A successive centralized circumcentered-reflection method for the convex feasibility problem

Roger Behling[1,4] · Yunier Bello-Cruz[2] · Alfredo Iusem[1] · Di Liu[3] ·
Luiz-Rafael Santos[4]

Abstract

In this paper, we present a successive centralization process for the circumcentered-reflection method with several control sequences for solving the convex feasibility problem in Euclidean space. Assuming that a standard error bound holds, we prove the linear convergence of the method with the most violated constraint control sequence. Moreover, under additional smoothness assumptions on the target sets, we establish the superlinear convergence. Numerical experiments confirm the efficiency of our method.

✉ Di Liu
di.liu@impa.br

Roger Behling
rogerbehling@gmail.com

Yunier Bello-Cruz
yunierbello@niu.edu

Alfredo Iusem
alfredo.iusem@fgv.br

Luiz-Rafael Santos
l.r.santos@ufsc.br

[1] School of Mathematics, Fundação Getúlio Vargas, Rio de Janeiro, RJ 22250-900, Brazil

[2] Department of Mathematical Sciences, Northern Illinois University, DeKalb, IL 60115-2828, USA

[3] Instituto de Matematica Pura e Aplicada, Rio de Janeiro, RJ 22460-320, Brazil

[4] Department of Mathematics, Federal University of Santa Catarina, Blumenau, SC 89065-300, Brazil

1 Introduction

The convex feasibility problem (CFP) aims to solve:

$$\text{find } x^* \in C := \bigcap_{i=1}^{m} C_i, \tag{1.1}$$

where each $C_i \subset \mathbb{R}^n$ is closed and convex, for $i = 1, 2, \ldots, m$. Moreover, we assume that $C \neq \emptyset$. Convex feasibility represents a modeling paradigm for solving many engineering and physics problems, e.g., image recovery [1], wireless sensor networks localization [2], and gene regulatory network inference [3].

Based on the orthogonal projections, a broad class of methods is available for solving problem (1.1); see, for instance, [4]. Two well-known algorithms among them are the Sequential Projection Method (SePM) and the Simultaneous Projection Method (SiPM), which only use the individual projections onto C_i's, P_{C_i}. The projection operator for each C_i, $P_{C_i} : \mathbb{R}^n \to C_i$, is given by

$$P_{C_i}(x) = \arg \min_{s \in C_i} \|x - s\|. \tag{1.2}$$

The SePM and the SiPM operators are defined as $\bar{P} = P_{C_m} \circ \cdots \circ P_{C_1}$ and $\hat{P} = \frac{1}{m} \sum_{i=1}^{m} P_{C_i}$, respectively. Given $s^0, y^0 \in \mathbb{R}^n$, we set $s^{k+1} = \bar{P}(s^k)$ and $y^{k+1} = \hat{P}(y^k)$ with $k \in \mathbb{N}$ the sequences generated by SePM and SiPM, respectively. These two iterations converge to a solution of problem (1.1) if $\bigcap_{i=1}^{m} C_i \neq \emptyset$. Moreover, it is well-known that under some error bound conditions, to be discussed later, SePM and SiPM have linear convergence rates. Further study of these two algorithms can be found in [4, 5].

In this paper we are going to use a circumcentered-reflection scheme for solving problem (1.1). The circumcentered-reflection method (CRM) has been proposed in [6] to solve problem (1.1) with two closed convex sets $A, B \subset \mathbb{R}^n$. CRM was first proposed to accelerate the Douglas–Rachford method (DRM) [7, 8], and Method of Alternating projections (MAP) [9, 10] (which coincides with SePM introduced above, if only two sets are considered). During the past four years, CRM has fascinated researchers in the field of continuous optimization, resulting in an avalanche of surprising results and improvements for the circumcenter scheme; see, for instance, [11–29].

The circumcenter of three points $x, y, z \in \mathbb{R}^n$, noted as $\text{circ}(x, y, z)$, is the point in $\mathbb{R}^n$ that lies in the affine space defined by x, y and z and is equidistant to these three points. CRM iterates by means of the operator $\mathcal{C}_{A,B}$ with respect to A, B defined as

$$\mathcal{C}_{A,B}(x) = \text{circ}(x, R_A(x), R_B(R_A(x))), \tag{1.3}$$

where $R_A = 2P_A - \text{Id}$ and $R_B = 2P_B - \text{Id}$, and Id is the identity operator in $\mathbb{R}^n$.

One of the limitations of CRM is that its convergence theory requires one of the sets to be a linear manifold. A counter-example for which CRM does not converge

for two general convex sets was found in [30]. It is worth noting that CRM can be used for solving the CFP with m general arbitrary closed convex sets by using Pierra's product space reformulation [31]. This method is called *CRM-Prod*, which is briefly described next. Define $\mathbf{W}:=C_1 \times C_2 \times \cdots \times C_m \subset \mathbb{R}^{nm}$ and $\mathbf{D}:=\{(x, x, \ldots, x) \in \mathbb{R}^{nm} \mid x \in \mathbb{R}^n\}$. One can easily see that

$$x^* \in C \Leftrightarrow \mathbf{z}^*:=(x^*, x^*, \ldots, x^*) \in \mathbf{W} \cap \mathbf{D}. \tag{1.4}$$

Due to (1.4), solving problem (1.1) corresponds to solve

$$\text{find } \mathbf{z}^* \in \mathbf{W} \cap \mathbf{D}.$$

Since $\mathbf{D}$ is an affine manifold, CRM operator (1.3) can be applied to the convex sets $\mathbf{D}$ and $\mathbf{W}$ in the product space $\mathbb{R}^{nm}$ given rise to the CRM-Prod iteration, i.e.,

$$\mathbf{z}^{k+1}:=\mathcal{C}_{\mathbf{W},\mathbf{D}}(\mathbf{z}^k) = \text{circ}(\mathbf{z}^{k+1}, R_{\mathbf{W}}(\mathbf{z}^k), R_{\mathbf{D}}(R_{\mathbf{W}}(\mathbf{z}^k))). \tag{1.5}$$

Unfortunately, the numerical evidence in [14] showed that the cost of introducing the product space is expensive.

More recently, an extension of CRM, called the centralized circumcentered-reflection method (cCRM), was introduced in [14] for overcoming the drawback of CRM, namely the request that one of the sets be an affine manifold. For describing cCRM, we need some notation. Suppose that $A, B \subset \mathbb{R}^n$ are both closed convex sets and define the *alternating projection operator* for SePM $Z_{A,B} : \mathbb{R}^n \to \mathbb{R}^n$ as

$$Z_{A,B}:=P_A \circ P_B,$$

and the *simultaneous projection operator* for SiPM $\tilde{Z}_{A,B} : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\tilde{Z}_{A,B}:=\frac{1}{2}(P_A + P_B), \tag{1.6}$$

with P_A, P_B as in (1.2). We also define operator $\bar{Z}_{A,B} : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\bar{Z}_{A,B}:=\frac{1}{2}(Z_{A,B} + P_B \circ Z_{A,B}) = \tilde{Z}_{A,B} \circ Z_{A,B}. \tag{1.7}$$

Finally, instead of using sequential reflections as in (1.3), cCRM relies on the *parallel circumcenter operator* $\mathcal{C}_{A,B}$ defined by circumcentering parallel reflections, i.e., for $x \in \mathbb{R}^n$ we have

$$\mathcal{C}_{A,B}(z):= \text{circ}(x, R_A(x), R_B(x)). \tag{1.8}$$

With this notation, we define the cCRM operator $T_{A,B} : \mathbb{R}^n \to \mathbb{R}^n$ for sets A and B at $x \in \mathbb{R}^n$ as

$$T_{A,B}(x) := \mathscr{C}_{A,B}(\bar{Z}_{A,B}(x))$$
$$= \mathrm{circ}(\bar{Z}_{A,B}(x), R_A(\bar{Z}_{A,B}(x)), R_B(\bar{Z}_{A,B}(x))). \tag{1.9}$$

Therefore, given $x^0 \in \mathbb{R}^n$, the cCRM method is defined by the iteration

$$x^{k+1} = T_{A,B}(x^k). \tag{1.10}$$

It has been proved in [14] that the sequence defined by cCRM converges to point $x^* \in A \cap B$ whenever $A \cap B \neq \emptyset$. Under an error bound assumption the sequence converges linearly. Moreover, under some additional smoothness hypotheses and an error bound condition, the sequence generated by cCRM was proved to have superlinear convergence [14, Thm. 3.13].

In this paper, we will extend the cCRM to the case of CFP with m sets. The natural way to generalize it is to choose a pair of sets among $\{C_1, C_2, \ldots, C_m\}$ at the iteration k and apply cCRM to this pair of sets. We define this pair as $\ell(k)$ and $r(k)$, with $\ell(k), r(k) \in \{1, \ldots, m\}$, and then apply the operator in (1.10) to this pair. Therefore, the *successive centralized circumcentered-reflection method* (s-cCRM) for CFP with m sets is defined as

$$z^{k+1} = T_{C_{r(k)}, C_{\ell(k)}}(z^k), \tag{1.11}$$

where $T_{C_{r(k)}, C_{\ell(k)}}$ is the cCRM operator defined in (1.9) w.r.t. $C_{\ell(k)}$ and $C_{r(k)}$.

The sequences $\{\ell(k)\}, \{r(k)\}$, determining which sets are used at the k-th iteration, are called *control sequences*. In all successive projection-type methods, control sequences considerably impact the algorithms' performance. Indeed, strategies using such sequences have been studied before for SePM; see, for instance, [32–34].

The following control sequences, which we will use in this paper, are classical; an in-depth treatment of them can be found in [33].

A natural one is the *cyclic control sequence*, i.e., at iteration k we choose

$$\ell(k) = 1, 2, 3, \ldots, m - 1, m, 1, 2, \ldots, \text{ and}$$
$$r(k) = 2, 3, 4, \ldots, m - 1, m, 1, 2, 3, \ldots. \tag{1.12}$$

This is the option found in the first approaches to these types of methods. A generalization of this control sequence is the *almost cyclical control sequence*, which requires that each set is used at least once in any cycle of iterations of some predetermined length. A limitation of cyclic and almost cyclic control sequences is that they do not use any information available at iteration k, e.g., the distances of the present iterate to the target sets.

An alternative option is the *most violated constraint control sequence (distance version)*, which chooses $\ell(k)$ as the set which lies the farthest away from z^k, with the goal of getting closer to the intersection set C. The drawback is that the distance from z^k to all sets C_i must be calculated to determine $\ell(k)$, which in general, is computationally expensive.

When the sets C_i are represented as sublevel sets of convex functions, that is, $C_i := \{x \in \mathbb{R}^n \mid f_i(x) \le 0\}$ where $f_i : \mathbb{R}^n \to \mathbb{R}$ is convex, for all $i = 1, 2, \ldots, m$. In this case, which happens frequently in applications, we have the option of the *most violated constraint control sequence (functional value version)*, where $\ell(k)$ is chosen so that $f_{\ell(k)}(z^k) \ge f_i(z^k)$ for all i, again with the expectation of getting closer to C.

These three options (almost cyclical and most violated constraint in distance version, or functional value version) are mutually independent. As explained above, the third one is expected to perform better than the second one, which is expected to perform better than the first one. Our numerical experiments confirm this behavior. Also, we are able to prove linear or superlinear convergence (under adequate assumptions) for the second and third option. We do not have results of this type for the almost cyclical control sequence.

Next, we formally define these three control sequences.

Definition 1.1 (Control sequences) We say that the sequence $\{\ell(k)\}$ is:

(i) *Almost cyclic*, if $1 \le \ell(k) \le m$ and there exists an integer $Q \ge m$ such that, for all $k \ge 0$ and $\{1, 2, \ldots, m\} \subset \{\ell(k+1), \ell(k+2), \ldots, \ell(k+Q)\}$. An almost cyclic control with $Q = m$ is called *cyclic*;

(ii) *Most violated constraint* (distance version), if

$$\ell(k) := \arg\max_{1 \le i \le m} \{\operatorname{dist}(z^k, C_i)\},$$
$$r(k) := \arg\max_{1 \le i \le m} \{\operatorname{dist}(P_{C_{\ell(k)}}(z^k), C_i)\}; \tag{1.13}$$

(iii) *Most violated constraint* (function value version), if we assume that the sets C_i in problem (1.1) are in the form

$$C_i := \{x \in \mathbb{R}^n \mid f_i(x) \le 0\}, \tag{1.14}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$ is convex for all $i = 1, 2, \ldots, m$, then the most violated constraint control sequence in the function value version is given by

$$\ell(k) := \arg\max_{1 \le i \le m} \{f_i(z^k)\},$$
$$r(k) := \arg\max_{1 \le i \le m} \{f_i(P_{C_{\ell(k)}}(z^k))\}. \tag{1.15}$$

when the sets in the CFP are presented as in (1.14), the control sequence of the most violated constraint (function value version) depends not only on the sets themselves but also on the specific functions f_i used to represent them, which are, of course, not unique. The control sequence will change if we change the functions f_i (keeping the same sets C_i). Here, we assume that the f_i's are fixed from the onset, so that the original problem can be seen as that of finding a point $\bar{x}$ which satisfies $f_i(\bar{x}) \le 0$ for all $i \in \{1, 2, \ldots m\}$.

We note that Borwein and Tam, in [35, 36], introduced and analyzed a cyclic Douglas–Rachford iteration scheme. In this paper, in addition to studying the cyclic version of cCRM, we are going to employ the other two control sequences above.

Let us define the terminology for addressing the different algorithmic choices of the control sequences used in s-cCRM iteration given in (1.11). *Algorithm 1* is s-cCRM with the almost cyclic control sequence presented in Definition 1.1(i); *Algorithm 2* stands for s-cCRM with the most violated constraint control sequence (distance version) introduced Definition 1.1(ii); and *Algorithm 3* considers the most violated constraint control sequence (function value version) set in Definition 1.1(iii) within s-cCRM.

The paper is organized as follows: In Sect. 2, we give definitions and preliminaries. In Sect. 3, we introduce and prove the global convergence of s-cCRM, under the three control sequences defined above. In Sect. 4, we prove linear convergence for both versions of the most violated constraint control sequence under a standard *error bound* assumption, and superlinear convergence under some additional smoothness assumptions. Section 5 presents numerical experiments comparing s-cCRM with SePM and CRM-Prod.

2 Preliminaries

Throughout this paper, we work in $\mathbb{R}^n$, and the norm $\|\cdot\|$ is the norm induced by the Euclidean scalar product $\langle \cdot, \cdot \rangle$. In this section we recall several basic results needed in our convergence analysis.

First, we introduce some orthogonal projection properties.

Lemma 2.1 (Properties of projection onto convex sets) *Let $C \subset \mathbb{R}^n$ be a nonempty closed convex set and P_C be the orthogonal projection onto set C defined in* (1.2). *Then, the following hold:*

(i) *For all $x, y \in \mathbb{R}^n$, $\|P_C(x) - P_C(y)\|^2 \leq \|x - y\|^2 - \|(P_C(x) - x) - (P_C(y) - y)\|^2$.*
(ii) *For all $x \in \mathbb{R}^n$ and $s \in C$, $\|P_C(x) - s\| \leq \|x - s\|$.*
(iii) *For all $x \in \mathbb{R}^n$ and $s \in C$, $\|P_C(x) - x\|^2 + \|P_C(x) - s\|^2 \leq \|x - s\|^2$.*
(iv) *If $C := \bigcap_{i=1}^{m} C_i$, with $C_i, i = 1 \ldots, m$ being closed convex sets of $\mathbb{R}^n$, then $\mathrm{dist}(P_{C_i}(x), C) \leq \mathrm{dist}(x, C)$, for all $x \in \mathbb{R}^n$.*

Proof Item (i) is by [37, Prop. 4.16], while items (ii) and (iii) are direct consequences of (i). Regarding item (iv), if $P_{C_i}(x) \in C$ we are done. Suppose it is not and let $\hat{s}, s \in C$ be the realizers of the distances between $P_{C_i}(x)$ and x to C, respectively. Then, taking into account that $s \in C_i$, we have

$$\mathrm{dist}(P_{C_i}(x), C) = \left\| P_{C_i}(x) - \hat{s} \right\| \leq \left\| P_{C_i}(x) - s \right\|$$
$$\leq \|x - s\| = \mathrm{dist}(x, C),$$

where the first inequality is by the definition of distance realizers, and the second inequality is by item (ii). $\qquad\square$

Lemma 2.1(i) means that projections onto convex sets are *firmly nonexpansive*. Note that this property is stronger than the well-known *nonexpansiveness* of projections, that is, $\|P_C(x) - P_C(y)\| \leq \|x - y\|$, for all $x, y \in \mathbb{R}^n$, with $C \subset \mathbb{R}^n$ being closed and convex. Lemma 2.1(ii) indicates that projections are *quasinonexpansive*, while Lemma 2.1(iii) says that projections are *firmly quasinonexpansive*; see [37, Def. 4.1].

We continue with several definitions and facts, beginning with the notion of *Fejér monotonicity*.

Definition 2.2 (Fejér monotonicity) Suppose that $M \subset \mathbb{R}^n$ is nonempty. Let $\{x^k\}$ be a sequence in $\mathbb{R}^n$. We say $\{x^k\}$ is *Fejér monotone* with respect to M if $\|x^{k+1} - s\| \leq \|x^k - s\|$, for all $s \in M$, and for all $k \in \mathbb{N}$.

The following lemma gives the properties of Fejér monotone sequences.

Lemma 2.3 (Fejér monotonicity properties [4, Thm. 2.16]) *Suppose that $M \subset \mathbb{R}^n$ is nonemtpy, and the sequence $\{x^k\}$ is Fejér monotone with respect to M. Then,*

(i) *$\{x^k\}$ is bounded.*
(ii) *For every $s \in M$, $\{\|x^k - s\|\}$ converges.*
(iii) *If there exists a cluster point x^* of $\{x^k\}$ such that $x^* \in M$, then $\{x^k\}$ converges to x^*.*

Next, we are going to present some results about the boundedness and approximation property of convex functions.

Definition 2.4 (Local Lipschitz continuity) Let $f : \mathbb{R}^n \to \mathbb{R}$ and $U \subset \mathbb{R}^n$. We say that f is *locally Lipschitz continuous* in U if for every $z \in U$ there exist a constant $L > 0$ and a neighborhood V of z, such that

$$\|f(x) - f(y)\| \leq L\|x - y\|,$$

for all $x, y \in V \cap U$.

It is well-known that convex functions are *locally Lipschitz continuous*; see [38, Thm. 2.1.12]. We now recall some properties on subgradients.

Definition 2.5 (Subgradient) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. We say that a vector $v \in \mathbb{R}^n$ is a *subgradient* of f at a point $x \in \mathbb{R}^n$ if, for all $y \in \mathbb{R}^n$,

$$f(y) \geq f(x) + \langle v, y - x \rangle.$$

The set of all *subgradients* of a convex function at $x \in \mathbb{R}^n$ is called the *subdifferential* of f at x, and is denoted by $\partial f(x)$. We present now a version of the mean value theorem for convex functions.

Lemma 2.6 (Mean value theorem for convex functions [39]) *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let x and y be vectors in $\mathbb{R}^n$. Then, there exists a vector $u \in \mathbb{R}^n$ and a subgradient $v(u) \in \partial f(u)$ such that*

$$f(y) = f(x) + \langle v(u), y - x \rangle,$$

where $u = \alpha x + (1 - \alpha)y$, and $\alpha \in (0, 1)$.

We end this section recalling that the subdifferential operator of the convex functions f, $\partial f : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, is *maximal monotone* [40, Cor. 31.5.2] and *locally bounded* [41, Thm. 3].

3 Convergence analysis of s-cCRM for the multiset case

We proceed to the convergence analysis of s-cCRM applied to the Convex Feasibility Problem with m convex sets.

3.1 The cCRM for two convex sets

In this subsection we are going to present some results from [14] regrading cCRM applied to two closed convex sets, in order to allow us to define the s-cCRM iteration to solve problem (1.1).

First, we present the notion of *centralized point* in connection with cCRM.

Definition 3.1 (Centralized point) Let $A, B \subset \mathbb{R}^n$ be two nonempty closed convex sets, and a point $z \in \mathbb{R}^n$ is said to be *centralized* with respect to A, B if

$$\langle R_A(z) - z, R_B(z) - z \rangle \leq 0.$$

Now, we show that if we apply the operator $\bar{Z}_{A,B} := \tilde{Z}_{A,B} \circ Z_{A,B}$, given in (1.7), to a point $z \in \mathbb{R}^n$, then the resulting point will be centralized with respect to A and B.

Lemma 3.2 (Centralization procedure [14, Lem. 2.2]) *Let $A, B \subset \mathbb{R}^n$ be two nonempty closed convex sets with nonempty intersection. For any $z \in \mathbb{R}^n$, then $\bar{Z}_{A,B}(z)$ is centralized w.r.t. A and B.*

We next state the firmly quasinonexpansiveness of a parallel circumcenter iteration taken from a centralized point. This result means that parallel circumcenter steps taken from centralized points move towards the solution of the convex feasibility problem involving two intersecting sets.

Lemma 3.3 (Firmly quasinonexpansiveness of circumcenters at centralized points [14, Lem. 2.5]) *Let $A, B \subset \mathbb{R}^n$ be two nonempty closed convex sets with nonempty intersection. Assume that $z \in \mathbb{R}^n$ is a centralized point with respect to A, B. Then, $\mathscr{C}_{A,B}(z)$ defined in (1.8) satisfies*

$$\|\mathscr{C}_{A,B}(z) - s\|^2 \leq \|z - s\|^2 - \|z - \mathscr{C}_{A,B}(z)\|^2,$$

for all $s \in A \cap B$.

Let us state the firmly quasinonexpansiveness of the cCRM operator. This lemma was the key result in [14] to prove the convergence of cCRM, when CFP consists of two sets.

Lemma 3.4 (Firmly quasinonexpansiveness of cCRM [14, Lem. 2.6]) *Let* A, $B \subset \mathbb{R}^n$ *be two nonempty closed convex sets with nonempty intersection. Let* $z \in \mathbb{R}^n$. *Then,* $T_{A,B}(z)$ *defined in* (1.9) *satisfies*

$$\|T_{A,B}(z) - s\|^2 \leq \|z - s\|^2 - \frac{1}{8}\|z - T_{A,B}(z)\|^2,$$

for all $s \in A \cap B$.

3.2 Successive cCRM for the multiset case

We present now the results that extend the cCRM for the multiset case, that is, to solve problem (1.1). We consider three options for the control sequence, as explained in Sect. 1, namely almost cyclic, most violated constraint (distance version) and most violated constraint (function value version), giving rise to Algorithms 1, 2 and 3, respectively.

Initially, Lemma 3.3 is extended to the multiset case.

Corollary 3.5 (Firmly quasinonexpansiveness of parallel circumcenters at centralized points) *Let* $C_1, \ldots, C_m \subset \mathbb{R}^n$ *be nonempty closed convex sets with nonempty intersection. Assume* $i, j \in \{1, 2, \ldots, m\}$ *and suppose* $z \in \mathbb{R}^n$ *is a centralized point with respect to* $C_i \cap C_j$. *Then,* $\mathscr{C}_{C_i,C_j}(z)$ *defined as in* (1.8) *satisfies*

$$\left\|\mathscr{C}_{C_i,C_j}(z) - s\right\| \leq \|z - s\| - \|z - \mathscr{C}_{C_i,C_j}(z)\|,$$

for all $s \in C := \bigcap_{i=1}^m C_i$.

Proof Since $C \subset C_i \cap C_j$ for any $i, j \in \{1, 2, \ldots, m\}$, the result is a direct consequence of Lemma 3.3. □

In sequel, we establish that SiPM iteration (1.6) is quasinonexpansive.

Lemma 3.6 (Quasinonexpansiveness of simultaneous projections) *Let* $C_1, \ldots, C_m \subset \mathbb{R}^n$ *be nonempty closed convex sets with* C *being their nonempty intersection. Assume that* $z \in \mathbb{R}^n$ *is an arbitrary point. Then, we have*

$$\|\tilde{Z}_{C_i,C_j}(z) - s\| \leq \|z - s\|,$$

for each $s \in C$ *and for every* $i, j \in \{1, \ldots, m\}$.

Proof Observe that, for $z \in \mathbb{R}^n$ and $s \in C$,

$$\begin{aligned}
\|\tilde{Z}_{C_i,C_j}(z) - s\| &= \|\frac{1}{2}(P_{C_i} + P_{C_j})(z) - s\| \\
&\leq \frac{1}{2}\|P_{C_i}(z) - s\| + \frac{1}{2}\|P_{C_j}(z) - s\| \\
&\leq \|z - s\|,
\end{aligned}$$

where the second inequality holds by Lemma 2.1(ii). □

Next, the firmly quasinonexpansiveness of the s-cCRM iteration (1.11) is stated as a corollary.

Corollary 3.7 (Firmly quasinonexpansiveness of s-cCRM) *Let $C_1, C_2, \ldots, C_m$ be nonempty closed convex sets with nonempty intersection. Then, for all $z \in \mathbb{R}^n$, we have*

$$\|T_{C_i, C_j}(z) - s\|^2 \le \|z - s\|^2 - \frac{1}{8}\|z - T_{C_i, C_j}(z)\|^2, \tag{3.1}$$

for all $s \in C := \bigcap_{i=1}^m C_i$, and for arbitrary $i, j \in \{1, \ldots, m\}$, with $i \ne j$.

Proof Using the fact that $C \subset C_i \cap C_j$ for any $i, j \in \{1, \ldots, m\}$, the result is directed yielded by Lemma 3.4. □

We state the Fejér monotonicity of s-cCRM, which follows from (3.1) immediately.

Corollary 3.8 (Fejér monotonicity of s-cCRM) *Let $C_1, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets with nonempty intersection C, and suppose that the sequence $\{z^k\} \subset \mathbb{R}^n$ is generated by s-cCRM defined in (1.11) with any of the control sequences given in Definition 1.1. Then $\{z^k\}$ is Fejér monotone with respect to C.*

Proof According to (3.1), we have

$$\|z^{k+1} - s\| = \|T_{C_i, C_j}(z^k) - s\|^2 \le \|z^k - s\|^2,$$

for all $s \in C$ and for arbitrary $i, j \in \{1, \ldots, m\}$, with $i \ne j$. Therefore, the sequence $\{z^k\}$ is Fejér monotone with respect to C. □

Now, the asymptotic convergence of s-cCRM is stated and proved.

Corollary 3.9 (Asymptotic convergence of s-cCRM) *Let $C_1, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and suppose $C := \bigcap_{i=1}^m C_i \ne \emptyset$. Suppose that the sequence $\{z^k\} \subset \mathbb{R}^n$ is generated by s-cCRM defined in (1.11) with any of the control sequences given in Definition 1.1. Then,*

$$\|z^{k+1} - z^k\| \to 0.$$

Proof By using Lemma 3.7, with $z = z^k$, we get

$$\frac{1}{8}\|T_{C_{r(k)}, C_{\ell(k)}}(z^k) - z^k\|^2 \le \|z^k - s\|^2 - \|T_{C_{r(k)}, C_{\ell(k)}}(z^k) - s\|^2,$$

for any $s \in C$. The definition of s-cCRM for CFP with m sets gives us

$$\frac{1}{8}\|z^{k+1} - z^k\|^2 \le \|z^k - s\|^2 - \|z^{k+1} - s\|^2.$$

From Corollary 3.8, the sequence $\{z^k\}$ is Fejér monotone with respect to C. Hence, Lemma 2.3(ii) establishes the result. □

The following lemma is keystone for derive convergence of our proposed algorithms.

Lemma 3.10 *Let $C_1, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and assume $C := \bigcap_{i=1}^{m} C_i \neq \emptyset$. Suppose that the sequence $\{z^k\}$ is generated by s-cCRM, as defined in* (1.11) *with any of the control sequences given in Definition* 1.1. *Then, we have*

$$\|z^{k+1} - s\| \leq \|P_{C_{\ell(k)}}(z^k) - s\|, \tag{3.2}$$

for all $s \in C$.

Proof Note that, for $s \in C$,

$$\begin{aligned}
\|z^{k+1} - s\| &= \|T_{C_{r(k)}, C_{\ell(k)}}(z^k) - s\| \leq \|\bar{Z}_{C_{r(k)}, C_{\ell(k)}}(z^k) - s\| \\
&\leq \|Z_{C_{r(k)}, C_{\ell(k)}}(z^k) - s\| = \|P_{C_{r(k)}}(P_{C_{\ell(k)}}(z^k)) - s\| \\
&\leq \|P_{C_{\ell(k)}}(z^k) - s\|,
\end{aligned} \tag{3.3}$$

where the first inequality follows from Lemma 3.2 and Corollary 3.5, the second inequality holds by Lemma 3.6, and the last one is due to Lemma 2.1(ii). $\qquad\square$

According to the last results, we only need to prove that there exists a cluster point of the s-cCRM sequence lying in the intersection of the underlying sets, in order to achieve the convergence of the proposed method. We establish this in sequel, for Algorithms 1, 2 and 3.

Theorem 3.11 (Convergence of Algorithm 1) *Let $C_1, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and assume $C := \bigcap_{i=1}^{m} C_i \neq \emptyset$. Suppose that z^0 is an arbitrary point in $\mathbb{R}^n$ and the sequence $\{z^k\}$ is generated by s-cCRM operator defined in* (1.11) *with almost cyclic control sequence. Then, there exists some point $x^* \in C := \bigcap_{i=1}^{m} C_i$ such that $z^k \to x^*$.*

Proof By Corollary 3.8 and Lemma 2.3(ii), we get that $\{z^k\}$ is a Fejér monotone sequence w.r.t. C, so it is bounded. Hence, there exist a subsequence $\{z^{k_j}\}$ of $\{z^k\}$ and a point $x^* \in \mathbb{R}^n$ such that $z^{k_j} \to x^*$. By the definition of almost cyclic control sequence, for each $q \in \{1, \ldots, m\}$, there exists a sequence $\{h_j\}$ which satisfies $k_j \leq h_j \leq k_j + Q$ such that $\ell(h_j) = q$, for each $j \geq 1$. By the triangle inequality,

$$\|z^{h_j} - z^{k_j}\| \leq \sum_{i=0}^{h_j - k_j - 1} \|z^{k_j + i + 1} - z^{k_j + i}\| \leq \sum_{i=0}^{Q-1} \|z^{k_j + i + 1} - z^{k_j + i}\|. \tag{3.4}$$

Note that the rightmost side of (3.4) is a sum of Q terms, and each one of them converges to 0 as $j \to \infty$ by Corollary 3.9. Hence, the whole summation goes to 0, so that

$$\lim_{j \to \infty} \|z^{h_j} - z^{k_j}\| = 0. \tag{3.5}$$

Since $z^{h_j} = z^{k_j} + (z^{h_j} - z^{k_j})$ and $z^{k_j} \to x^*$, by assumption, we conclude from (3.5) that $\lim_{j \to \infty} z^{h_j} - x^* = 0$. Note that

$$\|z^{h_j+1} - s\|^2 \leq \|P_{C_{\ell(h_j)}}(z^{h_j}) - s\|^2$$
$$\leq \|z^{h_j} - s\|^2 - \|P_{C_{\ell(h_j)}}(z^{h_j}) - z^{h_j}\|^2,$$

using Lemma 3.10 in the first inequality, and Lemma 2.1(iii) in the second one. Therefore,

$$\|P_{C_{\ell(h_j)}}(z^{h_j}) - z^{h_j}\|^2 \leq \|z^{h_j} - s\|^2 - \|z^{h_j+1} - s\|^2. \tag{3.6}$$

Since for all $\ell(h_j) = q$, in view of Lemma 2.3(ii), taking limit with $j \to \infty$ in (3.6), we get that the right side of (3.6) goes to 0. Consequently,

$$\|P_{C_q}(x^*) - x^*\|^2 \leq 0.$$

Hence, we obtain that $P_{C_q}(x^*) = x^*$. Since q is an arbitrary index, we have that $x^* \in C_q$ for all $q \in \{1, 2, \ldots, m\}$. Consequently, $x^* \in C$. By Lemma 2.3(iii), we get that $z^k \to x^* \in C$.

□

Remark 3.12 We know that the cyclic control sequence is a special case of the almost cyclic control sequence, so we conclude that the sequence $\{z^k\}$ generated by s-cCRM defined in (1.11) will converge to some point in C if we use a cyclic control sequence.

Theorem 3.13 (Convergence of Algorithm 2) *Suppose that z^0 is an arbitrary point in $\mathbb{R}^n$ and the sequence $\{z^k\}$ is generated by the s-cCRM defined in (1.11) with the most violated constraint control sequence in distance version (1.13). Then there exists some point $x^* \in C := \bigcap_{i=1}^m C_i$ such that $z^k \to x^*$.*

Proof Following the same lines of the proof of last theorem, Corollary 3.8 and Lemma 2.3(ii), imply that there exists a subsequence $\{z^{k_j}\}$ of z^k, and a point $x^* \in \mathbb{R}^n$ such that $z^{k_j} \to x^*$. Take any $i \in \{1, \ldots, m\}$, then

$$\text{dist}^2(z^{k_j}, C_i) \leq \text{dist}^2(z^{k_j}, C_{\ell(k_j)}) = \|z^{k_j} - P_{C_{\ell(k_j)}}(z^{k_j})\|^2$$
$$\leq \|z^{k_j} - s\|^2 - \|P_{C_{\ell(k_j)}}(z^{k_j}) - s\|^2 \tag{3.7}$$
$$\leq \|z^{k_j} - s\|^2 - \|z^{k_j+1} - s\|^2,$$

using the definition of the most violated constraint control sequence in the first inequality, Lemma 2.1(ii) in the second one and Lemma 3.10 in the third one. Take $j \to \infty$, and use Lemma 2.3(ii) for proving that the rightmost expression in (3.7) converges to 0. Hence,

$$\text{dist}(x^*, C_i) \leq 0, \ \forall i \in \{1, 2, \ldots, m\}.$$

Since i is an arbitrary index, we conclude that $x^* \in C_i$ for all $i \in \{1, \ldots, m\}$, so that $x^* \in C$. The result follows immediately by Lemma 2.3(iii). □

Theorem 3.14 (Convergence of Algorithm 3) *Consider problem* (1.1)*, suppose z^0 is an arbitrary point in $\mathbb{R}^n$ and assume that sequence $\{z^k\}$ is generated by the s-cCRM defined in* (1.11) *with the most violated constraint control sequence in function value version* (1.15)*. Then there exists some point $x^* \in C := \bigcap_{i=1}^m C_i$ such that $z^k \to x^*$.*

Proof Again, invoking Corollary 3.8 and Lemma 2.3(ii), we get a subsequence $\{z^{k_j}\}$ of z^k and a point $x^* \in \mathbb{R}^n$ such that $z^{k_j} \to x^*$. Using the locally Lipschitz continuity of the convex functions f_i's, for $i \in \{1, \ldots, m\}$, there exists a neighborhood V_i of x^* such that f_i is Lipschitz continuous in V_i with constant L_i. Take $V = \bigcap_{i=1}^m V_i$ and $L = \max_{1 \le i \le m} L_i$, so f_i is Lipschitz continuous in V with constant L for each $i = 1, 2, \ldots, m$.

For large enough k_j, we have

$$
\begin{aligned}
f_i(z^{k_j}) &\le f_{\ell(k_j)}(z^{k_j}) \le f_{\ell(k_j)}(z^{k_j}) - f_{\ell(k_j)}(P_{C_{\ell(k_j)}}(z^{k_j})) \\
&\le L\|z^{k_j} - P_{C_{\ell(k_j)}}(z^{k_j})\| \\
&\le L(\|z^{k_j} - s\|^2 - \|P_{C_{\ell(k_j)}}(z^{k_j}) - s\|^2)^{\frac{1}{2}} \\
&\le L(\|z^{k_j} - s\|^2 - \|z^{k_j+1} - s\|^2)^{\frac{1}{2}},
\end{aligned}
\tag{3.8}
$$

using the definition of the most violated control sequence in the first inequality, the definition of orthogonal projection and (1.14) in the second, the Lipschitz continuity in the third one, and Lemma 2.1(iii) and (3.2) in the fourth one. Taking $j \to \infty$, we have from Lemma 2.3(ii) that the rightmost expression in (3.8) converges to 0. Hence,

$$
f_i(x^*) \le 0.
$$

Since i is an arbitrary index, we get that $f_i(x^*) \le 0$ for all $i \in \{1, \ldots, m\}$, and hence $x^* \in C$. In view of Lemma 2.3(iii) and Corollary 3.8, $z^k \to x^* \in C$. □

4 Linear and superlinear convergence rate

In this section, we first introduce two options of error bounds for CFP. Then, we review the proofs of the linear convergence of SiPM and SePM under these error bounds. Next, we prove, also under these an error bound, the linear convergence of s-cCRM when using most violated control sequences, i.e., Algorithms 2 and 3. Finally, we prove that under a smoothness assumption on the sets C_i and a Slater condition, Algorithms 2 and 3 versions of s-cCRM achieve a superlinear convergence rate. These results are in agreement with those established in [14] for cCRM applied to CFP with two sets.

4.1 Error bounds for CFP

Error bound conditions are regularity assumptions under which convergence rates of projection-type schemes to solve problem (1.1) have been studied; see, for instance, [9, 42–45]. We start with Definition 4.1 below, regarding the error bound when two closed convex sets are considered. This error bound is also called *local (Lipschitz) linear regularity* [9, Def. 3.11], which in turn can be seen as *subtransversality* [46, Thm. 1].

Definition 4.1 *(Error bound for two sets)* Let A, $B \subset \mathbb{R}^n$ be closed convex and assume that $A \cap B \neq \emptyset$. We say that A and B *satisfy a local error bound condition* if for some point $\bar{z} \in X \cap Y$, there exist a real number $\omega \in (0, 1)$, and a neighborhood V of $\bar{z}$ such that

$$\omega \operatorname{dist}(z, A \cap B) \leq \max\{\operatorname{dist}(z, A), \operatorname{dist}(z, B)\},$$

for all $z \in V$.

Under this condition, a point in V cannot be too close to both A and B, and at the same time, far from $A \cap B$. This assumption was used in [11, 14] to prove the linear convergence rate for CRM and cCRM. Now we extend Definition 4.1 to the multi-set case.

Definition 4.2 (EB 1) Let $C_1, C_2, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and assume that $C := \bigcap_{i=1}^m C_i \neq \emptyset$. We say that $C_1, C_2, \ldots, C_m$ satisfy *the local error bound condition (EB1) at a point* $\bar{z} \in C$, if there exists a real number $\omega \in (0, 1)$, and a neighborhood V of $\bar{z}$ such that

$$\omega \operatorname{dist}(z, C) \leq \max_{1 \leq i \leq m} \operatorname{dist}(z, C_i), \tag{EB1}$$

for all $z \in V$.

If the underlying convex sets are defined by means of convex inequalities, we may consider the following error bound condition.

Definition 4.3 (EB 2) Let $C_1, C_2, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and assume that $C := \bigcap_{i=1}^m C_i \neq \emptyset$, with $C_i := \{x \in \mathbb{R}^n \mid f_i(x) \leq 0\}$, and $f_i : \mathbb{R}^n \to \mathbb{R}$ being convex, for all $i = 1, 2, \ldots, m$. We say that $C_1, C_2, \ldots, C_m$ satisfy *the local error bound condition 2 (EB2) at a point* $\bar{z} \in C$, if there exists a real number $\omega \in (0, 1)$, and a neighborhood V of $\bar{z}$ such that

$$\omega \operatorname{dist}(z, C) \leq \max_{1 \leq i \leq m} f_i(z), \tag{EB2}$$

for all $z \in V \setminus C$.

Remark 4.4 Using the definition of the most violated constraint control sequence (1.15), equation (EB2) becomes, for all $z \in V$,

$$\omega \operatorname{dist}(z, C) \leq f_\ell(z),$$

where $\ell := \arg\max\limits_{1 \le i \le m} \{f_i(z)\}$.

Note that (EB1) and (EB2) are clearly connected. Indeed, if (EB1) holds, we can get (EB2) satisfied by setting $f_i(z) := \mathrm{dist}(z, C_i)$, for $i = 1, \ldots, m$. Now, since convex functions are locally Lipschitz continuous, we get that (EB2) always implies (EB1).

Definition 4.5 *(Convergence rate)* Let $\{x^k\} \subset \mathbb{R}^n$ be a sequence converging to some point $\bar{x} \in \mathbb{R}^n$. Assume that $x^k \neq \bar{x}$ for all $k \in \mathbb{N}$. Define

$$\xi := \limsup_{k \to \infty} \frac{\|x^{k+1} - \bar{x}\|}{\|x^k - \bar{x}\|}, \text{ and } \rho := \limsup_{k \to \infty} \|x^k - \bar{x}\|^{\frac{1}{k}}.$$

Then, the convergence of $\{x^k\}$ is

1. *Q-linear* if $\xi \in (0, 1)$;
2. *Q-superlinear* if $\xi = 0$,
3. *R-linear* if $\rho \in (0, 1)$.

It is long-familiar that Q-linear convergence is a sufficient condition for R-linear convergence (with the same asymptotic constant), but the converse statement does not hold true [47].

The next lemma claims the linear convergence of Fejér monotone sequences.

Lemma 4.6 (Fejér monotonicity and linear convergence [14, Prop. 3.7]) *If the sequence $\{x^k\} \subset \mathbb{R}^n$ is Fejér Monotone with respect to a set $M \subset \mathbb{R}^n$, and the scalar sequence $\{\mathrm{dist}(x^k, M)\}$ converges Q-linearly to 0, then $\{x^k\}$ converges R-linearly to a point $\bar{x} \in M$.*

4.2 Linear convergence rate of Algorithm 2

Before proving the linear convergence rate of Algorithm 2, we recall the proof of the linear convergence for SePM and SiPM with the most violated constraint control related to (1.13). We note that, taking into account Definitions 4.2 and 4.3, we assume $\omega \in (0, 1)$.

Lemma 4.7 (Linear convergence of SePM and SiPM under EB1) *Let $C_1, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and suppose $C := \bigcap_{i=1}^{m} C_i \neq \emptyset$. Assume that (EB1) holds at $\bar{z} \in C$. Let B be a ball centered at $\bar{z}$ and contained in V. Let $z \in B$ and define $\beta := \sqrt{1 - \omega^2}$. Then,*

$$\mathrm{dist}(Z_{C_r, C_\ell}(z), C) \le \beta^2 \mathrm{dist}(z, C), \tag{4.1}$$

and

$$\mathrm{dist}(\tilde{Z}_{C_r, C_\ell}(z), C) \le \left(\frac{1 + \beta}{2}\right) \mathrm{dist}(z, C), \tag{4.2}$$

where $\ell := \arg\max\limits_{1 \le i \le m} \{\mathrm{dist}(z, C_i)\}$ *and* $r := \arg\max\limits_{1 \le i \le m} \{\mathrm{dist}(P_{C_\ell}(z), C_i)\}$.

Proof We start with the SePM case. Note that

$$
\begin{aligned}
\operatorname{dist}^2(z, C) = \|z - P_C(z)\|^2 &\geq \|P_{C_\ell}(z) - P_C(z)\|^2 + \|z - P_{C_\ell}(z)\|^2 \\
&\geq \operatorname{dist}^2(P_{C_\ell}(z), C) + \operatorname{dist}^2(z, C_\ell) \\
&= \operatorname{dist}^2(P_{C_\ell}(z), C) + \max_{1 \leq i \leq m} \operatorname{dist}^2(z, C_i) \\
&\geq \operatorname{dist}^2(P_{C_\ell}(z), C) + \omega^2 \operatorname{dist}^2(z, C),
\end{aligned}
$$

where the first inequality holds by Lemma 2.1(iii), in the second one we use the definition of P_{C_ℓ}, the last equality follows from the definition of ℓ, and the third inequality follows by (EB1). Hence,

$$
\operatorname{dist}(P_{C_\ell}(z), C) \leq \sqrt{1 - \omega^2}\, \operatorname{dist}(z, C) = \beta \operatorname{dist}(z, C). \tag{4.3}
$$

Since $\bar{z} \in C$, by the nonexpansiveness of P_{C_ℓ} and P_{C_r}, we have

$$
\|P_{C_r}(P_{C_\ell}(z)) - \bar{z}\| \leq \|P_{C_\ell}(z) - \bar{z}\| \leq \|z - \bar{z}\|, \tag{4.4}
$$

hence, $P_{C_r}(P_{C_\ell}(z)) \in B$. Consequently,

$$
\begin{aligned}
\operatorname{dist}^2(P_{C_\ell}(z), C) = \|P_{C_\ell}(z) - P_C(P_{C_\ell}(z))\|^2 \\
&\geq \|P_{C_r}(P_{C_\ell}(z)) - P_C(P_{C_\ell}(z))\|^2 + \|P_{C_\ell}(z) - P_{C_r}(P_{C_\ell}(z))\|^2 \\
&\geq \operatorname{dist}^2(P_{C_r}(P_{C_\ell}(z)), C) + \max_{1 \leq i \leq m} \operatorname{dist}^2(P_{C_i}(z), C_r) \\
&\geq \operatorname{dist}^2(P_{C_r}(P_{C_\ell}(z)), C) + \omega^2 \operatorname{dist}^2(P_{C_\ell}(z), C),
\end{aligned} \tag{4.5}
$$

where the first inequality follows from Lemma 2.1(iii), the second follows from the definition of r, and the third one from (EB1). From (4.5), we obtain

$$
\operatorname{dist}(P_{C_r}(P_{C_\ell}(z)), C) \leq \sqrt{1 - \omega^2}\, \operatorname{dist}(P_{C_\ell}(z), C) = \beta \operatorname{dist}(P_{C_\ell}(z), C). \tag{4.6}
$$

Now combining (4.4) and (4.6), we get

$$
\begin{aligned}
\operatorname{dist}(Z_{C_r, C_\ell}(z), C) = \operatorname{dist}(P_{C_r}(P_{C_\ell}(z)), C) &\leq \sqrt{1 - \omega^2}\, \operatorname{dist}(P_{C_\ell}(z), C) \\
&\leq (1 - \omega^2)\, \operatorname{dist}(z, C) = \beta^2 \operatorname{dist}(z, C),
\end{aligned}
$$

which establishes (4.1).

Next, we will establish the linear convergence for SiPM with (EB1). By the nonexpansiveness of P_{C_r}, we have that

$$
\operatorname{dist}(P_{C_r}(z), C) \leq \operatorname{dist}(z, C). \tag{4.7}
$$

Note that

$$
\begin{aligned}
\text{dist}(\tilde{Z}_{C_r, C_\ell}(z), C) &= \text{dist}\left(\frac{1}{2}[P_{C_\ell}(z) + P_{C_r}(z)], C\right) \\
&\leq \frac{1}{2}\left[\text{dist}(P_{C_\ell}(z), C) + \text{dist}(P_{C_r}(z), C)\right] \\
&\leq \left(\frac{1+\beta}{2}\right)\text{dist}(z, C),
\end{aligned}
$$

using the convexity of the distance function to C in the first inequality, and (4.3) and (4.7) in the second one. Hence, we get (4.2), and the results hold. □

Corollary 4.8 (Linear convergence of SePM and SiPM under EB1) *Let $C_1, C_2, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and assume that $C := \bigcap_{i=1}^m C_i \neq \emptyset$. Suppose $\{s^k\}$ and $\{y^k\}$ are sequences generated by SePM and SiPM starting from some $s^0 \in \mathbb{R}^n$ and $y^0 \in \mathbb{R}^n$, respectively. Assume also $\{s^k\}$ and $\{y^k\}$ are both infinite sequences. If (EB1) holds at the limit points $\bar{s}$ of $\{s^k\}$, $\bar{y}$ of $\{y^k\}$, then the sequences $\{s^k\}$, $\{y^k\}$ converge R-linearly, with asymptotic constants bounded above by β^2, $\frac{1+\beta}{2}$ respectively, where $\beta = \sqrt{1-\omega^2}$ and ω is the constant in Definition 4.2.*

Proof Convergence of $\{s^k\}$ and $\{y^k\}$ to points $\bar{s} \in C$ and $\bar{y} \in C$ follows from [4, Corollary 3.3(i)] and in [5, Theorem 3], respectively. Hence, for large enough k, s^k belongs to a ball centered at $\bar{s}$ contained in V, and y^k belongs to a ball centered at $\bar{y}$ contained in V.

In view of the definitions of the SePM and SiPM sequences, we get from Lemma 4.7,

$$
\frac{\text{dist}(s^{k+1}, C)}{\text{dist}(s^k, C)} \leq \beta^2, \quad \frac{\text{dist}(y^{k+1}, C)}{\text{dist}(y^k, C)} \leq \frac{1+\beta}{2}.
$$

Now, from Definition 4.5, we get

$$
\text{dist}(s^k, C) \leq \left(\frac{1+\beta}{2}\right)^k \text{dist}(s^0, C), \quad \text{and} \quad \text{dist}(y^k, C) \leq \beta^{2k} \text{dist}(y^0, C).
$$

Hence, both distance sequences $\{\text{dist}(s^k, C)\}$ and $\{\text{dist}(y^k, C)\}$ converge Q-linearly to 0, with asymptotic constants given by β^2, $\frac{1+\beta}{2}$, respectively, since $\beta \in (0, 1)$, because $\omega \in (0, 1)$.

The fact that $\{s^k\}$ and $\{y^k\}$ are Fejér monotone with respect to C is an immediate consequence of Lemma 2.1(ii) and Lemma 3.6, respectively. Then, the result follows then from Lemma 4.6. □

We now proceed to prove the linear convergence of s-cCRM. First, we show the linear rate for the most violated constraint control (Algorithm 2) related to (1.13), in view of (EB1).

Lemma 4.9 (Linear convergence of the distance for s-cCRM (Algorithm 2) under EB1)
Let $C_1, C_2, \ldots, C_m$ be nonempty closed convex sets, and suppose that $C := \bigcap_{i=1}^{m} C_i \neq \emptyset$. Assume that (EB1) at $\bar{z} \in C$. Let B be a ball centered at $\bar{z}$ and contained in V. Let $z \in B$ and define $\beta := \sqrt{1 - \omega^2}$. Then,

$$\operatorname{dist}(T_{C_r, C_\ell}(z), C) \leq \beta^2 \operatorname{dist}(z, C),$$

where $\ell := \arg\max_{1 \leq i \leq m}\{\operatorname{dist}(z, C_i)\}$ and $r := \arg\max_{1 \leq i \leq m}\{\operatorname{dist}(P_{C_\ell}(z), C_i)\}$.

Proof By (4.4), we know that $Z_{C_r, C_\ell}(z) \in B$. Using the definition of $\bar{Z}_{C_r, C_\ell}$, we have

$$
\begin{aligned}
\operatorname{dist}(\bar{Z}_{C_r, C_\ell}(z), C) &= \operatorname{dist}\left(\frac{1}{2}\left(Z_{C_r, C_\ell}(z) + P_{C_\ell}(Z_{C_r, C_\ell}(z))\right), C\right) \\
&\leq \frac{1}{2}\operatorname{dist}(Z_{C_r, C_\ell}(z), C) + \frac{1}{2}\operatorname{dist}(P_{C_\ell}(Z_{C_r, C_\ell}(z)), C) \\
&\leq \frac{1}{2}\beta^2 \operatorname{dist}(z, C) + \frac{1}{2}\beta^2 \operatorname{dist}(z, C) \\
&= \beta^2 \operatorname{dist}(z, C), \tag{4.8}
\end{aligned}
$$

where the first inequality follows from the convexity of the distance function, and the second from Lemma 2.1(iv) and (4.1).

By Corollary 3.5, we have

$$\|\mathscr{C}_{C_r, C_\ell}(z) - s\| \leq \|z - s\|,$$

for any centralized point $z \in B$ and for any $s \in C$. By Lemma 3.2 and (4.4), $\bar{Z}_{C_r, C_\ell}(z)$ is centralized with respect to C_ℓ and C_r and belongs to B. So, taking $\bar{Z}_{C_r, C_\ell}(z)$, we get

$$\|T_{C_r, C_\ell}(z) - s\| = \left\|\mathscr{C}_{C_r, C_\ell}(\bar{Z}_{C_r, C_\ell}(z)) - s\right\| \leq \|\bar{Z}_{C_r, C_\ell}(z) - s\|,$$

where the inequality follows from Lemma 3.3. Using the definition of distance between $T_{C_r, C_\ell}(z)$ and C, we get

$$\operatorname{dist}(T_{C_r, C_\ell}(z), C) \leq \|\bar{Z}_{C_r, C_\ell}(z) - s\|.$$

Take $s = P_C(\bar{Z}_{C_r, C_\ell}(z))$, then

$$\operatorname{dist}(T_{C_r, C_\ell}(z), C) \leq \operatorname{dist}(\bar{Z}_{C_r, C_\ell}(z), C), \tag{4.9}$$

for all $z \in B$. Combining (4.8) and (4.9), we obtain

$$\operatorname{dist}(T_{C_r, C_\ell}(z), C) \leq \beta^2 \operatorname{dist}(z, C),$$

which establishes the result. $\qquad\square$

Now, we are going to establish linear convergence of Algorithm 2, under (EB1).

Theorem 4.10 (Linear convergence of s-cCRM (Algorithm 2) under EB1) *Let $C_1, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and suppose that $C := \bigcap_{i=1}^m C_i \neq \emptyset$. Assume that sequence $\{z^k\}$ is generated by s-cCRM with the most violated constraint control sequence (distance version) as in (1.13), starting from some $z^0 \in \mathbb{R}^n$. Assume also that $\{z^k\}$ is an infinite sequence. If (EB1) holds at the limit $\bar{z}$ of $\{z^k\}$, then $\{z^k\}$ converges to $\bar{z} \in C$ R-linearly, with asymptotic constant bounded above β^2, where $\beta = \sqrt{1 - \omega^2}$ and ω is the constant from Definition 4.2.*

Proof The convergence of $\{z^k\}$ to a point $\bar{z} \in C$ follows from Theorem 3.13. Hence, for large enough k, z^k belongs to the ball centered at $\bar{z}$ and contained in V, whose existence is ensured in (EB1).

We recall that the s-cCRM sequence is defined as $z^{k+1} = T_{C_{r(k)}, C_{\ell(k)}}(z^k)$, so that it follows from Lemma 4.9 that

$$\frac{\text{dist}(z^{k+1}, C)}{\text{dist}(z^k, C)} \leq \beta^2. \tag{4.10}$$

Since $\omega \in (0, 1)$ implies that $\beta^2 \in (0, 1)$, it follows immediately from (4.10) that the scalar sequence $\{\text{dist}(z^k, C)\}$ converges Q-linearly to 0 with asymptotic constant bounded above by β^2.

Finally, recall that sequence $\{z^k\}$ is Fejér monotone with respect to C, due to Corollary 3.8. The R-linear convergence of $\{z^k\}$ to some point in C and the value of the upper bound of the asymptotic follow from Lemma 4.6. □

4.3 Linear convergence of Algorithm 3

Before proving the linear convergence of s-cCRM (Algorithm 3) under (EB2), we need some lemmas about the relationship between (EB1) and (EB2). First we provide a bound for the norm of subdifferentials under (EB2).

Lemma 4.11 (Bound of subdifferentials under EB2) *Let $C_1, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and suppose that $C := \bigcap_{i=1}^m C_i \neq \emptyset$. Assume that (EB2) holds at a point $\bar{z} \in C$. Then, there exists a ball B centered at $\bar{z}$ and a constant $\epsilon > 0$ such that*

$$\|v_i(u)\| \leq \epsilon,$$

for all $u \in B$, all $v_i(u) \in \partial f_i(u)$, and all $i \in \{1, \ldots, m\}$.

Proof Take an arbitrary point $u \in \mathbb{R}^n$, and let $v_i(u) \in \mathbb{R}^n$ be any subgradient of f_i at u. Now, recall that the convex function f_i is locally Lipschitz continuous in $\mathbb{R}^n$ for all $i \in \{1, \ldots, m\}$. Using the fact that the subdifferentials of f_i's are locally bounded in $\mathbb{R}^n$, we establish the result. □

Lemma 4.12 (Relation betweem EB1 and EB2) *Let $C_1, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and suppose that $C := \bigcap_{i=1}^m C_i \neq \emptyset$. Assume that (EB2) holds at point $\bar{z} \in C$. Then, there exists $\epsilon > 0$ such*

$$f_i(z) \leq \epsilon \operatorname{dist}(z, C_i), \tag{4.11}$$

for all $i \in \{1, \ldots, m\}$, and all z in the neighborhood V of $\bar{z}$ as defined in Definition 4.3. Moreover,

$$\operatorname{dist}(z, C) \leq \frac{\epsilon}{\omega} \operatorname{dist}(z, C_\ell), \tag{4.12}$$

where $\ell := \underset{1 \leq i \leq m}{\arg \max}\{f_i(z)\}$.

Proof By Lemma 4.11, there exists a ball B contained in V, centered at $\bar{z}$, and a constant ϵ such that

$$\|v_i(u)\| \leq \epsilon, \tag{4.13}$$

for all $v_i(u) \in \partial f_i(u)$, and for all $u \in B$, and for each index $i \in \{1, \ldots, m\}$. Take any $z \in B$ and let $z_i := P_{C_i}(z)$. Using Lemma 2.6, we get

$$f_i(z) = f_i(z_i) + \langle v_i(u_i), z - z_i \rangle, \tag{4.14}$$

for some u_i in the line segment between z and z_i and, $v_i(u_i) \in \partial f(u_i)$. Since $z_i \in C_i$, we have $f_i(z_i) = 0$, and it follows from (4.14) that

$$f_i(z) \leq \|v_i(u_i)\| \|z - z_i\| = \|v_i(u_i)\| \operatorname{dist}(z, C_i). \tag{4.15}$$

By the nonexpansiveness of the orthogonal projection, we have

$$\|P_{C_i}(z) - P_{C_i}(\bar{z})\| \leq \|z - \bar{z}\|.$$

The definition of z_i and $\bar{z} \in C$, yields

$$\|z_i - \bar{z}\| \leq \|z - \bar{z}\|,$$

which shows that $z_i \in B$. Hence, $u_i \in B$, by the convexity of the ball. In view of (4.13) and (4.15), we have

$$f_i(z) \leq \epsilon \operatorname{dist}(z, C_i),$$

for each $i = 1, \ldots, m$ and each $z \in B$. Therefore,

$$f_\ell(z) \leq \epsilon \operatorname{dist}(z, C_\ell),$$

so that, in view of (EB2), it holds that

$$\text{dist}(z, C) \leq \frac{\epsilon}{\omega} f_\ell(z) \leq \frac{\epsilon}{\omega} \text{dist}(z, C_\ell),$$

which establishes the result. $\square$

Remark 4.13 With arguments very similar to those used in the previous lemma, together with the nonexpansiveness of projection operator, we can easily get

$$\text{dist}(P_{C_\ell}(z), C) \leq \frac{\epsilon}{\omega} \text{dist}(P_{C_\ell}(z), C_r), \tag{4.16}$$

where $r := \underset{1 \leq i \leq m}{\arg\max}\{f_i(P_{C_\ell}(z))\}$.

Lemma 4.14 (Linear convergence of the distance for SePM and SiPM under EB2) *Let* $C_1, \ldots, C_m \subset \mathbb{R}^n$ *be nonempty closed convex sets, and suppose* $C := \bigcap_{i=1}^m C_i \neq \emptyset$. *Assume that* (EB2) *holds at* $\bar{z} \in C$, *and that B is a ball centered at* $\bar{z}$ *and contained in V. Define* $\beta := \sqrt{1 - \left(\frac{\omega}{\epsilon}\right)^2}$, *with* ω *being the constant in Definition 4.3, and* ϵ *being the constant in Lemma 4.12. Then,*

$$\text{dist}(Z_{C_r, C_\ell}(z), C) \leq \beta^2 \text{dist}(z, C), \tag{4.17}$$

for all $z \in B$, *and*

$$\text{dist}(\tilde{Z}_{C_r, C_\ell}(z), C) \leq \frac{1 + \beta}{2} \text{dist}(z, C), \tag{4.18}$$

for all $z \in B$, *where* $\ell := \underset{1 \leq i \leq m}{\arg\max}\{f_i(z)\}$ *and* $r := \underset{1 \leq i \leq m}{\arg\max}\{f_i(P_{C_\ell}(z))\}$.

Proof We start with proving (4.17). Take $z \in B$, and note that

$$\begin{aligned}
\text{dist}^2(z, C) &= \|z - P_C(z)\|^2 \\
&\geq \|P_{C_\ell}(z) - z\|^2 + \|P_{C_\ell}(z) - P_C(z)\|^2 \\
&\geq \text{dist}^2(z, C_\ell) + \text{dist}^2(P_{C_\ell}(z), C) \\
&\geq \text{dist}^2(P_{C_\ell}(z), C) + \frac{\omega^2}{\epsilon^2} \text{dist}^2(z, C),
\end{aligned}$$

using Lemma 2.1(iii) in the first inequality, the definition of orthogonal projection in the second one, and (4.12) in the third one. Hence,

$$\text{dist}(P_{C_\ell}(z), C) \leq \sqrt{1 - \frac{\omega^2}{\epsilon^2}} \text{dist}(z, C) = \beta \text{dist}(z, C). \tag{4.19}$$

By (4.4), we get $P_{C_r}(P_{C_\ell}(z)) \in B$. Therefore,

$$
\begin{aligned}
\text{dist}^2(P_{C_\ell}(z), C) &= \|P_{C_\ell}(z) - P_C(P_{C_\ell}(z))\|^2 \\
&\geq \|P_{C_r}(P_{C_\ell}(z)) - P_C(P_{C_\ell}(z))\|^2 \\
&\quad + \|P_{C_r}(P_{C_\ell}(z)) - P_{C_\ell}(z)\|^2 \\
&\geq \text{dist}^2(P_{C_r}(P_{C_\ell}(z)), C) + \text{dist}^2(P_{C_\ell}(z), C_r) \\
&\geq \text{dist}^2(P_{C_r}(P_{C_\ell}(z)), C) + \frac{\omega^2}{\epsilon^2}\text{dist}^2(P_{C_\ell}(z), C),
\end{aligned}
$$

using Lemma 2.1(iii) in the first inequality, the definition of orthogonal projection in the second one, and (4.16) in the third one. Thus, we obtain

$$
\text{dist}(P_{C_r}(P_{C_\ell}(z)), C) \leq \beta\,\text{dist}(P_{C_\ell}(z)), C).
$$

Together with (4.19), we have

$$
\text{dist}(Z_{C_r,C_\ell}(z), C) = \text{dist}(P_{C_r}(P_{C_\ell}(z)), C) \leq \beta^2\,\text{dist}(z, C).
$$

Next, we prove (4.18). By the nonexpansiveness of P_{C_r}, we have

$$
\text{dist}(P_{C_r}(z), C) \leq \text{dist}(z, C). \tag{4.20}
$$

Note that

$$
\begin{aligned}
\text{dist}(\tilde{Z}_{C_r,C_\ell}(z), C) &= \text{dist}(\frac{1}{2}[P_{C_\ell}(z) + P_{C_r}(z)], C) \\
&\leq \frac{1}{2}[\text{dist}(P_{C_\ell}(z), C) + \text{dist}(P_{C_r}(z), C)] \\
&\leq \left(\frac{1+\beta}{2}\right)\text{dist}(z, C).
\end{aligned}
$$

The first inequality holds by the convexity of the distance function, and the second one follows by (4.20) and (4.19). □

Corollary 4.15 (Linear convergence of SePM and SiPM under EB2) *Let $C_1, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and assume that $C := \bigcap_{i=1}^m C_i \neq \emptyset$. Let $\{s^k\}$ and $\{y^k\}$ be sequences generated by SePM and SiPM, starting from some $s^0 \in \mathbb{R}^n$ and $y^0 \in \mathbb{R}^n$, respectively. Assume also $\{s^k\}$ and $\{y^k\}$ are both infinite sequences. If (EB2) holds at the limit point $\bar{s}$ of $\{s^k\}$, $\bar{y}$ of $\{y^k\}$, then the sequences $\{s^k\}$, $\{y^k\}$ converge R-linearly, with asymptotic constants bounded above by β^2, $\frac{1+\beta}{2}$ respectively, where $\beta = \sqrt{1 - (\frac{\omega}{\epsilon})^2}$, ω is the constant in Definition 4.3, and ϵ is the constant in Lemma 4.12.*

Proof We invoke again [4, Corollary 3.3(i)] and [5, Theorem 3], to get the convergence of $\{s^k\}$ and $\{y^k\}$ to points $\bar{s} \in C$ and $\bar{y} \in C$, respectively. Hence, s^k belongs to a ball

centered at $\bar{s}$ contained in V, and y^k belongs to a ball centered at $\bar{y}$ contained in V and, for large enough k.

In view of the definitions of the SePM and SiPM sequences, from Lemma 4.14 we derive that

$$\frac{\text{dist}(s^{k+1}, C)}{\text{dist}(s^k, C)} \le \beta^2, \quad \text{and} \quad \frac{\text{dist}(y^{k+1}, C)}{\text{dist}(y^k, C)} \le \frac{1+\beta}{2}.$$

The limits of the above inequalities (with $k \to \infty$), together with Definition 4.5, yield that the sequences $\{\text{dist}(s^k, C)\}$ and $\{\text{dist}(y^k, C)\}$ converge Q-linearly to 0, with asymptotic constants given by β^2 and $\frac{1+\beta}{2}$, respectively, where $\beta = \sqrt{1 - (\frac{\omega}{\epsilon})^2}$, ω is the constant in (EB2), and ϵ is the constant in Lemma 4.12.

Remind that $\{s^k\}$ and $\{y^k\}$ are Fejér monotone with respect to C (by Lemma 2.1(ii) and Lemma 3.6, respectively). Hence, the result follows from Lemma 4.6. □

Now, we present a lemma that allows us to prove the linear convergence of Algorithm 3, under (EB2).

Lemma 4.16 (Linear convergence of distance for s-cCRM (Algorithm 3) under EB2) *Suppose $C_1, C_2, \ldots, C_m \subset \mathbb{R}^n$ nonempty closed convex sets and suppose $C := \bigcap_{i=1}^m C_i \neq \emptyset$. Assume that (EB2) at $\bar{z} \in C$. Let B be a ball centered at $\bar{z}$ and contained in V. Let $z \in B$ and define $\beta := \sqrt{1 - \left(\frac{\omega}{\epsilon}\right)^2}$, with ω being the constant in Definition 4.3, and ϵ being the constant in Lemma 4.12. Then,*

$$\text{dist}(T_{C_r, C_\ell}(z), C) \le \beta^2 \, \text{dist}(z, C),$$

for all $z \in B$.

Proof By (4.4), we know that $Z_{C_r, C_\ell}(z) \in B$. Using the definition of $\bar{Z}_{C_r, C_\ell}$, we have

$$\begin{aligned}
\text{dist}(\bar{Z}_{C_r, C_\ell}(z), C) &= \text{dist}\left(\frac{1}{2}[Z_{C_r, C_\ell}(z) + P_{C_\ell}(Z_{C_r, C_\ell}(z))], C\right) \\
&\le \frac{1}{2} \text{dist}(Z_{C_r, C_\ell}(z), C) + \frac{1}{2} \text{dist}(P_{C_\ell}(Z_{C_r, C_\ell}(z)), C) \\
&\le \frac{1}{2}\beta^2 \, \text{dist}(z, C) + \frac{1}{2}\beta^2 \, \text{dist}(z, C) \\
&= \beta^2 \, \text{dist}(z, C), \tag{4.21}
\end{aligned}$$

where the first inequality follows from the convexity of the distance function, and the second from Lemma 2.1(iv) and (4.17). Combining (4.21) and (4.9), we establish the result. □

We finalize this section stating and proving the linear convergence of s-cCRM with the most violated constraint control sequence (function value version) as in (1.15), under EB2.

Theorem 4.17 (Linear convergence of s-cCRM (Algorithm 3) under EB2) *Let $C_1, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and suppose that $C := \bigcap_{i=1}^m C_i \neq \emptyset$. Let the sequence $\{z^k\}$ be generated by s-cCRM with the most violated constraint control sequence (function value version) as in (1.15), starting from some $z^0 \in \mathbb{R}^n$. Assume also that $\{z^k\}$ is an infinite sequence. If (EB2) holds at the limit $\bar{z}$ of $\{z^k\}$, then $\{z^k\}$ converges to $\bar{z} \in C$ R-linearly, with asymptotic constant bounded above β^2, where $\beta = \sqrt{1 - (\frac{\omega}{\epsilon})^2}$, ω is the constant in Definition 4.3, and ϵ is the constant in Lemma 4.12.*

Proof Convergence of $\{z^k\}$ to a point $\bar{z} \in C$ follows from Theorem 3.14. Hence, for large enough k, z^k belongs to the ball centered at $\bar{z}$ and contained in V, whose existence is ensured in (EB2).

We recall that the s-cCRM sequence is defined as $z^{k+1} = T_{C_{r(k)}, C_{\ell(k)}}(z^k)$, so that it follows from Lemma 4.16 that

$$\frac{\text{dist}(z^{k+1}, C)}{\text{dist}(z^k, C)} \leq \beta^2. \tag{4.22}$$

Since $\beta^2 \in (0, 1)$, it follows immediately from (4.22) that the scalar sequence $\{\text{dist}(z^k, C)\}$ converges Q-linearly to zero with asymptotic constant bounded above by β^2.

Finally, recall that the sequence $\{z^k\}$ is Fejér monotone with respect to C, due to Corollary 3.8. The R-linear convergence of $\{z^k\}$ to some point in C and the value of the upper bound of the asymptotic follow from Lemma 4.6. □

4.4 Superlinear convergence of s-cCRM

In this subsection we prove superlinear convergence of Algorithms 2 and 3 versions of s-cCRM, assuming a Slater condition and a smoothness assumption: the boundaries of the sets C_i are differentiable manifolds (of codimension 1, due to the Slater condition) near the limit of the sequence. First, we need a lemma about differentiable manifolds.

Lemma 4.18 (Dimension of differentiable manifolds boundaries [48, Thm. 24.3]) *Let M be a k-dimensional manifold in $\mathbb{R}^n$, of class $\mathcal{C}^p$. If the boundary of M, $\text{bd}(M)$, is nonempty, then $\text{bd}(M)$ is a $(k-1)$-dimensional manifold without boundary in $\mathbb{R}^n$, of class $\mathcal{C}^p$.*

Now we can demonstrate the superlinear convergence of s-cCRM. We begin by the most violated constraint control sequence (distance version) as in (1.13) (Algorithm 2).

Lemma 4.19 (Superlinear convergence of the distance for s-cCRM (Algorithm 2)) *Let $C_1, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and suppose that $C := \bigcap_{i=1}^m C_i \neq \emptyset$. Let sequence the $\{z^k\}$ be generated by s-cCRM with the most violated constraint control sequence (distance version) as in (1.13), starting from some $z^0 \in \mathbb{R}^n$, and converging to a point $\bar{z} \in C$. Assume that the interior of C is nonempty and that*

the boundaries of C_i are differentiable manifolds in a neighborhood of $\bar{z}$ for each $i = 1, \ldots, m$. Then, the scalar sequence $\{\text{dist}(z^k, C)\}$ converges to zero superlinearly.

Proof It is trivial if sequence $\{z^k\}$ is finite. Hence, let's assume that it is infinite.

In order to prove the superlinear convergence rate, i.e.,

$$\lim_{k \to \infty} \frac{\text{dist}(z^{k+1}, C)}{\text{dist}(z^k, C)} = \lim_{k \to \infty} \frac{\text{dist}(T_{C_{r(k)}, C_{\ell(k)}}(z^k), C)}{\text{dist}(z^k, C)} = 0, \qquad (4.23)$$

it suffices to show that

$$\lim_{k \to \infty} \frac{\text{dist}(T_{C_{r(k)}, C_{\ell(k)}}(z^k), C)}{\text{dist}(\bar{Z}_{C_{r(k)}, C_{\ell(k)}}(z^k), C)} = 0, \qquad (4.24)$$

because, by (3.3) and the nonexpansiveness of orthogonal projection, we know that $\text{dist}(\bar{Z}_{C_{r(k)}, C_{\ell(k)}}(z^k), C) \leq \text{dist}(z^k, C)$, so that (4.23) follows from (4.24) immediately.

We claim that the assumption $\text{int}(C) \neq \emptyset$ implies (EB1). Indeed, by Corollary 5.14 in [4], there exists $\hat{k} \in N$ such that

$$\omega \, \text{dist}(z^k, C) \leq \max_{1 \leq i \leq m} \{\text{dist}(z^k, C_i)\} = \text{dist}(z^k, C_{\ell(k)}), \qquad (4.25)$$

for all $\hat{k} \geq k$. In addition, the nonemptyness of the interior of C, together with the hypothesis that the boundaries of C_i are locally differentiable manifolds, proves that these manifolds have dimension $n - 1$, in view of Lemma 4.18.

We remark that, when $M \subset \mathbb{R}^n$ is a differentiable manifold of dimension $n - 1$, we have that if $\bar{z}$ belongs to M and $z \in \mathbb{R}^n$ lies on the tangent hyperplane to M at $\bar{z}$, denoted here by $T_M(\bar{z})$, then

$$\lim_{\substack{z \to \bar{z} \\ z \in T_M(\bar{z})}} \frac{\text{dist}(z, M)}{\|z - \bar{z}\|} = 0. \qquad (4.26)$$

This limit is discussed in more detail in [14, Eq. 3.32].

Consider the $(n - 1)$ dimensional hyperplanes $H_{C_{\ell(k)}}^k$ and $H_{C_{r(k)}}^k$, which are tangent to the manifolds, respectively at

$$P_{C_{\ell(k)}}(\bar{Z}_{C_{r(k)}, C_{\ell(k)}}(z^k)) \text{ and } P_{C_{r(k)}}(\bar{Z}_{C_{r(k)}, C_{\ell(k)}}(z^k)).$$

Now, note that z^{k+1} is the circumcenter of $\{z^k, R_{C_{\ell(k)}}(z^k), R_{C_{r(k)}}(z^k)\}$. So, z^{k+1} lies in the intersection of the bisectors passing by $P_{C_{\ell(k)}}(z^k) \in H_{C_{\ell(k)}}^k$, and $P_{C_{r(k)}}(z^k) \in H_{C_{r(k)}}^k$, respectively. Each bisector is contained in the hyperplane $H_{C_{\ell(k)}}^k$ or $H_{C_{r(k)}}^k$, and hence $z^{k+1} \in H_{C_{\ell(k)}}^k \cap H_{C_{r(k)}}^k$. Therefore, we have, in view of (4.26),

$$\lim_{k \to \infty} \frac{\text{dist}(z^{k+1}, C_{\ell(k)})}{\|z^{k+1} - P_{C_{\ell(k)}}(\bar{Z}_{\ell(k), r(k)}(z^k))\|} = 0. \qquad (4.27)$$

Now, using the nonexpansiveness of projections onto $H_{C_{\ell(k)}}$ and $H_{C_{r(k)}}$, we get

$$\|P_{H_{C_{\ell(k)}}}(z^{k+1}) - P_{H_{C_{\ell(k)}}}(\bar{Z}_{C_{r(k)},C_{\ell(k)}}(z^k))\| \le \|z^{k+1} - \bar{Z}_{C_{r(k)},C_{\ell(k)}}(z^k)\|.$$

Since $z^{k+1} \in H_{C_{\ell(k)}}$ and $P_{H_{C_{\ell(k)}}}(\bar{Z}_{C_{r(k)},C_{\ell(k)}}(z^k)) = P_{C_{\ell(k)}}(\bar{Z}_{C_{r(k)},C_{\ell(k)}}(z^k))$, we get

$$\|z^{k+1} - P_{C_{\ell(k)}}(\bar{Z}_{C_{r(k)},C_{\ell(k)}}(z^k))\| \le \|z^{k+1} - \bar{Z}_{C_{r(k)},C_{\ell(k)}}(z^k)\|. \tag{4.28}$$

By Lemma 3.3, it holds that $\|z^{k+1} - \bar{Z}_{C_{r(k)},C_{\ell(k)}}(z^k)\| \le \text{dist}(\bar{Z}_{C_{r(k)},C_{\ell(k)}}(z^k), C)$, which combined with (4.28), implies that

$$\|z^{k+1} - P_{C_{\ell(k)}}(\bar{Z}_{C_{r(k)},C_{\ell(k)}}(z^k))\| \le \text{dist}(\bar{Z}_{C_{r(k)},C_{\ell(k)}}(z^k), C).$$

Thus, from (4.27), it follows that

$$\lim_{k \to \infty} \frac{\text{dist}(z^{k+1}, C_{\ell(k)})}{\text{dist}(\bar{Z}_{C_{r(k)},C_{\ell(k)}}(z^k), C)} = 0. \tag{4.29}$$

Moreover, (4.25) yields

$$\omega \frac{\text{dist}(z^{k+1}, C)}{\text{dist}(\bar{Z}_{C_{r(k)},C_{\ell(k)}}(z^k), C)} \le \frac{\text{dist}(z^{k+1}, C_{\ell(k)})}{\text{dist}(\bar{Z}_{C_{r(k)},C_{\ell(k)}}(z^k), C)}.$$

Taking limits now as $k \to \infty$, we get (4.24) and the proof is completed. □

The next key result allows us to use the superlinear rate of a scalar distance sequence to prove the superlinear rate of the underlying sequence.

Lemma 4.20 (Fejér monotonicity and superlinear convergence [14, Prop. 3.12]) *Take a sequence $\{z^k\} \subset \mathbb{R}^n$ which is Fejér monotone with respect to the closed convex set $M \subset \mathbb{R}^n$. If the scalar sequence $\{\text{dist}(z^k, M)\}$ converges superlinearly to 0, then $\{z^k\}$ converges superlinearly to a point $\bar{z} \in M$.*

Next, we present the superlinear convergence result for Algorithm 2.

Theorem 4.21 (Superlinear convergence of s-cCRM (Algorithm 2)) *Let $C_1, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and suppose that $C := \bigcap_{i=1}^m C_i$ is nonempty. Let $\{z^k\}$ be generated by s-cCRM with the most violated control sequence (distance version) as in (1.13), starting from some $z^0 \in \mathbb{R}^n$, and converging to a point $\bar{z} \in C$. Assume that the interior of C is nonempty, and that the boundary of C_i is a differentiable manifold in a neighborhood of $\bar{z}$ for each $i = 1, \ldots, m$. Then, $\{z^k\}$ converges to $\bar{z}$ superlinearly.*

Proof The result is a direct consequence of the Fejér monotonicity of $\{z^k\}$ with respect to C given in Corollary 3.8, together with Lemma 4.19 and Lemma 4.20. □

In the next two results we prove the superlinear convergence of Algorithm 3 under the same assumptions.

Lemma 4.22 (Superlinear convergence of the distance for s-cCRM (Algorithm 3)) *Let $C_1, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and suppose that $C := \bigcap_{i=1}^m C_i \neq \emptyset$. Let sequence $\{z^k\}$ be generated by s-cCRM with the most violated constraint control sequence (function value version) as in (1.15), starting from some $z^0 \in \mathbb{R}^n$, and converging to a point $\bar{z} \in C$. Assume that the interior of C is nonempty and that the boundaries of C_i are differentiable manifolds in a neighborhood of $\bar{z}$ for each $i = 1, \ldots, m$. Also, assume that (EB2) holds at $\bar{z}$. Then, the scalar sequence $\{\mathrm{dist}(z^k, C)\}$ converges to zero superlinearly.*

Proof Assume that $\{z^k\}$ is infinite. Using the definition of (EB1) and the fact that $z^k \to \bar{z}$, we conclude that there exists a positive integer $\hat{k}$ such that

$$\mathrm{dist}(z^k, C) \leq \frac{\omega}{\epsilon} \mathrm{dist}(z^k, C_{\ell(k)}), \tag{4.30}$$

for all $k \geq \hat{k}$, where ω and ϵ are the constants in EB2. By (4.30), we get

$$\frac{\mathrm{dist}(z^{k+1}, C)}{\mathrm{dist}(\bar{Z}_{C_{r(k)}, C_{\ell(k)}}(z^k), C)} \leq \frac{\omega \, \mathrm{dist}(z^{k+1}, C_{\ell(k)})}{\epsilon \, \mathrm{dist}(\bar{Z}_{C_{r(k)}, C_{\ell(k)}}(z^k), C)}. \tag{4.31}$$

Combining (4.29) and (4.31), we get (4.24) which is sufficient to guarantee (4.23). Then the result holds. □

Theorem 4.23 (Superlinear convergence of s-cCRM (Algorithm 3)) *Let $C_1, \ldots, C_m \subset \mathbb{R}^n$ be nonempty closed convex sets, and suppose that $C := \bigcap_{i=1}^m C_i$ is nonempty. Let $\{z^k\}$ be generated by s-cCRM with the most violated control sequence (1.15) starting from some $z^0 \in \mathbb{R}^n$, and converging to a point $\bar{z} \in C$. Assume that the interior of C is nonempty, and that the boundary of C_i is a differentiable manifold in a neighborhood of $\bar{z}$ for each $i = 1, \ldots, m$. Also, assume that (EB1) holds at $\bar{z}$. Then, $\{z^k\}$ converges to $\bar{z}$ superlinearly.*

Proof The result is a direct consequence of the Fejér monotonicity of $\{z^k\}$ with respect to C given in Corollary 3.8, together with Lemma 4.20 and Lemma 4.22. □

5 Numerical experiments

In this section, we present the results of the computational experiments comparing s-cCRM with SePM (referred as SePM) and CRM-Prod (presented in (1.5) and denoted by CRMprod). For s-cCRM we consider Algorithm 1 (denoted by Alg1) and Algorithm 3 (designated as Alg3). For Alg1 we use the cyclic control sequence given in (1.12). In view of Lemma 4.12, we do not present results concerning Algorithm 2, since Algorithm 2 is somehow equivalent to Algorithm 3 in the presence of error bound; see (4.11).

We apply the aforementioned four methods to the problem of finding a point in the intersection of m ellipsoids, i.e.,

$$\text{find } x^* \in \bigcap_{i=1}^{m} \xi_i.$$

Here, each ellipsoid ξ_i is a set given by

$$\xi_i := \{x \in \mathbb{R}^n \mid f_i(x) \leq 0\}, \text{ for } i = 1, 2, \ldots, m,$$

with $f_i : \mathbb{R}^n \to \mathbb{R}$ defined as

$$f_i(x) = \langle x, A_i x \rangle + 2 \left\langle x, b^i \right\rangle - c_i.$$

We consider A_i a symmetric positive definite matrix, b^i a vector, and c_i a positive scalar, for each $i = 1, \ldots, m$.

To construct the ellipsoids we follow the steps of [13]. First, we form the ellipsoid ξ_1 by generating a matrix A_1 of in the form of $A_1 = \gamma \operatorname{Id} + B_1^\top B_1$ with $B_1 \in \mathbb{R}^{n \times n}$, $\gamma \in \mathbb{R}_{++}$. Matrix B_1 is sparse with sparsity density $p = 2n^{-1}$, and with components sampled from the standard normal distribution. Vector b^1 is sampled from the uniform distribution in $[0, 1]$, and we enforce $\langle b^1, A_1 b^1 \rangle < c_1$, which ensures that 0 belongs to ξ_1.

Then the remaining ellipsoids, $\xi_2, \ldots, \xi_m$, are constructed in the following form:

$$\xi_i = \{x \in \mathbb{R}^n \mid \left\langle x - x_c^i, (A_i^\top A_i)^{-1}(x - x_c^i) \right\rangle \leq 1\},$$

where A_i is a positive definite matrix, and $x_c^i \in \mathbb{R}^n$ is the center of ξ_i for $i = 2, 3, \ldots, m$. To form ξ_2, first randomly generate x_c^2 of ξ_2 outside ξ_1. Define $d_2 := \lambda(P_{\xi_1}(c_2) - c_2)$ as the norm of the longest principal semi-axis of ξ_2, where $\lambda > 1$ is a constant which can decide the intersection is big or small. For ensuring this, we form a diagonal matrix $\Lambda_2 = \operatorname{diag}(\|d_2\|, u)$ where $u \in \mathbb{R}^{n-1}$ is a vector whose components are positive and have values less than $\|d_2\|$, and orthogonal matrix Q_2 where the first row or column is $\frac{d_2}{\|d_2\|}$. Define $A_2 = Q_2 \Lambda_2 Q_2^\top$, and then the ellipsoid ξ_2 is complete.

Before forming the remaining ellipsoids, we need to find a fixed point that lies in the intersection of the m ellipsoids. Take point $p = x_c^2 + d_2 \in \xi_1 \cap \xi_2$, and we will guarantee $p \in \xi_i$ for each $i = 3, 4, \ldots, m$ in the following steps.

Choose an arbitrary point $x_c^i \in \mathbb{R}^n$ which doesn't belong to $\cup_{j=1}^{i-1} \xi_j$, define $d^i = \lambda(p - x_c^i)$ as the norm of the longest semi-axis of ξ_i, and generate A_i similarly as was done for ξ_2 for each $i = 3, \ldots, m$. Repeat this process until we get all m ellipsoids.

The computational experiments were performed on an Intel Xeon W-2133 3.60GHz with 32GB of RAM running Ubuntu 20.04 using `Julia v1.8` [49], and are available at https://github.com/lrsantos11/CRM-CFP. The following conditions were used:

Table 1 The mean $\pm$ std CPU time (in seconds) comparison per dimension and number of sets

n	m	Alg1	Alg3	SePM	CRMprod
20	5	0.0198 ± 0.0389	0.037 ± 0.117	0.0334 ± 0.0585	0.0350 ± 0.0576
20	10	0.0098 ± 0.0102	0.0094 ± 0.0092	0.0192 ± 0.0232	0.0335 ± 0.0350
20	20	0.0096 ± 0.0087	0.0135 ± 0.0111	0.0203 ± 0.0243	0.0596 ± 0.0699
50	5	0.0417 ± 0.0280	0.0498 ± 0.0588	0.0686 ± 0.0532	0.0704 ± 0.0506
50	10	0.0372 ± 0.0293	0.0422 ± 0.0412	0.0679 ± 0.0596	0.0785 ± 0.0759
50	20	0.0372 ± 0.0233	0.0411 ± 0.0234	0.0676 ± 0.0546	0.212 ± 0.192
100	5	0.384 ± 0.611	0.354 ± 0.612	0.637 ± 0.909	0.533 ± 0.951
100	10	0.360 ± 0.620	y0.360 ± 0.625	0.623 ± 0.930	0.777 ± 0.989
100	20	60.4 ± 268.6	76.1 ± 338.5	73.7 ± 326.6	173.8 ± 772.3

1. A random initial point $x^0 \in \mathbb{R}^n$ is sampled for the standard normal distribution, ensuring that $x^0 \notin \xi_i$, for all $i = 1, \ldots, m$; note that for CRMprod, the initial point is $(x^0, x^0, \ldots, x^0) \in \mathbb{R}^{nm}$;
2. We use the method described in [50, Alg. 6] to compute the projections onto the ellipsoids. The number of projections onto ellipsoids per iteration that each algorithm requires differs: Alg1 involves 4, Alg3 asks for 5, while both SePM and CRMprod demand m.
3. We compare total number of projections until achieve precision and not number of iterations, and also register CPU time (in seconds) for each algorithm.
4. A limit of 30000 total number of projections is enforced.
5. After each iteration, we calculate the *current error* given by

$$e_k = \sum_{i=1}^{m} \| P_{C_i}(x^k) - x^k \|,$$

and we establish the *stopping criterion* as

$$\max\{\|x^{k+1} - x^k\|, e_k\} \leq \varepsilon,$$

where $\varepsilon = 10^{-6}$.
6. For each pair (n, m), for $n \in \{20, 50, 100\}$ and $m \in \{5, 10, 20\}$ we repeat the experiment 20 times.

Tables 1 and 2 summarize the results, in which we exhibit the mean and the standard deviation of CPU running time (in seconds) and total number of projections, respectively, for each algorithm. We also sum up our numerical findings in Figs. 1 and 2, by means of the so-called performance profiles from [51]. *Performance profiles* allow one to benchmark different methods on a set of problems with respect to a performance measure (in our case, CPU Time and total number of projections). The vertical axis indicates the percentage of problems solved, while the horizontal axis indicates the corresponding factor of the performance index used by the best solver.

Table 2 The mean ± std of total projections per dimension and number of sets

n	m	Alg1	Alg3	SePM	CRMprod
20	5	51.0 ± 13.7	13.50 ± 4.01	32.00 ± 26.03	66.00 ± 53.18
20	10	100.0 ± 30.4	13.00 ± 4.41	64.00 ± 65.49	183.5 ± 224.0
20	20	196.0 ± 54.9	13.50 ± 4.01	121.0 ± 120.6	588.0 ± 877.0
50	5	51.0 ± 12.1	14.25 ± 4.38	34.00 ± 25.37	65.50 ± 49.34
50	10	102.0 ± 27.5	14.00 ± 4.47	66.00 ± 54.91	142.5 ± 186.4
50	20	208.0 ± 54.4	14.50 ± 4.26	142.0 ± 116.8	827 ± 1016
100	5	58.0 ± 14.4	15.25 ± 4.13	45.50 ± 26.10	66.00 ± 60.17
100	10	120.0 ± 34.3	15.50 ± 4.26	96.50 ± 64.42	255.5 ± 272.2
100	20	764.0 ± 2381.6	57.75 ± 188.86	683.0 ± 2176.3	3005.0 ± 10 336.0
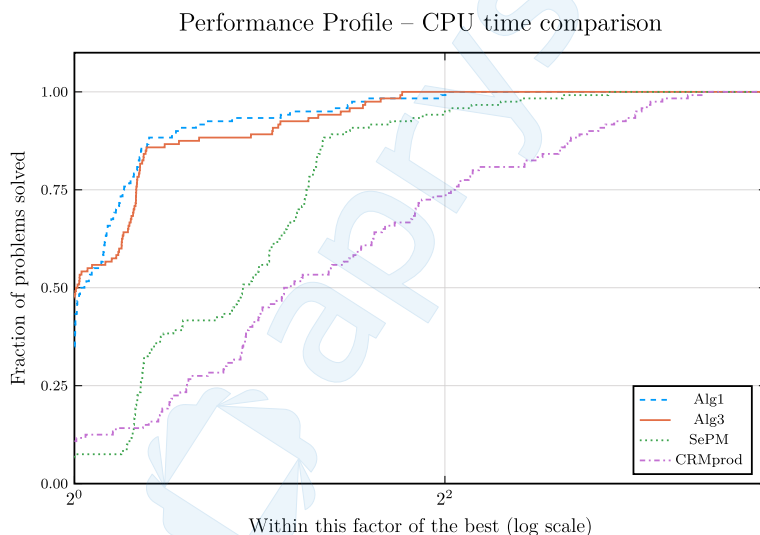
Fig. 1 Performance profile of the experiments considering CPU Time (in seconds)

We briefly comment on these results. Our numerical findings show that Alg1 and Alg3 are faster (in terms of CPU time) than their counterparts (see Table 1 and Fig. 1). Alg3 is the one with the less number of total projections to achieve the required tolerance, which is expected, as it uses only functional evaluation to determine the control sequence (see Table 2 and Fig. 2). We remark that Alg1 and Alg3 perform similarly in terms of CPU time, and when the dimension are higher, the difference between their performance with respect to SePM and CRMprod is more evident.
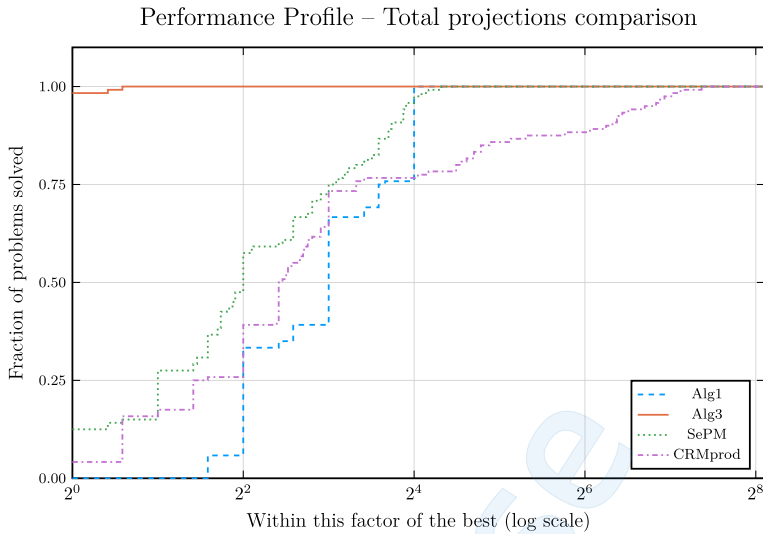
Fig. 2 Performance profile of the experiments considering total number of projections

6 Concluding remarks

In this paper we have presented the successive centralized circumcentered-reflection method (s-cCRM) extending cCRM for solving the multiset convex feasibility problem. Our theoretical analysis encompasses the global convergence of s-cCRM and, additionally under an error bound condition, we prove linear convergence of the method. Moreover, we have shown that the s-cCRM is superlinearly convergent under smoothness of the boundaries of the target sets. Furthermore, the numerical experiments illustrate the proposed version of s-cCRM have better performance than SePM and CRM-Prod. Extensions of novel circumcenter-based iterations by using more natural centralized procedures and possible applications to structured optimization problems are left for future research.

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

Ethical approval We certify that all authors are complying with the journal's ethical policies and that this manuscript has not been published or submitted simultaneously for publication elsewhere.

References

1. Combettes, P.L.: The convex feasibility problem in image recovery. In: Hawkes, P.W. (ed.) Advances in Imaging and Electron Physics, pp. 155–270. Elsevier, San Diego (1996). https://doi.org/10.1016/S1076-5670(08)70157-5
2. Hu, Y., Li, C., Yang, X.: On convergence rates of linearized proximal algorithms for convex composite optimization with applications. SIAM J. Optim. **26**(2), 1207–1235 (2016). https://doi.org/10.1137/140993090
3. Wang, J., Hu, Y., Li, C., Yao, J.-C.: Linear convergence of CQ algorithms and applications in gene regulatory network inference. Inverse Prob. **33**(5), 055017 (2017). https://doi.org/10.1088/1361-6420/aa6699
4. Bauschke, H.H., Borwein, J.M.: On projection algorithms for solving convex feasibility problems. SIAM Rev. **38**(3), 367–426 (1996). https://doi.org/10.1137/S0036144593251710
5. De Pierro, A.R., Iusem, A.N.: A simultaneous projections method for linear inequalities. Linear Algebra Appl. **64**, 243–253 (1985). https://doi.org/10.1016/0024-3795(85)90280-0
6. Behling, R., Bello-Cruz, Y., Santos, L.-R.: Circumcentering the Douglas–Rachford method. Numer. Algor. **78**(3), 759–776 (2018). https://doi.org/10.1007/s11075-017-0399-5. arXiv:1704.06737
7. Douglas, J., Rachford, H.H., Jr.: On the numerical solution of heat conduction problems in two and three space variables. Trans. Am. Math. Soc. **82**(2), 421–421 (1956). https://doi.org/10.1090/S0002-9947-1956-0084194-4
8. Bauschke, H.H., Bello-Cruz, J.-Y., Nghia, T.T.A., Phan, H.M., Wang, X.: The rate of linear convergence of the Douglas–Rachford algorithm for subspaces is the cosine of the Friedrichs angle. J. Approx. Theory **185**, 63–79 (2014). https://doi.org/10.1016/j.jat.2014.06.002
9. Bauschke, H.H., Borwein, J.M.: On the convergence of von Neumann's alternating projection algorithm for two sets. Set-Valued Anal. **1**(2), 185–212 (1993). https://doi.org/10.1007/BF01027691
10. Bauschke, H.H., Bello-Cruz, J.-Y., Nghia, T.T.A., Phan, H.M., Wang, X.: Optimal rates of linear convergence of relaxed alternating projections and generalized Douglas–Rachford methods for two subspaces. Numer. Algorithms **73**(1), 33–76 (2016). https://doi.org/10.1007/s11075-015-0085-4
11. Behling, R., Bello-Cruz, Y., Santos, L.-R.: On the linear convergence of the circumcentered-reflection method. Oper. Res. Lett. **46**(2), 159–162 (2018). https://doi.org/10.1016/j.orl.2017.11.018. arXiv:1711.08651
12. Behling, R., Bello-Cruz, Y., Santos, L.-R.: The block-wise circumcentered-reflection method. Comput. Optim. Appl. **76**(3), 675–699 (2020). https://doi.org/10.1007/s10589-019-00155-0. arXiv:1902.10866
13. Behling, R., Bello-Cruz, Y., Santos, L.-R.: On the Circumcentered-Reflection Method for the Convex Feasibility Problem. Numer. Algorithms **86**, 1475–1494 (2021). https://doi.org/10.1007/s11075-020-00941-6. arXiv:2001.01773
14. Behling, R., Bello-Cruz, Y., Iusem, A.N., Santos, L.-R.: On the centralization of the circumcentered-reflection method. Math. Programm. (2023). https://doi.org/10.1007/s10107-023-01978-w. arxiv:2111.07022
15. Behling, R., Bello-Cruz, Y., Lara-Urdaneta, H., Oviedo, H., Santos, L.-R.: Circumcentric directions of cones. Optim. Lett. **17**, 1069–1081 (2023). https://doi.org/10.1007/s11590-022-01923-4. arxiv:2112.08314
16. Araújo, G.H.M., Arefidamghani, R., Behling, R., Bello-Cruz, Y., Iusem, A., Santos, L.-R.: Circumcentering approximate reflections for solving the convex feasibility problem. Fixed Point Theory Algorithms Sci. Eng. **2022**(1), 30 (2022). https://doi.org/10.1186/s13663-021-00711-6. arXiv:2105.00497
17. Arefidamghani, R., Behling, R., Bello-Cruz, Y., Iusem, A.N., Santos, L.-R.: The circumcentered-reflection method achieves better rates than alternating projections. Comput. Optim. Appl. **79**(2), 507–530 (2021). https://doi.org/10.1007/s10589-021-00275-6. arXiv:2007.14466

18. Arefidamghani, R., Behling, R., Iusem, A.N., Santos, L.-R.: A circumcentered-reflection method for finding common fixed points of firmly nonexpansive operators. J. Appl. Numer. Optim. **(to appear)** (2023) arXiv:2203.02410
19. Bauschke, H.H., Ouyang, H., Wang, X.: On circumcenters of finite sets in Hilbert spaces. Linear Nonlinear Anal. **4**(2), 271–295 (2018). arXiv:1807.02093
20. Bauschke, H.H., Ouyang, H., Wang, X.: Circumcentered methods induced by isometries. Vietnam J. Math. **48**, 471–508 (2020). https://doi.org/10.1007/s10013-020-00417-z. arXiv:1908.11576
21. Bauschke, H.H., Ouyang, H., Wang, X.: Best approximation mappings in Hilbert spaces. Math. Program. (2021). https://doi.org/10.1007/s10107-021-01718-y. arXiv:2006.02644
22. Bauschke, H.H., Ouyang, H., Wang, X.: On circumcenter mappings induced by nonexpansive operators. Pure Appl. Funct. Anal. **6**(2), 257–288 (2021). arXiv:1811.11420
23. Bauschke, H.H., Ouyang, H., Wang, X.: On the linear convergence of circumcentered isometry methods. Numer. Algor. **87**, 263–297 (2021). https://doi.org/10.1007/s11075-020-00966-x. arXiv:1912.01063
24. Dizon, N.D., Hogan, J.A., Lindstrom, S.B.: Circumcentering reflection methods for nonconvex feasibility problems. Set-Valued Var. Anal (2022). https://doi.org/10.1007/s11228-021-00626-9
25. Dizon, N., Hogan, J., Lindstrom, S.: Circumcentered reflections method for wavelet feasibility problems. ANZIAMJ **62**, 98–111 (2022). https://doi.org/10.21914/anziamj.v62.16118. arXiv:2005.05687
26. Lindstrom, S.B.: Computable centering methods for spiraling algorithms and their duals, with motivations from the theory of Lyapunov functions. Comput. Optim. Appl. **83**(3), 999–1026 (2022). https://doi.org/10.1007/s10589-022-00413-8. arXiv:2001.10784
27. Ouyang, H., Wang, X.: Bregman circumcenters: basic theory. J. Optim. Theory Appl. **191**(1), 252–280 (2021). https://doi.org/10.1007/s10957-021-01937-5. arXiv:2104.03234
28. Ouyang, H.: Finite convergence of locally proper circumcentered methods. J. Convex Anal. **29**(3), 857–892 (2022). arXiv:2011.13512
29. Ouyang, H.: Bregman circumcenters: monotonicity and forward weak convergence. Optim. Lett. **17**, 121–141 (2023). https://doi.org/10.1007/s11590-022-01881-x. arxiv:2105.02308
30. Aragón Artacho, F.J., Campoy, R., Tam, M.K.: The Douglas–Rachford algorithm for convex and nonconvex feasibility problems. Math. Meth. Oper. Res. **91**, 201–240 (2020). https://doi.org/10.1007/s00186-019-00691-9
31. Pierra, G.: Decomposition through formalization in a product space. Math. Program. **28**(1), 96–115 (1984). https://doi.org/10.1007/BF02612715
32. McCormick, S.F.: The Methods of Kaczmarz and row orthogonalization for solving linear equations and least squares problems in Hilbert space. Indiana Univ. Math. J. **26**(6), 1137–1150 (1977)
33. Censor, Y.: Row-action methods for huge and sparse systems and their applications. SIAM Rev. **23**(4), 444–466 (1981). https://doi.org/10.1137/1023097
34. Martínez, J.M.: The projection method for solving nonlinear systems of equations under the "most violated constraint" control. Comput. Math. Appl. **11**(10), 987–993 (1985). https://doi.org/10.1016/0898-1221(85)90019-7
35. Borwein, J.M., Tam, M.K.: A cyclic Douglas–Rachford iteration scheme. J. Optim. Theory Appl. **160**(1), 1–29 (2014). https://doi.org/10.1007/s10957-013-0381-x
36. Borwein, J.M., Tam, M.K.: The cyclic Douglas–Rachford method for inconsistent feasibility problems. J. Nonlinear Convex Anal. **16**(4), 573–584 (2015). arXiv:1310.2195
37. Bauschke, H.H., Combettes, P.L.: Convex Analysis and Monotone Operator Theory in Hilbert Spaces, 2nd edn. CMS Books in Mathematics. Springer, Cham (2017). https://doi.org/10.1007/978-3-319-48311-5
38. Borwein, J.M., Vanderwerff, J.D.: Convex Functions: Constructions, Characterizations and Counterexamples. Encyclopedia of mathematics and its applications, vol. 109. Cambridge University Press, Cambridge(2010)
39. Wegge, L.L.: Mean value theorem for convex functions. J. Math. Econ. **1**(2), 207–208 (1974). https://doi.org/10.1016/0304-4068(74)90011-1
40. Rockafellar, R.T.: Convex Analysis, 10th edn. Princeton Landmarks in Mathematics and Physics. Princeton Univ. Press, Princeton (1997)
41. Qi, L.: Complete closedness of maximal monotone operators. Math. OR **8**(2), 315–317 (1983). https://doi.org/10.1287/moor.8.2.315

42. Beck, A., Teboulle, M.: Convergence rate analysis and error bounds for projection algorithms in convex feasibility problems. Optim. Methods Softw. **18**(4), 377–394 (2003). https://doi.org/10.1080/10556780310001604977
43. Bauschke, H.H., Borwein, J.M., Li, W.: Strong conical hull intersection property, bounded linear regularity, Jameson's property (G), and error bounds in convex optimization. Math. Program. **86**(1), 135–160 (1999). https://doi.org/10.1007/s101070050083
44. Behling, R., Bello-Cruz, Y., Santos, L.-R.: Infeasibility and error bound imply finite convergence of alternating projections. SIAM J. Optim. **31**(4), 2863–2892 (2021). https://doi.org/10.1137/20M1358669. arXiv:2008.03354
45. Liu, T., Lourenço, B.F.: Convergence analysis under consistent error bounds. Found. Comput. Math. (2022). https://doi.org/10.1007/s10208-022-09586-4
46. Kruger, A.Y., Luke, D.R., Thao, N.H.: Set regularities and feasibility problems. Math. Program. **168**(1), 279–311 (2018). https://doi.org/10.1007/s10107-016-1039-x
47. Ortega, J.M., Rheinboldt, W.C.: Iterative Solution of nonlinear equations in several variables, 1st edn. Classics in Applied Mathematics. SIAM, Philadelphia (2000)
48. Munkres, J.R.: Analysis on Manifolds, 1st edn. CRC Press, Boca Raton (1997). https://doi.org/10.1201/9780429494147
49. Bezanson, J., Edelman, A., Karpinski, S., Shah, V.B.: Julia: a fresh approach to numerical computing. SIAM Rev. **59**(1), 65–98 (2017). https://doi.org/10.1137/141000671
50. Jia, Z., Cai, X., Han, D.: Comparison of several fast algorithms for projection onto an ellipsoid. J. Comput. Appl. Math. **319**, 320–337 (2017). https://doi.org/10.1016/j.cam.2017.01.008
51. Dolan, E.D., Moré, J.J.: Benchmarking optimization software with performance profiles. Math. Program. **91**(2), 201–213 (2002). https://doi.org/10.1007/s101070100263