E(H')$ be such that $v_{j-1}, u_1 \in f$. Similarly, since $v_k \notin X$, we have $v_{k-1} \in X$ unless $k = i+1$, which we handle separately. Let $f' \in E(H')$ be such that $v_{k-1}, u_1 \in f'$, and observe that we can choose edges such that f, f' are distinct because u_1 is in $\binom{t'-1}{r-2} \geq 2$ edges with each vertex in X. Thus if $j < k$, we have the hamiltonian path

$$v_1, e_1, v_2, \ldots, v_{j-1}, f, u_1, f', v_{k-1}, e_{k-2}, v_{k-2}, \ldots, v_j, g, v_k, e_k, v_{k+1}, \ldots, v_s.$$

A similar path can be found for $j > k$ by symmetry. In the case $k = i+1$, replace f' in the above path with e_i to obtain the desired hamiltonian path.

Case 2.2: $g = e_m \in E(Q)$. Since $g \neq e_i$, we may assume by symmetry that $v_m \in X$, unless $m = k$. Let f be as in the previous case, and let $f' \in E(H')$ be such that $v_m, u_1 \in f'$.

Thus for $j < m$ we have the hamiltonian path

$$v_1, e_1, v_2, \ldots, v_{j-1}, f, u_1, f', v_m, e_{m-1}, v_{m-1}, \ldots, v_j, g, v_{m+1}, e_{m+1}, v_{m+2}, \ldots, v_s$$

(and similar for $j > m$). If $m = k$, then $v_{k+1} \in X$, so we let $f'', f''' \in E(H')$ be such that $v_{j+1}, u_1 \in f''$ and $v_{k+1}, u_1 \in f'''$. Then

$$v_1, e_1, v_2, \ldots, v_j, g, v_k, e_{k-1}, v_{k-1}, \ldots, v_{j+1}, f'', u_1, f''', v_{k+1}, e_{k+1}, v_{k+2}, \ldots, v_s$$

is hamiltonian if $j < k$ and we can find a similar path for $j > k$, ending the proof of Case 2.

Case 3: $b_1 \geq 2$. Then by (7)

$$1 + \binom{t'}{r-1} \leq b_1 + \binom{\lceil (n-1-b_1)/2 \rceil}{r-1} \leq b_1 + \binom{\lceil (n-3)/2 \rceil}{r-1} = b_1 + \binom{t'-1}{r-1}.$$

Hence $1 + \binom{t'}{r-1} - \binom{t'-1}{r-1} = 1 + \binom{t'-1}{r-2} \leq b_1 \leq n-3$ by Corollary 4.8. If $2 \leq r-2 \leq t'-3$, then we have $n - 4 \geq \binom{t'-1}{r-2} \geq \binom{t'-1}{2}$, a contradiction when $n \geq 12$. For $n \leq 11$, it is straightforward to check that $1 + \binom{t'}{r-1} > b_1 + \binom{\lceil (n-1-b_1)/2 \rceil}{r-1}$ in all cases.

For $r = 3$, we have $t' = 1 + \binom{t'-1}{r-2} \leq b_1$, so

$$1 + \binom{t'}{2} \leq b_1 + \binom{\lceil (n-1-b_1)/2 \rceil}{2} \leq b_1 + \binom{t' - \lfloor b_1/2 \rfloor}{2} \leq n - 3 + \binom{\lceil t'/2 \rceil}{2}.$$

This gives a contradiction when $n \geq 12$. For $n \leq 11$, it is straightforward to check that $1 + \binom{t'}{2} > b_1 + \binom{\lceil (n-1-b_1)/2 \rceil}{2}$ in all cases except $n = 7$, $b_1 \in \{3, 4\}$, which will be considered with the case $r = t' = 3$.

For $r = t'$, by (7) we have

$$1 + t' = 1 + \binom{t'}{r-1} \leq b_1 + \binom{\lceil (s-b_1)/2 \rceil}{r-1} \leq b_1 + 1,$$

since $s \leq n-1$ and $b_1 \geq 2$. Thus $b_1 \geq t'$. For $n \geq 8$, we have

$$|N_{H'}(u_1)| \leq \lceil (s-b_1)/2 \rceil \leq \lceil (n-1-t')/2 \rceil \leq \lceil (n-5)/2 \rceil = t' - 2 < r - 1.$$

This also holds for $n < 8$ if $s \leq n-2$, so we will handle the case $n < 8, s = n-1$ separately at the end of this subsection.

Since each edge in H' containing u_1 contains $r-1$ other vertices, $|N_{H'}(u_1)| < r-1$ gives that $|N_{H'}(u_1)| = 0$ and hence $1 + t' \leq b_1$. Notice also that $n \leq \delta(H)\frac{n}{t'} \leq |E(H)| = |E(Q)| + |E(H')| \leq n - 2 + |E(H')|$, so there are at least 2 edges in $E(H')$.

Case 3.1: There exists $e \in E(H')$ with $e \subseteq V(Q)$. By Claim 4.7, $b_1 \leq s - t' + 1 \leq n - 1 - t' + 1 \leq t' + 1$ with equality only if there exists $i < j$ such that $B_1 = \{e_i, \ldots, e_j\}$ and

$e = \{v_1, \ldots, v_i\} \cup \{v_{j+1}, \ldots, e_s\}$. Without loss of generality, $i \geq 2$. Then we can replace e_1 in Q with e to obtain another x, y-path Q' such that (Q', P) is also a best pair. As B_1 does not change for this new pair, e_1 must play the old role of e, i.e., $e_1 = \{v_1, \ldots, v_i\} \cup \{v_{j+1}, v_{j+3}, \ldots, v_s\}$, but then $e = e_1$, a contradiction.

Case 3.2: Every edge in H' contains exactly one vertex outside of $V(Q)$. Since $E(H') \neq \emptyset$ and u_1 is contained only in edges of Q, there must be at least one additional vertex outside of Q and hence $s \leq n-2$. Let $e \in E(H')$. Because Case 3.1 does not hold, $|e - V(Q)| \leq n-1-s$, so $|e \cap V(Q)| \geq t' - (n-1-s)$ with equality only if $e \cup V(Q) \cup \{u_1\} = V(H)$. By Claim 4.7, $1+t' \leq b_1 \leq s - |e \cap V(Q)| + 1 \leq s - (t' - (n-1-s)) + 1 \leq +n-t'$, so $2t'+1 \leq n$. We get a contradiction unless the "equality" part of Claim 4.7(i) holds. Then as in the previous subcase, there exists $i < j$ such that $e \cap V(Q) = \{v_1, \ldots, v_i\} \cup \{v_j, \ldots, v_s\}$. Moreover, since the choice of $e \in E(H')$ was arbitary, for each $e' \neq e$ in $E(H')$, $e' \cap V(Q) = e \cap V(Q)$. But since $V(H) = e \cup V(Q) \cup \{u_1\}$ and $u_1 \notin e'$, $e' - V(Q) = e - V(Q)$, hence $e = e'$, a contradiction.

Finally we handle the cases $6 \leq n \leq 7$, $r = t' = 3$, $s = n-1$, and $b_1 \in \{3, 4\}$. The average degree of H is

$$\sum_{v \in V(H)} d(v)/n = 3|E(H)|/n \geq \delta(H) \geq 4,$$

so $|E(H)| \geq \lceil 4n/3 \rceil$ which is equal to 8 when $n = 6$ and 10 when $n = 7$. In either case, there exists at least 3 edges in H'.

We will first show that B_1 is a set of b_1 consecutive edges in Q. If u_1 is not contained in any edges in H', then $b_1 \geq \delta(H) \geq 4$. Otherwise if u_1 belongs to an edge h of H', then Claim 4.7(ii) implies $b_1 = 3$ and $n = 7$. In both cases, the "equality" part of Claim 4.7 implies $B_1 = \{e_i, e_{i+1}, \ldots, e_{i+b_1-1}\}$ for some i.

If u_1 is not contained in any edges in H', then for any $e \in E(H'), e = \{v_1, \ldots, v_i\} \cup \{v_{i+b_1}, \ldots, v_s\}$. But this holds for all edges in H', a contradiction. Now suppose $n = 7$, $b_1 = 3$ and u_1 is contained in an edge e of H'. Since $N_{H'}(u_1)$ contains no consecutive vertices and is disjoint from $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}$, we have $i = 2$, and $e = \{v_1, v_6, u_1\}$. In particular, $d_{H'}(u_1) = 1$.

Let E'' be the set of edges in H not containing u_1. Since $|E''| = |E(H)| - d_H(u_1) \geq 10 - 4 = 6$, some edge $g \in E''$ does not contain $\{v_1, v_6\}$. By symmetry, we may assume $v_6 \notin g$. If $g = e_1 = \{v_1, v_2, v_h\}$, then we have a longer v_1, v_6-path $v_1, e, u_1, e_{h-1}, v_{h-1}, e_{h-2}, \ldots, v_2, e_1, v_h, e_h, v_{h+1}, \ldots, v_6$.

Otherwise, $g \in E(H')$. So, by Claim 4.6, $|g \cap \{v_2, v_3, v_4\}| \leq 1$ and $|g \cap \{v_3, v_4, v_5\}| \leq 1$. This is possible only if $g = \{v_1, v_2, v_5\}$. Then we have v_1, v_6-path $v_1, e, u_1, e_4, v_4, e_3, v_3, e_2, v_2, g, v_5, e_5, v_6$, a contradiction. □

6.2 Finishing the proof of Theorem 1.4 for $r \leq n/2$

Proof of Theorem 1.4 for $r \leq n/2$. Consider a best pair (Q, P) with $Q = v_1, e_1, \ldots, e_{s-1}, v_s$ and $P = u_1, f_1, \ldots, f_{\ell-1}, u_\ell$.

By symmetry, we may assume $b_\ell = |B_\ell| \geq |B_1| = b_1$. By Lemma 6.1, $\ell \geq 2$. Recall also that $s \geq n/2 + 1 \geq t' + 1$ and $\ell \leq t' - 1$.

By Claim 4.3 and Lemma 3.4, either

$$s \geq b_1 + b_\ell + \ell - 1, \tag{8}$$

or $B_1 = B_\ell$ and

$$s \geq 2 + \ell(b_1 - 1). \tag{9}$$

Recall that by the maximality of $V(P)$, all edges of H' containing u_1 or u_ℓ are contained in $V(Q) \cup V(P)$. For $j \in \{1, \ell\}$, define $A_j = N_{H'}(u_j) \cap V(Q)$ and $a_j = |A_j|$. By Claim 4.1, A_j contains no consecutive vertices of Q.

Case 1: $A_1 = \emptyset$. Then all edges in H' containing u_1 are contained in $V(P)$.

Case 1.1: $r = t'$. Since $\ell \leq t' - 1$, no edge can be contained entirely in $V(P)$. Thus u_1 must only be contained in edges of Q and P.
Then $b_1 \geq \delta(H) - |E(P)| = 1 + \binom{t'}{r-1} - (\ell - 1) = t' - \ell + 2$. If (8) holds, then

$$n - \ell \geq s \geq 2(t' - \ell + 2) + \ell - 1 = 2t' + 3 - \ell \geq n + 2 - \ell,$$

a contradiction.
If instead (9) holds, then

$$n \geq \ell + s \geq \ell + 2 + \ell((t' - \ell + 2) - 1) = 2 + \ell(t' - \ell + 2). \tag{10}$$

Since $2 \leq \ell \leq t' - 1$, $\ell(t' - \ell + 2) \geq 2(t' - 2 + 2) \geq n - 1$, contradicting (10).

Case 1.2: $3 \leq r \leq t' - 1$. The number of edges in H' containing u_1 and contained in $V(P)$ is at most $\binom{\ell-1}{r-1}$. Thus,

$$b_1 \geq \delta(H) - \binom{\ell-1}{r-1} - |E(P)| = 1 + \binom{t'}{r-1} - \binom{\ell-1}{r-1} - (\ell - 1)$$

$$\geq 1 + \binom{t'}{2} - \binom{\ell-1}{2} - (\ell - 1) = \frac{(t' + \ell - 2)(t' - \ell + 1)}{2} - \ell + 2.$$

If (8) holds, then

$$\frac{(t' + \ell - 2)(t' - \ell + 1)}{2} - \ell + 2 \leq \frac{s - \ell + 2}{2} \leq \frac{n}{2} - \ell + 1.$$

However, $\frac{(t'+\ell-2)(t'-\ell+1)}{2} \leq t' - 1$ implies that $0 \geq t'^2 - \ell^2 - 3t' + 3\ell = (t' - \ell)(t' + \ell - 3)$. This cannot hold because $2 \leq \ell \leq t' - 1$ and $t' \geq 3$, so $\frac{(t'+\ell-2)(t'-\ell+1)}{2} \geq t' > n/2 - 1$, a contradiction.
If instead (9) holds, then

$$\frac{(t' + \ell - 2)(t' - \ell + 1)}{2} - \ell + 2 \leq \frac{s - 2}{\ell} + 1 \leq \frac{n - 2}{\ell}.$$

However, we have $\ell + \frac{n-2}{\ell} \leq \frac{n+\ell^2-2}{2} \leq \frac{n}{2} + 1$, and thus $\frac{n-2}{\ell} \leq \frac{n}{2} - \ell + 1 < \frac{(t'+\ell-2)(t'-\ell+1)}{2} - \ell + 2$.
Case 2: $A_1 \neq \emptyset, B_\ell \neq \emptyset$. Let $B = \bigcup_{e_j \in B_\ell} \{v_j, v_{j+1}\}$. By Claim 4.5, Lemma 3.2 and the facts $s \leq n - \ell \leq 2t' + 1 - \ell$, $B_\ell \neq \emptyset$, $|B| \geq b_\ell + 1$, and $B \cap A_1 = \emptyset$, we have

$$a_1 \leq \lfloor \frac{(2t' + 1 - \ell) - b_\ell - 1 - \ell + 1}{2} \rfloor = t' - \ell + 1 - \lceil b_\ell/2 \rceil. \tag{11}$$

Recall that we assumed $b_1 \le b_\ell$. Therefore

$$d(u_1) \le \binom{a_1 + |V(P) - \{u_1\}|}{r-1} + b_1 + |E(P)| \le \binom{a_1 + \ell - 1}{r-1} + b_\ell + \ell - 1, \tag{12}$$

with equality only if u_1 belongs to every edge of P, and $b_1 = b_\ell$.

Combining (12) and (11), we obtain

$$\binom{t'}{r-1} + 1 \le d(u_1) \le \binom{t' - \lceil b_\ell/2 \rceil}{r-1} + b_\ell + \ell - 1. \tag{13}$$

Case 2.1: $r = t'$. Since $A_1 \ne \emptyset$, we need $a_1 \ge r - |V(P)| = t' - \ell$. By (11), $1 \le b_\ell \le 2$, and $a_1 + \ell - 1 = r - 1$. Then from (13), we get

$$t' + 1 = \binom{t'}{r-1} + 1 \le d(u_1) \le b_\ell + \ell \le 2 + \ell \le 2 + (t' - 1).$$

This gives a contradiction unless $b_1 = b_\ell = 2$ and $\ell = t' - 1$ (so $s = |V(Q)| \le t' + 2 = \ell + 3$). But then there is no way to fit two edges in B_1 and two edges in B_ℓ without violating Claim 4.3.

Case 2.2: $3 \le r \le t' - 1$. If $t' - \lceil b_\ell/2 \rceil \le r - 1$, then as in the previous subcase, $d(u_1) \le b_\ell + \ell$. Since $A_1 \ne \emptyset$ and $\ell \ge 2$, in order not to violate Claim 4.5 we need $b_\ell \le |E(Q)| - 2 = s - 3$. Therefore $d(u_1) \le s - 3 + \ell \le n - 3 \le 2t' - 2$. When $t' \ge 4$ ($n \ge 8$), we have $\binom{t'}{r-1} + 1 > 2t' - 2$, a contradiction. In the remaining cases $6 \le n \le 7$, $r \ge 3$ implies $r = 3 = t'$ which was handled in the previous subcase.

So we may assume $t' - \lceil b_\ell/2 \rceil > r - 1$ and therefore from (13) we get,

$$\binom{t'}{r-1} - \binom{t' - \lceil b_\ell/2 \rceil}{r-1} = \binom{t'-1}{r-2} + \ldots + \binom{t' - \lceil b_\ell/2 \rceil}{r-2} \le b_\ell + \ell - 2 \le b_\ell + t' - 3. \tag{14}$$

Here we use the fact that $\ell \le t' - 1$ and $\binom{t'}{r-1} - \binom{t'-c}{r-1} = \binom{t'-1}{r-2} + \binom{t'-2}{r-2} + \ldots + \binom{t'-c}{r-2}$ for any positive integer $c \le \lceil b_1/2 \rceil$.

Let $f(x) = \binom{t'-1}{r-2} + \ldots + \binom{t'-\lceil x/2 \rceil}{r-2}$ and $g(x) = x + t' - 3$. For $x \in \{1, 2\}$ and $r \ge 3$, $f(x) = \binom{t'-1}{r-2} \ge t' - 1 \ge g(x)$ with equality only if $r = 3$ and $x = 2$. For integers $2 < x \le b_\ell$, $g(x) = g(2) + (x - 2) \le g(2) + 2\lceil (x-2)/2 \rceil$, and

$$f(x) \ge f(2) + \binom{t'-2}{r-2} + \ldots + \binom{t' - \lceil x/2 \rceil}{r-2}.$$

Each of these terms is at least 2, so $f(x) \ge f(2) + 2\lceil x/2 \rceil$. So $f(b_\ell) > g(b_\ell)$ if $b_\ell \ne 3$, contradicting (14). The final case is $b_\ell = 2$. Moreover, we also get a contradiction if equalities in (12)–(14) do not hold. In this case, we must have $b_1 = b_\ell = 2$, and $\ell = t' - 1$ (so $s \le t' + 2$). But then there is no way to fit two edges in B_1 and two edges in B_ℓ without violating Claim 4.3. This finishes Case 2.

Case 3: $B_1, B_\ell = \emptyset$. Let us show that

$$\textit{if } B_1, B_\ell = \emptyset, \textit{ then } a_1, a_\ell \ge t' - \ell + 1 \ge 2. \tag{15}$$

Indeed, if $a_1 \le t' - \ell$, then $d_{H'}(u_1) \le \binom{t'-\ell+(|V(P)|-1)}{r-1} = \binom{t'-1}{r-1}$. So, since $B_1 = \emptyset$ and $\ell \le t'-1$,

$$d_H(u_1) = d_{H'}(u_1) + |E(P)| \le \binom{t'-1}{r-1} + \ell - 1 \le \binom{t'}{r-1},$$

a contradiction. The same argument works if $a_\ell \le t' - \ell$.

Similarly, if $B_1, B_\ell = \emptyset$ and $i \in \{1, \ell\}$, then at least two edges in H' containing u_i are not subsets of $V(P)$. Indeed, otherwise

$$d_H(u_1) = d_{H'}(u_1) + d_{H_P}(u_1) \le 1 + \binom{\ell-1}{r-1} + \ell - 1 < \binom{t'}{r-1}.$$

Let f, f' be distinct edges of H' such that for distinct i, j, we have $\{v_i, u_1\} \subset f, \{v_j, u_\ell\} \subset f'$. By Claim 4.4, $|j-i| \ge \ell+1$, and by Claim 4.1, A_1 contains no consecutive vertices in Q. Without loss of generality, there exists $v_i \in A_1$, $v_j \in A_\ell$ with $j \ge i + \ell + 1$. If $a_1 \ge t' - \ell + 2$, then

$$\begin{aligned} n - \ell \ge s &\ge |A_1| + |\{v_{k+1} : v_k \in A_1, k \ne i, s\}| + |\{v_{i+1}, \ldots, v_{i+\ell}\}| \\ &\ge 2(t' - \ell + 2) - 2 + \ell \ge 2\frac{n-1}{2} - \ell + 2 = n - \ell + 1, \end{aligned}$$

a contradiction. Hence by (15),

$$a_1 = a_\ell = t' - \ell + 1 \ge 2. \tag{16}$$

We also prove that

if $B_1, B_\ell = \emptyset$, then every $v_i \in A_1$ is contained in at least two common edges of H' with u_1, and similar with u_ℓ. (17)

Indeed, otherwise by (16),

$$\begin{aligned} d_H(u_1) &= d_{H'}(u_1) + d_{H_P}(u_1) \\ &\le \binom{|A_1 \cup V(P) - \{u_1\}|}{r-1} - |\{f \subseteq A_1 \cup V(P) : v_i \in f, |f| = r-1\}| + 1 + |E(P)| \\ &\le \binom{t' - \ell + 1 + (\ell - 1)}{r-1} - \binom{t' - \ell + 1 + (\ell-1) - 1}{r-2} + 1 + \ell - 1 \\ &\le \binom{t'}{r-1} - (t'-1) + 1 + \ell - 1 \\ &\le \binom{t'}{r-1} < \delta(H). \end{aligned}$$

This implies that for each $v_i \in A_1$ and $v_j \in A_\ell$, there exist distinct edges $f, f' \in H'$ such that $\{u_1, v_i\} \in f, \{u_\ell, v_j\} \in f'$, and hence $|j - i| \ge \ell + 1$ by Claim 4.4.

Case 3.1: $B_1, B_\ell = \emptyset$ and $A_1 \ne A_\ell$. Without loss of generality, $|A_\ell| \ge |A_1|$. Since A_1, A_ℓ are independent sets and $A_\ell - A_1$ is nonempty, by Lemma 3.1, $s \ge 2a_1 + 2|A_\ell - A_1| + (\ell+1) - 3$. Hence

$$a_1 \le \lfloor \frac{s - 2|A_\ell - A_1| - \ell + 2}{2} \rfloor \le \lfloor \frac{(2t' + 1 - \ell) - 2|A_\ell - A_1| - \ell + 2}{2} \rfloor = t' - \ell - |A_\ell - A_1| + 1. \tag{18}$$

We have

$$d(u_1) \le \binom{a_1+\ell-1}{r-1} + |E(P)| \le \binom{t'-|A_\ell - A_1|}{r-1} + \ell - 1.$$

Since $|A_\ell - A_1| \ge 1$ and $\ell \le t'-1$, this quantity is strictly less than $\binom{t'}{r-1}+1$, a contradiction.

Case 3.2: $B_1, B_\ell = \emptyset$ and $A_1 = A_\ell$. If H' has no edges containing both, u_1 and u_ℓ, then by the case and (16),

$$d_H(u_1) = d_{H'}(u_1) + d_{H_P}(u_1) \le \binom{t'}{r-1} - |\{f \subseteq A_1 \cup V(P) : |f| = r-1, u_\ell \in f\}| + |E(P)|$$

$$= \binom{t'}{r-1} - \binom{t'-1}{r-2} + (\ell-1) \le \binom{t'}{r-1},$$

a contradiction. So suppose there is $f_0 \in E(H')$ containing $\{u_1, u_\ell\}$. Let

$$P_j = u_j, f_{j-1}, \dots, f_1, u_1, f_0, u_\ell, f_{\ell-1}, \dots, u_{j+1}$$

denote the path obtained from P by adding f_0 and deleting f_j. By definition, for each $1 \le j \le \ell-1$, the pair (Q, P_j) is also a best pair. This yields that each f_j is contained in $V(Q) \cup V(P)$. Moreover, for each such j we have Case 3.2. By (17), deleting f_0 from H' does not change A_1. It follows that $A_1 = A_2 = \dots = A_\ell$, and hence each f_j is contained in $A_1 \cup V(P)$. So by (16), $d_H(u_1) \le \binom{(t'-\ell+1)+(\ell-1)}{r-1}$, a contradiction. $\square$

7 Proof of Theorem 1.4 for $r > n/2$

In this section, we complete the proof of Theorem 1.4 by showing that if $r > n/2 \ge 3$ and $\delta(H) \ge r-1$ or $r=3$, $n=5$ and $\delta(H) \ge 3$, then H is hamiltonian-connected.

Proof of Theorem 1.4 for $r > n/2$. Suppose that an r-graph H with $\delta(H) \ge r-1$ has no hamiltonian x,y-path for some $x, y \in V(G)$. Let (Q,P) be a best pair of two vertex-disjoint paths Q and P such that Q is a x,y-path.

It is straightforward to check that the theorem is satisfied when $n=4$, $r=3$, $\delta(H)=2$, so we may assume $\delta(H) \ge 3$.

Since by Lemma 5.2, $s \ge r+1$ and $r \ge \lceil \frac{n+1}{2} \rceil$, we have $\ell \le n-s \le \lfloor \frac{n-3}{2} \rfloor$ and

$$r - \ell \ge \left\lceil \frac{n+1}{2} \right\rceil - \left\lfloor \frac{n-3}{2} \right\rfloor \ge 2. \tag{19}$$

Case 1: $\ell = 1$. As in Section 6.1, in this case every edge $g \in H'$ contains at most one vertex outside of $V(Q)$.

Case 1.1: There are two edges $g, g' \in E(H')$ containing u_1. Then $|(g \cup g') \cap V(Q)| \ge r$, and no two vertices of $g \cup g'$ are consecutive on Q. It follows that $s \ge 2r-1 \ge 2\frac{n+1}{2} - 1 = n$, a contradiction to $s \le n - \ell$.

Case 1.2: There is exactly one edge $g \in E(H')$ containing u_1. Since $\delta(H) \geq 3$, at least two edges of H_Q contain u_1. By Claim 4.2, g does not intersect the sets $\{v_i, v_{i+1}\}$ such that $u_1 \in e_i$. On the other hand, since no two vertices in g are consecutive on Q, the $r-1$ vertices of $g \cap V(Q)$ intersect at least $2(r-1)-2 \geq n-3$ sets $\{v_i, v_{i+1}\}$. This contradicts the fact that Q has $s-1 \leq n-2$ pairs $\{v_i, v_{i+1}\}$.

Case 1.3: All edges containing u_1 are in B_1, and some edge $g \in H'$ is contained in $V(Q)$. Then $d(u_1) = b_1 \geq r-1$. By Claim 4.7, $r-1 \leq b_1 \leq s-r+1$ and therefore $n \leq 2r-1 \leq s+1 \leq n$. This implies that the "equality" part of Claim 4.7(i) holds, and so $g = \{v_1, \ldots, v_i\} \cup \{v_{j+1}, \ldots, v_s\}$ and $B_1 = \{e_i, \ldots, e_j\}$ for some $i < j$. In particular, by symmetry we may assume that $i > 1$. Let Q' be the path obtained by replacing e_1 with g. We get a new best pair (Q', P) with e_1 playing the old role of g. As B_1 does not change, Claim 4.7 asserts $g = e_1$, a contradiction.

Case 1.4: All edges containing u_1 are in B_1, and no edges in H' are contained in $V(Q)$. Again, $|B_1| \geq r-1$. Since $|E(H)| \geq n-1$, there is an edge $g \in E(H')$. Since $\ell = 1$ and Case 1.3 does not hold, $|g \cap V(Q)| = r-1$. Then g has a vertex w outside of $V(Q) \cup \{u_1\}$, so $s \leq n-2$.

If there is another edge $g' \in E(H')$ containing w, then there could not be consecutive vertices v_i, v_{i+1} in Q such that one of them is in g and the other in g'. Hence the sets $A = g \setminus \{w\}$ and $B = g' \setminus \{w\}$ satisfy condition (1) for $q = 2$ in Lemma 3.3. Since $g' \not\subseteq g$ and $g \not\subseteq g'$, Lemma 3.3(ii) for $q = 2$ yields $s \geq |A| + |B| + q - 1 \geq 2r-1 \geq n$. This contradicts the fact that $s \leq n-2$.

Otherwise, w belongs to some $r-2$ edges $e_{i_1}, \ldots, e_{i_{r-2}}$. Let $A = \bigcup_{j=1}^{r-2} \{v_{i_j}, v_{i_j+1}\}$. Then $|A| \geq r-1$. By Claim 4.2, $g \cap A = \emptyset$. Hence $s \geq (r-1) + (r-1) \geq 2\frac{n+1}{2} - 2 = n-1$, contradicting $s \leq n-2$.

Case 2: $2 \leq \ell \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$.

Case 2.1: $a_1 \geq 1$ and $b_\ell \geq 1$. Let $g \in E(H')$ contain u_1. Then $g \subset V(Q) \cup V(P)$ and $|g \cap V(Q)| \geq r - \ell$. Since $\ell \geq 2$, by Lemma 3.2 with $q = \ell$,

$$r - \ell \leq \frac{s-1-\ell+1}{2} \leq \frac{n}{2} - \ell,$$

contradicting $r > \frac{n}{2}$.

If $b_1 \leq 1$, then u_1 is contained in at least one edge in H' and so $a_1 \geq r - \ell \geq 2$ by (19). Either way, $a_1 + b_1 \geq 2$ and similarly $a_\ell + b_\ell \geq 2$. By symmetry, the following two subcases remain.

Case 2.2: $b_1 = b_\ell = 0$. Then $d_{H'}(u_1) \geq \delta(H) - |E(P)| \geq (r-1) - (\ell - 1) \geq 2$ and similarly $d_{H'}(u_\ell) \geq 2$. Let $g_1 \in E(H')$ contain u_1 and $g_\ell \in E(H') - g_1$ contain u_ℓ. Let $A = g_1 \cap V(Q)$ and $B = g_\ell \cap V(Q)$. Then A and B satisfy condition (1) for $q = 1 + \ell$ in Lemma 3.3.

Also, $|A| \geq r - \ell$ with equality only if $g_1 \supset V(P)$, and the same holds for B. If $|A| = |B| = r - \ell$, then $A \neq B$ because $g_1 \neq g_\ell$. In this case, by Lemma 3.3(ii) for $q = 1+\ell$, $s \geq 2(r-\ell) + (1+\ell) - 1 = 2r - \ell$. Since $s \leq n - \ell$, this contradicts $r > n/2$. Similarly, if $\max\{|A|, |B|\} \geq r - \ell + 1$ and $A \neq B$, then by Lemma 3.3(ii) for $q = 1 + \ell$, $s \geq (r-\ell) + (r-\ell+1) + (1+\ell) - 2 = 2r - \ell$. So, we get the same contradiction.

Finally, suppose $A = B$. Since $g_1 \neq g_\ell$, this implies $|A| \geq r - \ell + 1$. Hence by Lemma 3.3(i) for $q = 1 + \ell$, we get

$$n - \ell \geq s \geq 1 + (\ell+1)(r-\ell),$$

which yields

$$\ell(r - \ell) \leq n - r - 1. \tag{20}$$

Since $2 \le \ell \le r-2$, for fixed n and r, the LHS in (20) is at least $2(r-2)$. Thus (20) implies $3r \le n+3$. But $3r \ge (n+1)+r \ge n+4$, a contradiction.

Case 2.3: $a_1 = a_\ell = 0$. Similarly to Case 2.2, $b_1 \ge r-\ell$ and $b_\ell \ge r-\ell$. Let $A = \{v_i v_{i+1} : u_1 \in e_i\}$ and $B = \{v_j v_{j+1} : u_\ell \in e_j\}$. Then A and B satisfy condition (2) for $q = \ell$ in Lemma 3.4.

Also, $|A| \ge r-\ell$ with equality only if $u_1 \in f_j$ for all j, and the same holds for B (with u_ℓ in place of u_1). If $|A| = |B| = r-\ell$ and $A \ne B$, then by Lemma 3.4(ii) for $q=\ell$, $s-1 \ge 2(r-\ell)+\ell-1 = 2r-\ell-1$. Since $s \le n-\ell$, this contradicts $r > n/2$. Similarly, if $\max\{|A|,|B|\} \ge r-\ell+1$ and $A \ne B$, then by Lemma 3.4(ii) for $q=\ell$, $s-1 \ge (r-\ell)+(r-\ell+1)+\ell-2 = 2r-\ell-1$. So, we get the same contradiction.

Finally, suppose $A = B$. Let $B' = \bigcup_{\{j: u_1 \in e_j\}} \{v_j, v_{j+1}\}$. Since $A = B$, $|B'| \ge 2(r-\ell)$. Let $A' = f_1 \cap V(Q)$. If $A' = \emptyset$, then $|V(H)-V(Q)-V(P)| \ge |f_1 - V(P)| \ge r-\ell \ge 2$. Then by Lemma 3.4(i) for $q=\ell$, we get $n-\ell-2 \ge s \ge 2+\ell(r-\ell-1)$, which yields $\ell(r-\ell) \le n-4$. Since $2 \le \ell \le r-2$, the LHS of this inequality is at least $2(r-2) \ge n-3$, a contradiction. Thus $A' \ne \emptyset$. If $v_{i_1} \in A' \cap B'$, say $e_{i_1} \in B$, then we can replace edge e_{i_1} in Q by the path $v_{i_1}, f_1, u_1, e_{i_1}, v_{i_1+1}$, contradicting the choice of (Q,P). Thus $A' \cap B' = \emptyset$. Moreover, similarly if $i_1 < i_2 \le i_1+\ell-2$, $v_{i_1} \in A'$ and $e_{i_2} \in B'$, then we can replace the subpath $v_{i_1}, e_{i_1}, v_{i_1+1}, \ldots, v_{i_2}$ of Q with the longest path $v_{i_1}, f_1, u_2, f_2, u_3, \ldots, u_\ell, e_{i_2}, v_{i_2+1}$, a contradiction again. It follows that $s \ge |A'|+|B'|+\ell-2$. If $|A'| = 1$, then $s \le n-\ell-1$, and therefore $n-\ell \ge |B'|+\ell \ge 2r-\ell > n-\ell$, a contradiction. Otherwise if $|A'| \ge 2$, then $s \ge |B'|+\ell \ge 2r-\ell > n-\ell$ again.

□

8 Concluding remarks

1. A number of theorems on graphs, in particular, Theorem 1.1, give sufficient conditions for the existence of hamiltonian cycles in terms of $\sigma_2(G) = \min_{uv \notin E(G)} d(u)+d(v)$. Partially, this is because many proofs of bounds in terms of the minimum degree also work for $\sigma_2(G)$. It seems this is not the case for r-graphs when $r \ge 3$. Moreover the degree of a vertex in an r-graph can be interpreted in different ways: the number of edges containing the vertex or the number of vertices in its neighborhood. Defining a suitable analog of $\sigma_2(G)$ for hypergraphs is unclear. For example, if $n = 2r$, then there are n-vertex r-graphs with 6 edges in which every two vertices are in a common edge (e.g., a blow up of a K_4), so counting the sizes of the neighborhoods is not a useful parameter at least for large r. On the other hand for small r, the hypergraph consisting of a K^r_{n-1} and one additional edge satisfies $d(u)+d(v) \ge \binom{n-2}{r-1}+1$ for every pair of vertices and is not hamiltonian. While it is likely possible to prove an Ore-type theorem using this bound, this quantity is significantly larger than the sufficient minimum degree condition $\delta(H) \ge \binom{\lfloor (n-1)/2 \rfloor}{r-1}+1$ needed for hamiltonicity, and so such a result may not be very meaningful. It would be interesting to find some analog of $\sigma_2(G)$ for r-graphs that is both natural and nontrivial for a given range of r.

2. Given $k \ge 2$, a (hyper)graph G is *k-path-connected* if for any distinct $x, y \in V(G)$, there is an x,y-path with at least k vertices. In these terms, an n-vertex (hyper)graph is hamiltonian-connected exactly when it is n-path-connected. It would be interesting to find exact restrictions on the minimum degree of an n-vertex r-graph G providing that G is k-path-connected for $r < k < n/2$.

3. Call a graph G 1-*extendable* if for each edge $e \in E(G)$, G has a hamiltonian cycle containing e. Thus Theorem 1.1 yields that for $n \ge 3$ each n-vertex graph G with $\delta(G) \ge (n+1)/2$ is 1-

extendable. Also, one can define 1-extendable hypergraphs in several ways. One natural definition would be: An r-graph G 1-*extendable* if for each edge $e \in E(G)$ and any two vertices $u, w \in e$, G has a hamiltonian cycle $C = v_1, e_1, v_2, \dots, v_n, e_n, v_1$ such that $e_1 = e$, $v_1 = u$ and $v_2 = w$.

For $r = 2$, this definition coincides with the original definition of 1-extendable graphs, but for $r \geq 3$ the claim that each hamiltonian-connected r-graph is 1-extendable is not true: as we have seen in Section 2.2, hamiltonian-connected r-graphs do not need to be even just hamiltonian. On the other hand, trivially if each n-vertex r-graph with minimum degree at least d is hamiltonian-connected, then each n-vertex r-graph with minimum degree at least $d+1$ is 1-extendable. So, Theorem 1.4 yields the following.

Corollary 8.1. *Let $n \geq r \geq 3$. Suppose H is an n-vertex, r-graph such that (1) $r \leq n/2$ and $\delta(H) \geq \binom{\lfloor n/2 \rfloor}{r-1} + 2$, or (2) $r > n/2 \geq 3$ and $\delta(H) \geq r$, or (3) $r = 3$, $n = 5$ and $\delta(H) \geq 4$. Then H is* 1*-extendable.*

When $r > n/2 \geq 3$, the bound in Corollary 8.1 is exact, but when $3 \leq r \leq n/2$ or $r = 3$ and $n = 5$ it probably can be improved by 1.

4. Pósa [11] considered the following generalization of 1-extendable graphs. Given a linear forest (i.e., a set of vertex-disjoint paths) L, call a graph G L-*extendable* if $G \cup L$ has a hamiltonian cycle containing all edges of L. Pósa [11] proved that for each $n > \ell \geq 0$ and every linear forest L with ℓ edges, each n-vertex graph G with $\sigma_2(G) \geq n + \ell$ is L-extendable. This is a far reaching generalization of Theorem 1.1. It also implies that if $\sigma_2(G) \geq n + \ell$, then G is $\ell + 1$-hamiltonian-connected (take L to be a path on at most ℓ vertices).

One may consider different hypergraph definitions of being L-extendable for a given graph linear forest L. For example, given a positive integer ℓ, we can say that a hypergraph G is ℓ-*extendable* if for every choice of $\ell + 1$ vertices $u_1, \dots, u_{\ell+1}$ and ℓ edges $g_1, \dots, g_\ell$ in G such that $\{u_i, u_{i+1}\} \subset g_i$ for all $i \in [\ell]$, G has a hamiltonian cycle $C = v_1, e_1, v_2, \dots, v_n, e_n, v_1$ such that $v_i = u_i$ for all $i \in [\ell + 1]$ and $e_j = g_j$ for all $j \in [\ell]$. Exact bounds on the minimum degree providing that an n-vertex r-graph is ℓ-extendable seem difficult for general ℓ but probably are feasible for very small or very large ℓ.

5. Chartrand, Kapoor and Lick [2] proved analogs of Dirac's Theorem and its generalizations by Ore [9] and Pósa [11] for α-*hamiltonian graphs*, that is, the graphs that are hamiltonian after deleting any set of at most α vertices. If in this definition we replace "at most" with "exactly", the class of the graphs satisfying the definition may change. For example, after deleting any vertex from Petersen Graph, the remaining graph is hamiltonian. Lick [7] proved similar exact results for α-*hamiltonian-connected graphs*, that is, the graphs that are hamiltonian-connected after deleting any set of at most α vertices. The ideas and tricks in [6] and this paper may be used to try to find exact or close to exact bounds on minimum degree in an n-vertex r-graph G ensuring that G is α-hamiltonian/α-hamiltonian-connected. As with graphs, the answers for the definitions with "at most" and "exactly" may differ.

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