



Tutorial article

Stochastic linear-quadratic optimal control problems — Recent developments

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ABSTRACT

In this paper, we present a brief survey for some recent developments of stochastic linear-quadratic optimal controls. We mainly concentrate on the results obtained by the authors and their collaborators in the last decay.

1. Introduction

Let us consider the following controlled linear ordinary differential equation (ODE, for short)

$$\begin{cases} \dot{X}(t) = AX(t) + Bu(t), & t \in [0, T], \\ X(0) = x, \end{cases} \quad (1.1)$$

where $X(\cdot)$ is the *state trajectory* valued in $\mathbb{R}^n$ with x being the *initial state*, and $u(\cdot)$ is a *control function* valued in $\mathbb{R}^m$; $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are given constant matrices. We refer to the above as the *state equation*. Clearly, for any $x \in \mathbb{R}^n$ and $u(\cdot) \in \mathcal{U}[0, T]$, the set of all $\mathbb{R}^m$ -valued, square-integrable functions, the state equation (1.1) admits a unique solution $X(\cdot) \equiv X(\cdot; x, u(\cdot))$. To measure the performance of the control, one can introduce the following quadratic *cost functional*:

$$J(x; u(\cdot)) \triangleq \int_0^T [\langle QX(t), X(t) \rangle + \langle Ru(t), u(t) \rangle] dt + \langle GX(T), X(T) \rangle, \quad (1.2)$$

where $Q, G \in \mathbb{S}^n$, the set of all $(n \times n)$ symmetric real matrices, and $R \in \mathbb{S}^m$. The two terms on the right-hand side of the above are called the

running cost and the *terminal cost*, respectively. Then the classical linear-quadratic (LQ, for short) optimal control problem associated with the above linear state equation (1.1) and quadratic cost functional (1.2) can be formulated as follows.

Problem (DLQ). For any $x \in \mathbb{R}^n$, find a $\bar{u}(\cdot) \in \mathcal{U}[0, T]$ such that

$$J(x; \bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[0, T]} J(x; u(\cdot)). \quad (1.3)$$

If $\bar{u}(\cdot) \in \mathcal{U}[0, T]$ exists satisfying (1.3), we call it an *optimal control*, the corresponding $\bar{X}(\cdot) \equiv X(\cdot; x, \bar{u}(\cdot))$ and the pair $(\bar{X}(\cdot), \bar{u}(\cdot))$ are called an *optimal trajectory* and an *optimal pair*, respectively. The above problem is referred to as the deterministic *linear-quadratic optimal control problem* (LQ problem, for short). The following collects the most basic conclusions for Problem (DLQ).

Proposition 1.1. For Problem (DLQ), the following hold:

(i) Suppose that Problem (DLQ) admits an optimal pair $(\bar{X}(\cdot), \bar{u}(\cdot))$. Then it is necessary that

$$R \geq 0, \quad (1.4)$$

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and the cost functional $J(x; u(\cdot))$ is convex in the control function $u(\cdot)$. Moreover, there exists a solution to the following ODE, called the adjoint equation:

$$\begin{cases} \dot{\bar{Y}}(t) = -[A^\top \bar{Y}(t) + Q \bar{X}(t)], & t \in [0, T], \\ \bar{Y}(T) = G \bar{X}(T), \end{cases} \quad (1.5)$$

such that the following stationarity condition holds:

$$B^\top \bar{Y}(t) + R \bar{u}(t) = 0, \quad \forall t \in [0, T]. \quad (1.6)$$

(ii) Suppose that the optimality system

$$\begin{cases} \dot{\bar{X}}(t) = A \bar{X}(t) + B \bar{u}(t), \\ \dot{\bar{Y}}(t) = -[A^\top \bar{Y}(t) + Q \bar{X}(t)], \\ \bar{X}(0) = x, \quad \bar{Y}(T) = G \bar{X}(T), \\ B^\top \bar{Y}(t) + R \bar{u}(t) = 0 \end{cases} \quad (1.7)$$

admits a solution $(\bar{X}(\cdot), \bar{u}(\cdot), \bar{Y}(\cdot))$, and that the mapping $u(\cdot) \mapsto J(x; u(\cdot))$ is convex. Then Problem (DLQ) admits an optimal control.

(iii) Let $R > 0$ and suppose that the following differential Riccati equation has a solution $P(\cdot)$:

$$\begin{cases} \dot{P}(t) + P(t)A + A^\top P(t) - P(t)BR^{-1}B^\top P(t) + Q = 0, \\ P(T) = G. \end{cases} \quad (1.8)$$

Then the mapping $u(\cdot) \mapsto J(x; u(\cdot))$ is uniformly convex, and Problem (DLQ) admits a unique optimal control $\bar{u}(\cdot)$, which has the following closed-loop representation:

$$\bar{u}(t) = -R^{-1}B^\top P(t)\bar{X}(t), \quad t \in [0, T]. \quad (1.9)$$

The above will be the case if the following canonical condition holds:

$$G, Q \geq 0, \quad R > 0. \quad (1.10)$$

The study of LQ problems for ODEs began with the seminal works of Bellman, Glicksberg, and Gross (1958), Kalman (1960) and Letov (1961) appeared around 1960. Standard results for LQ theory of ODEs, including the above, can be found in Lee and Markus (1967), Anderson and Moore (1971), Willems (1971), Wonham (1979), and so on. See also Yong and Zhou (1999).

From the above results, we see that associated with the LQ problem, there are several notions closely related: Existence (and uniqueness) of optimal control, (uniform) convexity of the cost functional, solvability of the optimality system (two-point boundary value problem), well-posedness of the differential Riccati equation, possibility of the closed-loop representation of the optimal control. It seems that they are almost equivalent somehow. It is a desire to make these relations clear.

The stochastic LQ problem is concerned with making optimal control in the presence of randomness that is typically encountered in most real-life problems. Its rich mathematical theory has found numerous applications, within engineering, finance and economics, management sciences, and so on. A typical example of the stochastic LQ problem is the mean-variance model, which was formulated as a quadratic programming problem by Markowitz in his Nobel-prize-winning work (Markowitz, 1952) in 1952.

The study of stochastic LQ problems began with the works of Kushner (1962) and Wonham (1968) in the 1960s. See also Davis (1977), Bensoussan (1981), and so on, for classical stochastic LQ theory. In 1998, Chen, Li, and Zhou (1998) found that for stochastic LQ problems, the weighting matrices in the cost functional could be indefinite to some extent; in particular, (1.4) is not necessary for the existence of an optimal control for the corresponding stochastic LQ problem. See Yong and Zhou (1999) for some presentation of the updated theory by the end of the last century. Since the early 2010s, the authors of this paper, together with their collaborators, started to investigate the LQ problems for stochastic differential equations (SDEs, for short) from a different angle. Notions of finiteness, open-loop and closed-loop solvability were

introduced, together with the relationship among them and to the other relevant notions, as well as their characterizations. More precisely, the following have been established for stochastic LQ problems in finite time-horizons, denoted by Problem (SLQ) $_T$:

- If Problem (SLQ) $_T$ is finite, then the cost functional is convex in the control process.
- The closed-loop solvability implies the open-loop solvability, but not vice versa, in general.
- The open-loop solvability is equivalent to the solvability of the optimality system, which is now a forward-backward stochastic differential equation (FBSDE, for short), plus the convexity of the cost functional.
- The closed-loop solvability is equivalent the regular solvability of a differential Riccati equation, which implicitly implies that the cost functional is convex (in the control process). In this case, the problem is also open-loop solvable, and any open-loop optimal control admits a closed-loop representation (or called a state-feedback representation), which must be the outcome of some closed-loop optimal strategy.
- If the cost functional is uniformly convex in the control process, then Problem (SLQ) $_T$ is uniquely closed-loop solvable and hence also uniquely open-loop solvable.

For stochastic LQ problems in the infinite time-horizon, denoted by Problem (SLQ) $_\infty$, under the square integrability of the nonhomogeneous terms and the linear weight processes, the following results have been established:

- Problem (SLQ) $_\infty$ is well-formulated for every initial state if and only if the homogeneous controlled system is L^2 -stabilizable, which is equivalent to the existence of a positive definite solution to an algebraic Riccati equation (ARE, for short).
- The admissible control sets are characterized explicitly, for different initial conditions.
- Under the stabilizability condition, the following are equivalent:
 - * Problem (SLQ) $_\infty$ is open-loop solvable;
 - * Problem (SLQ) $_\infty$ is closed-loop solvable;
 - * An ARE admits a stabilizing solution, and an FBSDE over $[0, \infty)$ admits an L^2 -stable adapted solution satisfying a certain condition.

The purpose of this paper is to give a brief survey for the above developments. In the rest of the paper, we will first formulate the stochastic LQ problem in both finite and infinite time-horizons in Section 2. Then in Section 3, we investigate the finite time-horizon problem from both open-loop and closed-loop point of views. In Section 4, we further discuss the case of infinite time-horizon, and in Section 5 we give some conclusion remarks.

2. Stochastic LQ problems

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $W = \{W(t); t \geq 0\}$ is defined, whose natural filtration augmented by all the $\mathbb{P}$ -null sets is $\mathbb{F}$. For a random variable ξ , we write $\xi \in \mathcal{F}_t$ if ξ is $\mathcal{F}_t$ -measurable; and for a stochastic process X , we write $X \in \mathbb{F}$ if it is progressively measurable with respect to the filtration $\mathbb{F}$. Let $\mathbb{H}$ be a Euclidean space (which could be $\mathbb{R}^n$, $\mathbb{R}^{n \times m}$, etc.). We define for $0 \leq s < T < \infty$,

$$\begin{aligned} C([s, T]; \mathbb{H}) &\triangleq \left\{ \varphi : [s, T] \rightarrow \mathbb{H} \mid \varphi(\cdot) \text{ is continuous} \right\}, \\ L^2(s, T; \mathbb{H}) &\triangleq \left\{ \varphi : [s, T] \rightarrow \mathbb{H} \mid \int_s^T |\varphi(t)|^2 dt < \infty \right\}, \\ L^2_{\mathcal{F}_s}(\Omega; \mathbb{H}) &\triangleq \left\{ \xi : \Omega \rightarrow \mathbb{H} \mid \xi \in \mathcal{F}_s \text{ and } \mathbb{E}|\xi|^2 < \infty \right\}, \\ L^2_{\mathbb{F}}(s, T; \mathbb{H}) &\triangleq \left\{ \varphi : [s, T] \times \Omega \rightarrow \mathbb{H} \mid \varphi \in \mathbb{F} \text{ and } \mathbb{E} \int_s^T |\varphi(t)|^2 dt < \infty \right\}, \\ \mathcal{D} &\triangleq \left\{ (s, \xi) \mid s \in [0, \infty), \xi \in L^2_{\mathcal{F}_s}(\Omega; \mathbb{R}^n) \right\}. \end{aligned}$$

In particular, we let $\mathcal{U}[s, T] \triangleq L^2_{\mathbb{F}}(s, T; \mathbb{R}^m)$ and

$$\mathcal{U}[0, \infty) \triangleq \left\{ \varphi : [0, \infty) \times \Omega \rightarrow \mathbb{R}^m \mid \varphi \in \mathbb{F} \text{ and } \mathbb{E} \int_0^\infty |\varphi(t)|^2 dt < \infty \right\}.$$

Now, consider the following controlled linear stochastic differential equation (SDE, for short):

$$\begin{cases} dX(t) = [A(t)X(t) + B(t)u(t) + b(t)]dt \\ \quad + [C(t)X(t) + D(t)u(t) + \sigma(t)]dW(t), \quad t \geq s, \\ X(s) = \xi. \end{cases} \quad (2.1)$$

In the above, $X(\cdot)$ is the state process, valued in $\mathbb{R}^n$, $u(\cdot)$ is a control process taken from $\mathbb{R}^m$, and $(s, \xi) \in \mathcal{D}$ is called an *initial pair*. For the coefficients $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$, and the non-homogeneous terms $b(\cdot)$ and $\sigma(\cdot)$, we introduce the following hypothesis.

(H2.1) The coefficients $A, C : [0, \infty) \rightarrow \mathbb{R}^{n \times n}$, $B, D : [0, \infty) \rightarrow \mathbb{R}^{n \times m}$ are all deterministic, Lebesgue measurable and bounded; the nonhomogeneous terms $b, \sigma : [0, \infty) \times \Omega \rightarrow \mathbb{R}^n$ are $\mathbb{F}$ -progressively measurable and square integrable on $[0, \infty)$.

It is clear that under (H2.1), for any given initial pair $(s, \xi) \in \mathcal{D}$ and any control $u(\cdot) \in \mathcal{U}[s, T]$ ($0 \leq s < T < \infty$), the state equation (2.1) admits a unique strong solution $X(\cdot) \equiv X(\cdot; s, \xi, u(\cdot))$ on $[s, T]$. To measure the performance of the control $u(\cdot)$ over $[s, T]$, we may introduce the following quadratic cost functional:

$$\begin{aligned} J_T(s, \xi; u(\cdot)) \triangleq & \mathbb{E} \left\{ \int_s^T \left[\langle Q(t)X(t), X(t) \rangle + 2\langle S(t)X(t), u(t) \rangle \right. \right. \\ & + \langle R(t)u(t), u(t) \rangle + 2\langle q(t), X(t) \rangle + 2\langle r(t), u(t) \rangle \Big] dt \\ & \left. + \langle G X(T), X(T) \rangle + 2\langle g, X(T) \rangle \right\}, \end{aligned} \quad (2.2)$$

where we assume the following:

(H2.2) The quadratic weighting matrices $Q : [0, \infty) \rightarrow \mathbb{S}^n$, $S : [0, \infty) \rightarrow \mathbb{R}^{m \times n}$, $R : [0, \infty) \rightarrow \mathbb{S}^m$ are deterministic, Lebesgue measurable and bounded; the linear weighting processes $q(\cdot) : [0, \infty) \times \Omega \rightarrow \mathbb{R}^n$ and $r(\cdot) : [0, \infty) \times \Omega \rightarrow \mathbb{R}^m$ are $\mathbb{F}$ -progressively measurable and square integrable on $[0, \infty)$; $G \in \mathbb{S}^n$ and $g \in L_{F_T}^2(\Omega; \mathbb{R}^n)$.

It is easy to see that under (H2.1)–(H2.2), the functional $J_T(s, \xi; u(\cdot))$ is well-defined. The two terms on the right-hand side of (2.2) are called the *expected running cost* and the *expected terminal cost*, respectively. Correspondingly, the integrand appeared in (2.2) is called the *running cost rate function*. We point out that (H2.1) and (H2.2) can be slightly relaxed. But, we prefer not to pursue that type of generality.

In the case that the nonhomogeneous terms $b(\cdot)$ and $\sigma(\cdot)$ are identically zero, the control system is said to be *homogeneous*. The state process of the homogeneous system is denoted by $X^0(\cdot) \equiv X^0(\cdot; s, \xi, u(\cdot))$. Thus,

$$\begin{cases} dX^0(t) = [A(t)X^0(t) + B(t)u(t)]dt \\ \quad + [C(t)X^0(t) + D(t)u(t)]dW(t), \quad t \geq s, \\ X^0(s) = \xi. \end{cases} \quad (2.3)$$

In addition, if $q(\cdot)$, $r(\cdot)$, g are all identically zero, we denote the corresponding cost functional by $J_T^0(s, \xi; u(\cdot))$. Hence,

$$\begin{aligned} J_T^0(s, \xi; u(\cdot)) \triangleq & \mathbb{E} \left\{ \int_s^T \left[\langle Q(t)X^0(t), X^0(t) \rangle + 2\langle S(t)X^0(t), u(t) \rangle \right. \right. \\ & \left. + \langle R(t)u(t), u(t) \rangle \right] dt + \langle G X^0(T), X^0(T) \rangle \Big\}. \end{aligned} \quad (2.4)$$

Our finite time-horizon stochastic optimal control problem can be stated as follows.

Problem (SLQ)_T. For any given $(s, \xi) \in \mathcal{D}$ ($s < T$), with the state equation (2.1) and the cost functional (2.2), find a control $\tilde{u}(\cdot) \in \mathcal{U}[s, T]$ such that

$$J_T(s, \xi; \tilde{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[s, T]} J_T(s, \xi; u(\cdot)) \equiv V_T(s, \xi). \quad (2.5)$$

The problem for the homogeneous state equation (2.3) and the cost functional (2.4) is denoted by Problem (SLQ)_T⁰, called a homogeneous stochastic LQ problem (in finite time-horizon).

In the case $T = \infty$, we only consider the state equation

$$\begin{cases} dX(t) = [AX(t) + Bu(t) + b(t)]dt \\ \quad + [CX(t) + Du(t) + \sigma(t)]dW(t), \quad t \geq 0, \\ X(0) = x, \end{cases} \quad (2.6)$$

namely, the coefficients A, B, C, D are constant matrices, and the nonhomogeneous terms are random processes. The corresponding cost functional takes the following form:

$$\begin{aligned} J_\infty(x; u(\cdot)) \triangleq & \mathbb{E} \int_0^\infty \left[\langle QX(t), X(t) \rangle + 2\langle SX(t), u(t) \rangle + \langle R(t)u(t), u(t) \rangle \right. \\ & \left. + 2\langle q(t), X(t) \rangle + 2\langle r(t), u(t) \rangle \right] dt, \end{aligned} \quad (2.7)$$

with no terminal cost and the quadratic weights being constant matrices, but still allowing the linear weights to be random processes. Clearly, to pose the corresponding LQ problem rigorously, we had better to introduce the following set:

$$\mathcal{U}_{ad}(x) \triangleq \left\{ u(\cdot) \in \mathcal{U}[0, \infty) \mid \mathbb{E} \int_0^\infty |X(t; x, u(\cdot))|^2 dt < \infty \right\}. \quad (2.8)$$

Any element in the above set is called an *admissible control* with respect to x . Then, we may formulate the following LQ problem in the infinite time-horizon $[0, \infty)$.

Problem (SLQ)_∞. For any given $x \in \mathbb{R}^n$, with state equation (2.6) and the cost functional (2.7), find a control $\tilde{u}(\cdot) \in \mathcal{U}_{ad}(x)$ such that

$$J_\infty(x; \tilde{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}_{ad}(x)} J_\infty(x; u(\cdot)) \equiv V_\infty(x).$$

When the system is stabilizable (see below for precise definition), and the nonhomogeneous terms $b(\cdot)$, $\sigma(\cdot)$ in the state equation (2.6) as well as the linear weighting coefficients $q(\cdot)$, $r(\cdot)$ in the cost functional (2.7) are square integrable over $[0, \infty)$, the set $\mathcal{U}_{ad}(x)$ of admissible controls will be rich enough and one will have satisfactory results for Problem (SLQ)_∞.

Similar to the finite time-horizon case, we denote the LQ problem with $b(\cdot)$, $\sigma(\cdot)$, $q(\cdot)$, and $r(\cdot)$ being all identically zero by Problem (SLQ)_∞⁰, called a homogeneous stochastic LQ problem in the infinite time-horizon $[0, \infty)$.

3. Open-loop and closed-loop solvability — Finite time-horizon

In this section, we discuss the open-loop and closed-loop solvability of the finite time-horizon problem. First, we introduce the following definition.

Definition 3.1. (i) Problem (SLQ)_T is said to be *finite at the initial pair* $(s, \xi) \in \mathcal{D}$ if

$$V_T(s, \xi) = \inf_{u(\cdot) \in \mathcal{U}[s, T]} J_T(s, \xi; u(\cdot)) > -\infty.$$

If Problem (SLQ)_T is finite at (s, ξ) for every $\xi \in L_{F_s}^2(\Omega; \mathbb{R}^n)$, then we say that the problem is *finite at s*. We simply say that Problem (SLQ)_T is *finite* if it is finite at every $s \in [0, T]$. We call $V_T(\cdot, \cdot)$ the *value function* of Problem (SLQ)_T.

(ii) Problem (SLQ)_T is said to be *open-loop solvable at* $(s, \xi) \in \mathcal{D}$ if there exists a $\tilde{u}(\cdot) \in \mathcal{U}[s, T]$ such that (2.5) holds. In this case, $\tilde{u}(\cdot)$ is called an *open-loop optimal control* for the initial pair (s, ξ) , and $\bar{X}(\cdot) \equiv X(\cdot; s, \xi, \tilde{u}(\cdot))$ and $(\bar{X}(\cdot), \tilde{u}(\cdot))$ are called the corresponding *open-loop optimal state process* and the *open-loop optimal pair*, respectively. If Problem (SLQ)_T is open-loop solvable at (s, ξ) for every $\xi \in L_{F_s}^2(\Omega; \mathbb{R}^n)$, we say that the problem is *open-loop solvable at s*. We simply say that Problem (SLQ)_T is *open-loop solvable* if it is open-loop solvable at every $s \in [0, T]$.

Since the state equation is affine in the state-control pair $(X(\cdot), u(\cdot))$, and the cost functional consists of quadratic and linear terms in

($X(\cdot), u(\cdot)$), we see that, under (H2.1) and (H2.2), the cost functional has the following representation:

$$J_T(s, \xi; u(\cdot)) = \langle \Psi_2 u(\cdot), u(\cdot) \rangle + 2\langle \psi_1(\xi), u(\cdot) \rangle + \psi_0(\xi) \equiv \Psi(u(\cdot)) \quad (3.1)$$

for some bounded self-adjoint operator $\Psi_2 : \mathcal{U}[s, T] \rightarrow \mathcal{U}[s, T]$ and some process $\psi_1(\xi) \in \mathcal{U}[s, T]$, as well as some ξ -dependent scalar $\psi_0(\xi)$. Thus, we may regard the mapping $u(\cdot) \mapsto J_T(s, \xi; u(\cdot))$ as a functional consisting of quadratic and linear terms of $u(\cdot)$ in the Hilbert space $\mathcal{U}[s, T]$, and we want to minimize such a functional. Therefore, the following result gives us a natural path (see Mou and Yong (2006) or Sun and Yong (2020a) for a proof).

Proposition 3.2. *Let the functional $\Psi(u(\cdot))$ be given by (3.1) on the Hilbert space $\mathcal{U}[s, T]$.*

(i) *If $\inf_{u(\cdot) \in \mathcal{U}[s, T]} \Psi(u(\cdot)) > -\infty$, then*

$$\Psi_2 \geq 0, \quad (3.2)$$

i.e., Ψ_2 is a positive operator, or equivalently, $u(\cdot) \mapsto \Psi(u(\cdot))$ is convex.

(ii) *Let (3.2) hold. Then $\Psi(u(\cdot))$ admits a minimum over $\mathcal{U}[s, T]$ if and only if*

$$\psi_1(\xi) \in \mathcal{R}(\Psi_2),$$

i.e., $\psi_1(\xi)$ belongs to the range of Ψ_2 . In this case, $\bar{u}(\cdot) \in \mathcal{U}[s, T]$ is a minimum of Ψ if and only if

$$\Psi_2 \bar{u}(\cdot) + \psi_1(\xi) = 0.$$

Consequently, if instead of (3.2), the following holds:

$$\Psi_2 \geq \delta I \quad (3.3)$$

for some $\delta > 0$, then $\Psi(u(\cdot))$ admits a unique minimizer $\bar{u}(\cdot)$ given by

$$\bar{u}(\cdot) = -\Psi_2^{-1} \psi_1(\xi).$$

The condition (3.3) is equivalent to the following uniform convexity of the mapping $u(\cdot) \mapsto \Psi(u(\cdot))$:

$$\Psi(u(\cdot)) \geq \delta \langle u(\cdot), u(\cdot) \rangle + \lambda$$

for some constants $\delta > 0$ and λ . Directly applying the above result to our Problem (SLQ) $_T$, we have the following characterization of the open-loop solvability.

Theorem 3.3. *Let (H2.1)–(H2.2) hold.*

(i) *If Problem (SLQ) $_T$ is finite at some initial pair $(s, \xi) \in \mathcal{D}$, then the mapping $u(\cdot) \mapsto J_T^0(s, 0; u(\cdot))$ is convex over $\mathcal{U}[s', T]$ for every $s' \in [s, T]$ (or equivalently, $u(\cdot) \mapsto J_T(s', \xi; u(\cdot))$ is convex for all $(s', \xi) \in \mathcal{D}$ with $s' \in [s, T]$). If the problem is finite at s , then there exists a function $P(\cdot) : [s, T] \rightarrow \mathbb{S}^n$ such that*

$$V_T^0(t, \xi) \triangleq \inf_{u(\cdot) \in \mathcal{U}[t, T]} J_T^0(t, \xi; u(\cdot)) = \mathbb{E} \langle P(t) \xi, \xi \rangle, \quad (3.4)$$

$\forall (t, \xi) \in \mathcal{D}$ with $t \geq s$.

(ii) *Let $u(\cdot) \mapsto J_T(s, \xi; u(\cdot))$ be convex. Then Problem (SLQ) $_T$ is open-loop solvable at $(s, \xi) \in \mathcal{D}$ if and only if the following optimality system admits an adapted solution $(\bar{X}(\cdot), \bar{u}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot))$:*

$$\begin{cases} d\bar{X}(t) = [A(t)\bar{X}(t) + B(t)\bar{u}(t) + b(t)]dt \\ \quad + [C(t)\bar{X}(t) + D(t)\bar{u}(t) + \sigma(t)]dW(t), \\ d\bar{Y}(t) = -[A(t)^\top \bar{Y}(t) + C(t)^\top \bar{Z}(t) + Q(t)\bar{X}(t) \\ \quad + S(t)^\top \bar{u}(t) + q(t)]dt + \bar{Z}(t)dW(t), \\ \bar{X}(s) = \xi, \quad \bar{Y}(T) = G\bar{X}(T) + g, \\ B(t)^\top \bar{Y}(t) + D(t)^\top \bar{Z}(t) + S(t)\bar{X}(t) + R(t)\bar{u}(t) + r(t) = 0. \end{cases} \quad (3.5)$$

In this case, the pair $(\bar{X}(\cdot), \bar{u}(\cdot))$ is an open-loop optimal pair.

(iii) *In the case that the mapping $u(\cdot) \mapsto J_T^0(s, 0; u(\cdot))$ is uniformly convex over $\mathcal{U}[s, T]$, i.e., there exists a constant $\delta > 0$ such that*

$$J_T^0(s, 0; u(\cdot)) \geq \delta \mathbb{E} \int_s^T |u(t)|^2 dt, \quad \forall u(\cdot) \in \mathcal{U}[s, T], \quad (3.6)$$

Problem (SLQ) $_T$ is uniquely open-loop solvable at all $s' \in [s, T]$. Moreover, (3.4) holds with

$$R(t) + D(t)^\top P(t)D(t) \geq \delta I, \quad \text{a.e. } t \in [s, T]. \quad (3.7)$$

The second equation in (3.5) is called a *backward stochastic differential equation* (BSDE, for short), whose *adapted solution* is the pair of $\mathbb{F}$ -adapted processes $(\bar{Y}(\cdot), \bar{Z}(\cdot))$. System (3.5) is a coupled FBSDE, with the coupling through the last equality in (3.5), which is called the *stationarity condition*. Basically, part (ii) of Theorem 3.3 is called the *Pontryagin's maximum principle*, a set of necessary conditions for optimal controls. Since our problem is linear-quadratic, the necessary condition is also sufficient under the convexity condition. Hence, (ii) becomes a characterization of the open-loop solvability of Problem (SLQ) $_T$. Note that (ii) does not tell when the problem is actually open-loop solvable. Part (iii) gives a sufficient condition for that. Finally, it is easy to prove that (3.6) holds if and only if $u(\cdot) \mapsto J_T(s, \xi; u(\cdot))$ is uniformly convex for any $\xi \in L^2_{\mathcal{F}_s}(\Omega; \mathbb{R}^n)$. In principle, the uniform convexity of the mapping $u(\cdot) \mapsto J_T^0(s, 0; u(\cdot))$ is checkable.

We now present an interesting example.

Example 3.4. Consider the controlled SDE

$$\begin{cases} dX(t) = u(t)dW(t), & t \in [s, T], \\ X(s) = \xi, \end{cases}$$

with the cost functional

$$J_T(s, \xi; u(\cdot)) \triangleq \mathbb{E} \left[2|X(T)|^2 - \int_s^T |u(t)|^2 dt \right].$$

Clearly,

$$\mathbb{E}|X(t)|^2 = \mathbb{E} \left[|\xi|^2 + \int_s^t |u(\tau)|^2 d\tau \right], \quad t \in [s, T].$$

Thus,

$$J_T(s, \xi; u(\cdot)) = \mathbb{E} \left[\int_s^T |u(t)|^2 dt + 2|\xi|^2 \right].$$

Hence, the corresponding stochastic LQ problem has the unique open-loop optimal control $\bar{u}(\cdot) = 0$. However, in the current case, $R = -1 < 0$. This means that for Problem (SLQ) $_T$, the condition $R \geq 0$ is not necessary for the existence of an open-loop optimal control, unlike deterministic LQ problems.

Having the results about the open-loop solvability of Problem (SLQ) $_T$, we now move to the next step. It is not hard to see that the FB-SDE characterization (3.5) of open-loop optimal pair is not practically feasible. The reason is as follows: Suppose that $R(t)^{-1}$ exists. Then the open-loop optimal control can be obtained through the following:

$$\bar{u}(t) = -R(t)^{-1} [B(t)^\top \bar{Y}(t) + D(t)^\top \bar{Z}(t) + S(t)\bar{X}(t) + r(t)], \quad t \in [s, T].$$

The above shows that to determine the value $\bar{u}(t)$ of $\bar{u}(\cdot)$ at the current time t , besides the value $\bar{X}(t)$ of the open-loop optimal process $\bar{X}(\cdot)$, one also needs the value $(\bar{Y}(t), \bar{Z}(t))$ of the adapted solution $(\bar{Y}(\cdot), \bar{Z}(\cdot))$ to the BSDE in (3.5), which depends on $\bar{X}(T)$, a future value of $\bar{X}(\cdot)$. Hence, the above result has its mathematical value, while its practical value is very little.

To remedy the above, let $\Theta[s, T]$ denote the space of $\mathbb{R}^{m \times n}$ -valued, Lebesgue square integrable functions on $[s, T]$, i.e., $\Theta[s, T] = L^2(s, T; \mathbb{R}^{m \times n})$. We introduce the following definition.

Definition 3.5. Let $s \in [0, T]$ be given. Suppose that Problem (SLQ) $_T$ is open-loop solvable at (s, ξ) for all $\xi \in L^2_{\mathcal{F}_s}(\Omega; \mathbb{R}^n)$ with $\bar{u}(\cdot; \xi)$ (depending on ξ) being the open-loop optimal control. We say that such an optimal control $\bar{u}(\cdot; \xi)$ has a *closed-loop representation* if there exists a pair $(\Theta(\cdot), \nu(\cdot)) \in \Theta[s, T] \times \mathcal{U}[s, T]$, independent of ξ , such that

$$\bar{u}(t; \xi) = \Theta(t)\bar{X}(t) + \nu(t), \quad t \in [s, T],$$

where $\bar{X}(\cdot)$ is the solution to the following closed-loop system:

$$\begin{cases} d\bar{X}(t) = \{[A(t) + B(t)\Theta(t)]\bar{X}(t) + B(t)v(t) + b(t)\}dt \\ \quad + \{[C(t) + D(t)\Theta(t)]\bar{X}(t) + D(t)v(t) + \sigma(t)\}dW(t), \quad t \in [s, T], \\ \bar{X}(s) = \xi. \end{cases}$$

We now try an ansatz:

$$\bar{Y}(t) = P(t)\bar{X}(t) + \eta(t), \quad t \in [s, T], \quad (3.8)$$

for some differentiable deterministic $\mathbb{S}^n$ -valued function $P(\cdot)$ and an $\mathbb{F}$ -adapted solution $(\eta(\cdot), \zeta(\cdot))$ to the BSDE

$$d\eta(t) = \gamma(t)dt + \zeta(t)dW(t), \quad \eta(T) = g,$$

with $\gamma(\cdot)$ undetermined. We hope that $\bar{Z}(\cdot)$ will also be written in terms of $\bar{X}(\cdot)$ non-anticipatingly. Consequently, one could determine the value $\bar{u}(t)$ of $\bar{u}(\cdot)$ without using the future values of $\bar{X}(\cdot)$ so that $\bar{u}(\cdot)$ can be, in principle, practically determined. Such an idea is called the *invariant imbedding* (due to Bellman, Kalaba, and Wing 1960). This was also used by Ma–Protter–Yong for decoupling general FBSDEs; see Ma, Protter, and Yong (1994) and Ma and Yong (1999). We now make this precise.

By Itô's formula, we have (suppressing t)

$$\begin{aligned} & - (A^\top \bar{Y} + C^\top \bar{Z} + Q\bar{X} + S^\top \bar{u} + q)dt + \bar{Z}dW = d\bar{Y} \\ & = [\dot{P}\bar{X} + P(A\bar{X} + B\bar{u} + b) + \gamma]dt + [P(C\bar{X} + D\bar{u} + \sigma) + \zeta]dW \end{aligned} \quad (3.9)$$

Thus, by comparing the diffusion terms of the above, one has

$$\bar{Z} = P(C\bar{X} + D\bar{u} + \sigma) + \zeta. \quad (3.10)$$

On the other hand, the stationarity condition in (3.5) becomes

$$\begin{aligned} 0 &= B^\top \bar{Y} + D^\top \bar{Z} + S\bar{X} + R\bar{u} + r \\ &= B^\top (P\bar{X} + \eta) + D^\top [P(C\bar{X} + D\bar{u} + \sigma) + \zeta] + S\bar{X} + R\bar{u} + r \\ &= (B^\top P + D^\top PC + S)\bar{X} + (R + D^\top PD)\bar{u} + B^\top \eta + D^\top \zeta + D^\top P\sigma + r. \end{aligned}$$

Suppose that our choice of $P(\cdot)$ has the property that

$$R(t) + D(t)^\top P(t)D(t) \geq \delta I, \quad \text{a.e. } t \in [s, T], \quad (3.11)$$

for some constant $\delta > 0$. Then,

$$\bar{u} = -(R + D^\top PD)^{-1} [(B^\top P + D^\top PC + S)\bar{X} + B^\top \eta + D^\top \zeta + D^\top P\sigma + r]. \quad (3.12)$$

Next, by comparing the drift terms on both sides of (3.9) and noting (3.8) and (3.10), one has

$$\begin{aligned} 0 &= A^\top \bar{Y} + C^\top \bar{Z} + Q\bar{X} + S^\top \bar{u} + q + \dot{P}\bar{X} + P(A\bar{X} + B\bar{u} + b) + \gamma \\ &= (\dot{P} + PA + A^\top P + C^\top PC + Q)\bar{X} + (PB + C^\top PD + S^\top)\bar{u} \\ &\quad + A^\top \eta + C^\top \zeta + C^\top P\sigma + Pb + q + \gamma. \end{aligned}$$

Substituting (3.12) into the above, we obtain

$$\begin{aligned} 0 &= \left[\dot{P} + PA + A^\top P + C^\top PC + Q \right. \\ &\quad \left. - (PB + C^\top PD + S^\top)(R + D^\top PD)^{-1}(B^\top P + D^\top PC + S) \right] \bar{X} \\ &\quad + (A + B\Theta)^\top \eta + (C + D\Theta)^\top \zeta + (C + D\Theta)^\top P\sigma + \Theta^\top r + Pb + q + \gamma, \end{aligned}$$

where

$$\Theta \triangleq -(R + D^\top PD)^{-1}(B^\top P + D^\top PC + S).$$

Clearly, we should let $P : [s, T] \rightarrow \mathbb{S}^n$ be the solution to the following differential Riccati equation:

$$\begin{cases} \dot{P} + PA + A^\top P + C^\top PC + Q \\ \quad - (PB + C^\top PD + S^\top)(R + D^\top PD)^{-1}(B^\top P + D^\top PC + S) = 0, \\ P(T) = G, \end{cases} \quad (3.13)$$

and let $(\eta(\cdot), \zeta(\cdot))$ be the adapted solution to the following BSDE:

$$\begin{cases} d\eta(t) = - \left[(A + B\Theta)^\top \eta + (C + D\Theta)^\top \zeta + (C + D\Theta)^\top P\sigma \right. \\ \quad \left. + \Theta^\top r + Pb + q \right] dt + \zeta dW, \quad t \in [s, T], \\ \eta(T) = g. \end{cases} \quad (3.14)$$

To summarize, we have the following result.

Proposition 3.6. *Let (H2.1)–(H2.2) hold. Suppose that for a given $s \in [0, T)$, Problem (SLQ) $_T$ admits an open-loop optimal control $\bar{u}(\cdot)$ at (s, ξ) for each $\xi \in L^2_{F_s}(\Omega; \mathbb{R}^n)$. Moreover, suppose that the differential Riccati equation (3.13) admits a solution $P(\cdot)$ such that (3.11) holds for some constant $\delta > 0$, and that the BSDE (3.14) has an adapted solution $(\eta(\cdot), \zeta(\cdot))$. Then $\bar{u}(\cdot)$ admits the closed-loop representation (3.12).*

The above result suggests us to reconsider Problem (SLQ) $_T$ from a different angle. We now present that.

Recall $\Theta[s, T] = L^2(s, T; \mathbb{R}^{m \times n})$. For any initial pair $(s, \xi) \in \mathcal{D}$ and $(\Theta(\cdot), v(\cdot)) \in \Theta[s, T] \times \mathcal{U}[s, T]$, called a *closed-loop strategy*, consider the following SDE:

$$\begin{cases} dX(t) = \{[A(t) + B(t)\Theta(t)]X(t) + B(t)v(t) + b(t)\}dt \\ \quad + \{[C(t) + D(t)\Theta(t)]X(t) + D(t)v(t) + \sigma(t)\}dW(t), \\ X(s) = \xi. \end{cases}$$

This is called the *closed-loop system* under the closed-loop strategy $(\Theta(\cdot), v(\cdot))$. Clearly, under (H2.1), the above admits a unique solution $X^{\Theta, v}(\cdot) \equiv X(\cdot; s, \xi, \Theta(\cdot), v(\cdot))$. We define

$$u(\cdot) \triangleq \Theta(\cdot)X^{\Theta, v}(\cdot) + v(\cdot),$$

which is called the *outcome* of $(\Theta(\cdot), v(\cdot))$ associated with the initial pair (s, ξ) . Correspondingly, the cost functional reads

$$\begin{aligned} J_T(s, \xi; \Theta(\cdot)X^{\Theta, v}(\cdot) + v(\cdot)) &= \mathbb{E} \left\{ \int_s^T \left[\langle Q(t)X^{\Theta, v}(t), X^{\Theta, v}(t) \rangle + 2\langle S(t)X^{\Theta, v}(t), \Theta(t)X^{\Theta, v}(t) + v(t) \rangle \right. \right. \\ &\quad + \langle R(t)[\Theta(t)X^{\Theta, v}(t) + v(t)], \Theta(t)X^{\Theta, v}(t) + v(t) \rangle + 2\langle q(t), X^{\Theta, v}(t) \rangle \\ &\quad \left. \left. + 2\langle r(t), \Theta(t)X^{\Theta, v}(t) + v(t) \rangle \right] dt + \langle GX^{\Theta, v}(T), X^{\Theta, v}(T) \rangle \right. \\ &\quad \left. + 2\langle g, X^{\Theta, v}(T) \rangle \right\}. \end{aligned}$$

We now introduce the following definition.

Definition 3.7. A pair $(\bar{\Theta}(\cdot), \bar{v}(\cdot)) \in \Theta[s, T] \times \mathcal{U}[s, T]$ is called a *closed-loop optimal strategy* of Problem (SLQ) $_T$ on $[s, T]$ if the following holds:

$$\begin{aligned} J_T(s, \xi; \bar{\Theta}(\cdot)X^{\bar{\Theta}, \bar{v}}(\cdot) + \bar{v}(\cdot)) &\leq J_T(s, \xi; \Theta(\cdot)X^{\Theta, v}(\cdot) + v(\cdot)), \\ \forall (\Theta(\cdot), v(\cdot)) &\in \Theta[s, T] \times \mathcal{U}[s, T], \quad \forall \xi \in L^2_{F_s}(\Omega; \mathbb{R}^n). \end{aligned}$$

In this case, we say that Problem (SLQ) $_T$ is *closed-loop solvable* on $[s, T]$. If $(\bar{\Theta}(\cdot), \bar{v}(\cdot))$ is unique, we say that the problem is uniquely closed-loop solvable on $[s, T]$.

Remark 3.8. The closed-loop optimal strategy $(\bar{\Theta}(\cdot), \bar{v}(\cdot))$ is independent of the initial state ξ .

Proposition 3.9. *Let (H2.1)–(H2.2) hold and $s \in [0, T)$. Let $(\bar{\Theta}(\cdot), \bar{v}(\cdot)) \in \Theta[s, T] \times \mathcal{U}[s, T]$. Then the following are equivalent:*

- (i) $(\bar{\Theta}(\cdot), \bar{v}(\cdot))$ is a closed-loop optimal strategy of Problem (SLQ) $_T$ on $[s, T]$.
- (ii) For any $\xi \in L^2_{F_s}(\Omega; \mathbb{R}^n)$ and $v(\cdot) \in \mathcal{U}[s, T]$,

$$J_T(s, \xi; \bar{\Theta}(\cdot)X^{\bar{\Theta}, \bar{v}}(\cdot) + \bar{v}(\cdot)) \leq J_T(s, \xi; \Theta(\cdot)X^{\Theta, v}(\cdot) + v(\cdot)).$$

(iii) For any $\xi \in L^2_{\mathcal{F}_s}(\Omega; \mathbb{R}^n)$ and $u(\cdot) \in \mathcal{U}[s, T]$,

$$J_T(s, \xi; \bar{\theta}(\cdot)X^{\bar{\theta}, \bar{v}}(\cdot) + \bar{v}(\cdot)) \leq J_T(s, \xi; u(\cdot)).$$

From the above result, we see that for a closed-loop optimal strategy $(\bar{\theta}(\cdot), \bar{v}(\cdot))$ of Problem (SLQ) $_T$ on $[s, T]$, the following two conclusions hold:

- $\bar{v}(\cdot)$ is an open-loop optimal control of the stochastic LQ problem with the state equation

$$\begin{cases} dX(t) = [A^{\bar{\theta}}(t)X(t) + B(t)v(t) + b(t)]dt \\ \quad + [C^{\bar{\theta}}(t)X(t) + D(t)v(t) + \sigma(t)]dW(t), \\ X(s) = \xi \end{cases} \quad (3.15)$$

and the cost functional

$$\begin{aligned} J_T^{\bar{\theta}}(s, \xi; v(\cdot)) \triangleq \mathbb{E} \left\{ \int_s^T \left[\langle Q^{\bar{\theta}}(t)X(t), X(t) \rangle + 2\langle S^{\bar{\theta}}(t)X(t), v(t) \rangle \right. \right. \\ \left. \left. + \langle R(t)v(t), v(t) \rangle + 2\langle q^{\bar{\theta}}(t), X(t) \rangle + 2\langle r(t), v(t) \rangle \right] dt \right. \\ \left. + \langle GX(T), X(T) \rangle + 2\langle g, X(T) \rangle \right\} \end{aligned} \quad (3.16)$$

at (s, ξ) for any $\xi \in L^2_{\mathcal{F}_s}(\Omega; \mathbb{R}^n)$, where

$$\begin{cases} A^{\bar{\theta}}(t) \triangleq A(t) + B(t)\bar{\theta}(t), & C^{\bar{\theta}}(t) \triangleq C(t) + D(t)\bar{\theta}(t), \\ Q^{\bar{\theta}}(t) \triangleq Q(t) + \bar{\theta}(t)^\top S(t) + S(t)^\top \bar{\theta}(t) + \bar{\theta}(t)^\top R(t)\bar{\theta}(t), \\ S^{\bar{\theta}}(t) \triangleq S(t) + R(t)\bar{\theta}(t), & q^{\bar{\theta}}(t) \triangleq q(t) + \bar{\theta}(t)r(t). \end{cases} \quad (3.17)$$

- The outcome $\bar{u}(\cdot) \triangleq \bar{\theta}(\cdot)X^{\bar{\theta}, \bar{v}}(\cdot) + \bar{v}(\cdot)$ is an open-loop optimal control of Problem (SLQ) $_T$ at (s, ξ) .

Consequently, we have the following corollary.

Corollary 3.10. Let (H2.1)–(H2.2) hold. If Problem (SLQ) $_T$ is closed-loop solvable on $[s, T]$, then it is open-loop solvable at (s, ξ) for any $\xi \in L^2_{\mathcal{F}_s}(\Omega; \mathbb{R}^n)$.

The converse of the above is not true in general. Here is an example.

Example 3.11. Consider the following one-dimensional controlled SDE:

$$\begin{cases} dX(t) = u_1(t)dt + u_2(t)dW(t), & t \in [s, T], \\ X(s) = \xi, \end{cases}$$

with the cost functional

$$J_T(s, \xi; u_1(\cdot), u_2(\cdot)) \triangleq \mathbb{E} \left[\int_s^T |u_2(t)|^2 dt + |X(T)|^2 \right].$$

Here, $(u_1(\cdot), u_2(\cdot))^\top$ is the control process, valued in $\mathbb{R}^2$. Then the corresponding LQ problem is open-loop solvable. As a matter of fact, for any $\xi \in L^2_{\mathcal{F}_s}(\Omega; \mathbb{R})$ and any $0 < \delta < T - s$, if we define

$$\bar{u}_1^\delta(t) = -\frac{\xi}{\delta} \mathbf{1}_{[s, s+\delta]}(t), \quad \bar{u}_2(t) = 0, \quad t \in [s, T],$$

then $X(T) = 0$ and

$$J_T(s, \xi; \bar{u}_1^\delta(\cdot), \bar{u}_2(\cdot)) = 0 = \inf_{(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}[s, T]} J_T(s, \xi; u_1(\cdot), u_2(\cdot)).$$

Thus, such a control is an open-loop optimal control. We claim that the problem is not closed-loop solvable. Otherwise, let $(\bar{\theta}(\cdot), \bar{v}(\cdot))$ with

$$\bar{\theta}(\cdot) = \begin{pmatrix} \bar{\theta}_1(\cdot) \\ \bar{\theta}_2(\cdot) \end{pmatrix}, \quad \bar{v}(\cdot) = \begin{pmatrix} \bar{v}_1(\cdot) \\ \bar{v}_2(\cdot) \end{pmatrix}$$

be a closed-loop optimal strategy on $[s, T]$. Then, the closed-loop system reads (denoting $\bar{X}(\cdot) = X^{\bar{\theta}, \bar{v}}(\cdot)$)

$$\begin{cases} d\bar{X}(t) = [\bar{\theta}_1(t)\bar{X}(t) + \bar{v}_1(t)]dt + [\bar{\theta}_2(t)\bar{X}(t) + \bar{v}_2(t)]dW(t), \\ \bar{X}(s) = \xi, \end{cases}$$

such that the outcome

$$\bar{u}_1(t) \triangleq \bar{\theta}_1(t)\bar{X}(t) + \bar{v}_1(t), \quad \bar{u}_2(t) \triangleq \bar{\theta}_2(t)\bar{X}(t) + \bar{v}_2(t)$$

is an open-loop optimal control. Hence, we must have

$$\bar{\theta}_2(t)\bar{X}(t) + \bar{v}_2(t) = 0, \quad \bar{X}(T) = 0.$$

Then

$$d\bar{X}(t) = [\bar{\theta}_1(t)\bar{X}(t) + \bar{v}_1(t)]dt, \quad \bar{X}(s) = \xi, \quad \bar{X}(T) = 0.$$

Consequently,

$$0 = \bar{X}(T) = e^{\int_s^T \bar{\theta}_1(\tau) d\tau} \xi + \int_s^T e^{\int_s^\tau \bar{\theta}_1(\tau) d\tau} \bar{v}_1(\lambda) d\lambda,$$

which should be true for all $\xi \in L^2_{\mathcal{F}_s}(\Omega; \mathbb{R})$, by the definition of closed-loop optimal strategy. But, this is impossible. Hence, the problem is not closed-loop solvable on $[s, T]$.

Our next goal is to characterize the closed-loop optimal strategy. Let $(\bar{\theta}(\cdot), \bar{v}(\cdot))$ be a closed-loop optimal strategy. By Proposition 3.9 (ii), we know that $\bar{v}(\cdot)$ is an open-loop optimal control of the LQ problem with coefficients and weights modified by (3.17) (which is uniform in the initial state ξ). Hence, by Theorem 3.3 (ii), one has (with $\bar{X}(\cdot) = X^{\bar{\theta}, \bar{v}}(\cdot)$)

$$\begin{cases} d\bar{X}(t) = [A^{\bar{\theta}}(t)\bar{X}(t) + B(t)\bar{v}(t) + b(t)]dt \\ \quad + [C^{\bar{\theta}}(t)\bar{X}(t) + D(t)\bar{v}(t) + \sigma(t)]dW(t), \\ d\bar{Y}(t) = -[A^{\bar{\theta}}(t)^\top \bar{Y}(t) + C^{\bar{\theta}}(t)^\top \bar{Z}(t) + Q^{\bar{\theta}}(t)\bar{X}(t) + S^{\bar{\theta}}(t)^\top \bar{v}(t) + q^{\bar{\theta}}(t)]dt \\ \quad + \bar{Z}(t)dW(t), \\ \bar{X}(s) = \xi, \quad \bar{Y}(T) = G\bar{X}(T) + g, \\ B(t)^\top \bar{Y}(t) + D(t)^\top \bar{Z}(t) + S^{\bar{\theta}}(t)\bar{X}(t) + R(t)\bar{v}(t) + r(t) = 0. \end{cases}$$

To emphasize the dependence on the initial state ξ , let us denote by $(X^\xi(\cdot), Y^\xi(\cdot), Z^\xi(\cdot))$ the adapted solution to the above, where $(\bar{\theta}(\cdot), \bar{v}(\cdot))$ stays the same. By setting

$$\hat{X}(\cdot) \triangleq X^\xi(\cdot) - X^0(\cdot), \quad \hat{Y}(\cdot) \triangleq Y^\xi(\cdot) - Y^0(\cdot), \quad \hat{Z}(\cdot) \triangleq Z^\xi(\cdot) - Z^0(\cdot),$$

we see that the following holds:

$$\begin{cases} d\hat{X}(t) = A^{\bar{\theta}}(t)\hat{X}(t)dt + C^{\bar{\theta}}(t)\hat{X}(t)dW(t), \\ d\hat{Y}(t) = -[A^{\bar{\theta}}(t)^\top \hat{Y}(t) + C^{\bar{\theta}}(t)^\top \hat{Z}(t) + Q^{\bar{\theta}}(t)\hat{X}(t)]dt + \hat{Z}(t)dW(t), \\ \hat{X}(s) = \xi, \quad \hat{Y}(T) = G\hat{X}(T), \\ B(t)^\top \hat{Y}(t) + D(t)^\top \hat{Z}(t) + S^{\bar{\theta}}(t)\hat{X}(t) = 0. \end{cases}$$

By letting ξ run over $\mathbb{R}^n$, we see that the following equation for matrix-valued processes is well-posed:

$$\begin{cases} d\mathbb{X}(t) = A^{\bar{\theta}}(t)\mathbb{X}(t)dt + C^{\bar{\theta}}(t)\mathbb{X}(t)dW(t), \\ d\mathbb{Y}(t) = -[A^{\bar{\theta}}(t)^\top \mathbb{Y}(t) + C^{\bar{\theta}}(t)^\top \mathbb{Z}(t) + Q^{\bar{\theta}}(t)\mathbb{X}(t)]dt + \mathbb{Z}(t)dW(t), \\ \mathbb{X}(s) = I, \quad \mathbb{Y}(T) = G\mathbb{X}(T), \\ B(t)^\top \mathbb{Y}(t) + D(t)^\top \mathbb{Z}(t) + S^{\bar{\theta}}(t)\mathbb{X}(t) = 0. \end{cases}$$

It is clear that $\mathbb{X}(t)$ is invertible for each $t \in [s, T]$ and $\mathbb{X}(\cdot)^{-1}$ satisfies the following SDE:

$$\begin{cases} d[\mathbb{X}(t)^{-1}] = -\mathbb{X}(t)^{-1}[A^{\bar{\theta}}(t) - C^{\bar{\theta}}(t)^2]dt - \mathbb{X}(t)^{-1}C^{\bar{\theta}}(t)dW(t), \\ \mathbb{X}(s)^{-1} = I. \end{cases}$$

Define for $t \in [s, T]$,

$$P(t) \triangleq \mathbb{Y}(t)\mathbb{X}(t)^{-1}, \quad \Gamma(t) \triangleq \mathbb{Z}(t)\mathbb{X}(t)^{-1}.$$

Then by Itô's formula (suppressing t),

$$\begin{aligned} dP &= \left\{ -[(A^{\bar{\theta}})^\top \mathbb{Y} + (C^{\bar{\theta}})^\top \mathbb{Z} + Q^{\bar{\theta}}\mathbb{X}]\mathbb{X}^{-1} \right. \\ &\quad \left. - \mathbb{Y}\mathbb{X}^{-1}[A^{\bar{\theta}} - (C^{\bar{\theta}})^2] - \mathbb{Z}\mathbb{X}^{-1}C^{\bar{\theta}} \right\} dt \\ &\quad + (\mathbb{Z}\mathbb{X}^{-1} - \mathbb{Y}\mathbb{X}^{-1}C^{\bar{\theta}})dW \\ &= -\left\{ (A^{\bar{\theta}})^\top P + (C^{\bar{\theta}})^\top \Gamma + Q^{\bar{\theta}} + P[A^{\bar{\theta}} - (C^{\bar{\theta}})^2] + \Gamma C^{\bar{\theta}} \right\} dt \end{aligned}$$

$$+ (\Gamma - PC^\theta) dW.$$

By setting $\Lambda = \Gamma - PC^\theta$, the above becomes

$$dP = - \left[PA^\theta + (A^\theta)^\top P + (C^\theta)^\top PC^\theta + \Lambda C^\theta + (C^\theta)^\top \Lambda + Q^\theta \right] dt + \Lambda dW.$$

The above, together with

$$P(T) = G,$$

is a BSDE with deterministic coefficients and terminal state. Therefore, one must have $\Lambda(\cdot) = 0$. Hence, $P(\cdot)$ is the solution to the following ODE:

$$\begin{cases} \dot{P} + PA^\theta + (A^\theta)^\top P + (C^\theta)^\top PC^\theta + Q^\theta = 0, \\ P(T) = G. \end{cases}$$

On other hand, the stationarity condition is equivalent to

$$\begin{aligned} 0 &= B^\top P + D^\top \Gamma + S^\theta = B^\top P + D^\top [\Lambda + P(C + D\bar{\theta})] + S + R\bar{\theta} \\ &= B^\top P + D^\top PC + S + (R + D^\top PD)\bar{\theta}. \end{aligned}$$

Let $\mathcal{R}(M)$ and $M^\dagger$ denote the range and the generalized inverse of a matrix M , respectively. The above then implies that

$$\mathcal{R}(B^\top P + D^\top PC + S) \subseteq \mathcal{R}(R + D^\top PD),$$

and that

$$\bar{\theta} = -(R + D^\top PD)^\dagger (B^\top P + D^\top PC + S) + [I - (R + D^\top PD)^\dagger (R + D^\top PD)] \Pi,$$

for some $\Pi(\cdot) \in \Theta[s, T]$. Consequently,

$$\begin{aligned} PA^\theta + (A^\theta)^\top P + (C^\theta)^\top PC^\theta + Q^\theta &= P(A + B\bar{\theta}) + (A + B\bar{\theta})^\top P + (C + D\bar{\theta})^\top P(C + D\bar{\theta}) \\ &\quad + Q + S^\top \bar{\theta} + \bar{\theta}^\top S + \bar{\theta}^\top R\bar{\theta} \\ &= PA + A^\top P + C^\top PC + Q + (PB + C^\top PD + S^\top) \bar{\theta} \\ &\quad + \bar{\theta}^\top (B^\top P + D^\top PC + S) + \bar{\theta}^\top (R + D^\top PD) \bar{\theta} \\ &= PA + A^\top P + C^\top PC + Q \\ &\quad - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger (B^\top P + D^\top PC + S). \end{aligned}$$

This gives a derivation of the following differential Riccati equation (which generalizes (3.13)):

$$\begin{cases} \dot{P}(t) + P(t)A(t) + A(t)^\top P(t) + C(t)^\top P(t)C(t) + Q(t) \\ \quad - [P(t)B(t) + C(t)^\top P(t)D(t) + S(t)^\top][R(t) + D(t)^\top P(t)D(t)]^\dagger \\ \quad \times [B(t)^\top P(t) + D(t)^\top P(t)C(t) + S(t)] = 0, \\ P(T) = G. \end{cases} \quad (3.18)$$

We introduce the following definition.

Definition 3.12. A function $P(\cdot) \in C([s, T]; \mathbb{S}^n)$ is called a *regular solution* of the differential Riccati equation (3.18) on $[s, T]$ if (3.18) is satisfied and the following hold:

$$\begin{cases} R(t) + D(t)^\top P(t)D(t) \geq 0, \quad \text{a.e. } t \in [s, T], \\ \mathcal{R}(B(t)^\top P(t) + D(t)^\top P(t)C(t) + S(t)) \subseteq \mathcal{R}(R(t) + D(t)^\top P(t)D(t)), \quad \text{a.e. } t \in [s, T], \\ [R(\cdot) + D(\cdot)^\top P(\cdot)D(\cdot)]^\dagger [B(\cdot)^\top P(\cdot) + D(\cdot)^\top P(\cdot)C(\cdot) + S(\cdot)] \in \Theta[s, T]. \end{cases} \quad (3.19)$$

Now, we are ready to state the following result.

Theorem 3.13. Let (H2.1)–(H2.2) hold. Then Problem (SLQ)_T is closed-loop solvable on $[s, T]$ if and only if the following two conditions are satisfied:

(i) The differential Riccati equation (3.18) admits a regular solution $P(\cdot) \in C([s, T]; \mathbb{S}^n)$.

(ii) The adapted solution $(\eta(\cdot), \zeta(\cdot))$ to the BSDE

$$\begin{cases} d\eta(t) = - \left\{ [A(t) + B(t)\Theta(t)]^\top \eta(t) + [C(t) + D(t)\Theta(t)]^\top \zeta(t) \right. \\ \quad \left. + [C(t) + D(t)\Theta(t)]^\top P(t)\sigma(t) + \Theta(t)^\top r(t) + P(t)b(t) + q(t) \right\} dt \\ \quad + \zeta(t) dW(t), \\ \eta(T) = g \end{cases} \quad (3.20)$$

satisfies

$$\begin{aligned} \kappa(t) &\triangleq B(t)^\top \eta(t) + D(t)^\top \zeta(t) + D(t)^\top P(t)\sigma(t) + r(t) \\ &\in \mathcal{R}(R(t) + D(t)^\top P(t)D(t)), \\ v(\cdot) &\triangleq -[R(\cdot) + D(\cdot)^\top P(\cdot)D(\cdot)]^\dagger \kappa(\cdot) \in \mathcal{U}[s, T], \end{aligned}$$

where in (3.20),

$$\Theta(t) \triangleq -[R(t) + D(t)^\top P(t)D(t)]^\dagger [B(t)^\top P(t) + D(t)^\top P(t)C(t) + S(t)].$$

In this case, the closed-loop optimal strategy $(\bar{\theta}(\cdot), \bar{v}(\cdot))$ admits the following representation:

$$\begin{aligned} \bar{\theta}(t) &= \Theta(t) + \{I - [R(t) + D(t)^\top P(t)D(t)]^\dagger [R(t) + D(t)^\top P(t)D(t)]\} \Pi(t), \\ \bar{v}(t) &= v(t) + \{I - [R(t) + D(t)^\top P(t)D(t)]^\dagger [R(t) + D(t)^\top P(t)D(t)]\} \pi(t), \end{aligned}$$

with any $(\Pi(\cdot), \pi(\cdot)) \in \Theta[s, T] \times \mathcal{U}[s, T]$. Furthermore, the value function is given by

$$\begin{aligned} V_T(s, \xi) &= \mathbb{E} \left[\langle P(s)\xi, \xi \rangle + 2\langle \eta(s), \xi \rangle + \int_s^T \left(\langle P(t)\sigma(t), \sigma(t) \rangle + 2\langle \eta(t), b(t) \rangle \right. \right. \\ &\quad \left. \left. + 2\langle \zeta(t), \sigma(t) \rangle - \langle [R(t) + D(t)^\top P(t)D(t)]^\dagger \kappa(t), \kappa(t) \rangle \right) dt \right]. \end{aligned}$$

In the above, the BSDE for $(\eta(\cdot), \zeta(\cdot))$ is the same as (3.14) and it can be derived by defining

$$\begin{aligned} \eta(t) &\triangleq \bar{Y}(t) - P(t)\bar{X}(t), \\ \zeta(t) &\triangleq \bar{Z}(t) - P(t)\{[C(t) + D(t)\bar{\theta}(t)]\bar{X}(t) + D(t)\bar{v}(t) + \sigma(t)\}, \end{aligned}$$

and applying Itô's formula to $\eta(\cdot)$. This basically gives the necessity of the above result. The sufficiency can be proved by applying Itô's formula to $\langle P(\cdot)X(\cdot), X(\cdot) \rangle$ and completing the squares. We also note that the first condition in (3.19) guarantees the convexity of the functional $u(\cdot) \mapsto J_T(s, \xi; u(\cdot))$.

Note that Theorem 3.13 only gives an equivalence relation between the closed-loop solvability of Problem (SLQ)_T and the regular solvability of the differential Riccati equation (3.18). There is no guarantee so far that the stochastic LQ problem is closed-loop solvable. We now fill this gap.

Theorem 3.14. Let (H2.1)–(H2.2) hold. Then the following are equivalent:

- (i) The functional $u(\cdot) \mapsto J_T^0(s, 0; u(\cdot))$ is uniformly convex, i.e., (3.6) holds for some $\delta > 0$.
- (ii) The differential Riccati equation (3.18) admits a regular solution $P(\cdot)$ on $[s, T]$ such that (3.7) holds for some constant $\delta > 0$.
- (iii) There exists an $\mathbb{S}^n$ -valued function $P(\cdot)$ on $[s, T]$ such that (3.4) and (3.7) hold for some constant $\delta > 0$.

We come to the uniform convexity condition of functional $u(\cdot) \mapsto J_T^0(s, 0; u(\cdot))$ again. Therefore, the following result is a nice conclusion of this section.

Proposition 3.15. *Let the following standard condition hold:*

$$G \geq 0, \quad R(t) \geq \delta I, \quad Q(t) - S(t)^\top R(t)^{-1} S(t) \geq 0, \quad \text{a.e. } t \in [s, T] \quad (3.21)$$

for some $\delta > 0$. Then $u(\cdot) \mapsto J_T^0(s, 0; u(\cdot))$ is uniformly convex.

Hence, from the results presented earlier, we see that in the case of (3.21), the stochastic LQ problem is closed-loop (and open-loop) solvable.

Remark 3.16. In this section, we start with the open-loop solvability of Problem (SLQ)_T. After getting the characterization of that in terms of the solvability of an FBSDE, we find that the representation of open-loop optimal control is not practically feasible. Then we try to seek a closed-loop representation, which is non-anticipating. This amounts to decoupling the optimality system (an FBSDE). Such an idea serves as a bridge leading us to the closed-loop solvability of Problem (SLQ)_T. Next, we obtain the characterization of the closed-loop solvability in terms of the regular solvability of the differential Riccati equation. It turns out that if the problem is closed-loop solvable, then the problem is also open-loop solvable with each open-loop optimal control being the outcome of some closed-loop optimal strategy. Finally, we discuss the uniform convexity of the cost functional, which is sufficient for both the open-loop and closed-loop solvability of the stochastic LQ problem.

4. Infinite time-horizon cases

In this section, we would like to look at the situation where the time-horizon is infinite. For such a case, we only consider the following controlled SDE:

$$\begin{cases} dX(t) = [AX(t) + Bu(t) + b(t)]dt \\ \quad + [CX(t) + Du(t) + \sigma(t)]dW(t), \quad t \geq 0, \\ X(0) = x. \end{cases} \quad (4.1)$$

Let $L^2_{\mathbb{F}}(0, \infty; \mathbb{R}^n)$ be the space of $\mathbb{R}^n$ -valued, $\mathbb{F}$ -progressively measurable, and square integrable processes on $[0, \infty)$. For the coefficients and the nonhomogeneous terms in (4.1), we introduce the following hypothesis.

(H4.1) $A, C \in \mathbb{R}^{n \times n}$, $B, D \in \mathbb{R}^{n \times m}$, and $b(\cdot), \sigma(\cdot) \in L^2_{\mathbb{F}}(0, \infty; \mathbb{R}^n)$.

It is clear that under (H4.1), for any initial state $x \in \mathbb{R}^n$ and any control $u(\cdot) \in \mathcal{U}[0, \infty)$, the state equation (4.1) admits a unique solution $X(\cdot) \equiv X(\cdot; x, u(\cdot))$, which is locally square integrable but might not be square integrable over the entire $[0, \infty)$. To measure the performance of the control, we introduce the following cost functional:

$$J_{\infty}(x; u(\cdot)) \triangleq \mathbb{E} \int_0^{\infty} \left[\langle QX(t), X(t) \rangle + 2\langle SX(t), u(t) \rangle + \langle Ru(t), u(t) \rangle + 2\langle q(t), X(t) \rangle + 2\langle r(t), u(t) \rangle \right] dt. \quad (4.2)$$

For the above cost functional, we assume the following:

(H4.2) $Q \in \mathbb{S}^n$, $S \in \mathbb{R}^{m \times n}$, $R \in \mathbb{S}^m$, $q(\cdot) \in L^2_{\mathbb{F}}(0, \infty; \mathbb{R}^n)$, and $r(\cdot) \in \mathcal{U}[0, \infty)$.

It is not hard to see that (H4.1)–(H4.2) are not enough for the cost functional (4.2) to be well-defined, because the given integrand might not be integrable over $[0, \infty)$. Therefore, for any $x \in \mathbb{R}^n$, we recall

$$\mathcal{U}_{ad}(x) \triangleq \left\{ u(\cdot) \in \mathcal{U}[0, \infty) \mid \mathbb{E} \int_0^{\infty} |X(t; x, u(\cdot))|^2 dt < \infty \right\}.$$

We want to find conditions under which the set $\mathcal{U}_{ad}(x)$ is sufficiently large. In order to present a satisfactory theory, we first consider the following uncontrolled homogeneous SDE (i.e., set $u(\cdot) = 0$ and $b(\cdot) = \sigma(\cdot) = 0$ in (4.1)):

$$dX(t) = AX(t)dt + CX(t)dW(t), \quad t \geq 0.$$

We denote the above by $[A, C]$. The following definition is useful.

Definition 4.1. System $[A, C]$ is said to be L^2 -stable if for any initial state $x \in \mathbb{R}^n$, the solution $X(\cdot) \equiv X(\cdot; x)$ satisfies

$$\mathbb{E} \int_0^{\infty} |X(t)|^2 dt < \infty.$$

The following result gives a nice characterization of the L^2 -stability of $[A, C]$.

Proposition 4.2. System $[A, C]$ is L^2 -stable if and only if there exists a $P \in \mathbb{S}^n_+$, the set of positive definite matrices in $\mathbb{S}^n$, such that the following Lyapunov inequality holds:

$$PA + A^\top P + C^\top PC < 0.$$

In this case, for any $\Lambda \in \mathbb{S}^n$, the following Lyapunov equation

$$PA + A^\top P + C^\top PC + \Lambda = 0$$

admits a unique solution $P \in \mathbb{S}^n$, which is given by

$$P = \mathbb{E} \int_0^{\infty} \Phi(t)^\top \Lambda \Phi(t) dt,$$

where $\Phi(\cdot)$ is the solution to the matrix SDE

$$\begin{cases} d\Phi(t) = A\Phi(t)dt + C\Phi(t)dW(t), \quad t \geq 0, \\ \Phi(0) = I. \end{cases}$$

Furthermore, for the nonhomogeneous SDE

$$\begin{cases} dX(t) = [AX(t) + f(t)]dt + [CX(t) + g(t)]dW(t), \quad t \geq 0, \\ X(0) = x, \end{cases} \quad (4.3)$$

and the following BSDE in the infinite time-horizon:

$$dY(t) = -[A^\top Y(t) + C^\top Z(t) + h(t)]dt + Z(t)dW(t), \quad t \in [0, \infty), \quad (4.4)$$

we have the following natural result.

Proposition 4.3. Let $[A, C]$ be L^2 -stable. Then, for any $f(\cdot), g(\cdot), h(\cdot) \in L^2_{\mathbb{F}}(0, \infty; \mathbb{R}^n)$, the SDE (4.3) has a unique solution $X(\cdot) \in L^2_{\mathbb{F}}(0, \infty; \mathbb{R}^n)$, satisfying

$$\mathbb{E} \int_0^{\infty} |X(t)|^2 dt \leq K \left\{ |x|^2 + \mathbb{E} \int_0^{\infty} [|f(t)|^2 + |g(t)|^2] dt \right\},$$

and the BSDE (4.4) has a unique L^2 -stable adapted solution $(Y(\cdot), Z(\cdot))$ satisfying

$$\mathbb{E} \left[\sup_{0 \leq t < \infty} |Y(t)|^2 + \int_0^{\infty} |Z(t)|^2 dt \right] \leq K \mathbb{E} \int_0^{\infty} |h(t)|^2 dt,$$

where K is a positive constant independent of x , $f(\cdot)$, $g(\cdot)$, and $h(\cdot)$.

We now consider the following controlled homogeneous SDE:

$$dX(t) = [AX(t) + Bu(t)]dt + [CX(t) + Du(t)]dW(t), \quad t \geq 0,$$

which is denoted by $[A, C; B, D]$ for simplicity.

Definition 4.4. System $[A, C; B, D]$ is said to be stabilizable if there exists a matrix $\Theta \in \mathbb{R}^{m \times n}$ such that the uncontrolled homogeneous SDE $[A + B\Theta, C + D\Theta]$ is L^2 -stable. In this case, Θ is called a stabilizer of $[A, C; B, D]$. We denote by Θ_s the set of all stabilizers of $[A, C; B, D]$.

Note that we should distinguish Θ_s from $\Theta[s, T]$: the latter is the space of deterministic matrix-valued functions that are square integrable on $[s, T]$, whereas, the former is the class of constant matrices that stabilize the system $[A, C; B, D]$. Any element in $\Theta_s \times \mathcal{U}[0, \infty)$ is called a closed-loop strategy of Problem (SLQ)_∞. We now introduce the following crucial hypothesis.

(H4.3) System $[A, C; B, D]$ is stabilizable, i.e., $\Theta_s \neq \emptyset$.

We have the following proposition.

Proposition 4.5. Let (H4.1)–(H4.2) hold. Then the following are equivalent:

- (i) $\mathcal{U}_{ad}(x) \neq \emptyset$ for all $x \in \mathbb{R}^n$.
- (ii) $\Theta_s \neq \emptyset$.
- (iii) The following ARE admits a solution $P \in \mathbb{S}_+^n$:

$$PA + A^\top P + C^\top PC + I - (PB + C^\top PD)(I + D^\top PD)^{-1}(B^\top P + D^\top PC) = 0.$$

In the above case,

$$-(I + D^\top PD)^{-1}(B^\top P + D^\top PC) \in \Theta_s,$$

and for any $\Theta \in \Theta_s$,

$$\mathcal{U}_{ad}(x) = \left\{ \Theta X_x^{\Theta, v}(\cdot) + v(\cdot) \mid v(\cdot) \in \mathcal{U}[0, \infty) \right\}, \quad \forall x \in \mathbb{R}^n, \quad (4.5)$$

where $X_x^{\Theta, v}(\cdot)$ is the solution to the following closed-loop system:

$$\begin{cases} dX(t) = [(A + B\Theta)X(t) + Bv(t) + b(t)]dt \\ \quad + [(C + D\Theta)X(t) + Dv(t) + \sigma(t)]dW(t), \\ X(0) = x. \end{cases}$$

When $\mathcal{U}_{ad}(x) = \emptyset$, the corresponding LQ problem is meaningless. The above result tells the necessity of the condition (H4.3). Also, we see that as long as $\Theta_s \neq \emptyset$, by picking up any $\Theta \in \Theta_s$, one has (4.5). In this case,

$$\Theta X_x^{\Theta, v}(\cdot) + v(\cdot) \in \mathcal{U}[0, \infty),$$

which leads to $\mathcal{U}_{ad}(x) \subseteq \mathcal{U}[0, \infty)$. However, $\mathcal{U}_{ad}(x) \neq \mathcal{U}[0, \infty)$ in general. This can be easily seen from the following simple fact: If $[A, C]$ is not L^2 -stable, then $0 \notin \mathcal{U}_{ad}(x)$ for some $x \in \mathbb{R}^n$.

Under (H4.1)–(H4.3), $J_\infty(x; u(\cdot))$ is defined over a very large non-empty set $\mathcal{U}_{ad}(x)$ of admissible controls, and $J_\infty(x; \Theta X_x^{\Theta, v}(\cdot) + v(\cdot))$ is finite for all $(\Theta, v(\cdot)) \in \Theta_s \times \mathcal{U}[0, \infty)$. Thus, Problem (SLQ) $_\infty$ is well-formulated (see Section 2). Next, mimicking Problem (SLQ) $_T$, we introduce the following definition.

Definition 4.6. (i) Problem (SLQ) $_\infty$ is said to be *finite* at $x \in \mathbb{R}^n$ if

$$V_\infty(x) \triangleq \inf_{u(\cdot) \in \mathcal{U}_{ad}(x)} J_\infty(x; u(\cdot)) > -\infty.$$

If the problem is finite at every $x \in \mathbb{R}^n$, we simply say that Problem (SLQ) $_\infty$ is *finite*. The function $V_\infty(\cdot)$ is called the *value function* of Problem (SLQ) $_\infty$.

(ii) Problem (SLQ) $_\infty$ is said to be *open-loop solvable* at $x \in \mathbb{R}^n$ if there exists a $\bar{u}(\cdot) \in \mathcal{U}_{ad}(x)$ such that

$$V_\infty(x) = J_\infty(x; \bar{u}(\cdot)).$$

In this case, $\bar{u}(\cdot)$ is called an *open-loop optimal control* for the initial state x , and the corresponding $\bar{X}(\cdot) \equiv X(\cdot; x, \bar{u}(\cdot))$ and $(\bar{X}(\cdot), \bar{u}(\cdot))$ are called an *open-loop optimal process* and an *open-loop optimal pair* for x , respectively. If the problem is open-loop solvable at every $x \in \mathbb{R}^n$, we simply say that Problem (SLQ) $_\infty$ is open-loop solvable. Correspondingly, the cost functional and the value function for the (homogeneous) Problem (SLQ) $_\infty^0$ are denoted by $J_\infty^0(x; u(\cdot))$ and $V_\infty^0(\cdot)$, respectively.

(iii) Problem (SLQ) $_\infty$ is said to be *closed-loop solvable* if there exists a $(\bar{\Theta}, \bar{v}(\cdot)) \in \Theta_s \times \mathcal{U}[0, \infty)$ such that

$$J_\infty(x; \bar{\Theta} X_x^{\bar{\Theta}, \bar{v}}(\cdot) + \bar{v}(\cdot)) \leq J_\infty(x; \Theta X_x^{\Theta, v}(\cdot) + v(\cdot)), \quad \forall (x, \Theta, v(\cdot)) \in \mathbb{R}^n \times \Theta_s \times \mathcal{U}[0, \infty).$$

In this case, $(\bar{\Theta}, \bar{v}(\cdot))$ is called a *closed-loop optimal strategy* of Problem (SLQ) $_\infty$.

From (4.5), we see that for any $\Theta_1, \Theta_2 \in \Theta_s$,

$$\begin{aligned} & \left\{ \Theta_1 X_x^{\Theta_1, v}(\cdot) + v(\cdot) \mid v(\cdot) \in \mathcal{U}[0, \infty) \right\} \\ &= \left\{ \Theta_2 X_x^{\Theta_2, v}(\cdot) + v(\cdot) \mid v(\cdot) \in \mathcal{U}[0, \infty) \right\}, \quad \forall x \in \mathbb{R}^n. \end{aligned}$$

Thus, if $\Theta_s \neq \emptyset$, we need only to pick up one (convenient) $\Theta \in \Theta_s$. Now, let us state the main results of this section.

Theorem 4.7. Let (H4.1)–(H4.3) hold.

(i) Suppose that Problem (SLQ) $_\infty$ is finite. Then for some (or for all) $\Theta \in \Theta_s$, the mapping $v(\cdot) \mapsto J_\infty^0(0; \Theta X_0^{\Theta, v}(\cdot) + v(\cdot))$ is convex (or equivalently, $v(\cdot) \mapsto J_\infty(x; \Theta X_x^{\Theta, v}(\cdot) + v(\cdot))$ is convex for every initial state x). Moreover, there exists a $P \in \mathbb{S}^n$ such that

$$V_\infty^0(x) \triangleq \inf_{u(\cdot) \in \mathcal{U}_{ad}(x)} J_\infty^0(x; u(\cdot)) = \langle Px, x \rangle, \quad \forall x \in \mathbb{R}^n.$$

(ii) Let $v(\cdot) \mapsto J_\infty^0(0; \Theta X_0^{\Theta, v}(\cdot) + v(\cdot))$ be convex for some $\Theta \in \Theta_s$. Then an admissible control

$$\bar{u}(\cdot) \triangleq \Theta \bar{X}(\cdot) + \bar{v}(\cdot)$$

where $\bar{v}(\cdot) \in \mathcal{U}[0, \infty)$ and $\bar{X}(\cdot) \triangleq X_x^{\Theta, \bar{v}}(\cdot)$, is open-loop optimal for the initial state x if and only if the following BSDE

$$d\bar{Y}(t) = -[A^\top \bar{Y}(t) + C^\top \bar{Z}(t) + Q\bar{X}(t) + S^\top \bar{u}(t) + q(t)]dt + \bar{Z}(t)dW(t)$$

admits an L^2 -stable adapted solution $(\bar{Y}(\cdot), \bar{Z}(\cdot))$ such that

$$B^\top \bar{Y}(\cdot) + D^\top \bar{Z}(\cdot) + S\bar{X}(\cdot) + R\bar{u}(\cdot) + r(\cdot) = 0.$$

To state the next result, we need first introduce the following definition.

Definition 4.8. A matrix $P \in \mathbb{S}^n$ is called a *stabilizing solution* to the ARE

$$\begin{cases} PA + A^\top P + C^\top PC + Q \\ \quad - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger (B^\top P + D^\top PC + S) = 0, \\ \mathcal{R}(B^\top P + D^\top PC + S^\top) \subseteq \mathcal{R}(R + D^\top PD), \\ R + D^\top PD \geq 0, \end{cases} \quad (4.6)$$

if P satisfies the above and there exists a $\Pi \in \mathbb{R}^{m \times n}$ such that

$$-(R + D^\top PD)^\dagger (B^\top P + D^\top PC + S) + [I - (R + D^\top PD)^\dagger (R + D^\top PD)]\Pi \in \Theta_s.$$

Theorem 4.9. Let (H4.1)–(H4.3) hold. Then the following statements are equivalent:

- (i) Problem (SLQ) $_\infty$ is open-loop solvable.
- (ii) Problem (SLQ) $_\infty$ is closed-loop solvable.
- (iii) The ARE (4.6) admits a stabilizing solution $P \in \mathbb{S}^n$, and the BSDE

$$\begin{aligned} d\eta(t) = & \left\{ [A - B(R + D^\top PD)^\dagger (B^\top P + D^\top PC + S)]^\top \eta(t) \right. \\ & + [C - D(R + D^\top PD)^\dagger (B^\top P + D^\top PC + S)]^\top \zeta(t) \\ & + [C - D(R + D^\top PD)^\dagger (B^\top P + D^\top PC + S)]^\top P\sigma(t) \\ & - (B^\top P + D^\top PC + S)^\top (R + D^\top PD)^\dagger r(t) + Pb(t) + q(t) \Big\} dt \\ & + \zeta(t)dW(t) \end{aligned}$$

admits an L^2 -stable adapted solution $(\eta(\cdot), \zeta(\cdot))$ such that

$$\theta(t) \triangleq B^\top \eta(t) + D^\top \zeta(t) + D^\top P\sigma(t) + r(t) \in \mathcal{R}(R + D^\top PD),$$

a.s. $t \in [0, \infty)$, a.s.

Whenever (i), (ii), or (iii) is satisfied, all the closed-loop optimal strategies $(\bar{\Theta}, \bar{v}(\cdot))$ are given by

$$\begin{cases} \bar{\Theta} = -(R + D^\top PD)^\dagger (B^\top P + D^\top PC + S) \\ \quad + [I - (R + D^\top PD)^\dagger (R + D^\top PD)]\Pi, \\ \bar{v}(t) = -(R + D^\top PD)^\dagger \theta(t) + [I - (R + D^\top PD)^\dagger (R + D^\top PD)]\pi(t), \end{cases}$$

where $\Pi \in \mathbb{R}^{m \times n}$ is chosen so that $\bar{\Theta} \in \Theta_s$ and $\pi(\cdot) \in \mathcal{U}[0, \infty)$ is arbitrary. Further, every open-loop optimal control $\bar{u}(\cdot)$ for a given initial state x is given by

$$\bar{u}(t) = \bar{\Theta} X_x^{\bar{\Theta}, \bar{v}}(t) + \bar{v}(t), \quad t \geq 0,$$

for some closed-loop optimal strategy $(\bar{\theta}, \bar{v}(\cdot))$. Moreover, the value function is given by

$$V_\infty(x) = \langle Px, x \rangle + 2\langle \eta(0), x \rangle + \mathbb{E} \int_0^\infty \left[\langle P\sigma(t), \sigma(t) \rangle + 2\langle \eta(t), b(t) \rangle + 2\langle \zeta(t), \sigma(t) \rangle - \langle (R + D^\top PD)^\dagger \theta(t), \theta(t) \rangle \right] dt.$$

Similar to the finite time-horizon problems, the above only gives the equivalence among the three statements but does not tell when Problem $(\text{SLQ})_\infty$ is open-loop and/or closed-loop solvable. Now, we introduce the following condition.

(H4.4) For some $\theta \in \Theta_s$, the mapping $v(\cdot) \mapsto J_\infty^0(0; \theta X_0^{\theta, v}(\cdot) + v(\cdot))$ is uniformly convex, i.e.,

$$J_\infty^0(0; \theta X_0^{\theta, v}(\cdot) + v(\cdot)) \geq \delta \mathbb{E} \int_0^\infty |v(t)|^2 dt, \quad \forall v(\cdot) \in \mathcal{U}[0, \infty).$$

It is natural to have the following result.

Proposition 4.10. *Let (H4.1)–(H4.4) hold. Then Problem $(\text{SLQ})_\infty$ is open-loop solvable.*

5. Conclusions

In this paper, we have surveyed some works mainly done by the authors of the current paper for stochastic LQ optimal controls in the past decay, see Sun, Li and Yong (2016), Sun and Yong (2018), as well as Sun and Yong (2020a). There are a lots of more relevant results obtained, in addition to the above. To save some space and the time of the readers, we prefer to present only the most basic results and we now mention some results with the similar main ideas.

For two-person (nonzero-sum) stochastic LQ differential games in finite time-horizon, we have established the following results (see Sun and Yong 2019, and also Sun and Yong 2020b):

- The existence of an open-loop Nash equilibrium is equivalent to the solvability of a constrained system of FBSDEs, together with the convexity of the cost functionals.
- The existence of a closed-loop Nash equilibrium is equivalent to the regular solvability of a system of coupled symmetric differential Riccati equations.
- The existence of a closed-loop Nash equilibrium does not imply the existence of an open-loop Nash equilibrium, and vice-versa.
- The closed-loop representation of an open-loop Nash equilibrium does not have to be the outcome of an closed-loop Nash strategy.

For the two-person zero-sum stochastic LQ differential games, the above results remains true with Nash equilibrium replaced by saddle point, except the last item, which should be stated as follows:

- Suppose that the problem admits both open-loop and closed-loop saddle points. If the open-loop saddle point admits a closed-loop representation, then this representation must be the outcome of a closed-loop saddle point; see Sun and Yong (2014).
- If the performance functional satisfies a uniform convexity-concavity condition, then the associated differential Riccati equation has a unique strongly regular solution, in terms of which a closed-loop representation can be obtained for the open-loop saddle point; see Sun (2021).

For differential games in infinite time-horizon, we have pretty much similar results as those in finite time-horizon, with the following features (see Sun, Yong and Zhang 2016):

- Unlike the optimal control problem in the infinite time-horizon, we do not claim that the existence of closed-loop Nash equilibria (saddle points) implies that of the open-loop Nash equilibria (saddle points), or the other way around, or equivalent. The exact statement is open.
- Even for zero-sum differential games (in infinite time-horizon), there are several strange features which were shown by some examples:

- The algebraic Riccati equation might only have non-stabilizing solutions so that the saddle point does not exist;
- The game might have uncountably many saddle points;
- There might be some stabilizing solutions to the algebraic Riccati equation, which are not relevant to the saddle points.

One may further consider the case that both the state equation and the cost functional involve the expectation of the state and/or the control. Such kind of problems are referred to as LQ problems with mean-field, or mean-field LQ problems. One may pose optimal control problems and/or differential games, both in finite and infinite time-horizons. Then, open-loop and closed-loop optimal controls, saddle points, Nash equilibria can be introduced. By our main ideas above, some systematic theory can be (and actually have been) established (see Huang, Li, & Yong, 2015; Li, Shi, & Yong, 2021; Li, Sun, & Yong, 2016; Sun, 2017; Sun & Yong, 2020b; Tang, 2003, 2015; Wei, Yong, & Yu, 2019; Yong, 2013).

One might note that all in the above, the coefficients in the state equation and the quadratic weights in the cost functional are deterministic functions. If we let them to be random, the problems will become much harder, because the associated differential Riccati equation becomes a BSDE with quadratic growth:

$$\begin{cases} dP(t) = - \left[PA + A^\top P + C^\top PC + AC + C^\top \Lambda + Q \right. \\ \quad \left. - (PB + C^\top PD + AD + S^\top)(R + D^\top PD)^{-1} \right. \\ \quad \left. \times (B^\top P + D^\top PC + D^\top \Lambda + S) \right] dt + AdW(t), \\ P(T) = G. \end{cases} \quad (5.1)$$

It was shown in Sun, Xiong, and Yong (2021) that if the cost functional is uniformly convex in the control process, then the above stochastic Riccati equation is uniquely solvable and the open-loop optimal control admits a closed-loop representation. However, unlike the deterministic coefficient case, it is not clear whether the solvability of (5.1) implies the uniform convexity of the cost functional or not. There are also some other relevant results published (see Lü, Wang, and Zhang 2017, Tang 2015, and also Li, Wu, and Yu 2018). But it is far from complete.

One might also note that the stochastic LQ problems studied in the paper are finite-dimensional. For the infinite-dimensional case, we refer the reader to the nice works (Hafizoglu, Lasiecka, Levajković, Mena, & Tuffaha, 2017; Lü, 2019; Lü & Wang, 2023; Lü & Zhang, 2021), as well as the references cited therein, for some recent developments.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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