SPARSE CRITICAL GRAPHS FOR DEFECTIVE DP-COLORINGS

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Abstract. An interesting generalization of list coloring is so called DP-coloring (named after Dvořák and Postle). We study (i,j)-defective DP-colorings of simple graphs. Define $g_{DP}(i,j,n)$ to be the minimum number of edges in an n-vertex DP-(i,j)-critical graph. We prove sharp bounds on $g_{DP}(i,j,n)$ for $i=1,2$ and $j \geq 2i$ for infinitely many n.

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1. Introduction

A $(d_1,\ldots,d_k)$-*defective coloring* (or simply $(d_1,\ldots,d_k)$-*coloring*) of a graph G is a partition of $V(G)$ into sets $V_1,V_2,\ldots,V_k$ such that for every $i \in [k]$, each vertex in V_i has at most d_i neighbors in V_i. In particular, a $(0,0,\ldots,0)$-defective coloring is the ordinary proper k-coloring. Defective colorings have been studied in a number of papers, see e.g. [1, 9, 12, 14, 16, 21, 23, 24, 28].

In this paper we consider colorings with 2 colors. If $(i,j) \neq (0,0)$, then the problem to decide whether a graph G has an (i,j)-coloring is NP-complete. In view of this, a direction of study is to find how sparse can be graphs with no (i,j)-coloring for given i and j; see e.g. [3, 4, 5, 6, 7, 8, 19, 20]. A natural measure of "sparsity" of a graph is the *maximum average degree*, $mad(G) = \max_{G' \subseteq G} \frac{2|E(G')|}{|V(G')|}$. In such considerations, an important role plays the notion of (i,j)-*critical graphs*, that is, the graphs that do not have (i,j)-coloring but every proper subgraph of which has such a coloring. Let $f(i,j,n)$ denote the minimum number of edges in an (i,j)-critical n-vertex graph. Observe that for odd $n \geq 3$ we have $f(0,0,n) = n$. Indeed, for odd n the n-cycle is not bipartite, but every n-vertex graph with fewer than n edges has a vertex of degree at most 1 and thus cannot be $(0,0)$-critical. The reader can find interesting bounds on $f(i,j,n)$ in the papers cited above.

A k-*list* for a graph G is a function $L : V(G) \to \mathcal{P}(\mathbb{N})$ such that $|L(v)| = k$ for every $v \in V(G)$. A d-*defective* L-*coloring* of G is a function $\varphi : V(G) \to \bigcup_{v \in V(G)} L(v)$ such that $\varphi(v) \in L(v)$ for every $v \in V(G)$ and every vertex has at most d neighbors of the same color. If G has a d-defective L-coloring from every k-list assignment L, then G is called d-*defective* k-*choosable*. These notions were introduced in [13, 26] and studied in [27, 30, 15, 16]. A direction of study is showing that "sparse" graphs are d-defective k-choosable. The best known bounds on maximum average degree that guarantee that a graph is d-defective 2-choosable are due to Havet and Sereni [15]:

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Theorem A ([15])**.** *For every $d \geq 0$, if $mad(G) < \frac{4d+4}{d+2}$, then G is d-defective 2-choosable. On the other hand, for every $\epsilon > 0$, there is a graph G_ϵ with $mad(G_\epsilon) < 4 + \epsilon - \frac{2d+4}{d^2+2d+2}$ that is not (d, d)-colorable.*

Dvořák and Postle [11] introduced and studied so called *DP-coloring* which is more general than list coloring. Bernshteyn, Kostochka and Pron [2] extended this notion to multigraphs.

Definition 1. Let G be a multigraph. A *cover* of G is a pair $\mathscr{H} = (L, H)$, consisting of a graph H (called the *cover graph* of G) and a function $L \colon V(G) \to 2^{V(H)}$, satisfying the following requirements:

(1) the family of sets $\{L(u) : u \in V(G)\}$ forms a partition of $V(H)$;
(2) for every $u \in V(G)$, the graph $H[L(u)]$ is complete;
(3) if $E(L(u), L(v)) \neq \varnothing$, then either $u = v$ or $uv \in E(G)$;
(4) if the multiplicity of an edge $uv \in E(G)$ is k, then $H[L(u), L(v)]$ is the union of at most k matchings connecting $L(u)$ with $L(v)$.

A cover (L, H) of G is *k-fold* if $|L(u)| = k$ for every $u \in V(G)$.

Throughout this paper, we consider only 2-fold covers. And we call the vertices in the cover graph by "nodes", in order to distinguish them from the vertices in the original graph.

Definition 2. Let G be a multigraph and $\mathscr{H} = (L, H)$ be a cover of G. An *$\mathscr{H}$-map* is an injection $\varphi : V(G) \to V(H)$, such that $\varphi(v) \in L(v)$ for every $v \in V(G)$. The subgraph of H induced by $\varphi(V(G))$ is called the *φ-induced cover graph*, denoted by H_φ.

Definition 3. Let $\mathscr{H} = (L, H)$ be a cover of G. For $u \in V(G)$, let $L(u) = \{p(u), r(u)\}$, where $p(u)$ and $r(u)$ are called the *poor* and the *rich* nodes, respectively. Given $i, j \geq 0$ and $i \leq j$, an $\mathscr{H}$-map φ is an *(i, j)-defective-$\mathscr{H}$-coloring of G* if the degree of every poor node in H_φ is at most i, and the degree of every rich node in H_φ is at most j.

Definition 4. A multigraph G is *(i, j)-defective-DP-colorable* if for every 2-fold cover $\mathscr{H} = (L, H)$ of G, there exists an (i, j)-defective-$\mathscr{H}$-coloring. We say G is *(i, j)-defective-DP-critical*, if G is not (i, j)-defective-DP-colorable, but every proper subgraph of G is.

For brevity, in the rest of the paper, we call an (i, j)-defective-$\mathscr{H}$-coloring simply by an *$(i, j, \mathscr{H})$-coloring* (or '$\mathscr{H}$-coloring', if i and j are clear from the context). Similarly, instead of "(i, j)-defective-DP-colorable" and "(i, j)-defective-DP-critical" we will say "(i, j)-colorable" and "(i, j)-critical".

We say a 2-fold cover $\mathscr{H} = (L, H)$ of a simple graph G is *full* if for every edge $uv \in E(G)$, the matching in H corresponding to uv is perfect. To show that G is (i, j)-colorable, it suffices to show that G admits an $(i, j, \mathscr{H})$-coloring for all full 2-fold cover $\mathscr{H}$ of G. Hence below we consider only full covers.

Denote the minimum number of edges in an n-vertex (i, j)-*critical multigraph* by $f_{DP}(i, j, n)$, and the minimum number of edges in an n-vertex (i, j)-*critical simple graph* by $g_{DP}(i, j, n)$. By definition, $f_{DP}(i, j, n) \leq g_{DP}(i, j, n)$. Jing, Kostochka, Ma, Sittitrai and Xu [17] proved lower bounds on $f_{DP}(i, j, n)$ that are exact for infinitely many n for every choice of $i \leq j$.

Theorem B ([17])**.**

(1) If $i = 0$ and $j \geq 1$, then $f_{DP}(0, j, n) \geq n + j$. This is sharp for every $j \geq 1$ and every $n \geq 2j + 2$.

(2) If $i \geq 1$ and $j \geq 2i + 1$, then $f_{DP}(i, j, n) \geq \frac{(2i+1)n-(2i-j)}{i+1}$. This is sharp for each such pair (i, j) for infinitely many n.

(3) If $i \geq 1$ and $i + 2 \leq j \leq 2i$, then $f_{DP}(i, j, n) \geq \frac{2jn+2}{j+1}$. This is sharp for each such pair (i, j) for infinitely many n.

(4) If $i \geq 1$, then $f_{DP}(i, i+1, n) \geq \frac{(2i^2+4i+1)n+1}{i^2+3i+1}$. This is sharp for each $i \geq 1$ for infinitely many n.

(5) If $i \geq 1$, then $f_{DP}(i, i, n) \geq \frac{(2i+2)n}{i+2}$. This is sharp for each $i \geq 1$ for infinitely many n.

The bound in Part (1) is also sharp for simple graphs.

For $i > 0$ we know little if the bounds of Theorem B are sharp on simple graphs. In fact, we think that $g_{DP}(i, j, n) > f_{DP}(i, j, n)$ for $i > 0$. It follows from [22] that $g_{DP}(1, 1, n) > f_{DP}(1, 1, n)$ and $g_{DP}(2, 2, n) > f_{DP}(2, 2, n)$. Recently Jing, Kostochka, Ma and Xu [18] showed that when $i \geq 3$ and $j \geq 2i + 1$, the exact lower bound for $g_{DP}(i, j, n)$ differs from the bound of Theorem B(2) but only by 1.

Theorem C ([18])**.** *Let $i \geq 3$, $j \geq 2i + 1$ be positive integers, and let G be an (i, j)-critical simple graph. Then*

$$g_{DP}(i, j, n) \geq \frac{(2i + 1)n + j - i + 1}{i + 1}.$$

This is sharp for each such pair (i, j) for infinitely many n.

The goal of this paper is to extend Theorem C to $i = 1, 2$ and $j \geq 2i$, and to show that our bound is exact for infinitely many n for each such pair (i, j).

2. Results

The main result of this paper is the following.

Theorem 2.1. *Let $i = 1, 2$, $j \geq 2i$ be positive integers, and let G be an (i, j)-critical simple graph. Then*

$$g_{DP}(i, j, n) \geq \frac{(2i + 1)n + j - i + 1}{i + 1}.$$

This is sharp for each such pair (i, j) for infinitely many n.

Comparing Theorem 2.1 with Theorem B(3) we see that not only $g_{DP}(1, 2, n) > f_{DP}(1, 2, n)$ and $g_{DP}(2, 4, n) > f_{DP}(2, 4, n)$, but also the asymptotics when $n \to \infty$ are different.

Since every non-(i, j)-colorable graph contains an (i, j)-critical subgraph, Theorem 2.1 yields the following.

Corollary 2.2. *Let G be a simple graph. If $i = 1, 2$ and $j \geq 2i$ and for every subgraph H of G, $|E(H)| \leq \frac{(2i+1)|V(H)|+j-i}{i+1}$, then G is (i, j)-colorable. This is sharp.*

In the next section we introduce a more general framework to prove the lower bound of Theorem 2.1. In Section 4 we prove some useful lemmas that apply to all pairs of $i = 1, 2, j \geq 2i$. In Sections 5 and 6, we prove Theorem 2.1 for $i = 1, j \geq 2$, and for $i = 2, j \geq 4$, respectively. In the last section, we present constructions showing that our bounds are sharp for each pair of $i = 1, 2, j \geq 2i$ for infinitely many n.

3. Notation and a more general framework

For induction purposes, it will be useful to prove a more general result. Instead of (i, j)-colorings of a cover (L, H) of a graph G, we will consider (L, H)-maps φ with variable restrictions on the degrees of the vertices in H_φ.

Definition 5. A *capacity function* on G is a map $\mathbf{c} : V(G) \to \{-1, 0, \dots, i\} \times \{-1, 0, \dots, j\}$. For $u \in V(G)$, denote $\mathbf{c}(u)$ by $(\mathbf{c}_1(u), \mathbf{c}_2(u))$. We call such pair $(G, \mathbf{c})$ a *weighted pair*.

Below, let $(G, \mathbf{c})$ be a weighted pair, and $\mathscr{H} = (L, H)$ be a cover of G. For a subgraph G' of G, let $\mathscr{H}_{G'} = (L_{G'}, H_{G'})$ denote the subcover *induced* by G', i.e.,

(1) $L_{G'} = L|_{V(G')}$, where '$f|_S$' is the restriction of function f to subdomain S;

(2) $V(H_{G'}) = L(V(G'))$ and $L_{G'}(v) = L(v)$ for every $v \in V(G')$;

(3) $H_{G'}[L(u) \cup L(v)] = H[L(u) \cup L(v)]$ for every $uv \in E(G')$, and for x, y such that $xy \notin E(G')$, there is no edge between $L_{G'}(x)$ and $L_{G'}(y)$.

For a subset S of $V(G)$, let $\mathscr{H}_S = (L_S, H_S)$ denote the subcover induced by $G[S]$. If a capacity function is the restriction of $\mathbf{c}$ to some $S \subseteq V(G)$, we denote this capacity function by $\mathbf{c}$ instead of $\mathbf{c}|_S$, for simplicity. Let $N_G(S)$, or $N(S)$ when clear from the context, denote the vertices in $G - S$ having a neighbor in S. When $S = \{v\}$ for some single vertex v, we write $N(v)$ instead of $N(\{x\})$ for simplicity.

For two vertices/nodes x, y, we use $x \sim y$ to indicate that x is adjacent to y, and $x \nsim y$ to indicate that x is not adjacent to y.

Definition 6. A *$(\mathbf{c}, \mathscr{H})$-coloring of* G is an $\mathscr{H}$-map φ such that for each $u \in V(G)$, the degree of $p(u)$ in H_φ is at most $\mathbf{c}_1(u)$, and that of $r(u)$ is at most $\mathbf{c}_2(u)$. We call $\mathbf{c}_1(u)$ the *capacity* of $p(u)$ and $\mathbf{c}_2(u)$ the *capacity* of $r(u)$. If the capacity of some v in $V(H)$ is -1, then v is not allowed in the image of any $(\mathbf{c}, \mathscr{H})$-coloring of G. If for every cover $\mathscr{H}$ of G, there is a $(\mathbf{c}, \mathscr{H})$-coloring, we say that G is $\mathbf{c}$-*colorable*.

If $\mathbf{c}(v) = (i, j)$ for all $v \in V(G)$, then any $(\mathbf{c}, \mathscr{H})$-coloring of G is an $(i, j, \mathscr{H})$-coloring in the sense of Definition 3. So, Definition 6 is a refinement of Definition 3. Similarly, we say that G is $\mathbf{c}$-*critical* if G is not $\mathbf{c}$-colorable, but every proper subgraph of G is. For every node x in the cover graph, we slightly abuse the notation of $\mathbf{c}$ and denote the capacity of x by $\mathbf{c}(x)$. When $\mathscr{H}$ is clear from the context, say it is equal to $\mathscr{H}$ or some induced subcover, we drop the corresponding cover notation and say G has a $\mathbf{c}$-coloring for some capacity function $\mathbf{c}$, for simplicity.

Definition 7. For a vertex $u \in V(G)$, the $(i, j, \mathbf{c})$-*potential* of u is

$$\rho_{\mathbf{c}}(u) := i - j + 1 + \mathbf{c}_1(u) + \mathbf{c}_2(u).$$

The $(i, j, \mathbf{c})$-potential of a subgraph G' of G is

$$\rho_{G,\mathbf{c}}(G') := \sum\nolimits_{u \in V(G')} \rho_{\mathbf{c}}(u) - (i+1)|E(G')|. \tag{1}$$

For a subset $S \subseteq V(G)$, the $(i, j, \mathbf{c})$-potential of S, $\rho_{G,\mathbf{c}}(S)$, is the $(i, j, \mathbf{c})$-potential of $G[S]$. The $(i, j, \mathbf{c})$-potential of $(G, \mathbf{c})$ is defined by $\rho(G, \mathbf{c}) := \rho_{G,\mathbf{c}}(G)$.

When clear from context, we call the $(i, j, \mathbf{c})$-potential simply by potential.

If follows from the definition (see, e.g. Lemma 3.1 in [18]) that the potential function is submodular: For all $A, B \subseteq V(G)$,

$$\rho_{G,\mathbf{c}}(A) + \rho_{G,\mathbf{c}}(B) = \rho_{G,\mathbf{c}}(A \cup B) + \rho_{G,\mathbf{c}}(A \cap B) + (i+1)|E(A \setminus B, B \setminus A)|. \tag{2}$$

The main lower bound in this paper is the following.

Theorem 3.1. *Let $i = 1, 2$, $j \geq 2i$ be positive integers, and $(G, \mathbf{c})$ be a weighted pair such that G is $\mathbf{c}$-critical. Then $\rho(G, \mathbf{c}) \leq i - j - 1$.*

To deduce the lower bound in Theorem 2.1, simply set $\mathbf{c}(v) = (i, j)$ for every $v \in V(G)$. We will prove the lower bound of Theorem 3.1 in the next three sections and present a sharpness construction in Section 7.

4. Basic lemmas

Let $1 \leq i \leq 2$ and $j \geq 2i$. Suppose there exists a $\mathbf{c}$-critical graph G with $\rho(G, \mathbf{c}) \geq i - j$. Choose such $(G, \mathbf{c})$ with $|V(G)| + |E(G)|$ minimum. We say that G' is *smaller* than G if $|V(G')| + |E(G')| < |V(G)| + |E(G)|$. Let $\mathscr{H} = (L, H)$ be an arbitrary cover of G. In this section we show some properties of smallest counterexamples $(G, \mathbf{c})$.

Lemma 4.1. *Let S be a proper subset of $V(G)$. If $\rho_{G,\mathbf{c}}(S) \leq i - j$, then $S = \{x\}$ for some $x \in V(G)$ with $\rho_{\mathbf{c}}(x) = i - j$.*

Proof. Suppose the lemma fails. Let S be a maximal proper subset of $V(G)$ such that $\rho_{G,\mathbf{c}}(S) \leq i - j$ and $|S| \geq 2$. If $|N(v) \cap S| \geq 2$ for some $v \in V(G) \setminus S$, then

$$\rho_{G,\mathbf{c}}(S \cup \{v\}) \leq i - j + 2i + 1 - 2(i+1) = i - j - 1.$$

If $S \cup \{v\} \neq V(G)$, this contradicts the maximality of S, otherwise this contradicts the choice of G. Thus

$$|N(v) \cap S| \leq 1 \textit{ for every } v \in V(G) \setminus S. \tag{3}$$

Since G is $\mathbf{c}$-critical, $G[S]$ admits an $(\mathbf{c}, \mathscr{H}_S)$-coloring φ.

Construct G' from $G - S$ by adding a new vertex v^* adjacent to every $u \in V(G) - S$ that was adjacent to a vertex in S. Define a capacity function $\mathbf{c}'$ by letting $\mathbf{c}'(v^*) = (-1, 0)$ and $\mathbf{c}'(u) = \mathbf{c}(u)$ for $u \in V(G' - v^*)$.

By (3), G' is simple. Suppose $\rho_{G',\mathbf{c}'}(A) \leq i - j - 1$ for some $A \subseteq V(G')$. Since $G' - v^* \subseteq G$ and $\mathbf{c}'(u) = \mathbf{c}(u)$ for $u \in V(G' - v^*)$, $v^* \in A$, otherwise it contradicts the choice of G. Then using (2) and $\rho_{G,\mathbf{c}}(S) \leq i - j = \rho_{G',\mathbf{c}'}(v^*)$,

$$\rho_{G,\mathbf{c}}(S \cup (A - v^*)) = \rho_{G,\mathbf{c}}(S) + \rho_{G,\mathbf{c}}(A - v^*) - (i+1)|E_G(S, A - v^*)|$$

$$\leq \rho_{G',\mathbf{c}'}(v^*) + \rho_{G',\mathbf{c}'}(A - v^*) - (i+1)|E_{G'}(v^*, A - v^*)| = \rho_{G',\mathbf{c}'}(A) \leq i - j - 1.$$

Again, this contradicts either the maximality of S or the choice of G. This yields

$$\rho(G', \mathbf{c}') \geq i - j. \tag{4}$$

For every $x \in S$ and $y \in N(x) \setminus S$, denote the neighbor of $\varphi(x)$ in $L(y)$ by y_φ. Let $\mathscr{H}' = (L', H')$ be the cover of G' defined as follows:

1) $L'(v^*) = \{p(v^*), r(v^*)\}$, and $L'(u) = L(u)$ for every $u \in V(G) \setminus S$;

2) $y_\varphi \sim r(v^*)$ for every $y \in N(S)$, and $H'[\{u,w\}] = H[\{u,w\}]$ for every $u, w \in V(G'-v^*)$.

By (4) and the minimality of G, G' has a $(\mathbf{c}', \mathscr{H}')$-coloring ψ. Since $\mathbf{c}'(v^*) = (-1, 0)$,

(5) $$\psi(v^*) = r(v^*) \textit{ and } \psi(y) \neq y_\varphi \textit{ for every } y \in N(S).$$

Let θ be an $\mathscr{H}$-map such that $\theta|_S = \varphi$ and $\theta|_{V(G)\setminus S} = \psi|_{V(G'-v^*)}$. By (5), θ is a $(\mathbf{c}, \mathscr{H})$-coloring of G, a contradiction. $\square$

Lemma 4.1 implies that

(6) $$\text{for every } F \subsetneq V(G) \text{ with } |F| \geq 2,\ \rho_{G,\mathbf{c}}(F) \geq i - j + 1.$$

Claim 4.2. *There is no edge $xy \in E(H)$, such that $\mathbf{c}(x) = \mathbf{c}(y) = -1$.*

Proof. Suppose $uv \in E(G)$, $L(u) = \{u_1, u_2\}, L(v) = \{v_1, v_2\}$, and in H, $u_1 \sim v_1, u_2 \sim v_2$ in H, $\mathbf{c}(u_1) = \mathbf{c}(v_1) = -1$. Then by

$$i - j \leq \rho(G, \mathbf{c}) \leq \rho_{G,\mathbf{c}}(\{u,v\}) = \mathbf{c}(u_2) + \mathbf{c}(v_2) - 1 - 1 + 2(i-j+1) - (i+1),$$

we get $\mathbf{c}(u_2) + \mathbf{c}(v_2) \geq j+1$, which implies $\mathbf{c}(u_2), \mathbf{c}(v_2) \geq 1$, since $\mathbf{c}(x) \leq j$ for all $x \in V(H)$. Let $\mathbf{c}'$ differ from $\mathbf{c}$ only in that $\mathbf{c}'(u_2) = \mathbf{c}(u_2) - 1, \mathbf{c}'(v_2) = \mathbf{c}(v_2) - 1$. Then $\rho_{\mathbf{c}'}(G - uv) \geq i - j$. Suppose not, let $S \subset V(G-uv)$ be a set with $\rho_{G-uv,\mathbf{c}'}(S) \leq i-j-1$. Then $S \cap \{u,v\} \neq \emptyset$. If $u, v \in S$, then $\rho_{G,\mathbf{c}}(S) = \rho_{G-uv,\mathbf{c}'}(S) + 2 - (i+1) \leq i-j-1$, contradiction. If $u \in S, v \notin S$, then $\rho_{G,\mathbf{c}}(S) = \rho_{G-uv,\mathbf{c}'}(S) + 1 \leq i - j$. By Lemma 4.1, $S = \{u\}$, contradicts the fact that $\mathbf{c}(u_2) \geq 1$. Hence, by symmetry, we may assume that ϕ is a $\mathbf{c}'$-coloring of $G - uv$. Then ϕ is also a $\mathbf{c}$-coloring on G, a contradiction. $\square$

The argument in Claim 4.2 implies that

(7) *For each $uv \in E(G)$, with $\rho_{\mathbf{c}}(u), \rho_{\mathbf{c}}(v) \geq i-j+1$, if $\mathbf{c}'$ differs from $\mathbf{c}$ only in that for some $x \in L(u), y \in L(v)$, $\mathbf{c}'(x) = \mathbf{c}(x) - 1, \mathbf{c}'(y) = \mathbf{c}(y) - 1$, then $G - uv$ has a $\mathbf{c}'$-coloring.*

Claim 4.3. *$|V(G)| \geq 3$.*

Proof. If G has only one vertex, then for G to be $\mathbf{c}$-critical, the single vertex has to have capacity $(-1,-1)$, which implies that $\rho_{\mathbf{c}}(G) = i-j-1$, a contradiction.

Now suppose $V(G) = \{u, v\}$ and $G = K_2$. Say $L(u) = \{u_1, u_2\}, L(v) = \{v_1, v_2\}$, and $u_1 \sim v_1, u_2 \sim v_2$ in H. If $\mathbf{c}(u_k)$ and $\mathbf{c}(v_{3-k})$ are both non-negative for either $k = 1$ or 2, then we can color G by letting $\varphi(u) = u_k, \varphi(v) = v_{3-k}$. Thus we may assume $\mathbf{c}(u_2) = \mathbf{c}(v_2) = -1$. But this contradicts Claim 4.2. $\square$

Lemma 4.4. *For every $u \in V(G)$, $\mathbf{c_1}(u), \mathbf{c_2}(u) \geq 0$ and $d(u) \geq 2$.*

Proof. Suppose there is a vertex $u \in V(G)$ with $L(u) = \{u_1, u_2\}$, where $\mathbf{c}(u_1) = -1$. Then $\mathbf{c}(u_2) \geq 0$. Choose such u with the smallest $\mathbf{c}(u_2)$. Let $v \in N(u)$, $L(v) = \{v_1, v_2\}$ and $u_1v_1, u_2v_2 \in E(H)$. By Claim 4.2, $\mathbf{c}(v_1) \geq 0$. If $\mathbf{c}(v_2) = -1$, then let ϕ be a $\mathbf{c}$-coloring on $G - uv$. Since $\mathbf{c}(u_1) = \mathbf{c}(v_2) = -1$, $\phi(u) = u_2, \phi(v) = v_1$. Then ϕ is a $\mathbf{c}$-coloring on G, a contradiction.

Now suppose $\mathbf{c}(u_2) \geq 1$. Then by (7), $G - uv$ has a $\mathbf{c}'$-coloring ϕ, where $\mathbf{c}'(x) = \mathbf{c}(x) - 1$ for $x \in \{u_2, v_2\}$. Then ϕ is a $\mathbf{c}$-coloring on G, a contradiction.

Thus we may assume $\mathbf{c}(u_2) = 0$. If $d(v) = 1$, then we find a $\mathbf{c}$-coloring φ of $G - v$ since G is critical, and then let $\varphi(v) = v_1$.

So we may assume $d(v) \geq 2$. For a neighbor $w \neq u$ of v, denote $L(w) = \{w_1, w_2\}$ so that $w_1v_1, w_2v_2 \in E(H)$. If $\mathbf{c}(w_1) = -1$ for all $w \in N(v)$, then we can easily extend a $\mathbf{c}$-coloring φ of $G - v$ to G by letting $\varphi(v) = v_1$. Thus we may assume some neighbor w^* of v has $\mathbf{c}(w_1^*) \geq 0$. Form $(G', \mathbf{c}')$, such that $G' = G - vw^*$, $\mathbf{c}'$ differs from $\mathbf{c}$ by only $\mathbf{c}'(x_1) = \mathbf{c}(x_1) - 1$ for $x \in \{v, w^*\}$.

If G' has a $\mathbf{c}'$-coloring φ, then by definition, $\varphi(u) = u_2$, and since $\mathbf{c}(u_2) = 0$, $\phi(v) = v_1$. Then ϕ is a $\mathbf{c}$-coloring on G, a contradiction. Otherwise, by (7), $\rho_{\mathbf{c}}(w) = i - j$. Then $\rho_{G,\mathbf{c}}(\{u, v, w\}) = \rho_{\mathbf{c}}(u) + \rho_{\mathbf{c}}(v) + \rho_{\mathbf{c}}(w) - 2(i+1) \leq 2(i-j) + 2i + 1 - 2(i+1) = i - j - 1 + (i-j) < i - j - 1$, a contradiction to the choice of G.

This proves the first half of the statement. Now if some u has degree 1, since $\mathbf{c_1}(u), \mathbf{c_2}(u) \geq 0$, we can extend any coloring of $G - u$ to u greedily, a contradiction. □

Lemmas 4.1 and 4.4 imply that

(8) $$\text{for every } \emptyset \neq F \subsetneq V(G),\ \rho_{G,\mathbf{c}}(F) \geq i - j + 1.$$

Corollary 4.5. *Let $\emptyset \neq S \subset V(G)$. Then for all $A, B \subset V(G) \setminus S$ with $\rho(A) = \rho(B) = i - j + 1$,*

$$\rho(A \cup B) = \rho(A \cap B) = i - j + 1.$$

Proof. By submodularity, $\rho(A) + \rho(B) \geq \rho(A \cup B) + \rho(A \cap B)$. Also none of $A, B, A \cup B, A \cap B$ is equal to $V(G)$. So the claim follows from (8). □

A helpful notion in this and next sections is the notion of $B(S)$: For $S \subset V(G)$, denote by $B(S)$ the union of all the subsets in $V(G) \setminus S$ with potential equal to $i - j + 1$. When S consists of only a single vertex v, we write $B(v)$ instead of $B(\{v\})$ for simplicity. By Corollary 4.5, we have the following.

Corollary 4.6. *For every nonempty $S \subsetneq V(G)$, if $B(S) \neq \emptyset$, then $\rho_{G,\mathbf{c}}(B(S)) = i - j + 1$.*

Claim 1. *Let $xy \in E(G)$, $L(x) = \{x_1, x_2\}, L(y) = \{y_1, y_2\}$, $x_1 \sim y_1, x_2 \sim y_2$. For $h = 1, 2$, graph $G - xy$ has a $\mathbf{c}$-coloring ψ_h such that $\psi_h(x) = x_h$.*

Indeed, let $G' = G - xy$ and let $\mathbf{c}'$ differ from $\mathbf{c}$ only in that $\mathbf{c}'(x_{3-h}) = \mathbf{c}(x_{3-h}) - 1$ and $\mathbf{c}'(y_{3-h}) = \mathbf{c}(h_{3-h}) - 1$. By (8), if $A \subseteq V(G)$ and $|A \cap \{x, y\}| \leq 1$, then $\rho_{G',\mathbf{c}'}(A) \geq \rho_{G,\mathbf{c}}(A) - 1 \geq 2 - j$. On the other hand if $\{x, y\} \subseteq A$, then $\rho_{G',\mathbf{c}'}(A) \geq \rho_{G,\mathbf{c}}(A) - 2 + (i+1) \geq \rho_{G,\mathbf{c}}(A) \geq 2 - j$. So, by the minimality of G, G' has a $\mathbf{c}'$-coloring ψ_h.

If $\psi_h(x) = x_{3-h}$, then by the definition of $\mathbf{c}'$, ψ_h is a $\mathbf{c}$-coloring of G regardless of the value of $\psi_h(y)$, a contradiction. Thus, $\psi_h(x) = x_h$, as claimed. ⋈

We say a vertex $v \in V(G)$ is a $(d; c_1, c_2)$-*vertex* if $d(v) = d$, $\mathbf{c}_1(v) = c_1$, and $\mathbf{c}_2(v) = c_2$. We call a $(2; i, j)$-vertex a *surplus vertex*. All non-surplus vertices will be called *ordinary*. Let $V_0 = V_0(G, \mathbf{c})$ denote the set of surplus vertices in $(G, \mathbf{c})$.

Observation 4.7. *Surplus vertices in $(G, \mathbf{c})$ cannot be adjacent.*

Proof. Suppose $v_1, v_2 \in V_0(G)$ and $v_1v_2 \in E(G)$. Let v_j' be the neighbor of v_j distinct from v_{3-j}. Since G is $\mathbf{c}$-critical, $G - v_1 - v_2$ has a $\mathbf{c}$-coloring φ. Extend φ to v_1, v_2 by choosing $\varphi(v_j) \neq \varphi(v_j')$ for $j = 1, 2$. Then φ is a $\mathbf{c}$-coloring on G, a contradiction. □

To show that $(G,\mathbf{c})$ does not exist, we use discharging. It is done in two steps. At the beginning, the charge of each vertex v is $\rho_{\mathbf{c}}(v)$ and the charge of each edge is $-(i+1)$, so by (1), (6) and Lemma 4.4, the total sum of all charges is $\rho(G,\mathbf{c})$. On Step 1, each edge gives to each its end charge $-(i+1)/2$ and is left with charge 0. Note that each surplus vertex after Step 1 has charge $2i+1-2\frac{i+1}{2}=i$. On Step 2, each $u\in V_0$ gives $i/2$ to each of its two neighbors and is left with charge 0. The resulting charge, ch, can be nonzero only on vertices in $V(G)-V_0$, and the total charge over all vertices and edges does not change. Thus,

$$\sum_{v\in V(G)-V_0} ch(v)=\sum_{v\in V(G)} ch(v)=\rho(G,\mathbf{c}). \tag{9}$$

For an ordinary vertex v, if $d_1(v)$ denotes the number of ordinary neighbors of v and $d_2(v)$ denotes the number of surplus neighbors of v, then

$$ch(v)=\rho_{\mathbf{c}}(v)-\frac{i+1}{2}d_1(v)-\frac{1}{2}d_2(v)=\mathbf{c}_1(v)+\mathbf{c}_2(v)+i-j+1-\frac{i+1}{2}d_1(v)-\frac{1}{2}d_2(v). \tag{10}$$

The following lemma is the crucial final step of our proofs.

Lemma 4.8. *For $i\geq 1, j\geq 2i$, if $ch(v)\leq 0$ for all $v\in V(G)$, then G is $\mathbf{c}$-colorable.*

Proof. Suppose $ch(v)\leq 0$ for all $v\in V(G)$. Construct G_h and H_h from G and H as follows:

Forming G_h: for each surplus vertex v, say $N(v)=\{u,w\}$, we replace v with an edge connecting u and w. We call such edge a *half edge*. And if $L(x)=\{x_1,x_2\}$ for $x\in\{u,v,w\}$ so that $u_1v_1, w_1v_1\in E(H)$, then for the cover graph H_h we delete $L(v)$ from H and add edges u_1w_2, u_2w_1. Note that G_h and H_h are not necessarily simple graphs.

Since $\rho(G,\mathbf{c})=\sum_{v\in V(G_h)} ch(v)\geq i-j$ by assumption and all charges are nonpositive,

(11) *$ch(v)\geq i-j$ for every vertex v, and equality may hold for at most one vertex.*

Let φ be a $\mathscr{H}_{G_h}$-map (does not need to be a coloring). Define
$d^*_{\varphi}(v)=|\{uv: uv \text{ not a half edge}, \varphi(u)\sim\varphi(v)\}|+\frac{1}{2}|\{uv: uv \text{ is a half edge}, \varphi(u)\sim\varphi(v)\}|$,
$S(\varphi):=\sum_{v\in V(G)}\mathbf{c}(\varphi(v))-\frac{1}{2}\sum_{v\in V(G)} d^*_{\varphi}(v)$.

Let $S:=\max\{S(\varphi):\varphi \text{ is a } \mathscr{H}_{G_h}\text{-map}\}$, and ψ a $\mathscr{H}_{G_h}$-map with $S(\psi)=S$.

Suppose $\mathbf{c}(\psi(u))<d^*_{\psi}(u)$ for some $u\in V(G_h)$. Let ψ_u differ from ψ only on u. By the choice of ψ, $S(\psi_u)\leq S(\psi)$. Hence

$$0\geq S(\psi_u)-S(\psi)=\mathbf{c}(\psi_u(u))-d^*_{\psi_u}(u)-(\mathbf{c}(\psi(u))-d^*_{\psi}(u)).$$

So, $\mathbf{c}(\psi_u(u))-d^*_{\psi_u}(u)\leq\mathbf{c}(\psi(u))-d^*_{\psi}(u)$, and

$$\begin{aligned}2(\mathbf{c}(\psi(u))-d^*_{\psi}(u))&\geq\mathbf{c}(\psi(u))-d^*_{\psi}(u)+\mathbf{c}(\psi_u(u))-d^*_{\psi_u}(u)=\mathbf{c}_1(u)+\mathbf{c}_2(u)-d_1(u)-\frac{d_2(u)}{2}\\&=ch(u)-i+j-1+\frac{i-1}{2}d_1(u)\geq i-j-i+j-1+\frac{i-1}{2}d_1(u)\geq -1.\end{aligned}$$

Also since $2(\mathbf{c}(\psi(u))-d^*_{\psi}(u))$ is an integer, we must have $ch(u)=i-j$ and $\mathbf{c}(\psi(u))-d^*_{\psi}(u)=\mathbf{c}(\psi_u(u))-d^*_{\psi_u}(u)=-1/2$. And by (11), there is at most one such u.

We say an edge $xy\in E(G_h)$ is *ψ-conflicting* if $\psi(x)\sim\psi(y)$. Let G' be the spanning subgraph of G_h where $E(G')$ consists of only the ψ-conflicting half edges in G under ψ.

Since $\mathbf{c}(\psi(u))-d^*_\psi(u)=-1/2$, $d_{G'}(u)>0$ and is an odd number. Let C be the component in G' containing u. Then there is another vertex $v\in C$ with $d_{G'}(v)$ odd. And since u is the unique vertex with $\mathbf{c}(\psi(u))<d^*_\psi(u)$, $\mathbf{c}(\psi(v))-d^*_\psi(v)\geq 1/2$. Let P be a uv-path in G'. Let $G''=G'-E(P)$. Add a vertex v^* to G'' and add an edge between v^* and every odd-degree vertex in G''. Then we can decompose $E(G'')$ into cycles. Let τ be an Eulerian orientation on these cycles. Extend τ to $E(P)$ so that P is a directed path from v to u.

We extend ψ from G_h to G as follows. For a surplus vertex x, if it does not correspond to a ψ-conflicting edge in G_h, color x so that $\varphi(x)$ is adjacent to neither of its neighbors; if x corresponds to a ψ-conflicting vertex, then we color x so that $\psi(x)$ is not adjacent to the head of the corresponding half edge w.r.t. τ. Then ψ is a $\mathbf{c}$-coloring of G, a contradiction.

If $\mathbf{c}(\psi(x))-d^*_\psi(x)\geq 0$ for every $x\in V(G_h)$, then we extend ψ to G as above, just take $G''=G'$ in the above construction. □

In the following two sections, we will prove that for each $i=1,2, j\geq 2i$, $ch(v)\leq 0$ for every $v\in V(G)$ (assume for each pair of (i,j) with $i=1,2, j\geq 2i$, $(G,\mathbf{c})$ denotes the minimal counterexample). Then together with Lemma 4.8, we will get a contradiction to the choice of G.

5. The case of $i=1$ and $j\geq 2$

In this section, the potential of a vertex with capacity (c_1,c_2) is c_1+c_2+2-j, and the potential of an edge is -2. Our G has potential at least $1-j$. And by (8), the potential of each proper nonempty subset of $V(G)$ is at least $2-j$. To show that G has no vertices with positive charge, we first prove four claims on potentials of vertices with small degree.

Claim 5.1. *If $u\in V(G)$ is a degree 2 vertex, then $\rho_{\mathbf{c}}(u)\neq 2$.*

Proof. Suppose $N_G(u)=\{v,w\}$ and $\rho(u)=2$. By symmetry, we may assume $\mathbf{c}(u_2)\geq\mathbf{c}(u_1)$. For $x\in\{u,v,w\}$ let $L(x)=\{x_1,x_2\}$ be such that $y_1u_1, y_2u_2\in E(H)$ for $y\in N(u)$.

Case 1. $\mathbf{c}(u_1)=0, \mathbf{c}(u_2)=j$. Form $(G',\mathbf{c}')$, such that $G'=G-u$ and $\mathbf{c}'$ differs from $\mathbf{c}$ in G' by only $\mathbf{c}'(x_2)=\mathbf{c}(x_2)-1$ for $x\in N(u)$. If there is $A\subset V(G')$ with $\rho_{G',\mathbf{c}'}(A)\leq -j$, then by (8), $v,w\in A$. But then $\rho_{G,\mathbf{c}}(A\cup\{u\})=\rho_{G',\mathbf{c}'}(A)+2+2-2\cdot 2\leq -j$, a contradiction.

Case 2. $\mathbf{c}(u_1)=1, \mathbf{c}(u_2)=j-1$. For each $x\in N(u)$, form $(G',\mathbf{c_x})$ so that $G'=G-u$, $\mathbf{c_x}$ differs from $\mathbf{c}$ by only $\mathbf{c_x}(x_i)=\mathbf{c}(x_i)-1$ for $i=1,2, x\in N(u)$. If there is some $S_x\subset V(G')$ such that $\rho_{G',\mathbf{c_x}}(S_x)\leq -j$, then $x\in S_x$. If $N(u)\subset S_x$, then $\rho_{G,\mathbf{c}}(S_x\cup\{u\})=\rho_{G',\mathbf{c_x}}(S_x)+2+2-2\cdot 2\leq -j$, a contradiction. Thus $S_v\cap N(u)=\{v\}, S_w\cap N(u)=\{w\}$, and $\rho_{G,\mathbf{c}}(S_x)\leq 2-j$ for $x\in N(u)$. By submodularity and (8), $\rho_{G,\mathbf{c}}(S_v\cup S_w)\leq\rho_{G,\mathbf{c}}(S_v)+\rho_{G,\mathbf{c}}(S_w)\leq 2-j$. Then $\rho_{G,\mathbf{c}}(S_v\cup S_w\cup\{u\})\leq 2-j+2-2\cdot 2=-j$, a contradiction. □

Claim 5.2. *Let $u\in V(G)$ be a degree three vertex. If u has at least one surplus neighbor, then $\rho_{\mathbf{c}}(u)\leq 1$.*

Proof. Suppose $N(u)=\{x,y,v\}$, $N(v)=\{u,v'\}$, v is a surplus vertex and $\rho_{\mathbf{c}}(u)\geq 2$. For $w\in N(v)\cup N(u)$, let $L(w)=\{w_1,w_2\}$ so that $w_1z_1, w_2z_2\in E(H)$ whenever $w\sim z$ in G. By symmetry, assume $\mathbf{c}(u_1)\leq\mathbf{c}(u_2)$.

Case 1. $\mathbf{c}(u_2)\geq 2$. Form $(G',\mathbf{c}')$ as follows: $G'=G-u-v$ and $\mathbf{c}'$ differs from $\mathbf{c}$ only by $\mathbf{c}'(z_2)=\mathbf{c}(z_2)-1$ for $z\in\{x,y\}$. If some $S\subset V(G')$ has $\rho_{G',\mathbf{c}'}(S)\leq -j$, then

$x, y \in S$. But then $\rho_{G,\mathbf{c}}(S \cup \{u\}) = \rho_{G',\mathbf{c}'}(S) + 2 + \rho_{\mathbf{c}}(u) - 2 \cdot 2 \le 1 - j$ if $v' \notin S$, and $\rho_{G,\mathbf{c}}(S \cup \{u, v\}) \le \rho_{G',\mathbf{c}'}(S) + 2 + 3 \cdot 2 - 2 \cdot 4 \le -j$ if $v' \in S$. Neither case is possible by (8). Thus G' has a $\mathbf{c}'$-coloring φ. Extend φ to G: first let $\varphi(v) \nsim \varphi(v')$, and then let $\varphi(u) = u_2$, unless all three neighbors of u_2 are colored by φ, in which case let $\varphi(u) = u_1$.

Case 2. $\mathbf{c}(u_2) = 1$. Then $j = 2$, $\mathbf{c}(u_1) = 1$ and $\rho_{\mathbf{c}}(u) = 2$. Still, let $G' = G - u - v$. Define $\mathbf{c}''$ so that $\mathbf{c}''$ differs from $\mathbf{c}'$ by only $\mathbf{c}''(v_1') = \mathbf{c}'(v_1') - 1$. If some $S \subset V(G')$ has $\rho_{G',\mathbf{c}''}(S) \le -j$, then $|S \cap \{x, y, v'\}| \ge 2$. If $\{x, y, v'\} \subset S$, then $\rho_{G,\mathbf{c}}(S \cup \{u, v\}) = \rho_{G',\mathbf{c}''}(S) + 3 + \rho_{\mathbf{c}}(u) + \rho_{\mathbf{c}}(v) - 4 \cdot 2 \le -j$, a contradiction. If $S \cap \{x, y, v'\} = \{x, v'\}$, then $\rho_{G,\mathbf{c}}(S \cup \{u, v\}) = \rho_{G',\mathbf{c}''}(S) + 2 + \rho_{\mathbf{c}}(u) + \rho_{\mathbf{c}}(v) - 3 \cdot 2 \le 1 - j$, contradicting (8). The remaining possibilities $S \cap \{x, y, v'\} = \{y, v'\}$ and $S \cap \{x, y, v'\} = \{x, y\}$ are very similar.

Thus G' has a $\mathbf{c}''$-coloring ψ. Extend ψ to G: if $\psi(x) = x_1, \psi(y) = y_1$, let $\psi(u) = u_2$ and $\psi(v) \nsim \psi(v')$, then ψ is a $\mathbf{c}$-coloring on G. The case when $\psi(x) = x_2, \psi(y) = y_2$ is similar. Then $\psi(x) = x_1, \psi(y) = y_2$ or $\psi(x) = x_2, \psi(y) = y_1$. Let $\psi(u) = u_2$ and $\psi(v) = v_1$. Then ψ is a $\mathbf{c}$-coloring of G, a contradiction. $\square$

Claim 5.3. *A $(4; 1, j)$-vertex cannot have three surplus neighbors.*

Proof. Suppose the claim fails for some $(4; 1, j)$-vertex u. Let $N(u) = \{x, y, z, v\}$ where x, y, z are surplus vertices and x', y', z' are their other neighbors. Note that v may also be a surplus vertex. Suppose $\mathbf{c}(u_1) = 1, \mathbf{c}(u_2) = j$ For $w \in N(u)$, denote $L(w) = \{w_1, w_2\}$ so that $u_1w_1, u_2w_2 \in E(H)$, and for $w \in N(u) - v$, denote $L(w') = \{w_1', w_2'\}$ so that $w_1w_1', w_2w_2' \in E(H)$.

Case 1. $v \notin B(u)$. Form $(G', \mathbf{c}')$ so that $G' = G - \{u, x, y, z\}$ and $\mathbf{c}'$ differs from $\mathbf{c}$ by only $\mathbf{c}'(v_k) = \mathbf{c}(v_k) - 1$ for $k = 1, 2$. Since $v \notin B(u)$, $\rho_{G',\mathbf{c}'}(S) \ge 2 - j$ for each $S \subseteq V(G')$ by (8); thus by the minimality of G, G' has a $\mathbf{c}'$-coloring φ. We extend φ to G as follows. First, for $w \in \{x, y, z\}$ we let $\varphi(w) \nsim \varphi(w')$. Second, if the color of at most one vertex in $\{v, x, y, z\}$ conflicts with u_1, then we let $\varphi(u) = u_1$, else the color of at most two vertices in $\{v, x, y, z\}$ conflicts with u_2, and we let $\varphi(u) = u_2$.

Case 2. $\{v, x', y', z'\} \subseteq B(u)$. By Corollary 4.6, the following contradicts the choice of G:

$$\rho_{G,\mathbf{c}}(B(u) \cup \{u, x, y, z\}) \le \rho_{G,\mathbf{c}}(B(u)) + 4 \cdot 3 - 7 \cdot 2 \le (2 - j) - 2 = -j.$$

Case 3. $v \in B(u)$ and there is $w \in \{x, u, z\}$ such that $w' \notin B(u)$, say $x' \notin B(u)$. Then $x' \ne v$. Form $(G', \mathbf{c}'')$: let $\mathbf{c}''$ differ from $\mathbf{c}$ only by $\mathbf{c}''(v_2) = \mathbf{c}(v_2) - 1$ and $\mathbf{c}''(x_1') = \mathbf{c}(x_1') - 1$. Since $x' \notin B(u)$, $\rho_{G',\mathbf{c}''}(S) \ge 2 - j$ for each $S \subseteq V(G')$ by (8), and so G' has a $\mathbf{c}''$-coloring ψ. We extend ψ to G as follows. First, for $w \in \{y, z\}$ we let $\psi(w) \nsim \psi(w')$. If at most two $w \in \{v, y, z\}$ are colored with w_2, then we let $\psi(x) = x_1$ and $\psi(u) = u_2$, else we let $\psi(x) \nsim \psi(x')$ and $\psi(u) = u_1$. $\square$

Claim 5.4. *A $(5; 1, j)$-vertex cannot have five surplus neighbors.*

Proof. Suppose u is a $(5; 1, j)$-vertex with all its five neighbors being surplus vertices. For every $v \in N(u)$, let v' be the neighbor of v distinct from u. Let $T = N(u) \cup \{u\}$ and $T' = \{v' : v \in N(u)\}$. Consider the graph $G' = G - T$. If $T' \subseteq B(T)$, then $\rho_{G,\mathbf{c}}(B(T) \cup T) \le \rho_{G,\mathbf{c}}(B(T)) + 6 \cdot 3 - 10 \cdot 2 \le 2 - j - 2$, a contradiction. Thus, there is $x \in N(u)$ such that $x' \notin B(T)$. Denote $L(x') = \{x_1', x_2'\}$.

Define $\mathbf{c}'$ so that $\mathbf{c}'$ differs from $\mathbf{c}$ by only $\mathbf{c}'(x_k') = \mathbf{c}(x_k') - 1$ for $k = 1, 2$. Since $x' \notin B(T)$, $\rho_{G',\mathbf{c}'}(S) \ge 2 - j$ for each $S \subseteq V(G')$ by (8); thus G' has a $\mathbf{c}'$-coloring φ. For every

$w \in N(u) \setminus \{x\}$, extend φ to w so that $\varphi(w) \nsim \varphi(w')$. If $r(u)$ has at most two colored neighbors in $N(u) - x$, then we let $\varphi(u) = r(u)$, else $p(u)$ has at most $4 - 3 = 1$ colored neighbor in $N(u) - x$, and we let $\varphi(u) = p(u)$. In both cases, we let $\varphi(x) \nsim \varphi(u)$. By construction, we get a $\mathbf{c}$-coloring of G. □

Now we are ready to prove the main lemma of this section.

Lemma 5.5. *For every vertex $v \in V(G)$, $ch(v) \leq 0$.*

Proof. Let $v \in V(G)$ and $d_G(v) = d$. By Lemma 4.4, $d \geq 2$.

If $d = 2$, then by Claim 5.1, either v is a surplus vertex and has $ch(v) = 0$ by definition, or $\rho(v) \leq 1$ and $ch(v) \leq 1 - 2 \cdot (1/2) = 0$.

If $d = 3$, then by Claim 5.2, either v has no surplus neighbors and so $ch(v) \leq 3 - 3 = 0$, or $\rho(v) \leq 1$ and $ch(v) \leq 1 - 3 \cdot (1/2) = -1/2$.

If $d = 4$, then by Claim 5.3, either v has at most two surplus neighbors and so $ch(v) \leq 3 - 2 - 2 \cdot (1/2) = 0$, or $\rho(v) \leq 2$ and $ch(v) \leq 2 - 4 \cdot (1/2) = 0$.

If $d = 5$, then by Claim 5.4, either v has at most four surplus neighbors and so $ch(v) \leq 3 - 1 - 4 \cdot (1/2) = 0$, or $\rho(v) \leq 2$ and $ch(v) \leq 2 - 5 \cdot (1/2) = -1/2$.

If $d \geq 6$, then $ch(v) \leq 3 - 6 \cdot (1/2) = 0$. □

Now Lemma 4.8 together with Lemma 5.5 imply the part $i = 1$ of Theorem 3.1.

6. The case of $i = 2$ and $j \geq 4$.

In this section, the potential of a vertex with capacity (c_1, c_2) is $c_1 + c_2 + 3 - j$, and the potential of an edge is $-(i + 1) = -3$. Our G has potential at least $2 - j$. And by (8), the potential of each proper nonempty subset of $V(G)$ is at least $3 - j$.

Lemma 6.1. *Suppose there is a partition of $V(G)$ into nonempty sets F, S and R such that each vertex in S is a surplus vertex with one neighbor in F and one neighbor in R, and there is no edge connecting F with R. Then $\rho_{G,\mathbf{c}}(F) > 0$.*

Proof. Suppose $\rho_{G,\mathbf{c}}(F) \leq 0$. For every vertex $x \in S$, denote by x_F (respectively, x_R) the neighbor of x in F (respectively, in R). We say that $\alpha \in L(x_R)$ *is conflicting* with $\beta \in L(x_F)$ if their neighbors in $L(x)$ are distinct.

Since G is $\mathbf{c}$-critical, $G[F]$ has a $\mathbf{c}$-coloring φ. We obtain a new capacity function $\mathbf{c}'$ on R from $\mathbf{c}$ as follows. For every $x \in S$, decrease the capacity of the node in $L(x_R)$ conflicting with $\varphi(x_F)$ by 1. If a vertex $y \in R$ is adjacent to s vertices in S, then such decrease for nodes in $L(y)$ will happen s times. If $G[R]$ has a $\mathbf{c}'$-coloring φ', then we extend φ to each $x \in S$ by $\varphi(x) \nsim \varphi(x_F)$, and now $\varphi \cup \varphi'$ will be a $\mathbf{c}$-coloring of G by the choice of $\mathbf{c}'$. Thus $G[R]$ has no $\mathbf{c}'$-coloring.

By the minimality of G, there is some $R' \subset R$ with $\rho_{G,\mathbf{c}'}(R') \leq 1 - j$. Let $S' \subset S$ be the set of surplus vertices connecting R' with F in G. Since $\rho_{G,\mathbf{c}}(F) \leq 0$,

$$\rho_{G,\mathbf{c}}(F \cup R' \cup S') = \rho_{G,\mathbf{c}}(F) + \rho_{G,\mathbf{c}}(R') - |S'| \leq 0 + (1 - j + |S'|) - |S'| \leq 1 - j,$$

a contradiction. □

A set $A \subset V(G)$ in $(G, \mathbf{c})$ is *trivial* if $A = V(G)$ or $V(G) - A$ is a surplus vertex, and *nontrivial* otherwise.

Lemma 6.2. *For any nontrivial $F \subset V(G)$, $\rho_{G,\mathbf{c}}(F) \geq 4-j$, or F is a single vertex with potential $3-j$.*

Proof. Suppose there is a nontrivial $F \subset V(G)$ with $\rho_{G,\mathbf{c}}(F) \leq 3-j$ and $|F| > 1$. Choose a maximum such F. If some vertex $w \in V(G) - F$ has at least two neighbors in F, then consider $F' = F + w$. Since $\rho_{G,\mathbf{c}}(F') \leq \rho_{G,\mathbf{c}}(F) + \rho_{G,\mathbf{c}}(w) - 3d(w) < \rho_{G,\mathbf{c}}(F)$, the maximality of F implies that F' is trivial, which means $F' = V(G)$ or $F' = V(G) - z$ for some surplus vertex z. If $F' = V(G)$, then since F is nontrivial, $\rho_{G,\mathbf{c}}(V(G)) \leq \rho_{G,\mathbf{c}}(F) - 2 \leq 1-j$, a contradiction. If $F' = V(G) - z$ for some surplus vertex z, then $\rho_{G,\mathbf{c}}(F') \leq \rho_{G,\mathbf{c}}(F) - 1 \leq 2-j$, contradicting (8). Thus every vertex in $V(G) \setminus F$ has at most one neighbor in F. So, if all vertices in the set $S = N(F) - F$ were surplus vertices, then the set $R = V(G) \setminus (F \cup N(F))$ is nonempty, because S is independent. Thus, the sets F, S and R would contradict Lemma 6.1. Therefore, $N_G(F) - F$ has an ordinary vertex, say y.

Let x be the neighbor of y in F. We can change the names of the colors so that

$$L(x) = \{x_1, x_2\}, L(y) = \{y_1, y_2\},\ \mathbf{c}(y_2) \geq \mathbf{c}(y_1),\ x_1 \sim y_1 \text{ and } x_2 \sim y_2. \tag{12}$$

Claim 2. *Every neighbor of y outside of F is a surplus vertex.*

Indeed, suppose z is an ordinary neighbor of y outside of F, say $L(z) = \{z_1, z_2\}$, where $y_1 \sim z_1$. Since G is $\mathbf{c}$-critical, $G[F]$ has a $\mathbf{c}$-coloring φ; say $\varphi(x) = x_1$. Construct $G', \mathbf{c}', H'$ from $G, \mathbf{c}, H$ as follows:

Replace F by a single vertex v, where $L(v) = \{v_1, v_2\}$ and $\mathbf{c}'(v_1) = 0, \mathbf{c}'(v_2) = -1$. For every vertex $u \in N_G(F) - F$, denote $L(u) = \{u_1, u_2\}$ so that the neighbor of u_1 in H is colored by φ. In G', add an edge between v and each vertex in $N_G(F)$. In H', let $v_1u_1, v_2u_2 \in E(H')$. Remove edge yz from $E(G')$, also remove edges between $L(y)$ and $L(z)$ in H'. Let $\mathbf{c}'(y_2) = \mathbf{c}(y_2) - 1, \mathbf{c}'(z_2) = \mathbf{c}(z_2) - 1$, and let $G', \mathbf{c}', H'$ agree with G, c, H everywhere else.

If G' has a $\mathbf{c}'$-coloring ψ, then $\psi(v) = v_1$ and $\psi(u) = u_2$ for all $u \in N_G(F) - F$. Let θ be an H-map such that $\theta = \varphi$ on F and $\theta = \psi$ on $V(G) \setminus F$. Then θ is a $\mathbf{c}$-coloring on G, a contradiction. Thus G' has no $\mathbf{c}'$-coloring. Since G' is a smaller graph than G, there is $S \subset V(G')$ with $\rho_{G',\mathbf{c}'}(S) \leq 1-j$. Since $\rho_{G',\mathbf{c}'}(v) = 2-j \leq -1$, we may assume $v \in S$. Let $S' = (S-v) \cup F \subset V(G)$. If $y, z \notin S$, then $\rho_{G,\mathbf{c}}(S') \leq \rho_{G',\mathbf{c}'}(S) + 1 \leq 2-j$, a contradiction to (8). If exactly one of y, z is in S, then $\rho_{G,\mathbf{c}}(S') \leq \rho_{G',\mathbf{c}'}(S) + 1 + 1 \leq 3-j$, and $S' \supset F$. Since one of y, z is not in S', this means that S' is a larger than F nontrivial set with potential at most $3-j$, a contradiction. Thus $\{v, y, z\} \subseteq S$, and $\rho_{G,\mathbf{c}}(S') \leq \rho_{G',\mathbf{c}'}(S) + 1 + 1 + 1 - 3 \leq 1-j$, a contradiction again. ⋈

Let $u_1, \ldots, u_d \in N(y)$ be the surplus neighbors of y outside of F, and $u'_1, \ldots, u'_d$ be their other neighbors, where u'_is are not necessarily distinct.

Claim 3. $d > \mathbf{c}(y_2)$.

Indeed, suppose $d \leq \mathbf{c}(y_2)$. By Claim 1, there is a $\mathbf{c}$-coloring φ of $G - y$ with $\varphi(x) = x_1$. We recolor each u_h so that it has no conflict with u'_h, and then color y with y_2. This would give a $\mathbf{c}$-coloring on G, a contradiction. ⋈

Claim 4. $\rho_{\mathbf{c}}(y) = 5$*, and hence* $\mathbf{c}(y) = (2, j)$.

If $\rho_{\mathbf{c}}(y) \le 3$, then $\rho_{G,\mathbf{c}}(F+y) \le 3-j-3+3 = 3-j$, a contradiction to the maximality of $|F|$. Suppose $\rho_{\mathbf{c}}(y) = 4$. By Claim 1, there is a $\mathbf{c}$-coloring φ of $G[F]$ with $\varphi(x) = x_1$.

Since $\rho_{\mathbf{c}}(y) = 4$, $\mathbf{c}(y_2) \ge j-1 \ge 3$. Construct G'', c'' as follows: Let $G'' = G - F + v - y - u_1 - \cdots - u_d$, where v is the same as in the proof of Claim 2. Let $\mathbf{c}''$ differ from $\mathbf{c}$ only in that for each $1 \le h \le d - \mathbf{c}(y_2)$, the capacity of u'_h decreases by 1. This means that if some $u'_{h_1}, \ldots, u'_{h_s}$ coincide, then the capacity of the corresponding vertex decreases by s.

Suppose some $S \subset V(G'')$ has $\rho_{G'',\mathbf{c}''}(S) \le 1-j$. Let S be a maximal one with this property. Then $v \in S$. Let a be the number of indices $1 \le h \le d - \mathbf{c}(y_2)$ such that u'_hs in S and b be the number of indices $d - \mathbf{c}(y_2) + 1 \le h \le d$ such that u'_hs in S. Denote by S' the set obtained from $S - v + F + y$ by adding the surplus vertices in $N(y)$ connecting $S - v$ with y in G. Then $\rho_{G,\mathbf{c}}(S') \le 1 - j + a - a - b + \rho_{\mathbf{c}}(y) - 3 + 1 = \rho_{\mathbf{c}}(y) - b - 1 - j$. If $\rho_{\mathbf{c}}(y) = 4$, then $\rho_{G,\mathbf{c}}(S') \le 3-j$. By the maximality of $|F|$, S' is trivial. But then $b = \mathbf{c}(y_2) \ge 3$, $\rho_{G,\mathbf{c}}(S') \le -j$, a contradiction.

Thus by the minimality of G, graph G'' has a $\mathbf{c}''$-coloring ψ. By the definition of v, $\psi(v) = v_1$. We extend ψ to y and u_i's, so that $\psi(y) = y_2$, and y_2 has at most $\mathbf{c}(y_2)$ neighbors in $\psi(u_1), \ldots, \psi(u_d)$. Then $\psi \cup \varphi$ is a $\mathbf{c}$-coloring on G, a contradiction. $\bowtie$

Now we prove the lemma.

By Claim 4, $\mathbf{c}(y_2) = j$. By Claim 1, there is a $\mathbf{c}$-coloring φ of $G[F]$ with $\varphi(x) = x_1$.

We construct $G'', \mathbf{c}''$ as in Claim 4. Let $N'_1 \subset N'$ be the (multi)set of secondary neighbors whose capacity decreased, and let $N'_2 = N' \setminus N'_1$. By Claim 3, $d > j$. So, (as a multiset) $|N'_1| = d - j$, $|N'_2| = j$.

Note that $|N' \cap F| \le 1$, since otherwise $\rho_{G,\mathbf{c}}(F+y) \le 3 - j + 5 - 3 - 2 = 3 - j$, which contradicts the choice of F.

As in the proof of Claim 4, there is some $S \subset V(G'')$ with $\rho_{G'',\mathbf{c}''}(S) \le 1-j$. Define S', a, b as in Claim 4. Then $\rho_{G,\mathbf{c}}(S') \le 1 - j + a - a - b + \rho_{\mathbf{c}}(y) - 3 + 1 = 4 - j - b$. If $b > 0$, then $\rho_{G,\mathbf{c}}(S') \le 4 - j - b \le 3 - j$. So by the maximality of $|F|$, S' is trivial, and hence $b = \mathbf{c}(y_2) = j \ge 4$, implying $\rho_{G,\mathbf{c}}(S') \le 4 - j - b \le -j$ a contradiction. Thus $b = 0$, that is, $S' \cap N'_2 = \emptyset$, and $\rho_{G,\mathbf{c}}(S') \le 4 - j$.

By choosing different $N'_2 \subset N'$, we can form different corresponding S'. Let $\mathcal{S}$ denote the family of all $S' \subset V(G)$ satisfying: (i) $\rho_{G,\mathbf{c}}(S') \le 4-j$; (ii) $S' \supseteq F \cup \{y\}$; (iii) S' misses at least j vertices in N and the neighbors of these vertices distinct from y. By construction, the S''s obtained above are in $\mathcal{S}$.

Let $A \in \mathcal{S}$ contain fewest neighbors of y. If $A \cap N = \emptyset$, then $\rho_{G,\mathbf{c}}(A - y) \le 4 - j - 5 + 3 = 2 - j$, a contradiction. Thus we may assume $u_k, u'_k \notin A$ for $k \in [j]$, and $u_l, u'_l \in A$ for some $l > j$. By choosing $N'_2 = \{u'_1, u'_2, \ldots, u'_{j-1}, u'_l\}$, we can find a set B that is also in $\mathcal{S}$, where $u_k, u'_k \notin B$ for $k \in [j-1]$ and also $u_l, u'_l \notin B$.

Then $u_k, u'_k \notin A \cup B$ for $k \in [j-1]$, hence $A \cup B$ is nontrivial. By the maximality of $|F|$, $\rho_{\mathbf{c}}(A \cup B) \ge 4 - j$. By submodularity, $\rho(A \cap B) \le \rho(A \cup B) + \rho(A \cap B) \le \rho(A) + \rho(B) \le 2(4-j) \le 4-j$. Thus $A \cap B$ is also a member of $\mathcal{S}$. However, $u_l \notin A \cap B$, a contradiction to the choice of A. $\square$

Lemma 6.3. *Suppose $\emptyset \ne F \subsetneq V(G)$ is a nontrivial set, and $\rho_{G,\mathbf{c}}(F) \le 4 - j$. Then F is obtained from $V(G)$ by deleting two surplus vertices.*

Proof. Let F be a counterexample to the lemma maximum in size. By Lemma 6.1, F has an ordinary neighbor y. Let x be the neighbor of y in F, where $L(x) = \{x_1, x_2\}, L(y) = \{y_1, y_2\}$, and $x_1 \sim y_1, x_2 \sim y_2$. Fix a $\mathbf{c}$-coloring φ of $G[F]$ with $\varphi(x) = x_1$.

We construct G', H' and $\mathbf{c}'$ as follows. Replace F in G by a single vertex v, where $L(v) = \{v_1, v_2\}$. For every vertex $u \in N_G(F) - F - y$, denote $L(u) = \{u_1, u_2\}$ so that the neighbor of u_1 in H is colored by φ. In G', add an edge between v and each vertex in $N_G(F) - y$. In H', let $v_1u_1, v_2u_2 \in E(H')$. Let $\mathbf{c}'(v_1) = 0, \mathbf{c}'(v_2) = -1, \mathbf{c}'(y_1) = \mathbf{c}(y_1) - 1$, and let $\mathbf{c}'$ coincide with $\mathbf{c}$ for all other nodes of H'.

If G' has a $\mathbf{c}'$-coloring ψ, then $\psi(v) = v_1$ and $\varphi \cup \psi$ is a $\mathbf{c}$-coloring of G, a contradiction. Thus G' has no $\mathbf{c}'$-coloring, and by the minimality of G, there is a set $S \subset V(G')$ with $\rho_{G'\mathbf{c}'}(S) \leq 1 - j$.

Let S be such set maximal in size. Then $v \in S$. If $y \notin S$, then $\rho_{G,\mathbf{c}}((S-v) \cup F) \leq 1-j+2 = 3-j$. Since $y \notin (S-v) \cup F$, this contradicts Lemma 6.2. If $y \in S$, $\rho_{G,\mathbf{c}}((S-v) \cup F) \leq 1-j+2+1-3 = 1-j$, a contradiction again. □

Corollary 6.4. *Suppose all neighbors of a vertex $w \in V(G)$ are surplus vertices. If $r \geq 3$ and w is an $(m; 2, r)$-vertex, then $m \geq 8$ and if $r \geq 4$ and w is an $(m; 2, r)$-vertex, then $m \geq 10$.*

Proof. Suppose first that $r \geq 3$ and w is an $(m; 2, r)$-vertex where $m \leq 4 + r$. Denote $N = N(w) = \{u_1, \ldots, u_m\}$, and for $1 \leq h \leq m$, let u'_h be the neighbor of u_h distinct from w. Let $N' = \{u'_1, \ldots, u'_m\}$.

Let $\mathbf{c}'$ differ from $\mathbf{c}$ only in that $\mathbf{c}'(u'_1) = \mathbf{c}(u'_1) - (1,1)$. Then by Lemma 6.3 and the minimality of G, $G - w - N$ has a $\mathbf{c}'$-coloring φ. We extend φ to G: let $\varphi(u_k) \nsim \varphi(u'_k)$ for each $2 \leq k \leq m$. Since $m - 1 < (2+1) + (r+1)$, there is a node $\alpha(w) \in L(w)$, such that the number of its already colored neighbors is at most its capacity. Color w by $\alpha(w)$. Extend φ to u_1 so that $\varphi(u_1) \nsim \alpha(w)$. Then φ is a $\mathbf{c}$-coloring on G, a contradiction.

The only case not covered by the argument above is that $r = 4$ and $m = 9$. Let $S \subset V(G-w-N)$, and $N_S \subset N$ be the set of surplus vertices connecting S and w. Suppose there are $h, h' \in [9]$, such that every $S \subset V(G - w - N)$ containing u'_h and $u'_{h'}$ has potential at least two. Let $\mathbf{c}'$ differ from $\mathbf{c}$ only in that each node $\beta \in L(u'_h) \cup L(u'_{h'})$ has $\mathbf{c}'(\beta) = \mathbf{c}(\beta) - 1$. By the choice of h, h', $G - w - N$ has a $\mathbf{c}'$-coloring φ. We extend φ to $N \setminus \{u_h, u_{h'}\}$, so that $\varphi(u_k) \nsim \varphi(u'_k)$ for each $k \in [9] \setminus \{h, h'\}$. Then there is a node $\alpha(w) \in L(w)$, such that the number of its already colored neighbors is at most its capacity. Color w by $\alpha(w)$. Extend φ to $u_h, u_{h'}$, so that $\varphi(u_h), \varphi(u_{h'}) \nsim \alpha(w)$. Then φ is a $\mathbf{c}$-coloring on G, a contradiction.
Thus we may assume that for each pair of $h, h' \in [9]$, there is some $S \subset V(G - w - N)$ containing $u_h, u_{h'}$ with potential at most one. Let $\mathcal{F}$ be the family of all subsets of $V(G-w-N)$ whose potential is at most one. Take any $M \in \mathcal{F}$, and let N_M denote the set of surplus vertices connecting M and w. If $|N_M| = 9$, then $\rho_{G,\mathbf{c}}(M+w+N_M) \leq \rho_{G,\mathbf{c}}(M)+5-9 \leq -3$, a contradiction. If $|N_M| \geq 6$, then $\rho_{G,\mathbf{c}}(M+w+N_M) \leq \rho_{G,\mathbf{c}}(M)+5-6 \leq 0$, a contradiction to Lemma 6.3, since $N' \setminus (M + w + N_M) \neq \emptyset$. Let $M \in \mathcal{F}$ with $|N_M|$ maximum. We may assume $u'_1 \in M, u_2 \notin M$. Then there is some $M' \in \mathcal{F}$ containing u'_1, u'_2. By submodularity,

$$\rho_{G,\mathbf{c}}(M \cap M') + \rho_{G,\mathbf{c}}(M \cup M') \leq \rho_{G,\mathbf{c}}(M) + \rho_{G,\mathbf{c}}(M') \leq 1 + 1 = 2 \tag{13}$$

By Lemma 6.3, $\rho_{G,\mathbf{c}}(M \cap M') \geq 1$, so by (13), $\rho_{G,\mathbf{c}}(M \cup M') \leq 1$. But then $|N_{M \cup M'}| > |N_M|$, a contradiction to the choice of M. □

Lemma 6.5. *$ch(v) \le 0$ for all $v \in V(G)$.*

Proof. Suppose $ch(v) > 0$ for some $v \in V(G)$. Then we have

$$ch(v) = c_1 + c_2 + 3 - j - \frac{3}{2}d_1 - \frac{1}{2}d_2 \ge \frac{1}{2}. \tag{14}$$

Let N_1 denote the set of ordinary neighbors of v and N_2 the set of surplus neighbors of v. For $u \in N_2$, let $g(u)$ denote the neighbor of u distinct from v. Let $N_2' = \{g(u) : u \in N_2\}$.

For $u \in N_1$, a node $u_\alpha \in L(u)$ is *conflicting* with $v_\beta \in L(v)$ if $u_\alpha \sim v_\beta$ in H. For $u \in N_2$, a node $g(u)_\alpha \in L(u)$ is *u-conflicting* with $v_\beta \in L(v)$ if the neighbors of $g(u)_\alpha$ and v_β in $L(u)$ are distinct.

Vertices $x, y \in N_2$ with $g(x) = g(y) = u \in N_2'$ are *parallel*, if the x-conflicting node in $L(u)$ is also y-conflicting. Otherwise, we call (x, y) a *twisted* pair.

We aim to construct an auxiliary graph and either find a $\mathbf{c}$-coloring of this graph such that v is colored by $r(v)$ and extend this coloring to a $\mathbf{c}$-coloring of G, or find a low-potential set in the new graph, which leads to a contradiction to the choice of $(G, \mathbf{c})$.

Construct $G', \mathbf{c}', H'$ from $G, \mathbf{c}, H$ as follows:

Step 0: Initialize $G' = G, \mathbf{c}' = \mathbf{c}, H' = H$.

Step 1: For each $u \in N_1$, let $u_\alpha \in L(u)$ be conflicting with $r(v)$. Remove edge uv from G', remove edges between $L(u)$ and $L(v)$ from H', and decrease the capacity of u_α by one in $\mathbf{c}'$.

Step 2: For each $u' \in N_2'$, if there are $u_1, u_2 \in N_2$ connecting u' and v that form a twisted pair, then remove u_1, u_2 from G', and remove $L(u_1), L(u_2)$ from H'. Repeat this step until there are no such u', u_1, u_2.

Step 3: For each $u' \in N_2'$, if there are $u_1, u_2 \in N_2$ connecting u' and v that form a parallel pair, then let $u'_\alpha \in L(u')$ be u_1-conflicting with $r(v)$. Remove u_1, u_2 from G', and remove $L(u_1), L(u_2)$ from H'. Decrease the capacity of u'_α by one in $\mathbf{c}'$. Repeat this step until there are no such u', u_1, u_2.

At this point, each vertex in N_2' has at most one common neighbor with v. Denote the set consisting of vertices in N_2' having now exactly one neighbor in N_2 by N_2''.

Step 4.1: If $|N_2''|$ is odd, then take any $u_0' \in N_2''$, let u_0 be the surplus vertex connecting u_0' and v. Delete u_0 from G', and delete $L(u_0)$ from H'.

Step 4.2: Now we may assume that $|N_2''|$ is even. Take $w', u' \in N_2''$, let $w'_\alpha \in L(w')$ and $u'_\alpha \in L(u')$ be conflicting with $r(v)$. Delete the surplus neighbors u, w of v adjacent to w', u' from G', and delete their lists from H'. Add a surplus vertex z to G' adjacent to w' and u'. And in H', let $L(z) = \{z_1, z_2\}$, such that $z_1 \sim w'_\alpha, z_2 \sim u'_\alpha$. Repeat the above until N_2 is empty. Denote the set consisting of all newly added surplus vertices z by N_3. Then $|N_3| \le \lfloor \frac{|N_2'|}{2} \rfloor$.

Step 5: At this point, v is an isolated vertex in G'. Delete v from G', and delete $L(v)$ from H'. The resulting G', H' and $\mathbf{c}'$ are final.

Suppose G' has a $(\mathbf{c}', H')$-coloring ϕ. We now extend ϕ to the deleted vertices of G following the steps of constructing H' in the reversed order.

Step 5^-: Let $\phi(v) = r(v)$. Then $\phi(v)$ has at most $|N_1|$ neighbors in H_ϕ and each $u \in N_1$ gets at most one extra conflicting neighbor.

Step 4.2$^-$: For each $z \in N_3$ and its neighbors $w', u' \in N_2''$, let w, u be the surplus vertices in G connecting v with w' and u', respectively at the beginning of Step 4.2. We will delete z and assign colors to w and u as follows. If $\phi(z) \sim \phi(u')$, then we choose $\phi(u) \nsim r(v)$ and $\phi(w) \nsim \phi(w')$. In this way, the degrees of $\phi(u')$ and $\phi(w')$ in H_ϕ do not increase, and the degree of $r(v)$ increases by at most 1. If $\phi(z) \nsim \phi(u')$ but $\phi(z) \sim \phi(w')$, then we switch the roles of u and w. If $\phi(z) \nsim \phi(u')$ and $\phi(z) \nsim \phi(w')$, then by Step 4.2, either $\phi(u')$ is not u-conflicting with $r(v)$, or $\phi(w')$ is not w-conflicting with $r(v)$. Then after we choose $\phi(u) \nsim \phi(u')$ and $\phi(w) \nsim \phi(w')$, again the degree of $r(v)$ in H_ϕ increases only by 1 and the degrees of $\phi(u')$ and $\phi(w')$ in H_ϕ do not increase.

Step 4.1$^-$: If $|N_2''|$ is odd, then we let $\phi(u_0) \nsim \phi(u_0')$. So the degree of $r(v)$ in H_ϕ increases by at most 1 and the degree of $\phi(u_0')$ does not change.

Step 3$^-$: For every $u' \in N_2'$ and for each parallel pair $u_1, u_2 \in N_2$ connecting u' and v deleted on Step 3, we let $\phi(u_1) \nsim \phi(u')$ and $\phi(u_2) \nsim \phi(r(v))$. So, the degree of $r(v)$ in H_ϕ increases by at most 1 and the degree of $\phi(u')$ increases by at most 1.

Step 2$^-$: For every $u' \in N_2'$ and and for each twisted pair $u_1, u_2 \in N_2$ connecting u' and v deleted on Step 2, we let $\phi(u_1) \nsim \phi(u')$ and $\phi(u_2) \nsim \phi(u')$. Then the degree of $\phi(u')$ does not increase and since u_1, u_2 is a twisted pair, the degree of $r(v)$ in H_ϕ increases by at most 1.

Now by construction, the degree in H_ϕ of every u_α apart from possibly $r(v)$ is at most its capacity. Let $c_1 = \mathbf{c}_1(v), c_2 = \mathbf{c}_2(v), d_1 = |N_1|, d_2 = |N_2|$. By construction, $r(v)$ has at most $|N_1| + \lceil |N_2|/2 \rceil = d_1 + \lceil d_2/2 \rceil$ colored neighbors. Thus, if

$$c_2 = \mathbf{c}(r(v)) \geq d_1 + \lceil d_2/2 \rceil, \tag{15}$$

then ϕ is a $\mathbf{c}$-coloring of G. By (14) and $j \geq 4$,

$$d_1 + \frac{1}{2}d_2 \leq \frac{3}{2}d_1 + \frac{1}{2}d_2 \leq c_1 + c_2 + \frac{5}{2} - j \leq c_2 + \frac{1}{2}.$$

This implies that if d_2 is even, or d_1 is positive, or $c_1 \leq 1$, or $j \geq 5$, then (15) holds. So, assume that d_2 is odd, $d_1 = 0$, $c_1 = 2$ and $j = 4$. This means $\mathbf{c}(v) = (2, c_2)$, $d(v) = 2c_2 + 1$, $c_2 \leq j = 4$ and all neighbors of v are surplus vertices. If $c_1 + c_2 + 1 = c_2 + 3 \geq d(v) = 2c_2 + 1$, then we can extend any $\mathbf{c}$-coloring of $G - v$ to v greedily. The only remaining cases are $c_2 = 3, d(v) = d_2 = 7$ and $c_2 = 4, d(v) = d_2 = 9$. By Corollary 6.4, such v does not exist. Thus ϕ is a $\mathbf{c}$-coloring of G, a contradiction.

Therefore, we may assume that G' is not $\mathbf{c}'$-colorable. By the minimality of G, there is an $S \subset V(G')$ with $\rho_{G',\mathbf{c}'}(S) \leq 1 - j$. Let S be such a set of minimum potential and modulo this maximal in size.

Let $S_1 = S \cap N_1$, $s_2 = \sum_{w \in S \cap N_2'} (\mathbf{c}(w) - \mathbf{c}'(w))$, $S_3 = S \cap N_3$, and $S' = S - S_3$. Then $S' \subset V(G)$ and $v \notin S'$. So by Lemma 6.3, $\rho_{G,\mathbf{c}}(S') \geq 5 - j$.

Recall that while constructing $\mathbf{c}'$, if we decreased the capacity of a vertex, then either this vertex is in N_1 (in Step 1) or has two common neighbors with v (in Step 3). It follows that

$$5 - j \leq \rho_{G,\mathbf{c}}(S') = \rho_{G',\mathbf{c}'}(S) + |S_3| + |S_1| + s_2 \leq 1 - j + |S_3| + |S_1| + s_2. \tag{16}$$

Let $N_4 \subset N_2$ be the set of surplus vertices in G connecting v and $S \cap N_2'$, and $V' := (S' \cup \{v\} \cup N_4)$. Recall that if S contains a $z \in N_3$, then it also contains both its neighbors

u', w', each of which has a common neighbor with v in G (that belongs to N_4) and is adjacent to only one vertex in N_3 in G'. Also, while constructing $\mathbf{c}'$, every time when we decreased the capacity of a vertex $u' \in N_2'$ in Step 3, we have deleted two its common neighbors with v, and these vertices are now in N_4. It follows that $|N_4| \geq 2|S_3| + 2s_2$. By this and (16),

$$\rho_{G,\mathbf{c}}(V') = \rho_{G,\mathbf{c}}(S') + \rho_{\mathbf{c}}(v) - 3|S_1| - |N_4| \leq \rho_{G',\mathbf{c}'}(S) + \rho_{\mathbf{c}}(v) - 2|S_1| - \left\lceil \frac{|N_4|}{2} \right\rceil \tag{17}$$

$$\leq 1 - j + \rho_{\mathbf{c}}(v) - 2|S_1| - \left\lceil \frac{|N_4|}{2} \right\rceil .$$

Since S is maximal in size, we may assume that if $V' \neq V(G)$, then there is some ordinary vertex in $V(G) \setminus V'$, otherwise we can just include the surplus vertices outside of S into S to make its size larger. Thus by Lemma 6.3,

$$\rho_{G,\mathbf{c}}(V') \geq 5 - j \text{ when } V' \neq V(G). \tag{18}$$

So, if $V' \neq V(G)$, then (17) and (18) together yield

$$4 + 2|S_1| + \left\lceil \frac{|N_4|}{2} \right\rceil \leq \rho_{\mathbf{c}}(v) \leq 5.$$

This implies $S_1 = \emptyset$ and $|N_4| \leq 2$. But then $\rho_{G,\mathbf{c}}(S') \leq \rho_{G',\mathbf{c}'}(S) + 1 \leq 2 - j$, a contradiction to (16). Therefore, $V' = V(G)$, which means $S_1 = N_1$ and $N_4 = N_2$.

Since (16) yields $4 \leq |S_3| + |S_1| + s_2$, we infer from $|N_4| \geq 2|S_3| + 2s_2$ that

$$|N_1| + \left\lfloor \frac{|N_2|}{2} \right\rfloor \geq |S_3| + |S_1| + s_2 \geq 4. \tag{19}$$

Then, since $\rho_{G,\mathbf{c}}(V') \geq 2 - j$, (17) gives

$$2 - j \leq 1 - j + \rho_{\mathbf{c}}(v) - 2|N_1| - \left\lceil \frac{|N_2|}{2} \right\rceil \leq 1 - j + \rho_{\mathbf{c}}(v) - |N_1| - 4.$$

For this to happen, we need $\rho_{\mathbf{c}}(v) = 5$, $N_1 = \emptyset$ and $|N_2|$ be even. Now, (19) yields $|N_2| \geq 8$ and (14) yields $|N_2| \leq 9$. Since $|N_2|$ is even, we have $|N_2| = 8$. By Corollary 6.4, G has no such vertices. □

Now Lemma 4.8 together with Lemma 6.5 complete the proof of Theorem 3.1.

7. A construction

A construction in [18] shows that the bounds in Theorem 2.1 are sharp for each pair (i, j) satisfying the theorem for infinitely many n. For the convenience of the reader, we repeat this construction below, but do not present the proof of its properties. The interested readers may find it in Section 5 of [18].

Fix $i \in \{1, 2\}$ and $j \geq 2i$.

Definition 8. Given a vertex v in a graph G, a *flag at v* is a subgraph F of G with $i + 3$ vertices $v, x, u_1, \dots, u_{i+1}$, such that $d_G(x) = i + 2$, $vx \in E(F)$, and $u_1, \dots, u_{i+1}$ are 2-vertices adjacent to both v and x. We call v the *base vertex*, x the *top vertex*, and $u_1, \dots, u_{i+1}$ the *middle vertices* of the flag F.

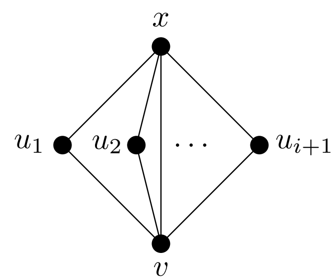

FIGURE 1. A flag at base vertex v.

We call a vertex being the base of k flags a k-*base* vertex. A graph with a k-base vertex v (and all the flags based at v) contains exactly $1+k(i+2)$ vertices and $k(1+2(i+1))$ edges.

We now define our critical graph G_m for given positive integer m: when $m=1$, let v be an $(i+j+2)$-base vertex; when $m \geq 2$, let $v_1, \ldots, v_m$ be a path, where v_1 is an $(i+1)$-base vertex, v_m is an $(i+j+1)$-base vertex, and v_k is an i-base vertex for all $k=2,\ldots,m-1$. One can easily check that $|V(G_m)| = (i+2)(mi+j+2)+m$ and $|E(G_m)| = (2i+3)(mi+j+2)+m-1$, thus

$$|E(G_m)| = \frac{(2i+1)|V(G_m)|+j-i+1}{i+1}.$$

For the cover graph H_m of G_m, we need the following definition:

Definition 9. A flag with base vertex v, top vertex x, and middle vertices $u_1, \ldots, u_{i+1}$ is called *parallel*, if $p(u) \sim p(w)$, $r(u) \sim r(w)$ for each edge uw in the flag; the flag is called *twisted* if $p(v) \sim r(x)$, $p(x) \sim r(v)$, $p(x) \sim p(u_k), r(x) \sim r(u_k)$, and $p(v) \sim r(u_k), p(u_k) \sim r(v)$ for each $k \in [i+1]$.

To construct a cover graph H_m that does not admit an (i,j)-coloring, we let all $i+1$ flags at v_1 and all i flags at v_k for $k=2,\ldots,m-1$ be twisted. Among the $i+j+1$ flags at v_m, let $i+1$ of them be twisted and the remaining j be parallel. For edge v_kv_{k+1} on the path, let $p(v_k) \sim r(v_{k+1}), r(v_k) \sim p(v_{k+1})$ for each $k=1,\ldots,m-1$.

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