Saturation for the 3-Uniform Loose 3-Cycle

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Dedicated to the memory of Landon Rabern

Abstract

Let F and H be k-uniform hypergraphs. We say H is F-saturated if H does not contain a subgraph isomorphic to F, but $H+e$ does for any hyperedge $e \notin E(H)$. The saturation number of F, denoted $\mathrm{sat}_k(n,F)$, is the minimum number of edges in a F-saturated k-uniform hypergraph H on n vertices. Let $C_3^{(3)}$ denote the 3-uniform loose cycle on 3 edges. In this work, we prove that

$$\left(\frac{4}{3}+o(1)\right)n \le \mathrm{sat}_3(n, C_3^{(3)}) \le \frac{3}{2}n + O(1).$$

This is the first non-trivial result on the saturation number for a fixed short hypergraph cycle.

1 Introduction

Let F and H be k-uniform hypergraphs. We say H is *F-free* if H does not contain F as a sub-hypergraph. One of the central problems in extremal combinatorics is to determine the *Turán number* of H, denoted $\mathrm{ex}_k(n,F)$, and defined

$$\mathrm{ex}_k(n,F) = \max\{|E(H)| : H \text{ is a } F\text{-free hypergraph on } n \text{ vertices}\}.$$

We say that H is F-saturated if H is F-free, but $H+e$ contains a copy of F for every hyperedge $e \notin E(H)$. Since each F-free graph H with $E(H) = \mathrm{ex}_k(|V(H)|, F)$ is F-saturated, Turán numbers can be defined in terms of maximizing the number of edges over F-saturated graphs rather than F-free graphs. A consequence of this phrasing of the definition is that it leads to a natural minimization problem related to Turán numbers. Originally introduced by Erdős, Hajnal and Moon [12] using different terminology, *the saturation number*, $\mathrm{sat}_k(n,F)$ is defined by

$$\mathrm{sat}_k(n,F) = \min\{|E(H)| : H \text{ is a } F\text{-saturated hypergraph on } n \text{ vertices}\}.$$

Kászonyi and Tuza [19] proved that $\mathrm{sat}_2(n,F) = O(n)$, and then Pikhurko [24] proved that for general k, $\mathrm{sat}_k(n,F) = O(n^{k-1})$.

In the seminal paper [12], Erdős, Hajnal and Moon determined the saturation numbers for graph cliques exactly. The first result for saturation of k-uniform hypergraphs is due to Bollobás [3], and in this work, the method known as the *set-pair method* [27] was first developed. In addition to complete graphs, different trees have received careful study (e.g. [13, 19]). In terms of hypergraphs, most of the specific saturation numbers determined outside of complete graphs involve forbidden families of hypergraphs, such as triangular families [25], intersecting hypergraphs [8], and very recently Berge hypergraphs (e.g. [1, 2, 10, 11, 17]). For a detailed dynamic survey on all aspects of saturation in graphs and hypergraphs, see [14].

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1.1 Saturation for Cycles

One of the families of graphs that have received the most attention in saturation literature is cycles. While cycles have received considerable attention, very few exact results have been obtained. For specific short graph cycles, the following bounds are known (We assume n is large enough for results below):

- $\mathrm{sat}_2(n, C_3) = \mathrm{sat}_2(K_3, n) = n - 1$, Erdős, Hajnal, and Moon [12],
- $\mathrm{sat}_2(n, C_4) = \lfloor \frac{3n-5}{2} \rfloor$, Ollmann [23],
- $\mathrm{sat}_2(n, C_5) = \lceil \frac{10}{7}(n-1) \rceil$, Chen [5],
- $\lceil \frac{7n}{6} \rceil - 2 \leq \mathrm{sat}_2(n, C_6) \leq \lceil \frac{10}{7}(n-1) \rceil$, Gould, Łuczak, Schmitt [18] and Zhang, Luo, Shigeno [28].

Aside from these small cases, the best-known bounds on $\mathrm{sat}_2(n, C_\ell)$ for fixed ℓ are obtained by Füredi and Kim [16]:

$$\left(1 + \frac{1}{\ell+2}\right) n - 1 < \mathrm{sat}_2(n, C_\ell) < \left(1 + \frac{1}{\ell-4}\right) n + \binom{\ell-2}{2}.$$

Of note is the fact that even for graphs, the asymptotics for saturation numbers of cycles is not known already for cycles of length 6 or more. Also of note, for the 5-cycle, through a technical feat, it was shown that there are exactly 29 distinct minimal constructions, some of which are specific graphs that only work for one value of n, others which constitute infinite families [6]. This highlights a difficulty in studying the saturation function in general and for cycles - one usually does not expect to prove a nice stability result when there are multiple different extremal examples.

Saturation numbers for the family of all cycles of length above a certain value ℓ have also been studied, with exact results determined for $3 \leq \ell \leq 6$ [15, 22]. In addition to these results involving cycles of short length, many results on the saturation numbers of Hamiltonian cycles have been studied, with numerous results leading up to proving that $\mathrm{sat}_2(n, C_n) = \lceil \frac{3n}{2} \rceil$ (upper bound given first in [4], while the lower bound can be found in [21]).

When passing from graphs to hypergraphs, there are many ways to generalize the notion of a cycle. One of the more general notions of a cycle in a hypergraph is an r-overlapping cycle. Namely, the *k-uniform r-overlapping cycle on ℓ edges* is the unique k-uniform hypergraph on $\ell(k-r)$ vertices and ℓ edges such that there exists an ordering of the vertex set, say $v_1, v_2, \ldots, v_{\ell(k-r)}$ such that $e_i = \{v_{(k-r)(i-1)+1}, v_{(k-r)(i-1)+2}, \ldots, v_{(k-r)(i-1)+k}\}$ is an edge for each $1 \leq i \leq \ell$ (indices taken modulo $\ell(k-r)$). When $r = 1$, we will simply call this hypergraph the k-uniform **loose** cycle on ℓ edges, and denote it by $C_\ell^{(k)}$.

Up until this work, the entire literature involving saturation for r-overlapping cycles has been for Hamiltonian cycles (for a k-uniform r-overlapping Hamiltonian cycle to exist in an n-vertex graph, we must have $(k-r) \mid n$). In this setting, due to the difficulty in such problems, the work has been mostly focused on determining the order of magnitude for which these saturation numbers grow (see e.g. [9, 20, 26] for some of the results in this direction).

1.2 Main Result

In this work, we study the saturation function for a short loose cycle, namely $C_3^{(3)}$. Our main result is as follows.

Theorem 1.1. *We have that*

$$\left(\frac{4}{3} + o(1)\right) n \leq \mathrm{sat}_3(n, C_3^{(3)}) \leq \frac{3}{2} n + O(1).$$

To the authors best knowledge, this is the first non-trivial result on saturation numbers for a specific hypergraph cycle of fixed length (the first author and others did provide some bounds on saturation for short

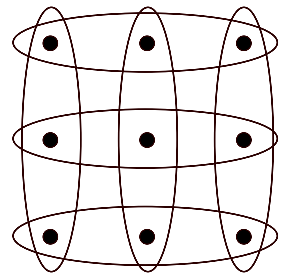

Figure 1: A $C_3^{(3)}$-saturated hypergraph with very few edges exhibiting that $\mathrm{sat}_3(9, C_3^{(3)}) \leq 6$.

Berge hypergraphs cycles in [11], but in general results involving families of hypergraphs tend to be easier than results involving a single hypergraph).

It is worth noting that at least for small values of n, the $o(1)$ in the lower bound is necessary, as for example one might note that $\mathrm{sat}_3(9, C_3^{(3)}) = 6$ (See Figure 1 for the optimal construction). The authors think that for large n, the upper bound is more likely to be asymptotically correct than the lower bound.

The proof of our main result is broken up as follows. In Section 2, we provide a relatively straightforward deterministic construction on $\frac{3}{2}n + O(1)$ edges and show that this construction is saturated. In the remaining sections of the paper we show the more involved lower bound. The proof uses the technique of **asymptotic discharging**. For a primer on discharging, see [7]. Normally in a discharging proof, it would be shown that certain small structures are *reducible*, i.e. they cannot exist in a minimum counterexample to a proposition. Here we allow many configurations to exist, but only in numbers small enough that their existence does not affect the leading term of the final bound. In Section 3, we provide a high-level proof sketch that goes through the main ideas of the proof without the technical details. Then in Sections 4, 5 and 6, we provide the main structural results necessary for the discharging proof, and finally in Section 7, we give our discharging scheme and provide the proof of the lower bound using this discharging scheme.

1.3 Notation and Definitions

We are working with 3-uniform hypergraphs which we will call 3-*graphs* for short. Given a cycle $C_3^{(3)}$, we will call the vertices of degree 2 the *core vertices* of the cycle. We may refer to $C_3^{(3)}$ as a *triangle*.

Vertices of degree i in a 3-graph G will be called i-*vertices*, and i-vertices adjacent to a vertex v will be called i-*neighbors* of v. For $A \subseteq V(G)$, $N(A)$ denotes the set of vertices $u \in V(G) \setminus A$ such that some edge of G contains u and some vertex in A, and $N[A] = A \cup N(A)$. Given a pair of vertices u and v, we write $d(uv)$ to denote the co-degree of the pair, i.e. the number of edges that contain both u and v. We may say that u is a double neighbor or triple neighbor of v if $d(uv) \geq 2$ or $d(uv) \geq 3$ respectively.

Given a 3-graph G and sets $A, B, C \subseteq V(G)$, we will say an edge $e = \{a, b, c\} \in E(G)$ is an (A, B, C) edge if (after possibly renaming) $a \in A$, $b \in B$ and $c \in C$. If one of the sets A, B or C is of the form $\{v \in V(G) \mid d(v) = d\}$ for some $d \in \mathbb{N}$, we will often just write the number d in place of the set. For example, we may say an edge e is an $(A, 4, 2)$ edge if e contains one vertex in the set A, one vertex of degree 4, and one vertex of degree 2. Finally, if one of the sets is of the form $\{v\}$ for some $v \in V(G)$, we will simply write v in place of the set.

For $u, v \in V(G)$, a u, v-*link* is a 2-edge loose path L from u to v. The common vertex of the two edges of L is the *center of* L. If there exists a u, v-link in G, we will say uv is a *good pair*, and if not, we will say uv is a *bad pair*.

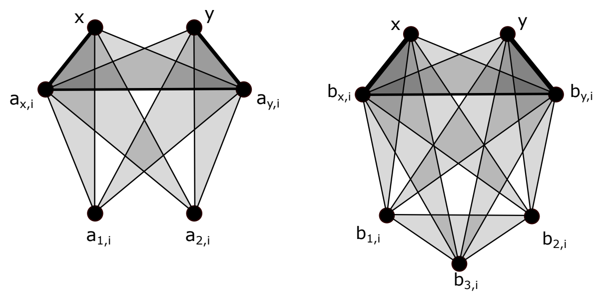

Figure 2: The bricks A_i and B_i from Construction 2.1.

2 Upper Bound - A Construction

Let $n \geq 14$ be an integer. We now present a construction, G_n, that has n vertices and $\frac{3}{2}n + O(1)$ edges.

Construction 2.1. *Let m and c be integers such that $2 \leq c \leq 5$ and $n = 4m + c$. For each i with $1 \leq i \leq m + 2 - c$, let A_i denote the 3-uniform hypergraph on 6 vertices and 6 edges with $V(A_i) = \{x, y, a_{x,i}, a_{y,i}, a_{1,i}, a_{2,i}\}$ and*

$$E(A_i) = \big\{\{x, a_{x,i}, a_{y,i}\}, \{y, a_{x,i}, a_{y,i}\}, \{x, a_{x,i}, a_{1,i}\}, \{x, a_{x,i}, a_{2,i}\}, \{y, a_{y,i}, a_{1,i}\}, \{y, a_{y,i}, a_{2,i}\}\big\}.$$

Furthermore, for each i with $1 \leq i \leq c - 2$, let B_i denote the hypergraph on 7 vertices and 9 edges such that $V(B_i) = \{x, y, b_{x,i}, b_{y,i}, b_{1,i}, b_{2,i}, b_{3,i}\}$ and

$$E(B_i) = \big\{\{x, b_{x,i}, b_{y,i}\}, \{y, b_{x,i}, b_{y,i}\}, \{b_{1,i}, b_{2,i}, b_{3,i}\}\big\} \cup \bigcup_{j=1}^{3} \big\{\{x, b_{x,i}, b_{j,i}\}, \{y, b_{y,i}, b_{j,i}\}\big\}.$$

Now, let G_n be union of the A_i's and B_i's, and note that

$$|V(G_n)| = 2 + 4(m + 2 - c) + 5(c - 2) = 4m + c = n,$$

while

$$|E(G_n)| = 6(m - 2 + c) + 9(c - 2) = \begin{cases} \frac{3}{2}n & \text{if } n \equiv 0 \mod 4, \\ \frac{3}{2}n + \frac{3}{2} & \text{if } n \equiv 1 \mod 4, \\ \frac{3}{2}n - 3 & \text{if } n \equiv 2 \mod 4, \\ \frac{3}{2}n - \frac{3}{2} & \text{if } n \equiv 3 \mod 4. \end{cases}$$

*We call each of the subgraphs A_i or B_i a **brick** of G_n. See Figure 2 for drawings of the two types of bricks in G_n.*

Theorem 2.2. *The hypergraph G_n from Construction 2.1 is $C_3^{(3)}$-saturated for all $n \geq 14$. Consequentially,*

$$\mathrm{sat}_3(n, C_3^{(3)}) \leq \frac{3}{2}n + O(1).$$

Proof. First we will show that G_n is $C_3^{(3)}$-free. Assume to the contrary that G_n contains a copy, T, of $C_3^{(3)}$, and note that all the core vertices of T must be contained in the same brick of G_n since they are adjacent. Furthermore, x and y cannot both be core vertices, so this implies T must be completely contained inside one brick of G_n. However it can be directly verified that neither type of brick contains a copy of $C_3^{(3)}$. Thus, G_n is $C_3^{(3)}$-free.

Now we will show that G_n is $C_3^{(3)}$-saturated. Let e be any triple of vertices such that $e \notin E(G_n)$. First note that every brick of G_n contains an x, y-link, so if both x and y are in e, then this creates a $C_3^{(3)}$, so we may assume at least one of x or y is not in e. Furthermore, under this assumption, it can be directly verified that if all three vertices in e are contained in a single brick of G_n, this creates a $C_3^{(3)}$ within that brick, so we may assume otherwise.

Case 1: e contains exactly one of x or y. Assume without loss of generality that $x \in e$. Let $u, v \in e$ be the other two vertices of e, and note that u and v must be in different bricks of G_n. However, every vertex in every brick is in an edge that contains y but not x. Thus e, along with the edge containing y and u, but not x, and the edge containing y and v, but not x, form a $C_3^{(3)}$.

Case 2: $\{x, y\} \cap e = \emptyset$ and e intersects only two bricks of G_n. Let $e = \{u, v, w\}$ where u and v are in the same brick. It can be readily verified that regardless of which vertices u and v are in this brick, there is an edge in G_n that contains exactly one of u or v and one of x or y, say that it contains u and x, but not v. This edge, along with any edge containing x and w, and e form a $C_3^{(3)}$ in G_n.

Case 3: $\{x, y\} \cap e = \emptyset$ and e intersects three distinct bricks. Let $u, v \in e$. Then e along with any edge containing u and x, and any edge containing v and x, form a $C_3^{(3)}$ in G.

Thus, in all cases, we find a $C_3^{(3)}$ in $G_n + e$, so G_n is saturated. $\square$

3 Lower Bound: Proof Sketch

We will assume for the rest of the paper that n is large, and that G is a $C_3^{(3)}$-saturated 3-uniform hypergraph on n vertices with $\mathrm{sat}_3(n, C_3^{(3)})$ edges. By Theorem 2.2, we can crudely assume

$$|E(G)| \leq 2n.$$

At the simplest level, the goal of our proof will be to show that the average degree of G is at least $4 - o(1)$. To do this, we will use a discharging scheme: every vertex will start with charge equal to its degree, and we will then move charge around according to certain rules until the following two things are satisfied:

- Every vertex has non-negative charge, and
- $n - o(n)$ vertices have charge at least 4.

In particular, charge will always be moved around via edges, i.e. any time our discharging rules move charge from one vertex to another, they will be adjacent.

Let $\ell = \ell(n) \ll \log(n)$ be a function that tends to infinity with n (but slowly). Let

$$L = \{v \in V(G) \mid d(v) < \ell\},$$

and

$$M = V(G) \setminus L.$$

We will call the vertices in L *low vertices* of G, and the vertices in M *non-low vertices*. Since $|E(G)| \leq 2n$, we have

$$|M| \leq \frac{3|E(G)|}{\ell} \leq \frac{6n}{\ell} = o(n).$$

Since there are few vertices in M, they can give all their charge to vertices in L. We also will need to keep track of vertices whose degree is almost linear in n. In particular, let

$$H = \{v \in V(G) \mid d(v) \geq n/\ell^2\}.$$

We will call vertices in H *high vertices*. Note that for a vertex being non-low is a much weaker condition than being high.

Furthermore, for $i \leq \ell$ we will denote by L_i the set of vertices of degree at most i. So, $L_{\ell-1} = L$. We will say a vertex $v \in L$ is *i-flat* if the total number of vertices in M in all edges containing v (counted with multiplicities) is exactly i, and $v \in L$ is i^+-flat if v is j-flat for some $j \geq i$.

One of the first results we will prove (Lemma 4.1) implies that there is a set of two vertices, x and y such that almost all vertices in L that are not adjacent to at least two vertices in H are in $N(\{x, y\})$. Among other things, this implies that almost all vertices in L are adjacent to at least one vertex in H, which will be very helpful.

3.1 Simple discharging rules that do not work

To motivate the rest of the proof, it is useful to mention a simple discharging scheme that does not work, but is the basis for the more complicated discharging rules we use.

The following simple discharging rules do not create or destroy charge, they simply move it (we add an asterisk to the label for these rules because they are not the final discharging rules we use, and are instead simply a heuristic to motivate our final rules):

(D1*) Every (M, L, L) edge $\{h, a, b\}$ with $h \in M$ removes charge 1 from h and gives charge 1/2 to each of a and b.

(D2*) Every (M, M, L) edge $\{h_1, h_2, a\}$ with $a \in L$ removes charge 1 from each of h_1 and h_2, and gives charge 2 to a.

Under this discharging scheme, every vertex in M ends up with non-negative charge while vertices in L only receive extra charge beyond the initial charge they had from their degree, making them more likely to end up above charge 4. However, it is possible that many vertices in L do not get up to charge 4.

Now by this simple discharging scheme, every i-flat vertex in L gets extra charge at least $i/2$ from vertices in M, but without more information, we cannot guarantee a vertex gets any more than this. Aside from 0-flat vertices (which there are few of by Lemma 4.1), this scheme leaves the following types of vertices in L below charge 4:

- 1-flat 3-vertices,
- 1-flat 2-vertices,
- 2-flat 2-vertices, and
- 1-vertices and 0-vertices.

We will show that there are very few 1-vertices and 0-vertices. However, it is possible that G contains $\Omega(n)$ vertices of any of the other three types listed above. To deal with this, we introduce more complicated discharging rules, both refining the rule (D1*) (i.e. still moving charge from vertices in M to vertices in L, but possibly not splitting charge evenly among the low vertices in an (M, L, L) edge), as well as introducing some rules which will move charge from vertices in L that have excess charge to other vertices in L that are not satisfied by their non-low neighbors.

3.2 Overview of how to deal with 1-flat 3-vertices

We wish to enact a discharging rule such as

(D1.1*) Every $(M, L, 3)$ edge $\{h, a, b\}$ with $h \in H$, $a \in L$ and b a 1-flat 3-vertex removes charge 1 from h and gives charge 1 to b.

If this rule were well-defined, then every 1-flat 3-vertex would receive an extra charge 1, bringing them up to charge 4. Thus, the goal of Section 5 will be to show that a rule very similar to this is well-defined, and that the rule does not cause any other vertices to end up with too little charge. In particular, we will establish claims that essentially imply the following:

- There are very few $(M, 3, 2)$ edges,
- There are very few $(M, 3, 3)$ edges containing either two 1-flat 3-vertices or a 1-flat 3-vertex and a 2-flat 3-vertex, and
- There are very few 3-flat 3-vertices that are in two or more $(M, 3, 3)$ edges with 1-flat 3-vertices.

Together these statements will imply that for almost all 1-flat 3-vertices, a rule like (D1.1*) will leave them with charge 4, while not having many 2^+-flat 3-vertices end up with charge less than 4.

3.3 Overview of how to deal with 2-vertices

The simple discharging scheme presented above leaves both 1-flat and 2-flat 2-vertices with charge less than 4. We again wish to refine the rule (D1*) to prioritize giving 2-vertices more charge. For example, we wish to enact a discharging rule similar to

(D1.2*) Every $(M, L, 2)$ edge $\{h, a, b\}$ with $h \in M$, $a \in L$, and $d(b) = 2$ removes charge 1 from h and gives charge 1 to b.

As with rule (D1.1*), it is not obvious that this rule is well-defined, and also it is possible this rule conflicts with (D1.1*). However we will show that there are very few $(M, 2, 2)$ or $(M, 3, 2)$ edges, which implies that for almost all 2-vertices, (D1.2*) is well-defined and does not conflict with (D1.1*).

(D1.2*) is enough for 2-flat 2-vertices to get charge 4, but 1-flat 2-vertices would still be left with only charge 3. To deal with this, we need to move charge from vertices in L to 1-flat 2-vertices. In Section 6, we will define the concept of a low vertex being "helpful", which essentially will imply that these low vertices have enough charge to give $1/2$ to each of their 1-flat 2-neighbors (with multiplicity), while still being satisfied. This will allow us to enact a discharging rule like

(D3*) Every $(V, L, 2)$ edge $\{z, a, b\}$ with $a \in L$ being a "helpful" vertex and b being a 1-flat 2-vertex will remove charge $1/2$ from a and give charge $1/2$ to b.

If we could show that almost every edge containing a 1-flat 2-vertex also contained a helpful vertex, (D3*) would be enough to satisfy almost all 1-flat 2-vertices, since they would get an extra charge $1/2$ from each of their two edges, bringing them up to charge 4. Unfortunately, it is unclear if this is true.

To deal with this, we use a second round of moving charge from vertices in L to 1-flat 2-vertices, this time asking for a little less from the "helping" vertex. In particular, we define "half-helpful" vertices, which essentially are vertices in L that may not have had enough charge to be "helpful", but can give $1/2$ to all the 1-flat 2-neighbors that weren't already satisfied by the charge moved via (D3*). This leads us to a rule like

(D4*) Every $(V(G), L, 2)$ edge $\{z, a, b\}$ with $a \in L$ being "half-helpful", and b being a 1-flat 2-vertex that still does not have charge 4 after the implementation of all previous rules, takes charge $1/2$ from a and gives charge $1/2$ to b.

In order to make sure that (D3*) and (D4*) are well-defined and satisfy almost all 2-vertices, we will need to prove claims that essentially imply the following:

- There are almost no $(L, 2, 2)$ edges, and
- Every $(V, L, 2)$ edge that contains a 1-flat 2-vertex also contains a "helpful" or "half-helpful" vertex.

All of this is done in Section 6.

3.4 One More Discharging Rule

In order to guarantee that we have enough "helpful" and "half-helpful" vertices in G, we will further refine (D1*) to move charge away from vertices that we expect to have a lot of extra charge (say vertices of degree more than 8) to other vertices. We will classify (M, L, L) edges as "rich" and name one of the low vertices in every "rich" edge as the "recipient" if we expect the second vertex of this "rich" edge to be "helpful" even without charge from this edge (for example if a vertex is degree 8 or more). Thus, we will enact a rule like:

(D1.3*) Every "rich" (M, L, L) edge $\{h, u, v\}$ with $h \in H$ and recipient v removes charge 1 from h and gives charge 1 to v.

These heuristic discharging rules capture most of the main ideas of the proof. In Section 7, we go through the actual discharging scheme we use.

The final main idea in our proof that has not been mentioned here is the use of "garbage sets", i.e. small sets of vertices with numerous "bad" properties, which we will show we can largely ignore. In particular, throughout this heuristic section we have used the phrase "almost all" quite loosely. In order to rigorously show that $n - o(n)$ vertices end up with charge at least 4, we will define a sequence of 10 "garbage sets", $R_1, R_2, \ldots, R_{10}$. We will show that the union of these sets is $o(n)$ and that all vertices in L outside of these sets indeed end up with charge at least 4 after the discharging rules take place.

As we define the "garbage sets" R_i, we often want to exclude not just the vertices in R_i, but low vertices that are neighbors of R_i as well. Some of these "garbage sets" could be around size n/ℓ, so we may not be able to include all low neighbors if a "garbage set" contains any vertices of degree close to ℓ. We will however usually include all low neighbors of the vertices of degree at most 8 inside a "garbage set". To that end, given a garbage set R_i, we will usually define a set $\mathbf{R}_i$ that contains all the previously defined garbage sets, and then a set $\mathbf{R}'_i$ which contains $\mathbf{R}_i$, along with any vertices in L that have a neighbor of degree at most 8 in $\mathbf{R}_i$.

4 Lower Bound: Preliminary Lemmas

Lemma 4.1. *Let $Q \subseteq N(H)$ denote the set of vertices which have at least two neighbors in H and let $S = V(G) \setminus (Q \cup H)$. There exist two vertices, $x, y \in V(G)$ with $x \in H$ such that*

$$|S \setminus N(\{x, y\})| \leq 20n/\ell.$$

Proof. We will prove a slightly stronger statement, which implies our result. Namely, we will prove the following: If $|S| \geq 20n/\ell$, then there either exists

- a pair of vertices $x, y \in H$ such that $|S \setminus N(\{x, y\})| \leq 4n/\ell$, or
- a single vertex $x \in H$ such that $|S \setminus N(\{x\})| \leq 8n/\ell$.

Let $\delta := 4/\ell$. We claim that there are no partitions $S = S_1 \cup S_2 \cup S_3$ and $H = H_1 \cup H_2 \cup H_3$ such that $|S_1| \geq |S_2| \geq |S_3| \geq \delta n$, and such that $S_i \cap N(H \setminus H_i) = \emptyset$ for $i \in [3]$.

Indeed, if we consider the $|S_1| \cdot |S_2| \cdot |S_3| := s \geq \delta^3 n^3 = \omega(n^2)$ triples that contain one vertex in each set S_i, $s - 2n = (1 + o(1))s$ of these must be non-edges, and thus must contain a good pair. Let uv be one such good pair. Then uv can cover at most $\max\{|S_1|, |S_2|, |S_3|\} = |S_1|$ of the $(1 + o(1))s$ non-edges, so there must be at least

$$(1 + o(1))|S_2||S_3| \geq (1 + o(1))\delta^2 n^2 \tag{1}$$

good pairs with endpoints in different S_i's. Note that no vertices in H can serve as the center of a u, v-link connecting any of these good pairs since the pairs contain vertices in two different S_i's, and since each other vertex v can be the center of a link including at most $4\binom{d(v)}{2}$ other vertices, we have that there are at most

$$\sum_{v \in V(G) \setminus H} 4\binom{d(v)}{2} \leq \frac{2}{\ell^2} n \sum_{v \in V(G) \setminus H} d(v) \leq \frac{6}{\ell^2} n \cdot |E(G)| \leq \frac{12}{\ell^2} n^2, \tag{2}$$

such good pairs, where the last inequality follows from Theorem 2.2. However, this and the value of δ contradict (1), so no such partition could exist.

Now, if there exist two vertices $x, y \in H$ with $|N(x) \cap S|, |N(y) \cap S| \geq \delta n$, then

$$|S \setminus N(\{x, y\})| < \delta n,$$

since otherwise $H = \{x\} \cup \{y\} \cup (H \setminus \{x, y\})$ would give us a partition that we know does not exist, so in this case we are done.

If there is exactly one vertex $x \in H$ with $|N(x) \cap S| \geq \delta n$, then $|S \setminus N(x)| \leq 3\delta n$, since otherwise we could partition $H \setminus \{x\}$ into two sets A_1, A_2 such that $\delta n \leq |N(A_1) \cap S| \leq 2\delta n$ by starting with all the vertices of $H \setminus \{x\}$ in A_2, and then moving them one at a time into A_1 until the first time the inequality is satisfied. This further implies that $|N(A_2) \cap S| \geq \delta n$, so $H = \{x\} \cup A_1 \cup A_2$ would be a partition that we know does not exist.

Finally, if no vertices in H are adjacent to at least δn vertices in S, then we can find a partition $H = A_1 \cup A_2 \cup A_3$ with $\delta n \leq |N(A_i) \cap S| \leq 2\delta n$ for $i = 1, 2$ and consequently $|N(A_3) \cap S| \geq \delta n$, again by starting with all the vertices in H in A_3, moving them one at a time into A_1 until the first inequality is satisfied, then one at a time into A_2 until the second is satisfied, giving us a partition we know does not exist. $\square$

Let $\mathbf{R}_1 = R_1$ be the set of vertices of degree at most 8 in the set $S \setminus N(\{x, y\})$ as defined in Lemma 4.1. Let $\mathbf{R}_1' = L \cap N[\mathbf{R}_1]$.

Remark 4.2. *Since $\mathbf{R}_1 \subseteq L_8$, Lemma 4.1 yields $|\mathbf{R}_1'| \leq 17|\mathbf{R}_1| \leq 340n/\ell$.*

We now focus on vertices that are in pairs with high codegree relative to their degree.

Claim 4.3. *Let $u, v \in V(G)$. If $d(uv) = d(v) \leq |V(G)| - 3$, then $d(u) \geq d(v) + 2$. Furthermore, if $d(uv) = d(v) = 2$, then v is the only degree 2 double neighbor of u.*

Proof. First, assume that $d(uv) = d(v) = d$ while $d(u) \leq d+1$. If $d(u) = d$, then for any $w \in V(G) \setminus N(u)$ the non-edge $\{u, v, w\}$ intersects every edge containing u or v in two vertices, so $G + \{u, v, w\}$ does not contain a $C_3^{(3)}$, a contradiction. Thus, we can assume $d(u) = d + 1$.

Suppose $\{u, v, a_1\}, \{u, v, a_2\}, \dots, \{u, v, a_d\} \in E(G)$, and let $\{u, b_1, b_2\}$ be the single edge that contains u but not v. If $b_1 \notin \{a_1, \dots, a_d\}$, then consider the non-edge $\{u, v, b_1\}$. This non-edge intersects every edge containing u or v in two vertices, so again $G + \{u, v, b_1\}$ is $C_3^{(3)}$-free, again a contradiction. Thus, we may assume $b_1 \in \{a_1, \dots, a_d\}$, and similarly $b_2 \in \{a_1, \dots, a_d\}$.

Now consider the non-edge $\{v, b_1, b_2\}$ and let T be the $C_3^{(3)}$ in $G + \{v, b_1, b_2\}$. A b_1, b_2-link that avoids v would also avoid u since v is in every edge that u is in except $\{u, b_1, b_2\}$, so $\{b_1, b_2\}$ cannot be the set of core vertices of T, thus v is a core vertex. Without loss of generality, assume b_1 is the second core vertex. Then any v, b_1-link must use some edge $\{u, v, a_i\}$ where $a_i \notin \{b_1, b_2\}$, and a second edge $\{a_i, b_1, z\}$ for some $z \notin \{u, v, b_1, b_2, a_i\}$. But then $\{u, b_1, b_2\}$, $\{u, v, a_i\}$ and $\{a_i, b_1, z\}$ form a $C_3^{(3)}$ in G, a contradiction. Thus, $d(u) \geq d(v) + 2$.

Now assume that $d(v) = 2$ and that there exists a vertex w with $d(uw) = d(w) = 2$ as well. If $\{u, v, w\} \notin E(G)$, then adding it cannot create a $C_3^{(3)}$ since this non-edge intersects all edges containing v and w in two vertices. Thus, we may assume $\{u, v, w\}$ is an edge of G.

Let $\{u, v, v'\}$ and $\{u, w, w'\}$ be the other edges containing v and w, respectively. If $v' = w'$, then the non-edge $\{v, w, v'\}$ again intersects all edges containing either v or w in two vertices, another contradiction. Thus we have $v' \neq w'$.

Now consider the non-edge $\{u, v, w'\}$. As this non-edge intersects every edge containing v in two vertices, there is a u, w'-link that avoids v. To avoid a $C_3^{(3)}$ in G with the edge $\{u, w, w'\}$, w must be contained in this link, but every edge containing w intersects $\{u, v, w'\}$ in two vertices, which is a contradiction. This proves the claim. $\square$

Claim 4.4. *Let $v \in V(G)$. No edge $e \in E(G)$ not containing v can have two 2-vertices in $N(v)$.*

Proof. Suppose to the contrary that $e = \{u_1, u_2, u_3\} \in E(G)$, where $u_1, u_2 \in N(v)$ and $d(u_1) = d(u_2) = 2$. By definition, for $i = 1, 2$, there is an edge $f_i = \{v, u_i, w_i\}$. By Claim 4.3, $w_i \neq u_{3-i}$ since 2-vertices cannot be double neighbors with each other.

Let $g = \{v, u_1, u_2\}$. Since $d(u_1) = 2$ and $u_1 \in e \cap f_1$, $g \notin E(G)$. Thus $G + g$ has a copy of $C_3^{(3)}$, $\mathcal{T}$ that contains the edge g. The edge g intersects all edges containing u_1 and u_2 in two vertices, so neither of these vertices can be core vertices in $\mathcal{T}$, a contradiction. □

Claim 4.5. *If $e = \{a, b, c\}$ is an edge in G, and ab is a good pair, then c is a double neighbor of a or b.*

Proof. If ab is a good pair, then there is an a, b-link. If this link does not contain c, then together with e they form a $C_3^{(3)}$ in G, a contradiction. Thus, one of the edges of the loose path contains c, which implies c is a double neighbor of a or b. □

Let R_2 be the set of vertices $z \in L_8 \setminus \mathbf{R}_1'$ such that there exists some $h \in M$ where zh is a bad pair and $N(z) \setminus \{h\} \subseteq L$. Let $\mathbf{R}_2 = R_2 \cup \mathbf{R}_1'$, and let $\mathbf{R}_2' = \mathbf{R}_2 \cup (L \cap N(L_8 \cap \mathbf{R}_2))$.

Lemma 4.6. $|R_2| \leq 12\ell^2$.

Proof. For a vertex $h \in M$, let $R_2(h)$ be the set of vertices $z \in L_8 \setminus \mathbf{R}_1'$ such that zh is a bad pair and $N(z) \setminus \{h\} \subseteq L$. For $z \in R_2(h)$, the condition that $N(z) \setminus \{h\} \subseteq L$ implies that h is the only vertex in M adjacent to z, and further the condition that $z \notin \mathbf{R}_1$ implies that $h \in \{x, y\}$. Thus $R_2 = R_2(x) \cup R_2(y)$.

Fix some $h \in \{x, y\}$ and let $z \in R_2(h)$. Let $A_1 = N(z) \setminus \{h\}$. Note that $|A_1| \leq 2\ell$ since $z \in L$. Let $A_2 = N(A_1)$. Since $A_1 \subseteq L$, $|A_2| \leq (2\ell + 1)|A_1| \leq 6\ell^2$. Then if $|R_2(h)| \geq 6\ell^2 + 1$, there exists some $z' \in R_2(h) \setminus A_2$. Adding the edge $\{h, z, z'\}$ gives a contradiction, since h cannot be a core vertex because hz and hz' are bad pairs, and z and z' are too far apart in $G - h$ to be core vertices together. Thus the number of choices for z adjacent to h is at most $6\ell^2$.

Since $R_2 = R_2(x) \cup R_2(y)$, this yields $|R_2| \leq 12\ell^2$. □

Let R_3 be the set of 1-flat vertices $u \in L_8 \setminus \mathbf{R}_2'$ such that there exists an (M, L, L) edge $\{h, u, v\}$ with $h \in M$ and $d(hv) = 1$. Let $\mathbf{R}_3 = R_3 \cup \mathbf{R}_2'$, and let $\mathbf{R}_3' = \mathbf{R}_3 \cup (L \cap N(L_8 \cap \mathbf{R}_3))$.

Lemma 4.7. $|R_3| \leq 16\ell^2$.

Proof. The only vertices in M that are adjacent to 1-flat vertices outside of $\mathbf{R}_2'$ are x and y. For $h \in \{x, y\}$, let $\mathcal{L}(h)$ be the family of edges $\{h, u, v\}$ such that $u, v \in L \setminus \mathbf{R}_2'$, u is a 1-flat vertex of degree at most 8, and $d(hv) = 1$. We will show that $|\mathcal{L}(h)| \leq 8\ell^2$, which will imply that $|R_3| \leq 16\ell^2$.

Suppose to the contrary that $|\mathcal{L}(h)| > 8\ell^2$ and let $e = \{h, u, v\} \in \mathcal{L}$. Let A_1 be the set of the vertices in $L \cap N[\{u, v\}]$, and $A_2 = N[A_1] \setminus \{h\}$. Since $u, v \in L$, $|A_1| < 4\ell$ and $|A_2| < 8\ell^2$. Since $d(hu) = d(hv) = 1$, each vertex adjacent to h can be in at most one edge in $\mathcal{L}(h)$ in the role of u. Hence A_2 contains less than $8\ell^2$ vertices in the role of u in $\mathcal{L}(h)$. Thus there is an edge $e' = \{h, u', v'\} \in \mathcal{L}(h)$ such that u' is 1-flat and is not in A_2.

Consider the non-edge $\{u, v, u'\}$. By Claim 4.5, since $d(uh) = d(vh) = 1$, uv is a bad pair, so there is a u', u-link avoiding v or a u', v-link avoiding u. This link cannot contain h since the only edge containing h and one of u or v contains both. So since u' is 1-flat, this link must use a low edge containing u', and some other edge connecting this to one of u or v. But then $u' \in A_2$, a contradiction. Thus, $|R_3| \leq 16\ell^2$. □

Let R_4 be the set of vertices $u \in L_8 \setminus \mathbf{R}_3'$ such that there exists an $h \in M$ with $d(hu) = d(u)$. Let $\mathbf{R}_4 = R_4 \cup \mathbf{R}_3'$, and let $\mathbf{R}_4' = \mathbf{R}_4 \cup (L \cap N(L_8 \cap \mathbf{R}_4))$.

Lemma 4.8. $|R_4| \leq 54\frac{n}{\ell}$.

Proof. Fix a vertex $h \in M$. Note that if u and u' are both vertices such that $d(hu) = d(u)$ and $d(hu') = d(u')$, then $\{h, u, u'\} \in E(G)$ since if not, adding $\{h, u, u'\}$ cannot create a $C_3^{(3)}$ in G as it intersects every edge containing u and u' in at least two vertices. We claim that h has at most 9 neighbors u with $d(u) \leq 8$ such that $d(hu) = d(u)$. Indeed, if h had 10 such neighbors, say $\{u_1, u_2, \ldots, u_{10}\}$, then u_{10} must be in 9 edges, namely $\{h, u_1, u_{10}\}, \{h, u_2, u_{10}\}, \ldots, \{h, u_9, u_{10}\}$, contradicting the fact that $d(u) \le 8$.

Since every vertex $h \in M$ has at most 9 such neighbors, the total number of vertices u such that $d(u) \le 8$ and $d(hu) = d(u)$ for some $h \in M$ is at most

$$9|M| \le 9 \cdot \frac{6n}{\ell} = 54\frac{n}{\ell}.$$

□

It is perhaps worth noting that all 0-vertices and 1-vertices end up in $\mathbf{R}_4'$ since they are either contained in R_1 or in R_4.

Lemma 4.9. *For each $h \in M$ and each neighbor $u \in L \setminus \mathbf{R}_4'$ of h with $d(u) \le 8$ and $d(hu) \ge 2$, one of the following holds:*

(a) there is an edge $\{h, u, w\}$ such that $d(w) \ge 4$ or w is not 1-flat and $d(w) \ge 3$, or

(b) there are two edges $\{h, u, w_1\}$ and $\{h, u, w_2\}$ such that neither of w_1 and w_2 is a 1-flat 2-vertex.

Proof. Suppose that for some $u \in L_8 \setminus \mathbf{R}_4'$ with $d(hu) \ge 2$ neither of (a) and (b) holds. This means that there are vertices w_1, w_2 such that for $i \in [2]$, $e_i = \{h, u, w_i\} \in E(G)$ and at least one of them, say w_1, is a 1-flat 2-vertex. Since $u \notin \mathbf{R}_2'$, $w_1 \notin \mathbf{R}_2$, and hence w_1h is a good pair. Let edges f_1 and f_2 form a w_1, h-link with $w_1 \in f_1$. In order to avoid a triangle with edges e_1, f_1 and f_2 in G, f_1 or f_2 contains u.

Case 1: $u \notin f_2$. Then $u \in f_1$, so that f_1 has the form $\{w_1, u, z\}$ and f_2 has the form $\{h, z, z'\}$ for some z, z'. If $w_2 \notin \{z, z'\}$, then G contains triangle with edges f_1, f_2 and $\{h, z, z'\}$, a contradiction. Otherwise, the co-degree of w_2h is at least 2. Since $u \notin \mathbf{R}_4'$, $w_2 \notin \mathbf{R}_4$, and hence w_2 cannot be a double neighbor of h of degree 2. Thus Part (a) of the claim of the lemma holds.

Case 2: $u \notin f_1$. Then $u \in f_2$, so that f_2 has the form $\{h, u, z\}$ and f_1 has the form $\{w_1, z, z'\}$ for some z, z'. By symmetry, we may assume $z = w_2$. If $d(w_2) \ge 4$ or w_2 is a not 1-flat 3-vertex, then Part (a) of the lemma holds. If $d(w_2) = 2$, then the non-edge $f = \{h, w_1, w_2\}$ intersects each of the edges containing w_1 or w_2 in two vertices. Thus neither of w_1 or w_2 could be a core vertex in a triangle in $G + f$. This yields $d(w_2) \ge 3$. So, if (a) does not hold, then

$$w_2 \textit{ is a 1-flat 3-vertex.} \tag{3}$$

Let $e_3 = \{w_1, u, w_2\}$. Since we know both edges containing w_1, $e_3 \notin E(G)$. So $G' = G + e_3$ contains a triangle, say with edges g_1, g_2 and e_3. Since each of the edges containing w_1 has two common vertices with e_3, g_1 and g_2 form a u, w_2-link. We may assume that $u \in g_1$.

Since G has no triangle with edges g_1, g_2 and e_2, $h \in g_1 \cup g_2$. Since by (3) the co-degree of hw_2 is 1, g_1 has the form $\{h, u, w_3\}$ and g_2 has the form $\{w_2, w_3, t\}$ for some t. If w_3 is not a 1-flat 2-vertex, then Part (b) of the the lemma holds. So suppose w_3 is a 1-flat 2-vertex.Then we know all edges containing at least one of w_1, w_2 and w_3. In particular, edge $e_4 = \{w_1, w_2, w_3\}$ is not in G, and hence $G_1 = G + e_4$ must contain a triangle. However, the edges containing at least one of w_1, w_2 and w_3 and sharing less than two vertices with e_4 are only e_1, e_2 and g_1. Any two of these edges share u and v, a contradiction.

Case 3: $u \in f_1 \cap f_2$. We may assume that $f_1 = \{w_1, u, z\}$ and $f_2 = e_2$. In particular, $z \ne w_2$. Consider e_3 and G' as in Case 2. Again, there are edges g_1 and g_2 forming a u, w_2-link. Since $u \notin \mathbf{R}_4'$, $w_2 \notin \mathbf{R}_4$; thus w_2 is 1-flat. Then $v \in g_1 - g_2$, so we may assume $g_1 = \{u, h, w_3\}$ and $g_2 = \{w_2, w_3, t\}$ for some $t \notin \{h, u\}$. So, we have Case 2 with w_3 in the role of w_1. □

4.1 Edges of type $(M, L, 2)$

We have two goals in this section. First, we want to prove a structural result which tells us essentially that in almost all $(M, L, 2)$ edges, either such an edge contains a good pair involving the vertex in M, or the two low vertices in the edge are connected via a tight path of length 2. Second, we will use this structural result to show that there are only few $(M,3,2)$ or $(M,2,2)$ edges in G.

Let R_5 be the set of vertices $u \in L_8 \setminus \mathbf{R}'_4$ such that there exists an edge $\{h,u,t\}$ with the following properties

- $h \in M$ and $d(t) = 2$,
- hu and ht are bad pairs, and
- there are no edges $\{u,a,b\}$ and $\{t,a,b\}$ for some $a, b \in V(G) \setminus \{h,u,t\}$.

Let $\mathbf{R}_5 = R_5 \cup \mathbf{R}'_4$, and let $\mathbf{R}'_5 = \mathbf{R}_5 \cup (L \cap N(L_8 \cap \mathbf{R}_5))$.

Lemma 4.10. $|R_5| \leq 2000\frac{n}{\ell}$.

Proof. Fix $h \in M$. We claim that h is in at most 314 edges $\{h,u,t\}$ that satisfy the conditions in the definition of R_5. To see this, assume to the contrary that h is in at least 315 such edges. Note that each such edge containing h can intersect at most 8 other edges in two vertices, possibly 7 other such edges containing hu and one containing ht. Thus, we can find a collection of $35 = 315/9$ such edges $\{h,u,t\}$ that only intersect in the vertex h.

Call these edges $e_1, e_2, \ldots, e_{35}$, and let $e_i = \{h, u_i, t_i\}$, where $d(t_i) = 2$. Furthermore, let $e'_i = \{t_i, a_i, b_i\}$ be the second edge containing t_i. Consider the non-edge $\{h, t_i, t_j\}$ for some $1 \leq i, j \leq 35$, $i \neq j$. Since ht_i and ht_j are bad pairs, there is a t_i, t_j-link that avoids h. Thus, this link consists of the edges e'_i and e'_j.

Now, fix t_1, and let U_a and U_b denote the sets of vertices t_j with $2 \leq j \leq 35$ such that $e'_1 \cap e'_j = \{a_1\}$ and $e'_1 \cap e'_j = \{b_1\}$ respectively. One of $|U_a|$ or $|U_b|$ must have size at least 18, so assume without loss of generality $|U_a| \geq 18$, and by reordering if necessary, we can assume that $a_i = a_j$ for $1 \leq i, j \leq 18$.

Now consider the non-edge $\{h, u_1, t_j\}$ for each $2 \leq j \leq 18$. By assumption, hu_1 and ht_j are bad pairs, so there is a u_1, t_j-link that avoids h. This link must use the edge $\{t_j, a_j, b_j\}$. Note that the second edge of this link cannot be of the form $\{u_1, a_j, z\}$ since if $z = b_1$, then recalling that $a_j = a_1$, this gives us that $\{u_1, a_1, b_1\}$ and $\{t_1, a_1, b_1\}$ are edges, contradicting the initial assumption of the lemma, and if $z \neq b_1$, then the edges $\{h, u_1, t_1\}$, $\{u_1, a_j, z\}$ and $\{t_1, a_1, b_1\}$ form a $C_3^{(3)}$ in G. Thus, there must be an edge $\{u_1, b_j, z\}$ where $z \neq a_j$. So, u_1 is adjacent to b_j for all $1 \leq j \leq 18$.

Now, since $e'_i \cap e'_j = \{a_1\}$ for all $2 \leq i, j \leq 18$ with $i \neq j$, we have that $b_i \neq b_j$ for all $2 \leq i, j \leq 18$, $i \neq j$. Thus, $|N(u_1)| \geq |\{b_2, b_3, \ldots, b_{18}\}| = 17$, which contradicts the fact that $d(u_1) \leq 8$ and consequently, $|N(u_1)| \leq 16$.

This establishes that indeed, every vertex $h \in M$ is in at most 314 such edges. Thus,

$$|R_5| \leq 314 \cdot |M| \leq 314\frac{6n}{\ell} < 2000\frac{n}{\ell}.$$

$\square$

We now use the above lemma to prove that there are very few $(M,2,2)$ and $(M,3,2)$ edges.

Lemma 4.11. *Let $h \in M$ and let $e_1 = \{h, v_1, v_2\}$ be a $(h,2,2)$ or $(h,2,3)$ edge, where $v_1, v_2 \notin \mathbf{R}'_5$. Then hv_1 and hv_2 are bad pairs.*

Proof. By Claim 4.3, v_1 and v_2 cannot be double neighbors since one of them is degree 2, and the other is degree at most 3.

Assume that hv_1 is a good pair. To avoid having a $C_3^{(3)}$ in G, any h, v_1-link must contain v_2. Since v_1 is not a double neighbor of v_2, this implies that we have edges $\{h, v_2, w\}$ and $\{v_1, w, z\}$ for some vertices $w, z \in V(G) \setminus \{h, v_1, v_2\}$. Since v_2 is a double neighbor of h and is not in $\mathbf{R}_4$, we have $d(v_2) = 3$, and consequently $d(v_1) = 2$.

Consider the non-edge $\{v_1, v_2, w\}$. If $d(v_2w) = 2$, then this non-edge intersects every edge containing v_1 and v_2 in two vertices, so $G + \{v_1, v_2, w\}$ cannot contain a $C_3^{(3)}$. Thus, $d(v_2w) = 1$. Furthermore, since

$\{v_1, v_2, w\}$ intersects every edge containing v_1 in two vertices, there exists a v_2, w-link that avoids v_1. To avoid a $C_3^{(3)}$ in G, h must be in this link. Since v_2 is not in R_4, $d(hv_2) < d(v_2) = 3$, so this v_2, w-link must contain edges of the form $\{h, w, a\}$ and $\{v_2, a, b\}$. But this creates a $C_3^{(3)}$ in G with $\{h, v_1, v_2\}$, a contradiction. □

Lemma 4.12. *Let $h \in M$ and let $\{h, v_1, v_2\}$ be a $(h, 2, 2)$ or a $(h, 3, 2)$ edge of G, where $v_1, v_2 \notin \mathbf{R}'_5$. Then for any $a, b \in V(G) \setminus \{h, v_1, v_2\}$, at most one of $\{v_1, a, b\}$ or $\{v_2, a, b\}$ is in $E(G)$.*

Proof. Suppose G contains all the edges $\{h, v_1, v_2\}$, $\{v_1, a, b\}$ and $\{v_2, a, b\}$ for some $a, b \in V(G) \setminus \{h, v_1, v_2\}$. If $d(v_1) = d(v_2) = 2$, then the edge $\{v_1, v_2, a\}$ is not an edge of G, but it intersects every edge containing either v_1 or v_2 in two vertices. So adding this edge to G does not create a $C_3^{(3)}$. Thus, we may assume that $\{h, v_1, v_2\}$ is a $(h, 3, 2)$ edge, say with $d(v_1) = 2$ and $d(v_2) = 3$. If $d(v_2a) = 2$, then the non-edge $e' = \{v_1, v_2, a\}$ intersects every edge containing v_1 and v_2 in two vertices, so $G + e'$ cannot contain a $C_3^{(3)}$, a contradiction. Thus $d(av_2) = 1$. Similarly, $d(bv_2) = 1$.

Suppose now that $d(hv_2) = 2$, say $\{h, v_2, z\} \in E(G)$, and note that $z \notin \{v_1, h, a, b\}$. Consider the non-edge $\{h, v_1, z\}$, and let T be a $C_3^{(3)}$ in $G + \{h, v_1, z\}$. If h and z are core vertices in T, then any h, z-link must contain v_1. But every edge that contains v_1 and one of h or z intersects $\{h, v_1, z\}$ in two vertices, so h and z cannot both be core vertices. Thus v_1 is a core vertex of T. So, $\{v_1, a, b\}$ is one of the edges of T. By Lemma 4.11, hv_1 is a bad pair, so z must be the second core vertex of T. Thus, there exists some edge containing z and exactly one of a or b, say without loss of generality $\{z, a, w\} \in E(G)$. Note that $w \notin \{h, v_1, v_2, b\}$, and thus $\{h, v_1, v_2\}$, $\{v_1, a, b\}$ and $\{h, a, w\}$ form a $C_3^{(3)}$ in G, a contradiction. Thus, $d(hv_2) = 1$.

Let $u_1, u_2 \in V(G)$ be such that $\{v_2, u_1, u_2\} \in E(G)$, and note that $u_1, u_2 \notin \{h, v_1, a, b\}$. Consider the non-edge $\{v_1, u_1, u_2\}$. By Claim 4.5, u_1u_2 is a bad pair since $d(u_1v_2) = d(u_2v_2) = 1$, and thus there is a v_1, u_1-link avoiding u_2 or a v_1, u_2-link avoiding u_1. Assume the former. If this v_1, u_1-link uses the edge $\{h, v_1, v_2\}$, then it must also use an edge $\{h, u_1, w\}$ for some $w \notin \{v_2, u_2\}$ since the only edge containing $\{v_2, u_1\}$ is $\{v_2, u_1, u_2\}$. But then $\{h, v_1, v_2\}$, $\{h, u_1, w\}$ and $\{v_2, u_1, u_2\}$ form a $C_3^{(3)}$ in G, a contradiction. If instead, the link uses the edge $\{v_1, a, b\}$, then it must also use an edge containing exactly one of a or b and u_1, but not u_2, say $\{a, u_1, w\}$ for some $w \notin \{b, v_2, u_2\}$. Again we reach a contradiction since $\{v_2, a, b\}$, $\{v_2, u_1, u_2\}$ and $\{a, u_1, w\}$ form a $C_3^{(3)}$ in G. Thus, in all cases we arrive at a contradiction. □

Lemma 4.12 together with the definition of R_5 yield the following.

Corollary 4.13. *For every $(h, 2, 2)$ or $(h, 3, 2)$ edge $\{h, v_1, v_2\}$ where $h \in M$, vertices v_1 and v_2 are in $\mathbf{R}'_5$.*

5 Lower Bound: 1-flat 3-vertices

Let R_6 be the set of 2-flat 3-vertices $u \in L \setminus \mathbf{R}'_5$ such that there exists an edge $\{h, u, v\}$ where $h \in M$ and v is a 1-flat 3-vertex. Let $\mathbf{R}_6 = R_6 \cup \mathbf{R}'_5$, and let $\mathbf{R}'_6 = \mathbf{R}_6 \cup (L \cap N(L_8 \cap \mathbf{R}_6))$.

Lemma 5.1. $|R_6| \leq 32\ell^2$.

Proof. For $h \in M$, let $\mathcal{F}(h)$ denote the set of 2-flat 3-vertices $u \in R_6$ contained in some edge $\{h, u, v\}$ where v is a 1-flat 3-vertex. Since v is a 1-flat 3-vertex and $u \notin \mathbf{R}'_1$, $v \notin \mathbf{R}_1$. So, $h \in \{x, y\}$. Assume that $|\mathcal{F}(h)| \geq 16\ell^2 + 1$.

We first describe a certain structure that all but 8ℓ vertices in $\mathcal{F}(h)$ are contained in. Let $e = \{h, u, v\}$ be an edge of G with $u \in \mathcal{F}(h)$ and v a 1-flat 3-vertex. Since $u \notin \mathbf{R}'_3$, $v \notin \mathbf{R}_3$. Hence $d(\{u, h\}) \geq 2$, so by the case, $d(\{u, h\}) = 2$. Let a be a vertex such that $\{h, u, a\} \in E(G)$. Since u is 2-flat, note that a is low.

Since $|N(u) \setminus \{h\}| \leq 4$ and every vertex in $N(u) \setminus \{h\}$ is low (and thus has at most 2ℓ neighbors), $|N(N(u) \setminus \{h\})| \leq 8\ell$. Thus, if there are at least $8\ell + 1$ vertices in $\mathcal{F}(h)$ that are in a bad pair with h, then we can find some $u' \in \mathcal{F}(h) \setminus N(N(u) \setminus \{h\})$ such that hu' is also a bad pair. Consider the non-edge $\{h, u, u'\}$. Since hu and hu' are both bad pairs, there must be a u, u'-link that avoids h. But since $u' \notin N(N(u) \setminus \{h\})$, there is no such link, a contradiction. Thus, there are at most 8ℓ vertices in $\mathcal{F}(h)$ that are in a bad pair with h.

Let $\mathcal{F}_{good}(h)$ denote the set of vertices in $\mathcal{F}(h)$ that are in a good pair with h. Then,

$$|\mathcal{F}_{good}(h)| \geq |\mathcal{F}(h)| - 8\ell \geq 8\ell^2 + 1.$$

Assume now that $u \in \mathcal{F}_{good}(h)$, with $\{h, u, v\}$ and $\{h, u, a\}$ edges of G with v a 1-flat 3-vertex. Since hu is a good pair, there exist edges $\{u, w_1, w_2\}$ and $\{w_2, w_3, h\}$ forming a u, h-link. Note that to avoid a $C_3^{(3)}$ in G, we must have $v, a \in \{w_1, w_2, w_3\}$. Since $d(\{h, v\}) = 1$, we have $v = w_1$.

If $a = w_2$, then consider the non-edge $\{h, v, a\}$. Note that any link between two of the vertices in $\{h, v, a\}$ that does not contain the third cannot contain u since every edge containing u contains two vertices in $\{h, v, a\}$. Thus, if there were such a link, say a z_1, z_2-link with $z_1, z_2 \in \{h, v, a\}$, then there would be a $C_3^{(3)}$ in G consisting of this link and the edge $\{u, z_1, z_2\}$, a contradiction. Thus, $a \neq w_2$, so $a = w_3$. For ease of notation, let us define $b := w_2$.

Let $\{v, s, s'\}$ denote the third edge of G containing v. We claim that $s, s' \notin \{u, a, b\}$. Indeed, $u \notin \{s, s'\}$ by Claim 4.3. If $a \in \{s, s'\}$, then the non-edge $\{u, v, a\}$ intersects every edge containing u and v in two vertices, and thus adding it to G does not create a $C_3^{(3)}$. If $b \in \{s, s'\}$, then since $u, a \notin \{s, s'\}$, the edges $\{h, u, v\}$, $\{v, s, s'\}$ and $\{h, a, b\}$ form a $C_3^{(3)}$ in G, a contradiction. Thus, $s, s' \notin \{u, a, b\}$ as claimed. Note that this also implies that uv is a bad pair since any u, v-link would necessarily use the edges $\{h, u, a\}$ and $\{v, s, s'\}$, but $\{h, a\} \cap \{s, s'\} = \emptyset$.

At this point, we define the *configuration* of $u \in \mathcal{F}_{good}(h)$ to be the collection of vertices $\{u, v, a, b, s, s'\}$. We wish to speak of the configurations associated with multiple vertices from $\mathcal{F}_{good}(h)$ simultaneously, so given $u \in \mathcal{F}_{good}(h)$, we will let v_u, a_u, b_u, s_u and s'_u denote the vertices in the configuration of u playing the roles of v, a, b, s and s' as described above. Note that while there is a unique choice of v_u, a_u and b_u, the vertices s_u and s'_u are interchangeable, so we may arbitrarily choose which vertex in $N(v_u) \setminus \{h, u, b\}$ is s_u and which is s'_u. We will label the configuration of u as $\mathcal{C}_u$. We have proven the following about $\mathcal{C}_u$ for any $u \in \mathcal{F}_{good}(h)$:

- The edges $\{h, u, v_u\}, \{h, u, a_u\}, \{h, a_u, b_u\}, \{u, v_u, b_u\}$ and $\{v_u, s_u, s'_u\}$ are all present in G,
- The vertices h, u, v_u, a_u, b_u, s_u and s'_u are all distinct, and
- The pair uv_u is a bad pair.

Now, given a vertex $w \in L$ and a vertex $u \in \mathcal{F}_{good}(h)$, note that w is adjacent in $G - h$ to a vertex in $\mathcal{C}_u$ if and only if either u or v_u are in $N[N(w) \cap L]$. Since a vertex can play the role of u or v_u in at most one configuration, this implies that w is adjacent to at most $|N[N(w) \cap L]| \leq 4\ell^2$ configurations.

We claim that for every $u \in \mathcal{F}_{good}(h)$, $\{h, s_u, s'_u\} \in E(G)$, and no other edges containing h and one of s_u or s'_u are in $E(G)$. Indeed, first consider the case where $s_u \notin N(h)$. Then s_u is adjacent to at most $4\ell^2$ configurations, so there exists a u' such that s_u is not adjacent to $\mathcal{C}_{u'}$. Consider the non-edge $\{s_u, u', v_{u'}\}$. Since $u'v_{u'}$ is a bad pair, there must be a loose path of length 2 from a vertex in $\{u', v_{u'}\}$ to s_u, but since s_u is not adjacent to any vertex in $\mathcal{C}_{u'} \cup \{h\} = N(\{u', v_{u'}\})$, this is a contradiction. Thus, $s_u \in N(h)$. Now consider the possibility that we have an edge $\{s_u, h, z\}$ for some $z \neq s'_u$. Then clearly $z \notin \{u, v_u\}$, and so $\{h, s_u, z\}$, $\{v_u, s_u, s'_u\}$ and $\{h, u, v_u\}$ form a $C_3^{(3)}$ in G, a contradiction. Thus, the only edge containing h and s_u is $\{h, s_u, s'_u\}$. By the symmetry between s_u and s'_u, this also is the only edge containing h and s'_u as claimed.

Now, given any $u \in \mathcal{F}_{good}(h)$, there are at most $8\ell^2 < |\mathcal{F}_{good}(h)|$ configurations adjacent to either s_u or s'_u in $G - h$, so there exists some $u' \in \mathcal{F}_{good}(h)$ such that neither s_u or s'_u are adjacent to $\mathcal{C}_{u'}$. Consider the non-edge $\{s_u, s'_u, u'\}$, and a triangle T in $G + \{s_u, s'_u, u'\}$. By the choice of u', u' cannot be a core vertex in T, and thus $s_u s'_u$ is a good pair, but no edge of G contains h and exactly one of these vertices, so the s_u, s'_u-link, along with the edge $\{s_u, s'_u, h\}$ gives a $C_3^{(3)}$ in G, a contradiction.

Thus for each $h \in \{x, y\}$, $|\mathcal{F}(h)| \leq 16\ell^2$. Therefore, $|R_6| \leq 32\ell^2$. $\square$

We will call a vertex $v \in L$ *supported* if G contains a (M, M, v) edge, and *unsupported* otherwise. Let R_7 be the set of unsupported 3-flat 3-vertices $u \in L \setminus \mathbf{R}'_6$ adjacent to at least two 1-flat 3-vertices. Let $\mathbf{R}_7 = R_7 \cup \mathbf{R}'_6$, and let $\mathbf{R}'_7 = \mathbf{R}_7 \cup (L \cap N(L_8 \cap \mathbf{R}_7))$.

Lemma 5.2. $|R_7| \leq 2n/\ell$.

Proof. Suppose $|R_7| > 2n/\ell$. Let $u \in R_7$ and v_1 and v_2 be 1-flat 3-neighbors of u. Since u is unsupported, and is contained in edges e_1, e_2 and e_3, where $e_i = \{u, v_i, h_i\}$, $h_1, h_2, h_3 \in M$ and v_1 and v_2 are 1-flat 3-vertices (possibly, some h_is and/or v_is coincide). In this case the set $U(u) = \{u, v_1, v_2, v_3\}$ will be called the *u-set*.

Since $v_1, v_2 \notin \mathbf{R}_2$, we have that $h_1, h_2 \in \{x, y\}$ and the pairs h_1v_1 and h_2v_2 are good. Since h_1v_1 is a good pair, the codegree of either v_1u or h_1u is at least 2. On the other hand, since v_1 is 1-flat and u is 3-flat and unsupported, the codegree of v_1u is 1. Thus the codegree of h_1u is at least 2. Similarly, the codegree of h_2u is at least 2. It follows that $h_1 = h_2$. If also $h_3 = h_1$, then $d(u) = d(h_1u)$, contradicting the fact that $u \notin \mathbf{R}_4$.

Let X be the set of $u \in R_7$ such that $h_1 = x$. Since $|R_7| > 2n/\ell$, by the symmetry between x and y we may assume that $|X| > n/\ell$.

Let $u \in X$. Since v_1 and v_2 are 1-flat, $v_3 \notin \{v_1, v_2\}$. This implies that ux is a bad pair. Since for every two distinct $u, u' \in X$ we can try to add the edge $\{x, u, u'\}$ and the pairs ux and $u'x$ are bad, so the pair uu' is good in $G - x$. Since v_1, v_2, v_3 are low, the number of $u' \in X$ such that h_3 is adjacent to u' is at least

$$|X| - 3(2\ell) > n/\ell - 6\ell.$$

Denote the set of such u' (including our u) by X_1.

If there is an edge $f = \{x, h_3, w\}$ containing both x and h_3, then there is a $u \in X_1$ such that w is not in the u-set, and so G has a triangle formed by e_1, e_3 and f, a contradiction. Thus $h_3 \notin N(x)$.

Now, if for some $u \in X_1$ there is an edge $g = \{x, v_3, w\}$ containing h_1 and v_3, then again there is a $u' \in X_1$ such that w is not in the u'-set, and so G has a triangle formed by e'_1, e'_3 and g, a contradiction. Thus $v_3 \notin N(x)$ for each $u \in X_1$.

Consider again a $u \in X_1$. Since $\{x, v_1\}$ is a good pair, there are edges g_1 and g_2 forming a v_1, x-link. In view of e_1, $u \in g_1 \cup g_2$. Let $v_1 \in g_1$. Since the codegree of v_1u is 1, $u \in g_2 - g_1$. It follows that $g_2 = e_2$ and $v_2 \in g_1$. So, we may assume $g_1 = \{v_1, v_2, w\}$.

We claim that

$$w \notin N(x). \tag{4}$$

Indeed, suppose G has edge $e_4 = \{x, w, w_1\}$. Since v_1 and v_2 are 1-flat, $w_1 \notin \{v_1, v_2\}$. Since codegree of ux is two, $w_1 \neq u$. But then G has a triangle formed by e_4, e_1 and g_1, a contradiction. This proves (4).

Let G' be obtained from G by adding edge $e_5 = \{x, v_1, v_3\}$. Then G' must have a triangle T formed by e_5 and some edges f_1 and f_2. If the core vertices of T are x and v_1, then in view of e_1, $u \in f_1 \cup f_2$. Let $v_1 \in f_1$. Since the codegree of $\{v_1, u\}$ is 1, $u \in f_2 - f_1$. But both edges in G containing $\{x, u\}$ have two common vertices with e_5, a contradiction. If the core vertices of T are x and v_2, then we get a similar contradiction. Hence the core vertices of T are v_1 and v_2. Let $v_1 \in f_1$. In view of g_1, $w \in f_1 \cup f_2$. By the symmetry between v_1 and v_2, we may assume $w \in f_1$, say $f_1 = \{v_1, w, w_1\}$.

We claim that

$$w_1 \notin N(x). \tag{5}$$

Indeed, suppose G has edge $e_6 = \{x, w_1, w_2\}$. By (4), $w_2 \neq w$. Since we know all edges incident to v_1 and u, $w_2 \notin \{v_1, u\}$. But then G has a triangle formed by e_6, e_1 and f_1, a contradiction. This proves (5).

By the definition of f_1 and f_2, $f_1 \cap f_2$ is either $\{w\}$ or $\{w_1\}$.

Case 1: $f_1 \cap f_2 = \{w\}$, say $f_2 = \{w, v_2, w_3\}$. Then we know all edges incident to v_2. Since $|X_1| > n/\ell - 6\ell$, v_1 is not adjacent to h_3, and w and w_1 are low vertices not adjacent to x (by (4) and (5)), there exists $u' \in X_1$ such that v'_1 is at distance at least 3 from $\{w, w_1\}$.

Let G' be obtained from G by adding edge $e_7 = \{w, w_1, v'_1\}$. Then G' must have a triangle T formed by e_7 and some edges j_1 and j_2. By the choice of v'_1, the core vertices of T in e_7 are w and w_1. Let j_1 contain

w_1. In view of f_1, $v_1 \in j_1 \cup j_2$. We know all edges containing v_1, in particular, we see that the codegree of $\{w_1, v_1\}$ is 1. Also, the only candidate for j_2 is g_1. It follows that $j_2 = g_1$ and $j_1 \cap j_2 = \{v_2\}$. But we also know all edges containing v_2, and none of them contains w_1, a contradiction.

Case 2: $f_1 \cap f_2 = \{w_1\}$, say $f_2 = \{w_1, v_2, w_3\}$. Again, we know all edges incident to v_2. Let G' be obtained from G by adding edge $e_8 = \{v_1, w_1, v_2\}$. Then G' must have a triangle T formed by e_8 and some edges j_1 and j_2.

Suppose first that the core vertices of T in e_8 are v_1 and v_2. Note that each of the edges g_1, f_1 and f_2 shares two vertices with e_8. Thus the only candidates for j_1 and j_2 are e_1 and e_2, but they have two common vertices, a contradiction.

Suppose now that the core vertices of T in e_8 are v_1 and w_1. Suppose that $v_1 \in j_1$. Again, the only candidate for j_1 is e_1. On the other hand, in view of f_1, $w \in j_1 \cup j_2$. Hence $w \in j_2 - j_1$. But the third vertex of j_2 cannot be x (by (4)) and cannot be u because we know all neighbors of u and w is not in this list.

Finally, suppose now that the core vertices of T in e_8 are v_2 and w_1. Suppose that $v_2 \in j_1$. Now the only candidate for j_1 is e_2. Similarly to the previous paragraph, in view of f_2, $w_3 \in j_1 \cup j_2$, and hence $w_3 \in j_2 - j_1$. Again, the third vertex of j_2 cannot be x (now by (5)) and cannot be u because we know all neighbors of u, and w_1 is not in this list. This finishes the proof of the lemma. $\square$

6 Lower Bound: Vertices of degree 2

The goal of this section is to provide the results necessary to show that almost all 2-vertices end up with charge at least 4. Note that 3^+-flat 2-vertices are automatically supported, and thus will receive charge at least 4, and 2-flat 2-vertices that are not supported should receive charge 1 from each of their non-low neighbors, again leaving them satisfied. Thus, our main obstacle is 1-flat 2-vertices.

As a reminder to the reader, we expect a 1-flat 2-vertex t to get charge 1 from its non-low neighbor, and charge 1/2 from two of its low neighbors, in particular, one low neighbor in each edge containing t (possibly the same vertex twice if t has a double neighbor). One result which will be helpful is that there are almost no edges containing two 2-vertices. Indeed, from Corollary 4.13 we already know each $(M, 2, 2)$ edge has two vertices in $\mathbf{R}'_5$. Now we will show there are almost no $(L, 2, 2)$ edges.

6.1 Edges of type $(L, 2, 2)$

For $j \geq 2$, we will say that a vertex u with $d(u) \leq j$ is a *j-far neighbor of a vertex v* if $u \in N(v)$, and

1. for each edge $\{u, u', u''\}$ containing u either $v \in \{u', u''\}$ or $\{u', u''\} \cap N(v) = \emptyset$; and
2. for each edge $e \in E(G - v)$ containing u, the degree in G of each vertex of e is at most j

Let R_8 be the set of ℓ-far neighbors $u \in L \setminus \mathbf{R}'_7$ of vertices in H. Let $\mathbf{R}_8 = R_8 \cup \mathbf{R}'_7$, and let $\mathbf{R}'_8 = \mathbf{R}_8 \cup (L \cap N(L_8 \cap \mathbf{R}_8))$.

Claim 6.1. *For each $j \geq 2$ every $v \in V(G)$ has at most $2j^2$ j-far neighbors. Consequently,*

$$|R_8| \leq 2\ell^2 |H|.$$

Proof. Suppose, v has at least $1 + 2j^2$ j-far neighbors, and u is one of them. Let u be in k edges of the form $\{u, v, a\}$, where a is also a j-far neighbor of v. Note that by Property (1) in the definition of j-far, u is only adjacent to other j-far neighbors of v through edges that contain v. Since $d(u) \leq j$, u has at most $2(j - k)$ neighbors in $G - v$. By Property (1), all these neighbors are not j-far neighbors of v, and again by Properties (1) and (2), each of them is a neighbor of at most $j - 1$ other j-far neighbors of v. There are at least

$$(1 + 2j^2) - 1 - 2(j - k)(j - 1) = 2j + 2kj - 2k > k$$

j-far neighbors of v at distance at least 3 from u in $G - v$, and since there are at most k j-far neighbors of v adjacent to u in G, there exists at least one j-far neighbor of v that is not adjacent to u in G, and is distance at least 3 from u in $G - v$. Let w be one such j-far neighbor of v.

Let $e = \{v, u, w\}$ and $G + e$. Assume that e together with edges e' and e'' form a triangle $\mathcal{T}$ in $G + e$. If the core vertices of $\mathcal{T}$ in e are u and w, then u and w are at distance at most 2 in $G - e$, a contradiction to the choice of w.

So by the symmetry between u and w, assume without loss of generality that the core vertices of $\mathcal{T}$ in e are u and v, and the edge e' of $\mathcal{T}$ contains u. But then one of the vertices in $e' \setminus \{u\}$ is in $N(v)$, a contradiction to Property (1) in the definition of j-far. □

Lemma 6.2. *All vertices of every $(L, 2, 2)$ edge are in $\mathbf{R}'_8$.*

Proof. Let $e = \{a_1, a_2, b\}$, where $d(a_1) = d(a_2) = 2$, and $d(b) < \ell$. We will assume to the contrary that not all vertices of e are in $\mathbf{R}'_8$. Thus, neither of a_1 and a_2 is in $\mathbf{R}_8$. Then a_1 and a_2 are not in $\mathbf{R}'_1$ and hence are 1-flat. So, there are vertices $h_1, h_2 \in H$ (indeed, actually $h_1, h_2 \in \{x, y\}$ since $a_1, a_2 \notin \mathbf{R}'_1$) such that $h_1 \in N(a_1)$ and $h_2 \in N(a_2)$. By Claim 4.4, $h_1 \notin N(a_2)$, and $h_2 \notin N(a_1)$. By Claim 4.3, b cannot be a double neighbor with both a_1 and a_2, so assume that $d(\{b, a_1\}) = 1$.

Since some vertex of e is not in $\mathbf{R}'_8$, a_1 cannot be a ℓ-far neighbor of h_1 (as otherwise this would give $a_1 \in \mathbf{R}_8$), so $b \in N(h_1)$, since otherwise $\{a_2, b\} \cap N(h_1) = \emptyset$, and thus a_1 would satisfy Properties (2) and (1) of the definition of an ℓ-far neighbor. Let $e_1 = \{a_1, h_1, z_1\}$ be the other edge containing a_1 in G, and let e'_1 be an edge that contains h_1 and b. Since $a_2 \notin N(h_1)$, $a_2 \notin e'_1$, so for e, e_1, and e'_1 to not be a triangle in G, we must have that $z_1 \in e'_1$. Let $e_2 = \{a_2, h_2, z_2\}$ be the second edge containing a_2, and note that z_2 may be equal to b.

Consider the edge $f = \{h_1, z_1, a_2\}$, and note that $f \notin E(G)$ since h_1 and a_2 are not adjacent. Then $G + f$ contains a $C_3^{(3)}$, say T. Note that $h_1 z_1$ is a bad pair by Claim 4.5 since a_1 is not a double neighbor of either vertex. Thus, a_2 is one of the core vertices of T. This implies that either e or e_2 is an edge of T. If e is such edge, then b must be in an edge with one of h_1 or z_1, but not both; this is a contradiction though as this edge, along with e and e_1 would give us a $C_3^{(3)}$ in G. If e_2 is an edge of T, then we must have an edge g in G containing exactly one vertex from $\{h_1, z_1\}$ and exactly one vertex from $\{h_2, z_2\}$. Furthermore, $a_1 \notin g$ since $d(a_1) = 2$ and so $g \neq e_1, e$. Consider two cases based on if $b = z_2$ or not.

Case 1: $b = z_2$. In this case, if g contains b and a vertex from $\{h_1, z_1\}$, this gives us a triangle in G with edges e, e_1 and g. If g contains h_2 and a vertex from $\{h_1, z_1\}$, then this also gives us a triangle in G, this time with edges e'_1, e_2 and g. Thus, we reach a contradiction.

Case 2: $z_2 \neq b$. Note that since a_2 is not a ℓ-far neighbor of h_2, $b \in N(h_2)$, we have some edge e'_2 containing both b and h_2. Since in this case a_2 does not have any double neighbors, and $a_1 \notin e_2$ or e'_2, for the edges e, e_2 and e'_2 to not form a triangle in G, we must have that $z_2 \in e'_2$. Now, since $|g \cap \{h_1, z_1\}| = |g \cap |\{h_2, z_2\}| = 1$, the only possible way that g, e'_1 and e'_2 do not form a triangle is if $b \in g$, but in this case, g, e_1 and e give us a triangle, and so we reach a contradiction. □

6.2 Helpful and Half-Helpful Vertices

We now provide definitions which will help us deal with 2-vertices.

Given vertices $h \in M$ and $u \in L$, we will call the pair hu *rich* if $d(hu) \geq 3$, and we will call a vertex $u \in L$ *rich* if u is in any rich pair. Let $\text{flat}(u)$ be the number of edges containing u and a 1-flat 2-vertex. We will call a $(M, L, 4)$ edge $\{h, v, u\}$ with $h \in M$ and $d(u) = 4$ *exceptional* if

- $d(hv) = 2$ and $d(v) \geq 4$, and
- hu is a rich pair and $\text{flat}(u) = 2$.

We will call v the *exception* in the exceptional edge $\{h, v, u\}$.

We now classify all (M, L, L) edges of G into three mutually exclusive types: needy edges, rich edges and reasonable edges.

Needy Edges:

Let $e = \{h, u, v\}$ be a (M, L, L) edge with $h \in M$. We will say e is a *needy edge with recipient* v if v is either a 2-vertex or a 1-flat 3-vertex, and u is not.

Rich Edges:

Let $e = \{h, u, v\}$ be a (M, L, L) edge with $h \in M$ that is not needy. We will say e is a *rich edge with recipient* v if

1. $\{h, u, v\}$ is not exceptional,
2. $d(v) \leq 7$, v is unsupported, and hv is not rich, and
3. at least one of the following occurs:
   - hu is a rich pair,
   - $d(u) \geq 3$ and u is supported, or
   - $d(u) \geq 8$.

Reasonable Edges:

We will call all (M, L, L) edges that are not needy or rich, *reasonable*.

Helpful and Half-Helpful Vertices:

We now can turn our attention to the low vertices which will give charge to 1-flat 2-vertices. Given a low vertex v let $\mathrm{rich}(v)$ denote the number of rich edges in which v is the recipient, let $\mathrm{reas}(v)$ denote the number of reasonable edges containing v, and let $\mathrm{supp}(v)$ denote the number of (M, M, v) edges, and finally recall that $\mathrm{flat}(v)$ is the number of edges containing v and a 1-flat 2-vertex.

Then, a low vertex v of degree at least 3 is *helpful* if

$$d(v) + 2\mathrm{supp}(v) + \mathrm{rich}(v) + \frac{1}{2}\mathrm{reas}(v) \geq \frac{1}{2}\mathrm{flat}(v) + 4. \tag{6}$$

The left-hand side of (6) will be exactly the amount of charge v ends up with before v gives any charge away. Then, the right-hand side is the amount of charge we want v to be able to give away, plus the amount of charge we want v to keep (namely 4). So, helpful vertices are exactly those low vertices that can give charge 1/2 to each of their 1-flat 2-neighbors (counted with multiplicity), and still have charge at least 4.

Since (almost) every edge contains at most one 1-flat 2-vertex, it would be nice if we could simply show that every edge containing a 1-flat 2-vertex also contains a helpful vertex, however it is not clear if this is the case. Instead, we will do a second round of "helping". We call a low vertex v a *k-donor* if v is in exactly k edges with 1-flat 2-vertices which are either type $(M, v, 2)$ or are type $(L, v, 2)$, where the other low vertex is not helpful. Let $\mathrm{donor}(v)$ denote the value of k for which v is a k-donor.

A low vertex v of degree at least 3 is *half-helpful* if

$$d(v) + 2\mathrm{supp}(v) + \mathrm{rich}(v) + \frac{1}{2}\mathrm{reas}(v) \geq \frac{1}{2}\mathrm{donor}(v) + 4. \tag{7}$$

Again, the left-hand side of the above inequality is exactly the charge that v has before it gives any away, and the right-hand side is the amount of charge we want v to give away, along with the charge we want v to keep. Note that since $\mathrm{donor}(v) \leq \mathrm{flat}(v)$, every helpful vertex is also half-helpful.

We now will show that almost every edge containing a 1-flat 2-vertex contains either a helpful or half-helpful vertex. In particular, we will show the following:

- Almost every $(V, L, 2)$ edge containing a 1-flat 2-vertex contains a vertex of degree at least 4, and

- Almost every vertex of degree at least 4 is helpful or half-helpful.

As we have already shown that there are almost no $(M, 2, 2)$ or $(M, 3, 2)$ edges, to prove the first bullet point above, we need to only deal with $(L, L, 2)$ edges.

Lemma 6.3. *Let $e = \{a, b, t\}$ be an $(L, L, 2)$ edge with a 1-flat 2-vertex t. If $\max\{d(a), d(b)\} \leq 3$, then $\{a, b, t\} \subseteq \mathbf{R}'_8$.*

Proof. Suppose $\max\{d(a), d(b)\} \leq 3$ and $\{a, b, t\} \not\subseteq \mathbf{R}'_8$. Then no vertex of e is in $\mathbf{R}_8$. By Lemma 6.2, $d(a) = d(b) = 3$. Since t is not in $\mathbf{R}_1$, it is contained in a high edge $\{h, u, t\}$, say $h \in H$. Since t is not in $\mathbf{R}'_2$, ht is a good pair. To avoid a $C_3^{(3)}$ in G, each h, t-link must contain u, say, $\{h, u, a\}$ is an edge of G. Consider the non-edges $\{h, t, a\}$ and $\{u, t, a\}$. Since these triples intersect every edge containing t in two vertices, there is an h, a-link and a u, a-link avoiding the vertex t.

Assume that there exist distinct vertices $z, z' \in \{h, u\}$ such that there is a z, a-link that has no edge containing both z and z'. Thus there exist vertices w_1 and w_2 such that $\{z, w_1, w_2\}$ and $\{w_1, a, z'\}$ are edges in G (note that z' must be in this second edge to avoid a triangle with $\{z, z', a\} = \{h, u, a\}$). However, then $\{z, w_1, w_2\}$, $\{w_1, a, z'\}$ and $\{z, z', t\}$ form a $C_3^{(3)}$ in G, a contradiction.

Thus, $\{h, u, s\}$ and $\{s, s', a\}$ are edges for some s, s' such that $\{s, s'\} \cap \{h, u, a, t\} = \emptyset$. We will consider cases based on whether $b \in \{s, s'\}$ or not.

Case 1: $s = b$. Under this assumption, $s' \notin \{a, b, u, t, h\}$ and $d(ab) \geq 2$. By Claim 4.3, we must have $d(ab) = 2$, and furthermore, $d(at) = d(bt) = 1$. Thus by Claim 4.5, ab is a bad pair. Consider the non-edge $\{a, b, h\}$. There must be an h, a-link or an h, b-link in $G + \{a, b, h\}$. However every edge containing a or b intersects $\{a, b, h\}$ in two vertices, so no such links exist. This proves the case.

Case 2: $s \neq b$. Note that we may have $b = s'$. Consider the non-edge $\{s, a, t\}$. The only edges containing a or t that do not intersect $\{s, a, t\}$ in two vertices are $\{a, u, h\}$ and $\{t, u, h\}$. Thus, at cannot be the pair of core vertices in the $C_3^{(3)}$ in $G + \{s, a, t\}$, so s must, and there has to be an edge in G of the form $\{s, x, y\}$ for some $x \in \{u, h\}$, $y \notin \{u, h, a, t\}$. If $y \neq s'$, then $\{s, x, y\}$, $\{a, u, h\}$ and $\{s, s', a\}$ form a $C_3^{(3)}$ in G, so we must have $y = s'$. Furthermore, if $b = s'$, then $\{s, s', x\}$, $\{h, u, t\}$ and $\{t, a, b\}$ form a $C_3^{(3)}$ in G, so $b \neq s'$. Furthermore, note that there is no edge in G containing exactly one vertex from $\{s, s'\}$ and exactly one from $\{h, u\}$ since any such edge could not contain a, and thus would create a $C_3^{(3)}$ in G with $\{s, s', a\}$ and $\{a, h, u\}$. Similarly, there is no edge containing b and exactly one vertex from $\{s, s'\}$.

Since a has no double neighbors, by Claim 4.5, ss' is a bad pair. Consider the non-edge $\{s, s', t\}$. There must be a t, q-link for some $q \in \{s, s'\}$. This link cannot use edge $\{t, a, b\}$ since no edge involving a can complete the path, and there is no edge in G containing b and exactly one vertex from $\{s, s'\}$. Similarly, the link cannot use the edge $\{t, h, u\}$ since there is no edge containing exactly one vertex from $\{h, u\}$ and one from $\{s, s'\}$. This contradiction completes the proof. $\square$

Now we focus on showing that almost all low vertices of degree at least 4 are half-helpful. Vertices of degree 6 or more and supported vertices are relatively easy to deal with, but first we need a helpful lemma.

Lemma 6.4. *Let $\{h, u, t\}$ be a $(H, L, 2)$ edge where $h \in H$ and t is a 1-flat 2-vertex. If neither u nor t is supported or belongs to $\mathbf{R}'_8$, then $d(hu) \geq 3$.*

Proof. Since t is 1-flat and not in $\mathbf{R}'_2$, ht is a good pair. We will consider two cases based on if $d(ut) = 1$ or not.

Case 1: $d(ut) = 1$. For ht to be a good pair and to avoid a triangle with $\{h, u, t\}$, there must be edges $\{h, u, w\}$ and $\{w, z, t\}$ for some $w, z \notin \{h, u, t\}$.

Consider the non-edge $\{h, w, t\}$. Since $\{h, w, t\}$ intersects every edge containing t in two vertices, hw must be a good pair, and furthermore, this h, w-link must avoid t. Furthermore, this link must use u to avoid a $C_3^{(3)}$ in G with $\{h, u, w\}$. If one of the edges of the link is $\{h, u, v\}$ for some $v \notin \{w, t\}$, then we are done since $d(hu) \geq 3$, so let us assume otherwise. Then the loose path must use edges of the form $\{w, u, v\}$ and $\{v, v_1, h\}$ for some $v, v_1 \notin \{h, w, u, t\}$, but then $\{h, u, t\}$, $\{u, w, v\}$ and $\{v, v_1, h\}$ form a $C_3^{(3)}$ in G, a contradiction.

Case 2: $d(ut) \geq 2$, say $\{u, v, t\}$ is an edge of G with $v \neq h$. Since t is not in $\mathbf{R}_3'$, the codegree of hu must be at least 2, say $\{h, u, z\}$ is an edge.

First, if $z = v$, then the non-edge $\{h, t, v\}$ intersects every edge of t in at least two vertices, so hv is a good pair. Any h, v-link must contain u to avoid a $C_3^{(3)}$ in G with $\{h, u, v\}$, but if one of the edges in this link contains both h and u, then $d(hu) \geq 3$, satisfying the hypothesis of the lemma, so we may assume that one of the edges in this path is $\{u, v, a\}$ and another is $\{h, a, b\}$ for some $a, b \notin \{h, u, v, t\}$. However, this creates a $C_3^{(3)}$ in G with edges $\{h, u, t\}$, $\{u, v, a\}$ and $\{h, a, b\}$. Thus, we may assume $z \neq v$.

Now consider the non-edge $\{u, t, z\}$. Since it intersects both edges containing t in two vertices, uz must be a good pair, and there must be a u, z-link that avoids t. Furthermore, this link must contain h to avoid a $C_3^{(3)}$ in G with the edge $\{h, u, z\}$. If this link contains an edge containing both h and u, this would satisfy the claim of the lemma, so the link consists of edges of the form $\{h, z, a\}$ and $\{u, a, b\}$ for some $a, b \notin \{h, u, t\}$. But this gives a $C_3^{(3)}$ in G with edges $\{h, u, t\}$, $\{u, a, b\}$ and $\{h, z, a\}$, a contradiction. $\square$

Lemma 6.5. *Let $v \in L \setminus \mathbf{R}_8'$ and $d(v) \geq 3$. If v is supported or $d(v) \geq 6$, then v is helpful.*

Proof. If v is supported, then $\mathrm{supp}(v) \geq 1$ and $\mathrm{flat}(v) \leq d(v) - 1$. So, since $d(v) \geq 3$,

$$d(v) + 2\mathrm{supp}(v) - \frac{1}{2}\mathrm{flat}(v) - 4 \geq \frac{1}{2}(d(v) + 1) - 2 \geq 0,$$

which yields (6).

Suppose now that some vertex $v \in L \setminus \mathbf{R}_8'$ with $d(v) \geq 6$ is not helpful. By above, $\mathrm{supp}(v) = 0$. Similarly, if v is contained in a reasonable edge or is a recipient in a rich edge, then this edge does not contain a 1-flat 3^--vertex, and hence

$$d(v) + \mathrm{rich}(v) + \frac{1}{2}\mathrm{reas}(v) - \frac{1}{2}\mathrm{flat}(v) - 4 \geq d(v) + \frac{1}{2} - \frac{1}{2}(d(v) - 1) - 4 = \frac{1}{2}d(v) - 3 \geq 0,$$

which again yields (6). Thus

$$\mathrm{rich}(v) + \mathrm{reas}(v) = 0. \tag{8}$$

Since $v \notin \mathbf{R}_1'$, there are a vertex $h \in H$ and a vertex $w \in L$ such that $\{h, v, w\}$ is an edge in G.

Case 1: $d(hv) \geq 2$. By Lemma 4.9, either $\{h, v\}$ is contained in an edge $e_1 = \{h, v, w_1\}$ where w_1 is neither a 2-vertex nor a 1-flat 3-vertex, or $\mathrm{flat}(v) \leq d(v) - 2$, in which case

$$d(v) - \frac{1}{2}\mathrm{flat}(v) - 4 \geq \frac{1}{2}d(v) + 1 - 4 \geq 0,$$

and hence (6) holds. Thus, assume the former. Since $d(v) > 3$, e_1 is either reasonable or rich. By (8), we conclude that vh is a rich pair. Since $d(w_1) \geq 3$, $\mathrm{flat}(v) \leq d(v) - 1$. Thus, if v is not helpful, then

$$d(v) = 6 \text{ and each edge containing } v \text{ apart from } e_1 \text{ contains a 1-flat 2-vertex.} \tag{9}$$

In particular, since vh is a rich pair, we have edges $e_2 = \{h, v, w_2\}$ and $e_3 = \{h, v, w_3\}$ where w_2 and w_3 are 1-flat 2-vertices.

Then the pair vh cannot be good: the first edge of any h, v-link must contain at least one of w_2 and w_3. Since $v \notin \mathbf{R}_2'$, v is adjacent to another vertex in M, say G has an edge $e_4 = \{h', v, w'\}$ where $h' \in M$. By (9), w' must be a 1-flat 2-vertex, thus $h' \in H$. Then by Lemma 6.4, $d(uw') \geq 3$, so by Lemma 4.9 some edge containing $\{h', u\}$ does not contain 1-flat 2-vertices. This contradicts (9).

Case 2: $d(hv) = 1$. Since $v \notin \mathbf{R}_3'$, $w \notin \mathbf{R}_3$. Then w is not 1-flat. Thus, w is a 2-flat 2-vertex. Since $w \notin \mathbf{R}_4$, the co-degree of hw is 1. If $d(vw) = 2$, say $e_2 = \{v, w, h_1\} \in E(G)$, then $\mathrm{flat}(v) \leq d(v) - 2$ and we are done. So $d(vw) = 1$, and hence hv is a bad pair.

Since $v \notin \mathbf{R}_2'$, v is adjacent to a vertex $h' \in M$ distinct from h, say $e_2 = \{v, h', w'\} \in E(G)$. If w' is not a 1-flat 2-vertex, then $\mathrm{flat}(v) \leq d(v) - 2$ and we are done again. Suppose w' is a 1-flat 2-vertex. Since $v \notin \mathbf{R}_2'$, $d(h'v) \geq 2$. Then by Lemma 4.9, some edge containing $\{h', v\}$ does not contain 1-flat 2-vertices, and we again get $\mathrm{flat}(v) \leq d(v) - 2$. $\square$

Let R_9 be the set of 1-flat 2-vertices $u \in L \setminus \mathbf{R}'_8$ that are double neighbors with an unsupported 5-vertex not in $\mathbf{R}'_8$. Let $\mathbf{R}_9 = R_9 \cup \mathbf{R}'_8$, and let $\mathbf{R}'_9 = \mathbf{R}_9 \cup (L \cap N(L_8 \cap \mathbf{R}_9))$.

Lemma 6.6. $|R_9| \le 60\ell^3$.

Proof. Each 1-flat 2-vertex in $L \setminus \mathbf{R}'_1$ is adjacent to x or y. For $h \in \{x, y\}$, let $\mathcal{F}(h)$ be the set of edges $\{h, u, t\}$ such that $u \in L \setminus \mathbf{R}'_8$ is an unsupported 5-vertex, $t \in L \setminus \mathbf{R}'_8$ is a 1-flat 2-vertex, and $d(ut) = 2$. By Lemma 6.4, $d(hu) \ge 3$ for each such $e \in \mathcal{F}(h)$. Choose one such edge $e_1 = \{h, u, t\}$. Let $e_2 = \{z, u, t\}$ be the other edge containing t, and $e_3 = \{h, u, v_1\}$ and $e_4 = \{h, u, v_2\}$ be other edges containing $\{h, u\}$. We consider cases based on whether $z \in \{v_1, v_2\}$ or not.

Case 1: For at least $10\ell^3$ edges $e_1 = \{h, u, t\} \in \mathcal{F}(h)$, $z \in \{v_1, v_2\}$, say $z = v_1$. Let G_1 be obtained from G by adding the non-edge $e_5 = \{h, t, v_1\}$. By definition, G_1 has a triangle T formed by e_5 and, say g_1 and g_2. Since both edges in G containing t have two common vertices with e_5, the core vertices of T in e_5 are h and v_1, say $h \in g_1$.

In order for e_3, g_1 and g_2 not to form a triangle in G, we need $u \in g_1 \cup g_2$. If $u \notin g_2$, then $u \in g_1 \setminus g_2$, and the edges g_1, g_2 and e_2 form a triangle in G, a contradiction. Thus we may assume $g_2 = \{v_1, u, a\}$. Similarly, if $u \notin g_1$, then $u \in g_2 \setminus g_1$, and the edges g_1, g_2 and e_1 form a triangle in G, a contradiction again. So we may assume $g_1 = e_4$, and in particular, $v_2 \ne a$. Thus the edges containing u are e_1, e_2, e_3, e_4, g_2.

Case 1.1: For some $e_1 = \{h, u, t\} \in \mathcal{F}(h)$, uh is a good pair, say edges f_1 and f_2 form a u, h-link. In order to avoid triangles, $f_1 \cup f_2$ must contain t, v_1 and v_2. Since we know all edges containing t or u, the only possibility for this is that $e_6 = \{h, v_1, v_2\} \in E(G)$. But then, since $a \ne v_2$, the edges e_6, g_2 and e_1 form a triangle in G.

Case 1.2: uh is a bad pair for at least $10\ell^3$ edges $e_1 = \{h, u, t\} \in \mathcal{F}(h)$ with $v_1 = z$. We can try to add the edge $\{u, h, u'\}$ for each two such edges e_1 and $e'_1 = \{h, u', t'\}$. By the case, the core vertices in each obtained triangle should be always u and u'. Since u is unsupported, all vertices u, t, v_1, v_2 are low. So the common neighbor of u with most of u' should be a, which then must participate in at least $3\ell^3$ configurations for corresponding $\{h, u, t\}$.

Case 1.2.1: This a is adjacent to h, say $e_7 = \{h, a, v_4\} \in E(G)$. Since each v_1 is low, there is a configuration in which $v_4 \ne v_1$. For this configuration, G will have a triangle with edges e_7, g_2 and e_1, a contradiction.

Case 1.2.2: This a is not adjacent to h. Let G_2 be obtained from G by adding new edge $e_8 = \{h, t, v_2\}$. By definition, G_1 has a triangle T formed by e_8 and, say f_1 and f_2.

If the core vertices of T were t and h, then the edge of T containing t, say f_1 can be only e_2 and hence $f_1 \cap f_2 \subseteq \{u, v_1\}$. Since we know all edges containing u and each of e_1 and g_1 shares two vertices with e_8, if $f_1 \cap f_2 = \{u\}$, then the only candidate for f_2 is e_3, but it shares two vertices with e_2, a contradiction. Thus in this case $f_1 \cap f_2 = \{v_1\}$, say $f_2 = \{v_1, h, v_5\}$. Since a is not adjacent to h, $v_5 \ne a$. Hence e_1, g_2 and f_2 form a triangle, a contradiction.

If the core vertices of T were t and v_2, then again the edge f_1 containing t can be only e_2 and hence $f_1 \cap f_2 \subseteq \{u, v_1\}$. Again, if $f_1 \cap f_2 = \{u\}$, then the only candidate for f_2 is e_3, but it shares two vertices with e_2, a contradiction. So, again $f_1 \cap f_2 = \{v_1\}$, say $f_2 = \{v_1, v_2, v_5\}$. Since $h \in e_8$ and $u \in e_2$, $v_5 \notin \{h, u\}$. Then f_2, e_2 and e_4 form a triangle in G.

The last possibility is that the core vertices of T are h and v_2. We may assume that $v_2 \in f_2$. Because of e_4, $u \in f_1 \cup f_2$. We know all edges containing u and among these edges only e_4 contains $\{u, v_2\}$. Hence $u \in f_1 - f_2$. In this case, the only candidate for f_1 is e_3, and so $f_1 \cap f_2 = \{v_1\}$. Thus as in the previous paragraph we may assume $f_2 = \{v_1, v_2, v_5\}$. Since $h \in e_8$, $v_5 \ne h$. And we know all edges containing u, so $v_5 \ne u$. Again, f_2, e_2 and e_4 form a triangle in G. This finishes Case 1.

Case 2: For at least $10\ell^3$ edges $e_1 = \{h, u, t\} \in \mathcal{F}(h)$, $z \ne v_1$ and $z \in N(h)$. Let an edge containing z and h be $e_9 = \{z, h, z'\}$. If $z' \notin \{u, v_1\}$, then edges e_9, e_2 and e_3 form a triangle. If $z' = u$, then we have Case 1 with z' in the role of v_1. So, $z' = v_1$.

Let G_3 be obtained from G by adding edge $e_{10} = \{u, t, v_1\}$. By definition, G_3 has a triangle T formed by e_{10} and, say g_1 and g_2. Since both edges in G containing t have two common vertices with e_{10}, the core vertices of T in e_{10} are u and v_1. We may assume $u \in g_1$. Because of e_3, $h \in g_1 \cup g_2$. Suppose $h \in g_1$, say $g_1 = \{u, h, v_6\}$. Since g_1 shares only one vertex with e_{10}, $v_6 \notin \{t, v_1\}$. Since Case 1 is already covered, $v_6 \neq z$. Then edges e_9, e_2 and e_{10} form a triangle. This contradiction implies $h \in g_2 - g_1$. In this case, edges g_1, g_2 and e_1 form a triangle. This finishes Case 2.

Case 3: For at least $10\ell^3$ edges $e_1 = \{h, u, t\} \in \mathcal{F}(h)$, $z \neq v_1$ and $z \notin N(h)$. As in Case 2, let G_3 be obtained from G by adding edge $e_{10} = \{u, t, v_1\}$. By definition, G_3 has a triangle T formed by e_{10} and, say g_1 and g_2. Again, the core vertices of T in e_{10} are u and v_1. We may assume $u \in g_1$. Again, $h \in g_1 \cup g_2$. If $h \notin g_1$, then edges g_1, g_2 and e_1 form a triangle. So, we may assume that $g_1 = \{u, h, v_2\}$.

By the case, G has no edges $\{u, t, v_1\}$ and $\{u, t, v_2\}$. Hence uh is a bad pair, and we have at least $10\ell^3$ such edges containing h. We can try to add the edge $\{u, h, u'\}$ for each two such edges. Since pairs uh and $u'h$ are bad, the core vertices in each obtained triangle should be always u and u'. So most of these vertices u must have a common neighbor, say w (of high degree). Since u is unsupported and t is 1-flat, all vertices u, v_1, v_2, z are low and so $w \notin \{u, z, v_1, v_2\}$. Let the edge containing u and w be $e_{11} = \{u, w, u_1\}$. Now we know all edges containing u apart from the fact that we do not know whether u_1 is one of v_1, v_2 or z.

If w is adjacent to h, then we simply repeat the argument of Subcase 1.2.1. So below we assume

$$w \notin N(h). \tag{10}$$

Case 3.1: $u_1 = z$. Then let G_4 be obtained from G by adding edge $e_{12} = \{u, t, w\}$. By definition, G_4 has a triangle T formed by e_{12} and, say f_1 and f_2. Again, the core vertices of T in e_{12} are u and w. We may assume $u \in f_1$. Now $z \in f_1 \cup f_2$. Since we know all edges containing u and three of these edges have 2 common vertices with e_{12}, the only candidates for f_1 are e_3 and e_4. Neither of them contains z, so $z \in f_2 - f_1$. The common vertex of f_1 and f_2 cannot be h because $w \notin N(h)$, so by symmetry, we may assume that $f_2 = \{u, u_1, v_1\}$. Then edges e_2, f_2 and e_3 form a triangle.

Case 3.2: $u_1 \neq z$, but there is an edge e_{13} containing z and w, say $e_{13} = \{z, w, z_1\}$. If $z_1 \neq u_1$, then edges e_{13}, e_2 and e_{11} form a triangle, so suppose $z_1 = u_1$. Furthermore, if $z_1 = v_i$ for some $i \in [2]$, then edges e_{13}, e_2 and e_{2+i} form a triangle. So below we assume

$$z_1 = u_1 \notin \{v_1, v_2\}. \tag{11}$$

As in Case 3.1, consider G_4 obtained from G by adding edge $e_{12} = \{u, t, w\}$. Again, G_4 has a triangle T formed by e_{12} and, say f_1 and f_2. Again, the core vertices of T in e_{12} are u and w. We may assume $u \in f_1$. Now $u_1 \in f_1 \cup f_2$. Since we know all edges containing u and three of these edges have 2 common vertices with e_{11}, the only candidates for f_1 are e_3 and e_4. By (11), neither of them contains u_1, so $u_2 \in f_2 - f_1$. The common vertex of f_1 and f_2 cannot be h because $w \notin N(h)$, so by symmetry, we may assume that $f_2 = \{u, u_1, v_1\}$.

We claim that

$$u_1 \notin N(h). \tag{12}$$

Indeed, suppose there is an edge $e_{14} = \{h, u_1, u_2\} \in E(G)$. Since $w \notin N(h)$, $u_2 \neq w$. Since we know all neighbors of t and u, we know that $u_2 \notin \{u, t\}$. Then edges e_{14}, e_1 and e_{10} form a triangle. This contradiction proves (12).

Now, consider G_5 obtained from G by adding edge $e_{15} = \{t, w, u_1\}$. Again, G_5 has a triangle T formed by e_{15} and, say j_1 and j_2. Suppose first that t is a core vertex of T and $t \in j_1$. Then $j_1 \in \{e_1, e_2\}$. So, the vertex $q \in j_1 \cap j_2$ is one of u, h or z. Since the only edge containing u and at least one of w and u_1 is e_{11}, and it has two common vertices with e_{15}, $q \neq u$. By (10) and (12), $q \neq h$. So, $q = z$. But j_2 must have exactly one common vertex with $\{w, u_1\}$. Then edges j_2, e_1 and e_{11} form a triangle. Therefore, the core vertices of T are w and u_1.

We may assume $w \in j_1$. Since the codegree of wu_1 is at least 3, $j_1 \cup j_2 = \{z, u, v_1\}$. But u is not in an edge in which one vertex is in $\{w, u_1\}$ and the other is in $\{z, v_1\}$. This contradiction finishes Case 3.2.

Case 3.3: $z \notin N(w)$. Note that h and w are the only high vertices adjacent to v or u. We have at least $3\ell^3$ such configurations containing h. Fix one such configuration. Among the remaining $3\ell^3 - 1$ similar configurations with the same vertices h and w, find one with vertices v', u', z', v_1', v_2' such that the distance from z' to $\{v, u\}$ is at least 3 (we can do it because z' is not adjacent to h or w). Consider G_6 obtained from G by adding edge $e_{16} = \{t, u, z'\}$. Again, G_6 has a triangle T formed by e_{16} and, say m_1 and m_2. By the choice of z', it cannot be a core vertex, so t and u must be. But both edges containing t share two vertices with e_{16}. This contradiction shows that there are less than $10\ell^3$ edges $e_1 = \{h, u, t\} \in \mathcal{F}(h)$ with $z \neq v_1$ and $z \notin N(h)$.

Together with Cases 1 and 2, this implies that $|\mathcal{F}(h)| \leq 30\ell^3$ for $h \in \{x, y\}$. This proves the lemma. $\square$

Lemma 6.7. *Let $h \in M$. Then for every edge $\{h, u, t\} \in E(G)$ such that neither of t and u is supported, $d(t) = 2$, $d(u) \leq 8$ and $d(hu) = 2$, we have $\{u, t\} \subseteq \mathbf{R}_9'$.*

Proof. Suppose to the contrary G contains an edge $e_1 = \{h, u, t\}$ such that $d(t) = 2$, $d(u) \leq 8$, $d(hu) = 2$, neither of t and u is supported and $|\{u, t\} \cap \mathbf{R}_9'| \leq 1$. Then $u, t \in L \setminus \mathbf{R}_9$ since if either one of u or t were in $\mathbf{R}_9$, then both would be in $\mathbf{R}_9'$. Let the second edge containing h and u be $e_2 = \{h, u, w\}$.

Since $t \notin \mathbf{R}_4'$,

$$d(ht) = 1. \tag{13}$$

Case 1: There is an edge $e_3 = \{t, w, z\} \in E(G)$ containing $\{t, w\}$. Let $e_4 = \{t, h, w\}$. By (13), $e_4 \notin E(G)$. Let $G' = G + e_4$. Then G' has a triangle containing e_4. Since both e_1 and e_3 have two common vertices with e_4, there is an h, w-link avoiding t formed by some edges g_1 and g_2, say $h \in g_1$. In view of e_2, $u \in g_1 \cup g_2$, but since $d(hu) = 2$, $u \notin g_1$. Thus $u \in g_2 - g_1$, and hence g_1 and g_2 form also an h, u-link. Then together with e_1 they form a triangle in G, a contradiction.

Case 2: $d(ut) > 1$. Since $d(t) = 2$, $d(ut) = 2$, and there is an edge $e_5 = \{t, u, u'\} \in E(G)$. By Case 1, $u' \neq w$, and $e_6 = \{t, w, u\}$ is not in $E(G)$. Let $G' = G + e_6$. Then G' has a triangle containing e_6. Since both e_1 and e_5 contain $\{u, t\}$, there is a u, w-link avoiding t, say with edges g_1 and g_2 where $u \in g_1$. In view of e_2, $h \in g_1 \cup g_2$, but since $d(hu) = 2$, $h \notin g_1$. Thus $h \in g_2 - g_1$, and hence g_1 and g_2 form also a u, h-link. Then together with e_1 they form a triangle in G, a contradiction.

Case 3: Cases 1 and 2 do not hold. If the pair uh is good, then because of e_1, the degree of ut or of ht would be at least 2. But by (13) and Case 2, neither holds. Hence uh is a bad pair.

Suppose now th is good. Let g_1 and g_2 be the two edges of a t, h-link in G with $t \in g_1$. In view of e_1, $u \in g_1 \cup g_2$, but since Case 2 does not hold, $u \notin g_1$. Thus $g_2 = e_2$, and the common vertex of g_1 and g_2 is w. But then Case 1 holds, a contradiction.

Since both pairs uh and th are bad and $t \notin \mathbf{R}_5'$, by Lemma 4.10, there is a pair $\{z, z'\}$ of vertices distinct from h such that $e_7 = \{t, z, z'\} \in E(G)$ and $e_8 = \{u, z, z'\} \in E(G)$. By the case, $w \notin \{z, z'\}$. Consider again $G' = G + e_6$. One of the pairs contained in e_6 must be good.

If ut is good, then in view of e_1 either $d(ht) \geq 2$ or $d(uh) \geq 3$. The former inequality is Case 2 and the latter contradicts the hypothesis of our lemma. If uw is good and g_1 and g_2 are the two edges of a u, w-link in G with $u \in g_1$, then in view of e_2, $h \in g_1 \cup g_2$, but since $d(hu) = 2$, $h \notin g_1$. Thus $h \in g_2 \setminus g_1$, and hence g_1 and g_2 form a u, h-link. Then together with e_1 they form a triangle in G, a contradiction.

The last possibility is that tw is good. Let g_1 and g_2 be the two edges of a t, w-link in G with $t \in g_1$. Then, since e_1 shares two vertices with e_6, $g_1 = e_7$. By the symmetry between z and z', we may assume that $g_2 = \{w, z, w'\}$. If $w' \neq h$, then G has a triangle with the edges g_2, e_8 and e_1. Finally, if $w' = h$, then G has a triangle with the edges g_2, e_7 and e_1. $\square$

Lemma 6.8. *No unsupported 4-vertex $u \in L \setminus \mathbf{R}_9'$ is a double neighbor of a 1-flat 2-vertex.*

Proof. Suppose an unsupported 4-vertex $u \in L \setminus \mathbf{R}_9'$ is a double neighbor of a 1-flat 2-vertex t, say $\{h, u, t\}, \{u, t, z\} \in E(G)$, where $h \in H$ and $z \in L$. By Lemma 6.4, $d(hu) \geq 3$, and since $u \notin \mathbf{R}_4'$, $d(hu) = 3$. Let $\{h, u, v_1\}$ and $\{h, u, v_2\}$ be the other two edges containing h and u, possibly $z \in \{v_1, v_2\}$.

Since $u \notin \mathbf{R}_2'$, hu is a good pair. Since $\{u, t, z\}$ is the only edge containing u but not h, this edge must be in each u, h-link. Furthermore, to avoid a $C_3^{(3)}$ in G with any of the edges $\{h, u, t\}$, $\{h, u, v_1\}$ and $\{h, u, v_2\}$, each u,h-link must contain all t, v_1, and v_2. This implies that actually $z \in \{v_1, v_2\}$. Assume without loss of generality $z = v_2$. The second edge of this path must be $\{h, z, v_1\}$.

Consider the non-edge $\{h, t, z\}$. Since this non-edge intersects every edge containing t in two vertices, there must be an h,z-link that avoids t. To avoid a $C_3^{(3)}$ in G with the edge $\{h, u, z\}$, this link must contain u. The only edge of G that contains u, does not contain t, and does not contain both h and z is $\{h, u, v_1\}$. Furthermore, the second edge in this path must be $\{v_1, z, a\}$ for some $a \notin \{h, u, v_1, z, t\}$, but then the edges $\{h, u, v_1\}$, $\{v_1, z, a\}$ and $\{u, a, t\}$ form a $C_3^{(3)}$ in G, a contradiction. □

Lemma 6.9. *Let $h \in M$ and $u \in L \setminus \mathbf{R}_9'$. If hu is a rich pair, then u is helpful.*

Proof. Since hu is rich, $d(hu) \geq 3$. Let $e_1 = \{h, u, v_1\}$, $e_2 = \{h, u, v_2\}$ and $e_3 = \{h, u, v_3\}$ be edges in G. If u is supported or $d(u) \geq 6$, then by Lemma 6.5, u is helpful. So we assume u is unsupported and $d(u) \leq 5$. Since $u \notin \mathbf{R}_4'$, $d(u) \geq 4$, thus $4 \leq d(u) \leq 5$. We will consider cases based on the degree of u and whether hu is a good pair or a bad pair.

Case 1: hu is a good pair. Since hu is a good pair, there is an h,u-link, and it must contain the vertices v_1, v_2 and v_3. By symmetry, we can assume the link contains the edges $e_4 = \{h, v_1, v_2\}$ and $e_5 = \{v_2, v_3, u\}$. This implies that neither v_1 nor v_2 is 1-flat. Since $d(uv_3) \geq 2$, by the definition of $\mathbf{R}_9'$ (when $d(u) = 5$) or Lemma 6.8 (when $d(u) = 4$, v_3 is not a 1-flat 2-vertex. It follows that none of e_1, e_2, e_3, e_5 contains a 1-flat vertex and hence $\mathrm{flat}(u) \leq d(u) - 4$. Then (6) holds.

Case 2: hu is a bad pair and $d(u) = 5$. By Lemma 4.9, one of the vertices v_i is not a 1-flat 2-vertex. Since $u \notin \mathbf{R}_2'$, u is adjacent to some other vertex $h' \in M$, say $\{u, h', z\} \in E(G)$, where $h' \neq h$, but z may be one of the v_i's. Since $d(u) = 5$, $1 \leq d(h'u) \leq 2$.

Case 2.1: $d(h'u) = 2$. By Lemma 4.9, either both edges containing h' and u are not 1-flat 2-vertices, implying $\mathrm{flat}(u) \leq 2$, or at least one of the edges containing h' and u is either reasonable or rich with recipient u, so $\mathrm{flat}(u) \leq 3$ and $\mathrm{rich}(u) + \frac{1}{2}\mathrm{reas}(u) \geq \frac{1}{2}$. In either case, (6) holds, so u is helpful.

Case 2.2: $d(h'u) = 1$. If $\{h', u, z\}$ is reasonable or rich with recipient u, then $\mathrm{flat}(u) \leq 3$, while $\mathrm{rich}(u) + \frac{1}{2}\mathrm{reas}(u) \geq \frac{1}{2}$, so (6) holds. Thus $\{h', u, z\}$ must contain either a 2-vertex or a 1-flat 3-vertex. However since $u \notin \mathbf{R}_3'$, z cannot be 1-flat, so z must be a 2-flat 2-vertex. This implies $\mathrm{flat}(u) \leq 3$.

Suppose first that $\{h, u, z\}$ is an edge, say $z = v_1$. In this case, $d(v_1) = 2$, so by Lemma 4.9, hu is contained in at least one edge that does not contain a 1-flat 2-vertex and is not $\{h, u, z\}$. This implies that $\mathrm{flat}(u) \leq 2$, so (6) holds. Thus $\{h, u, z\}$ is not an edge.

If $d(uz) = 2$, then since z is a 2-flat 2-vertex, again $\mathrm{flat}(u) \leq 2$, and so (6) holds. Thus $d(uz) = 1$. In this case, since $u \notin \mathbf{R}_5'$, $h'u$ and $h'z$ are bad pairs. So by Lemma 4.10, there exist vertices $a, b \in V(G)$ such that $\{u, a, b\}$ and $\{z, a, b\}$ are edges. Since z is 2-flat, one of a or b must be in M, say $a \in M$. Furthermore, since $d(ab) \geq 2$ and $u \notin \mathbf{R}_4'$, $d(b) \geq 3$. So $\{u, a, b\}$ is either reasonable or rich with recipient u. In either case, $\mathrm{rich}(u) + \frac{1}{2}\mathrm{reas}(u) \geq \frac{1}{2}$ while $\mathrm{flat}(u) \leq 3$ (as neither $\{h', u, z\}$ nor $\{u, a, b\}$ contain a 1-flat 2-vertex), so (6) holds, i.e., u is helpful.

Case 3: hu is a bad pair and $d(u) = 4$. Since $u \notin \mathbf{R}_4'$, u is adjacent to a vertex in M that is not h, say $\{u, h', z\}$ is an edge, where $h' \in M \setminus \{h\}$. Since $d(h'u) = 1$ and $u \notin \mathbf{R}_3'$, z cannot be 1-flat.

Case 3.1: z is a 2-flat 2-vertex. We claim that

$$h'z \text{ and } uz \text{ are bad pairs.} \tag{14}$$

Indeed, since $z \in N(u)$ and $u \notin \mathbf{R}_4'$, $d(h'z) = 1$. Hence if $d(uz) = 1$, then by Claim 4.5, $h'u$ and $h'z$ are bad pairs.

Suppose now $d(uz) = 2$, say $z = v_1$. Then h and h' are not adjacent since any edge containing h and h' cannot contain u or z and misses one of v_2 or v_3, so such an edge would create a $C_3^{(3)}$ in G with $\{h', u, z\}$ and either $\{h, u, v_2\}$ or $\{h, u, v_3\}$. If there was an h',z-link, it would have to use the edge $\{z, u, h\}$, but there is no edge connecting u or h to h' that does not contain z. Thus $h'z$ is bad. If there was an h',u-link, then it would contain z to avoid a $C_3^{(3)}$ in G with the edge $\{h', u, z\}$, and so the edge $\{h, u, z\}$ must be contained in the path, but since h and h' are not adjacent, there is no edge that can connect h' to one of h or z that does not contain u. Thus, $h'u$ is also bad. This proves (14).

Since $u \notin \mathbf{R}'_5$, by (14), there exist edges $\{u, a, b\}$ and $\{z, a, b\}$ for some $a, b \in V(G)$. As we already know all edges containing u, $\{u, a, b\} = \{h, u, v_i\}$ for some $i \in \{1, 2, 3\}$, assume without loss of generality $\{u, a, b\} = \{h, u, v_1\}$. Then consider the non-edge $\{h, u, z\}$, and note that this intersects both edges containing z in two vertices, so there should be an h, u-link. But hu is a bad pair, a contradiction.

Case 3.2: z is not a 2-flat 2-vertex. Then $\{h', u, z\}$ is either reasonable or rich with recipient u. By Lemma 4.9, one of the vertices v_i is not a 1-flat 2-vertex, so $\text{flat}(u) \leq 2$, and $\text{rich}(u) + \frac{1}{2}\text{reas}(u) \geq \frac{1}{2}$. Thus, (6) holds unless $\text{flat}(u) = 2$, $\text{rich}(u) = 0$, and $\text{reas}(u) = 1$. By symmetry, assume that v_1 is not a 1-flat 2-vertex, while v_2 and v_3 are. By Lemma 6.8, $d(uv_2) = d(uv_3) = 1$. Furthermore, Since $u \notin \mathbf{R}'_2$, hv_2 an hv_3 are good pairs. Consider first an h, v_2-link. To avoid a $C_3^{(3)}$ in G with $\{h, u, v_2\}$, u must be in this link. Since $d(uv_2) = 1$, either $\{h, u, v_1\}$ or $\{h, u, v_3\}$ must be in this link, but if $\{h, u, v_3\}$ was in this link, then the second edge would need to contain v_2 and v_3, which contradicts Lemma 6.2. Thus, the h, v_2-link must contain the edge $\{h, u, v_1\}$ and an edge $\{v_1, v_2, w_1\}$. Similarly, there is an h, v_3-link containing the edges $\{h, u, v_1\}$ and $\{v_1, v_3, w_2\}$ for some vertices w_1 and w_2, possibly equal. This gives us all the edges incident with v_2 or v_3.

Consider the non-edge $\{v_1, v_2, v_3\}$, and let T be a $C_3^{(3)}$ in $G + \{v_1, v_2, v_3\}$. Note that v_2 and v_3 cannot both be core vertices in T since the only edges containing v_2 or v_3 that do not contain two vertices in $\{v_1, v_2, v_3\}$ are $\{h, u, v_2\}$ and $\{h, u, v_3\}$, which share two vertices. Thus, v_1 is a core vertex in T, and we may assume by symmetry that v_2 is as well. Since v_2 is a core vertex, $\{h, u, v_2\}$ must be an edge of T, which implies that v_1 is in an edge containing one of h or u, but not both. If v_1 is in an edge containing u but not h, this edge must be $\{h', u, z\}$ (i.e. $v_1 = z$), but then $\{h, u, v_2\}$, $\{v_1, v_2, w_1\}$ and $\{v_1, u, h'\}$ form a $C_3^{(3)}$ in G, a contradiction. So there must be an edge containing v_1 and h, but not u. This implies that $d(hv_1) \geq 2$. If $d(hv_1) \geq 3$, then hv_1 is a rich pair so $\{h, u, v_1\}$ is reasonable, which makes $\text{reas}(u) = 2$, giving us that u helpful. Similarly, if $d(hv_1) = 2$, then $\{h, u, v_1\}$ is exceptional and thus reasonable, so again u is helpful. □

Claim 6.10. *Let $\{h, u, v\}$ be an exceptional edge such that $h \in M$, and $v \in L \setminus \mathbf{R}'_9$ is the exception in this edge. Then v is helpful.*

Proof. Let $\{h, v, z\}$ be the second edge in G that contains h and v, and let $\{h, u, w_1\}$ and $\{h, u, w_2\}$ be the remaining edges of G that contain the rich pair hu, and note that w_1 and w_2 must be 1-flat 2-vertices. By Lemma 6.8, $d(uw_1) = d(uw_2) = 1$. If at least one of hw_1 and hw_2 is a bad pair, then some of w_1 and w_2 is in $\mathbf{R}'_2$. In this case, $u \in \mathbf{R}'_3$ and hence $v \in \mathbf{R}'_4$, a contradiction. Thus hw_1 and hw_2 are good pairs. Since Lemma 6.2 implies that w_1 and w_2 are not adjacent, and any h, w_1-link must contain u to avoid a $C_3^{(3)}$ in G with the edge $\{h, u, w_1\}$, any h, w_1-link must use the edge $\{h, u, v\}$ and an edge $\{v, w_1, z_1\}$. Similarly, any h, w_2-link must use the edge $\{h, u, v\}$ and an edge $\{v, w_2, z_2\}$. We claim that $z = z_1 = z_2$. If $z \neq z_1$, then $\{h, v, z\}$, $\{h, u, w_1\}$ and $\{v, w_1, z_1\}$ for a $C_3^{(3)}$ in G, a contradiction. Similarly if $z \neq z_2$, then $\{h, v, z\}$, $\{h, u, w_2\}$ and $\{v, w_2, z_2\}$ form a $C_3^{(3)}$ in G, again a contradiction. Thus, $z = z_1 = z_2$. This implies that $d(z) \geq 3$. We will consider cases based on $d(z)$.

Case 1: $d(z) \geq 4$. Then $\{h, v, z\}$ is either reasonable or rich with recipient v. Along with the reasonable edge $\{h, u, v\}$, this gives us that $\frac{1}{2}\text{reas}(v) + \text{rich}(v) \geq 1$, while $\text{flat}(v) \leq d(v) - 2$, which along with $d(v) \geq 4$ is sufficient to satisfy (6), so v is helpful.

Case 2: $d(z) = 3$. Then $d(vz) = d(z)$, so by Lemma 4.8, $d(z) \geq 5$. Since $\{h, v, z\}$ does not contain a 1-flat 2-vertex and $\{h, u, z\}$ is reasonable, $\text{flat}(v) \leq d(v) - 2$, while $\text{reas}(v) \geq 1$ and $d(v) \geq 5$, which implies (6), so v is helpful. □

Lemma 6.11. *All 2^+-flat 5-vertices outside of $\mathbf{R}'_9$ are helpful.*

Proof. Suppose there is a 2^+-flat 5-vertex $u \notin \mathbf{R}'_9$ that is not helpful. By Lemma 6.5, u is unsupported. By Lemma 6.9 and Claim 6.10, we can assume u is not rich or exceptional. Furthermore, by Lemma 6.4, any non-low edge containing u must not contain a 1-flat 2-vertex, since this would imply u is rich. Thus, if u is 3^+-flat, then $\text{flat}(u) \leq 2$, and (6) is satisfied. Thus, we may assume u is 2-flat. Furthermore, since $\text{flat}(u) \leq 3$, if u is in at least one reasonable edge or a rich edge with recipient u, then (6) is satisfied, so we must have $\text{reas}(u) = \text{rich}(u) = 0$. We will consider cases based on if u has one or two neighbors in M.

Case 1: There exists a vertex $h \in M$ such that $d(uh) = 2$. Let $\{h, u, v_1\}$ and $\{h, u, v_2\}$ be edges of G. Note that neither v_1 nor v_2 are 1-flat 2-vertices, and furthermore by Lemma 6.7, v_1 and v_2 are not 2-flat 2-vertices. Then since $\text{reas}(u) = \text{rich}(u) = 0$, v_1 and v_2 must be 1-flat 3-vertices. Note that if $\{u, v_1, v_2\}$ is an edge, then $\text{flat}(u) \leq 2$, and hence u is helpful, so $\{u, v_1, v_2\} \notin E(G)$.

Since u is 2-flat, by the case, h is the only non-low neighbor of u, so by Lemma 4.6 hu must be a good pair. Any h, u-link L must contain both v_1 and v_2, but $\{v_1, v_2, u\}$ is not an edge of G, so the edge in L that contains h must contain v_1 or v_2, contradicting the fact that they both are 1-flat.

Case 2: There exist distinct vertices $h, h' \in M$ such that $d(hu) = d(h'u) = 1$. Let $\{h, u, v_1\}$ and $\{h', u, v_2\}$ be edges in G, noting that we may have $v_1 = v_2$. Since $u \notin \mathbf{R}'_3$, v_1 cannot be 1-flat, and so v_1 is a 2-flat 2-vertex. Similarly, v_2 is a 2-flat 2-vertex.

Case 2.1: $v_1 = v_2$. Let $\{u, t_1, w_1\}$, $\{u, t_2, w_2\}$ and $\{u, t_3, w_3\}$ be the low edges containing u with t_1, t_2 and t_3 being 1-flat 2-vertices. Note that while all t_i are distinct, the w_i's do not need be. First consider if $d(uw_i) = 1$ for some $i \in \{1, 2, 3\}$. In this case, consider the non-edge $\{v_1, u, t_i\}$. It intersects all edges containing v_1 in two vertices, so there is a u, t_i-link . Let $\{u, t_{i'}, w_{i'}\}$ be the edge of this link containing u, and note that the second edge must be of the form $\{t_i, w_{i'}, z\}$ since t_i and $t_{i'}$ are not adjacent by Lemma 6.2 and the fact that t_i and $t_{i'}$ are not in $\mathbf{R}'_5$. However, in this case, $\{u, t_i, w_i\}$, $\{u, t_{i'}, w_{i'}\}$ and $\{t_i, w_{i'}, z\}$ form a $C_3^{(3)}$ in G, where all these vertices are distinct because $d(uw_i) = 1$, and z must actually be a vertex in M. Thus, we cannot have $d(uw_i) = 1$ for any $i \in \{1, 2, 3\}$. This implies that $w_1 = w_2 = w_3$. In this case, consider the non-edge $\{v_1, u, w_1\}$. It intersects every edge containing v_1 or u in at least two vertices, so adding this non-edge cannot create a $C_3^{(3)}$ in G, a contradiction.

Case 2.2: $v_1 \neq v_2$. In this case, $d(uv_1) = d(hv_1) = 1$, so by Claim 4.5, hu and hv_1 are bad pairs. Then since $u \notin \mathbf{R}'_5$, there exist vertices $a, b \in V(G)$ such that $\{u, a, b\}$ and $\{v_1, a, b\}$ are edges in G. Since u is 2-flat and v_1 is a 2-flat 2-vertex, $\{u, a, b\}$ must be the edge $\{h', u, v_2\}$. But then $d(h'v_2) = 2 = d(v_2)$, which contradicts the fact that $u \notin \mathbf{R}'_4$. $\square$

Lemma 6.12. *All 2^+-flat 4-vertices outside of $\mathbf{R}'_9$ are helpful.*

Proof. Suppose there is a 2^+-flat 4-vertex $u \notin \mathbf{R}'_9$ that is not helpful. By Lemma 6.5, u is unsupported. By Lemma 6.9 and Claim 6.10, we are done if u is rich or exceptional, so we may assume u is neither. Let $h \in M$ be a neighbor of u.

Case 1: u is 4-flat. In this case, u is only in high edges, so if u has a 1-flat 2-neighbor, u is rich by Lemma 6.4, contradicting our earlier assumption. Thus u has no 1-flat 2-neighbors, and is trivially helpful.

Case 2: u is 3-flat. Since u is not special, it has a neighbor $h \in H$. If h is the only non-low neighbor of u, then hu is a rich pair, and we are done by Lemma 6.9. Thus assume u is adjacent to at least two vertices in M, and consequently $d(h'u) = 1$ for some $h' \in M$ (possibly $h = h'$), say $\{h', u, z\}$ is an edge for some $z \in L$. Furthermore, none of the non-low edges containing u can contain a 1-flat 2-vertex, so only the single low edge containing u may contain a 1-flat 2-vertex. Thus, since u is not helpful, $\text{flat}(u) = 1$ and $\text{rich}(u) = \text{reas}(u) = 0$. This implies that z is either a 2-vertex or a 1-flat 3 vertex, but since $u \notin \mathbf{R}'_3$, z is not 1-flat. So z is a 2-flat 2-vertex. By Claim 6.7 $d(uz) = 1$, and by Claim 4.3 $d(h'z) = 1$, so by Claim 4.5, $h'u$ and $h'z$ are bad pairs. Thus since $u \notin \mathbf{R}'_5$, there exist vertices a and b such that $\{u, a, b\}$ and $\{z, a, b\}$ are edges. Since z is 2-flat, one of a and b is in M, say $a \in M$. Then $d(b) \geq 3$ by Claim 4.3 since $d(ab) \geq 2$, and thus edge $\{u, a, b\}$ is either reasonable or rich with recipient u. In both cases we get a contradiction with the fact that $\text{rich}(u) = \text{reas}(u) = 0$.

Case 3: u is 2-flat and $d(hu) = 2$. Let $\{h, u, v_1\}$ and $\{h, u, v_2\}$ be the edges of G containing h and u.

Case 3.1: $\{h, v_1, v_2\} \in E(G)$. Then neither v_1 nor v_2 is 1-flat. Moreover, since $u \notin \mathbf{R}'_4$, $d(v_1), d(v_2) \geq 3$. Thus, the edges $\{h, u, v_1\}$ and $\{h, u, v_2\}$ are either reasonable or rich with recipient u, so we have $\text{rich}(u) + \frac{1}{2}\text{reas}(u) \geq \frac{1}{2}\text{flat}(u)$. Then (6) holds and u is helpful. Thus, below we assume $\{h, v_1, v_2\} \notin E(G)$.

Case 3.2: $\{u, v_1, v_2\} \in E(G)$. Since u is unsupported, by Lemma 6.8 neither v_1 nor v_2 has degree 2.

Suppose both v_1 and v_2 are 1-flat 3-vertices. Since $u \notin \mathbf{R}'_4$, $d(uv_1) < d(v_1) = 3$ and $d(uv_2) < d(v_2) = 3$. Hence $d(uv_1) = d(uv_2) = 2$. Since $u \notin \mathbf{R}'_2$, hv_1 is a good pair, and any h, v_1-link needs to contain u to avoid a $C_3^{(3)}$ in G. The only edges containing u that could be in this link are $\{h, u, v_2\}$ and $\{u, v_1, v_2\}$. Edge $\{u, v_1, v_2\}$ cannot be there since neither u nor v_2 is in an edge with h that contains only one of these vertices, and if $\{h, u, v_2\}$ is in this link, then the other edge of the link must be $\{v_1, v_2, z\}$ for some $z \notin \{h, u, v_1, v_2\}$. But then the non-edge $\{h, v_1, v_2\}$ intersects all edges containing either v_1 or v_2 in at least two vertices, so neither can be a core vertex in the $C_3^{(3)}$ in $G + \{h, v_1, v_2\}$, a contradiction. Thus, at least one of v_1 or v_2 is not a 1-flat 3-vertex, say v_1 is not.

Then $\{h, v_1, u\}$ is either reasonable or a rich edge with recipient u, and $\text{flat}(u) \le 1$ since $\{h, v_1, u\}$, $\{h, v_2, u\}$ and $\{u, v_1, v_2\}$ all do not contain a 1-flat 2-vertex. Thus u is helpful since $\text{rich}(u) + \frac{1}{2}\text{reas}(u) \ge \frac{1}{2}\text{flat}(u)$, i.e., (6) is satisfied.

Case 3.2: $\{u, v_1, v_2\} \notin E(G)$. Since $u \notin \mathbf{R}'_2$, hu is a good pair. In particular, there is an h, u-link, say with edges e_1 and e_2, where $h \in e_1$ and $u \in e_2$. To avoid a $C_3^{(3)}$ in G with $\{h, u, v_1\}$ and $\{h, u, v_2\}$, v_1 and v_2 must both be in this link. Since $\{h, v_1, v_2\}, \{u, v_1, v_2\} \notin E(G)$, we can assume without loss of generality that $e_1 = \{h, v_1, z\}$ and $e_2 = \{z, v_2, u\}$ for some $z \in V(G) \setminus \{h, u, v_1, v_2\}$.

If z is a 1-flat 2-vertex, consider the non-edge $\{h, u, z\}$. Since this non-edge intersects all edges containing z in two vertices, there is an h, u-link that avoids z. Again, this link must contain v_1 and v_2 to avoid a $C_3^{(3)}$ in G. If this link has edges $\{h, v_1, z'\}$ and $\{z', v_2, u\}$ for $z' \ne z$, then $d(hv_1) \ge 3$ so v_1 is not a 2-vertex or 1-flat 3-vertex, and $d(uv_2) \ge 3$. So by Claim 4.3, $d(v_2) \ge 4$ since $d(u) = 4$, and thus the edges $\{h, u, v_1\}$ and $\{h, u, v_2\}$ are either rich with recipient u or reasonable. The other possible h, u-link that avoids z is a path with edges $\{h, v_2, z'\}$ and $\{z', u, v_1\}$ for some $z' \ne z$. In this case, $d(hv_1) \ge 2$ and $d(hv_2) \ge 2$, so since $u \notin \mathbf{R}'_4$, v_1 and v_2 are not 2-vertices. Since they also are not 1-flat, $\{h, v_1, u\}$ and $\{h, v_2, u\}$ are either rich with recipient u or reasonable. Furthermore, in either of these two cases, $\text{flat}(u) \le 2$, so u is helpful since $\text{rich}(u) + \frac{1}{2}\text{reas}(u) \ge \frac{1}{2}\text{flat}(u)$ implies (6). Thus, we are done if z is a 1-flat 2-vertex.

If z is not a 1-flat 2-vertex, then $\text{flat}(u) \le 1$, and since $d(hv_1) \ge 2$ and $u \notin \mathbf{R}'_4$, v_1 is not a 2-vertex and not 1-flat. So $\{h, v_1, u\}$ is either rich with recipient u or reasonable. Therefore, again u is helpful since $\text{rich}(u) + \frac{1}{2}\text{reas}(u) \ge \frac{1}{2}\text{flat}(u)$.

Case 4: u is 2-flat and not a double neighbor with any vertices in M. Let $h, h' \in M$ be neighbors of u and let $\{h, u, z\}$ and $\{h', u, z'\}$ be edges in G. If either of these edges are rich with recipient u or if both of them are reasonable, then $\text{rich}(u) + \frac{1}{2}\text{reas}(u) \ge \frac{1}{2}\text{flat}(u)$, and so u is helpful. Thus we may assume at least one of z or z' is either a 2-vertex or a 1-flat 3-vertex, assume z is such a vertex. Since $d(hu) = 1$ and since $u \notin \mathbf{R}'_3$, z is not 1-flat, and thus must be a 2-flat 2-vertex.

Case 4.1: $d(uz) = 2$. For u to not be helpful, we must have $\text{flat}(u) \ge 1$. Let $\{u, t, w\}$ be a low edge containing u with t a 1-flat 2-vertex, and let $\{u, a_1, a_2\}$ be the other low edge containing u. Note that $t, w \notin \{h, h', z\}$, and by Lemma 6.8, $t \notin \{a_1, a_2\}$. Consider the non-edge $\{u, t, z\}$. Since this non-edge intersects every edge containing z in two vertices, there a u, t-link. The only edge containing u that can be in this link is $\{u, a_1, a_2\}$. So the other edge must connect a_1 or a_2 to t, say this edge is $\{t, a_1, h''\}$ (note that $h'' \in M$ since t is adjacent to some vertex in M), but then we may assume $w \in \{a_1, a_2\}$ since otherwise $\{u, t, w\}$, $\{u, a_1, a_2\}$ and $\{t, a_1, h''\}$ form a $C_3^{(3)}$ in G. Then consider the non-edge $\{u, t, w\}$, and note that this non-edge intersects every edge containing either u or t in two vertices, so adding $\{u, t, w\}$ to G does not create a $C_3^{(3)}$, a contradiction.

Case 4.2: $d(uz) = 1$. Since $u \notin \mathbf{R}'_4$, $d(zh) = 1$, so by Claim 4.5, hu and hz are bad pairs, and thus since $u \notin \mathbf{R}'_5$, there exist vertices $a, b \in V(G)$ such that $\{u, a, b\}$ and $\{z, a, b\}$ are edges of G. Since u is 2-flat and z is a 2-flat 2-vertex, $\{u, a, b\}$ must be the edge $\{h', u, z'\}$. This implies that $d(h'z) \ge 2$. If $d(h'z) \ge 3$ or if $d(z) \ge 8$, then $\text{rich}(u) \ge 1$, and $\text{flat}(u) \le 2$, so (6) holds. On the other hand, if $d(h'z) = 2$ and $d(z) \le 7$, then this contradicts Lemma 6.7. □

Let R_{10} be the set of 1-flat 2-vertices t such that the low edge $\{u, v, t\}$ containing t satisfies the following:

(a) none of u, v or t is in $\mathbf{R}'_9$;

(b) for some $4 \le k \le 5$, u is a 1-flat unsupported k-vertex;

(c) v is not helpful;

(d) the high vertex h_1 adjacent to t is distinct from the high vertex h_2 adjacent to u.

Let $\mathbf{R}_{10} = R_{10} \cup \mathbf{R}'_9$, and let $\mathbf{R} = \mathbf{R}_{10} \cup (L \cap N(L_8 \cap \mathbf{R}_{10}))$.

Lemma 6.13. *$|R_{10}| \leq n/\ell$.*

Proof. Let $\mathcal{F}$ be the set of low edges $\{u, v, t\}$ satisfying (a)–(d) for some 1-flat 2-vertex t. Let $\mathcal{F}_1$ be the set of edges in $\mathcal{F}$ where $h_1 = x$ and $h_2 = y$. If the lemma does not hold, then by the symmetry between x and y, we may assume that $|\mathcal{F}_1| \geq \frac{n}{2\ell}$. For $e_1 = \{u, v, t\} \in \mathcal{F}_1$, let $e_2 = \{x, t, v_1\}$ be the unique high edge containing t and $e_3 = \{y, u, u_1\}$ be the unique high edge containing u. By Lemma 6.4,

$$v_1 \textit{ is rich or supported.} \tag{15}$$

So if $v_1 = v$, then v is rich or supported, and thus helpful by either Lemma 6.9 or Lemma 6.5, contradicting (c). So, $v_1 \neq v$. Since $x \notin N(u)$, $v_1 \neq u$. If $v_1 = u_1$, then since $v_1 \neq v$, the edges e_1, e_2 and e_3 form a triangle, a contradiction. Thus, $v_1 \notin \{v, u, u_1\}$.

Since $t \notin \mathbf{R}'_2$, xt is a good pair. Let edges g_1 and g_2 form a x, t-link. Since $d(t) = 2$, $g_1 = e_1$. Since $x \notin N(u)$, g_2 has the form $\{v, x, w\}$ for some vertex w. If $w \neq v_1$, then e_1, e_2 and g_2 form a triangle in G. Hence $w = v_1$. Since each of e_2 and g_2 contains only one vertex in M, (15) yields that $d(v_1) \geq 3$.

If $v = u_1$, then v is a 2^+-flat vertex with $\mathrm{flat}(v) \leq d(v) - 2$ (because of e_3 and g_2) and $\mathrm{rich}(v) \geq 1$ (by (15)). Moreover, if $d(v) = 3$, then e_3 is reasonable. This contradicts the condition that v is not helpful. Thus $v \neq u_1$.

Suppose pair vt is good and edges f_1 and f_2 form a v, t-link. Since $d(t) = 2$, $f_1 = \{x, t, v_1\}$. In order for the edges f_1, f_2 and e_1 not to form a triangle, $u \in f_1 \cup f_2$. Since $u \notin N(x)$, $x \in f_2 \setminus f_1$, which yields $f_2 = \{x, v, v_1\}$. Now, consider adding the non-edge $e_5 = \{u, t, v_1\}$. Since e_5 shares two vertices with each of the two edges containing t, G must have edges f_3 and f_4 forming a u, v_1-link. In view of f_2, $v \in f_3 \cup f_4$. Since $u \notin N(x)$, $f_4 \neq g_2$. It follows that $d(v) \geq 4$ (because v is contained in e_1, f_2, g_2 and at least one of f_3 and f_4), $\mathrm{flat}(v) \leq d(v) - 2$ (because of f_2 and g_2), and $\mathrm{rich}(v) \geq 1$ (because of g_2). So,

$$d(v) + \mathrm{rich}(v) - \frac{1}{2}\mathrm{flat}(v) - 4 \geq \frac{1}{2}d(v) + 1 + 1 - 4 \geq 0, \tag{16}$$

contradicting (c). Thus for each $e_1 = \{u, v, t\} \in \mathcal{F}_1$, the pair vt is bad.

Since $u \notin \mathbf{R}'_4$, either $d(u_1) > 3$ or $d(u_1) > d(yu_1)$.

If G has an edge $e_6 = \{y, v, w_2\}$ containing y and v, then in order not to have triangle with edges e_6, e_1 and e_3, $w_2 \in \{u, t, u_1\}$. On the other hand, we know both edges of G containing t, so $w_2 \neq t$, and we know that unique edge containing $\{y, u\}$ does not contain w. Thus, $w_2 = u_1$. Since $d(yu_1) \geq 2$, $d(u_1) \geq 3$. Therefore, $\mathrm{flat}(v) \leq d(v) - 2$ (because of e_6 and g_2), and $\mathrm{rich}(v) \geq 1$ (because of g_2). So, if $d(v) \geq 4$, then as in (16), v is helpful, contradicting (c). Moreover, if $d(v) = 3$, then v is not rich, and hence either e_6 is reasonable or u_1 is rich, so

$$d(v) + \mathrm{rich}(v) + \frac{1}{2}\mathrm{reas}(v) - \frac{1}{2}\mathrm{flat}(v) - 4 \geq \frac{1}{2}d(v) + 1 + 1 + \frac{1}{2} - 4 = 0.$$

It follows that $y \notin N(v)$.

Since u is 1-flat, at most 9ℓ vertices not adjacent to y are at distance at most 2 from u. Recall that for each $e'_1 = \{u', v', t'\} \in \mathcal{F}_1$, v' and t' are not adjacent to y. So, since $|\mathcal{F}_1| \geq \frac{n}{2\ell}$, there is an edge $e'_1 = \{u', v', t'\} \in \mathcal{F}_1$ such that the distance from u to $\{v', t'\}$ is at least 3. Then, since $v't'$ is a bad pair, adding the non-edge $\{v', t', u\}$ to G does not create a triangle, a contradiction. $\square$

For ease of notation, let us define $\mathbf{R} = \mathbf{R}_{10}$.

Lemma 6.14. *The number of vertices in $\mathbf{R}$ is $o(n)$.*

Proof. By definition, $\mathbf{R}$ is a subset of the set formed by $\bigcup_{i=1}^{10} R_i$ and the vertices in L which we can reach from this union by paths of length at most 10 via vertices in L_8. According to Remark 4.2, Claim 6.1 and Lemmas 4.6, 4.7, 4.8, 4.10, 5.1, 5.2, 6.6, and 6.13, $|R_i| \leq 2000n/\ell$ for each $1 \leq i \leq 10$.

Each vertex of degree at most 8 has at most 16 neighbors. Hence the total number of vertices reachable from $\bigcup_{i=1}^{10} R_i$ via paths of length at most 10 in which all vertices apart from the last ones are in L_8 is at most

$$\left|\bigcup_{i=1}^{10} R_i\right| \cdot \sum_{j=0}^{10} 16^j \leq 20000(n/\ell)16^{11} = o(n).$$

□

Lemma 6.15. *Every 1-flat vertex $u \notin \mathbf{R}$ of degree 4 or 5 is half-helpful.*

Proof. Assume that a 1-flat vertex $u \notin \mathbf{R}$ of degree 4 or 5 is not half-helpful. If u is in a $(M, u, 2)$ edge with a 1-flat 2-vertex, then by Lemma 6.4, u is not 1-flat. So the only edges that contain u and a 1-flat 2-vertex are low edges. If u is in no such low edges, then $\text{donor}(u) = 0$, so u satisfies (7) trivially. Thus we may assume u is in at least one edge with a 1-flat 2-vertex, and a second vertex that is not helpful. Say $\{u, w, t\} \in E(G)$, where t is a 1-flat 2-vertex and w is not helpful. By Lemma 6.13, u and t must both be adjacent to x or both be adjacent to y, say $h \in \{x, y\}$ is this common neighbor of u and t.

Let $\{h, t, z\}$ be the second edge of G containing t, and note that hz is a rich pair by Lemma 6.4. Thus, if $z \in \{u, w\}$, this would imply either hu or hw is a rich pair, and thus $\{h, w, t\}$ contains a helpful vertex by Lemma 6.9. So, we may assume $z \notin \{u, w\}$. Let $\{h, u, a\}$ be the non-low edge containing u. To avoid a $C_3^{(3)}$ in G with $\{h, t, z\}$ and $\{u, t, w\}$, we must have $a \in \{z, w\}$.

Case 1: $a = z$. Then $\{h, u, z\}$ is a rich edge with recipient u, so $\text{rich}(u) \geq 1$. If $d(u) = 5$, then u satisfies (7) even if $\text{donor}(u) = 4$, thus $d(u) = 4$. Furthermore, even if $d(u) = 4$, u satisfies (7) unless $\text{donor}(u) = 3$. Let $\{u, t', w'\}$ be a second edge containing u, a 1-flat 2-vertex t' and a non-helpful vertex w'. By Lemma 6.13, t' and u must share a high neighbor, in particular h. Let $\{h, t', z'\} \in E(G)$.

If $z' = w'$, then since $t' \notin \mathbf{R}'_5$, $d(w') \geq 4$. Furthermore, as in this case $d(w't') = 2$, by Lemmas 6.8 and 6.6, we actually have $d(w') \geq 6$. However, then by Lemma 6.5 w' is helpful, contradicting our earlier assumption. Thus $z' \neq w'$.

Then the only way the edges $\{h, u, z\}$, $\{h, t', z'\}$ and $\{u, w', t'\}$ do not form a $C_3^{(3)}$ in G is that $z' = z$. Consider the non-edge $\{u, t, t'\}$, and let T be a $C_3^{(3)}$ in $G + \{u, t, t'\}$. Note that t and t' cannot both be core vertices in T since the only edges containing t and t' that do not intersect $\{u, t, t'\}$ in two vertices are $\{h, z, t\}$ and $\{h, z, t'\}$, which do not form a link. Thus, u is a core vertex of T, along with one of t and t', say with t. In this case, the edge $\{h, t, z\}$ must be one of the edges of T, and the second edge must contain u and exactly one vertex in $\{h, z\}$. As u is 1-flat, this second edge does not contain h, so there must be some edge $\{u, z, b\}$. Since $\text{donor}(u) = 3$, then we must actually have that b is a 1-flat 2-vertex and z is not helpful. But hz is a rich pair, so we arrive at a contradiction to Lemma 6.9.

Case 2: $a = w$. Since $u \notin \mathbf{R}'_3$, $d(hw) \geq 2$, say $\{h, w, b\}$ is an edge. To avoid a $C_3^{(3)}$ in G, we must have $b = z$. Since w is not helpful, by Lemma 6.9, $d(hw) = 2$. Furthermore, since w is 2^+-flat, by Lemmas 6.11, 6.12 and 6.5, w is helpful unless $d(w) \leq 3$. Furthermore, if $d(w) \leq 3$, then by Lemma 6.2, $d(w) = 3$. However, in this case, the non-edge $\{h, w, t\}$ intersects every edge containing either w or t in two vertices, so adding $\{h, w, t\}$ to G does not create a $C_3^{(3)}$, a contradiction. □

7 Lower Bound: Discharging and Final Proof

We are now ready to present our discharging rules and prove the lower bound. Recall that every vertex in G starts with charge equal to their degree. We then move charge around in G according to the following rules:

(D1) A (M, M, L) edge $\{h, h', u\}$ with $u \in L$ removes charge 1 from each of h and h' and gives charge 2 to u.

(D2) A needy (M, L, L) edge $\{h, u, t\}$, where $h \in M$ and t is a 2-vertex or a 1-flat 3-vertex removes charge 1 from h and gives charge 1 to t.

(D3) A rich (M, L, L) edge $\{h, u, v\}$ with $h \in M$ and v the recipient removes charge 1 from h and gives charge 1 to v.

(D4) A reasonable (M, L, L) edge $\{h, u, v\}$ with $h \in M$ removes charge 1 from h and gives charge 1/2 to each of u and v.

(D5) All helpful vertices give charge 1/2 to each of their 1-flat 2-neighbors (with multiplicity).

(D6) All vertices that are not helpful but are half-helpful give charge 1/2 to each of their 1-flat 2-neighbors that are not at charge at least 4 after the application of rules (D1) through (D5).

Now we will use the above discharging scheme to bound the number of edges in G.

Theorem 7.1. *The saturation number*

$$\mathrm{sat}_3(n, C_3^{(3)}) \geq \frac{4n}{3} - o(n).$$

Proof. We first prove that the average degree of G is at least $4 - o(1)$. Note that since $\ell = \omega(n)$, $|E(G)| \leq 2n$ and M only contains vertices with degree at least ℓ, $|M| = o(n)$, and consequentially $|L| = n - o(n)$.

We claim that after applying (D1)-(D6), no vertices end up with negative charge. Indeed, first note that vertices in M only lose charge 1 for each edge they are in, so since the initial charge is their degree, vertices in M end up with non-negative charge. Furthermore, vertices in L that are not helpful or half-helpful do not give away charge, so end up with charge at least equal to their degree. Finally, vertices in L that are helpful or half-helpful end up with at least charge 4 by the way they are defined. Thus, no vertex ends up with negative charge after applying our discharging scheme.

Now, we claim that all vertices in $L \setminus \mathbf{R}$ end up with charge at least 4. Let $v \in L \setminus \mathbf{R}$. Note that (D1)-(D6) only removes charge from low vertices if they are helpful or half-helpful, and by the definitions of helpful and half-helpful, these vertices are left with at least charge 4, so we may assume v is not helpful or half-helpful, and thus v does not give out any charge. Now, let us consider cases based on $d(v)$.

Case 1: $d(v) \leq 1$. Then v is in R_1 or in R_4'.

Case 2: $d(v) = 2$. We need to show that v receives charge at least 2. If v is supported, (D1) gives v at least charge 2. If v is not supported but is 2-flat, then v receives charge 1 from each non-low edge containing v via (D2). If v is 0-flat, then v is in R_1. Finally, if v is 1-flat and not in $\mathbf{R}$, then the non-low edge containing v first gives v charge 1 via (D2), and then by Lemmas 6.4 and 6.9, this non-low edge contains a helpful vertex, and thus it also gives v charge 1/2 via (D5). The final 1/2 of charge v needs comes from the low edge containing v. Indeed, by Lemma 6.3, this low edge contains a vertex of degree at least 4, and by Lemma 6.5 and Lemmas 6.12, 6.11 and 6.15, this vertex is helpful or half-helpful so gives v charge 1/2 via (D4) or (D5). This gives v total charge at least 4.

Case 3: $d(v) = 3$. Then v needs to get charge at least 1. If v is 1-flat, then v gets charge 1 via its non-low edge by (D2). If v is supported, then it gets charge 2 via (D1), if v is 2-flat and not supported, then by Corollary 4.13, v is not in a non-low edge with any 2-vertices, and since $v \notin \mathbf{R}_6'$, v is not in a non-low edge with a 1-flat 3-vertex. So both non-low edges containing v are either reasonable or rich with recipient v, thus each of these edges gives v charge 1/2 by (D3) or (D4). Finally if v is 3-flat and not supported, then by Corollary 4.13, v has no 2-neighbors, and since $v \notin \mathbf{R}_7'$, v has at most one 1-flat 3-neighbor. Thus, v is in at least two non-low edges that do not contain 1-flat 3-vertices. These edges will each give charge at least 1/2 to v as long as they are not rich with a recipient that is not v. But since v is not supported and $d(v) < 8$, the only way such an edge could be rich with a recipient that is not v is if hv is a rich pair for some $h \in M$. This is not the case since $v \notin \mathbf{R}_4'$. Thus, v gets charge at least 1/2 via the two non-low edges that contain v and no 1-flat 3-vertex by either (D3) or (D4).

Case 4: $d(v) \geq 4$. Since $v \notin R_1$, v is 1^+-flat, and thus by Lemma 6.5 and Lemmas 6.12, 6.11 and 6.15, v is either helpful or half-helpful. By the definition of helpful or half-helpful, even after donating some charge via (D5) or (D6), v is left with at least charge 4.

Thus, every vertex in $L \setminus \mathbf{R}$ ends up with charge at least 4 and all other vertices end up with non-negative charge. Since the total charge is equal to the total degree, we have that the average degree of G is at least

$$\frac{4|L \setminus \mathbf{R}|}{n} \geq \frac{4((n - o(n)) - o(n))}{n} = 4 - o(1).$$

Consequently,

$$\mathrm{sat}(n, C_3^{(3)}) = |E(G)| \geq \frac{(4 - o(1))n}{3} = \frac{4n}{3} - o(n).$$

□

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References

[1] B. Austhof and S. English. Nearly-regular hypergraphs and saturation of Berge stars. *Electron. J. Combin.*, 26(4):Paper No. 4.49, 11, 2019.

[2] M. Axenovich and C. Winter. A note on saturation for Berge-G hypergraphs. *Graphs Combin.*, 35(4):933–939, 2019.

[3] B. Bollobás. On generalized graphs. *Acta Math. Acad. Sci. Hungar.*, 16:447–452, 1965.

[4] J. Bondy. Variations on the Hamiltonian theme. *Canad. Math. Bull.*, 15:57–62, 1972.

[5] Y. Chen. Minimum C_5-saturated graphs. *J. Graph Theory*, 61(2):111–126, 2009.

[6] Y. Chen. All minimum C_5-saturated graphs. *J. Graph Theory*, 67(1):9–26, 2011.

[7] D. Cranston and D. West. An introduction to the discharging method via graph coloring. *Discrete Math.*, 340(4):766–793, 2017.

[8] S. Dow, D. Drake, Z. Füredi, and J. Larson. A lower bound for the cardinality of a maximal family of mutually intersecting sets of equal size. In *Proceedings of the sixteenth Southeastern international conference on combinatorics, graph theory and computing (Boca Raton, Fla., 1985)*, volume 48, pages 47–48, 1985.

[9] A. Dudek and A. Żak. On hamiltonian chain saturated uniform hypergraphs. *Discrete Math. Theor. Comput. Sci.*, 14(1):21–28, 2012.

[10] S. English, D. Gerbner, A. Methuku, and M. Tait. Linearity of saturation for Berge hypergraphs. *European J. Combin.*, 78:205–213, 2019.

[11] S. English, P. Gordon, N. Graber, A. Methuku, and E. Sullivan. Saturation of Berge hypergraphs. *Discrete Math.*, 342(6):1738–1761, 2019.

[12] P. Erdős, A. Hajnal, and J. Moon. A problem in graph theory. *Amer. Math. Monthly*, 71:1107–1110, 1964.

[13] J. Faudree, R. Faudree, R. Gould, and M Jacobson. Saturation numbers for trees. *Electron. J. Combin.*, 16(1):Research Paper 91, 19, 2009.

[14] J. Faudree, R. Faudree, and J. Schmitt. A survey of minimum saturated graphs. *Electron. J. Combin.*, DS19(Dynamic Surveys):Paper No. DS19, 36, 2011.

[15] M. Ferrara, M. Jacobson, K. Milans, C. Tennenhouse, and P. Wenger. Saturation numbers for families of graph subdivisions. *J. Graph Theory*, 71(4):416–434, 2012.

[16] Z. Füredi and Y. Kim. Cycle-saturated graphs with minimum number of edges. *J. Graph Theory*, 73(2):203–215, 2013.

[17] D. Gerbner, B. Patkós, Zs. Tuza, and M. Vizer. On saturation of Berge hypergraphs. *European J. Combin.*, 102:Paper No. 103477, 7, 2022.

[18] R. Gould, T. Ł uczak, and J. Schmitt. Constructive upper bounds for cycle-saturated graphs of minimum size. *Electron. J. Combin.*, 13(1):Research Paper 29, 19, 2006.

[19] L. Kászonyi and Zs. Tuza. Saturated graphs with minimal number of edges. *J. Graph Theory*, 10(2):203–210, 1986.

[20] G. Katona and H. Kierstead. Hamiltonian chains in hypergraphs. *J. Graph Theory*, 30(3):205–212, 1999.

[21] X. Lin, W. Jiang, C. Zhang, and Y. Yang. On smallest maximally non-Hamiltonian graphs. *Ars Combin.*, 45:263–270, 1997.

[22] Y. Ma, X. Hou, D. Hei, and J. Gao. Minimizing the number of edges in $\mathcal{C}_{\geq r}$-saturated graphs. *Discrete Math.*, 344(11):Paper No. 112565, 14, 2021.

[23] L. Ollmann. $K_{2,2}$ saturated graphs with a minimal number of edges. In *Proceedings of the Third Southeastern Conference on Combinatorics, Graph Theory, and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1972)*, pages 367–392, 1972.

[24] O. Pikhurko. The minimum size of saturated hypergraphs. *Combin. Probab. Comput.*, 8(5):483–492, 1999.

[25] O. Pikhurko. Results and open problems on minimum saturated hypergraphs. *Ars Combin.*, 72:111–127, 2004.

[26] A. Ruciński and A. Żak. Upper bounds on the minimum size of Hamilton saturated hypergraphs. *Electron. J. Combin.*, 23(4):Paper 4.12, 26, 2016.

[27] Zs. Tuza. Applications of the set-pair method in extremal hypergraph theory. In *Extremal problems for finite sets (Visegrád, 1991)*, volume 3 of *Bolyai Soc. Math. Stud.*, pages 479–514. János Bolyai Math. Soc., Budapest, 1994.

[28] M. Zhang, S. Luo, and M. Shigeno. On the number of edges in a minimum C_6-saturated graph. *Graphs Combin.*, 31(4):1085–1106, 2015.