Finance and Stochastics (2024) 28:553–613
https://doi.org/10.1007/s00780-024-00532-6

Quadratic expansions in optimal investment with respect to perturbations of the semimartingale model

Oleksii Mostovyi[1] · **Mihai Sîrbu**[2]

Received: 10 February 2022 / Accepted: 21 April 2023 / Published online: 27 March 2024

Abstract

We study the response of the optimal investment problem to small changes of the stock price dynamics. Starting with a multidimensional semimartingale setting of an incomplete market, we suppose that the perturbation process is also a general semimartingale. We obtain second-order expansions of the value functions, first-order corrections to the optimisers, and provide the adjustments to the optimal control that match the objective function up to the second order. We also give a characterisation in terms of the risk-tolerance wealth process, if it exists, by reducing the problem to the Kunita–Watanabe decomposition under a change of measure and numéraire. Finally, we illustrate the results by examples of base models that allow closed-form solutions, but where this structure is lost under perturbations of the model where our results allow an approximate solution.

Keywords Asymptotic analysis · Semimartingale · Incomplete market · Duality theory

Mathematics Subject Classification 93E20 · 91G10 · 91G15 · 60H30

JEL Classification C61 · G11

Oleksii Mostovyi has been supported by the National Science Foundation under grant No. DMS-1848339 (2019-2024). Mihai Sîrbu has been supported by the National Science Foundation under grant No. DMS-1908903 (2019-2023). Any opinions, findings and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation. The authors would like to thank Martin Schweizer, whose questions, comments and suggestions led to a substantial improvement of the paper.

✉ O. Mostovyi
oleksii.mostovyi@uconn.edu

M. Sîrbu
sirbu@math.utexas.edu

1 University of Connecticut, Department of Mathematics, Storrs, CT 06269, USA

2 The University of Texas at Austin, Department of Mathematics, Austin, TX 78712, USA

1 Introduction

For the problem of optimal investment from expected terminal wealth, we study the second-order dependence of value functions and the first-order dependence of optimisers with respect to small perturbations of the stock price dynamics. More precisely, if the (return) of the stock price dynamics depends on a small parameter ε as

$$R^{\varepsilon} = R + \varepsilon \widetilde{R},$$

we study the following questions:

1) How does the value function (indirect utility) change up to the first order in ε?

2) What are the derivatives of the optimal wealth process in ε, and do they exist?

3) What are the corrections to the optimal strategy needed to match the indirect utility up to the second order?

By considering both R (the base model return) and $\widetilde{R}$ (the perturbation) to be general semimartingales, our model allows in a unified piece of analysis

(a) perturbations of both the finite-variation and the local martingale parts, including new sources of randomness in $\widetilde{R}$;

(b) both the base model R and the perturbation processes $\widetilde{R}$ to have jumps;

(c) general multidimensional models.

These questions appear important because while utility maximisation theory is mathematically well understood, the exact statistics of the stock price model is mostly a theoretical assumption. In other words, as an investor does not know exactly the model, understanding how optimal investment decisions change in the neighbourhood of a base model is desirable.

To the best of our knowledge, the existing answers in similar studies are partial, and they are obtained under particular forms of both R and $\widetilde{R}$ by considering only one-dimensional models with *continuous* paths *and* allowing only perturbations of the market price of risk (or, closely related, the volatility); see for example Mostovyi and Sîrbu [37].

Mathematically and even on the heuristic level, establishing asymptotic analysis results for processes with jumps and general perturbations is more difficult than in continuous settings, which is itself nontrivial compared to only perturbing the market price of risk or volatility.

Our results include a second-order expansion of the value function, first-order corrections to the optimisers, and explicit formulas for the corrections of the optimal strategies. The latter are particularly difficult to obtain. We also work with a general utility function, as in Horst et al. [17], Kramkov and Sîrbu [31], Santacroce and Trivellato [43]. For this reason, we have to study the dependence on the initial wealth x and the model parameter ε jointly by analysing the two-dimensional value function $u(x, \varepsilon)$ and the corresponding optimal investment strategies. We show that the response of the value function is linear up to the first order in ε, whereas up to the second order, it is nonlinear.

Four auxiliary quadratic stochastic control problems govern the second-order expansion of $u(x, \varepsilon)$. These quadratic optimisation problems appear naturally. Indeed, as the indirect utility depends jointly on (x, ε), the dual value function $v(y, \varepsilon)$ depends on (y, ε), where the Lagrange multiplier y has the natural meaning of the dual state in the model ε. The four quadratic optimisation problems describe the local second-order dependence of the primal value function on x and ε and of the dual value function on y and ε, respectively. This joint structure allows second-order asymptotic expansion formulas for both value functions. Moreover, we give a characterisation of the asymptotic expansions in terms of the *risk-tolerance wealth process*. In the case when it exists, the problem can be reduced to the *Kunita–Watanabe decomposition*.

One of the applications of our results is to study perturbations of models which allow explicit solutions and typically lose this property under perturbations, which can result for example from statistical inference/calibration procedures. For models which allow closed-form solutions, we refer to Kallsen [22], Zariphopoulou [46], Goll and Kallsen [10], Hu et al. [18], Liu [33], Guasoni and Robertson [13], Horst et al. [17], Kallsen and Muhle-Karbe [24], Kramkov and Sîrbu [31], Santacroce and Trivellato [43] and Bank and Körber [3]. A special structure is also exhibited by asymptotically complete models; see Robertson [42] and Robertson et al. [1].

Mathematically, to establish results, we have to deviate from the existing papers where primal–dual expansions are used. We first develop elements of the calculus of numéraire changes for both primal and dual problems to understand how the dual and primal domains change with respect to (x, ε) and (y, ε), respectively. This dependence is described by some key auxiliary processes that are represented in terms of the semimartingale characteristics of the driving processes. Based on the explicit dependence of the dual and primal domains on the parameters, we construct candidate nearly optimal processes explicitly establishing one-sided bounds for both primal and dual problems. This allows building the corrections to the optimal strategies that match the value function up to the second order, relying on auxiliary estimates and reformulations. A primal–dual approach for asymptotic analysis in mathematical finance has been introduced in Henderson [14], Henderson and Hobson [15] and Kallsen [23].

To obtain the four quadratic minimisation problems that govern the second-order corrections in tractable forms, we need to make some additional reformulations. These are not needed for the analysis of perturbations of the market price of risk or volatility in continuous-process settings which allowed a special structure. For our general class of base and perturbed models, these technical steps are at the core of our analysis, and they make the domains of the four quadratic minimisation problems to be sets of martingales under an appropriate change of measure and numéraire. The quadratic minimisation problems are related to the ones in Gouriéroux et al. [12], Laurent and Pham [32], Pham et al. [38], Kramkov and Sîrbu [30, 31], Černý and Kallsen [5], Czichowsky and Schweizer [6], Jeanblanc et al. [21], Monoyios [34], Mostovyi [36], Mostovyi and Sîrbu [37].

The change of both measure and numéraire is needed to make the natural domains of the four quadratic optimisation problems into sets of martingales which exhibit an orthogonal and complementary structure in the sense below. Indeed, a change of measure is needed to obtain martingales. However, as one could attempt to only use such a change of measure, a simultaneous change of numéraire is also needed because technically, the dual optimal elements for the base model may fail to be martingales, which complicates the analysis for this approach.

As mentioned, our approach relies on an increase of dimensionality of the value functions $u(x,\varepsilon)$ and $v(y,\varepsilon)$ that is necessary to handle the general utility, similarly to Mostovyi and Sîrbu [37]. The key process which governs the derivative in ε of the dual optimiser is characterised via *implicit differentiation* lemmas, which were pivotal in handling the multidimensional stock-price case (in contrast to the settings in [37]). The integrability condition on the perturbation processes is related to *entropic submartingales* introduced in Barrieu and El Karoui [4] in continuous settings.

The asymptotic analysis of stochastic control problems under perturbations of the model is a challenging problem. The majority of such existing results for optimal investment are obtained under the *assumption of continuity of the stock price*, and they only include particular changes of the driving controlled process, such as *perturbations of the finite-variation part*. For example, the results in Herrmann et al. [16], Veraguas and Silva [2] and [37] neither include models where R or $\widetilde{R}$ allow jumps nor can they handle the situation when the martingale part of $\widetilde{R}$ has a component orthogonal to the martingale part of R. Also, the existing results study *one-dimensional* models; see [16] and [37].

We point out that the semimartingale property of R^ε is necessary for the absence of arbitrage; see Delbaen and Schachermayer [7] for NFLVR, Karatzas and Kardaras [26] for the case of NUPBR and Karatzas and Kardaras [27, Chap. 2] for an overview. In other words, one cannot consider a more general perturbed model.

The remainder of the paper is organised as follows. In Sect. 2, we introduce the model, and in Sect. 3, we provide some elements of the calculus of numéraire changes which is central to the analysis. Section 4 contains the expansion theorems, whose proofs are given in Sect. 5. We build a connection of the expansion results to the Kunita–Watanabe decomposition in Sect. 6. (One can also describe the correction terms via utility-based prices. This characterisation is omitted for brevity of exposition.) In Sect. 7, we provide an example of perturbations of the Black–Scholes model and a Lévy process-based model, and we discuss connections to other models allowing explicit solutions. Finally, in the Appendix, we give a technical characterisation of the approximating sets for the primal and dual domains which is important for our analysis.

2 The model

Consider a complete stochastic basis $(\Omega,\mathcal{F},\mathbb{F}=(\mathcal{F}_t)_{t\in[0,T]},\mathbb{P})$, where $T\in(0,\infty)$ is the time horizon, $\mathbb{F}$ satisfies the usual conditions and $\mathcal{F}_0$ is trivial. We suppose that there are $d+1$ traded securities, d stocks and a bank account with zero interest rate. For the base model, we suppose that the *returns* of d stocks satisfy the structure condition from Föllmer and Schweizer [9], that is, they are given by a special semimartingale R whose dynamics is

$$R=M+\int d\langle M\rangle\lambda,\qquad R_0=0,\tag{2.1}$$

where M is a locally $\mathbb{P}$-square-integrable local $\mathbb{P}$-martingale and λ a predictable process such that

$$\int_0^T \lambda_s^\top d\langle M\rangle_s \lambda_s < \infty \qquad \mathbb{P}\text{-a.s.} \tag{2.2}$$

Note that the absolute continuity of the finite-variation part with respect to $\langle M\rangle$ in the semimartingale decomposition for R is known as the *structure condition*, a notion going back to Schweizer [44] and formally introduced in Schweizer [45].

2.1 Parametrisation of perturbations

The family of perturbed models has returns of the form

$$R^\varepsilon = R + \varepsilon\widetilde{R}, \qquad R_0^\varepsilon = 0, \varepsilon \in (-\varepsilon_0, \varepsilon_0), \tag{2.3}$$

where $\widetilde{R}$ is another semimartingale of the form (2.5) below and $\varepsilon_0 > 0$ is a constant.

2.1.1 Preliminary discussion

Adopting the notations in Jacod and Shiryaev [19, Chap. I] for stochastic integration, we start by assuming that

$$\widetilde{R} = \phi \cdot M + M^\perp + FV, \tag{2.4}$$

for some predictable matrix-valued process ϕ and some local martingale $M^\perp$ orthogonal to M and an adapted process FV of finite variation. By orthogonality in a multidimensional setting, we mean that every component of M is orthogonal to every component of $M^\perp$. The form of $\widetilde{R}$ in (2.4) is consistent with the GKW decomposition.

In view of no-arbitrage considerations, it is natural to suppose that

$$FV = \int d\langle M\rangle\zeta + \int d\langle M^\perp\rangle\zeta^\perp$$

for some processes ζ and $\zeta^\perp$ which are predictable and integrable with respect to $\langle M\rangle$ and $\langle M^\perp\rangle$, respectively. Therefore, we can rewrite (2.3) as

$$R^\varepsilon = R + \varepsilon\Big(\phi \cdot M + M^\perp + \int d\langle M\rangle\zeta + \int d\langle M^\perp\rangle\zeta^\perp\Big).$$

Note that for a bounded ϕ and every ε sufficiently close to 0, on the set

$$\{0 = d\langle (I+\varepsilon\phi)\cdot M + \varepsilon M^\perp\rangle\} = \{0 = (I+\varepsilon\phi)d\langle M\rangle(I+\varepsilon\phi)^\top + \varepsilon^2 d\langle M^\perp\rangle\},$$

where I is the identity matrix, we have $d\,\langle M\rangle = d\langle M^\perp\rangle = 0$ so that

$$d\Big(\int d\langle M\rangle(\lambda + \varepsilon\zeta) + \varepsilon\int d\langle M^\perp\rangle\zeta^\perp\Big) = 0.$$

In view of the absolute continuity of $\langle M^{\perp}\rangle$ with respect to $\langle M\rangle$ imposed in Assumption 4.5, which is key for the expansions below, we can further suppose that $\zeta^{\perp}\equiv 0$ without loss of generality. Therefore the *structure condition* from Föllmer and Schweizer [9] holds for R^{ε}, i.e., the finite-variation part of R^{ε} is absolutely continuous with respect to the predictable quadratic covariation of its martingale part.

2.1.2 The exact form of $\widetilde{R}$

To summarise, in view of the discussion above and Assumption 4.5 below, we suppose that

$$\widetilde{R} = \phi\cdot M + M^{\perp} + \int d\langle M\rangle\zeta, \qquad \widetilde{R}^{\varepsilon}_0 = 0,\, \varepsilon\in(-\varepsilon_0,\varepsilon_0), \tag{2.5}$$

where $M^{\perp}$ is a locally $\mathbb{P}$-square-integrable local $\mathbb{P}$-martingale orthogonal to M and ζ is a predictable process such that for some constant $C'>0$, we have

$$|\zeta_t|\le C'|\lambda_t|, \qquad t\in[0,T],\, \mathbb{P}\text{-a.s.} \tag{2.6}$$

Note that (2.2) and (2.6) imply that

$$\int_0^T \zeta_s^{\top} d\langle M\rangle_s\zeta_s<\infty \qquad \mathbb{P}\text{-a.s.}$$

2.2 Primal problem

The family of admissible wealth processes is defined, for $(x,\varepsilon)\in(0,\infty)\times(-\varepsilon_0,\varepsilon_0)$ (where $\varepsilon_0>0$ is specified in (2.3)), as

$$\mathcal{X}(x,\varepsilon) := \{X\ge 0 : X = x+\tilde{H}\cdot R^{\varepsilon},\ \tilde{H} \text{ is } R^{\varepsilon}\text{-integrable}\}.$$

Assumption 2.1 The utility function U on $(0,\infty)$ is strictly increasing, strictly concave, twice continuously differentiable, and its relative risk aversion is bounded away from 0 and ∞, that is, there exist positive constants c_1 and c_2 such that

$$c_1\le A(x) := -\frac{xU''(x)}{U'(x)}\le c_2, \qquad x>0. \tag{2.7}$$

The function A is the relative risk aversion. The family of indirect utility functions is given by

$$u(x,\varepsilon) := \sup_{X\in\mathcal{X}(x,\varepsilon)} \mathbb{E}[U(X_T)], \qquad (x,\varepsilon)\in(0,\infty)\times(-\varepsilon_0,\varepsilon_0). \tag{2.8}$$

We use the convention

$$\mathbb{E}[U(X_T)] := -\infty \qquad \text{if } \mathbb{E}[U^{-}(X_T)]=\infty,$$

where U^{-} denotes the negative part of U.

2.3 Dual problem

For every $(y, \varepsilon) \in (0, \infty) \times (-\varepsilon_0, \varepsilon_0)$, we first specify the dual feasible set as

$$\begin{aligned}\mathcal{Y}(y, \varepsilon) := \{Y \geq 0 : Y \text{ is a supermartingale such that } Y_0 = y \\ \text{and } XY = (X_t Y_t)_{t\in[0,T]} \text{ is a supermartingale} \\ \text{for every } X \in \mathcal{X}(1, \varepsilon)\}.\end{aligned}$$

Note that for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, this is the usual formulation of the dual domain as in Kramkov and Schachermayer [28, 29].

Next, we define the convex conjugate of U as

$$V(y) := \sup_{x>0} \big(U(x) - xy\big), \qquad y > 0,$$

and we note that it follows from (2.7) that

$$B(y) := -\frac{yV''(y)}{V'(y)}, \qquad y > 0,$$

is well defined and satisfies

$$\frac{1}{c_2} \leq B(y) \leq \frac{1}{c_1}, \qquad y > 0.$$

We also have

$$A(x) = \frac{1}{B(U'(x))}, \qquad x > 0,$$

so that $B(U'(x))$ is the relative risk-tolerance of U computed at x, and

$$V''\big(U'(x)\big) = -\frac{1}{U''(x)}, \qquad x > 0.$$

The dual value function is defined as

$$v(y, \varepsilon) := \inf_{Y\in\mathcal{Y}(y,\varepsilon)} \mathbb{E}[V(Y_T)], \qquad (y, \varepsilon) \in (0, \infty) \times (-\varepsilon_0, \varepsilon_0). \tag{2.9}$$

Here we use the convention

$$\mathbb{E}[V(Y_T)] := \infty \qquad \text{if } \mathbb{E}[V^+(Y_T)] = \infty,$$

where V^+ is the positive part of V.

2.4 Assumptions on the base model

As our main results provide asymptotics around $\varepsilon = 0$ and some $x > 0$, we need to impose conditions on the base model that allow expansion results as well as some conditions on perturbations. In order for the base model to be well defined, we assume that

$$\mathcal{Y}(1, 0) \neq \emptyset. \tag{2.10}$$

We also need the following condition.

Assumption 2.2 The dual value function for the base model is finite-valued, that is,

$$v(y,0) < \infty, \qquad y > 0.$$

Condition (2.10) is necessary for the absence of arbitrage in the sense of NUPBR introduced in Karatzas and Kardaras [26], whereas Assumption 2.2 is necessary for the standard conclusions of utility maximisation theory as in Kramkov and Schachermayer [29]. We also refer to the abstract theorems in Mostovyi [35] for the case when the condition of existence of local martingale measures fails. Under (2.10) and Assumption 2.2, and with the utility function satisfying Assumption 2.1, we deduce the existence and uniqueness of the optimisers to (2.8) and (2.9) for $\varepsilon = 0$, denoted by $\widehat{X}(x,0)$ and $\widehat{Y}(y,0)$, respectively, for every positive x and y, and we get continuous differentiability of $u(\cdot,0)$ and $v(\cdot,0)$ as well as the duality relation

$$\widehat{Y}_T\big(u_x(x,0),0\big) = U'\big(\widehat{X}_T(x,0)\big), \qquad x > 0.$$

We note that $u_x(\cdot,0)$ is well defined by the abstract theorems in [35]. Also, the product $\widehat{X}(x,0)\widehat{Y}(u_x(x,0),0)$ is a martingale, which allows defining a new probability measure $\mathbb{R}(x,0)$ via

$$\frac{d\mathbb{R}(x,0)}{d\mathbb{P}} = \frac{\widehat{X}_T(x,0)\widehat{Y}_T(u_x(x,0),0))}{xu_x(x,0)}.$$

As usual $\mathbb{R}(x,0)$ plays an important role in the second-order expansions of the primal and dual value functions; see Kramkov and Sîrbu [30, 31], Mostovyi and Sîrbu [37] and Mostovyi [36], among others. Below, we fix $x > 0$ and set $y = u_x(x,0)$, which is well defined in the present setting. We also write

$$\widehat{X}(x,0) = x\mathcal{E}(\pi \cdot R) \tag{2.11}$$

for some R-integrable process π. As computations get technically involved, we denote for brevity $\widehat{X}(x,0)$ by X and $\widehat{Y}(u_x(x,0),0) = \widehat{Y}(y,0)$ by Y below. Likewise, we denote $\mathbb{R}(x,0)$ by $\mathbb{R}$.

Assumption 2.3 We assume that

$$\widehat{Y}(y,0) = Y = y\mathcal{E}(-\lambda \cdot M + \beta \cdot M^{\perp} + L) \tag{2.12}$$

for some predictable process β with $\int_0^T \beta_s^{\top} d\langle M^{\perp}\rangle_s \beta_s < \infty$ $\mathbb{P}$-a.s. and for some $L \in \mathcal{H}^2_{\mathrm{loc}}(\mathbb{P})$ which is orthogonal to every component of both M and $M^{\perp}$.

We also set

$$H := -\lambda \cdot M + \beta \cdot M^{\perp} + L \tag{2.13}$$

so that $Y = y\mathcal{E}(H)$. We remark that the form of Y in (2.12) is fairly natural. It is consistent with most models in the literature from the introduction, whereas building a model where such a form of Y fails to hold requires a special effort.

3 Elements of the change of numéraire calculus

Recall from Sect. 2.4 the primal and dual optimisers $X = \widehat{X}(x,0) = x\mathcal{E}(\pi \cdot R)$ and $Y = \widehat{Y}(y,0) = y\mathcal{E}(H)$; see (2.11) and (2.12). The key role in the second-order analysis is played by the sets $\mathcal{M}^2$ and $\mathcal{N}^2$ of complementary and orthogonal martingales under $\mathbb{R}$. These sets are introduced in Kramkov and Sîrbu [30] in a two-step procedure. First, while the original assets are $(1, \mathcal{E}(R^1), \dots, \mathcal{E}(R^d))$, we change numéraire to

$$S^X := \left(\frac{x}{X}, \frac{x\mathcal{E}(R^1)}{X}, \dots, \frac{x\mathcal{E}(R^d)}{X}\right), \tag{3.1}$$

and second, we define

$$\mathcal{M}^2 := \{\widetilde{M} \in \mathcal{H}_0^2(\mathbb{R}) : \widetilde{M} = \bar{H} \cdot S^X \text{ for some } \bar{H}\},$$

where $\mathcal{H}_0^2(\mathbb{R})$ is the set of square-integrable martingales under $\mathbb{R}$ with initial value 0. The complement of $\mathcal{M}^2$ in $\mathcal{H}_0^2(\mathbb{R})$ is denoted by $\mathcal{N}^2$, that is,

$$\mathcal{N}^2 := \{N \in \mathcal{H}_0^2(\mathbb{R}) : \widetilde{M}N \text{ is an } \mathbb{R}\text{-martingale for every } \widetilde{M} \in \mathcal{M}^2\}.$$

Following [30], let us denote by $\mathcal{M}^\infty$ the family of uniformly bounded wealth processes under the numéraire X with initial value 0, that is, the family of semimartingales M such that for some $\delta = \delta(M) > 0$, we have

$$X(1 + \delta M) \in \mathcal{X}(x,0) \qquad \text{and} \qquad X(1 - \delta M) \in \mathcal{X}(x,0).$$

By $\mathcal{N}^\infty$, we denote the family of semimartingales N such that for some $\delta = \delta(N) > 0$, we have

$$Y(1 + \delta N) \in \mathcal{Y}(y,0) \qquad \text{and} \qquad Y(1 - \delta N) \in \mathcal{Y}(y,0).$$

The Appendix provides a characterisation of these sets in the present setting.

We need a representation of R in terms of its *predictable characteristics*. We follow Jacod and Shiryaev [19, Sect. II.2] and fix the truncation function $h(x) = x1_{\{|x|\le 1\}}$ and denote by R^c the continuous martingale part of R, by $\widetilde{B}$ the predictable finite-variation part of R corresponding to the truncation function h, by

μ the jump measure of R, that is, the random counting measure on $[0, T] \times \mathbb{R}^d$ defined by

$$\mu([0,t] \times E) := \sum_{0<s\leq t} 1_{\{E\setminus\{0\}\}}(\Delta R_s), \qquad t \in [0,T], E \subseteq \mathbb{R}^d,$$

where 1_E is the indicator function of a set E and ν is the predictable compensator of μ, that is, a predictable random measure on $[0, T] \times \mathbb{R}^d$ such that in particular, $(x1_{\{|x|\leq 1\}}) * (\mu - \nu)$ is a purely discontinuous local martingale. Defining the quadratic covariation process $C := [R^c, R^c]$ of R^c, we call $(\widetilde{B}, C, \eta)$ the triplet of predictable characteristics of R associated with the truncation function h. By [19, Theorem II.2.34], the semimartingale R can be represented in terms of (B, C, η) as

$$R = R^c + \widetilde{B} + (x1_{\{|x|\leq 1\}}) * (\mu - \nu) + (x1_{\{|x|>1\}}) * \mu.$$

Define the predictable scalar-valued locally integrable increasing process $\widetilde{A}$ as

$$\widetilde{A} := \sum_{i\leq d} \mathrm{Var}(\widetilde{B}^i) + \sum_{i\leq d} C^{i,i} + \big(\min(1, |x|^2)\big) * \nu,$$

where $\mathrm{Var}(\widetilde{B}^i)$ denotes the variation process of $\widetilde{B}^i$, $i = 1, \ldots, d$. Then $\widetilde{B}$, C and ν are absolutely continuous with respect to $\widetilde{A}$, and therefore we have

$$\widetilde{B} = b \cdot \widetilde{A}, \qquad C = c \cdot \widetilde{A}, \qquad \nu = \eta \cdot \widetilde{A},$$

where b is a predictable $\mathbb{R}^d$-valued process, c is a predictable process with values in the set of positive semidefinite matrices, and ν is a predictable Lévy-measure-valued process. Further, since one can write $X = x\mathcal{E}(\pi \cdot R)$ for some R-integrable process π, following Karatzas and Kardaras [26, Lemma 3.4], let us set

$$R^\pi = R - (c\pi) \cdot \widetilde{A} - \left(\frac{\pi^\top x}{1 + \pi^\top x} x\right) * \mu.$$

One can see that R^π is a semimartingale. It is shown in Mostovyi [36, Lemma 4.1] that every element of $\mathcal{M}^\infty$ can be represented as a stochastic integral with respect to R^π by relying on the formula

$$\frac{\mathcal{E}(\alpha \cdot R)}{\mathcal{E}(\pi \cdot R)} = \mathcal{E}\big((\alpha - \pi) \cdot R^\pi\big), \tag{3.2}$$

which holds in particular for every predictable and R-integrable process α such that the left-hand side of (3.2) is bounded and positive. Equivalently, the process R^π can be described as follows. For any semimartingale K, let us consider the transformation

$$K^\pi = K - [K^c, \pi \cdot R^c] - \sum_{0<s\leq \cdot} \Delta K_s \frac{\pi_s \Delta R_s}{1 + \pi_s \Delta R_s}.$$

Particularly important cases correspond to $K = \pi' \cdot R$ for some predictable and R-integrable process π', in which case we have

$$\pi' \cdot R^{\pi} = \pi' \cdot R - [\pi' \cdot R^c, \pi \cdot R^c] - \sum_{0<s\leq \cdot} \pi'_s \Delta R_s \frac{\pi_s \Delta R_s}{1+\pi_s \Delta R_s},$$

and to $K = \widetilde{\pi} \cdot \widetilde{R}$, where $\widetilde{\pi}$ is predictable and $\widetilde{R}$-integrable, so that

$$\widetilde{\pi} \cdot \widetilde{R}^{\pi} = \widetilde{\pi} \cdot \widetilde{R} - [\widetilde{\pi} \cdot \widetilde{R}^c, \pi \cdot R^c] - \sum_{0<s\leq \cdot} \widetilde{\pi}_s \Delta \widetilde{R}_s \frac{\pi_s \Delta R_s}{1+\pi_s \Delta R_s}. \tag{3.3}$$

While the dual problem does not have a numéraire, a very similar transformation is needed; we call it the *dual numéraire change*. It can be described as follows. For a semimartingale K, with H being defined in (2.13) and where $\frac{Y}{y} = \mathcal{E}(H)$ is the dual numéraire, we set

$$K^H := K - [K^c, H^c] - \sum_{0<s\leq \cdot} \Delta K_s \frac{\Delta H_s}{1+\Delta H_s},$$

which is also a semimartingale.

Remark 3.1 If $\mathcal{E}(H+K)$ is non-vanishing, K^H can be thought of as excess return of K^H under the (dual) numéraire $\mathcal{E}(H)$, that is,

$$\mathcal{E}(K^H) = \frac{\mathcal{E}(K+H)}{\mathcal{E}(H)}.$$

The transformations $\cdot^H$ and $\cdot^\pi$ are central in our analysis. If K is a continuous process of finite variation, we have $K^H = K^\pi = K$. We provide additional characterisations below, in particular in Lemma 5.13.

4 Expansion theorems

4.1 Assumptions on the perturbations

We begin by introducing the remaining assumptions needed for the second-order asymptotics. With $\kappa := \sum_{i=1}^d \langle M^i \rangle$, we have $\langle M \rangle = \bar{A} \cdot \kappa$ for some process $\bar{A}$.

Assumption 4.1 We suppose that $\bar{A}_t$ is invertible for every $t \in [0, T]$, $\mathbb{P}$-a.s.

Assumption 4.2 The processes R^π, $\widetilde{R}^\pi$, M^H and $M^{\perp,H}$ are sigma-bounded.

Remark 4.3 Requiring R^π to be sigma-bounded is exactly the sigma-boundedness condition from Kramkov and Sîrbu [30, Assumption 2], which is needed to ensure that $u(\cdot, 0)$ allows a second-order expansion in x.

Remark 4.4 A sufficient condition for Assumption 4.2, which ensures that every semimartingale on the probability space is sigma-bounded, see [30, Theorem 3], can be formulated as follows. There is a d-dimensional local martingale $\bar{M}$ such that any bounded, purely discontinuous martingale N is a stochastic integral with respect to $\bar{M}$, that is,

$$N = N_0 + \tilde{H} \cdot \bar{M}$$

for some predictable and $\bar{M}$-integrable $\tilde{H}$. We recall that this condition was introduced in [30, Assumption 4]. We note that it is invariant with respect to an equivalent choice of reference probability measure; see [30, Remark 3].

We impose a structure-type condition on the orthogonal local martingale $M^{\perp}$ of the perturbed model.

Assumption 4.5 We suppose that

$\langle M^{\perp} \rangle = \nu \cdot \langle M \rangle$ for some bounded and predictable matrix-valued process ν,

and that M (and therefore also $M^{\perp}$) is quasi-left-continuous.

Remark 4.6 In Mostovyi and Sîrbu [37] and the volatility uncertainty part in Herrmann et al. [16], Assumption 4.5 holds due to the special parametrisations of perturbations. We also note that Assumption 4.5 *typically* holds in the standard case of Itô semimartingales, where both $\langle M \rangle$ and $\langle M^{\perp} \rangle$ are absolutely continuous with respect to time.

Assumption 4.7 We suppose that ϕ is bounded, where ϕ is introduced in (2.5).

A central role in our analysis is played by the process

$$g_t := \bar{A}_t^{-1} \nu_t \bar{A}_t \beta_t - \bar{A}_t^{-1} \phi_t \bar{A}_t \lambda_t + \zeta_t, \qquad t \in [0, T], \tag{4.1}$$

We characterise g in Lemma 5.2 below; see also Remark 5.3.

Remark 4.8 If $d = 1$, that is, if there is only one risky asset, g reduces to

$$g = \nu\beta - \phi\lambda + \zeta.$$

For the perturbations, we need to impose the following condition, which is needed to ensure that candidate nearly optimal elements of the primal and dual domains have enough integrability. It is consistent with the integrability condition in [37].

Assumption 4.9 We suppose that $\int_0^T \pi_s^{\top} d\, \langle M \rangle_s\, \pi_s < \infty$ so that under the assumptions above, $\pi \cdot \widetilde{R}^{\pi}$ and $g \cdot M^H$ are well defined. Further, we suppose that there exists $c > 0$ such that

$$\mathbb{E}_{\mathbb{R}}\big[\exp\big(c(|\pi \cdot \widetilde{R}^{\pi}_T| + [\pi \cdot \widetilde{R}^{\pi}]_T + [g \cdot M^H]_T)\big)\big] < \infty.$$

The following assumption is needed to ensure the admissibility of the candidate nearly optimal elements of the primal and dual domains.

Assumption 4.10 The jumps of $\pi \cdot \widetilde{R}^{\pi}$ and of $g \cdot M^H$ are bounded.

We state below the first-order expansion in Theorem 4.11 under the same assumptions as the second-order expansion in Theorem 4.15. This is for brevity, even though some assumptions in Theorem 4.11 can be relaxed, such as the sigma-boundedness in Assumption 4.2.

Theorem 4.11 *Let $x > 0$ be fixed and suppose M and $M^{\perp}$ are in $\mathcal{H}^2_{\mathrm{loc}}(\mathbb{P})$ and that* (2.2), (2.6) *and* (2.10) *and Assumptions* 2.1–2.3, 4.1, 4.2, 4.5, 4.7, 4.9 *and* 4.10 *hold true. Set $y = u_x(x, 0)$, which is well defined by the abstract theorems in Mostovyi* [35]. *Then there exists $\bar{\varepsilon}_0 > 0$ such that for every $\varepsilon \in (-\bar{\varepsilon}_0, \bar{\varepsilon}_0)$, we have*

$$u(x, \varepsilon) \in \mathbb{R}, \quad x > 0, \qquad \textit{and} \qquad v(y, \varepsilon) \in \mathbb{R}, \quad y > 0.$$

Further, u and v are jointly differentiable (*and thus continuous*) *at $(x, 0)$ and $(y, 0)$, respectively. We also have*

$$\nabla u(x, 0) = \begin{pmatrix} y \\ u_{\varepsilon}(x, 0) \end{pmatrix}, \qquad \nabla v(y, 0) = \begin{pmatrix} -x \\ v_{\varepsilon}(y, 0) \end{pmatrix}, \tag{4.2}$$

where

$$u_{\varepsilon}(x, 0) = xy\mathbb{E}_{\mathbb{R}}[\pi \cdot \widetilde{R}^{\pi}_T] = v_{\varepsilon}(y, 0) = xy\mathbb{E}_{\mathbb{R}}[g \cdot M^H_T] \tag{4.3}$$

and g is defined in (4.1).

Remark 4.12 This remark provides a characterisation of $\pi \cdot \tilde{R}^{\pi}_T$ as an excess return. Similarly to Remark 3.1, we can further characterise $\pi \cdot \widetilde{R}^{\pi}_T$ as the stochastic logarithm of $\frac{\mathcal{E}(\pi \cdot (\widetilde{R}+R))}{\mathcal{E}(\pi \cdot R)}$, provided that the latter exists. If $\Delta(\pi \cdot (\varepsilon \widetilde{R} + R)) \neq -1$ for some constant ε, we have

$$\varepsilon\pi \cdot \widetilde{R}^{\pi}_T = \mathcal{L}og\left(\frac{\mathcal{E}(\pi \cdot (\varepsilon \widetilde{R} + R))}{\mathcal{E}(\pi \cdot R)}\right)_T.$$

This expression provides the return of $\mathcal{E}(\pi \cdot (\varepsilon \widetilde{R} + R))$ under the numeraire $\mathcal{E}(\pi \cdot R)$, where $\mathcal{L}og$ denotes the stochastic logarithm. If $\Delta(\pi \cdot (\widetilde{R} + R)) \neq -1$, (4.3) reads

$$u_{\varepsilon}(x, 0) = xy\mathbb{E}_{\mathbb{R}}\left[\mathcal{L}og\left(\frac{\mathcal{E}(\pi \cdot (\widetilde{R} + R))}{\mathcal{E}(\pi \cdot R)}\right)_T\right].$$

Remark 4.13 Theorem 4.11 provides the first-order in ε (thus linear) response of the value functions to small perturbations of the stock price. Thus in particular, perturbations of the drift, volatility and the orthogonal martingale part could be considered separately, and the proof of Theorem 4.11 could be implemented by matching the associated terms in the primal and dual representations for the first-order derivatives given by (4.3) (under sufficient integrability). The proof of Theorem 4.15 is significantly more involved as it provides a second-order expansion, which corresponds to the quadratic in ε (thus nonlinear) response of the value functions to perturbations of the stock price.

4.2 Minimisation problems for u_{xx} and $u_{\varepsilon\varepsilon}$

Having specified the structure of $\mathcal{M}^2$, we are ready to state the auxiliary minimisation problems that govern the second-order expansion terms for u. We define

$$a(x,x) := \inf_{\widetilde{M}\in\mathcal{M}^2} \mathbb{E}_{\mathbb{R}}[A(X_T)(1+\widetilde{M}_T)^2], \tag{4.4}$$

$$\widetilde{a}(\varepsilon,\varepsilon) := \inf_{\widetilde{M}\in\mathcal{M}^2} \mathbb{E}_{\mathbb{R}}[A(X_T)(\widetilde{M}_T + x\pi\cdot\widetilde{R}^{\pi}_T)^2 - 2\widetilde{M}_T(xg\cdot M^H_T)], \tag{4.5}$$

$$\begin{aligned} T_1 := -\mathbb{E}_{\mathbb{R}}\bigg[&\bigg(\int_0^T \pi_s^\top d\langle M\rangle_s g_s\bigg)^2 \\ &+ 2\bigg(\int_0^T \pi_s^\top d\langle M\rangle_s g_s\bigg) \\ &\times \bigg(\pi\cdot\Big(\phi\cdot R + M^\perp - \int d\langle M^\perp\rangle\beta\Big)^\pi_T\bigg)\bigg], \end{aligned} \tag{4.6}$$

$$a(\varepsilon,\varepsilon) := \widetilde{a}(\varepsilon,\varepsilon) + x^2 T_1. \tag{4.7}$$

Using standard techniques of the calculus of variations, one can show the existence and uniqueness of the optimisers to (4.4) and (4.5), which we denote by M^x and M^ε, respectively. Let us now set

$$a(x,\varepsilon) := \mathbb{E}_{\mathbb{R}}[A(X_T)(M^\varepsilon_T + x\pi\cdot\widetilde{R}^\pi_T)(M^x_T+1) - (xg\cdot M^H_T)(M^x_T+1)]. \tag{4.8}$$

4.3 Minimisation problems for v_{yy} and $v_{\varepsilon\varepsilon}$

We recall g from (4.1) and set

$$b(y,y) := \inf_{N\in\mathcal{N}^2} \mathbb{E}_{\mathbb{R}}[B(Y_T)(1+N_T)^2], \tag{4.9}$$

$$\widetilde{b}(\varepsilon,\varepsilon) := \inf_{N\in\mathcal{N}^2} \mathbb{E}_{\mathbb{R}}[B(Y_T)(N_T - yg\cdot M^H_T)^2 + 2N_T(y\pi\cdot\widetilde{R}^\pi_T)], \tag{4.10}$$

$$\begin{aligned} T_2 := \mathbb{E}_{\mathbb{R}}\bigg[&\bigg(\int_0^T \pi_s^\top d\langle M\rangle_s g_s\bigg)^2 - 2\bigg(\int_0^T \pi_s^\top d\langle M\rangle_s g_s\bigg)(g\cdot M^H_T) \\ &- 2\int_0^T \pi_s^\top \phi_s d\langle M\rangle_s g_s\bigg], \end{aligned} \tag{4.11}$$

$$b(\varepsilon,\varepsilon) := \widetilde{b}(\varepsilon,\varepsilon) + y^2 T_2. \tag{4.12}$$

With N^y and N^ε denoting the unique solutions to (4.9) and (4.10), respectively, we define

$$b(y,\varepsilon) := \mathbb{E}_{\mathbb{R}}[B(Y_T)(N_T^\varepsilon - yg \cdot M_T^H)(N_T^y + 1) + (y\pi \cdot \widetilde{R}_T^\pi)(N_T^y + 1)]. \quad (4.13)$$

We remark that the minimisation problems (4.4) and (4.9) have been introduced in Kramkov and Sîrbu [30], whereas the minimisation problems (4.5) and (4.10) that govern the corrections in ε are new to the best of our knowledge.

4.4 The joint structure of four auxiliary value functions and their optimisers

The following theorem establishes relations between the auxiliary value functions and between the optimisers to auxiliary minimisation problems.

Theorem 4.14 *Fix $x > 0$ and set $y = u_x(x,0)$. Under the conditions of Theorem 4.11 and with M^x, M^ε, N^y and N^ε denoting the solutions to (4.4), (4.5), (4.9) and (4.10), respectively, we have*

$$\begin{pmatrix} a(x,x) & 0 \\ a(x,\varepsilon) & -\frac{x}{y} \end{pmatrix} \begin{pmatrix} b(y,y) & 0 \\ b(y,\varepsilon) & -\frac{y}{x} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (4.14)$$

and

$$\frac{y}{x}a(\varepsilon,\varepsilon) + \frac{x}{y}b(\varepsilon,\varepsilon) = a(x,\varepsilon)b(y,\varepsilon). \quad (4.15)$$

Further, we have

$$A(X_T)\begin{pmatrix} 1 + M_T^x \\ x\pi \cdot \widetilde{R}_T^\pi + M_T^\varepsilon \end{pmatrix} = \begin{pmatrix} a(x,x) & 0 \\ a(x,\varepsilon) & -\frac{x}{y} \end{pmatrix} \begin{pmatrix} 1 + N_T^y \\ -yg \cdot M_T^H + N_T^\varepsilon \end{pmatrix} \quad (4.16)$$

or, equivalently,

$$B(Y_T)\begin{pmatrix} 1 + N_T^y \\ -yg \cdot M_T^H + N_T^\varepsilon, \end{pmatrix} = \begin{pmatrix} b(y,y) & 0 \\ b(y,\varepsilon) & -\frac{y}{x} \end{pmatrix} \begin{pmatrix} 1 + M_T^x \\ x\pi \cdot \widetilde{R}_T^\pi + M_T^\varepsilon \end{pmatrix}. \quad (4.17)$$

4.5 Second-order expansions of the value functions

The following theorem establishes second-order expansions of the value functions.

Theorem 4.15 *Fix $x > 0$ and set $y = u_x(x,0)$. Assume the conditions of Theorem 4.11 and recall (4.2). Then with*

$$H_u(x,0) := -\frac{y}{x}\begin{pmatrix} a(x,x) & a(x,\varepsilon) \\ a(x,\varepsilon) & a(\varepsilon,\varepsilon) \end{pmatrix}$$

and $a(x,x)$, $a(x,\varepsilon)$, $a(\varepsilon,\varepsilon)$ given by (4.4), (4.8) and (4.7), respectively, we have

$$\begin{aligned} u(x+\Delta x,\varepsilon) &= u(x,0) + (\Delta x \quad \varepsilon)\nabla u(x,0) \\ &\quad + \frac{1}{2}(\Delta x \quad \varepsilon)H_u(x,0)\begin{pmatrix} \Delta x \\ \varepsilon \end{pmatrix} + o(\Delta x^2 + \varepsilon^2), \end{aligned}$$

where Δx^2 is a shorthand for $(\Delta x)^2$. Likewise, with

$$H_v(y,0) := \frac{x}{y}\begin{pmatrix} b(y,y) & b(y,\varepsilon) \\ b(y,\varepsilon) & b(\varepsilon,\varepsilon)\end{pmatrix}$$

and $b(y,y)$, $b(y,\varepsilon)$, $b(\varepsilon,\varepsilon)$ given by (4.9), (4.13) and (4.12), respectively, we have

$$\begin{aligned} v(y+\Delta y,\varepsilon) &= v(y,0) + (\Delta y \quad \varepsilon)\nabla v(y,0) \\ &\quad + \frac{1}{2}(\Delta y \quad \varepsilon) H_v(y,0)\begin{pmatrix}\Delta y \\ \varepsilon\end{pmatrix} + o(\Delta y^2 + \varepsilon^2). \end{aligned}$$

4.6 Derivatives of the optimisers

The following theorem provides derivatives of the optimisers.

Theorem 4.16 *Fix $x > 0$ and set $y = u_x(x,0)$. Assume the conditions of Theorem 4.11. Then we have, with convergence in $\mathbb{P}$-probability, that*

$$\begin{aligned} 0 = \lim_{|\Delta x|+|\varepsilon|\to 0} &\frac{1}{|\Delta x|+|\varepsilon|} \\ &\times \Bigg| \widehat{X}_T(x+\Delta x,\varepsilon) \\ &\quad - \frac{\widehat{X}_T(x,0)}{x}\left(x + \Delta x(1+M^x_T) + \varepsilon(M^\varepsilon_T + x\pi\cdot \widetilde{R}^\pi_T)\right)\Bigg| \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} 0 = \lim_{|\Delta y|+|\varepsilon|\to 0} &\frac{1}{|\Delta y|+|\varepsilon|} \\ &\times \Bigg| \widehat{Y}_T(y+\Delta y,\varepsilon) \\ &\quad - \frac{\widehat{Y}_T(y,0)}{y}\left(y + \Delta y(1+N^y_T) + \varepsilon(N^\varepsilon_T - yg\cdot M^H_T)\right)\Bigg|. \end{aligned} \tag{4.19}$$

4.7 Corrections to the optimal strategies

With $M^x \in \mathcal{M}^2$ and $M^\varepsilon \in \mathcal{M}^2$ denoting the solutions to (4.4) and (4.7), respectively, we can approximate these optimisers by bounded processes in $\mathcal{M}^\infty$ such that

$$\lim_{n\to\infty} \bar{M}^{x,n}_T = M^x_T \quad \text{and} \quad \lim_{n\to\infty} \bar{M}^{\varepsilon,n}_T = M^\varepsilon_T \qquad \mathbb{P}\text{-a.s.}$$

We also refer to Lemma 5.10 below, where a construction of this type is made explicitly. Without loss of generality, we may suppose that $\bar{M}^{x,n}$ is bounded by n, $n \in \mathbb{N}$, and then we can further localise $\bar{M}^{x,n}$ by taking

$$T^n_k = \inf\{t \geq 0 : [\bar{M}^{x,n}]_t \geq k\}, \qquad k \in \mathbb{N}, n \in \mathbb{N}.$$

Then in particular, $[\bar{M}^{x,n}]_{t\wedge T_k^n} \leq k + 4n^2$. Thus if we set

$$\widetilde{M}_t^{x,n} := \bar{M}_{t\wedge T_n^n}^{x,n}, \qquad t \in [0,T], n \in \mathbb{N},$$

then $\widetilde{M}^{x,n}$ is bounded by n, its quadratic variation is bounded by $n + 4n^2$ and its jumps are bounded by $2n$. Further, by this construction, we have

$$\lim_{n\to\infty} \widetilde{M}_T^{x,n} = M_T^x \qquad \mathbb{P}\text{-a.s.}$$

We can construct a similar approximating sequence for M^ε, which we denote by $\widetilde{M}^{\varepsilon,n}$, $n \in \mathbb{N}$. Now Mostovyi [36, Lemma 4.1] implies the existence of predictable R^π-integrable processes $\pi^{x,n}$ and $\pi^{\varepsilon,n}$ such that

$$\pi^{x,n} \cdot R^\pi = \frac{\widetilde{M}^{x,n}}{x}, \quad \pi^{\varepsilon,n} \cdot R^\pi = \frac{\widetilde{M}^{\varepsilon,n}}{x}, \qquad n \in \mathbb{N}.$$

Remark 4.17 In the above representations, the dimensions match in the sense that, on the one hand, the elements of $\mathcal{M}^\infty$ are stochastic integrals with respect to the $(d+1)$-dimensional process S^X defined in (3.1). On the other hand, the elements of $\mathcal{M}^\infty$ can also be represented as stochastic integrals with respect to the d-dimensional process R^π; see [36, Lemma 4.1].

With these preliminaries, define the family $(\widetilde{X}^{\Delta x,\varepsilon,n})_{(\Delta x,\varepsilon,n)\in(-x,\infty)\times(-\varepsilon_0,\varepsilon_0)\times\mathbb{N}}$ of semimartingales by

$$\widetilde{X}^{\Delta x,\varepsilon,n} := (x+\Delta x)\mathcal{E}\left((\pi + \Delta x \pi^{x,n} + \varepsilon\pi^{\varepsilon,n})\cdot R^\varepsilon\right). \tag{4.20}$$

Theorem 4.18 *Fix $x > 0$. Under the conditions of Theorem 4.11, we have*:

1) *For every $n \in \mathbb{N}$, there exists $\varepsilon = \varepsilon(n) > 0$ such that*

$$\widetilde{X}^{\Delta x,\varepsilon,n} \in \mathcal{X}(x+\Delta x,\varepsilon), \qquad (\Delta x,\varepsilon) \in B_{\varepsilon(n)}(0,0),$$

where $B_{\varepsilon(n)}(0,0)$ denotes the ball of radius $\varepsilon(n)$ centered at $(0,0)$.

2) *There exists a function $n : (-x,\infty)\times(-\varepsilon_0,\varepsilon_0) \to \mathbb{N}$, $(\Delta x,\varepsilon) \mapsto n(\Delta x,\varepsilon)$ such that*

$$\mathbb{E}[U(\widetilde{X}_T^{\Delta x,\varepsilon,n(\Delta x,\varepsilon)})] = u(x+\Delta x,\varepsilon) - o(\Delta x^2 + \varepsilon^2).$$

3) *With $n = n(\Delta x,\varepsilon)$, the process $\widetilde{X}^{\Delta x,\varepsilon,n}$ invests in the corresponding traded assets the proportions*

$$\pi + \Delta x\pi^{x,n} + \varepsilon\pi^{\varepsilon,n}.$$

Remark 4.19 We note that the results of Theorem 4.18 are consistent with Mostovyi and Sîrbu [37, Theorem 3.1], where

$$X^{\Delta x,\Delta,\varepsilon} = (x+\Delta x)\mathcal{E}\Big(\big(\pi + \Delta x\gamma^{0,\varepsilon} + \varepsilon(\nu + \gamma^{1,\varepsilon})\big)\cdot R^\varepsilon\Big). \tag{4.20}$$

If we consider perturbations as in [37] and a continuous one-dimensional stock (also as in [37]), the difference between (4.20) and (4.19) is due to the difference in the notations. In particular, we can obtain that $\pi^{\varepsilon,n} = \nu + \gamma^{1,\varepsilon_n}$ (modulo a slightly different localisation procedure).

5 Proofs of the expansion theorems

5.1 Preliminary results

We begin with the following result, where we establish characterisations needed in the proofs of Lemmas 5.6, 5.8 and 5.21 below.

Lemma 5.1 *Consider a local martingale $\widetilde{M}$ with $\widetilde{M}_0 = 0$, with jumps bounded by a constant A', i.e., $|\Delta\widetilde{M}| \leq A'$, and whose quadratic variation has some exponential moments, i.e., there exists $c > 0$ such that*

$$\mathbb{E}[e^{c[\widetilde{M}]_T}] < \infty. \tag{5.1}$$

Then $\widetilde{M}$ is a martingale, and for $c' = \min(\frac{1}{4A'}, \frac{\sqrt{c}}{2}) > 0$, we have

$$\mathbb{E}[e^{c'|\widetilde{M}_T|}] < \infty.$$

Proof First observe that the choice of c' allows ensuring that the jumps of $2c'\widetilde{M}$ are bounded by $\frac{1}{2}$, and thus we have both

$$\mathcal{E}(2c'\widetilde{M}) > 0 \qquad \mathbb{P}\text{-a.s.} \tag{5.2}$$

and

$$\sum_{0<s\leq T} |\log(1 + 2c'\Delta\widetilde{M}_s) - 2c'\Delta\widetilde{M}_s| \leq \sum_{0<s\leq T} (2c'\Delta\widetilde{M}_s)^2 \qquad \mathbb{P}\text{-a.s.} \tag{5.3}$$

Let us observe that

$$\begin{aligned}\mathbb{E}[e^{c'\widetilde{M}_T}] = \mathbb{E}\Big[&e^{c'\widetilde{M}_T - [c'\widetilde{M}^c]_T + \frac{1}{2}\sum_{0<s\leq T}(\log(1+2c'\Delta\widetilde{M}_s) - 2c'\Delta\widetilde{M}_s)} \\ &\times e^{[c'\widetilde{M}^c]_T - \frac{1}{2}\sum_{0<s\leq T}(\log(1+2c'\Delta\widetilde{M}_s) - 2c'\Delta\widetilde{M}_s)}\Big].\end{aligned}$$

Applying the Cauchy–Schwarz inequality, we get

$$\begin{aligned}\mathbb{E}[e^{c'\widetilde{M}_T}] \leq\ & \mathbb{E}\Big[e^{2c'\widetilde{M}_T - \frac{1}{2}[2c'\widetilde{M}^c]_T + \sum_{0<s\leq T}(\log(1+2c'\Delta\widetilde{M}_s) - 2c'\Delta\widetilde{M}_s)}\Big]^{\frac{1}{2}} \\ &\times \mathbb{E}\Big[e^{2[c'\widetilde{M}^c]_T - \sum_{0<s\leq T}(\log(1+2c'\Delta\widetilde{M}_s) - 2c'\Delta\widetilde{M}_s)}\Big]^{\frac{1}{2}}.\end{aligned} \tag{5.4}$$

We observe that the first factor on the right-hand side of (5.4) is bounded by 1, as

$$e^{2c'\widetilde{M}_T - \frac{1}{2}[2c'\widetilde{M}^c]_T + \sum_{0<s\leq T}(\log(1+2c'\Delta\widetilde{M}_s) - 2c'\Delta\widetilde{M}_s)} = \mathcal{E}(2c'\widetilde{M})_T,$$

which in view of (5.2) is the terminal value of a nonnegative local martingale, thus supermartingale, starting from 1. To bound the second factor in (5.4), we use (5.3) to deduce that

$$\begin{aligned}\mathbb{E}\Big[e^{2[c'\widetilde{M}^c]_T - \sum_{0<s\leq T}(\log(1+2c'\Delta\widetilde{M}_s) - 2c'\Delta\widetilde{M}_s)}\Big] &\leq \mathbb{E}\Big[e^{2[c'\widetilde{M}^c]_T + \sum_{0<s\leq T}(2c'\Delta\widetilde{M}_s)^2}\Big] \\ &\leq \mathbb{E}[e^{4(c')^2[\widetilde{M}]_T}] \\ &\leq \mathbb{E}[e^{c[\widetilde{M}]_T}] < \infty.\end{aligned}$$

We deduce that in (5.4), we have

$$\mathbb{E}[e^{c'\widetilde{M}_T}] \leq \mathbb{E}[e^{c[\widetilde{M}]_T}]^{\frac{1}{2}} < \infty.$$

Similarly, we can obtain

$$\mathbb{E}[e^{-c'\widetilde{M}_T}] < \infty.$$

We conclude that

$$\mathbb{E}[e^{c'|\widetilde{M}_T|}] \leq \mathbb{E}[e^{c'\widetilde{M}_T}] + \mathbb{E}[e^{-c'\widetilde{M}_T}] < \infty.$$

Finally, from (5.1) via Protter [40, Corollary II.3], we deduce that $\widetilde{M}$ is a true martingale. □

5.1.1 Implicit differentiation

The following Lemmas 5.2 and 5.4 are crucial for handling the multidimensionality of the stock price process in the context of the proofs of the results from Sect. 4. The following lemma provides the first-order implicit differentiation formulas.

Lemma 5.2 *Fix $x > 0$, set $y = u_x(x, 0)$ and assume the conditions of Theorem 4.11. Then there exists $\widetilde{\varepsilon} > 0$ such that for every $\varepsilon \in (-\widetilde{\varepsilon}, \widetilde{\varepsilon})$, the families of vector-valued processes λ^ε and matrix-valued processes G^ε that are given (implicitly) via*

$$\int d\langle M\rangle(\lambda + \varepsilon\zeta) = \int (I + \varepsilon\phi) d\,\langle M\rangle\,\lambda^\varepsilon,$$

$$\int (I + \varepsilon\phi) d\,\langle M\rangle\,(G^\varepsilon)^\top = \varepsilon \int \nu d\,\langle M\rangle \tag{5.5}$$

are well defined. Further, for every predictable process a such that

$$\int a^\top d\langle M\rangle\zeta, \qquad \int a^\top \phi d\,\langle M\rangle\,\lambda, \qquad \int a^\top \nu d\,\langle M\rangle\,\beta$$

are well defined and finite-valued $\mathbb{P}$-a.s., we have

$$\int a^\top d\,\langle M\rangle\,(\lambda^0)' = \int a^\top d\langle M\rangle\zeta - \int a^\top\phi d\,\langle M\rangle\,\lambda,$$

$$\int a^\top d\,\langle M\rangle\,(G^0)'^\top\beta = \int a^\top\nu d\,\langle M\rangle\,\beta \qquad \mathbb{P}\text{-}a.s., \tag{5.6}$$

where the (implicit) derivatives (with respect to ε) above are given by

$$(\lambda_t^0)' = -(\bar{A}_t)^{-1}\phi_t\bar{A}_t\lambda_t + \zeta_t, \qquad (G_t^0)'^\top = \bar{A}_t^{-1}\nu_t\bar{A}_t, \qquad t\in[0,T], \tag{5.7}$$

and β is the process from (2.12). Further, we have

$$\int \pi^\top d\,\langle M\rangle\,\big((\lambda^0)' + (G^0)'^\top\beta\big) = \int \pi^\top d\,\langle M\rangle\,g. \tag{5.8}$$

Remark 5.3 In view of Assumption 4.1 and the assertions of Lemma 5.2, in particular (5.8), the process g can be characterised in the (convenient) form

$$g_t = (G_t^0)'^\top\beta_t + (\lambda_t^0)', \qquad t\in[0,T],$$

where $(G_t^0)'$ and $(\lambda_t^0)'$ are given by (5.7).

Proof of Lemma 5.2 First we observe that at $\varepsilon = 0$,

$$\lambda^0 \equiv \lambda \qquad \text{and} \qquad G^0 \equiv 0$$

satisfy (5.5), which shows that for square matrices with small off-diagonal elements, the eigenvalues are close to the diagonal elements. From the boundedness of ϕ and the Gershgorin theorem (see Golub and Van Loan [11, Theorem 7.2.1]), we deduce that there exists $\widetilde{\varepsilon} > 0$ such that for every $\varepsilon \in (-\widetilde{\varepsilon},\widetilde{\varepsilon})$, $I+\varepsilon\phi$ is invertible and $(I+\varepsilon\phi)^{-1}$ is bounded. Using Assumption 4.1 or the vector and matrix-valued versions of the Radon–Nikodým theorem, see e.g. Robertson and Rosenberg [41, Theorem 5.1], ensures that for every $\varepsilon \in (-\widetilde{\varepsilon},\widetilde{\varepsilon})$, the vector-valued process λ^ε and the matrix-valued process G^ε given through (5.5) are well defined $\mathbb{P}$-a.s. Now combining (5.5) and Assumption 4.1, we deduce (5.7).

Next, by invertibility of $I+\varepsilon\phi$, one can rewrite (5.5) for every $\varepsilon\in(-\widetilde{\varepsilon},\widetilde{\varepsilon})$ as

$$\int (I+\varepsilon\phi)^{-1}d\langle M\rangle(\lambda+\varepsilon\zeta) = \int d\,\langle M\rangle\,\lambda^\varepsilon,$$

$$\int d\,\langle M\rangle\,(G^\varepsilon)^\top = \varepsilon\int (I+\varepsilon\phi)^{-1}\nu d\,\langle M\rangle\,. \tag{5.9}$$

Let ε_n, $n\in\mathbb{N}$, be a sequence converging to 0. Considering the quotients of the form

$$\int a^\top d\,\langle M\rangle\,\frac{\lambda^{\varepsilon_n}-\lambda}{\varepsilon_n}, \qquad \int a^\top d\,\langle M\rangle\,\frac{(G^{\varepsilon_n})^\top}{\varepsilon_n}$$

and using (5.9), we obtain (5.6). Equation (5.8) can be obtained similarly, where we note that $\mathbb{P}$-a.s. finiteness of the right-hand side of (5.8) follows from Assumption 4.9 and Lemma 5.6. $\square$

The proof of the following result, which provides the second-order implicit differentiation formulas, is similar to that of Lemma 5.2; it is skipped for brevity.

Lemma 5.4 *Fix* $x > 0$, *set* $y = u_x(x, 0)$ *and assume the conditions of Theorem* 4.11. *Then for every predictable process* a *such that*

$$\int a^\top \phi d\langle M\rangle \zeta, \qquad \int a^\top \phi\phi d\left\langle M\right\rangle \lambda, \qquad \int a^\top \phi\nu d\left\langle M\right\rangle \beta$$

are well defined and finite-valued $\mathbb{P}$-*a.s., and with* λ^ε *and* G^ε *given* (*implicitly*) *via* (5.5), *we have*

$$\int a^\top d\left\langle M\right\rangle (\lambda^0)'' = -2\int a^\top \phi d\langle M\rangle \zeta + 2\int a^\top \phi\phi d\left\langle M\right\rangle \lambda,$$

$$\int a^\top d\left\langle M\right\rangle ({G^0})''^\top \beta = -2\int a^\top \phi\nu d\left\langle M\right\rangle \beta \qquad \mathbb{P}\text{-}a.s.$$

and thus

$$\int a^\top d\left\langle M\right\rangle \big(({G^0})''^\top \beta + (\lambda^0)''\big) = \int a^\top \phi d\left\langle M\right\rangle g.$$

The following result gives a representation used in the proof of Lemma 5.20 below.

Lemma 5.5 *Under the conditions of Lemma* 5.2, *we have*

$$\begin{aligned}
&\int_0^T \pi_s^\top d\left\langle M\right\rangle_s \big(({G_s^0})''^\top \beta_s + (\lambda_s^0)''\big)\\
&= -2\int_0^T \pi_s^\top \phi_s(\nu_s d\left\langle M\right\rangle_s \beta_s + d\left\langle M\right\rangle_s \zeta_s - \phi_s d\left\langle M\right\rangle_s \lambda_s)\\
&= -2\int_0^T \pi_s^\top \phi_s d\left\langle M\right\rangle_s g_s.
\end{aligned} \tag{5.10}$$

As a consequence, we have

$$\mathbb{E}_{\mathbb{R}}\big[\big((\lambda^0)'' + ({G^0})''^\top \beta\big) \cdot M_T^H\big] = -2\mathbb{E}_{\mathbb{R}}\left[\int_0^T \pi_s^\top \phi_s d\left\langle M\right\rangle_s g_s\right]. \tag{5.11}$$

Proof First, (5.10) can be proved similary to Lemma 5.2. For (5.11), we observe that under the conditions of Lemma 5.2, one can show the integrability of the random variable $((\lambda^0)'' + ({G^0})''^\top \beta) \cdot M_T^H$ and that $((\lambda^0)'' + ({G^0})''^\top \beta) \cdot (M - \int d\left\langle M\right\rangle \pi)^H$ is an $\mathbb{R}$-martingale. Then (5.11) follows from completing $((\lambda^0)'' + ({G^0})''^\top \beta) \cdot M^H$ to an $\mathbb{R}$-martingale. □

The following result provides characterisations of the key terms appearing in Theorem 4.11, see (4.3), and Theorem 4.16, see (4.18) and (4.19). These terms also govern the auxiliary value functions; see (4.5), (4.8), (4.10) and (4.13).

Lemma 5.6 *Fix $x > 0$ and set $y = u_x(x, 0)$. Under the conditions of Theorem 4.11, we have*

$$\pi \cdot \widetilde{R}^{\pi} = \pi \cdot \left(\phi \cdot R + M^{\perp} - \int d\langle M^{\perp}\rangle\beta\right)^{\pi} + \int \pi^{\top} d\,\langle M\rangle\, g, \tag{5.12}$$

where each term is well defined, $\pi \cdot (\phi \cdot R + M^{\perp} - \int d\langle M^{\perp}\rangle\beta)^{\pi}$ is an $\mathbb{R}$-martingale and there exists $\bar{c}'' > 0$ such that

$$\begin{aligned}\mathbb{E}_{\mathbb{R}}\Bigg[\exp\Bigg(\bar{c}''\Bigg(\Big|\int_0^T g_s^{\top} d\,\langle M\rangle_s\, \pi_s\Big| + |g \cdot M_T^H| \\ + \Big|\pi \cdot \left(\phi \cdot R + M^{\perp} - \int d\langle M^{\perp}\rangle\beta\right)_T^{\pi}\Big|\Bigg)\Bigg)\Bigg] < \infty\end{aligned} \tag{5.13}$$

and $|g \cdot (M - \int d\,\langle M\rangle\,\pi)_T^H|$ also has exponential moments under $\mathbb{R}$.

Remark 5.7 Assumption 4.9 and Lemma 5.6 allow us to characterise $c\pi \cdot R^{\pi}$ and $c\pi \cdot (\phi \cdot R + M^{\perp} - \int d\langle M^{\perp}\rangle\beta)^{\pi}$ as $\mathbb{R}$-*entropic submartingales* in the terminology of Barrieu and El Karoui [4, Sect. 3], for a sufficiently small and positive constant c. We recall that a process X with sufficient exponential integrability is called an entropic submartingale if for all stopping times σ and τ with $\sigma \le \tau \le T$, we have $X_\sigma \le \rho_\sigma(X_\tau)$ $\mathbb{P}$-a.s., where $\rho_s(X_\tau) = \log \mathbb{E}_{\mathbb{R}}[\exp(X_\tau)|\mathcal{F}_\sigma]$. The operator ρ is known as the entropic process and is studied in the context of risk measures; see the references in [4]. Further, following [4], our integrability assumption can be restated in terms of the key driving entropic (sub)martingales being in the class of $\mathbb{R}$-martingales whose stochastic exponential is a uniformly integrable martingale. In turn, this can also be characterised, through the class $L^1_{\exp}(\mathbb{R})$ of random variables Z with $\exp(|Z|) \in L^1(\mathbb{R})$, that is, the random variables appearing in Assumption 4.9 must be in $L^1_{\exp}(\mathbb{R})$. Again, we refer to [4, Sect. 3] for more details.

Proof of Lemma 5.6 Using Lemma 5.2, by direct computations, $\pi \cdot \widetilde{R}^{\pi}$ can be represented as in (5.12), where each term is well defined. Thus with quasi-left-continuity of $\langle M\rangle$, we have

$$[\pi \cdot \widetilde{R}^{\pi}] = \left[\pi \cdot \left(\phi \cdot R + M^{\perp} - \int d\langle M^{\perp}\rangle\beta\right)^{\pi}\right], \tag{5.14}$$

and together with Assumption 4.10, we deduce that the jumps of the process L defined by $L := \pi \cdot (\phi \cdot R + M^{\perp} - \int d\langle M^{\perp}\rangle\beta)^{\pi}$ are bounded. By Assumption 4.2, L can be written as a stochastic integral with respect to a sigma-bounded local martingale. Therefore, together with the boundedness of the jumps as above, we deduce via the Ansel–Stricker theorem, see Delbaen and Schachermayer [8, Theorem 7.3.7], that L is a local martingale. As a result, Assumption 4.9, (5.14) and Protter [40, Corollary II.3] imply that L is an $\mathbb{R}$-martingale. As $[\pi \cdot \widetilde{R}^{\pi}]_T$ has exponential moments under $\mathbb{R}$ by Assumption 4.9, Assumption 4.10, (5.14) and Lemma 5.1 imply that $|L_T|$ also has exponential moments under $\mathbb{R}$. This further implies via Hölder's inequality from Assumption 4.9 and (5.12) that $|\int_0^T g_s^\top d\langle M\rangle_s \pi_s|$ has exponential moments under $\mathbb{R}$.

Similarly, representing

$$g \cdot M^H = g \cdot \left(M - \int d\langle M\rangle \pi\right)^H + \int g^\top d\langle M\rangle \pi$$

and using Assumptions 4.9, 4.10 and Lemma 5.1, we deduce that the random variable $|g \cdot (M - \int d\langle M\rangle \pi)_T^H|$ has exponential moments under $\mathbb{R}$. Consequently, via Assumption 4.9 and Hölder's inequality, we deduce that $|g \cdot M_T^H|$ also has exponential moments under $\mathbb{R}$. Thus (5.13) holds. □

The following result establishes an equality (see (5.15)) that is needed for Theorem 4.11; see (4.3).

Lemma 5.8 *Fix $x > 0$ and set $y = u_x(x, 0)$. Under the conditions of Theorem 4.11, we have*

$$\mathbb{E}_{\mathbb{R}}[\pi \cdot \widetilde{R}_T^{\pi}] = \mathbb{E}_{\mathbb{R}}[g \cdot M_T^H]. \tag{5.15}$$

Proof Let us observe that

$$\pi \cdot \widetilde{R}^{\pi} = \pi \cdot \phi \cdot R^{\pi} + \pi \cdot \left(M^{\perp} - \int d\langle M^{\perp}\rangle\beta\right)^{\pi} + \int \pi^\top d\langle M\rangle g.$$

Assumption 4.9 implies that

$$\pi \cdot \left(\phi \cdot R + M^{\perp} - \int d\langle M^{\perp}\rangle\beta\right)^{\pi} \in \mathcal{H}^2(\mathbb{R}), \qquad \int_0^T \pi_s^\top d\langle M\rangle_s g_s \in L^2(\mathbb{R}).$$

We deduce that

$$\begin{aligned}\mathbb{E}_{\mathbb{R}}[\pi \cdot \widetilde{R}_T^{\pi}] &= \mathbb{E}_{\mathbb{R}}\left[\pi \cdot \left(\phi \cdot R + M^{\perp} - \int d\langle M^{\perp}\rangle\beta\right)_T^{\pi}\right] \\ &\quad + \mathbb{E}_{\mathbb{R}}\left[\int_0^T \pi_s^\top d\langle M\rangle_s g_s\right] \\ &= \mathbb{E}_{\mathbb{R}}\left[\int_0^T g_s^\top d\langle M\rangle_s \pi_s\right].\end{aligned} \tag{5.16}$$

Now, as $g \cdot M^H_T \in L^2(\mathbb{R})$ by Lemma 5.6, we can rewrite the latter expression as

$$\mathbb{E}_{\mathbb{R}}\left[\int_0^T \pi_s^\top d\langle M\rangle_s\, g_s\right] = \mathbb{E}_{\mathbb{R}}\Bigg[g\cdot\left(\int d\langle M\rangle\,\pi - M\right)^H_T + g\cdot M^H_T\Bigg]$$
$$= \mathbb{E}_{\mathbb{R}}[g\cdot M^H_T], \tag{5.17}$$

where we have used that $g\cdot(\int d\langle M\rangle\,\pi - M)^H \in \mathcal{H}^2(\mathbb{R})$ due to Assumption 4.9 and Lemma 5.1. Comparing (5.16) and (5.17), we deduce (5.15). □

Remark 5.9 The equality $u_\varepsilon(x,0) = v_\varepsilon(x,0)$ will follow from the proof of Theorem 4.15. Here we discuss the implications of the two matching representations in (4.3), in particular the linear structure of $u_\varepsilon(x,0) = v_\varepsilon(x,0)$ with respect to the individual components entering the process $\widetilde{R}$, that is, $\phi\cdot M$, $M^\perp$ and the finite-variation term $\int d\langle M\rangle\,\zeta$.

From Lemma 5.6, we have

$$\mathbb{E}_{\mathbb{R}}\left[\left|\int_0^T g_s^\top d\langle M\rangle_s\,\pi_s\right|\right] < \infty.$$

and that $g\cdot(M - \int d\langle M\rangle\pi)^H$ is a true martingale under $\mathbb{R}$. This allows rewriting $v_\varepsilon(y,0)$ as

$$\begin{aligned}\frac{v_\varepsilon(y,0)}{xy} &= \mathbb{E}_{\mathbb{R}}\Bigg[g\cdot\left(M - \int d\langle M\rangle\pi\right)^H_T\Bigg] + \mathbb{E}_{\mathbb{R}}\left[\int_0^T g_s^\top d\langle M\rangle_s\pi_s\right]\\ &= \mathbb{E}_{\mathbb{R}}\left[\int_0^T g_s^\top d\langle M\rangle_s\pi_s\right]\\ &= \mathbb{E}_{\mathbb{R}}\Bigg[\int_0^T \pi_s^\top d\langle M\rangle_s\big((\lambda_s^0)' + (G_s^0)'^\top\beta_s\big)\Bigg],\end{aligned} \tag{5.18}$$

where in the last equality, we have used Lemma 5.2; see (5.8). For the purpose of giving the intuition in this remark, assuming the needed integrability of the individual components of g, using Lemma 5.2, we can rewrite (5.18) as

$$\begin{aligned}\frac{v_\varepsilon(y,0)}{xy} &= \mathbb{E}_{\mathbb{R}}\left[\int_0^T \pi_s^\top d\langle M\rangle_s(G_s^0)'^\top\beta_s\right] + \mathbb{E}_{\mathbb{R}}\left[\int_0^T \pi_s^\top d\langle M\rangle_s(\lambda_s^0)'\right]\\ &= \mathbb{E}_{\mathbb{R}}\left[\int_0^T \pi_s^\top \nu_s d\langle M\rangle_s\beta_s\right] + \mathbb{E}_{\mathbb{R}}\left[\int_0^T \pi_s^\top d\langle M\rangle_s\zeta_s\right]\\ &\quad - \mathbb{E}_{\mathbb{R}}\left[\int_0^T \pi_s^\top\phi_s d\langle M\rangle_s\lambda_s\right].\end{aligned} \tag{5.19}$$

Likewise, assuming the needed integrability, we can rewrite $u_\varepsilon(x,0)$ as

$$
\begin{aligned}
\frac{u_\varepsilon(x,0)}{xy} &= \mathbb{E}_{\mathbb{R}}[\pi \cdot \widetilde{R}^\pi_T] \\
&= \mathbb{E}_{\mathbb{R}}\left[(\phi^\top \pi)\cdot M^\pi_T + \int_0^T \pi_s^\top d\langle M\rangle_s \zeta_s + \pi \cdot M^{\perp,\pi}_T\right] \\
&= \mathbb{E}_{\mathbb{R}}\left[(\phi^\top \pi)\cdot \left(M + \int d\langle M\rangle \lambda\right)^\pi_T\right] - \mathbb{E}_{\mathbb{R}}\left[\int_0^T \pi_s^\top \phi_s d\langle M\rangle_s \lambda_s\right] \\
&+ \mathbb{E}_{\mathbb{R}}\left[\int_0^T \pi_s^\top d\langle M\rangle_s \zeta_s\right] + \mathbb{E}_{\mathbb{R}}\left[\pi \cdot \left(M^\perp - \int d\langle M^\perp\rangle \beta\right)^\pi_T\right] \\
&+ \mathbb{E}_{\mathbb{R}}\left[\int_0^T \pi_s^\top d\langle M^\perp\rangle_s \beta_s\right] \\
&= \mathbb{E}_{\mathbb{R}}\left[\int_0^T \pi_s^\top \nu_s d\langle M\rangle_s \beta_s\right] + \mathbb{E}_{\mathbb{R}}\left[\int_0^T \pi_s^\top d\langle M\rangle_s \zeta_s\right] \\
&\quad - \mathbb{E}_{\mathbb{R}}\left[\int_0^T \pi_s^\top \phi_s d\langle M\rangle_s \lambda_s\right].
\end{aligned} \tag{5.20}
$$

Comparing the last expressions in (5.19) and (5.20), we conclude that heuristically, we can get $v_\varepsilon(y,0) = u_\varepsilon(x,0)$ by matching terms corresponding to three different types of perturbations, represented by an orthogonal martingale term ν, a perturbation ϕ of the volatility term, and a perturbation ζ of the finite-variation term. The contributions of ν, ϕ and ζ in $v_\varepsilon(y,0)$ and $u_\varepsilon(x,0)$, on a heuristic level, can be matched separately. This ends Remark 5.9.

5.2 Constructing a second-order bound for the primal problem

Below, we construct a family of admissible wealth processes that provides an asymptotic lower bound, quadratic in Δx and ε, for the primal value function, in the sense of Lemma 5.17. For bounded and predictable π^0 and π', let us set

$$
\begin{aligned}
K^{\Delta x,\varepsilon} &:= (\pi + \Delta x \pi^0 + \varepsilon \pi')\cdot R^\varepsilon - \pi \cdot R \\
&= (\Delta x \pi^0 + \varepsilon \pi')\cdot R + \varepsilon \pi \cdot \widetilde{R} + \varepsilon(\Delta x \pi^0 + \varepsilon \pi')\cdot \widetilde{R}.
\end{aligned} \tag{5.21}
$$

We drop the superscript $\Delta x, \varepsilon$ in $K^{\Delta x,\varepsilon}$ and write K instead for brevity of notation. We note that the boundedness and predictability of π^0 and π' ensure that $\pi^0 \cdot \tilde{R}^\pi$ and $\pi' \cdot \tilde{R}^\pi$ are well defined.

Lemma 5.10 *Fix $x > 0$ and set $y = u_x(x,0)$. Under the conditions of Theorem 4.11, consider bounded and predictable processes π^0 and π' such that the following processes are bounded*:

1) $\pi^0 \cdot R^\pi$ *and* $\pi' \cdot R^\pi$;
2) $[\pi^0 \cdot R^\pi]$ *and* $[\pi' \cdot R^\pi]$;
3) $\pi^0 \cdot \widetilde{R}^\pi$, $\pi' \cdot \widetilde{R}^\pi$, $(\phi^\top \pi') \cdot R^\pi$, $\pi' \cdot (M^\perp - \int d\langle M^\perp\rangle \beta)^\pi$, $\int \zeta^\top d\langle M\rangle \pi'$, $\int \pi'^\top \phi d\langle M\rangle \lambda$ *and* $\int \pi'^\top d\langle M^\perp\rangle \beta$;
4) $[\pi^0 \cdot \widetilde{R}^\pi]$, $[(\phi^\top \pi') \cdot R^\pi]$, $[\pi' \cdot (M^\perp - \int d\langle M^\perp\rangle \beta)^\pi]$;
5) $[\pi \cdot \widetilde{R}^\pi, \pi^0 \cdot R^\pi]$ *and* $[\pi \cdot \widetilde{R}^\pi, \pi' \cdot R^\pi]$;
6) $[\pi \cdot \widetilde{R}^\pi, \pi^0 \cdot \widetilde{R}^\pi]$ *and* $[\pi \cdot \widetilde{R}^\pi, \pi' \cdot \widetilde{R}^\pi]$.

Then there exists a constant $\delta > 0$ *such that for every* $(\Delta x, \varepsilon) \in B_\delta(0,0)$, *we have*

$$\frac{X}{x}\mathcal{E}(K^{\Delta x,\varepsilon,\pi}) \in \mathcal{X}(1,\varepsilon),$$

where K *is given by* (5.21) *and, similarly to* (3.3), $K^{\Delta x,\varepsilon,\pi}$ *is given by*

$$K^{\Delta x,\varepsilon,\pi} = K^{\Delta x,\varepsilon} - [K^{\Delta x,\varepsilon,c}, \pi \cdot R^c] - \sum_{0<s\le \cdot} \Delta K_s^{\Delta x,\varepsilon} \frac{\pi_s \Delta R_s}{1+\pi_s \Delta R_s}. \tag{5.22}$$

Proof It follows from the calculus of stochastic exponentials that

$$\frac{\mathcal{E}((\pi + \Delta x \pi^0 + \varepsilon \pi') \cdot R^\varepsilon)}{\mathcal{E}(\pi \cdot R)} = \mathcal{E}(K^{\Delta x,\varepsilon,\pi}),$$

where the positivity of $\mathcal{E}(K^{\Delta x,\varepsilon,\pi})$ follows from Assumption 4.10 and the choice of δ in $B_\delta(0,0)$. □

Now, under the transformation (5.22), for π^0 and π' as in Lemma 5.10, we have

$$K^{\Delta x,\varepsilon,\pi} = (\Delta x \pi^0 + \varepsilon \pi') \cdot R^\pi + \varepsilon(\Delta x \pi^0 + \varepsilon \pi') \cdot \widetilde{R}^\pi + \varepsilon \pi \cdot \widetilde{R}^\pi,$$

where $(\Delta x \pi^0 + \varepsilon \pi') \cdot R^\pi \in \mathcal{M}^\infty$, $(\Delta x \pi^0 + \varepsilon \pi') \cdot \widetilde{R}^\pi$ is bounded and $\pi \cdot \widetilde{R}^\pi$ has exponential moments under $\mathbb{R}$ and bounded jumps by Assumptions 4.9 and 4.10.

We need the following lemma from Mostovyi and Sîrbu [37, Corollary 4.13].

Lemma 5.11 *Under Assumption* 2.1, *for every* $z > 0$ *and* $x > 0$, *we have*

$$U'(zx) \le \max(z^{-c_2}, 1)U'(x) \le (z^{-c_2}+1)U'(x),$$

$$-V'(zx) \le \max(z^{-\frac{1}{c_1}}, 1)\big(-V'(x)\big) \le (z^{-\frac{1}{c_1}}+1)\big(-V'(x)\big).$$

The following result provides auxiliary representation formulas needed in the proof of Lemma 5.16.

Lemma 5.12 *Fix $x > 0$ and set $y = u_x(x, 0)$. Under the conditions of Theorem 4.11, let both π^0 and π' satisfy the assumptions of Lemma 5.10. Then we have*

$$\begin{aligned}&\mathbb{E}\Big[U''(X_T)\Big(X_T\Big(\frac{1}{x}+\pi^0\cdot R^{\pi}_T\Big)\Big)^2\\&\qquad+U'(X_T)X_T\Big(\Big(\frac{1}{x}+\pi^0\cdot R^{\pi}_T\Big)^2-\frac{1}{x^2}-[\pi^0\cdot R^{\pi}]_T\Big)\Big]\\&=-\frac{y}{x}\mathbb{E}\Big[A(X_T)\Big(1+x\pi^0\cdot R^{\pi}_T\Big)^2\Big]\end{aligned}\tag{5.23}$$

as well as

$$\begin{aligned}&\mathbb{E}\big[U''(X_T)(\pi'\cdot R^{\pi}_T+\pi\cdot\widetilde{R}^{\pi}_T)^2\\&\quad+U'(X_T)X_T(\pi'\cdot R^{\pi}_T+\pi\cdot\widetilde{R}^{\pi}_T)^2+2\pi'\cdot\widetilde{R}^{\pi}_T-[\pi'\cdot R^{\pi}+\pi\cdot\widetilde{R}^{\pi}]_T\big]\\&=-xy\mathbb{E}_{\mathbb{R}}\big[\big(A(X_T)-1\big)(\pi'\cdot R^{\pi}_T+\pi\cdot\widetilde{R}^{\pi}_T)^2\\&\qquad\qquad-2\pi'\cdot\widetilde{R}^{\pi}_T+[\pi\cdot\widetilde{R}^{\pi}+\pi'\cdot R^{\pi}]_T\big].\end{aligned}\tag{5.24}$$

Proof As $\pi^0\cdot R^{\pi}$ is a bounded martingale under $\mathbb{R}$, we deduce that

$$\begin{aligned}&\mathbb{E}\Big[U'(X_T)X_T\Big(\Big(\frac{1}{x}+\pi^0\cdot R^{\pi}_T\Big)^2-\frac{1}{x^2}-[\pi^0\cdot R^{\pi}]_T\Big)\Big]\\&=xy\mathbb{E}_{\mathbb{R}}\Big[\frac{2}{x}\pi^0\cdot R^{\pi}_T+(\pi^0\cdot R^{\pi}_T)^2-[\pi^0\cdot R^{\pi}]_T\Big]\\&=0.\end{aligned}\tag{5.25}$$

Using the definition of $\mathbb{R}$, we also get

$$\mathbb{E}\Big[U''(X_T)\Big(X_T\Big(\frac{1}{x}+\pi^0\cdot R^{\pi}_T\Big)\Big)^2\Big]=-\frac{y}{x}\mathbb{E}_{\mathbb{R}}[A(X_T)(1+x\pi^0\cdot R^{\pi}_T)^2].\tag{5.26}$$

Combining (5.25) and (5.26), we deduce (5.23). (5.24) can be obtained similarly. □

5.3 Preliminaries for representing $a(\varepsilon,\varepsilon)$ and $a(x,\varepsilon)$

To obtain the representations in (4.7) and (4.8), we will need some auxiliary results. The following lemma provides an auxiliary representation needed in the proof of Lemma 5.14.

Lemma 5.13 *Fix $x > 0$ and set $y = u_x(x, 0)$. Under the conditions of Theorem 4.11, let π' satisfy the assumptions of Lemma 5.10. Then we have*

$$\mathbb{E}_{\mathbb{R}}\left[\int_0^T\pi'^{\top}_s d\langle M\rangle_s g_s\right]=\mathbb{E}_{\mathbb{R}}\Big[(\pi'\cdot R^{\pi}_T)\Big(g\cdot\Big(M-\int d\langle M\rangle\pi\Big)^H_T\Big)\Big].\tag{5.27}$$

Proof First, let us observe that $\pi' \cdot M$ and $g \cdot M$ are in $\mathcal{H}^2_{\mathrm{loc}}(\mathbb{P})$. As $\frac{X_- Y_-}{xy}$ is left-continuous, by Protter [40, Theorem III.29], the process

$$\frac{X_- Y_-}{xy} \cdot \left(\int \pi'^{\top} d[M^d] g - \int \pi'^{\top} d\langle M^d \rangle g \right) \tag{5.28}$$

is a local $\mathbb{P}$-martingale, where $M = M^c + M^d$ is the decomposition of M into continuous and purely discontinuous parts; see e.g. Jacod and Shiryaev [19, Theorem I.4.18]. Let τ_n, $n \in \mathbb{N}$, be a localising sequence for the local $\mathbb{P}$-martingale in (5.28) and such that $(\int \pi'^{\top} d\langle M^c \rangle g)^{\tau_n}$ and $(\int \pi'^{\top} d\langle M^d \rangle g)^{\tau_n}$ are bounded for each $n \in \mathbb{N}$. The existence of such a sequence follows from Assumption 4.5. Fix $n \in \mathbb{N}$, consider an arbitrary stopping time σ and set $\tau := \sigma \wedge \tau_n$. Next, recalling the definitions of the transformations $\cdot^{\pi}$ and $\cdot^{H}$ and using the Kunita–Watanabe inequality, we get

$$\begin{aligned}
&\mathbb{E}_{\mathbb{R}}\left[(\pi' \cdot R^{\pi}_{\tau}) \left(g \cdot \left(M - \int d\langle M \rangle \pi \right)^{H}_{\tau} \right) \right] \\
&= \mathbb{E}_{\mathbb{R}}\Bigg[\Bigg[\pi' \cdot R - [\pi' \cdot R^c, \pi \cdot R^c] - \sum_{0<s\le \cdot} \pi'_s \Delta M_s \frac{\pi_s \Delta R_s}{1 + \pi_s \Delta R_s}, \\
&\qquad g \cdot \left(M - \int d\langle M \rangle \pi \right) - \left[g \cdot \left(M - \int d\langle M \rangle \pi \right)^{c}, H^c \right] \\
&\qquad - \sum_{0<s\le \cdot} g_s \Delta M_s \frac{\Delta H_s}{1 + \Delta H_s} \Bigg]_{\tau} \Bigg] \\
&= \mathbb{E}_{\mathbb{R}}\Bigg[\Bigg[\pi' \cdot M - \sum_{o<s\le \cdot} \pi'_s \Delta M_s \frac{\pi_s \Delta R_s}{1 + \pi_s \Delta R_s}, \\
&\qquad g \cdot M - \sum_{0<s\le \cdot} g_s \Delta M_s \frac{\Delta H_s}{1 + \Delta H_s} \Bigg]_{\tau} \Bigg].
\end{aligned} \tag{5.29}$$

One can rewrite the last expectation in (5.29) as

$$\mathbb{E}_{\mathbb{R}}\left[\int_0^{\tau} \pi'^{\top}_s d\langle M^c \rangle_s g_s + \sum_{0<s\le\tau} \frac{\pi'_s \Delta M_s}{1 + \pi_s \Delta R_s} \frac{g_s \Delta M_s}{1 + \Delta H_s} \right] =: \widetilde{T}_1 + \widetilde{T}_2.$$

The computations above show that both $\widetilde{T}_1$ and $\widetilde{T}_2$ are finite. Let us observe that by Lemma 5.6, both $\pi' \cdot R^{\pi}$ and $g \cdot (M - \int d\langle M \rangle \pi)^H$ are in $\mathcal{H}^2(\mathbb{R})$, and thus we have

$$\mathbb{E}_{\mathbb{R}}\left[\sum_{0<s\le\tau} \left(\frac{\pi'_s \Delta M_s}{1 + \pi_s \Delta R_s} \right)^2 \right] \le \mathbb{E}_{\mathbb{R}}\big[[\pi' \cdot R^{\pi}]_{\tau} \big] < \infty \tag{5.30}$$

and

$$\mathbb{E}_{\mathbb{R}}\left[\sum_{0<s\le\tau} \left(\frac{g_s \Delta M_s}{1 + \Delta H_s} \right)^2 \right] \le \mathbb{E}_{\mathbb{R}}\left[\left[g \cdot \left(M - \int d\langle M \rangle \pi \right)^{H} \right]_{\tau} \right] < \infty. \tag{5.31}$$

Further, since

$$1 + \pi_s \Delta R_s = \frac{X_s}{X_{s-}}, \qquad 1 + \Delta H_s = \frac{Y_s}{Y_{s-}},$$

and using (5.30), (5.31), localisation and integration by parts, one can rewrite $\widetilde{T}_2$ as

$$\widetilde{T}_2 = \mathbb{E}\bigg[\sum_{0<s\le\tau} \frac{X_{s-}Y_{s-}}{xy}\pi'_s \Delta M_s g_s \Delta M_s\bigg]. \tag{5.32}$$

By Protter [40, Theorem II.28], we have

$$\int \pi'^{\top} d[M^d] g = [\pi' \cdot M^d, g \cdot M^d] = \sum_{0<s\le\cdot} \pi'_s \Delta M_s g_s \Delta M_s. \tag{5.33}$$

As the process in (5.28) is a true $\mathbb{P}$-martingale on $[0, \tau]$, from (5.33), one can further rewrite $\widetilde{T}_2$ in (5.32) as

$$\widetilde{T}_2 = \mathbb{E}\bigg[\int_0^\tau \frac{X_{s-}Y_{s-}}{xy}\pi_s'^{\top} d\langle M^d\rangle_s g_s\bigg].$$

Next, using localisation and [19, Theorem I.4.49] (noting that $\int_0^\tau \pi_s'^{\top} d\langle M^d\rangle_s g_s$ is $\mathbb{P}$-integrable by the construction of τ), one can further rewrite $\widetilde{T}_2$ as

$$\widetilde{T}_2 = \mathbb{E}_{\mathbb{R}}\bigg[\int_0^\tau \pi_s'^{\top} d\langle M^d\rangle_s g_s\bigg].$$

We recapitulate that $\mathbb{E}_{\mathbb{R}}[(\pi' \cdot R^\pi_\tau)(g \cdot (M - \int d\langle M\rangle \pi)^H_\tau)]$ in (5.29) can be rewritten as

$$\begin{aligned}
&\mathbb{E}_{\mathbb{R}}\bigg[(\pi' \cdot R^\pi_\tau)\bigg(g \cdot \bigg(M - \int d\langle M\rangle \pi\bigg)^H_\tau\bigg)\bigg] \\
&= \widetilde{T}_1 + \widetilde{T}_2 \\
&= \mathbb{E}_{\mathbb{R}}\bigg[\int_0^\tau \pi_s'^{\top} d\langle M^c\rangle_s g_s\bigg] + \mathbb{E}_{\mathbb{R}}\bigg[\int_0^\tau \pi_s'^{\top} d\langle M^d\rangle_s g_s\bigg] \\
&= \mathbb{E}_{\mathbb{R}}\bigg[\int_0^\tau \pi_s'^{\top} d\,\langle M\rangle_s\, g_s\bigg].
\end{aligned}$$

As τ is an arbitrary stopping time valued in $[0, \tau_n]$ and τ_n, $n \in \mathbb{N}$, is a localising sequence, we conclude that $\int_0^\tau \pi'^{\top} d\,\langle M\rangle\, g$ is the predictable quadratic covariation of the pair $(\pi' \cdot R^\pi, g \cdot (M - \int d\langle M\rangle \pi)^H)$ (under $\mathbb{R}$).

Now, using Lemma 5.6, we observe that both $\pi' \cdot R^\pi$ and $g \cdot (M - \int d\langle M\rangle \pi)^H$ are in $\mathcal{H}^2(\mathbb{R})$. Therefore [19, Theorem I.4.2] asserts that

$$(\pi' \cdot R^\pi)\bigg(g \cdot \bigg(M - \int d\langle M\rangle \pi\bigg)^H\bigg) - \int \pi'^{\top} d\langle M\rangle g$$

is a true martingale under $\mathbb{R}$, which implies (5.27). □

The following result provides a representation needed in the proof of Lemma 5.15.

Lemma 5.14 *Fix $x > 0$ and set $y = u_x(x, 0)$. Under the conditions of Theorem 4.11, let π' be as in Lemma 5.10. Then we have*

$$\mathbb{E}_{\mathbb{R}}[\pi' \cdot \widetilde{R}^{\pi}_T] = \mathbb{E}_{\mathbb{R}}\left[(\pi' \cdot R^{\pi}_T)\left(g \cdot \left(M - \int d\langle M\rangle \pi\right)^H_T\right)\right]. \tag{5.34}$$

Proof From Lemmas 5.6 and 5.2, we get

$$\mathbb{E}_{\mathbb{R}}[\pi' \cdot \widetilde{R}^{\pi}_T] = \mathbb{E}_{\mathbb{R}}\left[\pi' \cdot (\phi \cdot R + M^{\perp} - \beta \cdot \langle M^{\perp}\rangle)^{\pi}_T + \int_0^T \pi_s'^{\top} d\,\langle M\rangle_s\, g_s\right]. \tag{5.35}$$

The construction of π' implies that $\pi' \cdot (\phi \cdot R + M^{\perp} - \beta \cdot \langle M^{\perp}\rangle)^{\pi} \in \mathcal{H}^2(\mathbb{R})$, and thus

$$\mathbb{E}_{\mathbb{R}}[\pi' \cdot (\phi \cdot R + M^{\perp} - \beta \cdot \langle M^{\perp}\rangle)^{\pi}_T] = 0.$$

Consequently, in (5.35), we obtain

$$\mathbb{E}_{\mathbb{R}}[\pi' \cdot \widetilde{R}^{\pi}_T] = \mathbb{E}_{\mathbb{R}}\left[\int_0^T \pi_s'^{\top} d\,\langle M\rangle_s\, g_s\right].$$

Comparing (2.11) to (5.27) in Lemma 5.13 gives (5.34). □

Lemma 5.15 *Fix $x > 0$ and set $y = u_x(x, 0)$. Under the conditions of Theorem 4.11, let π' be as in Lemma 5.10. Then we have*

$$\begin{aligned} &- \mathbb{E}_{\mathbb{R}}\big[\big(1 - A(X_T)\big)(\pi' \cdot R^{\pi}_T + \pi \cdot \widetilde{R}^{\pi}_T)^2 + 2\pi' \cdot \widetilde{R}^{\pi}_T - [\pi \cdot \widetilde{R}^{\pi} + \pi' \cdot R^{\pi}]_T\big] \\ &= \mathbb{E}_{\mathbb{R}}[A(X_T)(\pi' \cdot R^{\pi}_T + \pi \cdot \widetilde{R}^{\pi}_T)^2 - 2\pi' \cdot R^{\pi}_T g \cdot M^H_T] + T_1, \end{aligned} \tag{5.36}$$

where T_1 is given in (4.6) and g in (4.1). For π^0 as in Lemma 5.10, we also have

$$\begin{aligned} 0 = \mathbb{E}_{\mathbb{R}}\Bigg[& xg \cdot M^H_T + (x\pi^0 \cdot R^{\pi}_T)\left(x \int_0^T g_s^{\top} d\,\langle M\rangle_s\, \pi_s\right) \\ &- (1 + x\pi^0 \cdot R^{\pi}_T)(x\pi' \cdot R^{\pi}_T + x\pi \cdot \widetilde{R}^{\pi}_T) \\ &+ [1 + x\pi^0 \cdot R^{\pi}, x\pi' \cdot R^{\pi} + x\pi \cdot \widetilde{R}^{\pi}]_T\Bigg]. \end{aligned} \tag{5.37}$$

(*This is used later for the representation of $a(x, \varepsilon)$.*)

Proof First, using the representation in Lemma 5.6 and the square-integrability of

$$\left(\pi' \cdot R^{\pi}_T + \pi \cdot \left(\phi \cdot R + M^{\perp} - \int d\,\langle M\rangle^{\perp} \beta\right)^{\pi}_T\right)$$

under $\mathbb{R}$, we get, using (4.6), that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{R}}\left[2(\pi' \cdot R^{\pi}_T)\int_0^T g_s^\top d\langle M\rangle_s \pi_s - (\pi' \cdot R^{\pi}_T + \pi \cdot \widetilde{R}^{\pi}_T)^2 + [\pi' \cdot R^{\pi} + \pi \cdot \widetilde{R}^{\pi}]_T\right] \\
&= \mathbb{E}_{\mathbb{R}}\bigg[2(\pi' \cdot R^{\pi}_T)\int_0^T g_s^\top d\langle M\rangle_s \pi_s - \left(\int_0^T g_s^\top d\langle M\rangle_s \pi_s\right)^2 \\
&\qquad - 2\int_0^T g_s^\top d\langle M\rangle_s \pi_s\left(\pi' \cdot R^{\pi}_T + \pi \cdot \left(\phi \cdot R + M^{\perp} - \int d\langle M\rangle^{\perp}\beta\right)^{\pi}_T\right)\bigg] \\
&= -\mathbb{E}_{\mathbb{R}}\bigg[\left(\int_0^T g_s^\top d\langle M\rangle_s \pi_s\right)^2 \\
&\qquad + 2\left(\int_0^T g_s^\top d\langle M\rangle_s \pi_s\right)\left(\pi \cdot \left(\phi \cdot R + M^{\perp} - \int d\langle M\rangle^{\perp}\beta\right)^{\pi}_T\right)\bigg] \\
&= T_1.
\end{aligned} \tag{5.38}$$

Note that the representation in (5.38) does not depend on the choice of π' (which still has to be as in Lemma 5.10). Next, consider the left-hand side of (5.36). Using Lemma 5.14, we can rewrite it as

$$\begin{aligned}
&-\mathbb{E}_{\mathbb{R}}\big[\big(1 - A(X_T)\big)(\pi' \cdot R^{\pi}_T + \pi \cdot \widetilde{R}^{\pi}_T)^2 + 2\pi' \cdot \widetilde{R}^{\pi}_T - [\pi \cdot \widetilde{R}^{\pi} + \pi' \cdot R^{\pi}]_T\big] \\
&= \mathbb{E}_{\mathbb{R}}[A(X_T)(\pi' \cdot R^{\pi}_T + \pi \cdot \widetilde{R}^{\pi}_T)^2 - 2(\pi' \cdot R^{\pi}_T)(g \cdot M^H_T)] \\
&\quad + \mathbb{E}_{\mathbb{R}}\left[2(\pi' \cdot R^{\pi}_T)\int_0^T g_s^\top d\langle M\rangle_s \pi_s - (\pi' \cdot R^{\pi}_T + \pi \cdot \widetilde{R}^{\pi}_T)^2 + [\pi' \cdot R^{\pi} + \pi \cdot \widetilde{R}^{\pi}]_T\right] \\
&= \mathbb{E}_{\mathbb{R}}[A(X_T)(\pi' \cdot R^{\pi}_T + \pi \cdot \widetilde{R}^{\pi}_T)^2 - 2(\pi' \cdot R^{\pi}_T)(g \cdot M^H_T)] + T_1,
\end{aligned}$$

where the second equality uses (5.38). The computations for (5.37) are similar. □

The following result establishes a quadratic (in Δx and ε) expansion associated with a family of wealth processes parametrised by Δx and ε as in Lemma 5.10.

Lemma 5.16 *Fix $x > 0$ and set $y = u_x(x, 0)$. Under the conditions of Theorem 4.11, let π^0 and π' be as in Lemma 5.10; the associated K is defined in (5.21). Then consider*

$$\psi(\Delta x, \varepsilon) := \left(1 + \frac{\Delta x}{x}\right)\mathcal{E}(K^{\Delta x, \varepsilon, \pi})_T, \qquad (\Delta x, \varepsilon) \in B_\delta(0, 0),$$

where $\delta > 0$ is chosen to be sufficiently close to 0 so that for every $(\Delta x, \varepsilon) \in B_\delta(0, 0)$, we have $\frac{X}{x}\mathcal{E}(K^{\Delta x, \varepsilon, \pi}) \in \mathcal{X}(1, \varepsilon)$ by Lemma 5.10 and the jumps of $K^{\Delta x, \varepsilon, \pi}$ take *values in* $[-\frac{1}{2}, \frac{1}{2}]$. *Define*

$$w(\Delta x, \varepsilon) := \mathbb{E}\big[U\big(X_T\psi(\Delta x, \varepsilon)\big)\big], \qquad (\Delta x, \varepsilon) \in B_\delta(0,0).$$

Then w *admits at* $(0,0)$ *the expansion*

$$w(\Delta x, \varepsilon) = w(0,0) + (\Delta x \quad \varepsilon)\nabla w(0,0) + \frac{1}{2}(\Delta x \quad \varepsilon)H_w\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix} + o(\Delta x^2 + \varepsilon^2),$$

where

$$\begin{aligned} w_{\Delta x}(0,0) &= u_x(x,0),\\ w_\varepsilon(0,0) &= xy\mathbb{E}_{\mathbb{R}}[\pi \cdot \widetilde{R}^\pi_T],\\ H_w &:= \begin{pmatrix} w_{\Delta x\Delta x}(0,0) & w_{\Delta x\varepsilon}(0,0)\\ w_{\Delta x\varepsilon}(0,0) & w_{\varepsilon\varepsilon}(0,0)\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned} w_{\Delta x\Delta x}(0,0) &= -\frac{y}{x}\mathbb{E}_{\mathbb{R}}[A(X_T)(1 + x\pi^0 \cdot R^\pi_T)^2],\\ w_{\Delta x\varepsilon}(0,0) &= -\frac{y}{x}\mathbb{E}_{\mathbb{R}}[A(X_T)(1 + x\pi^0 \cdot R^\pi_T)(x\pi^1 \cdot R^\pi_T + x\pi \cdot \widetilde{R}^\pi_T)\\ &\qquad - (xg \cdot M^H_T)(x\pi^0 \cdot R^\pi_T + 1)],\\ w_{\varepsilon\varepsilon}(0,0) &= -\frac{y}{x}\mathbb{E}_{\mathbb{R}}[A(X_T)(x\pi^1 \cdot R^\pi_T + x\pi \cdot \widetilde{R}^\pi_T)^2\\ &\qquad - 2(x\pi^1 \cdot R^\pi_T)(xg \cdot M^H_T) + x^2 T_1].\end{aligned}$$

Proof Let us consider

$$\begin{aligned}\mathcal{E}(K^{\Delta x,\varepsilon,\pi})_T = \exp\Bigg(&K^{\Delta x,\varepsilon,\pi}_T - \frac{1}{2}[K^{\Delta x,\varepsilon,\pi}]^c_T\\ &+ \sum_{0<s\le T}\big(\log(1 + \Delta K^{\Delta x,\varepsilon,\pi}_s) - \Delta K^{\Delta x,\varepsilon,\pi}_s\big)\Bigg).\end{aligned} \tag{5.39}$$

As $|\log(1+x) - x| \le x^2$ for every $x \in [-\frac{1}{2}, \frac{1}{2}]$, we observe that in (5.39), the series $\sum_{0<s\le T}(\log(1 + \Delta K^{\Delta x,\varepsilon,\pi}_s) - \Delta K^{\Delta x,\varepsilon,\pi}_s)$ converges absolutely for every $(\Delta x, \varepsilon) \in B_\delta(0,0)$, $\mathbb{P}$-a.s., and we have

$$\sum_{0<s\le T}\big(\log(1 + \Delta K^{\Delta x,\varepsilon,\pi}_s) - \Delta K^{\Delta x,\varepsilon,\pi}_s\big) \le [K^{\Delta x,\varepsilon,\pi}]_T.$$

Hence one can find a constant $C > 0$ such that for every $(\Delta x, \varepsilon) \in B_\delta(0,0)$, we have

$$\mathcal{E}(K^{\Delta x,\varepsilon,\pi})_T \le C\exp\big(|\varepsilon| C(|\pi \cdot \tilde{R}^\pi_T| + [\pi \cdot \tilde{R}^\pi]_T)\big) \qquad \mathbb{P}\text{-a.s.}$$

We observe that the series of term-by-term partial derivatives of

$$\sum_{0<s\leq T}\left(\log(1+\Delta K_s^{\Delta x,\varepsilon,\pi})-\Delta K_s^{\Delta x,\varepsilon,\pi}\right)$$

converges $\mathbb{P}$-a.s. uniformly in $(\Delta x,\varepsilon)\in B_\delta(0,0)$, where additionally the term-by-term partial derivatives of $(\log(1+\Delta K_s^{\Delta x,\varepsilon,\pi})-\Delta K_s^{\Delta x,\varepsilon,\pi})$ are $\mathbb{P}$-a.s. continuous in $(\Delta x,\varepsilon)\in B_\delta(0,0)$. We deduce that

$$\begin{aligned}
&\frac{\partial}{\partial\Delta x}\sum_{0<s\leq T}\left(\log(1+\Delta K_s^{\Delta x,\varepsilon,\pi})-\Delta K_s^{\Delta x,\varepsilon,\pi}\right)\\
&=-\sum_{0<s\leq T}\frac{\Delta K_s^{\Delta x,\varepsilon,\pi}}{1+\Delta K_s^{\Delta x,\varepsilon,\pi}}\pi_s^0\Delta R_s^{\varepsilon,\pi},\\
&\frac{\partial}{\partial\varepsilon}\sum_{0<s\leq T}\left(\log(1+\Delta K_s^{\Delta x,\varepsilon,\pi})-\Delta K_s^{\Delta x,\varepsilon,\pi}\right)\\
&=-\sum_{0<s\leq T}\frac{\Delta K_s^{\Delta x,\varepsilon,\pi}}{1+\Delta K_s^{\Delta x,\varepsilon,\pi}}(\pi_s'\Delta R_s^\pi+\pi_s\Delta\tilde{R}_s^\pi+\Delta x\pi_s^0\Delta\tilde{R}_s^\pi+2\varepsilon\pi_s'\Delta\tilde{R}_s^\pi).
\end{aligned}$$

Fix Δx and ε and set $\bar{I}:=K^{\Delta x,\varepsilon,\pi}$. Since direct computations give

$$\frac{\psi_{\Delta x}(\Delta x,\varepsilon)}{\psi(\Delta x,\varepsilon)}=\left(\frac{1}{x+\Delta x}+\frac{\partial\bar{I}}{\partial\Delta x}-\left[\bar{I}^c,\frac{\partial\bar{I}^c}{\partial\Delta x}\right]-\sum_{0<s\leq\cdot}\frac{\frac{\partial\Delta\bar{I}_s}{\partial\Delta x}\Delta\bar{I}_s}{1+\Delta\bar{I}_s}\right)_T$$

as well as

$$\frac{\psi_{\varepsilon}(\Delta x,\varepsilon)}{\psi(\Delta x,\varepsilon)}=\left(\frac{\partial\bar{I}}{\partial\varepsilon}-\left[\bar{I}^c,\frac{\partial\bar{I}^c}{\partial\varepsilon}\right]-\sum_{0<s\leq\cdot}\frac{\frac{\partial\Delta\bar{I}_s}{\partial\varepsilon}\Delta\bar{I}_s}{1+\Delta\bar{I}_s}\right)_T,$$

we deduce that

$$\begin{aligned}
\psi_{\Delta x}(\Delta x,\varepsilon)|_{(\Delta x,\varepsilon)=(0,0)}&=\frac{1}{x}+\pi^0\cdot R_T^\pi,\\
\psi_{\varepsilon}(\Delta x,\varepsilon)|_{(\Delta x,\varepsilon)=(0,0)}&=\pi'\cdot R_T^\pi+\pi\cdot\tilde{R}_T^\pi.
\end{aligned}$$

Likewise, one can show that the series of term-by-term second-order partial derivatives of

$$\sum_{0<s\leq T}\left(\log(1+\Delta K_s^{\Delta x,\varepsilon,\pi})-\Delta K_s^{\Delta x,\varepsilon,\pi}\right)$$

converges uniformly in $(\Delta x, \varepsilon) \in B_\delta(0,0)$, where additionally the term-by-term second-order partial derivatives of $(\log(1+\Delta K_s^{\Delta x,\varepsilon,\pi}) - \Delta K_s^{\Delta x,\varepsilon,\pi})$ are continuous in $(\Delta x, \varepsilon) \in B_\delta(0,0)$. Therefore, with $\bar{I} := K^{\Delta x,\varepsilon,\pi}$, we get

$$
\begin{aligned}
\frac{\psi_{\Delta x \Delta x}(\Delta x,\varepsilon)}{\psi(\Delta x,\varepsilon)} = &\left(\left(\frac{1}{x+\Delta x} + \frac{\partial \bar{I}}{\partial \Delta x} - \left[\bar{I}^c, \frac{\partial \bar{I}^c}{\partial \Delta x}\right] - \sum_{0<s\le\cdot} \frac{\frac{\partial \Delta \bar{I}_s}{\partial \Delta x}\Delta \bar{I}_s}{1+\Delta \bar{I}_s}\right)_T\right)^2 \\
&- \frac{1}{(x+\Delta x)^2} + \frac{\partial^2 \bar{I}}{\partial \Delta x^2} - \left[\bar{I}^c, \frac{\partial^2 \bar{I}^c}{\partial \Delta x^2}\right] - \left[\frac{\partial \bar{I}^c}{\partial \Delta x}\right] \\
&- \sum_{0<s\le\cdot} \frac{\frac{\partial^2 \Delta \bar{I}_s}{\partial \Delta x^2}\Delta \bar{I}_s + (\frac{\partial \Delta \bar{I}_s}{\partial \Delta x})^2}{1+\Delta \bar{I}_s} + \sum_{0<s\le\cdot}\left(\frac{\frac{\partial \Delta \bar{I}_s}{\partial \Delta x}}{1+\Delta \bar{I}_s}\right)^2 \Delta \bar{I}_s
\end{aligned}
$$

and thus

$$
\begin{aligned}
\psi_{\Delta x \Delta x}|_{(\Delta x,\varepsilon)=(0,0)} &= \left(\frac{1}{x} + \pi^0 \cdot R_T^\pi\right)^2 - \frac{1}{x^2} - [\pi^0 \cdot R^\pi]_T^c - \sum_{0<s\le T} (\pi_s^0 \Delta R_s^\pi)^2 \\
&= \left(\frac{1}{x} + \pi^0 \cdot R_T^\pi\right)^2 - \frac{1}{x^2} - [\pi^0 \cdot R^\pi]_T.
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
\frac{\psi_{\Delta x\varepsilon}}{\psi} = &\left(\frac{1}{x+\Delta x} + \frac{\partial \bar{I}}{\partial \Delta x} - \left[\bar{I}^c, \frac{\partial \bar{I}^c}{\partial \Delta x}\right] - \sum_{0<s\le\cdot} \frac{\frac{\partial \Delta \bar{I}_s}{\partial \Delta x}\Delta \bar{I}_s}{1+\Delta \bar{I}_s}\right)_T \\
&\times \left(\frac{\partial \bar{I}}{\partial \varepsilon} - \left[\bar{I}^c, \frac{\partial \bar{I}^c}{\partial \varepsilon}\right] - \sum_{0<s\le\cdot} \frac{\frac{\partial \Delta \bar{I}_s}{\partial \varepsilon}\Delta \bar{I}_s}{1+\Delta \bar{I}_s}\right)_T + \frac{\partial^2 \bar{I}_T}{\partial \Delta x \partial \varepsilon} - \left[\bar{I}^c, \frac{\partial^2 \bar{I}^c}{\partial \Delta x \partial \varepsilon}\right]_T \\
&- \left[\frac{\partial \bar{I}^c}{\partial \varepsilon}, \frac{\partial \bar{I}^c}{\partial \Delta x}\right]_T - \sum_{0<s\le T} \frac{\frac{\partial^2 \Delta \bar{I}_s}{\partial \Delta x \partial \varepsilon}\Delta \bar{I}_s + \frac{\partial \Delta \bar{I}_s}{\partial \Delta x}\frac{\partial \Delta \bar{I}_s}{\partial \Delta x}}{1+\Delta \bar{I}_s} + \sum_{0<s\le T} \frac{\frac{\partial \Delta \bar{I}_s}{\partial \Delta x}\frac{\partial \Delta \bar{I}_s}{\partial \varepsilon}\Delta \bar{I}_s}{(1+\Delta \bar{I}_s)^2},
\end{aligned}
$$

and thus

$$
\begin{aligned}
\psi_{\Delta x\varepsilon}|_{(\Delta x,\varepsilon)=(0,0)} = &\left(\frac{1}{x} + \pi^0 \cdot R_T^{0,\pi}\right)(\pi' \cdot R_T^{0,\pi} + \pi \cdot \tilde{R}_T^\pi) \\
&+ \pi^0 \cdot \tilde{R}_T^\pi - \left[\frac{1}{x} + \pi^0 \cdot R^{0,\pi}, \pi' \cdot R^{0,\pi} + \pi \cdot \tilde{R}^\pi\right]_T.
\end{aligned}
$$

Continuing in this manner, we get

$$\begin{aligned}\frac{\psi_{\varepsilon\varepsilon}(\Delta x,\varepsilon)}{\psi(\Delta x,\varepsilon)} &= \left(\frac{\partial \bar I}{\partial\varepsilon} - \left[\bar I^c, \frac{\partial \bar I^c}{\partial\varepsilon}\right] - \sum_{0<s\le\cdot} \frac{\frac{\partial\Delta\bar I_s}{\partial\varepsilon}\Delta\bar I_s}{1+\Delta\bar I_s}\right)^2_T \\ &\quad + \frac{\partial^2\bar I_T}{\partial\varepsilon^2} - \left[\bar I^c, \frac{\partial^2\bar I^c}{\partial\varepsilon^2}\right]_T - \left[\frac{\partial\bar I^c}{\partial\varepsilon}\right]_T \\ &\quad - \sum_{0<s\le T}\frac{\frac{\partial^2\Delta\bar I_s}{\partial\varepsilon^2}\Delta\bar I_s + (\frac{\partial\Delta\bar I_s}{\partial\varepsilon})^2}{1+\Delta\bar I_s} + \sum_{0<s\le T}\left(\frac{\frac{\partial\Delta\bar I_s}{\partial\varepsilon}}{1+\Delta\bar I_s}\right)^2\Delta\bar I_s\end{aligned}$$

and consequently

$$\psi_{\varepsilon\varepsilon}(\Delta x,\varepsilon)|_{(\Delta x,\varepsilon)=(0,0)} = (\pi'\cdot R^\pi_T + \pi\cdot\widetilde R^\pi_T)^2 + 2\pi'\cdot\widetilde R^\pi_T - [\pi'\cdot R^\pi + \pi\cdot\widetilde R^\pi]_T.$$

Now fix $(\Delta x,\varepsilon)\in B_\delta(0,0)$ and define

$$\widetilde\psi(z) := \psi(z\Delta x, z\varepsilon), \qquad z\in(-1,1).$$

We observe that

$$\begin{aligned}\widetilde\psi'(z) &= \psi_{\Delta x}(\Delta x,\varepsilon)\Delta x + \psi_\varepsilon(\Delta x,\varepsilon)\varepsilon,\\ \widetilde\psi''(z) &= \psi_{\Delta x\Delta x}(z\Delta x,z\varepsilon)\Delta x^2 + 2\psi_{\Delta x\varepsilon}(z\Delta x,z\varepsilon)\Delta x\varepsilon + \psi_{\varepsilon\varepsilon}(z\Delta x,z\varepsilon)\varepsilon^2.\end{aligned}$$

Let us set

$$W(z) := U\big(X_T\widetilde\psi(z)\big), \qquad z\in(-1,1).$$

By direct computations, we get

$$\begin{aligned}W'(z) &= U'\big(X_T\widetilde\psi(z)\big)X_T\widetilde\psi'(z),\\ W''(z) &= U''\big(X_T\widetilde\psi(z)\big)\big(X_T\widetilde\psi'(z)\big)^2 + U'\big(X_T\widetilde\psi(z)\big)X_T\widetilde\psi''(z).\end{aligned}$$

Let us define

$$J := 1 + |\pi\cdot\widetilde R^\pi_T| + [\pi\cdot\widetilde R^\pi]_T.$$

Now one can deduce the existence of a constant $b_1>0$ such that

$$|\widetilde\psi'(z)| \le b_1 J\exp(b_1\delta J), \quad \widetilde\psi(z)^{-c_2}+1 \le b_1\exp(b_1\delta J), \qquad z\in(-1,1).$$

Using Lemma 5.11, we get from the computations above that

$$\begin{aligned}\sup_{z\in(-1,1)}|W'(z)| &\le \sup_{z\in(-1,1)} U'(X_T)X_T\big(\widetilde\psi(z)^{-c_2}+1\big)|\widetilde\psi'(z)| \\ &\le U'(X_T)X_T b_1^2 J\exp(2b_1\delta J). \end{aligned} \tag{5.40}$$

Likewise, from Assumption 2.1 and using Lemma 5.11, we obtain the existence of a constant $b_2 > 0$ such that

$$\sup_{z\in(-1,1)} |W''(z)| \leq b_2 U'(X_T) X_T J^2 \exp(b_2 \delta J). \tag{5.41}$$

From (5.40) and (5.41), we get

$$\begin{aligned}&\sup_{z\in(-1,1)} |W'(z)| + \sup_{z\in(-1,1)} |W''(z)| \\ &\quad \leq U'(X_T) X_T b_1^2 J \exp(2b_1\delta J) + b_2 U'(X_T) X_T J^2 \exp(b_2\delta J).\end{aligned}$$

As $1 \leq J \leq J^2$, we deduce the existence of a constant $b > 0$ such that for all z_1 and z_2 in $(-1, 1)$, we get

$$\left|\frac{W(z_1) - W(z_2)}{z_1 - z_2}\right| + \left|\frac{W'(z_1) - W'(z_2)}{z_1 - z_2}\right| \leq U'(X_T) X_T b J^2 \exp\left(b\delta J\right). \tag{5.42}$$

By passing to a smaller δ if needed, we deduce via Hölder's inequality that the right-hand side in (5.42) is integrable. Further, as the bound in (5.42) is uniform in $(\Delta x, \varepsilon) \in B_\delta(0, 0)$, we deduce the assertion of Lemma 5.16 from the dominated convergence theorem and the representation formulas from Lemmas 5.12 and 5.15. □

The following result gives an asymptotic second-order (in Δx and ε) lower bound for the primal value function.

Lemma 5.17 *Fix $x > 0$ and set $y = u_x(x, 0)$. Under the conditions of Theorem 4.15, we have*

$$\begin{aligned}u(x + \Delta x, \varepsilon) \geq u(x, 0) &+ \Delta x y + \varepsilon x y \mathbb{E}_{\mathbb{R}}[\pi \cdot \widetilde{R}^\pi_T] \\ &+ \frac{1}{2}(\Delta x \quad \varepsilon) H_u(x, 0) \begin{pmatrix}\Delta x \\ \varepsilon\end{pmatrix} + o(\Delta x^2 + \varepsilon^2).\end{aligned}$$

Proof Using Assumption 4.2, we can approximate the optimisers to (4.4) and (4.7) by elements of $\mathcal{M}^\infty$. By Mostovyi [36, Lemma 4.1], these elements of $\mathcal{M}^\infty$ can be represented as stochastic integrals with respect to R^π. In turn, by stopping, we can assume that the corresponding integrands satisfy the assumptions of Lemma 5.10. The result now follows from Lemma 5.16 via an approximation of elements of $\mathcal{M}^2$ by those in $\mathcal{M}^\infty$ as in the Appendix. □

5.4 Constructing a second-order bound for the dual problem

The following construction of an asymptotic second-order (in Δy and ε) upper bound for the dual value function has a similar structure, yet with some differences, to the one for the primal value function in Sects. 5.2 and 5.3. For every $\varepsilon \in (-\widetilde{\varepsilon}, \widetilde{\varepsilon})$, where $\widetilde{\varepsilon} > 0$ is given in Lemma 5.2, we recall that λ^ε and G^ε are given by the solutions to (5.5), where Assumption 4.1 allows explicit representations of λ^ε and G^ε. For bounded and predictable β^0 and β', we set

$$\begin{aligned} J^{\Delta y,\varepsilon} &:= -(\lambda^\varepsilon - \lambda^0)\cdot M - \big((G^\varepsilon)^\top \beta\big)\cdot M + (\Delta y \beta^0 + \varepsilon\beta')\cdot(-G^\varepsilon \cdot M + M^\perp) \\ &\quad + (\varepsilon + \Delta y \widetilde{\phi})\cdot \widetilde{L} + \Delta y \bar{L}, \qquad (\varepsilon, \Delta y) \in (-\widetilde{\varepsilon}, \widetilde{\varepsilon}) \times \mathbb{R}, \end{aligned} \tag{5.43}$$

where $\bar{L}$ and $\widetilde{L}$ are in $\mathcal{H}^2_{\mathrm{loc}}(\mathbb{P})$ and orthogonal to every component of both M and $M^\perp$ and to each other and are such that $\bar{L}^H$ and $\widetilde{L}^H$ are bounded, and $\widetilde{\phi}$ is predictable and such that $\int_0^T \widetilde{\phi}^2_s d\langle \widetilde{L}\rangle_s < \infty$ $\mathbb{P}$-a.s. One can show that for bounded and predictable processes β^0 and β', under the Assumptions of Theorem 4.11, the stochastic integrals in (5.43) are well defined.

The following result as well as its proof is similar to Lemma 5.10. Therefore we skip the proof for brevity of exposition.

Lemma 5.18 *Fix $x > 0$ and set $y = u_x(x, 0)$. Assume the conditions of Theorem 4.11 and let β^0 and β' be bounded predictable processes such that the following processes are bounded*:

1) $\beta^0 \cdot M^{\perp,H}$ *and* $\beta' \cdot M^{\perp,H}$;
2) $[\beta^0 \cdot M^{\perp,H}]$ *and* $[\beta' \cdot M^{\perp,H}]$;
3) $(\nu^\top\beta^0)\cdot M^H$, $(\nu^\top\beta')\cdot M^H$, $(\nu^\top\beta')\cdot(M - \int d\langle M\rangle\pi)^H$ *and* $\int \beta'^\top \nu d\langle M\rangle \pi$;
4) $[(\nu^\top\beta^0)\cdot M^H]$ *and* $[(\nu^\top\beta')\cdot M^H]$, $\int(\beta^0)^\top d\langle M\rangle\beta^0$, $\int \beta'^\top d\langle M\rangle \beta'$;
5) $[g\cdot M^H, (\nu^\top\beta^0)\cdot M^H]$ *and* $[g\cdot M^H, (\nu^\top\beta')\cdot M^H]$;
6) $[g\cdot M^H, \beta^0\cdot M^{\perp,H}]$ *and* $[g\cdot M^H, \beta'\cdot M^{\perp,H}]$.

Then there exists a constant $\delta > 0$ such that for every $(\Delta y, \varepsilon) \in B_\delta(0,0)$, we have

$$\frac{Y}{y}\mathcal{E}(J^{\Delta y,\varepsilon,H}) \in \mathcal{Y}(1,\varepsilon).$$

The following result provides a representation needed in the proof of Lemma 5.20.

Lemma 5.19 *Fix $x > 0$ and set $y = u_x(x, 0)$. Assume the conditions of Theorem 4.11, let β' satisfy the assumptions of Lemma 5.18, and suppose $\widetilde{L} \in \mathcal{H}^2_{\mathrm{loc}}(\mathbb{P})$ is orthogonal to every component of both M and $M^\perp$ and such that $\widetilde{L}^H$ is bounded. With $N := \beta'\cdot M^{\perp,H} + \widetilde{L}^H$, we have*

$$\begin{aligned} \mathbb{E}_{\mathbb{R}}\big[\big((G^0)'^\top\beta'\big)\cdot M^H_T\big] &= \mathbb{E}_{\mathbb{R}}\left[N_T\left(\pi\cdot\left(\phi\cdot R + M^\perp - \int d\langle M^\perp\rangle\beta\right)^\pi_T\right)\right] \\ &= \mathbb{E}_{\mathbb{R}}[N_T(\pi\cdot\widetilde{R}^\pi_T)] - \mathbb{E}_{\mathbb{R}}\left[N_T\int_0^T \pi_s^\top d\langle M\rangle_s g_s\right]. \end{aligned} \tag{5.44}$$

Proof Because β' satisfies the assumption of Lemma 5.18 and because the process $((G^0)'^\top\beta')\cdot(M - \int d\langle M\rangle\pi)^H$ is an $\mathbb{R}$-martingale, we get

$$\mathbb{E}_{\mathbb{R}}\left[\big((G^0)'^\top\beta'\big)\cdot\left(M - \int d\langle M\rangle\pi\right)^H_T\right] = 0.$$

Therefore we obtain

$$\begin{aligned}
\mathbb{E}_{\mathbb{R}}\Big[\big((G^0)'^{\top}\beta'\big)\cdot M^H_T\Big] &= \mathbb{E}_{\mathbb{R}}\Bigg[\big((G^0)'^{\top}\beta'\big)\cdot\Big(M-\int d\,\langle M\rangle\,\pi\Big)^H_T\Bigg] \\
&\quad + \mathbb{E}_{\mathbb{R}}\Bigg[\int_0^T \beta_s'^{\top}(G^0_s)'d\,\langle M\rangle_s\,\pi_s\Bigg] \\
&= \mathbb{E}_{\mathbb{R}}\Bigg[\int_0^T \beta_s'^{\top}(G^0_s)'d\,\langle M\rangle_s\,\pi_s\Bigg] \\
&= \mathbb{E}_{\mathbb{R}}\Bigg[\int_0^T \pi_s^{\top}d\,\langle M\rangle_s\,(G^0_s)'^{\top}\beta_s'\Bigg] \\
&= \mathbb{E}_{\mathbb{R}}\Bigg[\int_0^T \pi_s^{\top}\nu_s d\,\langle M\rangle_s\,\beta_s'\Bigg],
\end{aligned}$$

where the last equality can be established along the lines of the proof of Lemma 5.2. Further, using Assumption 4.5, we get

$$\begin{aligned}
\mathbb{E}_{\mathbb{R}}\Big[\big((G^0)'^{\top}\beta'\big)\cdot M^H\Big] &= \mathbb{E}_{\mathbb{R}}\Bigg[\int_0^T \pi_s^{\top}\nu_s d\,\langle M\rangle_s\,\beta_s'\Bigg] \\
&= \mathbb{E}_{\mathbb{R}}\Bigg[\int_0^T \pi_s^{\top}d\langle M^{\perp}\rangle_s\beta_s'\Bigg]. \qquad (5.45)
\end{aligned}$$

By Lemma 5.6, $\pi\cdot\widetilde{R}^{\pi}_T$ admits the representation (5.12), where both random variables $|\pi\cdot(\phi\cdot R+M^{\perp}-\int d\langle M^{\perp}\rangle\beta)^{\pi}_T|$ and $|\int_0^T\pi_s^{\top}d\,\langle M\rangle_s\,g_s|$ have exponential moments under $\mathbb{R}$, i.e., satisfy (5.13).

Next, with N as above, using that $\pi\cdot(\phi\cdot R+M^{\perp}-\int d\langle M^{\perp}\rangle\beta)^{\pi}$ and N are in $\mathcal{H}^2(\mathbb{R})$ by Lemma 5.6 and the assumption of Lemma 5.19, and $\pi\cdot M^{\perp}$ and $\beta'\cdot M^{\perp}$ are in $\mathcal{H}^2_{\mathrm{loc}}(\mathbb{P})$, one can show similarly to Lemma 5.13 that the last expression in (5.45) can be represented as

$$\mathbb{E}_{\mathbb{R}}\Bigg[\int_0^T \pi_s^{\top}d\langle M^{\perp}\rangle_s\beta_s'\Bigg] = \mathbb{E}_{\mathbb{R}}\Bigg[\Big(\pi\cdot\Big(\phi\cdot R+M^{\perp}-\int d\langle M^{\perp}\rangle\beta\Big)^{\pi}_T\Big)N_T\Bigg].$$

Combining this with (5.45), we conclude that (5.44) holds. □

The following result provides a representation needed in the proof of Lemma 5.22.

Lemma 5.20 *Fix $x>0$ and set $y=u_x(x,0)$. Assume the conditions of Theorem 4.11, let β' satisfy the assumptions of Lemma 5.18, and suppose $L\in\mathcal{H}^2_{\mathrm{loc}}(\mathbb{P})$ is orthogonal to every component of both M and $M^{\perp}$ and such that L^H is bounded. Then with*

$N := \beta' \cdot M^{\perp,H} + L^H$, we have

$$\mathbb{E}_{\mathbb{R}}\big[\big(B(Y_T)-1\big)(-g\cdot M_T^H+N_T)^2 \\ +\big((\lambda^0)''+(G^0)''^{\top}\beta+2(G^0)'^{\top}\beta'\big)\cdot M_T^H+[-g\cdot M^H+N]_T\big] \\ =\mathbb{E}_{\mathbb{R}}[B(Y_T)(-g\cdot M_T^H+N_T)^2+2N_T(\pi\cdot\widetilde{R}_T^{\pi})]+T_2, \tag{5.46}$$

where T_2 is defined in (4.11).

Proof Using Lemmas 5.5 and 5.19, we have

$$\mathbb{E}_{\mathbb{R}}\big[\big((\lambda^0)''+(G^0)''^{\top}\beta+2(G^0)'^{\top}\beta'\big)\cdot M_T^H\big] \\ =-2\mathbb{E}_{\mathbb{R}}\left[\int_0^T \pi_s^{\top}\phi_s d\langle M\rangle_s g_s\right] \\ +2\mathbb{E}_{\mathbb{R}}[N_T(\pi\cdot\widetilde{R}_T^{\pi})]-2\mathbb{E}_{\mathbb{R}}\left[N_T\int_0^T\pi_s^{\top}d\langle M\rangle_s g_s\right].$$

This allows to rewrite the left-hand side in (5.46) as

$$\mathbb{E}_{\mathbb{R}}\big[\big(B(Y_T)-1\big)(-g\cdot M_T^H+N_T)^2 \\ +\big((\lambda^0)''+(G^0)''^{\top}\beta+2(G^0)'^{\top}\beta'\big)\cdot M_T^H+[-g\cdot M^H+N]_T\big] \\ =\mathbb{E}_{\mathbb{R}}\Bigg[B(Y_T)(-g\cdot M_T^H+N_T)^2+2N_T(\pi\cdot\widetilde{R}_T^{\pi}) \\ -(-g\cdot M_T^H+N_T)^2+[-g\cdot M^H+N]_T \\ -2\int_0^T\pi_s^{\top}\phi_s d\langle M\rangle_s g_s-2N_T\int_0^T\pi_s^{\top}d\langle M\rangle_s g_s\Bigg] \\ =\mathbb{E}_{\mathbb{R}}\Bigg[B(Y_T)(-g\cdot M_T^H+N_T)^2+2N_T(\pi\cdot\widetilde{R}_T^{\pi}) \\ +\left(\int_0^T\pi_s^{\top}d\langle M\rangle_s g_s\right)^2-2\left(\int_0^T\pi_s^{\top}d\langle M\rangle_s g_s\right)(g\cdot M_T^H-N_T) \\ -2\int_0^T\pi_s^{\top}\phi_s d\langle M\rangle_s g_s-2N_T\int_0^T\pi_s^{\top}d\langle M\rangle_s g_s\Bigg] \\ =\mathbb{E}_{\mathbb{R}}\Bigg[B(Y_T)(-g\cdot M_T^H+N_T)^2+2N_T(\pi\cdot\widetilde{R}_T^{\pi}) \\ +\left(\int_0^T\pi_s^{\top}d\langle M\rangle_s g_s\right)^2-2\left(\int_0^T\pi_s^{\top}d\langle M\rangle_s g_s\right)(g\cdot M_T^H) \\ -2\int_0^T\pi_s^{\top}\phi_s d\langle M\rangle_s g_s\Bigg] \\ =\mathbb{E}_{\mathbb{R}}[B(Y_T)(-g\cdot M_T^H+N_T)^2+2N_T(\pi\cdot\widetilde{R}_T^{\pi})]+T_2.$$

□

The following result provides an integrability condition needed in the proof of Lemma 5.22.

Lemma 5.21 *Fix $x > 0$ and set $y = u_x(x, 0)$. Under the conditions of Theorem 4.11, set*

$$h^\varepsilon := \lambda^\varepsilon - \lambda^0 + (G^\varepsilon)^\top \beta, \qquad \varepsilon \in (-\widetilde{\varepsilon}, \widetilde{\varepsilon}),$$

where $\widetilde{\varepsilon}$ is as in Lemma 5.2. Then there exist $c'' > 0$ and $\bar{\varepsilon} > 0$ such that for every $\varepsilon \in (-\bar{\varepsilon}, \bar{\varepsilon})$, we have

$$\mathbb{E}_{\mathbb{R}}[\exp(c''|h^\varepsilon \cdot M^H_T| + c''[h^\varepsilon \cdot M^H]_T)] < \infty. \tag{5.47}$$

Proof For a sufficiently small positive c, set $\widetilde{M} := cg \cdot (M - \int d\langle M\rangle_s \pi_s)^H$. Then Lemma 5.1, applied under the measure $\mathbb{R}$, implies that $\mathbb{E}_{\mathbb{R}}[\exp(c|\widetilde{M}_T|)] < \infty$. Further, from boundedness of ν and ϕ, we observe that there exists a constant $\tilde{c}' > 0$ that does not depend on $\varepsilon \in (-\bar{\varepsilon}, \bar{\varepsilon})$ such that

$$\left| \int_0^T (h^\varepsilon_s)^\top d\,\langle M\rangle_s\,\pi_s \right| \leq \tilde{c}' \left| \int_0^T g_s^\top d\,\langle M\rangle_s\,\pi_s \right| \qquad \mathbb{P}\text{-a.s.},$$

which in view of Lemma 5.6 (see (5.13)) implies that there exists $c' > 0$ such that

$$\mathbb{E}_{\mathbb{R}}\left[e^{c'|\int_0^T (h^\varepsilon_s)^\top d\langle M\rangle_s \pi_s|}\right] < \infty, \qquad \varepsilon \in (-\bar{\varepsilon}, \bar{\varepsilon}). \tag{5.48}$$

Similarly, from the assumptions of Lemma 5.21, we have

$$[h^\varepsilon \cdot M^H] = [h^\varepsilon \cdot M^{H,c}] + \sum_{0<s\leq\cdot} (h^\varepsilon_s \Delta M^H_s)^2 \leq \bar{c}[g \cdot M^H] \qquad \mathbb{P}\text{-a.s.} \tag{5.49}$$

for some $\bar{c} > 0$ and every $\varepsilon \in (-\bar{\varepsilon}, \bar{\varepsilon})$. Therefore, from (5.49) and Assumption 4.9, we deduce that there exists $\bar{c}' > 0$ such that

$$\mathbb{E}_{\mathbb{R}}[e^{\bar{c}'[h^\varepsilon \cdot M^H]_T}] < \infty, \qquad \varepsilon \in (-\bar{\varepsilon}, \bar{\varepsilon}). \tag{5.50}$$

Next, from (5.50), using Lemma 5.1 with $\widetilde{M} = h^\varepsilon \cdot (M - \int d\langle M\rangle \pi)^H$ under $\mathbb{R}$, we conclude from (5.50) that there exists $\bar{c}'' > 0$ such that

$$\mathbb{E}_{\mathbb{R}}[e^{\bar{c}''|\widetilde{M}_T|}] = \mathbb{E}_{\mathbb{R}}[e^{\bar{c}''|h^\varepsilon \cdot (M - \int d\langle M\rangle_s \pi_s)^H_T|}] < \infty, \qquad \varepsilon \in (-\bar{\varepsilon}, \bar{\varepsilon}). \tag{5.51}$$

Now from (5.48) and (5.51), using Hölder's inequality gives the existence of $\tilde{c}'' > 0$ such that

$$\mathbb{E}_{\mathbb{R}}[e^{\tilde{c}''|h^\varepsilon \cdot M^H_T|}] < \infty, \qquad \varepsilon \in (-\bar{\varepsilon}, \bar{\varepsilon}).$$

This inequality and (5.50) imply (5.47). $\square$

The following result gives a second-order expansion (in both Δy and ε) associated with a family of dual elements parametrised by Δy and ε as in Lemma 5.18.

Lemma 5.22 *Fix $x > 0$ and set $y = u_x(x, 0)$. Assume the conditions of Theorem 4.11 and let β^0 and β' be as in Lemma 5.18. Then consider*

$$\widetilde{\psi}(\Delta x, \varepsilon) := \left(1 + \frac{\Delta y}{y}\right)\mathcal{E}(J^{\Delta y,\varepsilon,H})_T, \qquad (\Delta y, \varepsilon) \in B_\delta(0,0),$$

where $\delta > 0$ is chosen sufficiently close to 0 so that for every $(\Delta y, \varepsilon) \in B_\delta(0,0)$, the jumps of $J^{\Delta y,\varepsilon,H}$ in (5.43) take values in $[-\frac{1}{2}, \frac{1}{2}]$, and define

$$\widetilde{w}(\Delta x, \varepsilon) := \mathbb{E}\left[V\left(Y_T\widetilde{\psi}(\Delta y, \varepsilon)\right)\right], \qquad (\Delta x, \varepsilon) \in B_\delta(0,0).$$

Then w admits at $(0,0)$ the expansion

$$\widetilde{w}(\Delta x, \varepsilon) = \widetilde{w}(0,0) + (\Delta y \quad \varepsilon)\nabla\widetilde{w}(0,0) + \frac{1}{2}(\Delta y \quad \varepsilon)H_{\widetilde{w}}\begin{pmatrix}\Delta y\\ \varepsilon\end{pmatrix} + o(\Delta y^2 + \varepsilon^2),$$

where

$$\begin{aligned}\widetilde{w}_{\Delta y}(0,0) &= v_y(y,0),\\ \widetilde{w}_{\varepsilon}(0,0) &= xy\mathbb{E}_{\mathbb{R}}[g \cdot M^H_T],\\ H_{\widetilde{w}} &:= \begin{pmatrix}\widetilde{w}_{\Delta y\Delta y}(0,0) & \widetilde{w}_{\Delta y\varepsilon}(0,0)\\ \widetilde{w}_{\Delta y\varepsilon}(0,0) & \widetilde{w}_{\varepsilon\varepsilon}(0,0),\end{pmatrix},\end{aligned}$$

and

$$\begin{aligned}\widetilde{w}_{\Delta y\Delta y}(0,0) &= \frac{x}{y}\mathbb{E}_{\mathbb{R}}[B(Y_T)(1 + \bar{N}^y_T)^2],\\ \widetilde{w}_{\Delta y\varepsilon}(0,0) &= \frac{x}{y}\mathbb{E}_{\mathbb{R}}[B(Y_T)(1 + \bar{N}^y_T)(\bar{N}^\varepsilon_T - yg\cdot M^H_T) + y\pi\cdot\widetilde{R}^\pi_T(1 + \bar{N}^y_T)],\\ \widetilde{w}_{\varepsilon\varepsilon}(0,0) &= \frac{x}{y}\mathbb{E}_{\mathbb{R}}[B(Y_T)(\bar{N}^\varepsilon_T - yg\cdot M^H_T)^2 + 2\bar{N}^\varepsilon_T(y\pi\cdot\widetilde{R}^\pi_T)] + y^2T_2,\end{aligned}\qquad (5.52)$$

where

$$\bar{N}^y = y(\beta^0 \cdot M^{\perp,H} + \bar{L}^H) \qquad \text{and} \qquad \bar{N}^\varepsilon = y(\beta' \cdot M^{\perp,H} + \widetilde{L}^H),$$

and $\bar{L}$ and $\widetilde{L}$ are in $\mathcal{H}^2_{\mathrm{loc}}(\mathbb{P})$, orthogonal to every component of both M and $M^\perp$ and such that $\bar{N}^y$ and $\bar{N}^\varepsilon$ are bounded. (The Appendix contains more explanations behind this construction.)

Proof The proof parallels that of Lemma 5.16. In the context of the dual problem, there are some computational differences, and we have to establish the appropriate admissibility, some representations and (exponential) integrability. The admissibility of the approximating dual elements is established in Lemma 5.18; the integrability is provided by Lemma 5.21. Following the expansion, which is similar to that in Lemma 5.16, to represent $\widetilde{w}_{\varepsilon\varepsilon}(0,0)$ as in (5.52), one has to use Lemma 5.20. For the representation of $\widetilde{w}_{\Delta y \varepsilon}(0,0)$ in (5.52), similarly to the proof of Lemma 5.20, one can show that

$$\begin{aligned}&\mathbb{E}_{\mathbb{R}}\Big[-(1+\bar{N}^y_T)(-yg\cdot M^H_T+\bar{N}^\varepsilon_T)+y^2\big((G^0)'^\top\beta^0\big)\cdot M^H_T\\&\qquad+[1+\bar{N}^y,-yg\cdot M^H+\bar{N}^\varepsilon]_T\Big]\\&=\mathbb{E}_{\mathbb{R}}[y\pi\cdot\widetilde{R}^\pi_T(1+\bar{N}^y_T)].\end{aligned}$$

With these comments, the details are omitted for brevity as they are similar to the proof of Lemma 5.16. □

The following result gives an upper bound for the asymptotic behaviour of the dual value function.

Lemma 5.23 *Fix $x>0$ and set $y=u_x(x,0)$. Under the conditions of Theorem 4.11, we have*

$$\begin{aligned}v(y+\Delta y,\varepsilon)\le v(y,0)-\Delta yx+\varepsilon xy\mathbb{E}_{\mathbb{R}}[g\cdot M^H_T]\\+\frac{1}{2}(\Delta y\quad\varepsilon)H_v(y,0)\begin{pmatrix}\Delta y\\ \varepsilon\end{pmatrix}+o(\Delta y^2+\varepsilon^2),\end{aligned}$$

where $\nabla v(y,0)=(y\ \ v_\varepsilon(y,0))^\top$, $v_\varepsilon(y,0)$ is given by Theorem 4.11 and

$$H_v(y,0)=\frac{x}{y}\begin{pmatrix}b(y,y) & b(y,\varepsilon)\\ b(y,\varepsilon) & b(\varepsilon,\varepsilon)\end{pmatrix},$$

where $b(y,y)$, $b(y,\varepsilon)$, $b(\varepsilon,\varepsilon)$ are given by (4.9), (4.13) and (4.12), respectively.

The proof of Lemma 5.23 is skipped as it follows the structure of the proof of Lemma 5.17 with the corresponding modifications based on Lemmas 5.21 and 5.18.

5.5 Closing the duality gap up to the second order

Lemma 5.24 *Fix $x>0$ and set $y=u_x(x,0)$. Under the conditions of Theorem 4.11, T_1 and T_2 defined in (4.6) and (4.11), respectively, satisfy*

$$\mathbb{E}_{\mathbb{R}}\left[\frac{1}{2}(T_1+T_2)+(g\cdot M^H_T)(\pi\cdot\widetilde{R}^\pi_T)\right]=0.$$

Proof With $c:=\int_0^T\pi_s^\top d\langle M\rangle_s g_s$, direct computations give

$$\begin{aligned}&\mathbb{E}_{\mathbb{R}}\left[\frac{1}{2}(T_1+T_2)+(g\cdot M^H_T)(\pi\cdot\widetilde{R}^\pi_T)\right]\\&=\mathbb{E}_{\mathbb{R}}\Bigg[-\frac{1}{2}c^2-c\Big(\pi\cdot\Big(\phi\cdot R+M^\perp-\int d\langle M\rangle^\perp\beta\Big)^\pi_T\Big)\\&\qquad+\frac{1}{2}c^2-c(g\cdot M^H_T)-\int_0^T\pi_s^\top\phi_s d\langle M\rangle_s g_s+(g\cdot M^H_T)(\pi\cdot\widetilde{R}^\pi_T)\Bigg].\end{aligned}$$

Cancelling the $\frac{1}{2}c^2$ terms and collecting the terms $-c(g \cdot M_T^H)$ and $(g \cdot M_T^H)(\pi \cdot \widetilde{R}_T^\pi)$, we can rewrite the latter expectation as

$$\mathbb{E}_{\mathbb{R}}\Bigg[-c\bigg(\pi \cdot \Big(\phi \cdot R + M^\perp - \int d\langle M\rangle^\perp \beta\Big)_T^\pi\bigg) - \int_0^T \pi_s^\top \phi_s d\langle M\rangle_s g_s \\ + (g \cdot M_T^H)\bigg(\pi \cdot \Big(\phi \cdot R + M^\perp - \int d\langle M\rangle^\perp \beta\Big)_T^\pi\bigg)\Bigg]. \tag{5.53}$$

Next, after adding the terms $(g \cdot M_T^H)(\pi \cdot (\phi \cdot R + M^\perp - \int d\langle M\rangle^\perp \beta)_T^\pi)$ and $-c(\pi \cdot (\phi \cdot R + M^\perp - \int d\langle M\rangle^\perp \beta)_T^\pi)$, we can further rewrite (5.53) as

$$\mathbb{E}_{\mathbb{R}}\Bigg[\bigg(g \cdot \Big(M - \int d\langle M\rangle \pi\Big)_T^H\bigg)\bigg(\pi \cdot \Big(\phi \cdot R + M^\perp - \int d\langle M\rangle^\perp \beta\Big)_T^\pi\bigg) \\ - \int_0^T \pi_s^\top \phi_s d\langle M\rangle_s g_s\Bigg] = 0,$$

where the proof of the latter equality is entirely similar to the one of Lemma 5.13. It is therefore skipped for brevity. Here we remark that both $(g \cdot (M - \int d\langle M\rangle \pi)_T^H)$ and $(\pi \cdot (\phi \cdot R + M^\perp - \int d\langle M\rangle^\perp \beta)_T^\pi)$ are in $\mathcal{H}^2(\mathbb{R})$ by Lemma 5.6, and $\pi \cdot M$ and $g \cdot M$ are in $\mathcal{H}^2_{\mathrm{loc}}(\mathbb{P})$, which follows from the assumptions of Lemma 5.24. $\square$

We can now state the proof of Theorem 4.14.

Proof of Theorem 4.14 Using standard techniques of the calculus of variations, we get

$$\begin{aligned} A(X_T)(M_T^\varepsilon + x\pi \cdot \widetilde{R}_T^\pi) - xg \cdot M_T^H &= c + \widetilde{N}_T, \\ B(Y_T)(N_T^\varepsilon - yg \cdot M_T^H) + y\pi \cdot \widetilde{R}_T^\pi &= d + \widetilde{M}_T \end{aligned} \tag{5.54}$$

for some constants c and d and some $\widetilde{M} \in \mathcal{M}^2$ and $\widetilde{N} \in \mathcal{N}^2$. To compute c and d, we multiply the equations in (5.54) by $M_T^x + 1$ and $N_T^y + 1$, respectively, and take expectations under $\mathbb{R}$ to deduce that

$$\begin{aligned} c &= \mathbb{E}_{\mathbb{R}}[A(X_T)(M_T^\varepsilon + x\pi \cdot \widetilde{R}_T^\pi)(M_T^x + 1) - (xg \cdot M_T^H)(M_T^x + 1)] \\ &= a(x, \varepsilon), \\ d &= \mathbb{E}_{\mathbb{R}}[B(Y_T)(N_T^\varepsilon - yg \cdot M_T^H)(N_T^y + 1) + (y\pi \cdot \widetilde{R}_T^\pi)(N_T^y + 1)] \\ &= b(y, \varepsilon). \end{aligned} \tag{5.55}$$

From the second equation in (5.54), we deduce that

$$\begin{aligned} N_T^\varepsilon - yg \cdot M_T^H &= A(X_T)(-y\pi \cdot \widetilde{R}_T^\pi + d + \widetilde{M}_T) \\ &= dA(X_T)(1 + M_T^x) + A(X_T)(\widetilde{M}_T - dM_T^x - y\pi \cdot \widetilde{R}_T^\pi). \end{aligned} \tag{5.56}$$

Let us recall that

$$\begin{aligned} A(X_T)(M^x_T+1) &= a(x,x)(N^y_T+1),\\ B(Y_T)(N^y_T+1) &= b(y,y)(M^x_T+1), \end{aligned} \tag{5.57}$$

as proved in Kramkov and Sîrbu [30, Lemma 1]. This and (5.56) allow obtaining

$$\begin{aligned} &A(X_T)(dM^x_T-\widetilde{M}_T+y\pi\cdot\widetilde{R}^\pi_T)-yg\cdot M^H_T\\ &=da(x,x)+da(x,x)N^y_T-N^\varepsilon_T. \end{aligned} \tag{5.58}$$

Comparing to (5.54) and since the only element in $\mathcal{M}^2$ that satisfies (5.54) is M^ε, we deduce from (5.58) that

$$M^\varepsilon_T=\frac{x}{y}(dM^x_T-\widetilde{M}_T),\qquad c=d\frac{x}{y}a(x,x),\qquad \widetilde{N}_T=d\frac{x}{y}a(x,x)N^y_T-\frac{x}{y}N^\varepsilon_T.$$

Comparing with (5.55), we get

$$a(x,\varepsilon)=a(x,x)\frac{x}{y}b(y,\varepsilon). \tag{5.59}$$

Since $a(x,x)b(y,y)=1$ by [30, Lemma 1], we deduce (4.14). Next, from (5.58), plugging the expressions for c and $\widetilde{N}_T$ back into (5.54), we get

$$\begin{aligned} A(X_T)(M^\varepsilon_T+x\pi\cdot\widetilde{R}^\pi_T) &= xg\cdot M^H_T+a(x,\varepsilon)+a(x,\varepsilon)N^y_T-\frac{x}{y}N^\varepsilon_T,\\ &= a(x,\varepsilon)(1+N^y_T)-\frac{x}{y}(N^\varepsilon_T-yg\cdot M^H_T). \end{aligned}$$

Combining this with (5.57), we get (4.16), which also implies (4.17) in view of (4.14).

Let us now prove (4.15). In order to do this, let us denote

$$\begin{aligned} \bar{M}^x_T &:= M^x+1, & \bar{N}^y_T &:= N^y_T+1,\\ \bar{M}^\varepsilon_T &:= M^\varepsilon_T+x\pi\cdot\widetilde{R}^\pi_T, & \bar{N}^\varepsilon_T &:= N^\varepsilon_T-yg\cdot M^H_T. \end{aligned}$$

This and (4.16) allow writing

$$\begin{aligned} \frac{y}{x}a(\varepsilon,\varepsilon) &= \mathbb{E}_{\mathbb{R}}\left[\frac{y}{x}a(x,\varepsilon)\bar{N}^y_T\bar{M}^\varepsilon_T-\bar{N}^\varepsilon_T\bar{M}^\varepsilon_T+2M^\varepsilon_T\bar{N}^\varepsilon_T+xyT_1\right],\\ \frac{x}{y}b(\varepsilon,\varepsilon) &= \mathbb{E}_{\mathbb{R}}\left[\frac{x}{y}b(y,\varepsilon)\bar{M}^x_T\bar{N}^\varepsilon_T-\bar{M}^\varepsilon_T\bar{N}^\varepsilon_T+2N^\varepsilon_T\bar{M}^\varepsilon_T+xyT_2\right]. \end{aligned}$$

Let us denote

$$\begin{aligned} J_1 &:= \mathbb{E}_{\mathbb{R}}\left[\frac{y}{x}a(x,\varepsilon)\bar{N}^y_T\bar{M}^\varepsilon_T+\frac{x}{y}b(y,\varepsilon)\bar{M}^x_T\bar{N}^\varepsilon_T\right],\\ J_2 &:= \mathbb{E}_{\mathbb{R}}[-\bar{N}^\varepsilon_T\bar{M}^\varepsilon_T+2M^\varepsilon_T\bar{N}^\varepsilon_T+xyT_1-\bar{M}^\varepsilon_T\bar{N}^\varepsilon_T+2N^\varepsilon_T\bar{M}^\varepsilon_T+xyT_2]. \end{aligned}$$

Then we have

$$\frac{y}{x}a(\varepsilon,\varepsilon)+\frac{x}{y}b(\varepsilon,\varepsilon)=J_1+J_2. \tag{5.60}$$

One can rewrite J_1 as

$$\begin{aligned}J_1&=a(x,\varepsilon)\mathbb{E}_{\mathbb{R}}\left[\frac{y}{x}\bar{N}^y_T\bar{M}^\varepsilon_T+\frac{x}{y}\frac{b(y,\varepsilon)}{a(x,\varepsilon)}\bar{M}^x_T\bar{N}^\varepsilon_T\right]\\&=a(x,\varepsilon)\mathbb{E}_{\mathbb{R}}\left[\frac{y}{x}\bar{N}^y_T\bar{M}^\varepsilon_T+b(y,y)\bar{M}^x_T\bar{N}^\varepsilon_T\right]\\&=a(x,\varepsilon)\mathbb{E}_{\mathbb{R}}\left[\frac{y}{x}\bar{N}^y_T\bar{M}^\varepsilon_T+B(Y_T)\bar{N}^y_T\bar{N}^\varepsilon_T\right],\end{aligned}$$

where the second equality uses (5.59) and $a(x,x)b(y,y)=1$, and the third uses (5.57). As $b(y,\varepsilon)=\mathbb{E}_{\mathbb{R}}[\frac{y}{x}\bar{N}^y_T\bar{M}^\varepsilon_T+B(Y_T)\bar{N}^y_T\bar{N}^\varepsilon_T]$, we now can rewrite J_1 as

$$J_1=a(x,\varepsilon)b(y,\varepsilon). \tag{5.61}$$

Let us consider J_2. Using Lemma 5.24, we get

$$\begin{aligned}J_2&=\mathbb{E}_{\mathbb{R}}[-\bar{N}^\varepsilon_T\bar{M}^\varepsilon_T+2M^\varepsilon_T\bar{N}^\varepsilon_T-\bar{M}^\varepsilon_T\bar{N}^\varepsilon_T+2N^\varepsilon_T\bar{M}^\varepsilon_T-2xy(g\cdot M^H_T)(\pi\cdot\widetilde{R}^\pi_T)]\\&=2\mathbb{E}_{\mathbb{R}}[-\bar{N}^\varepsilon_T\bar{M}^\varepsilon_T+M^\varepsilon_T\bar{N}^\varepsilon_T+N^\varepsilon_T\bar{M}^\varepsilon_T-xy(g\cdot M^H_T)(\pi\cdot\widetilde{R}^\pi_T)]\\&=2\mathbb{E}_{\mathbb{R}}[-\bar{N}^\varepsilon_T(x\pi\cdot\widetilde{R}^\pi_T)-xy(g\cdot M^H_T)(\pi\cdot\widetilde{R}^\pi_T)+N^\varepsilon_T\bar{M}^\varepsilon_T]\\&=2\mathbb{E}_{\mathbb{R}}[-(x\pi\cdot\widetilde{R}^\pi_T)(\bar{N}^\varepsilon_T+g\cdot M^H_T)+N^\varepsilon_T\bar{M}^\varepsilon_T]\\&=2\mathbb{E}_{\mathbb{R}}[-(x\pi\cdot\widetilde{R}^\pi_T)N^\varepsilon_T+N^\varepsilon_T\bar{M}^\varepsilon_T]\\&=2\mathbb{E}_{\mathbb{R}}[(\bar{M}^\varepsilon_T-x\pi\cdot\widetilde{R}^\pi_T)N^\varepsilon_T]\\&=2\mathbb{E}_{\mathbb{R}}[M^\varepsilon_TN^\varepsilon_T]=0.\end{aligned}$$

As $J_2=0$, we get (4.15) from (5.60) and (5.61). □

The following result establishes a relationship between H_u and H_v (in (5.63)) that is needed to close the duality gap up to the second order in Δx and ε.

Lemma 5.25 *Fix $x>0$ and set $y=u_x(x,0)$. Assume the conditions of Theorem 4.15. Then for*

$$\Delta y=-\frac{y}{x}a(x,x)\Delta x-\frac{b(y,\varepsilon)}{b(y,y)}\varepsilon, \tag{5.62}$$

we have

$$2\Delta x\Delta y+\begin{pmatrix}\Delta y\\\varepsilon\end{pmatrix}^\top H_v\begin{pmatrix}\Delta y\\\varepsilon\end{pmatrix}=\begin{pmatrix}\Delta x\\\varepsilon\end{pmatrix}^\top H_u\begin{pmatrix}\Delta x\\\varepsilon\end{pmatrix}. \tag{5.63}$$

Proof First we recall that from Theorem 4.14, we have

$$b(\varepsilon,\varepsilon)=\frac{y^2}{x^2}\left(\frac{(a(x,\varepsilon))^2}{a(x,x)}-a(\varepsilon,\varepsilon)\right),\qquad b(y,\varepsilon)=\frac{a(x,\varepsilon)}{a(x,x)}\frac{y}{x}. \tag{5.64}$$

Therefore, we get

$$\begin{aligned}
&\frac{x}{y}\left(b(\varepsilon,\varepsilon)-\frac{(b(y,\varepsilon))^2}{b(y,y)}\right)\\
&=\frac{x}{y}\left(\frac{y^2}{x^2}\Big(\frac{(a(x,\varepsilon))^2}{a(x,x)}-a(\varepsilon,\varepsilon)\Big)-\frac{(b(y,\varepsilon))^2}{b(y,y)}\right)\\
&=-\frac{y}{x}a(\varepsilon,\varepsilon)+\frac{x}{y}\left(\frac{y^2}{x^2}\frac{(a(x,\varepsilon))^2}{a(x,x)}-\frac{(b(y,\varepsilon))^2}{b(y,y)}\right)\\
&=-\frac{y}{x}a(\varepsilon,\varepsilon)+\frac{x}{y}\left(\frac{y^2}{x^2}\frac{(a(x,\varepsilon))^2}{a(x,x)}-\frac{y^2}{x^2}\frac{(a(x,\varepsilon))^2}{(a(x,x))^2b(y,y)}\right)\\
&=-\frac{y}{x}a(\varepsilon,\varepsilon),
\end{aligned} \tag{5.65}$$

where the last equality uses $a(x,x)b(y,y)=1$.

Now consider the left-hand side in (5.63) and rewrite it as

$$\begin{aligned}
&2\Delta x\Delta y+\begin{pmatrix}\Delta y\\ \varepsilon\end{pmatrix}^{\top}H_v\begin{pmatrix}\Delta y\\ \varepsilon\end{pmatrix}\\
&=\frac{x}{y}b(y,y)\Delta y^2+2\frac{x}{y}b(y,\varepsilon)\Delta y\varepsilon+\frac{x}{y}b(\varepsilon,\varepsilon)\varepsilon^2+2\Delta x\Delta y\\
&=\Delta y\left(\frac{x}{y}b(y,y)\Delta y+2\frac{x}{y}b(y,\varepsilon)\varepsilon+2\Delta x\right)+\frac{x}{y}b(\varepsilon,\varepsilon)\varepsilon^2.
\end{aligned} \tag{5.66}$$

Using (5.62), we can rewrite the last line in (5.66) as

$$\begin{aligned}
&\Delta y\left(\frac{x}{y}b(y,y)\Big(-\frac{y}{x}a(x,x)\Delta x-\frac{b(y,\varepsilon)}{b(y,y)}\varepsilon\Big)+2\frac{x}{y}b(y,\varepsilon)\varepsilon+2\Delta x\right)\\
&+\frac{x}{y}b(\varepsilon,\varepsilon)\varepsilon^2.
\end{aligned} \tag{5.67}$$

As $a(x,x)b(y,y)=1$, we can simplify (5.67) to

$$\begin{aligned}
&\Delta y\left(-\Delta x-\frac{x}{y}b(y,\varepsilon)\varepsilon+2\frac{x}{y}b(y,\varepsilon)\varepsilon+2\Delta x\right)+\frac{x}{y}b(\varepsilon,\varepsilon)\varepsilon^2\\
&=-\frac{y}{x}\left(a(x,x)\Delta x+\frac{x}{y}\frac{b(y,\varepsilon)}{b(y,y)}\varepsilon\right)\left(\Delta x+\frac{x}{y}b(y,\varepsilon)\varepsilon\right)+\frac{x}{y}b(\varepsilon,\varepsilon)\varepsilon^2.
\end{aligned} \tag{5.68}$$

Rearranging terms on the right-hand side and using (5.64), we rewrite (5.68) as

$$
\begin{aligned}
&-\frac{y}{x}\left(a(x,x)\Delta x^2 + 2a(x,x)\frac{x}{y}b(y,\varepsilon)\Delta x\varepsilon + \frac{x^2}{y^2}\frac{(b(y,\varepsilon))^2}{b(y,y)}\varepsilon^2\right) + \frac{x}{y}b(\varepsilon,\varepsilon)\varepsilon^2\\
&= -\frac{y}{x}\left(a(x,x)\Delta x^2 + 2a(x,\varepsilon)\Delta x\varepsilon\right) + \frac{x}{y}\left(-\frac{(b(y,\varepsilon))^2}{b(y,y)} + b(\varepsilon,\varepsilon)\right)\varepsilon^2.
\end{aligned}
$$

Now applying (5.65), we can finally restate the right-hand side above as

$$
-\frac{y}{x}(a(x,x)\Delta x^2 + 2a(x,\varepsilon)\Delta x\varepsilon + a(\varepsilon,\varepsilon)\varepsilon^2),
$$

which is precisely the right-hand side of (5.63). □

We can now prove Theorem 4.15.

Proof of Theorem 4.15 The biconjugacy relations imply that

$$
\begin{aligned}
&u(x+\Delta x,\varepsilon) - u(x,0) - \Delta xy - \varepsilon xy\mathbb{E}_{\mathbb{R}}[\pi\cdot\widetilde{R}^{\pi}_T] - \frac{1}{2}\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}^{\top} H_u \begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}\\
&\le v(y+\Delta y,\varepsilon) + (x+\Delta x)(y+\Delta y) - v(y,0) - xy\\
&\quad -\Delta xy - \varepsilon xy\mathbb{E}_{\mathbb{R}}[\pi\cdot\widetilde{R}^{\pi}_T] - \frac{1}{2}\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}^{\top} H_u \begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}\\
&= v(y+\Delta y,\varepsilon) - v(y,0) + x\Delta y - \varepsilon xy\mathbb{E}_{\mathbb{R}}[\pi\cdot\widetilde{R}^{\pi}_T]\\
&\quad + \Delta x\Delta y - \frac{1}{2}\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}^{\top} H_u \begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}.
\end{aligned}
\tag{5.69}
$$

Further, since $-v_y(y,0) = x$ and using Lemma 5.8 gives

$$
x\Delta y - \varepsilon xy\mathbb{E}_{\mathbb{R}}[\pi\cdot\widetilde{R}^{\pi}_T] = -v_y(y,0)\Delta y - \varepsilon xy\mathbb{E}_{\mathbb{R}}[g\cdot M^H_T],
$$

we can rewrite the right-hand side of (5.69) as

$$
\begin{aligned}
&v(y+\Delta y,\varepsilon) - v(y,0) + x\Delta y - \varepsilon xy\mathbb{E}_{\mathbb{R}}[\pi\cdot\widetilde{R}^{\pi}_T]\\
&\quad + \Delta x\Delta y - \frac{1}{2}\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}^{\top} H_u \begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}\\
&= v(y+\Delta y,\varepsilon) - v(y,0) + \Delta yx - \varepsilon xy\mathbb{E}_{\mathbb{R}}[g\cdot M^H_T]\\
&\quad + \Delta x\Delta y - \frac{1}{2}\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}^{\top} H_u \begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}.
\end{aligned}
\tag{5.70}
$$

Now using Lemma 5.25 and picking Δy as in that result, we obtain

$$\begin{aligned}&v(y+\Delta y,\varepsilon)-v(y,0)-(-\Delta yx+\varepsilon xy\mathbb{E}_{\mathbb{R}}[g\cdot M_T^H])\\&\quad+\Delta x\Delta y-\frac{1}{2}\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}^\top H_u\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}\\&=v(y+\Delta y,\varepsilon)-v(y,0)-(-\Delta yx+\varepsilon xy\mathbb{E}_{\mathbb{R}}[g\cdot M_T^H])\\&\quad-\frac{1}{2}\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}^\top H_v\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix},\end{aligned}\tag{5.71}$$

and thus (5.69)–(5.71) give

$$\begin{aligned}&u(x+\Delta x,\varepsilon)-u(x,0)-\Delta xy-\varepsilon xy\mathbb{E}_{\mathbb{R}}[\pi\cdot\widetilde{R}_T^\pi]-\frac{1}{2}\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}^\top H_u\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}\\&\le v(y+\Delta y,\varepsilon)-v(y,0)-(-\Delta yx+\varepsilon xy\mathbb{E}_{\mathbb{R}}[g\cdot M_T^H])\\&\quad-\frac{1}{2}\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}^\top H_v\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}.\end{aligned}\tag{5.72}$$

Lemma 5.23 implies that

$$\begin{aligned}0&\ge v(y+\Delta y,\varepsilon)-v(y,0)-(-\Delta yx+\varepsilon xy\mathbb{E}_{\mathbb{R}}[g\cdot M_T^H])\\&\quad-\frac{1}{2}\begin{pmatrix}\Delta y\\ \varepsilon\end{pmatrix}^\top H_v\begin{pmatrix}\Delta y\\ \varepsilon\end{pmatrix}+o(\Delta y^2+\varepsilon^2),\end{aligned}\tag{5.73}$$

and from Lemma 5.17, we get

$$\begin{aligned}0&\le u(x+\Delta x,\varepsilon)-u(x,0)-\Delta xy-\varepsilon xy\mathbb{E}_{\mathbb{R}}[\pi\cdot\widetilde{R}_T^\pi]\\&\quad-\frac{1}{2}\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}^\top H_u\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix}+o(\Delta x^2+\varepsilon^2).\end{aligned}\tag{5.74}$$

Finally, (5.72)–(5.74) imply that the inequalities (5.73) and (5.74) are equalities. □

We now can state the proofs of Theorems 4.16 and 4.18.

Proof of Theorem 4.16 This proof parallels that of Mostovyi and Sîrbu [37, Theorem 4.8]; see [37, Sect. 4]. We remark that L_n^ε in [35, Sect. 4] should be replaced by $\mathcal{E}(J^{\Delta y_n,\varepsilon_n,H})$, where $\mathcal{E}(J^{\Delta y_n,\varepsilon_n,H})$ is defined in (5.43). We omit the proof for brevity of exposition. □

Proof of Theorem 4.18 This proof parallels that in Mostovyi and Sîrbu [37, Theorem 3.1] (see also the proof of [36, Theorem 4.2]). It is skipped for brevity of exposition. □

6 Characterisation of M^ε and N^ε through the Kunita–Watanabe decomposition

Instead of minimising (4.7) and (4.12), one can characterise M^ε and N^ε through a Kunita–Watanable decomposition in the case when a risk-tolerance wealth process exists. We recall that for the base model at a fixed $x > 0$, a risk-tolerance wealth process is a wealth process $\mathcal{R}$ which is maximal, i.e., its terminal value cannot be dominated by that of any nonnegative wealth process with the same initial value, and which is such that

$$\mathcal{R}_T = -\frac{U'(X_T)}{U''(X_T)}. \tag{6.1}$$

Again, similarly to the notations of the previous sections, we drop $(x, 0)$ and write $\mathcal{R}$ as in (6.1), which represents the more cumbersome but more precise expression

$$\mathcal{R}_T(x,0) = -\frac{U'(\widehat{X}_T(x,0))}{U''(\widehat{X}_T(x,0))}.$$

We refer to Kramkov and Sîrbu [31, Theorem 4] for equivalent conditions for the existence of $\mathcal{R}$. We also point out that even though NFLVR was imposed in [31], the proof of [31, Theorem 4] goes through also under NUPBR, a no-arbitrage type condition that was proved in Karatzas and Kardaras [26] to be equivalent to our standing assumption (2.10).

If the risk-tolerance wealth process for the base model exists, one can change measure and numéraire via

$$\frac{d\widetilde{\mathbb{R}}}{d\mathbb{P}} := \frac{\mathcal{R}_T Y_T}{\mathcal{R}_0 y}, \qquad S^{\mathcal{R}} := \left(\frac{\mathcal{R}_0}{\mathcal{R}}, \frac{\mathcal{R}_0\mathcal{E}(R^1)}{\mathcal{R}}, \dots, \frac{\mathcal{R}_0\mathcal{E}(R^d)}{\mathcal{R}}\right), \tag{6.2}$$

which leads to defining the orthogonal and complementary sets of square-integrable martingales under the measure $\widetilde{\mathbb{R}}$ and numéraire $S^{\mathcal{R}}$ as

$$\widetilde{\mathcal{M}}^2 := \{M \in \mathcal{H}_0^2(\widetilde{\mathbb{R}}) : M = \tilde{H} \cdot S^{\mathcal{R}} \text{ for some } S^{\mathcal{R}}\text{-integrable } \tilde{H}\}, \tag{6.3}$$

$$\widetilde{\mathcal{N}}^2 \text{ is the orthogonal complement of } \widetilde{M}^2 \text{ in } \mathcal{H}_0^2(\widetilde{\mathbb{R}}).$$

Similarly to Mostovyi and Sîrbu [37, Lemma 9.1], one can prove the following result.

Lemma 6.1 *Fix $x > 0$ and set $y = u_x(x, 0)$. Assume that (2.10) and Assumptions 2.1 and 2.2 hold and that the risk-tolerance wealth process $\mathcal{R}$ for the base model at x exists. Then we have*

$$M \in \mathcal{M}^2 \qquad \text{if and only if} \qquad M\frac{X}{\mathcal{R}} \in \widetilde{\mathcal{M}}^2$$

and

$$\widetilde{\mathcal{N}}^2 = \mathcal{N}^2.$$

The following result characterises the solutions to (4.7) and (4.12) through a Kunita–Watanabe decomposition obtained by embedding the random variable

$$g \cdot M^H_T - (\pi \cdot \widetilde{R}^\pi_T)\frac{X_T}{\mathcal{R}_T} \tag{6.4}$$

into a square-integrable martingale under the measure $\widetilde{\mathbb{R}}$ (and numéraire $\mathcal{R}$). We recall that $\tilde{R}$, defined in (2.5), has three components: the finite-variation part, the local martingale part orthogonal to M, and the local martingale part "parallel" to M. Likewise, g defined in (4.1) includes these three components of the perturbation process $\tilde{R}$. So both terms in (6.4) represent the joint effect of perturbations from the three parts of R on the perturbed problem (2.8) asymptotically at $\varepsilon = 0$. The determination of the solutions to (4.5) and (4.10) could be reduced to the best approximation of the "contingent claim" in (6.4) in the sense of the following result.

Proposition 6.2 *Fix $x > 0$ and set $y = u_x(x, 0)$. Assume the conditions of Theorem 4.11 and that the risk-tolerance wealth process $\mathcal{R}$ for the base model at x exists. Then*

$$P_t := x\mathbb{E}^{\widetilde{\mathbb{R}}}\left[\left(g \cdot M^H_T - (\pi \cdot \widetilde{R}^\pi_T)\frac{X_T}{\mathcal{R}_T}\right)\middle|\mathcal{F}_t\right], \qquad t \in [0, T],$$

satisfies $P \in \mathcal{H}^2(\widetilde{\mathbb{R}})$. Consider the Kunita–Watanabe decomposition of P given by

$$P = -a(x, \varepsilon) + \widetilde{M}^\varepsilon + \widetilde{N}^\varepsilon, \qquad \text{where } \widetilde{M}^\varepsilon \in \widetilde{\mathcal{M}}^2 \text{ and } \widetilde{N}^\varepsilon \in \widetilde{N}^2. \tag{6.5}$$

Then the solutions to (4.7) and (4.12) can be obtained by reverting to the original numérarie according to Lemma 6.1 through the identities

$$M^\varepsilon_t = \widetilde{M}^\varepsilon_t \frac{\mathcal{R}_t}{X_t}, \quad N^\varepsilon_t = \frac{y}{x}\widetilde{N}^\varepsilon_t, \qquad t \in [0, T]. \tag{6.6}$$

Proof Let us consider $\widetilde{a}(\varepsilon, \varepsilon)$ given by (4.5). By direct computations, we have

$$\begin{aligned}
&\mathbb{E}_{\mathbb{R}}[A(X_T)(M^\varepsilon_T + x\pi \cdot \widetilde{R}^\pi_T)^2 - 2M^\varepsilon_T x(g \cdot M^H_T)] \\
&= \frac{\mathcal{R}_0}{x}\mathbb{E}_{\widetilde{\mathbb{R}}}[(P_T - \widetilde{M}^\varepsilon_T)^2] + \frac{\mathcal{R}_0}{x}\mathbb{E}_{\widetilde{\mathbb{R}}}\left[x^2 g \cdot M^H_T\left(2(\pi \cdot \widetilde{R}^\pi_T)\frac{X_T}{\mathcal{R}_T} - g \cdot M^H_T\right)\right].
\end{aligned} \tag{6.7}$$

Similarly, we can rewrite $\widetilde{b}(\varepsilon, \varepsilon)$ in (4.10) as

$$\begin{aligned}
\widetilde{b}(\varepsilon, \varepsilon) &= \mathbb{E}_{\mathbb{R}}[B(Y_T)(N^\varepsilon_T - yg \cdot M^H_T)^2 + 2N_T y(\pi \cdot \widetilde{R}^\pi_T)] \\
&= \frac{\mathcal{R}_0}{x}\mathbb{E}_{\widetilde{\mathbb{R}}}\left[\left(N^\varepsilon_T - \frac{y}{x}P_T\right)^2\right] \\
&\quad + \frac{\mathcal{R}_0}{x}\mathbb{E}_{\widetilde{\mathbb{R}}}\left[y^2(\pi \cdot \widetilde{R}^\pi_T)\frac{X_T}{\mathcal{R}_T}\left(2g \cdot M^H_T - (\pi \cdot \widetilde{R}^\pi_T)\frac{X_T}{\mathcal{R}_T}\right)\right].
\end{aligned} \tag{6.8}$$

As $A(X_T) = \frac{X_T}{\mathcal{R}_T}$, taking expectations in (4.16) under $\widetilde{\mathbb{R}}$ and using Lemma 6.1 gives

$$P_0 = -a(x, \varepsilon). \tag{6.9}$$

From (6.7)–(6.9), we get (6.5). In turn, (6.6) follows from (6.7), (6.8) and Lemma 6.1. □

Remark 6.3 If the risk-tolerance wealth process does not exist, one may still attempt to choose a suitable numéraire[1] (wealth process) $\bar{\mathcal{R}}$ following the interesting ideas and construction from Kallsen and Muhle-Karbe [25]. A risk-tolerance *process* (which is not even necessarily a martingale under the dual measure, i.e., the probability measure whose density process is given by the normalised dual minimiser, provided that the latter is a true martingale, let alone a wealth process) can be defined as

$$\mathcal{R}_t := -\frac{\tilde{u}'(t, X_t)}{\tilde{u}''(t, X_t)}, \qquad t \in [0, T],$$

where $\tilde{u}$ is the indirect utility of the unperturbed problem if one starts with x at time t. Up to a drift term, under the dual measure, $\mathcal{R}$ can be decomposed into a stochastic integral with respect to the price process of the risky asset and an orthogonal part. The latter can be used to define a positive martingale $\bar{Z}$ with $\bar{Z}_0 = 1$ and such that $\bar{\mathcal{R}} := \mathcal{R}\bar{Z}$ is a *wealth process* and $\bar{\mathcal{R}}\bar{Z}$ is a martingale (under the dual measure). In other words, we have a wealth process satisfying

$$\frac{\bar{\mathcal{R}}_T}{\bar{Z}_T} = -\frac{U'(X_T)}{U''(X_T)} \qquad \text{and} \qquad \bar{\mathcal{R}}\bar{Z}Y,\ \bar{Z}Y \text{ are martingales under } \mathbb{P}.$$

One may now specify the probability measure $\tilde{\mathbb{R}}$ similarly to (6.2) and use $\tilde{\mathcal{R}}$ as numéraire. While a completion of this program would reduce our quadratic minimisation problem to Kunita–Watanabe decompositions, a rigorous analysis along these lines in the general case remains a subject for future investigation.

7 Examples

In this section, we study some example models.

7.1 Perturbations of the Black–Scholes model (with a general utility function)

We suppose that the vector-valued discounted return process R for the base model is given by

$$R_t = \mu t + \sigma^T B_t$$

[1] We should like to thank a referee for pointing out these ideas.

for some $\mu \in \mathbb{R}^d$ and an invertible matrix σ. Here, B is a d-dimensional Brownian motion. With $\widetilde{\lambda} := (\sigma^T)^{-1}\mu$, one can express B_T in terms of R_T as

$$B_T = (\sigma^T)^{-1}(R_T - \mu T) = (\sigma^T)^{-1}R_T - \widetilde{\lambda}T.$$

Then we can represent the density of the minimal martingale measure as

$$Z_T = \exp\Big(- \widetilde{\lambda}^T B_T - \frac{1}{2}|\widetilde{\lambda}|^2 T\Big) = \exp\Big(- \widetilde{\lambda}^T(\sigma^T)^{-1}R_T + \frac{1}{2}|\widetilde{\lambda}|^2 T\Big).$$

The latter formula suggests that instead of a d-dimensional state process R, we can use a one-dimensional mutual fund given by

$$\bar{R} := \widetilde{\lambda}^T(\sigma^T)^{-1}R.$$

Then we can represent $\bar{R}$ as

$$\bar{R}_t = \widetilde{\lambda}^T(\sigma^T)^{-1}(\mu t + \sigma^T B_t) = \widetilde{\lambda}^T(\widetilde{\lambda}t + B_t) = \widetilde{\lambda}^T B_t^{\mathbb{Q}}, \qquad t \in [0,T],$$

where $B^{\mathbb{Q}}$ is a Brownian motion under the minimal martingale measure $\mathbb{Q}$. In particular, $\bar{R}$ is a martingale under this measure. For a utility function satisfying Assumption 2.1 and with $y = u_x(x,0)$, we aim to replicate

$$-V'(yZ_T) = -V'\big(y\exp(-\bar{R}_T + |\widetilde{\lambda}|^2 T/2)\big).$$

Let us introduce

$$\bar{g}(z) = -V'\big(y\exp(-z + |\widetilde{\lambda}|^2 T/2)\big), \qquad z \in \mathbb{R},$$

and denote by Φ the one-dimensional heat kernel, that is,

$$\Phi(z,t) := \frac{1}{\sqrt{4\pi s}}\exp\Big(-\frac{z^2}{4s}\Big), \qquad z \in \mathbb{R}, s > 0.$$

With Φ_x denoting the partial derivative of Φ with respect to the first argument, the dynamics of $\widehat{X}(x,0)$ is given by

$$\begin{aligned}
&d\widehat{X}_t(x,0)\\
&= \Big(\int_{\mathbb{R}} \Phi_x\Big(\frac{\sqrt{2}\bar{R}_t}{|\widetilde{\lambda}|} - z, T-t\Big)\frac{\sqrt{2}}{|\widetilde{\lambda}|}\bar{g}\Big(\frac{|\widetilde{\lambda}|}{\sqrt{2}}z\Big)dz\Big)d\bar{R}_t\\
&= \widetilde{\lambda}^T(\sigma^T)^{-1}\Big(\int_{\mathbb{R}} \Phi_x\Big(\frac{\sqrt{2}\widetilde{\lambda}^T(\sigma^T)^{-1}R_t}{|\widetilde{\lambda}|} - z, T-t\Big)\frac{\sqrt{2}}{|\widetilde{\lambda}|}\bar{g}\Big(\frac{|\widetilde{\lambda}|}{\sqrt{2}}z\Big)dz\Big)dR_t.
\end{aligned}$$

In turn, the optimal proportions invested in the risky assets are

$$\begin{aligned}
\pi_t^{\top} &= \frac{\widetilde{\lambda}^T(\sigma^T)^{-1}}{\widehat{X}_T(x,0)}\\
&\quad\times\Big(\int_{\mathbb{R}} \Phi_x\Big(\frac{\sqrt{2}\widetilde{\lambda}^T(\sigma^T)^{-1}R_t}{|\widetilde{\lambda}|} - z, T-t\Big)\frac{\sqrt{2}}{|\widetilde{\lambda}|}\bar{g}\Big(\frac{|\widetilde{\lambda}|}{\sqrt{2}}z\Big)dz\Big)1_{[0,T)}(t).
\end{aligned}$$

Then the associated wealth process is given by

$$\widehat{X}(x,0) = x\mathcal{E}(\pi \cdot R).$$

Further, the risk-tolerance wealth process exists. With

$$\widetilde{h}(z) := y\exp(-z + |\widetilde{\lambda}|^2 T/2) V''\big(y\exp(-z + |\widetilde{\lambda}|^2 T/2)\big), \qquad z \in \mathbb{R},$$

the dynamics of the risk-tolerance wealth process can be represented as

$$\begin{aligned} d\mathcal{R}_t &= \bigg(\int_{\mathbb{R}} \Phi_x\Big(\frac{\sqrt{2}\bar{R}_t}{|\widetilde{\lambda}|} - z, T-t\Big)\frac{\sqrt{2}}{|\widetilde{\lambda}|}\bar{h}\Big(\frac{|\widetilde{\lambda}|}{\sqrt{2}}z\Big)dz\bigg)d\bar{R}_t \\ &= \widetilde{\lambda}^T(\sigma^T)^{-1}\bigg(\int_{\mathbb{R}} \Phi_x\Big(\frac{\sqrt{2}\widetilde{\lambda}^T(\sigma^T)^{-1}R_t}{|\widetilde{\lambda}|} - z, T-t\Big)\frac{\sqrt{2}}{|\widetilde{\lambda}|}\bar{h}\Big(\frac{|\widetilde{\lambda}|}{\sqrt{2}}z\Big)dz\bigg)dR_t, \end{aligned}$$

and the proportions invested in the risky assets are

$$\begin{aligned} \rho_t^\top &= \frac{\widetilde{\lambda}^T(\sigma^T)^{-1}}{\mathcal{R}_t} \\ &\quad \times \bigg(\int_{\mathbb{R}} \Phi_x\Big(\frac{\sqrt{2}\widetilde{\lambda}^T(\sigma^T)^{-1}R_t}{|\widetilde{\lambda}|} - z, T-t\Big)\frac{\sqrt{2}}{|\widetilde{\lambda}|}\bar{h}\Big(\frac{|\widetilde{\lambda}|}{\sqrt{2}}z\Big)dz\bigg)1_{[0,T)}(t) \end{aligned}$$

so that we have

$$\mathcal{R} = \mathcal{R}_0\mathcal{E}(\rho \cdot R).$$

For the sensitivity analysis, we consider general perturbations of R of the form

$$\widetilde{R} = \varphi \cdot B + M^\perp + \int \xi_s ds$$

for some predictable and sufficiently integrable processes φ and ξ, where ξ is $\mathbb{R}^d$-valued and φ is $d \times d$-matrix-valued and for some martingale $M^\perp$ which is orthogonal to B. To make the dynamics of $\widetilde{R}$ more consistent with the previous sections, we set $\phi := \varphi\sigma^{-1}$ and $\zeta := (\sigma^\top\sigma)^{-1}\xi$. Then we rewrite $\widetilde{R}$ as

$$\widetilde{R} = \phi \cdot (\sigma^\top B) + M^\perp + \sigma^\top\sigma \int \zeta_s ds,$$

observe that (2.2), (2.10) and Assumptions 2.2, 2.3 and 4.1 are satisfied and suppose that the remaining assumptions of Theorem 4.11 hold. In Proposition 6.2, the key role is played by the random variable

$$P_T = x\left(g \cdot M_T^H - (\pi \cdot \widetilde{R}_T^\pi)\frac{X_T}{\mathcal{R}_T}\right),$$

where in the present setting, we have $\kappa_t = \mathrm{tr}(\sigma^\top\sigma)t$, $\bar{A}_t = \frac{1}{\mathrm{tr}(\sigma^\top\sigma)}\sigma^\top\sigma$ which is invertible, and thus Assumption 4.1 holds. By direct computations, we get

$$(\lambda_t^0)' = \zeta_t - (\sigma^\top\sigma)^{-1}\phi_t\mu = (\sigma^\top\sigma)^{-1}(\xi_t - \varphi_t\sigma^{-1}\mu).$$

As $\widehat{Y}(y,0) = yZ$, $y > 0$, we deduce that $\beta = 0$. Consequently, we have

$$
\begin{aligned}
g_t &= (\lambda_t^0)' = \zeta_t - (\sigma^\top \sigma)^{-1}\phi_t \mu, \qquad t \in [0,T],\\
M_t^H &= \sigma^\top B_t - [\sigma^\top B, -\widetilde{\lambda}^\top B]_t = \sigma^\top B_t + \sigma^\top (\sigma^\top)^{-1}\mu t = R_t, \qquad t \in [0,T],\\
\widetilde{R}^\pi &= \widetilde{R} - [\widetilde{R}^c, \pi \cdot R^c] = \widetilde{R} - \int \phi_s \sigma^\top \sigma \pi_s ds,
\end{aligned}
$$

and thus we can rewrite P_T as

$$
P_T = x\bigg(\big(\zeta_t - (\sigma^\top\sigma)^{-1}\phi_t\mu\big) \cdot R_T - (\pi \cdot \widetilde{R}_T^\pi)\frac{X_T}{\mathcal{R}_T}\bigg).
$$

Similarly to the proof of Mostovyi [36, Lemma 4.1], one can show that the process

$$
R^\rho := R - [R^c, \rho \cdot R^c] - \sum_{0<s\le \cdot} \Delta R_s \frac{\rho_s \Delta R_s}{1 + \rho_s \Delta R_s}
$$

can be used to represent the elements of $\widetilde{\mathcal{M}}^2$, where $\widetilde{\mathcal{M}}^2$ is defined in (6.3). We note that in the setting of this example, the evolution of R^ρ reduces to

$$
R^\rho = R - [R, \rho \cdot R]
$$

because R is continuous. The decomposition (6.5) can be written as

$$
P = P_0 + \alpha \cdot R^\rho + N
$$

for some process α and some $N \in \widetilde{\mathcal{N}}^2$. Using (6.6) in Proposition 6.2, we can represent π^ε as a solution to

$$
\pi^\varepsilon \cdot R^\pi = \frac{1}{x}(\alpha \cdot R^\rho)\frac{\mathcal{R}_0 \mathcal{E}(\rho \cdot R)}{x\mathcal{E}(\pi \cdot R)} = \frac{\mathcal{R}_0}{x^2}(\alpha \cdot R^\rho)\mathcal{E}\big((\rho - \pi)\cdot R^\pi\big)
$$

so that

$$
\pi_t^\varepsilon = \frac{R}{x^2}\mathcal{E}\big((\rho - \pi)\cdot R^\pi\big)_t \left(\alpha_t + (\alpha \cdot R_t^\rho)(\rho_t - \pi_t)\right), \qquad t \in [0,T].
$$

In turn, π^x can be represented as

$$
\pi_t^x = \frac{\mathcal{R}_t}{\mathcal{R}_0 X_T}(\rho_t - \pi_t), \qquad t \in [0,T],
$$

and for the nearly optimal processes in the sense of Theorem 4.18, we need to truncate π^x and π^ε, that is, to use $\pi^x 1_{[\![0,\tau_n]\!]}$ and $\pi^\varepsilon 1_{[\![0,\sigma_n]\!]}$ for appropriate localising sequences of stopping times τ_n, $n \in \mathbb{N}$, and σ_n, $n \in \mathbb{N}$.

In particular, by assuming particular forms of $\widetilde{R}$, we obtain the corrections to perturbations of the finite-variation part (as in Mostovyi and Sîrbu [37]) or the martingale part or the orthogonal martingale part, which does not have to be continuous.

7.2 Distortions of exponential Lévy models

Closed-form solutions to the optimal investment problem with exponential Lévy models have been obtained in Kallsen [22]. However, once the dynamics of the underlying stock price processes is perturbed, the closed-form solutions cease to exist in general. We recall that the dynamics of the returns of d discounted risky assets in [22] is given by a Lévy process R with the characteristic triplet (b, c, F) relative to some truncation function $h\colon \mathbb{R}^d \to \mathbb{R}^d$. Below, we use $h(x) = x1_{\{|x|\leq 1\}}$. With $U(x) = \frac{x^p}{p}$, $x > 0$, where $p \in (-\infty, 0) \cup (0, 1)$, and assuming the conditions of [22, Theorem 3.2], the optimal π is constant-valued and characterised there as a solution to

$$b - (1-p)c\pi + \int_{\mathbb{R}^d} \left(\frac{x}{(1+\pi^\top x)^{1-p}} - h(x) \right) F(dx) = 0,$$

where we note that additional conditions are needed to ensure finiteness of the value function and admissibility of $\pi \cdot R$ (see conditions 1 and 2 in [22, Theorem 3.2]). Further, the structure of the dual problem is investigated in Jeanblanc et al. [20] and the proof of [22, Theorem 3.2].

With power utility, the risk-tolerance wealth process exists for every $x > 0$. Up to a multiplicative constant, $\mathcal{R}$ is equal to $\widehat{X}(1,0)$, where in turn, $\widehat{X}(x,0) = x\widehat{X}(1,0)$, $x > 0$, that is, the optimiser depends on the initial wealth in a trivial manner. Therefore, assuming the conditions of Theorem 4.11, one can see that $\widetilde{\mathbb{R}} = \mathbb{R}$ do not depend on $x > 0$. Below, we suppose that $x = 1$ for simplicity of notation. In this case, we have

$$\frac{X_T}{\mathcal{R}_T} = 1 - p.$$

We write R as in (2.1) and assume that the perturbation process has the dynamics

$$\widetilde{R} = \phi \cdot M + M^\perp + \int d\langle M \rangle \zeta,$$

for some predictable bounded matrix-valued process ϕ, orthogonal martingale $M^\perp$ and a bounded process ζ. To obtain the Kunita–Watanabe decomposition as in Proposition 6.2, under the conditions of that result, we represent P_T as

$$P_T = g \cdot M_T^H - (\pi \cdot \widetilde{R}_T^\pi)(1-p), \tag{7.1}$$

where

$$\pi \cdot \widetilde{R}^\pi = \pi \cdot \widetilde{R} - [(\phi^\top \pi) \cdot (\sigma B), \pi \cdot (\sigma B)] - \sum_{0<s\leq \cdot} \pi_s \Delta \widetilde{R}_s \frac{\pi_s \Delta R_s}{1 + \pi_s \Delta R_s}.$$

To ensure that the (local) martingale M is locally $\mathbb{P}$-square-integrable, assume that $\int_{\mathbb{R}^d} x^\top x F(dx) < \infty$, which implies that M is $\mathbb{P}$-square-integrable. Further, we have

$$\langle M \rangle_t = \left(c + \int_{\mathbb{R}^d} x x^\top F(dx) \right) t$$

and thus

$$R_t = \big(\sigma B_t + x*(\mu^R-\nu^{\mathbb{P}})_t\big) + \left(b + \int_{\mathbb{R}^d}\big(x-h(x)\big)F(dx)\right)t = M_t + \int_0^t d\,\langle M\rangle_s\,\lambda_s,$$

where $\nu^{\mathbb{P}}(dx,dt) = F(dx)dt$ is the predictable $\mathbb{P}$-compensator of μ^R, the random measure associated with the jumps of R. Therefore, assuming that the matrix $\Sigma := (c + \int_{\mathbb{R}^d} xx^\top F(dx))$ is invertible, λ can be represented as

$$\lambda = \Sigma^{-1}\left(b + \int_{\mathbb{R}^d}\big(x-h(x)\big)F(dx)\right). \tag{7.2}$$

The form of H comes from Jeanblanc et al. [20, Theorem 2.7], which asserts that

$$H = \bar{\beta}\cdot(\sigma B) + (\bar{Y}-1)*(\mu^R-\nu^{\mathbb{P}}),$$

where $\bar{\beta}\in\mathbb{R}^d$ and $\bar{Y}:\mathbb{R}^d\to\mathbb{R}_+$ are solutions to the constrained minimisation problem from [20, Theorem 2.7]. The quantities $\bar{\beta}$ and $\bar{Y}$ are also characterised in the proof of [22, Theorem 3.2]. Note that we do not suppose that the jumps of M and $M^\perp$ are not simultaneous, as was done in some of the early works on orthogonality of martingales; see e.g. Protter [39, Theorem 5.2]. Next, we have $\kappa_t = (\mathrm{tr}\Sigma)\,t$, and $\langle M\rangle_t = \Sigma t = \bar{A}\cdot\kappa_t$ produces $\bar{A} = \frac{\Sigma}{\mathrm{tr}\Sigma}$; therefore Assumption 4.1 holds. With $M^\perp$ allowing bounded $(\nu_t\beta_t)_{t\in[0,T]}$, where β is given through Assumption 2.3 and ν through Assumption 4.5, we get for g the representation

$$g_t = -\Sigma^{-1}\phi_t\Sigma\lambda_t + \zeta_t + \Sigma^{-1}\nu_t\Sigma\beta_t, \qquad t\in[0,T].$$

Therefore, using the characterisation of H in the proof of [22, Theorem 3.2], we get

$$g\cdot M^H = g\cdot M - [g\cdot(\sigma B), \bar{\beta}\cdot(\sigma B)] - \sum_{0<s\le\cdot} g_s\Delta M_s\big(1-(1+\pi_s\Delta R_s)^{1-p}\big).$$

Having bounded ζ, ϕ, $\nu\beta$ and λ, where the latter is given by (7.2), to ensure that Assumption 4.10 holds, it suffices to suppose that the jumps of both M and $M^\perp$ are bounded, plus that $1+\pi\Delta R \ge \delta > 0$.

Let us observe that $\widetilde{\mathbb{R}} = \mathbb{R}$ and P_T is given in (7.1). Assuming sufficient integrability so that Assumption 4.9 holds, we can reduce the decomposition to

$$P = P_0 + \alpha\cdot R^\pi + N$$

for some process α, and using Proposition 6.2 and Theorem 4.18, we have

$$\pi^\varepsilon = \frac{\alpha}{1-p}, \qquad \pi^x = 0,$$

where for power utility, the optimal strategy does not depend on the initial wealth and so $\pi^x = 0$, which can also be obtained by solving (4.4) for power utility. To specify the nearly optimal wealth processes in the sense of Theorem 4.18, we need to approximate π^ε along the lines of Sect. 4.7.

We conclude this section by pointing out that a similar analysis can be performed for models driven by processes with conditionally independent increments, relying on Kallsen and Muhle-Karbe [24]. Further, as mentioned in the introduction, many more stock price process models allow solutions that can be characterised explicitly. Some are developed in Goll and Kallsen [10], Guasoni and Robertson [13], Horst et al. [17], Hu et al. [18], Kallsen and Muhle-Karbe [24], Kramkov and Sîrbu [31], Liu [33], Robertson [42], Robertson et al. [1], Santacroce and Trivellato [43], Zariphopoulou [46], among others, and in some cases with general utility settings. Our results provide an approach for approximating solutions to these models even when the perturbations include jumps.

Appendix: The structure of $\mathcal{M}^\infty$ and $\mathcal{N}^\infty$

We recall that Mostovyi [36, Lemma 4.1] shows that every element of $\mathcal{M}^\infty$ can be represented as a stochastic integral with respect to R^π. The following lemma establishes the opposite direction.

Lemma A.1 *Fix $x > 0$ and set $y = u_x(x, 0)$. Suppose $M \in \mathcal{H}^2_{\mathrm{loc}}(\mathbb{P})$, that (2.2) and Assumptions 2.1 and 2.2 hold and that R^π is sigma-bounded. Then we have*

$$\mathcal{M}^\infty = \{\alpha \cdot R^\pi : \alpha \textit{ is predictable}, R^\pi\textit{-integrable and such that } \alpha \cdot R^\pi \textit{ is bounded}\}.$$

Remark A.2 The proof goes through without the sigma-boundedness assumption. The latter is imposed to ensure that $\mathcal{M}^\infty$ is non-degenerate and that the closure of $\mathcal{M}^\infty$ in $\mathcal{H}^2_0(\mathbb{R})$ is equal to $\mathcal{M}^2$. Also, the proof goes through with NUPBR or, equivalently, (2.10), instead of $M \in \mathcal{H}^2_{\mathrm{loc}}(\mathbb{P})$ and (2.2), and with the Inada conditions instead of Assumption 2.1; all we need is that together with Assumption 2.2, the standard assertions of utility maximisation theory hold.

Proof of Lemma A.1 Let α be predictable and R^π-integrable and such that $\alpha \cdot R^\pi$ is bounded. Then there exists a constant $C > 0$ such that $C + \alpha \cdot R^\pi$ is strictly positive. By Jacod and Shiryaev [19, Theorem II.8.3], there exists a predictable R^π-integrable process $\widetilde{\alpha}$ such that

$$C + \alpha \cdot R^\pi = C\mathcal{E}(\widetilde{\alpha} \cdot R^\pi) = C\frac{\mathcal{E}\left((\pi + \widetilde{\alpha}) \cdot R\right)}{\mathcal{E}\left(\pi \cdot R\right)},$$

where the second equality uses (3.2). We deduce that the bounded process $\alpha \cdot R^\pi$ admits the representation

$$\alpha \cdot R^\pi = C\frac{\mathcal{E}\left((\pi + \widetilde{\alpha}) \cdot R\right) - \mathcal{E}\left(\pi \cdot R\right)}{\mathcal{E}\left(\pi \cdot R\right)}$$

which is an element of $\mathcal{M}^\infty$ by the definition of $\mathcal{M}^\infty$. As α was arbitrary, the proof is complete. □

Lemma A.3 *Fix $x > 0$ and set $y = u_x(x, 0)$. Impose the assumptions of Lemma A.1 and that both M and H are in $\mathcal{H}^2_{\mathrm{loc}}(\mathbb{P})$. Recall that H is defined in Assumption 2.3 and is such that $Y = y\mathcal{E}(H)$. Then we have*

$$\mathcal{N}^\infty = \{N^H : N^H \text{ is bounded, } N \in \mathcal{H}^2_{\mathrm{loc}}(\mathbb{P}) \\ \text{and } N \text{ is orthogonal to each component of } M\}.$$

Proof Take $N \in \mathcal{H}^2_{\mathrm{loc}}(\mathbb{P})$ such that N^H is bounded and fix $\widetilde{M} \in \mathcal{M}^\infty$. By [36, Lemma 4.1], we have $\widetilde{M} = \alpha \cdot R^\pi$ for some predictable R^π-integrable process α. Let us approximate α by the M-integrable processes

$$\alpha^n := (-n \vee \alpha \wedge n)1_{[\![0,\tau_n]\!]}, \qquad n \in \mathbb{N},$$

where every component of α is truncated from above by n and below by $-n$ and where τ_n, $n \in \mathbb{N}$, is a localising sequence for both M and N. Then for a fixed $n \in \mathbb{N}$ and every stopping time τ, similarly to Lemma 5.13, we get

$$\mathbb{E}_{\mathbb{R}}[(\alpha^n \cdot R^\pi_\tau)N^H_\tau] = \mathbb{E}_{\mathbb{R}}[\langle \alpha^n \cdot M, N\rangle_{\tau_n \wedge \tau}] = 0. \tag{A.1}$$

As N^H is bounded and $\alpha^n \cdot R^\pi$, $n \in \mathbb{N}$, converges to $\alpha \cdot R^\pi$ in $\mathcal{H}^2(\mathbb{R})$, we deduce from (A.1) that N^H is orthogonal to $\alpha \cdot R^\pi$. Now from Lemma A.1, we deduce that N^H is orthogonal to $\mathcal{M}^\infty$. Since additionally, the closure of $\mathcal{M}^\infty$ in $\mathcal{H}^2_0(\mathbb{R})$ is equal to $\mathcal{M}^2$ by Kramkov and Sîrbu [30, Lemma 6], we get

$$\mathcal{N}^\infty \supseteq \{N^H : N^H \text{ is bounded, } N \in \mathcal{H}^2_{\mathrm{loc}}(\mathbb{P}) \\ \text{and } N \text{ is orthogonal to each component of } M\}.$$

To show the opposite inclusion, we proceed as follows. We fix $K \in \mathcal{N}^\infty$ and set $N := K + [K, H] = K^{-H}$. Then N is locally square-integrable under $\mathbb{P}$ because K is bounded and H is locally $\mathbb{P}$-square-integrable. We suppose that $\mathcal{E}(K) > 0$, as otherwise we may multiply K by a sufficiently small constant ε and conduct the proof for εK.

For $\alpha = -\pi$, as $\mathcal{E}(\alpha \cdot R^\pi) = \frac{1}{\mathcal{E}(\pi \cdot R)} > 0$, using the sigma-boundedness of R^π and the Ansel–Stricker theorem (see [8, Corollary 7.3.8]), we deduce that $\mathcal{E}(\alpha \cdot R^\pi)$ is a local martingale under $\mathbb{R}$ and hence in $\mathcal{H}^1_{\mathrm{loc}}(\mathbb{R})$. Let σ_n, $n \in \mathbb{N}$, be a localising sequence for $\mathcal{E}(\alpha \cdot R^\pi)$ such that on $[\![0, \sigma_n]\!]$, $\mathcal{E}(\alpha \cdot R^\pi)$ is in $\mathcal{H}^1(\mathbb{R})$, where by boundedness of K, we also suppose that $\mathcal{E}(K)$ is bounded on $[\![0, \sigma_n]\!]$. Further, as R^π is sigma-bounded, we can use [30, Theorem 4] to approximate $\mathcal{E}(\alpha \cdot R^\pi)$ on $[\![0, \sigma_n]\!]$ in $\mathcal{H}^1(\mathbb{R})$ by some bounded stochastic integrals with respect to R^π. These integrals are in $\mathcal{M}^\infty$ by Lemma A.1. By convergence in $\mathcal{H}^1(\mathbb{R})$ and boundedness of $\mathcal{E}(K)$ on $[\![0, \sigma_n]\!]$, we obtain

$$\begin{aligned}\mathbb{E}_{\mathbb{R}}[\mathcal{E}(\alpha \cdot R^\pi)_{t\wedge\sigma_n}\mathcal{E}(K)_{t\wedge\sigma_n}|\mathcal{F}_{s\wedge\sigma_n}] &= \mathcal{E}(\alpha \cdot R^\pi)_{s\wedge\sigma_n}\mathcal{E}\,(K)_{s\wedge\sigma_n} \\ &= \frac{\mathcal{E}\,(N+H)_{s\wedge\sigma_n}}{\mathcal{E}(H)_{s\wedge\sigma_n}\mathcal{E}(\pi \cdot R)_{s\wedge\sigma_n}},\end{aligned} \tag{A.2}$$

and by a change of measure, we have

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{R}}[\mathcal{E}(\alpha \cdot R^{\pi})_{t\wedge\sigma_n}\mathcal{E}(K)_{t\wedge\sigma_n}|\mathcal{F}_{s\wedge\sigma_n}] \\
&= \mathbb{E}\left[\frac{\mathcal{E}(H)_{t\wedge\sigma_n}\mathcal{E}(\pi \cdot R)_{t\wedge\sigma_n}}{\mathcal{E}(H)_{s\wedge\sigma_n}\mathcal{E}(\pi \cdot R)_{s\wedge\sigma_n}}\mathcal{E}(\alpha \cdot R^{\pi})_{t\wedge\sigma_n}\mathcal{E}(N^H)_{t\wedge\sigma_n}\middle|\mathcal{F}_{s\wedge\sigma_n}\right] \\
&= \mathbb{E}\left[\frac{\mathcal{E}(H)_{t\wedge\sigma_n}}{\mathcal{E}(H)_{s\wedge\sigma_n}\mathcal{E}(\pi \cdot R)_{s\wedge\sigma_n}}\frac{\mathcal{E}(N+H)_{t\wedge\sigma_n}}{\mathcal{E}(H)_{t\wedge\sigma_n}}\middle|\mathcal{F}_{s\wedge\sigma_n}\right] \\
&= \frac{1}{\mathcal{E}(H)_{s\wedge\sigma_n}\mathcal{E}(\pi \cdot R)_{s\wedge\sigma_n}}\mathbb{E}[\mathcal{E}(N+H)_{t\wedge\sigma_n}|\mathcal{F}_{s\wedge\sigma_n}].
\end{aligned}
\tag{A.3}
$$

Comparing (A.2) and (A.3), we deduce that $\mathcal{E}(N+H)$ is a local martingale under $\mathbb{P}$. As $\mathcal{E}(N+H)$ and $\mathcal{E}(N+H)_-$ are both non-vanishing by construction because $\mathcal{E}(N+H) = \mathcal{E}(K)\mathcal{E}(H)$, the stochastic logarithm of $\mathcal{E}(N+H)$ is well defined by [19, Theorem II.8.3] and is equal to $N+H$ by [19, Corollary II.8.7]. Further, from [19, Theorem II.8.3] and Protter [40, Theorem III.29], we conclude that $N+H$ is a local martingale under $\mathbb{P}$. Moreover, N is locally $\mathbb{P}$-square-integrable as seen in the preceding paragraph and $H \in \mathcal{H}^2_{\mathrm{loc}}(\mathbb{P})$ by assumption, and so we conclude that $N \in \mathcal{H}^2_{\mathrm{loc}}(\mathbb{P})$.

Second, we show that N is orthogonal to M. For this, take an element of $\mathcal{M}^{\infty}$ of the form $\alpha \cdot R^{\pi}$, where by sigma-boundedness, we suppose that each component of α takes values in $(0,1]$. Choose a stopping time σ such that $\mathbb{E}[\int_0^{\sigma} \alpha_s^{\top} d\langle M\rangle_s \alpha_s] < \infty$ and $\mathbb{E}[\langle N\rangle_{\sigma}] < \infty$. Then as $K = N^H \in \mathcal{N}^{\infty}$, similarly to the proof of Lemma 5.13, we have

$$
0 = \mathbb{E}_{\mathbb{R}}\big[[\alpha \cdot R^{\pi}, N^H]_{\sigma}\big] = \mathbb{E}_{\mathbb{R}}[\langle \alpha \cdot M, N\rangle_{\sigma}]. \tag{A.4}
$$

From (A.4), we deduce that $\langle \alpha \cdot M, N\rangle$ is an $\mathbb{R}$-martingale on $[\![0,\sigma]\!]$. Further, since $\langle \alpha \cdot M, N\rangle$ is predictable and of finite variation, we deduce that $\langle \alpha \cdot M, N\rangle \equiv 0$ on $[\![0,\sigma]\!]$. As M and N are locally square-integrable and each component of α in the previous paragraph was $(0,1]$-valued, thus non-vanishing, we can deduce by localisation that each component of M is orthogonal to N on $[0,T]$. □

Declarations

Competing Interests The authors declare no competing interests.

References

1. Anthropelos, M., Robertson, S., Spiliopoulos, K.: The pricing of contingent claims and optimal positions in asymptotically complete markets. Ann. Appl. Probab. **27**, 1778–1830 (2017)
2. Backhoff Veraguas, J., Silva, F.: Sensitivity analysis of expected utility maximization in incomplete Brownian market models. Math. Financ. Econ. **12**, 387–411 (2018)
3. Bank, P., Körber, L.: Merton's optimal investment problem with jump signals. SIAM J. Financ. Math. **13**, 1302–1325 (2022)
4. Barrieu, P., El Karoui, N.: Monotone stability of quadratic semimartingales with applications to unbounded general quadratic BSDEs. Ann. Probab. **41**, 1831–1863 (2013)

5. Černý, A., Kallsen, J.: On the structure of general mean–variance hedging strategies. Ann. Probab. **35**, 1479–1531 (2007)
6. Czichowsky, C., Schweizer, M.: Cone-constrained continuous-time Markowitz problems. Ann. Appl. Probab. **23**, 764–810 (2013)
7. Delbaen, F., Schachermayer, W.: A general version of the fundamental theorem of asset pricing. Math. Ann. **300**, 463–520 (1994)
8. Delbaen, F., Schachermayer, W.: The Mathematics of Arbitrage. Springer, Berlin (2006)
9. Föllmer, H., Schweizer, M.: The minimal martingale measure. In: Cont, R. (ed.) Encyclopedia of Quantitative Finance, pp. 1200–1204. Wiley, New York (2010)
10. Goll, T., Kallsen, J.: A complete explicit solution to the log-optimal portfolio problem. Ann. Appl. Probab. **13**, 774–799 (2003)
11. Golub, G.H., Van Loan, C.F.: Matrix Computations, 3rd edn. Johns Hopkins University Press, Baltimore (1996)
12. Gouriéroux, C., Laurent, J., Pham, H.: Mean–variance hedging and numéraire. Math. Finance **8**, 179–200 (1998)
13. Guasoni, P., Robertson, S.: Static fund separation of long-term investments. Math. Finance **25**, 789–826 (2015)
14. Henderson, V.: Valuation of claims on nontraded assets using utility maximization. Math. Finance **12**, 351–373 (2002)
15. Henderson, V., Hobson, D.: Real options with constant relative risk aversion. J. Econ. Dyn. Control **27**, 329–355 (2002)
16. Herrmann, S., Muhle-Karbe, J., Seifried, F.T.: Hedging with small uncertainty aversion. Finance Stoch. **21**, 1–64 (2017)
17. Horst, U., Hu, Y., Imkeller, P., Réveillac, A., Zhang, J.: Forward–backward systems for expected utility maximization. Stoch. Process. Appl. **124**, 1813–1848 (2014)
18. Hu, Y., Imkeller, P., Müller, M.: Utility maximization in incomplete markets. Ann. Appl. Probab. **15**, 1691–1712 (2005)
19. Jacod, J., Shiryaev, A.N.: Limit Theorems for Stochastic Processes, 2nd edn. Springer, Berlin (2003)
20. Jeanblanc, M., Klöppel, S., Miyahara, Y.: Minimal f^q-martingale measure for exponential Lévy processes. Ann. Appl. Probab. **17**, 1615–1638 (2007)
21. Jeanblanc, M., Mania, M., Santacroce, M., Schweizer, M.: Mean–variance hedging via stochastic control and BSDEs for general semimartingales. Ann. Appl. Probab. **22**, 2388–2428 (2012)
22. Kallsen, J.: Optimal portfolios for exponential Lévy processes. Math. Methods Oper. Res. **51**, 357–374 (2000)
23. Kallsen, J.: Derivative pricing based on local utility maximization. Finance Stoch. **6**, 115–140 (2002)
24. Kallsen, J., Muhle-Karbe, J.: Utility maximization in models with conditionally independent increments. Ann. Appl. Probab. **20**, 2162–2177 (2010)
25. Kallsen, J., Muhle-Karbe, J.: The general structure of optimal investment and consumption with small transaction costs. Math. Finance **27**, 659–703 (2017)
26. Karatzas, I., Kardaras, C.: The numéraire portfolio in semimartingale financial models. Finance Stoch. **11**, 447–493 (2007)
27. Karatzas, I., Kardaras, C.: Portfolio Theory and Arbitrage: A Course in Mathematical Finance. Am. Math. Soc., Providence (2021)
28. Kramkov, D., Schachermayer, W.: The asymptotic elasticity of utility functions and optimal investment in incomplete markets. Ann. Appl. Probab. **9**, 904–950 (1999)
29. Kramkov, D., Schachermayer, W.: Necessary and sufficient conditions in the problem of optimal investment in incomplete markets. Ann. Appl. Probab. **13**, 1504–1516 (2003)
30. Kramkov, D., Sîrbu, M.: On the two-times differentiability of the value functions in the problem of optimal investment in incomplete markets. Ann. Appl. Probab. **16**, 1352–1384 (2006)
31. Kramkov, D., Sîrbu, M.: Sensitivity analysis of utility-based prices and risk-tolerance wealth process. Ann. Appl. Probab. **16**, 2140–2194 (2006)
32. Laurent, J.P., Pham, H.: Dynamic programming and mean–variance hedging. Finance Stoch. **3**, 83–110 (1999)
33. Liu, J.: Portfolio selection in stochastic environments. Rev. Financ. Stud. **20**, 1–39 (2007)
34. Monoyios, M.: Malliavin calculus method for asymptotic expansion of dual control problems. SIAM J. Financ. Math. **4**, 884–915 (2013)
35. Mostovyi, O.: Necessary and sufficient conditions in the problem of optimal investment with intermediate consumption. Finance Stoch. **19**, 135–159 (2015)

36. Mostovyi, O.: Asymptotic analysis of the expected utility maximization problem with respect to perturbations of the numéraire. Stoch. Process. Appl. **130**, 4444–4469 (2020)
37. Mostovyi, O., Sîrbu, M.: Sensitivity analysis of the utility maximization problem with respect to model perturbations. Finance Stoch. **23**, 595–640 (2019)
38. Pham, H., Rheinländer, T., Schweizer, M.: Mean–variance hedging for continuous processes: new proofs and examples. Finance Stoch. **2**, 173–198 (1998)
39. Protter, P.: A comparison of stochastic integrals. Ann. Probab. **7**, 276–289 (1979)
40. Protter, P.: Stochastic Integration and Differential Equations. Springer, Berlin (2004)
41. Robertson, J., Rosenberg, M.: The decomposition of matrix-valued measures. Mich. Math. J. **15**, 353–368 (1968)
42. Robertson, S.: Pricing for large positions in contingent claims. Math. Finance **27**, 746–778 (2017)
43. Santacroce, M., Trivellato, B.: Forward backward semimartingale systems for utility maximization. SIAM J. Control Optim. **52**, 3517–3537 (2014)
44. Schweizer, M.: Martingale densities for general asset prices. J. Math. Econ. **21**, 363–378 (1992)
45. Schweizer, M.: On the minimal martingale measure and the Föllmer–Schweizer decomposition. Stoch. Anal. Appl. **13**, 573–599 (1995)
46. Zariphopoulou, T.: A solution approach to valuation with unhedgeable risks. Finance Stoch. **5**, 61–82 (2001)