Equitable coloring of planar graphs with maximum degree at least eight

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April 7, 2024

Abstract

The Chen-Lih-Wu Conjecture states that each connected graph with maximum degree $\Delta \geq 3$ that is not the complete graph $K_{\Delta+1}$ or the complete bipartite graph $K_{\Delta,\Delta}$ admits an equitable coloring with Δ colors. For planar graphs, the conjecture has been confirmed for $\Delta \geq 13$ by Yap and Zhang and for $9 \leq \Delta \leq 12$ by Nakprasit. In this paper, we present a proof that confirms the conjecture for graphs embeddable into a surface with non-negative Euler characteristic with maximum degree $\Delta \geq 9$ and for planar graphs with maximum degree $\Delta \geq 8$.

Keyword: equitable coloring, planar graphs.
Mathematics Subject Classification: 05C07, 05C10, 05C15.

1 Introduction

For a graph G, $\Delta(G)$ denotes the maximum degree of G. An *equitable coloring* of a graph is a proper vertex coloring such that for any two color classes V_i and V_j, we have that $||V_i| - |V_j|| \leq 1$. A graph G is *equitably k-colorable* if it has an equitable coloring with k colors.

The Hajnal-Szemerédi Theorem [2] states that every graph G is equitably k-colorable for any $k \geq \Delta(G)+1$. The bound is sharp for complete graphs $K_{\Delta+1}$ and for complete bipartite graphs $K_{\Delta,\Delta}$ when Δ is odd. Chen, Lih and Wu [1] conjectured the following strengthening of the Hajnal-Szemerédi Theorem.

Conjecture (**Chen-Lih-Wu Conjecture** [1])**.** *If G is an r-colorable graph with $\Delta(G) \leq r$, then either G has an equitable r-coloring, or r is odd and $K_{r,r} \subseteq G$.*

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Rese arch is supported in part by NSF Grant DMS-2153507 and by NSF RTG Grant DMS-1937241.
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Lih and Wu [6] proved the conjecture for bipartite graphs. Chen, Lih and Wu [1] themselves proved the conjecture for $r = 3$ and for $r \geq |V(G)|/2$. Kierstead and Kostochka proved the conjecture in [3] for $r = 4$ and in [4] for $r > |V(G)|/4$. Yap and Zhang [12] proved that the conjecture holds for planar graphs when $r \geq 13$ and Nakprasit [7, 8] confirmed the conjecture for planar graphs when $9 \leq r \leq 12$. These two results together can be stated as follows.

Theorem 1 (Yap and Zhang [12] and Nakprasit [7, 8])**.** *If $r \geq 9$ and G is a planar graph with $\Delta(G) \leq r$, then G has an equitable r-coloring.*

Zhang [11] proved the conjecture for the wider class of 1-planar graphs (and more generally, for graphs with maximum average degree less than 8) but with the stronger restriction on r: for $r \geq 17$.

For lower maximum degrees, Chen-Lih-Wu Conjecture was proved for planar graphs with extra restrictions, mainly with restrictions on cycle structure. In 2008, Zhu and Bu [13] proved that the conjecture holds for C_3-free planar graph with maximum degree $\Delta \geq 8$. It also holds for C_4, C_5-free planar graphs with maximum degree $\Delta \geq 7$. In 2009, Li and Bu [5] proved that the conjecture holds for C_4, C_6-free planar graph with maximum degree $\Delta \geq 6$. In 2012, Nakprasit and Nakprasit [9] proved that the conjecture holds for C_3-free planar graphs with maximum degree $\Delta \geq 6$, C_4-free planar graphs with maximum degree $\Delta \geq 7$, and planar graphs with maximum degree $\Delta \geq 5$ and girth at least 6.

The aim of this paper is twofold. First, we present a significantly shorter proof of Theorem 1. In fact, we prove it for a slightly broader class of graphs embeddable into a surface with non-negative Euler characteristic. For simplicity, we call such graphs *semi-planar*.

Theorem 2. *If $r \geq 9$ and G is a semi-planar graph with $\Delta(G) \leq r$, then G has an equitable r-coloring.*

Our second goal is to extend Theorem 1 to planar graphs with maximum degree 8:

Theorem 3. *If $r \geq 8$ and G is a planar graph with $\Delta(G) \leq r$, then G has an equitable r-coloring.*

The structure of the paper is as follows. In the next section we introduce notation, cite a known lemma and set up the proofs of both theorems. In Section 3 we prove the easier Theorem 2, and in the longer Section 3 we prove Theorem 3.

2 Preliminaries and setup of proofs

Most notation used in the paper is standard. For a graph G, let $\Delta(G)$ denote the maximum degree of G, $\delta(G)$ denote the minimum degree of G and $\delta^*(G)$ denote the minimum degree over non-isolated vertices in G. For a vertex subset $V \subseteq V(G)$ and some vertex $x \in V$ and $u \notin V$, we use $V - x$ to denote $V \setminus \{x\}$ and $V + u$ to denote $V \cup \{u\}$. For an edge $xy \in E(G)$, $G - xy$ denotes the graph obtained by removing xy from G. For two vertex subsets $X, Y \subseteq V(G)$, we use $E_G(X, Y)$ to denote the set of edges connecting X with Y.

For a graph G, $|G|$ denotes the number of vertices of G and $||G||$ denotes the number of edges of G.

Euler's Formula yields the following simple claim.

Lemma 4. *(a) For each planar graph G with $n \geq 3$ vertices, $\|G\| \leq 3n-6$ and $\delta(G) \leq 5$. For each semi-planar graph G with $n \geq 3$ vertices, $\|G\| \leq 3n$ and $\delta(G) \leq 6$.*

(b) For each bipartite planar graph G with $n \geq 3$ vertices, $\|G\| \leq 2n-4$ and $\delta(G) \leq 3$. For each bipartite semi-planar graph G with $n \geq 3$ vertices, $\|G\| \leq 2n$ and $\delta(G) \leq 4$.

We now show that it is sufficient to only consider graphs of order rs for some integer s.

Lemma 5. *It is enough to prove Theorems 2 and 3 for graphs F with $|F|$ divisible by r.*

Proof. Suppose the theorem holds for graphs F with $|F|$ divisible by r. Let G be a semi-planar (or planar) graph with $|G| = n = rs - p$, where $0 < p < r$. If $1 \leq p \leq 4$, then set $G' = G + K_p$. In this case, G' remains semi-planar (or planar). By construction, $|G'| = n + p$ is divisible by r and $\Delta(G') \leq r$. So G' has an equitable r-coloring f'. All vertices of the added K_p have different colors in f', and hence the restriction of f' to G is an equitable r-coloring of G.

Suppose now $p \geq 5$. By Lemma 4(a), either G is 6-regular or G has a vertex v_1 of degree at most 5. In the first case, the theorem follows from the Hajnal-Szemerédi Theorem. In the second case, we can order the vertices of G as $v_1, \ldots, v_n$ so that for each $2 \leq i < n$, $d_{G-\{v_1,\ldots,v_{i-1}\}}(v_i) \leq 6$. Let $G'' = G - \{v_1, \ldots, v_{r-p}\}$. Again, G'' is semi-planar (and planar if G is planar) and $|G''| = n - r + p$ is divisible by r, so G'' has an equitable r-coloring f'. For $j = r-p, r-p-1, \ldots, 1$, we color v_j with color α_j distinct from the colors of its colored neighbors and from $\alpha_{j+1}, \alpha_{j+2}, \ldots, \alpha_{r-p}$. Since $p \geq 5$, for $j \geq 2$, v_j has at most 6 colored neighbors, and the number of already used α_i is $r - p - j \leq r - p - 2$, we can find such α_j for each $j \geq 2$. For $j = 1$, we have $d(v_1) \leq 5$ and the number of already used α_i is $r - p - 1$. Thus, we get an equitable r-coloring of G. $\square$

We now describe the common setup for proofs of both Theorems 2 and 3. By Lemma 5, it is enough to consider graphs with $n = rs$ vertices for some $s \geq 1$. We use induction on $\|G\|$. If G has no edges, the claim is trivial. So, let G be an edge-minimal n-vertex semi-planar (or planar) graph G with $\Delta(G) \leq r$ that is not equitably r-colorable. It may have isolated vertices. Let V_0 denote the set of such vertices and $n_0 = |V_0|$. Let x be a vertex of a minimum degree in $G - V_0$ (we say that $d(x) = \delta^*(G)$) and let y be any neighbor of x. By Lemma 4(a), either $d(x) \leq 5$ or $\Delta(G) = 6$. As in the proof of Lemma 5, if $\Delta(G) = 6$, then we are done by the Hajnal-Szemerédi Theorem, so we may assume $d(x) \leq 5$.

By induction hypothesis, $G - xy$ has an equitable r-coloring, say φ. If vertices x and y are in different color classes, then φ is also an equitable r-coloring of G. Thus, we may assume that the color classes of $G - x$ are $V_1, \ldots, V_r$, where $|V_2| = \ldots = |V_r| = s$, $|V_1| = s - 1$, and $y \in V_1$. We call such (partial) colorings of G *almost equitable*.

Define an auxiliary digraph $\mathcal{H}$ with the vertex set $\{V_1 \ldots, V_r\}$ where a directed edge V_iV_j exists if and only if some vertex $v \in V_i$ has no neighbor in V_j. In order not to mix up vertices and edges in $\mathcal{H}$ and G, we will call the vertices in $\mathcal{H}$ *classes* and edges in $\mathcal{H}$ *arcs*. We say that v *witnesses* the arc V_iV_j, and vertex v is *movable to* V_j. A class V_i is *reachable* from class V_j if $\mathcal{H}$ contains a path from V_j to V_i. Naturally, a class V_i is *reachable* from a set $\mathcal{F}$ of classes, if it is reachable from at least one of classes in $\mathcal{F}$. Call a class V_j *accessible* if V_1 is reachable from V_j, i.e., $\mathcal{H}$ contains a path from V_j to V_1. Let $\mathcal{A}$ be the set of accessible classes in $\mathcal{H}$, and $\mathcal{B}$ be the set of classes not in $\mathcal{A}$. Among all almost equitable colorings, choose a coloring φ with maximum $|\mathcal{A}|$.

Set $a = |\mathcal{A}|$, $b = |\mathcal{B}|$, $A = \bigcup \mathcal{A}$ and $B = \bigcup \mathcal{B}$. Then $a + b = r$. Also for each $U \in \mathcal{B}$ and each $V \in \mathcal{A}$, every $u \in U$ has a neighbor in V, and hence

$$\textit{for each } U \in \mathcal{B} \textit{ and each } V \in \mathcal{A}, \qquad |E_{G-x}(U, V)| \geq |U| = s. \tag{1}$$

By Lemma 4(b) applied to the bipartite graph formed by the edges of $G - x$ connecting A with B, this yields

$$a \cdot b \cdot s \leq |E_{G-x}(B, A)| \leq 2(|A| + |B|) = 2(rs - 1). \tag{2}$$

For distinct classes $X, Y \in \mathcal{A}$, we say X *blocks* Y if V_1 is not reachable from Y in $\mathcal{H} - X$. A class in $\mathcal{A}$ is *terminal* if it blocks no any other class in $\mathcal{A}$. In particular, if $\mathcal{A} = \{V_1\}$, then V_1 is terminal. Let $\mathcal{A}'$ be the set of terminal classes in $\mathcal{A}$, $A' = \bigcup \mathcal{A}'$ and $a' = |A'|$.

Let $\mathcal{D}(x)$ be the set of classes with no neighbors of x. Since $d(x) \leq 5$, $|\mathcal{D}(x)| \geq r - 5$. If $V_i \in \mathcal{A} \cap \mathcal{D}(x)$, then $\mathcal{H}$ contains a V_i, V_1-path, say $V_{i_1}, V_{i_2}, \ldots, V_{i_t}$, where $i_1 = i$ and $i_t = 1$. Moving x into V_i, and each witness v_{i_j} of $V_{i_j}V_{i_{j+1}}$ to $V_{i_{j+1}}$ along the path yields an equitable r-coloring of G. So, $\mathcal{D}(x) \subseteq \mathcal{B}$; in particular

$$b = |\mathcal{B}| \geq r - 5. \tag{3}$$

For an edge $vu \in E_G(A, B)$ with $v \in V \in \mathcal{A}$ and $u \in B$, if $N_V(u) = \{v\}$, then we say that u and v are *solo neighbors* of each other, and each of them is a *solo vertex*.

For $v \in A$, let $\mathcal{F}_0(v)$ be the set of classes in $\mathcal{B}$ that do not have neighbors of v. Call a vertex $u \in V_i \in \mathcal{A}'$ *ordinary* if some $u' \in V_i - u$ is movable to another class in $\mathcal{A}$ or $a \leq 2$.

For $v \in A$, let $Q(v)$ denote the set of solo neighbors of v in B and let $q(v) = |Q(v)|$. Let $Q'(v)$ denote the set of vertices $u \in Q(v)$ that have non-neighbors in $Q(v) - u$ and let $q'(v) = |Q'(v)|$. We will use the following fact.

Lemma 6. *Let $v \in V_i \in \mathcal{A}'$ be an ordinary vertex. Let $u \in Q'(v)$, say $u \in W_j \in \mathcal{B}$.*
(a) $|N(v) \cap W_j| \neq 1$.
(b) If $\mathcal{F}_0(v) \neq \emptyset$, then W_j is not reachable from $\mathcal{F}_0(v)$.

Proof. Since $u \in Q'(v)$, there is some $u' \in Q'(v)$ not adjacent to u, say $u' \in W_{j'} \in \mathcal{B}$.

Suppose first that (a) does not hold, i.e., $N(v) \cap W_j = \{u\}$. If some $v' \in V_i - v$ is movable to another class in $\mathcal{A}$ or $a = 1$, then we let coloring φ' be obtained from φ by moving v to W_j and u to V_i. Each class in $\mathcal{A} - V_i$ remains accessible as V_i is a terminal class. And by the case, the class $V_i - v + u$ is still accessible. Moreover, now the class W_j' containing u' is also accessible with u' becoming a witness, which contradicts the maximality of a.

If $a = 2$ and no $v' \in V_i - v$ is movable to another class in $\mathcal{A}$, then since $V_i \in \mathcal{A}'$, $i = 2$ and v is the unique vertex in V_2 movable to V_1. Then we consider φ'' obtained from φ by moving v to V_1. In this coloring, $V_2 - v$ is the small class, and v is a witness that $V_1 + v$ is accessible. Moreover, both W_j and $W_{j'}$ are now also accessible. This contradiction proves (a).

The proof of (b) is similar. Moreover, the case when $a = 2$ and no $v' \in V_i - v$ is movable to another class in $\mathcal{A}$ word by word repeats the previous paragraph. So suppose (b) does not hold and either some $v' \in V_i - v$ is movable to another class in $\mathcal{A}$ or $a = 1$. This means there is $W_1 \in \mathcal{F}_0$ and $\mathcal{H}$ contains a directed W_1, W_j-path P. If $W_{j'}$ is a vertex in P distinct from W_j, then we switch the roles of u and u'; thus we assume this is not the case. By renaming the classes in $\mathcal{B}$, we may assume $P = W_1, W_2, \ldots, W_\ell$. For $h = 1, 2, \ldots, \ell - 1$, let u_h be a witness for the arc $W_h W_{h+1}$.

Change φ as follows. Move v to W_1, then for $h = 1, 2, \ldots, \ell - 1$, move u_h from W_h to W_{h+1}, and finally move u to V_i. Call the resulting coloring ψ. See Figure 1. Each class in $\mathcal{A} - V_i$ remains accessible as V_i is a terminal class. And by the case, the class $V_i - v + u$ is still accessible. Moreover, if $j' \neq j$ then class $W_{j'}$ is also accessible, and if $j' = j$ then class $W_j - u + u_{\ell-1}$ is accessible with u' being a witness in both cases. This proves Lemma 6. $\square$

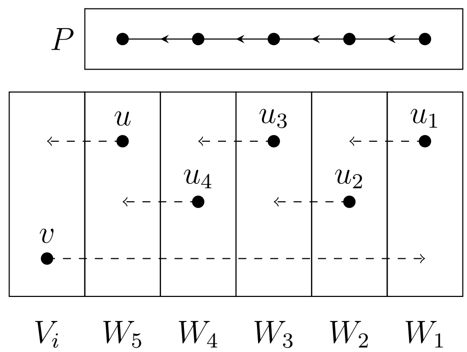

Figure 1: Obtaining ψ in the proof of Lemma 6(b), with $\ell = 5$

For an arbitrary class $V \in \mathcal{A}$ and a vertex $u \in B$, let $\|V, u\|$ denote the number of edges incident to u and a vertex in V. For each $u \in B$ and $v \in V \in \mathcal{A}$, define the weights

$$w(v, u) = \frac{1}{\|V, u\|} \qquad \textit{and} \quad w(v) = \sum\nolimits_{uv \in E(G): u \in B} w(v, u). \tag{4}$$

By definition,

$$\sum\nolimits_{v \in V} w(v) = \sum\nolimits_{v \in V, u \in B} w(v, u) = |B| = bs. \tag{5}$$

3 Proof of Theorem 2

For semi-planar G, we provide a bound on $q'(v)$ in terms of $q(v)$.

Claim 3.1. *If $q(v) \geq 8$, then $q'(v) \geq 5$. Also, if $q(v) = 7$, then $q'(v) \geq 4$.*

Proof of claim. Let $q = q(v)$ and $q' = q'(v)$. Consider graph $F = G[Q(v) \cup \{v\}]$. Vertices in $Q'(v)$ are those having degree less than $q - 1$ in F. Then $|E(F)| \geq \binom{q+1}{2} - \binom{q'}{2}$. So, if $q' \leq 4$ and $q \geq 8$, then

$$|E(F)| \geq \frac{q}{2}(q+1) - \binom{4}{2} \geq 4(q+1) - 6 > 3(q+1),$$

contradicting Lemma 4(a). If $q' \leq 3$ and $q = 7$, then similarly $|E(F)| \geq \frac{7}{2}(q+1) - 3 = 3.5(q+1) - 3 > 3(q+1)$, a contradiction again. This proves the claim.

When $r \geq 9$, by (3), $b \geq r - 5 \geq 4$; thus $a \leq 5$. If $3 \leq a \leq 5$, then

$$abs - 2(rs-1) = a(r-a)s - 2rs + 2 = (a-2)rs - a^2 s + 2 > 0,$$

contradicting (2). Thus, $a \leq 2$.

We now show a helpful property of the auxiliary digraph $\mathcal{H}$.

Claim 3.2. *The digraph $\mathcal{H}[\mathcal{B}]$ has a strong component of order at least $r-2$.*

Proof. Since $|\mathcal{B}| \geq r - 2 \geq 7$, if each strong component of $\mathcal{H}[\mathcal{B}]$ has at most $r-3$ vertices, then the union $\mathcal{U}$ of some strong components of $\mathcal{H}$ has at least 3 and at most $r-3$ vertices.

Suppose $|\mathcal{U}| = m$. Then for every pair (U_i, W_j) where $U_i \in \mathcal{U}$ and $W_j \in \mathcal{B} - \mathcal{U}$, either $U_i W_j$ is not an arc or $W_j U_i$ is not an arc in $\mathcal{H}$. By the construction of $\mathcal{H}$, either every vertex in W_j has at least one neighbor in U_i, or every vertex in U_i has at least one neighbor in W_j. In both cases, $|E_{G-x}(U_i, W_j)| \geq \min\{|W_j|, |U_i|\} = s$. Also by (1), $|E_{G-x}(U_i, A_j)| \geq s$ for each $A_j \in \mathcal{A}$.

It follows that denoting $U = \bigcup_{U_i \in \mathcal{U}} U_i$ and $W = V(G) - U - x$, we have

$$|E_{G-x}(U, W)| \geq ms(r-m) \geq 3s(r-3) = 2rs + (r-9)s.$$

For $r \geq 9$ this is greater than $2(rs-1)$, which contradicts Lemma 4(b) applied to the bipartite graph formed by the edges of $G - x$ connecting U with W. $\square$

Now we can prove the theorem. Recall that $1 \leq a \leq 2$.

Case 1: $a = 2$. Let $\mathcal{A} = \{V_1, V_2\}$ and $\mathcal{B} = \{W_1, ..., W_{r-2}\}$. First, we show that

$$\textit{if some } v \in V_2 \textit{ has a solo neighbor in } B\textit{, then } v \textit{ also has a neighbor in } V_1. \tag{6}$$

Indeed if $v \in V_2$ has a solo neighbor $u \in W_j \in \mathcal{B}$, then we consider a new coloring φ' obtained from φ by moving v to V_1. The new almost equitable coloring has the small class $V_2 - v$, and this class is reachable in the corresponding digraph $\mathcal{H}'$ from $V_1 + v$ (with a witness v) and from W_j (with a witness u). This contradiction to the maximality of $\mathcal{A}$ in φ proves (6).

Since $V_2 \in \mathcal{A}$, it contains a vertex u with no neighbors in V_1. By (6), u has no solo neighbors in B, and hence $w(u) \leq d(u)/2 < r-2$. Since by (5), the average weight of vertices in V_2 is $r-2$, this implies, that for some $v_0 \in V_2$ we have $w(v_0) > r-2$. By definition,

$$w(v_0) \leq q(v_0) + \frac{1}{2}(|N(v_0) \cap B| - q(v_0)) = \frac{1}{2}|N(v_0) \cap B| + \frac{1}{2}q(v_0)$$

Again by (6), v_0 has a neighbor in V_1 and so $|N(v_0) \cap B| \leq r-1$. Hence, in order to have $w(v_0) > r-2$, we need $|N(v_0) \cap B| = r-1$ and $q(v_0) \geq r-2$.

Let $\mathcal{F}_0 = \mathcal{F}_0(v)$ is the set of classes in $\mathcal{B}$ that do not have neighbors of v_0. By Lemma 6(a) and Claim 3.1, among the $r-1$ neighbors of v_0 in B, at least 4 vertices are not unique neighbors of v_0 in their color classes. It follows that

$$|\mathcal{F}_0| \geq (r-2) - (r-1-\frac{4}{2}) = 1. \tag{7}$$

By Claim 3.2, every color class in $\mathcal{B}$ is reachable from $\mathcal{F}_0(v)$. But there is some $u \in Q'(v_0) \cap W$ where $W \in \mathcal{B}$, and with W reachable from $\mathcal{F}_0(v)$, we have a contradiction to Lemma 6(b).

Case 2: $a = 1$. This case is similar to Case 1, but more complicated. We may assume $\mathcal{A} = \{V_1\}$ and $\mathcal{B} = \{W_1, ..., W_{r-1}\}$. Since $|V_1| = s-1$, by (5), the average weight of a vertex in V_1 is $\frac{(r-1)s}{s-1} > r-1$. Fix a vertex $v_0 \in V_1$ with $w(v_0) > r-1$. For this we need $d(v_0) = r$ and $q(v_0) \geq r-1$. By Claim 3.1, $q'(v_0) \geq 5$.

Recall $\mathcal{F}_0$ as in Case 1. By Lemma 6(a) and Claim 3.1, among the r neighbors of v_0 in B, at least 5 vertices are not unique neighbors of v_0 in their color classes. So, similarly to (7), we get

$$|\mathcal{F}_0| \geq (r-1) - (r - \left\lceil \frac{5}{2} \right\rceil) = 2. \tag{8}$$

By Claim 3.2 and (8), we have the following cases.

Case 2.1: Every color class in $\mathcal{B}$ is reachable from $\mathcal{F}_0$. There is some $u \in Q'(v_0) \cap W$ where $W \in \mathcal{B}$, and with W reachable from $\mathcal{F}_0$, we have a contradiction to Lemma 6(b).

Case 2.2: Exactly one color class in $\mathcal{B}$, say W_{r-1} is not reachable from $\mathcal{F}_0$. If there is $u \in Q'(v_0) \cap W$ where $W \in \mathcal{B} \setminus \{W_{r-1}\}$, then we again have a contradiction to Lemma 6(b). So, assume $Q'(v_0) \subseteq W_{r-1}$. Consider the following new weight function w'.

For each $u \in B - W_{r-1}$ and $v \in V_1$,define the weight $w'(v,u) = w(v,u) = \frac{1}{\|V_1,u\|}$, but for $u \in W_{r-1}$ and $v \in V_1$ we let $w'(v,u) = \frac{1}{2}w(v,u) = \frac{1}{2\|V_1,u\|}$. Then for each $v \in V_1$, define

$$w'(v) = \sum\nolimits_{uv \in E(G): u \in B} w'(v,u).$$

By definition,

$$\sum\nolimits_{v \in V_1} w'(v) = \sum\nolimits_{v \in V_1, u \in B} w'(v,u) = (r-1.5)s. \tag{9}$$

Since $|V_1| = s-1$, the average new weight of a vertex in V_1 is $\frac{(r-1)s}{s-1} > r-1.5$. Fix a vertex $v' \in V_1$ with $w'(v') > r-1.5$. Since $Q'(v_0) \subseteq W_{r-1}$ and $q'(v_0) \geq 5$, we have $w'(v_0) \leq (q(v_0) - q'(v_0)) + \frac{1}{2}(r - q(v_0) + q'(v_0)) \leq r - 2.5$. Thus $v' \neq v_0$. Since $a = 1$, v' is ordinary.

By Lemma 3.2, we may assume the following:

$$\textit{For all } W_i, W_j \textit{ such that } 1 \leq i,j \leq r-2,\ \mathcal{H} \textit{ has a } W_i, W_j\textit{-path.} \tag{10}$$

Let $\hat{Q}(v') = Q(v') - W_{r-1}$. If $|\hat{Q}(v')| = m \leq r-3$, then $w'(v') \leq m + \frac{1}{2}(r-m) \leq r-3+\frac{3}{2}$, a contradiction. Thus $|\hat{Q}(v')| \geq r-2 \geq 7$. So, by Claim 3.1, $|Q'(v')| \geq 4$. Since by Lemma 6(a), among the r neighbors of v' in B, at least 4 vertices are not unique neighbors of v' in their color classes, similar to (7), $|\mathcal{F}_0(v')| \geq 1$. Choose a smallest m such that $W_m \in \mathcal{F}_0(v')$.

Case 2.2.1: $1 \leq m \leq r-2$. If some 4 vertices in $Q'(v')$ are in W_{r-1}, then $w'(v') \leq r - 4(1/2) = r-2$, a contradiction. Thus some $u \in Q'(v')$ is not in W_{r-1}. Say $u \in W_j \in \mathcal{B} - \mathcal{F}_0(v')$. Then by Lemma 6(b), W_j is not reachable from W_m, but this is a contradiction to (10).

Case 2.2.2: $m = r-1$. By the minimality of m and by Lemma 6(a), in this case $q'(v') = 4$, and these four vertices are in exactly two color classes. Since $q(v') \geq r-2 \geq 7$, there is $z_0 \in Q(v') - Q'(v')$. This z_0 is adjacent to v' and to at least $r-3$ vertices in $Q(v')$, and hence has at most 2 neighbors in W_{r-1}. Recall that at least 5 vertices in $Q'(v_0)$, say $z_1, \dots, z_5$, are in W_{r-1}. So, we may assume that z_0 is not adjacent to z_1, z_2 and z_3.

By (8), we may assume that v_0 has no neighbors in W_1. Let $W(z_0)$ be the class containing z_0. By (10), $\mathcal{H}$ has a $W_1, W(z_0)$-path, say $W_1, W_2, \dots, W_\ell$, where $W_\ell = W(z_0)$. For $j = 1, 2, \dots, \ell-1$, let u_j be a witness for the arc W_jW_{j+1}.

Consider a new coloring φ' obtained as follows. Move v' to W_{r-1}, then z_1 to $V_1 - v'$, then v_0 to W_1, then for $j = 1, 2, \dots, \ell-1$, move u_j from W_j to W_{j+1}, and finally move z_0 to V_1. Since $z_0z_1 \notin E(G)$ and v' has no neighbors in W_{r-1}, φ' is an almost equitable coloring of $G - x$. But now the class $W_{r-1} - z_1 + v'$ is accessible with a witness z_2, contradicting the maximality of a. □

4 Proof of Theorem 3

By Theorem 2, it is enough to consider the case $r = 8$. Since G is planar, we can give a better bound on $q'(v)$ in terms of $q(v)$.

Claim 4.1. *If $q(v) \geq 5$, then $q'(v) \geq q(v) - 1$.*

Proof. Assume that $q'(v) \leq q(v) - 2$. Then there are two solo neighbors u_1, u_2 of v adjacent to all other vertices in $Q(v)$. In particular, G contains $K_{3,q(v)-2}$ with parts $\{v, u_1, u_2\}$ and $Q(v) - \{u_1, u_2\}$, a contradiction to planarity of G. □

We now prove an analogue of Claim 3.2 on strong components of $\mathcal{H}$.

Claim 4.2. *Suppose $a = |\mathcal{A}| \leq 4$.*

(i) No union of some strong components of $\mathcal{H}$ has exactly 4 vertices.

(ii) Digraph $\mathcal{H}$ either has a strong component of size at least 5, or has two strong components of size 3 and one strong component of size 2.

Proof. Suppose (i) does not hold, and the union of some strong components of $\mathcal{H}$ consists of exactly 4 classes, say this union is $\mathcal{U} = \{U_1, U_2, U_3, U_4\}$. Let $\mathcal{W} = V(\mathcal{H}) - \mathcal{U} = \{W_1, W_2, W_3, W_4\}$. Then as in the proof of Claim 3.2, $|E_G(U_j, W_i)| \geq \min\{|W_i|, |U_j|\}$. Without loss of generality, assume that $|U_1| = |V_1| = s-1$. Denoting $U = \bigcup_{i=1}^{4} U_i$ and

$W = \bigcup_{j=1}^{4} W_j$, we have

$$|E_G(U,W)| = |E_G(U_1,W)| + \sum_{i=2}^{4} |E_G(U_i,W)| \geq 4(s-1) + 12s = 16s - 4 > 2(8s-1) - 4,$$

contradicting Lemma 4(b). Thus (i) holds.

Let the sizes of the strong components of $\mathcal{H}$ be $a_1, \ldots, a_m$ and $a_1 \geq a_2 \geq \ldots \geq a_m$. Then $a_1 + \ldots + a_m = 8$. If (ii) does not hold, then $a_1 \leq 4$. Moreover, by (i), $a_1 \leq 3$ and no sums of several a_i equal to 4. This is possible only if $a_1 = a_2 = 3$ and $a_3 = 2$. $\square$

Notice that by the way we define $\mathcal{A}$ and $\mathcal{B}$, each strong component in $\mathcal{H}$ should be contained in either $\mathcal{A}$ or $\mathcal{B}$.

With $r = 8$, by (3) we have $b \geq 3$. So $a = r - b \leq 5$. By Claim 4.2, $a = 4$ would lead to a contradiction. Thus it suffices to consider the cases when $a = 1, 2, 3$ and 5.

4.1 Proof of the case $a = 1$

Recall the weight functions $w(v,u)$ and $w(v)$ defined by (4). By (5), with $b = r - a = 7$ and $|A| = |V_1| = s-1$, there is some $v_0 \in V_1$ with $d(v_0) \geq w(v_0) \geq 7s/(s-1) > 7$. Thus $d(v_0) = 8$. Note that $N(v_0) \subseteq B$. If $q(v_0) \leq 6$, then $w(v_0) \leq q(v_0) + (d(v_0) - q(v_0))/2 = 4 + q(v_0)/2 \leq 7$, a contradiction, so $q(v_0) \geq 7$ and $q'(v_0) \geq 6$.

Let $\mathcal{F}_0$ denote the set of classes in $\mathcal{B}$ that do not have neighbors of v_0, $\mathcal{F}$ denote the set of classes reachable in $\mathcal{H}$ from $\mathcal{F}_0$, $f = |\mathcal{F}|$ and $F = \bigcup \mathcal{F}$. Notice that every color class V_i is trivially reachable from itself in $\mathcal{H}$, so $\mathcal{F}_0 \subseteq \mathcal{F}$. By Lemma 6(a) with $q'(v_0) \geq 6$, at least 6 vertices in $N(v_0)$ are not unique neighbors of v_0 in their color classes. It follows that

$$7 \geq f \geq |\mathcal{F}_0| \geq (r-1) - (r - \frac{6}{2}) = 2. \tag{11}$$

Case 1.1: $f = 2$, say $\mathcal{F} = \{V_2, V_3\}$. In this case, by (11), $\mathcal{F} = \mathcal{F}_0$. Then $q'(v_0) = 6$ and there are three classes V_6, V_7, V_8 such that $Q'(v_0) = N(v_0) \cap (V_6 \cup V_7 \cup V_8)$. Specifically, by Lemma 6(a), we get $|N(v_0) \cap V_i| = |Q'(v_0) \cap V_i| = 2$ for $i \in \{6, 7, 8\}$. Since $q(v_0) \geq 7 > q'(v_0)$, some vertex $v' \in Q(v_0)$ is adjacent to all of $Q'(v_0)$.

Let $N(v_0) \cap V_8 = \{w, w'\}$. Consider the coloring φ'' of $G - x$ obtained from φ by moving v_0 into V_8 and moving w and w' into $V_1 - v_0$. Denote $V_1' = (V_1 - v_0) \cup \{w, w'\}$ and $V_8' = (V_8 - \{w, w'\}) \cup \{v_0\}$. If x is not adjacent to V_8', then we extend φ'' to G by moving x into V_8'. This extension is an equitable coloring of G as $|V_1'| = |V_8' \cup \{x\}| = s$ while other color classes remain unchanged. Thus we may assume that x has a neighbor y' in V_8'.

Note that φ'' is an almost equitable coloring of $G - x$ with the small class V_8'. By the maximality of a, every vertex in $V(G) - V_8'$ has a neighbor in V_8'. Thus

$$|E_G(V_8', V_i)| \geq |V_i| = s \qquad \text{for all } i \in [7] - \{1\}. \tag{12}$$

Now we count the edges between $X = V_1 \cup V_8 \cup F$ and $Y = V(G) - x - X = V_4 \cup V_5 \cup V_6 \cup V_7$. Since $a = 1$ and $f = 2$, for color classes $F_i \in \mathcal{F}$ and $B_j \in \mathcal{B} \setminus \mathcal{F}$, there is no edge of the form

F_iB_j or B_jV_1 in $\mathcal{H}$. Thus

$$|E_G(V_1 \cup F, Y)| \geq |E_G(V_1, Y)| + |E_G(F, Y)| \geq 4s + 8s = 12s. \tag{13}$$

Further notice that

$$|E_G(V_8', Y) \cap E_G(V_1, Y)| = |E_G(v_0, N(v) - \{w, w'\})| = 6, \tag{14}$$

and that

$$|E_G(v', \{w, w'\}) \cap (E_G(V_8', Y) \cup E_G(V_1, Y))| = 0.$$

Thus by (12), we get

$$|E_G(V_8, Y)| \geq |E_G(V_8', Y)| - |E_G(v_0, N(v) - \{w, w'\})| + |E_G(v', \{w, w'\})| \geq 4s - 6 + 2 = 4s - 4.$$

Combining this with (13), we obtain

$$|E_G(X, Y)| \geq 12s + 4s - 4 = 16s - 4 > 2(8s - 1) - 4,$$

a contradiction to Lemma 4(b).

Case 1.2: $f \in \{3, 4\}$. In this case we do not have a strong component of size at least 5 in $\mathcal{H}$, and V_1 forms a strong component of size 1 by itself. Then we have a contradiction to Claim 4.2.

Case 1.3: $f = 5$. Let $\mathcal{B} - \mathcal{F} = \{V_2, V_3\}$ and $C = V_2 \cup V_3$. Similarly to Case 2.2 in Section 3, we have $Q'(v_0) \subseteq C$. Consider the following new weight function w'.

For each $u \in B \setminus C = F$ and $v \in V_1$, define $w'(v, u) = w(v, u) = \frac{1}{||V_1, u||}$, but for $u \in C$ and $v \in V_1$, let $w'(v, u) = \frac{1}{2} w(v, u) = \frac{1}{2||V_1, u||}$. For each $v \in V_1$, define

$$w'(v) = \sum\nolimits_{uv \in E(G): u \in B} w'(v, u).$$

By definition, $\sum_{v \in V_1} w'(v) = \sum_{v \in V_1, u \in B} w'(v, u) = 6s$.

Since $|V_1| = s - 1$, the average new weight of a vertex in V_1 is $6s/(s - 1) > 6$. Pick a vertex $v' \in V_1$ with $w'(v') > 6$. Let $Q_1(v') = Q(v') \cap F$ and $q_1(v') = |Q_1(v')|$. By definition, for $u \in Q(v') - Q_1(v')$, $w'(v', u) \leq \frac{1}{2}$. Thus

$$6 < w'(v') \leq q_1(v') + \frac{1}{2}(8 - q_1(v')) = 4 + \frac{q_1(v')}{2},$$

so $q_1(v') \geq 5$. Denote by $Q_1'(v')$ the set of vertices $u \in Q_1(v')$ that have non-neighbors in $Q_1(v') - u$ and $q_1'(v') = |Q_1'(v')|$.

Case 1.3.1: $|N(v') \cap C| \geq 1$. Suppose first that every class in $\mathcal{F}$ has a neighbor of v'. Let $\mathcal{F}'(v')$ denote the set of classes in $\mathcal{F}$ that contain vertices in $Q_1'(v')$ and no other neighbors of v'. Since $q(v') \geq q_1(v') \geq 5$, repeating the argument of Claim 4.1, we get $q_1'(v') \geq q_1(v') - 1$. So since $|N(v') \cap F| \leq 7$, $|\mathcal{F}'(v')| \geq 2$. By Lemma 6(a), each class in $\mathcal{F}'(v')$ has at least two vertices from $Q_1'(v')$. If each of them has at least 3 such vertices, then $N(v') \cap F$ has at least $3|\mathcal{F}'(v')| + (5 - |\mathcal{F}'(v')|) = 2|\mathcal{F}'(v')| + 5$ vertices. But this contradicts the fact that $|N(v') \cap F| \le 7$ and $|\mathcal{F}'(v')| \ge 2$. Thus, some color class $V_8 \subseteq F$ satisfies $|V_8 \cap Q'_1(v')| = |V_8 \cap N(v')| = 2$, say $V_8 \cap Q'_1(v') = \{z, z'\}$.

Similarly to Case 1.1, we consider a coloring φ'' of $G-x$ obtained from φ by moving v' into V_8 and moving z and z' into $V_1 - v$. Denote $V'_1 = (V_1 - v') \cup \{z, z'\}$ and $V'_8 = (V_8 - \{z, z'\}) \cup \{v'\}$. As in Case 1.1, φ'' is an equitable coloring of $G-x$ with the small class V'_8. By the maximality of a, every vertex in $V(G) - V'_8$ has a neighbor in V'_8. Thus (12) holds again.

Now we count the edges between $X = V_1 \cup V_8 \cup C$ and $Y = V(G) - x - X = V_4 \cup V_5 \cup V_6 \cup V_7$. Similarly to (13), we get

$$|E_G(V_1 \cup C, Y)| \ge 12s.$$

As v' has a neighbor in C, $|N(v') \cap F| \le 7$. So similarly to (14), we have

$$|E_G(V'_8, Y) \cap E_G(V_1, Y)| = |E_G(v', (N(v') \cap F) - \{z, z'\})| \le 5.$$

Hence

$$|E_G(V_8, Y)| \ge |E_G(V'_8, Y) - E_G(V_1, Y)| \ge 4s - 5.$$

Therefore,

$$|E_G(V_1 \cup C \cup V_8, Y)| \ge 16s - 5 > 2(8s - 1) - 4,$$

a contradiction to Lemma 4(b).

Thus, we may assume that some class $U \in \mathcal{F}$ contains no neighbors of v'. Since $a = 1$, by Claim 4.2, $\mathcal{H}$ has a strong component $\mathcal{H}_1$ of size at least 5. Since $\mathcal{H}$ has no edges from $\mathcal{F}$ to V_1, V_2 or V_3, the vertex set of $\mathcal{H}_1$ is $\mathcal{F}$. Hence every class in $\mathcal{F}$ is reachable from U. In particular, there is some vertex $u \in Q'_1(v')$ that is contained in some class V_j and V_j is reachable from U. However, as $a \le 2$, v' is ordinary and this contradicts Lemma 6(b).

Case 1.3.2: $|N(v') \cap C| = 0$. Using the argument of Claim 4.1, we can show that as $q_1(v') \ge 5$,

$$q'_1(v') = |Q'_1(v')| \ge q_1(v') - 1.$$

So, there is at most one class in $\mathcal{F}$ containing the vertex from $Q_1(v') \setminus Q'_1(v')$ (if exists), at most 3 classes containing vertices from $N(v') \setminus Q_1(v')$, and hence there is a class $V_8 \in \mathcal{F}$ with $V_8 \cap (N(v') - Q'_1(v')) = \emptyset$.

If V_8 has no neighbors of v' at all, then we can denote the class as U and apply the argument at the end of Case 1.3.1 again. Otherwise, by Lemma 6(a), V_8 has at least two vertices from $Q'_1(v')$, say $\{w_1, w_2\} \subseteq V_8 \cap Q'_1(v')$.

Recall that $q'(v) \ge 6$. Without loss of generality, assume that $\{v_1, v_2, v_3\} \subseteq V_2 \cap Q'(v)$. Since G is planar, it is $K_{3,3}$-free, so by symmetry we can assume that w_1 and v_1 are not adjacent in G. Take $W_1 \in \mathcal{F}_0 \subseteq \mathcal{F}$. By Claim 4.2, $\mathcal{H}$ contains a W_1, V_8-path P. Let $P = W_1, W_2, \dots, W_\ell$ where $W_\ell = V_8$. For $j = 1, 2, \dots, \ell - 1$, let u_j be a witness for the arc $W_j W_{j+1}$.

Change φ as follows. Move v_0 to W_1, then for $j = 1, 2, \dots, \ell - 1$, move u_j from W_j to W_{j+1}, move u to V_2, move v_1 to V_1 and finally move w_1 to V_1. Class $V_1 - \{v_0, u\} + \{v_1, w_1\}$ remains accessible, but now $V_2 - v_1 + u$ is also accessible witnessed by v_2, contradicting the maximality of a.

Case 1.4: $f = 6$. Similarly to the argument of Case 2.1 in Section 3, suppose $B - F = V_2$. Then $Q'(v) \subseteq V_2$. We pick two arbitrary sets $X_1, X_2 \in \mathcal{F}$. Let $\mathcal{X}$ be the collection of classes in $\mathcal{H}$ reachable from X_1 and X_2. Then $2 \leq |\mathcal{X}| \leq 6$, since both V_1 and V_2 are not reachable from X_1 and X_2. Consider these cases.

Case 1.4.1: $2 \leq |\mathcal{X}| \leq 4$. As in Case 1.2, we do not have a strong component of size at least 5 in $\mathcal{H}$, and V_1 must form a strong component of size 1 by itself. Then we have a contradiction to Claim 4.2.

Case 1.4.2: $|\mathcal{X}| = 5$. Assume that $V_3 \in \mathcal{F} \setminus \mathcal{X}$. Since V_1 forms a strong component in $\mathcal{H}$, by Claim 4.2, $\mathcal{H}[\mathcal{X}]$ is strongly connected.

As in Case 1.3, let $C = V_2 \cup V_3$. Let $w'(v,u) = w(v,u) = \frac{1}{||V_1,u||}$ for each $u \in B \setminus C$ and $v \in V_1$, but for $u \in C$ and $v \in V_1$, let $w'(v,u) = \frac{1}{2}w(v,u) = \frac{1}{2||V_1,u||}$. For each $v \in V_1$, define

$$w'(v) = \sum\nolimits_{uv \in E(G): u \in B} w'(v,u).$$

By definition $\sum_{v \in V_1} w'(v) = \sum_{v \in V_1, u \in B} w'(v,u) = 6s$.

Since $|V_1| = s-1$, the average new weight of a vertex in V_1 is $6s/(s-1) > 6$. Pick vertex $v' \in V_1$ with $w'(v') > 6$. We claim that we can repeat the argument from Case 1.3 with v' and C defined identically. Thus, both $|N(v') \cap C| \geq 1$ and $|N(v') \cap C| = 0$ would lead to a contradiction.

Case 1.4.3: $|\mathcal{X}| = 6$. Since X_1, X_2 were picked arbitrarily,

$$\mathcal{H}[\mathcal{F}] \textit{ is strongly connected.} \tag{15}$$

We again use the function $w(v,u) = \frac{1}{||V,u||}$ defined by (4). For each $v \in V_1$, define

$$w_6(v) = \sum\nolimits_{uv \in E(G): u \in F \setminus V_3} w(v,u).$$

By definition $\sum_{v \in V_1} w_6(v) = \sum_{v \in V_1, u \in F \setminus V_3} w(v,u) = 6s$. Since $a \leq 2$, u is ordinary.

Since $|V_1| = s-1$, the average weight of a vertex in V_1 is $6s/(s-1) > 6$. Pick vertex $u \in V_1$ with $w_6(u) > 6$. Notice that $w_6(v_0) < |N(v_0) \setminus Q'(v_0)| \leq 2$, so u and v_0 are distinct. Let $Q_6(u) = Q(u) \cap (F \setminus V_3)$ and $q_6(u) = |Q_6(u)|$. Denote $Q'_6(u)$ the set of vertices $w \in Q_6(u)$ that have non-neighbors in $Q'_6(u) - w$.

If $|N(u) \cap V_2| \geq 1$, then $q_6(u) \geq 6$. By Claim 4.1, there is at most 1 class in $\mathcal{F}$ containing vertex from $Q_6(u) \setminus Q'_6(u)$, and at most 1 class containing vertices from $N(u) \setminus Q_6(u)$. Thus by Lemma 6(a), $|\mathcal{F}_0(u)| \geq 2$. Pick $z \in Q'_6(u)$ and the color class of z is $W(z)$. Then by (15), $W(z)$ is reachable from $\mathcal{F}_0(u)$, but this is a contradiction to Lemma 6(b).

If $|N(u) \cap V_2| = 0$, then $q_6(u) \geq 5$. Since G is planar, it is $K_{3,3}$-free. Then there is some $u' \in Q'_6(u)$ that is not adjacent to some $v_1 \in V_2 \cap Q'(v)$. Let the color class of u' be U'. Take $W_1 \in \mathcal{F}_0 \subseteq \mathcal{F}$. By (15), $\mathcal{H}$ contains a W_1, U'-path P. Let $P = W_1, W_2, \ldots, W_\ell$ where $W_\ell = U'$. For $j = 1, 2, \ldots, \ell-1$, let u_j be a witness for the arc W_jW_{j+1}.

Change φ as follows. Move v_0 to W_1, then for $j = 1, 2, \ldots, \ell-1$, move u_j from W_j to W_{j+1}, move u to V_2, move v_1 to V_1 and finally u' to V_1. Class $V_1 - \{v_0, u\} + \{v_1, u'\}$ remains accessible, but now $V_2 - v_1 + u$ is also accessible, witnessed by some $v_2 \in V_2 \cap Q'(v)$ distinct from v_1. This contradicts the maximality of a.

Case 1.5: $f = 7$. There is some $u \in Q'(v_0) \cap W$ where $W \in \mathcal{B} = \mathcal{F}$. But with W reachable from $\mathcal{F}_0$, we have a contradiction to Lemma 6(b).

4.2 Proof of the case $a = 2$

Let $\mathcal{A} = \{V_1, V_2\}$. For each $u \in B$ and $v \in V_2$, let $w(v,u) = \frac{1}{\|V,u\|}$. Then for each $v \in V_2$, let

$$w_2(v) = \sum\nolimits_{uv \in E(G): u \in B} w(u,v).$$

By definition $\sum_{v \in V_2} w_2(v) = \sum_{v \in V_2, u \in B} w(v,u) = 6s$.

There is a movable vertex $v' \in V_2$, and by (6), v' has no solo neighbors in B. So $w_2(v') \le 8 \cdot \frac{1}{2} = 4$. Then there is a vertex $v_0 \in V_2$ with $w_2(v_0) > 6$. Notice that such v_0 should not be movable in A, so $|N(v_0) \cap B| \le 7$. To have $w_2(v_0) > 6$, we need $q(v_0) = |Q(v_0)| \ge 6$. For each class $U \in \mathcal{B}$, by Lemma 6(a), if $|Q'(v_0) \cap U| \ne 0$, then $|Q'(v_0) \cap U| \ge 2$. Thus there are distinct color classes $U_1, U_2 \in \mathcal{F}_0(v_0)$. Let $\mathcal{U}$ be the collection of classes in $\mathcal{B}$ reachable from $\mathcal{F}_0(v_0)$. Then as $a = 2$, $2 \le |\mathcal{U}| \le 6$. If $|\mathcal{U}| = 2$, then $|\mathcal{A} \cup \mathcal{U}| = 4$, contradicting Claim 4.2(i). For the same reason, $|\mathcal{U}| \ne 4$. The remaining cases are as follows.

Case 2.1: $|\mathcal{U}| = 3$, say $\mathcal{U} = \{U_1, U_2, U_3\}$. Let $U = \bigcup \mathcal{U}$, $\mathcal{W} = \mathcal{B} \setminus \mathcal{U} = \{W_1, W_2, W_3\}$ and $W = \bigcup \mathcal{W}$. For $i = 1, 2$, let M_i denote the set of vertices in V_i movable to V_{r-i}. If $m_2 \ge m_1 + 2$, we move a vertex from M_2 to V_1, and relabel V_1 as V_2' and V_2 as V_1'. Then there are $m_2 - 1$ vertices movable from V_1' to V_2' and $m_1 + 1$ movable from V_2' to V_1'. So, we may assume

$$m_2 \le m_1 + 1. \tag{16}$$

Since no vertex in U_i is movable to W_j for $1 \le i, j \le 3$,

$$|E_G(U, W)| \ge |\mathcal{U}||\mathcal{W}|s = 9s. \tag{17}$$

Suppose that there are k_2 isolated vertices in V_2. Since $d(x) = \delta^*(G) \ge 2$,

$$|E_G(M_2, B)| \ge 2(m_2 - k_2).$$

By the symmetry between $\mathcal{U}$ and $\mathcal{W}$, we can assume

$$|E_G(M_2, U)| \ge m_2 - k_2.$$

If for every vertex $z \in U$, $|N(z) \cap (V_2 \setminus M_2)| \ge 1$, then

$$|E_G(V_2, U)| = |E_G(V_2 \setminus M_2, U)| + |E_G(M_2, U)| \ge 3s + m_2 - k_2. \tag{18}$$

Otherwise there is a vertex $z \in U$ with $|N(z) \cap (V_2 \setminus M_2)| = 0$. Since z is not movable to V_2, it is adjacent to some vertices in M_2. If there is any $v_1 \in M_1$ that is adjacent to y and $v_2 \in M_2$ that is not adjacent to z, then we can switch v_1 and v_2 to increase $|N(z) \cap M_2|$. When $|N(z) \cap M_2|$ is maximized in this way, we either have $M_2 \subseteq N(z)$ or $|N(z) \cap M_2| < m_2$ and $|N(z) \cap M_1| = 0$. In the latter case, we can switch $N(z) \cap M_2$ with equal number of vertices in M_1 since $m_2 \le m_1 + 1$. The switched vertices remain movable to the other class. However, z would become movable to V_2 , since z has no neighbor in V_2 after the switch, a contradiction to the maximality of a. So $M_2 \subseteq N(z)$, and hence $|N(z) \cap M_2| = m_2$. Let Z be the collection of all such $z \in U$ that $|N(z) \cap (V_2 \setminus M_2)| = 0$ and $|N(z) \cap M_2| = m_2$. If $k = |Z|$, then

$$|E_G(V_2, U)| \ge 3s - k + km_2, \tag{19}$$

where $k, m_2 \geq 1$.

Now we count the edges between $V_2 \cup W$ and $V_1 \cup U$, with k_2 isolated vertices in V_2 removed, so there should be $8s-1-k_2$ vertices. When $k=0$, we use the bounds (17), (18), $|E_G(V_2, V_1)| \geq s - m_2$ and $|E_G(W, V_1)| \geq 3s$ to derive that

$$|E_G(V_2 \cup W, V_1 \cup U)| = |E_G(V_2 \cup W, U)| + |E_G(V_2 \cup W, V_1)|$$

$$\geq (9s + 3s + m_2 - k_2) + (3s + s - m_2) = 16s - k_2 > 2(8s - 1 - k_2) - 4.$$

When $k \geq 1$, we use the bound (19) instead of (18):

$$|E_G(V_2 \cup W, V_1 \cup U)| = |E_G(V_2 \cup W, U)| + |E_G(V_2 \cup W, V_1)|$$

$$\geq (9s + 3s - k_2 + km_2) + (3s + s - m_2) \geq 16s - k_2 > 2(8s - 1 - k_2) - 4.$$

In both cases we get a contradiction to Lemma 4(b).

Case 2.2: $|\mathcal{U}| = 5$. Denote $\{V_3\} = \mathcal{B} \setminus \mathcal{U}$. We should have $Q'(v_0) \subseteq V_3$. Consider a new weight function w'_2 where $w'_2(v,u) = w_2(v,u) = \frac{1}{||V_2,u||}$ for $v \in V_2$ and $u \in B \setminus V_3$, but for $u \in V_3$, $w'_2(v,u) = \frac{1}{2||V_2,u||}$. For each $v \in V_2$, define

$$w'_2(v) = \sum_{uv \in E(G): u \in B} w'_2(v,u).$$

By definition $\sum_{v \in V_2} w'_2(v) = \sum_{v \in V_2, u \in B} w'_2(v,u) = \frac{11}{2}s$. Note that $w'_2(v_0) \leq 5 \cdot \frac{1}{2} + 2 = \frac{9}{2} < \frac{11}{2}$. Hence there is some $u \in V_2 - v_0$ with $w'_2(u) > \frac{11}{2}$. Then $|Q(u) \setminus V_3| \geq 5$, so $|N(u) \cap V_1| \geq 1$ and by Claim 4.1, $q'(u) \geq q(u) - 1$.

Case 2.2.1: $|N(u) \cap U_3| = 0$ for some $U_3 \in \mathcal{U}$. We have 2 classes V_1, V_2 not reachable from the other 6 classes and class V_3 not reachable from the remaining 5 classes. So, $\{V_3\}$ forms a strong component in $\mathcal{H}$. Hence by Claim 4.2, $\mathcal{H}(\mathcal{U})$ is strongly connected. Take some $z \in Q'(u) \setminus V_3$ with color class $W(z)$. Then in particular $W(z)$ is reachable from $\mathcal{F}_0(v_0)$, but this is a contradiction to Lemma 6(b).

Case 2.2.2: $|N(u) \cap U| \geq 1$ for every $U \in \mathcal{U}$. As $|N(u) \setminus V_1| \leq 7$, $|Q(u) \setminus V_3| \geq 5$ and $|Q'(u) \setminus V_3| \geq |Q(u) \setminus V_3| - 1 \geq 4$, at most 3 classes in $\mathcal{U}$ contain vertices in $N(u) \setminus Q'(u)$. So, by Lemma 6(a), some two classes in $\mathcal{U}$ contain at least two vertices in $Q'(u)$ each; thus at least 4 together. For this to happen, we need $|N(u) \setminus V_1| = 7$, $|Q(u) \cap V_3| = 0$ and $Q(u) \setminus Q'(u') \neq \emptyset$, say $z \in Q(u) \setminus Q'(u')$. Note that $|N(z) \cap (B \setminus V_3)| \geq 4$, and z is adjacent to u and some vertex in V_1 by definition. Thus $|N(z) \cap Q'(v)| \leq 2$, so there are $v_1, v_2 \in Q'(v)$ that are not adjacent to z. Let the color class of z be $W(z)$. Pick $U_1 \in \mathcal{F}_0(v_0)$. Then there is a $U_1, W(z)$ path P. Let $P = W_1, W_2, \ldots, W_\ell$ where $W_1 = U_1, W_\ell = W(z)$. For $j = 1, 2, \ldots, \ell - 1$, let u_j be a witness for the arc $W_j W_{j+1}$.

Now we change φ as follows. Move v_0 to U_1, then for $j = 1, 2, \ldots, \ell - 1$, move u_j from W_j to W_{j+1}, move z and v_1 to V_2, and finally u to V_3. Now $V_2 - \{v_0, u\} + \{v_1, z\}$ remains accessible as both v_0 and u are not movable, but now in addition $V_3 - v_1 + u$ becomes accessible with witness v_2, a contradiction to maximality of a.

Case 2.3: $|\mathcal{U}| = 6$. As $q(v_0) \geq 6$, by Claim 4.1, there are $z \in Q'(v_0)$ with color class $W(z)$. By the case $W(z)$ is reachable from $\mathcal{F}_0(v_0)$, but this contradicts Lemma 6(b).

4.3 Proof of the case $a = 3$

4.3.1 Setup

Let $\mathcal{A} = \{V_1, V_2, V_3\}$, $\mathcal{B} = \{W_1, \ldots, W_5\}$.

We first show that V_2 and V_3 can be chosen to be terminal classes. Assume not, say V_2 blocks V_3. Then there is a vertex $v_2 \in V_2$ movable to V_1 and a vertex $v_3 \in V_3$ movable to V_2. We move v_2 to V_1, so V_2 becomes the smaller class. Notice that v_2 is movable from $V_1 + v_2$ to $V_2 - v_2$ and v_3 remains movable from V_3 to $V_2 - v_2$. So in the new $\mathcal{H}$, both $V_1 + v_2$ and V_3 are terminal and $|V_2 - v_2| = s - 1$ while $|V_1 + v_2| = s$. Thus, we can assume that both V_2 and V_3 are terminal classes.

Lemma 7. *For $2 \le j \le 3$, each solo vertex $v \in V_j$ has neighbors in V_1 and V_{5-j}, and thus is ordinary.*

Proof. Since both V_2 and V_3 are terminal classes, without loss of generality we can assume that there is a movable $v \in V_2$ that has a solo neighbor $u \in W(u) \in \mathcal{B}$, and $v' \in V_3$ witnesses the directed edge V_3V_1 in $\mathcal{H}$.

If v is movable to V_3, then we move v' to V_1 and v to V_3. In the new coloring φ', $V_2 - v$ as the smaller class, $V_3 - v' + v$ and $W(u)$ are accessible with regard to $V_2 - v$. No other class in $\mathcal{B}$ is accessible otherwise we get a larger a. $V_1 + v'$ should not be in a strong component with classes other in $\mathcal{H}$ since V_1 is not. However, if $V_1 + v'$ can reach $V_2 - v$, we also get a larger a. Thus $V_1 + v'$ must be in a strong component by itself in the auxiliary digraph regarding the new coloring, but this contradicts Claim 4.2 as we would have no strong component of size 5 and one strong component of size 1.

If v is movable to V_1, then we move v to V_1. Now we take $V_2 - v$ as the smaller class, then V_1 and $W(u)$ are accessible with regard to $V_2 - v$. Again, no other class in $\mathcal{B}$ is accessible otherwise we get a larger a. V_3 would be in a strong component in the auxiliary digraph regarding the new coloring, but this contradicts Claim 4.2 as we would have no strong component of size 5 and one strong component of size 1. □

Denote the size of a largest strong component of $\mathcal{H}$ contained in $\mathcal{B}$ by b_0. By Claim 4.2, either $b_0 = 3$ or $b_0 = 5$.

Case 3.1: $b_0 = 3$. By Claim 4.2, we may assume that the vertex sets of strong components of $\mathcal{H}$ contained in $\mathcal{B}$ are $\mathcal{B}_1 = \{W_1, W_2\}$ and $\mathcal{B}_2 = \{W_3, W_4, W_5\}$. Recall that V_0 denotes the set of isolated vertices in G, and $n_0 = |V_0|$. By the definition of B, $V_0 \subset A$. Let $n'_0 = |V_0 - V_1|$.

Consider the following discharging procedure DP.

At the beginning, each edge of $G - x$ has charge 1, so the sum of all charges is $|E(G - x)|$. Then each edge $e = uv \in E(G - x)$ shares its charge among its ends according to the rules below.

(R1) if $v \in V_1$, then the edge sends all charge to u;

(R2) if $v \in A - V_1$ and u is its solo neighbor in B, then the e sends all charge to u;

(R3) in all other cases, e sends $\frac{1}{2}$ to each endpoint.

So, denoting the charge of a vertex $v \in V(G)$ by $ch(v)$, we have

$$\sum_{v\in V(G)} ch(v) = |E(G-x)|. \tag{20}$$

If a non-isolated vertex $v \in A - V_1$ has a solo neighbor in B, then by Lemma 7 it has a neighbor in each of the other two classes in $\mathcal{A}$, thus by rules (R2) and (R3) its charge is at least $\frac{1}{2} + 1 = \frac{3}{2}$. If this non-isolated $v \in A - V_1$ has no solo neighbors, then again by (R3) or (R2), v receives charge at least $\frac{1}{2}$ from each incident edge, and hence $ch(v) \geq \frac{3}{2}$.

Each vertex $u \in B$ receives at least 3 from the edges connecting u with A. Since $\mathcal{B}_1$ and $\mathcal{B}_2$ are vertex sets of disjoint strong components of $\mathcal{H}$, at least s edges connect any class in $\mathcal{B}_1$ with any class in $\mathcal{B}_2$. Hence the vertices of B receive total charge at least $6s$ from these edges. Thus,

$$\sum_{v\in V(G)} ch(v) \geq (2s-1-n_0') \cdot \frac{3}{2} + 5s \cdot 3 + 6s = 24s - n_0' - \frac{3}{2} > 3(8s-1-n_0') - 6.$$

Together with (20), this contradicts Lemma 4(b).

Case 3.2: $b_0 = 5$. For each $u \in B$ and $v \in V_2$, define the weight $w_3(v,u) = \frac{1}{\|V,u\|}$. Then for each $v \in V_2$, define

$$w_3(v) = \sum_{uv\in E(G):u\in B} w_3(v,u).$$

By definition, $\sum_{v\in V_2} w_3(v) = \sum_{v\in V_2, u\in B} w_3(v,u) = 5s$.

Since V_2 is accessible, there is some $v \in V_2$ movable to V_1. Then by Lemma 7, v has no solo neighbor, so $w_3(v) \leq 8 \cdot \frac{1}{2} = 4$. Thus there is some $v' \in V_2$ with $w_3(v') > 5$.

Now we know that v' has a neighbor in V_1 and a neighbor in V_3, so $|N(v') \cap B| \leq 6$. In order to achieve $w_3(v') > 5$, we need $q(v') \geq 5$, and hence by Claim 4.1, $q'(v') \geq 4$. By Lemma 6(a), each neighbor of v' in $Q'(v')$ must be in a class containing some other neighbor of v', so there is some class $W' \in \mathcal{B}$ that is not adjacent to v'. Then we pick some $z \in Q'(v')$ with color class $W(z)$. By the case, $W(z)$ is reachable from W', but by Lemma 7, v' is ordinary, and this leads to a contradiction to Lemma 6(b).

4.4 Proof of the case $a = 5$

In this case, since $d(x) \leq 5$ and x has a neighbor in each class of $\mathcal{A}$, we have $d(x) = 5$ and x has no neighbors in B. First, we take a closer look at $\mathcal{H}[\mathcal{A}]$.

We call $\mathcal{H}[\mathcal{A}]$ *nice*, if every accessible class other than V_1 blocks at most one accessible class. All 5-vertex nice in-trees rooted at V_1 are listed in Figure 2. The two 5-vertex in-trees rooted at V_1 with $d^-_{\mathcal{H}[\mathcal{A}]}(V_1) \geq 2$ that are not nice are listed in Figure 3.

Lemma 8. *If $a = 5$, then we can choose an almost equitable coloring φ so that $\mathcal{H}[\mathcal{A}]$ is nice.*

Proof. Note that if $d^-_{\mathcal{H}[\mathcal{A}]}(V_1) \geq 3$, then $\mathcal{H}[\mathcal{A}]$ is nice. So, we have the following cases.

Case 1: $d^-_{\mathcal{H}[\mathcal{A}]}(V_1) = 2$. Then $\mathcal{H}[\mathcal{A}]$ contains one of the two digraphs in Figure 3.

Case 1.1: $\mathcal{H}[\mathcal{A}]$ contains $\overrightarrow{T'_{3,1}}$. Let φ' be obtained from φ by moving a witness v_3 of the arc V_3V_1 into V_1. Then $V_3 - v_3$ is the new small class, and the arcs $V_4(V_3 - v_3)$, $V_5(V_3 - v_3)$ and

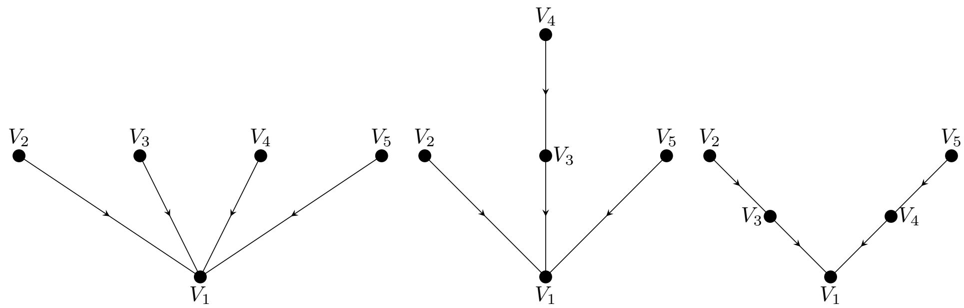

Figure 2: Nice digraphs: $\overrightarrow{K_{4,1}}$, $\overrightarrow{T_3}$ and $\overrightarrow{T_{2,2}}$.

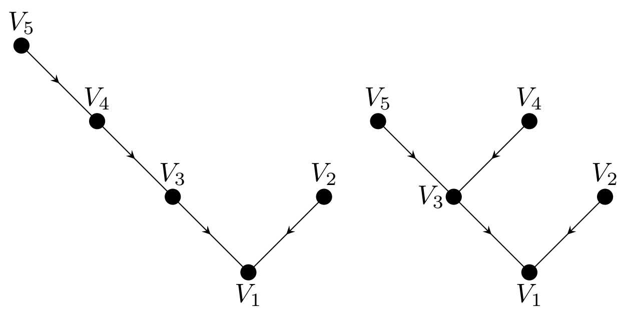

Figure 3: Digraphs with $d^-_{\mathcal{H}[\mathcal{A}]}(V_1) \geq 2$ that are not nice: $\overrightarrow{T_{3,1}}$ and $\overrightarrow{T'_{3,1}}$.

$(V_1+v_3)(V_3-v_3)$ are present in the new $\mathcal{H}$. So, if at least one of $V_1+v_3, V_3-v_3, V_4, V_5$ is an out-neighbor of V_2 in the new $\mathcal{H}$, then the new $\mathcal{H}[\mathcal{A}]$ is nice. Otherwise, $|E(V_2, V_1\cup V_3\cup V_4\cup V_5)| \geq 4s$, and hence

$$|E(V_2 \cup V_6 \cup V_7 \cup V_8, V_1 \cup V_3 \cup V_4 \cup V_5)| \geq 4s + 4|V_6 \cup V_7 \cup V_8| \geq 16s = 2n,$$

contradicting Lemma 4(b).

Case 1.2: $\mathcal{H}[\mathcal{A}]$ contains $\overrightarrow{T_{3,1}}$. If $V_5V_2 \in E(\mathcal{H})$, then $\mathcal{H}[\mathcal{A}]$ is nice, a contradiction. So, $|E(V_2, V_5)| \geq s$. Again, let φ' be obtained from φ by moving a witness v_3 of V_3V_1 into V_1. Again, V_3-v_3 is the new small class, and the arcs $V_4(V_3-v_3)$, $V_5(V_3-v_3)$ and $(V_1+v_3)(V_3-v_3)$ are present in the new $\mathcal{H}$. So, if one of V_1+v_3, V_3-v_3 is an out-neighbor of V_2 in the new $\mathcal{H}$, then $\mathcal{H}[\mathcal{A}]$ is nice, and if $V_2V_4 \in E(\mathcal{H})$, then we get Case 1.1. Otherwise, as in Case 1.1,

$$|E(V_2 \cup V_6 \cup V_7 \cup V_8, V_1 \cup V_3 \cup V_4 \cup V_5)| \geq s + 3s + 4|V_6 \cup V_7 \cup V_8| \geq 16s = 2n,$$

contradicting Lemma 4(b).

Case 2: $d^-_{\mathcal{H}[\mathcal{A}]}(V_1) = 1$. Suppose $V_2V_1 \in E(\mathcal{H})$ and $v_2 \in V_2$ a witness of this arc. Since each vertex in $\mathcal{A}$ is accessible, $d^-_{\mathcal{H}[\mathcal{A}]}(V_2) \geq 1$, say $V_3V_2 \in E(\mathcal{H})$. Let φ' be obtained from φ by moving v_2 into V_1. Then $V_2 - v_2$ is the new small class, all classes in $\mathcal{A}$ are still accessible, and $V_2 - v_2$ has at least two in-neighbors in the new $\mathcal{H}$. So either the new $\mathcal{H}$ is nice or we have Case 1. □

Lemma 9. *If $\mathcal{H}[\mathcal{A}]$ is nice, then each solo vertex $v \in V_i \in \mathcal{A} - V_1$ has a neighbor in each class of $\mathcal{A} - V_i$. In particular, v is ordinary.*

Proof. Suppose $v \in V_i \in \mathcal{A} - V_1$ has a solo neighbor $u \in W \in \mathcal{B}$ and has no neighbor in V_j for some $V_j \in \mathcal{A} - V_i$. If $\mathcal{H} - V_i$ has a V_j, V_1-path P, say $P = W_1, W_2, \dots, W_\ell$, where $W_1 = V_j$, $W_\ell = V_1$ and w_h is a witness of $W_h W_{h+1}$ for $h = 1, \dots, \ell - 1$, then we change φ as follows. Since x has no neighbors in B, move it into the class of u, then move u to V_i, v to $V_j = W_1$, and then for $h = 1, 2, \dots, \ell - 1$, move w_h from W_h to W_{h+1}. This would yield an equitable coloring on G, so assume that $\mathcal{H} - V_i$ has no such path.

This means that V_i blocks V_j. Since $\mathcal{H}[\mathcal{A}]$ is nice, V_j is the unique vertex in $\mathcal{H}[\mathcal{A}]$ blocked by V_i, and v has neighbors in each class of $\mathcal{A} - V_j - V_i$. Since V_i is the only out-neighbor of V_j in $\mathcal{H}[\mathcal{A}]$, we have $|E_G(V_j, A - V_i)| \geq 3s$.

If u is not adjacent to some vertex v' that is movable from V_j to V_i, then we can move v to V_j and v' to V_i. Since $\mathcal{H}[\mathcal{A}]$ is nice, all classes of $\mathcal{A}$ remain accessible, but now the class of u also becomes accessible, contradicting the maximality of a. Thus u is adjacent to all vertices movable from V_j to V_i. Let M be the set of these movable vertices and $m = |M|$.

Now we count the edges connecting $A \setminus V_j - v + u$ and $B \cup V_k + v - u$. Since v is adjacent to each class in $\mathcal{A} - V_j - V_i$ and to u, at most 4 edges connect v to $B - u$. No vertex in $B - u$ is movable to $A - V_j$, thus

$$|E_G(B + v - u, A - V_j - v + u)| \geq 4(3s - 1) - 4 + 3 + 1 = 12s - 4. \tag{21}$$

Since $|E_G(V_j, A - V_i - V_j)| \geq 3s$, we get

$$|E_G(V_j, A - V_j - v + u)| = |E_G(V_j, A - V_i - V_j)| + |E_G(V_j, V_i + u)| \geq 3s + s - m + m = 4s. \tag{22}$$

Summing (21) with (22) gives $16s - 4$ edges in a bipartite planar graph with $8s - 1$ vertices, a contradiction to Lemma 4(b). □

Suppose now that φ satisfies Lemma 8. Recall that V_0 denotes the set of isolated vertices in G, and $n_0 = |V_0|$. By the definition of B, $V_0 \subset A$. Let $n_0' = |V_0 - V_1|$. Consider the discharging procedure DP described in Case 3.1 of Subsection 4.3.1. We will show that the new charges of vertices of G satisfy

$$ch(u) \geq 5 \textit{ for each } u \in B, \textit{ and } ch(v) \geq 2.5 \textit{ for each } v \in A - V_1 - V_0, \tag{23}$$

which would imply that

$$E(G - x) = \sum_{w \in V(G) - V_1 - V_0} ch(w) \geq 5(3s) + 2.5(4s - n_0') = 25s - 2.5n_0' > 3(|V(G)| - n_0).$$

Together with (20), this contradicts Lemma 4(a). Thus, it remains to prove (23).

For $u \in B$ and $V_i \in \mathcal{A}$, u has a neighbor in V_i. If it is a unique neighbor of u in V_i, then u gets 1 from uv by (R2), otherwise at least two edges connect u to V_i and u gets 1/2 from each of them. This proves the first part of (23).

If $v \in A - V_1$ has a solo neighbor in B, then by Lemma 9, it has an edge to V_1 (from which it gets 1 by (R1)) and at least 3 edges to other classes in $\mathcal{A}$ (from each of which it gets 1/2 by (R3)). Thus in this case the second part of (23) holds.

Finally, if $v \in A - V_1 - V_0$ has no solo neighbors in B, then v receives by (R3) a charge of $\frac{1}{2}$ from each incident edge, and by the case, there are at least 5 of them. This proves (23) and hence finishes the proof of Theorem 3. □

Acknowledgment. We thank Dan Cranston, Hal Kierstead, Kittikorn Nakprasit and an anonymous referee for helpful comments.

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