SIAM J. CONTROL OPTIM.
Vol. 62, No. 1, pp. 752–775

TURNPIKE PROPERTIES FOR MEAN-FIELD LINEAR-QUADRATIC OPTIMAL CONTROL PROBLEMS*

JINGRUI SUN† AND JIONGMIN YONG‡

Abstract. This paper is concerned with an optimal control problem for a mean-field linear stochastic differential equation with a quadratic functional in the infinite time horizon. Under suitable conditions, including the stabilizability, the (strong) exponential, integral, and mean-square turnpike properties for the optimal pair are established. The keys are to correctly formulate the corresponding static optimization problem and find the equations determining the correction processes. These have revealed the main feature of the stochastic problems which are significantly different from the deterministic version of the theory.

Key words. strong turnpike property, mean-field, stochastic optimal control, linear-quadratic, static optimization, stabilizability, Riccati equation

MSC codes. 49N10, 49N80, 93E15, 93E20

DOI. 10.1137/22M1524187

1. Introduction.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space on which a standard one-dimensional Brownian motion $W = \{W(t); t \geqslant 0\}$ is defined, and denote by $\mathbb{F} = \{\mathscr{F}_t\}_{t\geqslant 0}$ the usual augmentation of the natural filtration generated by W. Consider the following controlled linear mean-field stochastic differential equation (SDE):

$$\begin{cases} dX(t) = \big\{AX(t) + \bar{A}\mathbb{E}[X(t)] + Bu(t) + \bar{B}\mathbb{E}[u(t)] + b\big\}dt \\ \qquad\quad + \big\{CX(t) + \bar{C}\mathbb{E}[X(t)] + Du(t) + \bar{D}\mathbb{E}[u(t)] + \sigma\big\}dW(t), \quad t \geqslant 0, \\ X(0) = x, \end{cases} \tag{1.1}$$

and the quadratic cost functional

$$\begin{aligned} J_T(x; u(\cdot)) \triangleq \mathbb{E}\int_0^T \bigg[&\bigg\langle \begin{pmatrix} Q & S^\top \\ S & R \end{pmatrix} \begin{pmatrix} X(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} X(t) \\ u(t) \end{pmatrix} \bigg\rangle + 2\bigg\langle \begin{pmatrix} q \\ r \end{pmatrix}, \begin{pmatrix} X(t) \\ u(t) \end{pmatrix} \bigg\rangle \\ &+ \bigg\langle \begin{pmatrix} \bar{Q} & \bar{S}^\top \\ \bar{S} & \bar{R} \end{pmatrix} \begin{pmatrix} \mathbb{E}[X(t)] \\ \mathbb{E}[u(t)] \end{pmatrix}, \begin{pmatrix} \mathbb{E}[X(t)] \\ \mathbb{E}[u(t)] \end{pmatrix} \bigg\rangle \bigg] dt, \end{aligned} \tag{1.2}$$

where $A, \bar{A}, C, \bar{C}, Q, \bar{Q} \in \mathbb{R}^{n\times n}$, $B, \bar{B}, D, \bar{D} \in \mathbb{R}^{n\times m}$, $S, \bar{S} \in \mathbb{R}^{m\times n}$, and $R, \bar{R} \in \mathbb{R}^{m\times m}$ are constant matrices with $Q, \bar{Q}, R, \bar{R}$ being symmetric; the superscript $\top$ denotes the transpose of matrices; $\langle \cdot, \cdot \rangle$ denotes the inner product of two vectors; $b, \sigma, q \in \mathbb{R}^n$ and

*Received by the editors September 23, 2022; accepted for publication (in revised form) September 13, 2023; published electronically February 16, 2024.
https://doi.org/10.1137/22M1524187
Funding: The first author was supported by NSFC grants 12271242 and 12322118, and by Shenzhen Fundamental Research General Program grant JCYJ20220530112814032. The second author was supported by NSF grant DMS-2305475.
†Department of Mathematics and SUSTech International Center for Mathematics, Southern University of Science and Technology, Shenzhen, Guangdong, 518055, China (sunjr@sustech.edu.cn).
‡Department of Mathematics, University of Central Florida, Orlando, FL 32816 USA (jiongmin.yong@ucf.edu).

$r \in \mathbb{R}^m$ are constant vectors; $x \in \mathbb{R}^n$ is called an *initial state,* and the process $u(\cdot)$, called a *control,* is selected from the space

$$\mathscr{U}[0,T] = \Big\{ u : [0,T] \times \Omega \to \mathbb{R}^m \;\Big|\; u \text{ is } \mathbb{F}\text{-progressively measurable, and } \mathbb{E}\int_0^T |u(t)|^2 dt < \infty \Big\}.$$

We introduce the following problem.

Problem (MFLQ)$_T$. For a given initial state $x \in \mathbb{R}^n$, find a control $u_T(\cdot) \in \mathscr{U}[0,T]$ such that

$$J_T(x; u_T(\cdot)) = \inf_{u(\cdot) \in \mathscr{U}[0,T]} J_T(x; u(\cdot)) \equiv V_T(x). \tag{1.3}$$

The above problem is referred to as a *mean-field linear-quadratic (LQ) optimal control problem* over the finite time horizon $[0,T]$, whose homogenous version (i.e., the case of $b, \sigma, q = 0$, $r = 0$) was initially studied by Yong [22] (in which the coefficients of the state equation and the weighting matrices of the cost functional are allowed to be time-dependent) under the standard condition in LQ theory. The results of Yong [22] were later generalized by Sun [18] to the case of uniformly convex cost functionals with nonhomogenous terms and by Huang, Li, and Yong [11] to the infinite time horizon case. Along this line, many researchers investigated the mean-field LQ problem from various points of view. For example, Basei and Pham [1] studied the mean-field LQ problem using a weak martingale approach; Li, Li, and Yu [12] introduced the notion of relax compensators to deal with indefinite problems; and Lü [13] considered a mean-field LQ problem for stochastic evolution equations.

For a fixed time horizon, finite or infinite, Problem (MFLQ)$_T$ has been well studied in recent years, whose optimal control usually admits a closed-loop representation via the solutions of two related Riccati differential/algebra equations (see [19, Chapter 3]). However, the limiting behavior of the optimal pair as the time horizon tends to infinity has not yet been fully discussed.

In the case that the diffusion term of the state equation (1.1) and the mean-field terms $\mathbb{E}[X(t)], \mathbb{E}[u(t)]$ are absent, Problem (MFLQ)$_T$ reduces to a deterministic LQ problem, denoted by Problem (LQ)$_T$, for which Porretta and Zuazua [16] and Trélat and Zuazua [21] obtained the following result: The optimal pair exponentially converges in the transient time (as $T \to \infty$) to the minimum point of the corresponding *static optimization problem,* under the controllability assumption on the state equation and the observability assumption on the cost functional. More precisely, it was shown in [16, 21] that there exist constants $K, \lambda > 0$, independent of T, such that the optimal pair $(X_T^*(\cdot), u_T^*(\cdot))$ of Problem (LQ)$_T$ satisfies

$$|X_T^*(t) - x^*| + |u_T^*(t) - u^*| \leqslant K\Big[e^{-\lambda t} + e^{-\lambda(T-t)}\Big] \quad \forall t \in [0,T], \tag{1.4}$$

where (x^*, u^*) is the solution to the following static optimization problem:

$$\begin{cases} \text{Minimize} \quad F_0(x,u) \equiv \langle Qx, x\rangle + 2\langle Sx, u\rangle + \langle Ru, u\rangle + 2\langle q, x\rangle + 2\langle r, u\rangle, \\ \text{subject to} \quad Ax + Bu + b = 0. \end{cases} \tag{1.5}$$

Clearly, (1.4) implies that

$$|X_T^*(t) - x^*| + |u_T^*(t) - u^*| \leqslant 2Ke^{-\lambda\delta T} \quad \forall t \in [\delta T, (1-\delta)T] \tag{1.6}$$

for any $\delta \in (0, \frac{1}{2})$. This means that in the major part of the time interval $[0,T]$, the optimal pair $(X_T^*(\cdot), u_T^*(\cdot))$ of Problem $(\mathrm{LQ})_T$ stays close to the solution (x^*, u^*) of the static optimization problem (1.5). Such a phenomenon is very similar to the *turnpike* in the US highway system. This is the main reason of the name "turnpike property" was made. Mathematically, we refer to (1.4) as the *exponential turnpike property* of the (deterministic) LQ optimal control problem.

The turnpike property was first realized by von Neumann [15] for infinite time horizon deterministic optimal growth problems.[1] The name "turnpike" was first coined by Dorfman, Samuelson, and Solow [7] in 1958. See [5] for an excellent survey. In the past several decades, the turnpike properties have attracted attentions of many researchers as such a property often gives people an essential picture of the optimal pair without solving it analytically and leads to a significant simplification in numerical methods for solving such a kind of optimal control problems. A large number of papers have been published for finite- and infinite-dimensional problems in the context of discrete-time and continuous-time deterministic systems; see, for example, [5, 23, 6, 25, 9, 14, 24, 2, 10, 17, 8] and the references therein.

Other than considering deterministic optimal control problems, we now look at the stochastic case. To begin with, we first assume that

$$\bar{A} = \bar{C} = 0, \quad \bar{B} = \bar{D} = 0, \quad \begin{pmatrix} \bar{Q} & \bar{S}^\top \\ \bar{S} & \bar{R} \end{pmatrix} = 0,$$

i.e., the problem does not involve mean-field terms. We denote the corresponding optimal control problem by Problem $(\mathrm{SLQ})_T$. For such a problem, to discuss the turnpike properties of the optimal pair, the major difficulty is to correctly formulate the corresponding static optimization problem. Note that intuitively mimicking the situation of Problem $(\mathrm{LQ})_T$ (the deterministic LQ problem) will lead to an incorrect static optimization problem. Recently, Sun, Wang, and Yong [20] investigated the turnpike properties for Problem $(\mathrm{SLQ})_T$ and found that the correct static optimization problem takes the following form:

$$\begin{cases} \text{Minimize} \quad F_1(x,u) \equiv F_0(x,u) + \langle P(Cx+Du+\sigma), Cx+Du+\sigma \rangle, \\ \text{subject to} \quad Ax+Bu+b=0, \end{cases} \tag{1.7}$$

where $P > 0$ is the solution to the following algebraic Riccati equation:

$$\begin{aligned} &PA + A^\top P + C^\top PC + Q \\ &\quad - (PB + C^\top PD + S^\top)(R + D^\top PD)^{-1}(B^\top P + D^\top C + S) = 0, \end{aligned} \tag{1.8}$$

whose solvability is guaranteed under some mild conditions. It was shown that the *expectation* of the optimal pair exhibits similar turnpike properties as the deterministic case, that is, for some positive constants $K, \lambda > 0$ independent of T,

$$\big|\mathbb{E}[X_T^*(t) - x^*]\big| + \big|\mathbb{E}[u_T^*(t) - u^*]\big| \leqslant K\Big[e^{-\lambda t} + e^{-\lambda(T-t)}\Big] \quad \forall t \in [0,T], \tag{1.9}$$

where (x^*, u^*) is the solution to the static optimization problem (1.7). It is worth noting that (1.9) only tells us $(\mathbb{E}[X_T^*(\cdot)], \mathbb{E}[u_T^*(\cdot)])$ converges exponentially in the transient time. In general, however, we could not expect the following:

$$\mathbb{E}|X_T^*(t) - x^*| + \mathbb{E}|u_T^*(t) - u^*| \leqslant K\Big[e^{-\lambda t} + e^{-\lambda(T-t)}\Big] \quad \forall t \in [0,T]. \tag{1.10}$$

[1] Later, these problems have been formulated as infinite time horizon optimal control problems.

But the behavior of the path $t \mapsto (X_T^*(t), u_T^*(t))$ seems to be more important and it is a crucial step to establishing general nonlinear theory.

In this paper, we shall investigate the turnpike properties for Problem $(\text{MFLQ})_T$. The main novelty and contributions of this paper can be briefly summarized as follows.

• We are concerned with stochastic mean-field LQ problems. By including the expectations of the state and the control, our Problem $(\text{MFLQ})_T$ substantially extends the ones studied previously. The significance is that we have successfully found the correct formulation of the static optimization problem for the current case, merely under the stabilizability of the state equation, which covers that for Problem $(\text{SLQ})_T$ (without mean-field terms) presented in [20].

• As (1.10) could not be expected in general, it needs to be corrected. We find the proper replacement of (x^*, u^*). It turns out that there exist stochastic processes $\boldsymbol{X}^*(\cdot)$ and $\boldsymbol{u}^*(\cdot)$ independent of T, which can be determined explicitly, such that the optimal pair $(X_T^*(\cdot), u_T^*(\cdot))$ of Problem $(\text{MFLQ})_T$ satisfies the following *strong turnpike property*:

$$\mathbb{E}|X_T^*(t) - \boldsymbol{X}^*(t)|^2 + \mathbb{E}|u_T^*(t) - \boldsymbol{u}^*(t)|^2 \leqslant K\Big[e^{-\lambda t} + e^{-\lambda(T-t)}\Big] \quad \forall t \in [0,T], \tag{1.11}$$

for some constants $K, \lambda > 0$ independent of T. The processes $\boldsymbol{X}^*(\cdot)$ and $\boldsymbol{u}^*(\cdot)$ have time-invariant means x^* and u^*, respectively, and (x^*, u^*) is exactly the solution to the corresponding static optimization problem.

• We point out that for deterministic LQ problems, the controllability is normally assumed to establish the turnpike properties; see, for example, [16, 21, 14]. We find that it suffices to assume the stabilizability condition for the state equation, which is much weaker than the controllability condition. As a matter of fact, for controlled stochastic linear SDEs (with or without mean-field terms), the former is much natural and easy to check than the latter.

The rest of this paper is organized as follows. In section 2, we make some necessary preliminaries by introducing some notation, assumptions, and a number of basic results that will be needed later. In section 3, we introduce the static optimization problem associated with Problem $(\text{MFLQ})_T$ and state the main results of this paper. Section 4 is devoted to the stability of Riccati equations, which plays a central role in studying the turnpike properties. In section 5, we give the proofs of the main results stated in section 3.

2. Preliminaries. In this section, we introduce the basic notation and assumptions which will be used throughout this paper. We also collect a number of basic results on stochastic LQ optimal control problems and prove a simple matrix inequality for later use.

For notational simplicity, we let

$$\begin{aligned} &\widehat{A} = A + \bar{A}, \quad &&\widehat{B} = B + \bar{B}, \quad &&\widehat{C} = C + \bar{C}, \quad &&\widehat{D} = D + \bar{D}, \\ &\widehat{Q} = Q + \bar{Q}, \quad &&\widehat{S} = S + \bar{S}, \quad &&\widehat{R} = R + \bar{R}. \end{aligned} \tag{2.1}$$

Denote by $\mathbb{S}^n$ the space of symmetric $n \times n$ real matrices. For $P, \Pi \in \mathbb{S}^n$, we let

$$\begin{aligned} &\mathcal{Q}(P) = PA + A^\top P + C^\top PC + Q, \quad &&\widehat{\mathcal{Q}}(P,\Pi) = \Pi\widehat{A} + \widehat{A}^\top \Pi + \widehat{C}^\top P\widehat{C} + \widehat{Q}, \\ &\mathcal{S}(P) = B^\top P + D^\top PC + S, \quad &&\widehat{\mathcal{S}}(P,\Pi) = \widehat{B}^\top \Pi + \widehat{D}^\top P\widehat{C} + \widehat{S}, \\ &\mathcal{R}(P) = R + D^\top PD, \quad &&\widehat{\mathcal{R}}(P) = \widehat{R} + \widehat{D}^\top P\widehat{D}. \end{aligned} \tag{2.2}$$

For matrices $M, N \in \mathbb{S}^n$, we will write $M \geqslant N$ (respectively, $M > N$) if $M - N$ is positive semidefinite (respectively, positive definite). The following basic assumptions will be imposed throughout this paper.

(A1) The weighting matrices in (1.2) satisfy

$$R, \widehat{R} > 0, \quad Q - S^\top R^{-1} S > 0, \quad \widehat{Q} - \widehat{S}^\top \widehat{R}^{-1} \widehat{S} > 0.$$

(A2) The controlled ordinary differential equation (ODE)

$$\dot{X}(t) = \widehat{A} X(t) + \widehat{B} u(t), \quad t \geqslant 0, \tag{2.3}$$

is stabilizable, i.e., there exists a $\widehat{\Theta} \in \mathbb{R}^{m \times n}$ such that all the eigenvalues of $\widehat{A} + \widehat{B}\widehat{\Theta}$ have negative real parts. In this case, $\widehat{\Theta}$ is called a *stabilizer* of (2.3). The controlled SDE

$$dX(t) = [AX(t) + Bu(t)]dt + [CX(t) + Du(t)]dW(t), \quad t \geqslant 0, \tag{2.4}$$

is L^2-stabilizable, i.e., there exists a $\Theta \in \mathbb{R}^{m \times n}$ such that for every initial state x, the solution $X(\cdot)$ of

$$\begin{cases} dX(t) = (A + B\Theta)X(t)dt + (C + D\Theta)X(t)dW(t), \quad t \geqslant 0, \\ X(0) = x \end{cases}$$

satisfies $\mathbb{E}\int_0^\infty |X(t)|^2 dt < \infty$. In this case, Θ is called a *stabilizer* of (2.4).

Let $C([0,T]; \mathbb{S}^n)$ be the space of $\mathbb{S}^n$-valued continuous functions on $[0,T]$. With the notation (2.1) and (2.2), the differential Riccati equations associated with Problem $(\text{MFLQ})_T$ can be, respectively, written as

$$\begin{cases} \dot{P}_T(t) + \mathcal{Q}(P_T(t)) - \mathcal{S}(P_T(t))^\top \mathcal{R}(P_T(t))^{-1} \mathcal{S}(P_T(t)) = 0, \\ P_T(T) = 0 \end{cases} \tag{2.5}$$

and

$$\begin{cases} \dot{\Pi}_T(t) + \widehat{\mathcal{Q}}(P_T(t), \Pi_T(t)) - \widehat{\mathcal{S}}(P_T(t), \Pi_T(t))^\top \widehat{\mathcal{R}}(P_T(t))^{-1} \widehat{\mathcal{S}}(P_T(t), \Pi_T(t)) = 0, \\ \Pi_T(T) = 0. \end{cases} \tag{2.6}$$

We present the following result concerning the existence and uniqueness of solutions to (2.5) and (2.6). For a proof, we refer to Yong [22, section 4].

LEMMA 2.1. *Let* (A1) *hold. Then for any* $T > 0$*, the differential Riccati equations* (2.5) *and* (2.6) *admit a unique solution pair* $(P_T(\cdot), \Pi_T(\cdot)) \in C([0,T]; \mathbb{S}^n) \times C([0,T]; \mathbb{S}^n)$ *satisfying* $P_T(t), \Pi_T(t) \geqslant 0$ *for all* $t \in [0,T]$.

When considering mean-field LQ optimal control problems over an infinite time horizon, we encounter *algebraic* Riccati equations (AREs) instead of differential ones. The following result establishes the unique solvability of the associated AREs, whose proof can be found in [11, Theorem 5.1]; see also [19, Chapter 3].

LEMMA 2.2. *Let* (A1)–(A2) *hold. Then the AREs*

$$\mathcal{Q}(P) - \mathcal{S}(P)^\top \mathcal{R}(P)^{-1} \mathcal{S}(P) = 0, \tag{2.7}$$

$$\widehat{\mathcal{Q}}(P, \Pi) - \widehat{\mathcal{S}}(P, \Pi)^\top \widehat{\mathcal{R}}(P)^{-1} \widehat{\mathcal{S}}(P, \Pi) = 0 \tag{2.8}$$

admit a unique solution pair (P,Π) *satisfying* $P,\Pi>0$. *Moreover, the matrix*

$$\widehat{\Theta}\triangleq-\widehat{\mathcal{R}}(P)^{-1}\widehat{\mathcal{S}}(P,\Pi)$$

is a stabilizer of system (2.3), *and the matrix*

$$\Theta\triangleq-\mathcal{R}(P)^{-1}\mathcal{S}(P)$$

is a stabilizer of system (2.4).

Comparing (2.7) with (2.5), and (2.8) with (2.6), we may guess that there are some connections between P and $P_T(\cdot)$, and between Π and $\Pi_T(\cdot)$. The following result, proved in [20, Theorem 4.1], establishes the connection between P and $P_T(\cdot)$. In section 4, we will establish the connection between Π and $\Pi_T(\cdot)$.

LEMMA 2.3. *Let* (A1)–(A2) *hold. Let* $P_T(\cdot)$ *and* P *be the unique solutions of* (2.5) *and* (2.7), *respectively. Then there exist constants* $K,\lambda>0$, *independent of* T, *such that*

$$|P_T(t)-P|\leqslant Ke^{-\lambda(T-t)}\quad\forall t\in[0,T].$$

We conclude this section with a simple matrix inequality that will be needed later.

LEMMA 2.4. *Let* $M\in\mathbb{R}^{n\times m}$ *and* $K\in\mathbb{S}^m$. *If* $K>0$, *then*

$$M(K+M^\top M)^{-1}M^\top\leqslant I.$$

Proof. Let $L\in\mathbb{R}^{m\times m}$ be an invertible matrix such that $K=L^\top L$. Then

$$M(K+M^\top M)^{-1}M^\top=ML^{-1}[I+(ML^{-1})^\top(ML^{-1})]^{-1}(ML^{-1})^\top.$$

Thus, without loss of generality we may assume $K=I$. Observe that

$$M(I+M^\top M)^{-1}M^\top=MM^\top(I+MM^\top)^{-1}.$$

For any $\varepsilon>0$, $N_\varepsilon\triangleq\varepsilon I+MM^\top>0$ and hence

$$N_\varepsilon(I+N_\varepsilon)^{-1}=(N_\varepsilon^{-1}+I)^{-1}\leqslant I.$$

Letting $\varepsilon\to0$ yields the desired result. ∎

3. Main results. We present the main results of the paper in this section. The rigorous proofs are deferred to the subsequent sections.

First, let us introduce the static optimization problem associated with Problem $(\mathrm{MFLQ})_T$. From the previous section we know (recall Lemma 2.2 and the notation of (2.1) and (2.2)) that under (A1) and (A2), the ARE (2.7) has a unique positive definite solution P. Let

$$\mathscr{V}\triangleq\big\{(x,u)\in\mathbb{R}^n\times\mathbb{R}^m\mid\widehat{A}x+\widehat{B}u+b=0\big\},\tag{3.1}$$

and define a continuous function $F:\mathscr{V}\to\mathbb{R}$ by

$$\begin{aligned}F(x,u)=&\langle\widehat{Q}x,x\rangle+\langle\widehat{R}u,u\rangle+2\langle\widehat{S}x,u\rangle+2\langle q,x\rangle+2\langle r,u\rangle\\&+\langle P(\widehat{C}x+\widehat{D}u+\sigma),\widehat{C}x+\widehat{D}u+\sigma\rangle.\end{aligned}\tag{3.2}$$

The static optimization problem associated with Problem $(\mathrm{MFLQ})_T$ can be stated as follows.

Problem (O). Find a pair $(x^*,u^*)\in\mathscr{V}$ such that

$$F(x^*,u^*)=\min_{(x,u)\in\mathscr{V}}F(x,u)\equiv V.$$

For the solvability of Problem (O), we have the following result.

PROPOSITION 3.1. *Let* (A1)–(A2) *hold. Then Problem* (O) *has a unique solution. Moreover,* (x^*, u^*) *is the solution of Problem* (O) *if and only if there exists a unique* $\lambda^* \in \mathbb{R}^n$ *such that*

$$\begin{cases} \widehat{A}x^* + \widehat{B}u^* + b = 0, \\ \widehat{A}^\top \lambda^* + \widehat{Q}x^* + \widehat{C}^\top P(\widehat{C}x^* + \widehat{D}u^* + \sigma) + \widehat{S}^\top u^* + q = 0, \\ \widehat{B}^\top \lambda^* + \widehat{R}u^* + \widehat{D}^\top P(\widehat{C}x^* + \widehat{D}u^* + \sigma) + \widehat{S}x^* + r = 0. \end{cases} \tag{3.3}$$

The proof of Proposition 3.1 is similar to the case without mean-field terms. We omit it here and refer the reader to [20, Proposition 3.2] for details.

Let (x^*, u^*) be the solution of Problem (O), and let $\lambda^* \in \mathbb{R}^n$ be the vector in (3.3). Set

$$\sigma^* = \widehat{C}x^* + \widehat{D}u^* + \sigma, \quad \Theta = -\mathcal{R}(P)^{-1}\mathcal{S}(P), \tag{3.4}$$

where P is the solution to the ARE (2.7). Let $X^*(\cdot)$ be the solution to the SDE

$$\begin{cases} dX^*(t) = (A + B\Theta)X^*(t)dt + \big[(C + D\Theta)X^*(t) + \sigma^*\big]dW(t), \quad t \geqslant 0, \\ X^*(0) = 0, \end{cases} \tag{3.5}$$

and define

$$\boldsymbol{X}^*(t) \triangleq X^*(t) + x^*, \quad \boldsymbol{u}^*(t) \triangleq \Theta X^*(t) + u^*. \tag{3.6}$$

Clearly, $\mathbb{E}[X^*(t)] = 0$. Hence, one has $\mathbb{E}[\boldsymbol{X}^*(t)] = x^*$ and $\mathbb{E}[\boldsymbol{u}^*(t)] = u^*$ for all $t \geqslant 0$.

We now state the main result of this paper, which establishes the exponential turnpike property of Problem $(\text{MFLQ})_T$. Recall from Proposition 3.1, Theorem 5.2, and Proposition 5.3 in [18] that under (A1), Problem $(\text{MFLQ})_T$ admits a unique optimal pair for every initial state x.

THEOREM 3.2. *Let* (A1)–(A2) *hold. Let* $(X_T^*(\cdot), u_T^*(\cdot))$ *be the optimal pair of Problem* $(\text{MFLQ})_T$ *for the initial state* x*. Then there exist constants* $K, \lambda > 0$*, independent of* T*, such that*

$$\mathbb{E}|X_T^*(t) - \boldsymbol{X}^*(t)|^2 + \mathbb{E}|u_T^*(t) - \boldsymbol{u}^*(t)|^2 \leqslant K\Big[e^{-\lambda t} + e^{-\lambda(T-t)}\Big] \quad \forall t \in [0, T]. \tag{3.7}$$

We call (3.7) the *strong exponential turnpike property* for the optimal pair $(X_T^*(\cdot), u_T^*(\cdot))$. The above result has several consequences. The first corollary establishes the *strong integral* and the *mean-square turnpike properties* for Problem $(\text{MFLQ})_T$, whose proof is direct.

COROLLARY 3.3. *Let* (A1)–(A2) *hold. Then,*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{E}\Big[|X_T^*(t) - \boldsymbol{X}^*(t)|^2 + |u_T^*(t) - \boldsymbol{u}^*(t)|^2\Big]dt = 0.$$

Consequently,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \Big[\big|\mathbb{E}[X_T^*(t)] - x^*\big|^2 + \big|\mathbb{E}[u_T^*(t)] - u^*\big|^2\Big]dt = 0.$$

The second corollary shows that for any initial state x, the value $V_T(x)$ of Problem $(\text{MFLQ})_T$ converges to the minimum of Problem (O) in the time-average sense.

COROLLARY 3.4. *Let* (A1)–(A2) *hold. Then*

$$\lim_{T\to\infty}\frac{1}{T}V_T(x)=V\quad\forall x\in\mathbb{R}^n.$$

Let $(X_T^*(\cdot),u_T^*(\cdot))$ be the optimal pair of Problem $(\mathrm{MFLQ})_T$ for the initial state x. By the results of [3] and [4], the following mean-field backward SDE admits a unique adapted solution $(Y_T^*(\cdot),Z_T^*(\cdot))$ (called the *adjoint process* of $X_T^*(\cdot)$):

$$\begin{cases} dY_T^*(t)=-\big\{A^\top Y_T^*(t)+\bar A^\top\mathbb{E}[Y_T^*(t)]+C^\top Z_T^*(t)+\bar C^\top\mathbb{E}[Z_T^*(t)]+QX_T^*(t)\\ \qquad\qquad +\bar Q\mathbb{E}[X_T^*(t)]+S^\top u_T^*(t)+\bar S^\top\mathbb{E}[u_T^*(t)]+q\big\}dt+Z_T^*(t)dW(t),\\ Y_T^*(T)=0. \end{cases}\tag{3.8}$$

The next corollary shows that a strong exponential turnpike property also holds for the adjoint process.

COROLLARY 3.5. *Let* (A1)–(A2) *hold. Let*

$$\boldsymbol{Y}^*(t)\triangleq PX^*(t)+\lambda^*,\quad \boldsymbol{Z}^*(t)\triangleq P(C+D\Theta)X^*(t)+P\sigma^*.\tag{3.9}$$

Then for some constants $K,\lambda>0$ independent of T,

$$\mathbb{E}|Y_T^*(t)-\boldsymbol{Y}^*(t)|^2+\mathbb{E}|Z_T^*(t)-\boldsymbol{Z}^*(t)|^2\leqslant K\Big[e^{-\lambda t}+e^{-\lambda(T-t)}\Big]\quad\forall t\in[0,T].\tag{3.10}$$

4. Stability of Riccati equations. As mentioned in section 2, there is a certain connection between the solutions to the differential Riccati equation (2.6) and the ARE (2.8). We now examine this connection. The main result of this section is as follows, which plays a central role in the study of turnpike properties.

THEOREM 4.1. *Let* (A1)–(A2) *hold. Let $\Pi_T(\cdot)$ and Π be the unique solutions of* (2.6) *and* (2.8)*, respectively. Then there exist positive constants $K,\lambda>0$ such that*

$$|\Pi_T(t)-\Pi|\leqslant Ke^{-\lambda(T-t)}\quad\forall\,0\leqslant t\leqslant T<\infty.\tag{4.1}$$

Before going further, let us point out that in order to prove Theorem 4.1, we can assume without loss of generality that

$$S=0,\quad \bar S=0.\tag{4.2}$$

Indeed, let

$$\mathcal{A}=A-BR^{-1}S,\quad \mathcal{C}=C-DR^{-1}S,\quad \mathcal{Q}=Q-S^\top R^{-1}S,\tag{4.3}$$

and let $\mathcal{P}(\cdot)$ be the positive semidefinite solution to the Riccati equation

$$\begin{cases} \dot{\mathcal{P}}+\mathcal{P}\mathcal{A}+\mathcal{A}^\top\mathcal{P}+\mathcal{C}^\top\mathcal{P}\mathcal{C}+\mathcal{Q}\\ \quad-(\mathcal{P}B+\mathcal{C}^\top\mathcal{P}D)(R+D^\top\mathcal{P}D)^{-1}(B^\top\mathcal{P}+D^\top\mathcal{P}\mathcal{C})=0,\\ \mathcal{P}(T)=0. \end{cases}\tag{4.4}$$

We have

$$\begin{aligned} &\mathcal{P}\mathcal{A}+\mathcal{A}^\top\mathcal{P}+\mathcal{C}^\top\mathcal{P}\mathcal{C}+\mathcal{Q}\\ &\quad=\mathcal{P}A+A^\top\mathcal{P}+C^\top\mathcal{P}C+Q-(\mathcal{P}B+C^\top\mathcal{P}D+S^\top)R^{-1}S\\ &\qquad-S^\top R^{-1}(B^\top\mathcal{P}+D^\top\mathcal{P}C+S)+S^\top R^{-1}(R+D^\top\mathcal{P}D)R^{-1}S. \end{aligned}\tag{4.5}$$

On the other hand,

$$\mathcal{P}B+\mathcal{C}^\top\mathcal{P}D=(\mathcal{P}B+C^\top\mathcal{P}D+S^\top)-S^\top R^{-1}(R+D^\top\mathcal{P}D),$$

and hence

$$\begin{aligned}&(\mathcal{P}B+\mathcal{C}^\top\mathcal{P}D)(R+D^\top\mathcal{P}D)^{-1}(B^\top\mathcal{P}+D^\top\mathcal{P}\mathcal{C})\\&\quad=(\mathcal{P}B+C^\top\mathcal{P}D+S^\top)(R+D^\top\mathcal{P}D)^{-1}(B^\top\mathcal{P}+D^\top\mathcal{P}C+S)\\&\qquad-(\mathcal{P}B+C^\top\mathcal{P}D+S^\top)R^{-1}S-S^\top R^{-1}(B^\top\mathcal{P}+D^\top\mathcal{P}C+S)\\&\qquad+S^\top R^{-1}(R+D^\top\mathcal{P}D)R^{-1}S.\end{aligned}\tag{4.6}$$

Substituting (4.5) and (4.6) into (4.4), we see that $\mathcal{P}(\cdot)$ satisfies the same equation as $P_T(\cdot)$, which implies $\mathcal{P}(\cdot)=P_T(\cdot)$. By a similar argument, we can show that $\Pi_T(\cdot)$ satisfies

$$\begin{cases}\dot\Pi_T+\Pi_T\widehat{\mathcal{A}}+\widehat{\mathcal{A}}^\top\Pi_T+\widehat{\mathcal{C}}^\top P_T\widehat{\mathcal{C}}+\widehat{\mathcal{Q}}\\\qquad-(\Pi_T\widehat{B}+\widehat{\mathcal{C}}^\top P_T\widehat{D})(\widehat{R}+\widehat{D}^\top P_T\widehat{D})^{-1}(\widehat{B}^\top\Pi_T+\widehat{D}^\top P_T\widehat{\mathcal{C}})=0,\\\Pi_T(T)=0,\end{cases}\tag{4.7}$$

where

$$\widehat{\mathcal{A}}=\widehat{A}-\widehat{B}\widehat{R}^{-1}\widehat{S},\quad\widehat{\mathcal{C}}=\widehat{C}-\widehat{D}\widehat{R}^{-1}\widehat{S},\quad\widehat{\mathcal{Q}}=\widehat{Q}-\widehat{S}^\top\widehat{R}^{-1}\widehat{S}.\tag{4.8}$$

Thus, by making transformations (4.3) and (4.8), we can assume without loss of generality that (4.2) holds.

To prove Theorem 4.1, we need the following lemma.

LEMMA 4.2. *Let* (A1) *hold. Then for any positive semidefinite matrix* $\Delta\geqslant0$,

$$\begin{pmatrix}\widehat{C}^\top\Delta\widehat{C}+\widehat{Q}&\widehat{C}^\top\Delta\widehat{D}\\\widehat{D}^\top\Delta\widehat{C}&\widehat{\mathcal{R}}(\Delta)\end{pmatrix}\geqslant0,$$

or equivalently,

$$\widehat{C}^\top\Delta\widehat{C}+\widehat{Q}-\Big(\widehat{C}^\top\Delta\widehat{D}\Big)\widehat{\mathcal{R}}(\Delta)^{-1}\Big(\widehat{D}^\top\Delta\widehat{C}\Big)\geqslant0.$$

Proof. Since $\widehat{R}>0$ and $\Delta\geqslant0$, we have

$$\widehat{\mathcal{R}}(\Delta)=\widehat{R}+\widehat{D}^\top\Delta\widehat{D}\geqslant\widehat{R}>0.$$

Let $M\in\mathbb{R}^{n\times n}$ be such that $\Delta=M^\top M$. Then by Lemma 2.4,

$$I-\Big(M\widehat{D}\Big)\widehat{\mathcal{R}}(\Delta)^{-1}\Big(\widehat{D}^\top M^\top\Big)=I-\Big(M\widehat{D}\Big)\Big[\widehat{R}+\Big(M\widehat{D}\Big)^\top\Big(M\widehat{D}\Big)\Big]^{-1}\Big(M\widehat{D}\Big)^\top\geqslant0.$$

Consequently,

$$\begin{aligned}&\widehat{C}^\top\Delta\widehat{C}+\widehat{Q}-\big(\widehat{C}^\top\Delta\widehat{D}\big)\widehat{\mathcal{R}}(\Delta)^{-1}\big(\widehat{D}^\top\Delta\widehat{C}\big)\\&\quad=\widehat{Q}+\widehat{C}^\top\Big[\Delta-\big(\Delta\widehat{D}\big)\widehat{\mathcal{R}}(\Delta)^{-1}\big(\widehat{D}^\top\Delta\big)\Big]\widehat{C}\\&\quad=\widehat{Q}+\big(M\widehat{C}\big)^\top\Big[I-\big(M\widehat{D}\big)\widehat{\mathcal{R}}(\Delta)^{-1}\big(\widehat{D}^\top M^\top\big)\Big]\big(M\widehat{C}\big)\geqslant0.\end{aligned}$$

The proof is now complete. ∎

In the following proof and in what follows, we shall denote by K and λ two generic positive constants, which do not depend on T and may vary from line to line.

Proof of Theorem 4.1. As mentioned previously, we may assume $S=0$ and $\bar{S}=0$ to simplify the proof. Define for $t\in[0,\infty)$,

$$\Sigma(t)\triangleq P_T(T-t),\quad \Upsilon(t)\triangleq \Pi_T(T-t)\quad \text{if } t\leqslant T.$$

It is shown in [20, Proposition 4.1 and Theorem 4.1] that the definition of $\Sigma(t)$ does not depend on the choice of $T\geqslant t$, and

$$0<\Sigma(s)\leqslant\Sigma(t),\quad |\Sigma(t)-P|\leqslant Ke^{-\lambda t}\quad \forall\, 0\leqslant s\leqslant t<\infty \tag{4.9}$$

for some positive constants $K,\lambda>0$. A similar argument shows that the definition of $\Upsilon(t)$ does not depend on the choice of $T\geqslant t$ either, and $\Upsilon(\cdot)$ satisfies the following ODE:

$$\begin{cases}\dot{\Upsilon}(t)=\widehat{\mathcal{Q}}(\Sigma(t),\Upsilon(t))-\widehat{\mathcal{S}}(\Sigma(t),\Upsilon(t))^\top\widehat{\mathcal{R}}(\Sigma(t))^{-1}\widehat{\mathcal{S}}(\Sigma(t),\Upsilon(t)),\\ \Upsilon(0)=0.\end{cases} \tag{4.10}$$

Step 1. We first show that $\lim_{t\to\infty}\Upsilon(t)=\Pi$.

Consider the deterministic LQ optimal control problem with state equation

$$\begin{cases}\dot{X}(t)=\widehat{A}X(t)+\widehat{B}u(t),\quad t\in[0,T],\\ X(0)=x,\end{cases}$$

and cost functional

$$\mathcal{J}_T(x;u(\cdot))=\int_0^T\left\langle\begin{pmatrix}\widehat{C}^\top P_T(t)\widehat{C}+\widehat{Q} & \widehat{C}^\top P_T(t)\widehat{D}\\ \widehat{D}^\top P_T(t)\widehat{C} & \widehat{\mathcal{R}}(P_T(t))\end{pmatrix}\begin{pmatrix}X(t)\\ u(t)\end{pmatrix},\begin{pmatrix}X(t)\\ u(t)\end{pmatrix}\right\rangle dt.$$

Since $P_T(t)>0$,

$$\widehat{\mathcal{R}}(P_T(t))=\widehat{R}+\widehat{D}^\top P_T(t)\widehat{D}\geqslant\widehat{R}>0, \tag{4.11}$$

and by Lemma 4.2,

$$\begin{pmatrix}\widehat{C}^\top P_T(t)\widehat{C}+\widehat{Q} & \widehat{C}^\top P_T(t)\widehat{D}\\ \widehat{D}^\top P_T(t)\widehat{C} & \widehat{\mathcal{R}}(P_T(t))\end{pmatrix}\geqslant 0. \tag{4.12}$$

Because of (4.11) and (4.12), the above deterministic LQ problem is uniquely solvable with value function given by $\langle\Pi_T(0)x,x\rangle$. Note that $\mathcal{J}_T(x;u(\cdot))$ can be written as

$$\mathcal{J}_T(x;u(\cdot))=\int_0^T\left\langle\begin{pmatrix}\widehat{C}^\top \Sigma(T-t)\widehat{C}+\widehat{Q} & \widehat{C}^\top \Sigma(T-t)\widehat{D}\\ \widehat{D}^\top \Sigma(T-t)\widehat{C} & \widehat{\mathcal{R}}(\Sigma(T-t))\end{pmatrix}\begin{pmatrix}X(t)\\ u(t)\end{pmatrix},\begin{pmatrix}X(t)\\ u(t)\end{pmatrix}\right\rangle dt.$$

For $T_2\geqslant T_1\geqslant 0$, set

$$\Delta(t)=\Sigma(T_2-t)-\Sigma(T_1-t),\quad 0\leqslant t\leqslant T_1.$$

Since $\Sigma(\cdot)$ is nondecreasing, $\Delta(t)\geqslant 0$ and hence by Lemma 4.2,

$$\begin{pmatrix}\widehat{C}^\top \Delta(t)\widehat{C}+\widehat{Q} & \widehat{C}^\top \Delta(t)\widehat{D}\\ \widehat{D}^\top \Delta(t)\widehat{C} & \widehat{\mathcal{R}}(\Delta(t))\end{pmatrix}\geqslant 0.$$

Thus, for any square-integrable $u:[0,T_2]\to\mathbb{R}^m$,

$$\begin{aligned}
&\mathcal{J}_{T_2}(x;u(\cdot))-\mathcal{J}_{T_1}(x;u(\cdot))\\
&\quad\geqslant\int_0^{T_1}\left\langle\begin{pmatrix}\widehat{C}^\top\Sigma(T_2-t)\widehat{C}+\widehat{Q} & \widehat{C}^\top\Sigma(T_2-t)\widehat{D}\\ \widehat{D}^\top\Sigma(T_2-t)\widehat{C} & \widehat{\mathcal{R}}(\Sigma(T_2-t))\end{pmatrix}\begin{pmatrix}X(t)\\ u(t)\end{pmatrix},\begin{pmatrix}X(t)\\ u(t)\end{pmatrix}\right\rangle dt\\
&\qquad-\int_0^{T_1}\left\langle\begin{pmatrix}\widehat{C}^\top\Sigma(T_1-t)\widehat{C}+\widehat{Q} & \widehat{C}^\top\Sigma(T_1-t)\widehat{D}\\ \widehat{D}^\top\Sigma(T_1-t)\widehat{C} & \widehat{\mathcal{R}}(\Sigma(T_1-t))\end{pmatrix}\begin{pmatrix}X(t)\\ u(t)\end{pmatrix},\begin{pmatrix}X(t)\\ u(t)\end{pmatrix}\right\rangle dt\\
&\quad=\int_0^{T_1}\left\langle\begin{pmatrix}\widehat{C}^\top\Delta(t)\widehat{C}+\widehat{Q} & \widehat{C}^\top\Delta(t)\widehat{D}\\ \widehat{D}^\top\Delta(t)\widehat{C} & \widehat{\mathcal{R}}(\Delta(t))\end{pmatrix}\begin{pmatrix}X(t)\\ u(t)\end{pmatrix},\begin{pmatrix}X(t)\\ u(t)\end{pmatrix}\right\rangle dt\\
&\quad\geqslant 0.
\end{aligned}$$

This implies that for $0\leqslant T_1\leqslant T_2$,

$$\langle\Pi_{T_1}(0)x,x\rangle\leqslant\langle\Pi_{T_2}(0)x,x\rangle\quad\forall x\in\mathbb{R}^n.$$

Since x is arbitrary, $\Upsilon(t)=\Pi_t(0)$ is nondecreasing in t. On the other hand, take $\widehat{\Theta}$ to be a stabilizer of the system (2.3). By (4.9), $P_T(t)\leqslant P$ and hence (recalling Lemma 4.2)

$$\begin{aligned}
\langle\Pi_T(0)x,x\rangle&\leqslant\int_0^T\left\langle\begin{pmatrix}\widehat{C}^\top P_T(t)\widehat{C}+\widehat{Q} & \widehat{C}^\top P_T(t)\widehat{D}\\ \widehat{D}^\top P_T(t)\widehat{C} & \widehat{\mathcal{R}}(P_T(t))\end{pmatrix}\begin{pmatrix}X(t)\\ \widehat{\Theta}X(t)\end{pmatrix},\begin{pmatrix}X(t)\\ \widehat{\Theta}X(t)\end{pmatrix}\right\rangle dt\\
&\leqslant\int_0^T\left\langle\begin{pmatrix}\widehat{C}^\top P\widehat{C}+\widehat{Q} & \widehat{C}^\top P\widehat{D}\\ \widehat{D}^\top P\widehat{C} & \widehat{\mathcal{R}}(P)\end{pmatrix}\begin{pmatrix}X(t)\\ \widehat{\Theta}X(t)\end{pmatrix},\begin{pmatrix}X(t)\\ \widehat{\Theta}X(t)\end{pmatrix}\right\rangle dt\\
&\leqslant\int_0^\infty\left\langle\begin{pmatrix}\widehat{C}^\top P\widehat{C}+\widehat{Q} & \widehat{C}^\top P\widehat{D}\\ \widehat{D}^\top P\widehat{C} & \widehat{\mathcal{R}}(P)\end{pmatrix}\begin{pmatrix}X(t)\\ \widehat{\Theta}X(t)\end{pmatrix},\begin{pmatrix}X(t)\\ \widehat{\Theta}X(t)\end{pmatrix}\right\rangle dt<\infty.
\end{aligned}$$

Thus, $\Upsilon(t)=\Pi_t(0)$ is bounded above, which, together with the monotonicity of $\Upsilon(t)$, implies that

$$\Upsilon_\infty\triangleq\lim_{t\to\infty}\Upsilon(t)$$

exists and is finite. From (4.10) we have

$$\Upsilon(t+1)-\Upsilon(t)=\int_t^{t+1}\widehat{\mathcal{Q}}(\Sigma(s),\Upsilon(s))-\widehat{\mathcal{S}}(\Sigma(s),\Upsilon(s))^\top\widehat{\mathcal{R}}(\Sigma(s))^{-1}\widehat{\mathcal{S}}(\Sigma(s),\Upsilon(s))ds.$$

Letting $t\to\infty$ yields

$$\widehat{\mathcal{Q}}(P,\Upsilon_\infty)-\widehat{\mathcal{S}}(P,\Upsilon_\infty)^\top\widehat{\mathcal{R}}(P)^{-1}\widehat{\mathcal{S}}(P,\Upsilon_\infty)=0.$$

By uniqueness, $\Upsilon_\infty=\Pi$.

Step 2. Set $\Delta(t)\triangleq\Upsilon(t)-\Pi$ and $\Lambda(t)\triangleq\Sigma(t)-P$. We show that $\Delta(\cdot)$ satisfies the following ODE:

$$\dot{\Delta}(t)=\Delta(t)\bar{\mathcal{A}}+\bar{\mathcal{A}}^\top\Delta(t)+f(\Delta(t),\Lambda(t))+g(\Lambda(t)),\tag{4.13}$$

where

$$\begin{aligned}
\bar{\mathcal{A}} &\triangleq \widehat{A} + \widehat{B}\widehat{\Theta} \quad \text{with} \quad \widehat{\Theta} \triangleq -\widehat{\mathcal{R}}(P)^{-1}\widehat{\mathcal{S}}(P,\Pi),\\
f(\Delta(t),\Lambda(t)) &\triangleq -\widehat{\mathcal{S}}(\Lambda(t),\Delta(t))^\top \widehat{\mathcal{R}}(\Sigma(t))^{-1}\widehat{\mathcal{S}}(\Lambda(t),\Delta(t))\\
&\quad - \Big[(\widehat{D}\widehat{\Theta})^\top \Lambda(t)\widehat{D}\widehat{\mathcal{R}}(\Sigma(t))^{-1}\widehat{B}^\top \Delta(t)\Big]^\top\\
&\quad - \Big[(\widehat{D}\widehat{\Theta})^\top \Lambda(t)\widehat{D}\widehat{\mathcal{R}}(\Sigma(t))^{-1}\widehat{B}^\top \Delta(t)\Big],\\
g(\Lambda(t)) &\triangleq \widehat{C}^\top \Lambda(t)\widehat{C} - (\widehat{D}\widehat{\Theta})^\top \Lambda(t)\widehat{D}\widehat{\mathcal{R}}(\Sigma(t))^{-1}\widehat{\mathcal{S}}(P,\Pi)\\
&\quad - \Big[\widehat{\mathcal{S}}(P,\Pi)^\top \widehat{\mathcal{R}}(\Sigma(t))^{-1}\widehat{D}^\top \Lambda(t)\widehat{C}\Big]^\top\\
&\quad - \Big[\widehat{\mathcal{S}}(P,\Pi)^\top \widehat{\mathcal{R}}(\Sigma(t))^{-1}\widehat{D}^\top \Lambda(t)\widehat{C}\Big].
\end{aligned}$$

To verify (4.13), we first observe that

$$\begin{aligned}
\dot{\Delta}(t) &= \Delta(t)\widehat{A} + \widehat{A}^\top \Delta(t) + \widehat{C}^\top \Lambda(t)\widehat{C}\\
&\quad - \Big[\widehat{B}^\top \Upsilon(t) + \widehat{D}^\top \Sigma(t)\widehat{C}\Big]^\top \Big[\widehat{R} + \widehat{D}^\top \Sigma(t)\widehat{D}\Big]^{-1}\Big[\widehat{B}^\top \Upsilon(t) + \widehat{D}^\top \Sigma(t)\widehat{C}\Big],\\
&\quad + \Big[\widehat{B}^\top \Pi + \widehat{D}^\top P\widehat{C}\Big]^\top \Big[\widehat{R} + \widehat{D}^\top P\widehat{D}\Big]^{-1}\Big[\widehat{B}^\top \Pi + \widehat{D}^\top P\widehat{C}\Big].
\end{aligned} \tag{4.14}$$

Noting that

$$\widehat{B}^\top \Upsilon + \widehat{D}^\top \Sigma(t)\widehat{C} = \Big[\widehat{B}^\top \Delta(t) + \widehat{D}^\top \Lambda(t)\widehat{C}\Big] + \Big[\widehat{B}^\top \Pi + \widehat{D}^\top P\widehat{C}\Big],$$

we have

$$\begin{aligned}
&\Big[\widehat{B}^\top \Upsilon(t) + \widehat{D}^\top \Sigma(t)\widehat{C}\Big]^\top \Big[\widehat{R} + \widehat{D}^\top \Sigma(t)\widehat{D}\Big]^{-1}\Big[\widehat{B}^\top \Upsilon(t) + \widehat{D}^\top \Sigma(t)\widehat{C}\Big]\\
&\quad = \Big[\widehat{B}^\top \Delta(t) + \widehat{D}^\top \Lambda(t)\widehat{C}\Big]^\top \Big[\widehat{R} + \widehat{D}^\top \Sigma(t)\widehat{D}\Big]^{-1}\Big[\widehat{B}^\top \Delta(t) + \widehat{D}^\top \Lambda(t)\widehat{C}\Big]\\
&\qquad + \Big[\widehat{B}^\top \Pi + \widehat{D}^\top P\widehat{C}\Big]^\top \Big[\widehat{R} + \widehat{D}^\top \Sigma(t)\widehat{D}\Big]^{-1}\Big[\widehat{B}^\top \Pi + \widehat{D}^\top P\widehat{C}\Big]\\
&\qquad + \Big[\widehat{B}^\top \Delta(t) + \widehat{D}^\top \Lambda(t)\widehat{C}\Big]^\top \Big[\widehat{R} + \widehat{D}^\top \Sigma(t)\widehat{D}\Big]^{-1}\Big[\widehat{B}^\top \Pi + \widehat{D}^\top P\widehat{C}\Big]\\
&\qquad + \Big[\widehat{B}^\top \Pi + \widehat{D}^\top P\widehat{C}\Big]^\top \Big[\widehat{R} + \widehat{D}^\top \Sigma(t)\widehat{D}\Big]^{-1}\Big[\widehat{B}^\top \Delta(t) + \widehat{D}^\top \Lambda(t)\widehat{C}\Big]\\
&\quad \equiv ① + ② + ③ + ④\,.
\end{aligned}$$

For ② and ④, we note that

$$\begin{aligned}
&\Big[\widehat{R} + \widehat{D}^\top \Sigma(t)\widehat{D}\Big]^{-1} - \Big[\widehat{R} + \widehat{D}^\top P\widehat{D}\Big]^{-1}\\
&\quad = \Big[\widehat{R} + \widehat{D}^\top P\widehat{D}\Big]^{-1}\Big[\Big(\widehat{R} + \widehat{D}^\top P\widehat{D}\Big) - \Big(\widehat{R} + \widehat{D}^\top \Sigma(t)\widehat{D}\Big)\Big]\Big[\widehat{R} + \widehat{D}^\top \Sigma(t)\widehat{D}\Big]^{-1}\\
&\quad = -\Big[\widehat{R} + \widehat{D}^\top P\widehat{D}\Big]^{-1}\widehat{D}^\top \Lambda(t)\widehat{D}\Big[\widehat{R} + \widehat{D}^\top \Sigma(t)\widehat{D}\Big]^{-1}.
\end{aligned}$$

Thus,

$$\begin{aligned}
②&=\Big[\widehat{B}^\top\Pi+\widehat{D}^\top P\widehat{C}\Big]^\top\Big[\widehat{R}+\widehat{D}^\top P\widehat{D}\Big]^{-1}\Big[\widehat{B}^\top\Pi+\widehat{D}^\top P\widehat{C}\Big]\\
&\quad-\Big[\widehat{B}^\top\Pi+\widehat{D}^\top P\widehat{C}\Big]^\top\Big[\widehat{R}+\widehat{D}^\top P\widehat{D}\Big]^{-1}\widehat{D}^\top\Lambda(t)\widehat{D}\Big[\widehat{R}+\widehat{D}^\top\Sigma(t)\widehat{D}\Big]^{-1}\Big[\widehat{B}^\top\Pi+\widehat{D}^\top P\widehat{C}\Big]\\
&=\Big[\widehat{B}^\top\Pi+\widehat{D}^\top P\widehat{C}\Big]^\top\Big[\widehat{R}+\widehat{D}^\top P\widehat{D}\Big]^{-1}\Big[\widehat{B}^\top\Pi+\widehat{D}^\top P\widehat{C}\Big]\\
&\quad+(\widehat{D}\widehat{\Theta})^\top\Lambda(t)\widehat{D}\widehat{\mathcal{R}}(\Sigma(t))^{-1}\widehat{\mathcal{S}}(P,\Pi)
\end{aligned}$$

and

$$\begin{aligned}
④&=\Big[\widehat{B}^\top\Pi+\widehat{D}^\top P\widehat{C}\Big]^\top\Big[\widehat{R}+\widehat{D}^\top\Sigma(t)\widehat{D}\Big]^{-1}\widehat{D}^\top\Lambda(t)\widehat{C}\\
&\quad+\Big[\widehat{B}^\top\Pi+\widehat{D}^\top P\widehat{C}\Big]^\top\Big[\widehat{R}+\widehat{D}^\top\Sigma(t)\widehat{D}\Big]^{-1}\widehat{B}^\top\Delta(t)\\
&=\Big[\widehat{B}^\top\Pi+\widehat{D}^\top P\widehat{C}\Big]^\top\Big[\widehat{R}+\widehat{D}^\top\Sigma(t)\widehat{D}\Big]^{-1}\widehat{D}^\top\Lambda(t)\widehat{C}\\
&\quad+\Big[\widehat{B}^\top\Pi+\widehat{D}^\top P\widehat{C}\Big]^\top\Big[\widehat{R}+\widehat{D}^\top P\widehat{D}\Big]^{-1}\widehat{B}^\top\Delta(t)\\
&\quad-\Big[\widehat{B}^\top\Pi+\widehat{D}^\top P\widehat{C}\Big]^\top\Big[\widehat{R}+\widehat{D}^\top P\widehat{D}\Big]^{-1}\widehat{D}^\top\Lambda(t)\widehat{D}\Big[\widehat{R}+\widehat{D}^\top\Sigma(t)\widehat{D}\Big]^{-1}\widehat{B}^\top\Delta(t)\\
&=\widehat{\mathcal{S}}(P,\Pi)^\top\widehat{\mathcal{R}}(\Sigma(t))^{-1}\widehat{D}^\top\Lambda(t)\widehat{C}-(\widehat{B}\widehat{\Theta})^\top\Delta(t)+(\widehat{D}\widehat{\Theta})^\top\Lambda(t)\widehat{D}\widehat{\mathcal{R}}(\Sigma(t))^{-1}\widehat{B}^\top\Delta(t).
\end{aligned}$$

Substituting these into (4.14) and noting that $③=④^\top$, we obtain

$$\begin{aligned}
\dot\Delta(t)&=\Delta(t)\widehat{A}+\widehat{A}^\top\Delta(t)+\widehat{C}^\top\Lambda(t)\widehat{C}\\
&\quad-\Big[\widehat{B}^\top\Delta(t)+\widehat{D}^\top\Lambda(t)\widehat{C}\Big]^\top\Big[\widehat{R}+\widehat{D}^\top\Sigma(t)\widehat{D}\Big]^{-1}\Big[\widehat{B}^\top\Delta(t)+\widehat{D}^\top\Lambda(t)\widehat{C}\Big]\\
&\quad-(\widehat{D}\widehat{\Theta})^\top\Lambda(t)\widehat{D}\widehat{\mathcal{R}}(\Sigma(t))^{-1}\widehat{\mathcal{S}}(P,\Pi)\\
&\quad-\Big[\widehat{\mathcal{S}}(P,\Pi)^\top\widehat{\mathcal{R}}(\Sigma(t))^{-1}\widehat{D}^\top\Lambda(t)\widehat{C}-(\widehat{B}\widehat{\Theta})^\top\Delta(t)\\
&\quad+(\widehat{D}\widehat{\Theta})^\top\Lambda(t)\widehat{D}\widehat{\mathcal{R}}(\Sigma(t))^{-1}\widehat{B}^\top\Delta(t)\Big]^\top\\
&\quad-\Big[\widehat{\mathcal{S}}(P,\Pi)^\top\widehat{\mathcal{R}}(\Sigma(t))^{-1}\widehat{D}^\top\Lambda(t)\widehat{C}-(\widehat{B}\widehat{\Theta})^\top\Delta(t)\\
&\quad+(\widehat{D}\widehat{\Theta})^\top\Lambda(t)\widehat{D}\widehat{\mathcal{R}}(\Sigma(t))^{-1}\widehat{B}^\top\Delta(t)\Big]\\
&=\Delta(t)\Big(\widehat{A}+\widehat{B}\widehat{\Theta}\Big)+\Big(\widehat{A}+\widehat{B}\widehat{\Theta}\Big)^\top\Delta(t)+\widehat{C}^\top\Lambda(t)\widehat{C}\\
&\quad-\Big[\widehat{B}^\top\Delta(t)+\widehat{D}^\top\Lambda(t)\widehat{C}\Big]^\top\Big[\widehat{R}+\widehat{D}^\top\Sigma(t)\widehat{D}\Big]^{-1}\Big[\widehat{B}^\top\Delta(t)+\widehat{D}^\top\Lambda(t)\widehat{C}\Big]\\
&\quad-\Big[(\widehat{D}\widehat{\Theta})^\top\Lambda(t)\widehat{D}\widehat{\mathcal{R}}(\Sigma(t))^{-1}\widehat{B}^\top\Delta(t)\Big]^\top-\Big[(\widehat{D}\widehat{\Theta})^\top\Lambda(t)\widehat{D}\widehat{\mathcal{R}}(\Sigma(t))^{-1}\widehat{B}^\top\Delta(t)\Big]\\
&\quad-(\widehat{D}\widehat{\Theta})^\top\Lambda(t)\widehat{D}\widehat{\mathcal{R}}(\Sigma(t))^{-1}\widehat{\mathcal{S}}(P,\Pi)\\
&\quad-\Big[\widehat{\mathcal{S}}(P,\Pi)^\top\widehat{\mathcal{R}}(\Sigma(t))^{-1}\widehat{D}^\top\Lambda(t)\widehat{C}\Big]^\top-\Big[\widehat{\mathcal{S}}(P,\Pi)^\top\widehat{\mathcal{R}}(\Sigma(t))^{-1}\widehat{D}^\top\Lambda(t)\widehat{C}\Big],
\end{aligned}$$

which is exactly (4.13).

Step 3. Next, we show that there exist positive constants $K,\lambda>0$ such that

$$|\Upsilon(t)-\Pi|\leqslant Ke^{-\lambda t}\quad\forall\,0\leqslant t\leqslant T<\infty, \tag{4.15}$$

which is equivalent to (4.1).

By the variation of constants formula, for $t \geqslant s \geqslant 0$,

$$\begin{aligned}\Delta(t) &= e^{\bar{A}^\top (t-s)}\Delta(s)e^{\bar{A}(t-s)} \\ &\quad + \int_s^t e^{\bar{A}^\top (t-r)}\Big[f(\Delta(r),\Lambda(r))+g(\Lambda(r))\Big]e^{\bar{A}(t-r)}dr.\end{aligned} \tag{4.16}$$

Clearly, there exists a constant $\rho > 0$ such that (noting that $|\Lambda(t)|$ is bounded so that $|\Lambda(t)|^2 \leqslant K|\Lambda(t)|$)

$$|f(\Delta(t),\Lambda(t))| + |g(\Lambda(t))| \leqslant \rho\Big[|\Delta(t)|^2 + |\Lambda(t)|\Big]. \tag{4.17}$$

By Lemma 2.2, $\bar{\mathcal{A}}$ is stable, so

$$|e^{\bar{\mathcal{A}}t}| \leqslant \alpha_1 e^{-\beta_1 t} \quad \forall\, t \geqslant 0$$

for some constants $\alpha_1, \beta_1 > 0$. Also, by Lemma 2.3,

$$|\Lambda(t)| = |\Sigma(t) - P| \leqslant \alpha_2 e^{-3\beta_2 t} \quad \forall t \geqslant 0$$

for some positive constants $\alpha_2, \beta_2 > 0$. Take $\alpha = \alpha_1 \vee \alpha_2$ and $\beta = \beta_1 \wedge \beta_2$ so that

$$|e^{\bar{\mathcal{A}}t}| \leqslant \alpha e^{-\beta t}, \quad |\Lambda(t)| \leqslant \alpha e^{-3\beta t} \quad \forall t \geqslant 0. \tag{4.18}$$

From (4.16), (4.17), and (4.18) we get

$$\begin{aligned}|\Delta(t)| &\leqslant \alpha^2 e^{-2\beta(t-s)}\left\{|\Delta(s)| + \rho\int_s^t e^{2\beta(r-s)}\Big[|\Delta(r)|^2 + |\Lambda(r)|\Big]dr\right\} \\ &\leqslant \alpha^2 e^{-2\beta(t-s)}\left\{|\Delta(s)| + \rho\int_s^t e^{2\beta(r-s)}\Big[|\Delta(r)|^2 + \alpha e^{-3\beta r}\Big]dr\right\} \\ &= \alpha^2 e^{-2\beta(t-s)}\left\{|\Delta(s)| + \rho\alpha e^{-2\beta s}\beta^{-1}\Big[e^{-\beta s} - e^{-\beta t}\Big] + \rho\int_s^t e^{2\beta(r-s)}|\Delta(r)|^2 dr\right\} \\ &\leqslant \alpha^2 e^{-2\beta(t-s)}\left\{|\Delta(s)| + \rho\alpha e^{-2\beta s}\beta^{-1} + \rho\int_s^t e^{2\beta(r-s)}|\Delta(r)|^2 dr\right\}.\end{aligned}$$

For fixed $s \geqslant 0$, set

$$h(t) = e^{2\beta(t-s)}|\Delta(t)|, \quad k(s) = \alpha^2|\Delta(s)| + \rho\alpha^3 e^{-2\beta s}\beta^{-1}.$$

Then

$$h(t) \leqslant k(s) + \rho\alpha^2\int_s^t e^{-2\beta(r-s)}[h(r)]^2 dr, \quad t \geqslant s.$$

Set

$$\theta(t) = k(s) + \rho\alpha^2\int_s^t e^{-2\beta(r-s)}[h(r)]^2 dr, \quad p(t) = \frac{1}{\theta(t)}; \quad t \geqslant s.$$

Then

$$p'(t) = -\frac{\theta'(t)}{[\theta(t)]^2} = -\frac{\rho\alpha^2 e^{-2\beta(t-s)}[h(t)]^2}{[\theta(t)]^2} \geqslant -\rho\alpha^2 e^{-2\beta(t-s)}.$$

Since $\lim_{s\to\infty} k(s)=0$, for s large enough such that

$$p(s)=\frac{1}{\theta(s)}=\frac{1}{k(s)}\geqslant 1+\frac{\rho\alpha^2}{2\beta},$$

we have

$$p(t)\geqslant p(s)-\rho\alpha^2\int_s^t e^{-2\beta(r-s)}dr=p(s)-\frac{\rho\alpha^2}{2\beta}\Big[1-e^{2\beta(s-t)}\Big]\geqslant 1$$

and hence

$$h(t)\leqslant\theta(t)=\frac{1}{p(t)}\leqslant 1\quad\forall t\geqslant s.$$

It follows that

$$|\Delta(t)|=e^{-2\beta(t-s)}h(t)\leqslant Ke^{-\lambda t}\quad\forall t\geqslant 0$$

for some constants $K,\lambda>0$. ∎

5. The turnpike property. In this section we prove the main results stated in section 3. According to [18, Theorem 2.3], under the assumption (A1), Problem $(\mathrm{MFLQ})_T$ admits a unique optimal control for every initial state x. Moreover, a pair $(X_T^*(\cdot),u_T^*(\cdot))$ is optimal for x if and only if it satisfies the following optimality system:

$$\begin{cases} dX_T^*(t)=\big\{AX_T^*(t)+\bar{A}\mathbb{E}[X_T^*(t)]+Bu_T^*(t)+\bar{B}\mathbb{E}[u_T^*(t)]+b\big\}dt\\ \qquad\quad+\big\{CX_T^*(t)+\bar{C}\mathbb{E}[X_T^*(t)]+Du_T^*(t)+\bar{D}\mathbb{E}[u_T^*(t)]+\sigma\big\}dW(t),\\ dY_T^*(t)=-\big\{A^\top Y_T^*(t)+\bar{A}^\top\mathbb{E}[Y_T^*(t)]+C^\top Z_T^*(t)+\bar{C}^\top\mathbb{E}[Z_T^*(t)]+QX_T^*(t)\\ \qquad\quad+\bar{Q}\mathbb{E}[X_T^*(t)]+S^\top u_T^*(t)+\bar{S}^\top\mathbb{E}[u_T^*(t)]+q\big\}dt+Z_T^*(t)dW(t),\\ X_T^*(0)=x,\quad Y_T^*(T)=0,\\ B^\top Y_T^*(t)+\bar{B}^\top\mathbb{E}[Y_T^*(t)]+D^\top Z_T^*(t)+\bar{D}^\top\mathbb{E}[Z_T^*(t)]+SX_T^*(t)+\bar{S}\mathbb{E}[X_T^*(t)]\\ \qquad\quad+Ru_T^*(t)+\bar{R}\mathbb{E}[u_T^*(t)]+r=0. \end{cases}\tag{5.1}$$

Let $(X_T^*(\cdot),u_T^*(\cdot))$ be the optimal pair of Problem $(\mathrm{MFLQ})_T$ for the initial state x and $(Y_T^*(\cdot),Z_T^*(\cdot))$ the adapted solution to the corresponding adjoint equation in (5.1). Let (x^*,u^*) be the unique solution of Problem (O) and $\lambda^*\in\mathbb{R}^n$ the vector in (3.3). Define

$$\widetilde{X}_T(\cdot)=X_T^*(\cdot)-x^*,\quad \tilde{u}_T(\cdot)=u_T^*(\cdot)-u^*,\quad \widetilde{Y}_T(\cdot)=Y_T^*(\cdot)-\lambda^*.\tag{5.2}$$

Subtracting (3.3) from (5.1) yields

$$\begin{cases} d\widetilde{X}_T(t)=\big\{A\widetilde{X}_T(t)+\bar{A}\mathbb{E}[\widetilde{X}_T(t)]+B\tilde{u}_T(t)+\bar{B}\mathbb{E}[\tilde{u}_T(t)]\big\}dt\\ \qquad\quad+\big\{C\widetilde{X}_T(t)+\bar{C}\mathbb{E}[\widetilde{X}_T(t)]+D\tilde{u}_T(t)+\bar{D}\mathbb{E}[\tilde{u}_T(t)]+\sigma^*\big\}dW,\\ d\widetilde{Y}_T(t)=-\big\{A^\top\widetilde{Y}_T(t)+\bar{A}^\top\mathbb{E}[\widetilde{Y}_T(t)]+C^\top Z_T^*(t)+\bar{C}^\top\mathbb{E}[Z_T^*(t)]+Q\widetilde{X}_T(t)\\ \qquad\quad+\bar{Q}\mathbb{E}[\widetilde{X}_T(t)]+S^\top\tilde{u}_T(t)+\bar{S}^\top\mathbb{E}[\tilde{u}_T(t)]-\widehat{C}^\top P\sigma^*\big\}dt+Z_T^*(t)dW,\\ \widetilde{X}_T(0)=x-x^*,\quad \widetilde{Y}_T(T)=-\lambda^*,\\ B^\top\widetilde{Y}_T(t)+\bar{B}^\top\mathbb{E}[\widetilde{Y}_T(t)]+D^\top Z_T^*(t)+\bar{D}^\top\mathbb{E}[Z_T^*(t)]+S\widetilde{X}_T(t)+\bar{S}\mathbb{E}[\widetilde{X}_T(t)]\\ \qquad\quad+R\tilde{u}_T(t)+\bar{R}\mathbb{E}[\tilde{u}_T(t)]-\widehat{D}^\top P\sigma^*=0, \end{cases}\tag{5.3}$$

where $\sigma^* = \widehat{C}x^* + \widehat{D}u^* + \sigma$. From (5.3) we see that $(\widetilde{X}_T(\cdot), \tilde{u}_T(\cdot))$ is an optimal pair of the mean-field stochastic LQ problem with state equation

$$\begin{cases} dX(t) = \big\{AX(t) + \bar{A}\mathbb{E}[X(t)] + Bu(t) + \bar{B}\mathbb{E}[u(t)]\big\}dt \\ \qquad\quad + \big\{CX(t) + \bar{C}\mathbb{E}[X(t)] + Du(t) + \bar{D}\mathbb{E}[u(t)] + \sigma^*\big\}dW(t), \\ X(0) = x - x^*, \end{cases} \tag{5.4}$$

and cost functional

$$\begin{aligned} J(x;u) = \mathbb{E}\Big\{ & -2\langle \lambda^*, X(T)\rangle + \int_0^T \Big[\Big\langle \begin{pmatrix} Q & S^\top \\ S & R \end{pmatrix}\begin{pmatrix} X(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} X(t) \\ u(t) \end{pmatrix}\Big\rangle \\ & + \Big\langle \begin{pmatrix} \bar{Q} & \bar{S}^\top \\ \bar{S} & \bar{R} \end{pmatrix}\begin{pmatrix} \mathbb{E}[X(t)] \\ \mathbb{E}[u(t)] \end{pmatrix}, \begin{pmatrix} \mathbb{E}[X(t)] \\ \mathbb{E}[u(t)] \end{pmatrix}\Big\rangle \\ & - 2\Big\langle \begin{pmatrix} C^\top P\sigma^* \\ D^\top P\sigma^* \end{pmatrix}, \begin{pmatrix} X(t) \\ u(t) \end{pmatrix}\Big\rangle - 2\Big\langle \begin{pmatrix} \bar{C}^\top P\sigma^* \\ \bar{D}^\top P\sigma^* \end{pmatrix}, \begin{pmatrix} \mathbb{E}[X(t)] \\ \mathbb{E}[u(t)] \end{pmatrix}\Big\rangle\Big] dt\Big\}. \end{aligned}$$

Thus, by [18, Theorem 5.2] we have the following result.

PROPOSITION 5.1. *Let* (A1)–(A2) *hold. Let* $P_T(\cdot)$ *be the solution to* (2.5)*, and let* $\Pi_T(\cdot)$ *be the solution to* (2.6)*. Define*

$$\Theta_T(t) \triangleq -\mathcal{R}(P_T(t))^{-1}\mathcal{S}(P_T(t)), \tag{5.5}$$

$$\widehat{\Theta}_T(t) \triangleq -\widehat{\mathcal{R}}(P_T(t))^{-1}\widehat{\mathcal{S}}(P_T(t), \Pi_T(t)), \tag{5.6}$$

and let $\widehat{\varphi}_T(\cdot)$ *be the solution to*

$$\begin{cases} \dot{\widehat{\varphi}}_T(t) + [\widehat{A} + \widehat{B}\widehat{\Theta}_T(t)]^\top \widehat{\varphi}_T(t) + [\widehat{C} + \widehat{D}\widehat{\Theta}_T(t)]^\top [P_T(t) - P]\sigma^*, \\ \widehat{\varphi}_T(T) = -\lambda^*. \end{cases} \tag{5.7}$$

Then the process $\tilde{u}_T(\cdot)$ *defined in* (5.2) *is given by*

$$\tilde{u}_T(t) = \Theta_T(t)\{\widetilde{X}_T(t) - \mathbb{E}[\widetilde{X}_T(t)]\} + \widehat{\Theta}_T(t)\mathbb{E}[\widetilde{X}_T(t)] + \widehat{\theta}_T(t), \tag{5.8}$$

where

$$\widehat{\theta}_T(t) = -\widehat{\mathcal{R}}(P_T(t))^{-1}\big[\widehat{B}^\top \widehat{\varphi}_T(t) + \widehat{D}^\top (P_T(t) - P)\sigma^*\big]. \tag{5.9}$$

LEMMA 5.2. *Let* (A1)–(A2) *hold. Then there exist positive constants* $K, \lambda > 0$*, independent of* T*, such that*

$$|\widehat{\varphi}_T(t)| + |\widehat{\theta}_T(t)| \leqslant Ke^{-\lambda(T-t)} \quad \forall t \in [0, T].$$

Proof. Proceeding similarly to the proof of [20, Lemma 5.1], we can show that

$$|\widehat{\varphi}_T(t)| \leqslant Ke^{-\lambda(T-t)} \quad \forall t \in [0, T]$$

for some positive constants $K, \lambda > 0$ independent of T. The desired result then follows from the fact that

$$\widehat{\mathcal{R}}(P_T(t)) \geqslant \widehat{R} > 0$$

and Lemma 2.3. ∎

If we substitute (5.8) into (5.4), we see that the process $\widetilde{X}_T(\cdot)$ defined in (5.2) satisfies the closed-loop system

$$\begin{cases} d\widetilde{X}_T(t) = \big\{(A+B\Theta_T)(\widetilde{X}_T-\mathbb{E}[\widetilde{X}_T])+(\widehat{A}+\widehat{B}\widehat{\Theta}_T)\mathbb{E}[\widetilde{X}_T]+\widehat{B}\widehat{\theta}_T\big\}dt \\ \qquad\qquad + \big\{(C+D\Theta_T)(\widetilde{X}_T-\mathbb{E}[\widetilde{X}_T])+(\widehat{C}+\widehat{D}\widehat{\Theta}_T)\mathbb{E}[\widetilde{X}_T]+\widehat{D}\widehat{\theta}_T+\sigma^*\big\}dW, \\ \widetilde{X}_T(0) = x - x^*. \end{cases} \tag{5.10}$$

We now provide an estimate for $\mathbb{E}[\widetilde{X}_T(t)]$.

PROPOSITION 5.3. *Let* (A1)–(A2) *hold. Then there exist constants* $K, \lambda > 0$, *independent of* T, *such that*

$$|\mathbb{E}[\widetilde{X}_T(t)]| \leqslant K\Big[e^{-\lambda t} + e^{-\lambda(T-t)}\Big] \quad \forall t \in [0,T]. \tag{5.11}$$

Proof. The function $t \mapsto \mathbb{E}[\widetilde{X}_T(t)]$ satisfies the following ODE:

$$\begin{cases} \dfrac{d}{dt}\mathbb{E}[\widetilde{X}_T(t)] = [\widehat{A} + \widehat{B}\widehat{\Theta}_T(t)]\mathbb{E}[\widetilde{X}_T(t)] + \widehat{B}\widehat{\theta}_T(t), \\ \mathbb{E}[\widetilde{X}_T(0)] = x - x^* \equiv \tilde{x}. \end{cases} \tag{5.12}$$

Using the notation

$$\widehat{\mathcal{A}}_T(t) = \widehat{A} + \widehat{B}\widehat{\Theta}_T(t), \quad \widehat{\Theta} = -\widehat{\mathcal{R}}(P)^{-1}\widehat{\mathcal{S}}(P,\Pi), \quad \widehat{\mathcal{A}} = \widehat{A} + \widehat{B}\widehat{\Theta},$$

we have by the variation of constants formula that

$$\mathbb{E}[\widetilde{X}_T(t)] = e^{\widehat{\mathcal{A}}t}\tilde{x} + \int_0^t e^{\widehat{\mathcal{A}}(t-s)}\Big[\Big(\widehat{\mathcal{A}}_T(s) - \widehat{\mathcal{A}}\Big)\mathbb{E}[\widetilde{X}_T(s)] + \widehat{B}\widehat{\theta}_T(s)\Big]ds.$$

By Lemma 2.2, $\widehat{\mathcal{A}}$ is stable, so

$$|e^{\widehat{\mathcal{A}}t}| \leqslant Ke^{-\lambda t} \quad \forall t \geqslant 0$$

for some constants $K, \lambda > 0$ independent of T. Since

$$\begin{aligned} \widehat{\Theta} - \widehat{\Theta}_T(t) &= \Big(\widehat{R} + \widehat{D}^\top P_T(t)\widehat{D}\Big)^{-1}\Big(\widehat{B}^\top \Pi_T(t) + \widehat{D}^\top P_T(t)\widehat{C} + \widehat{S}\Big) \\ &\quad - \Big(\widehat{R} + \widehat{D}^\top P\widehat{D}\Big)^{-1}\Big(\widehat{B}^\top \Pi + \widehat{D}^\top P\widehat{C} + \widehat{S}\Big) \\ &= \Big(\widehat{R} + \widehat{D}^\top P_T(t)\widehat{D}\Big)^{-1}\Big(\widehat{B}^\top [\Pi_T(t) - \Pi] + \widehat{D}^\top [P_T(t) - P]\widehat{C}\Big) \\ &\quad + \Big[\Big(\widehat{R} + \widehat{D}^\top P_T(t)\widehat{D}\Big)^{-1} - \Big(\widehat{R} + \widehat{D}^\top P\widehat{D}\Big)^{-1}\Big]\Big(\widehat{B}^\top \Pi + \widehat{D}^\top P\widehat{C} + \widehat{S}\Big) \\ &= \Big(\widehat{R} + \widehat{D}^\top P_T(t)\widehat{D}\Big)^{-1}\Big(\widehat{B}^\top [\Pi_T(t) - \Pi] + \widehat{D}^\top [P_T(t) - P]\widehat{C}\Big) \\ &\quad + \Big(\widehat{R} + \widehat{D}^\top P_T(t)\widehat{D}\Big)^{-1}\widehat{D}^\top [P_T(t) - P]\widehat{D}\widehat{\Theta}, \end{aligned}$$

we have by Lemma 2.3 and Theorem 4.1

$$\begin{aligned} |\widehat{\mathcal{A}}_T(t) - \widehat{\mathcal{A}}| &\leqslant K|\widehat{\Theta}_T(t) - \widehat{\Theta}| \leqslant K\Big[|\Pi_T(t) - \Pi| + |P_T(t) - P|\Big] \\ &\leqslant Ke^{-\lambda(T-t)} \quad \forall t \in [0,T]. \end{aligned} \tag{5.13}$$

Also recalling Lemma 5.2, we obtain

$$\begin{aligned}|\mathbb{E}[\widetilde{X}_T(t)]| &\leqslant Ke^{-\lambda t} + K\int_0^t e^{-\lambda(t-s)}e^{-\lambda(T-s)}\Big[\mathbb{E}[\widetilde{X}_T(s)]+1\Big]ds\\ &\leqslant K\Big[e^{-\lambda t}+e^{-\lambda(T-t)}\Big] + K\int_0^t e^{-\lambda(T+t-2s)}\mathbb{E}[\widetilde{X}_T(s)]ds\\ &\leqslant K\Big[e^{-\lambda t}+e^{-\lambda(T-t)}\Big] + K\int_0^t e^{-\lambda(T-s)}\mathbb{E}[\widetilde{X}_T(s)]ds.\end{aligned}$$

The desired result then follows from Gronwall's inequality. ∎

Recall the SDE (3.5). We have the following result.

PROPOSITION 5.4. *Let* (A1)–(A2) *hold. Then there exists a constant* $K>0$ *such that the solution* $X^*(\cdot)$ *to the SDE* (3.5) *satisfies*

$$\mathbb{E}|X^*(t)|^2 \leqslant K \quad \forall t \geqslant 0. \tag{5.14}$$

Proof. Let P be the solution to the ARE (2.7), and let $\Theta = -\mathcal{R}(P)^{-1}\mathcal{S}(P)$. By Itô's rule,

$$\begin{aligned}&\frac{d}{dt}\mathbb{E}\langle PX^*(t),X^*(t)\rangle\\ &\quad=\mathbb{E}\Big\{2\langle PX^*(t),(A+B\Theta)X^*(t)\rangle\\ &\qquad+\langle P[(C+D\Theta)X^*(t)+\sigma^*],(C+D\Theta)X^*(t)+\sigma^*\rangle\Big\}\\ &\quad=\mathbb{E}\Big\{\langle[P(A+B\Theta)+(A+B\Theta)^\top P+(C+D\Theta)^\top P(C+D\Theta)]X^*(t),X^*(t)\rangle\\ &\qquad+2\langle P(C+D\Theta)X^*(t),\sigma^*\rangle+\langle P\sigma^*,\sigma^*\rangle\Big\}\\ &\quad=\mathbb{E}\Big[-\langle(Q+\Theta^\top R\Theta+S^\top\Theta+\Theta^\top S)X^*(t),X^*(t)\rangle\\ &\qquad+2\langle P(C+D\Theta)X^*(t),\sigma^*\rangle+\langle P\sigma^*,\sigma^*\rangle\Big].\end{aligned}$$

Note that by (A1),

$$Q+\Theta^\top R\Theta+S^\top\Theta+\Theta^\top S = Q-S^\top R^{-1}S+(\Theta+R^{-1}S)^\top R(\Theta+R^{-1}S)>0.$$

Let $\mu>0$ be the largest eigenvalue of P, and let $\nu>0$ be the smallest eigenvalue of $Q+\Theta^\top R\Theta+S^\top\Theta+\Theta^\top S$. Then

$$\begin{aligned}&\frac{d}{dt}\mathbb{E}\langle PX^*(t),X^*(t)\rangle\\ &\quad\leqslant\mathbb{E}\left[-\nu|X^*(t)|^2+2|X^*(t)|\cdot|(C+D\Theta)^\top P\sigma^*|+\langle P\sigma^*,\sigma^*\rangle\right]\\ &\quad\leqslant\mathbb{E}\left[-\frac{\nu}{2}|X^*(t)|^2+\frac{2}{\nu}|(C+D\Theta)^\top P\sigma^*|^2+\langle P\sigma^*,\sigma^*\rangle\right]\\ &\quad\leqslant\mathbb{E}\left[-\frac{\nu}{2\mu}\langle PX^*(t),X^*(t)\rangle+\frac{2}{\nu}|(C+D\Theta)^\top P\sigma^*|^2+\langle P\sigma^*,\sigma^*\rangle\right]\\ &\quad=-\frac{\nu}{2\mu}\mathbb{E}\langle PX^*(t),X^*(t)\rangle+\frac{2}{\nu}|(C+D\Theta)^\top P\sigma^*|^2+\langle P\sigma^*,\sigma^*\rangle.\end{aligned}$$

It follows from the Gronwall inequality that

$$\begin{aligned}\mathbb{E}\langle PX^*(t),X^*(t)\rangle &\leqslant \left[\frac{2}{\nu}|(C+D\Theta)^\top P\sigma^*|^2+\langle P\sigma^*,\sigma^*\rangle\right]\int_0^t \exp\left[\frac{\nu}{2\mu}(s-t)\right]ds\\ &\leqslant \frac{2\mu}{\nu}\left[\frac{2}{\nu}|(C+D\Theta)^\top P\sigma^*|^2+\langle P\sigma^*,\sigma^*\rangle\right].\end{aligned}$$

This implies (5.14), since $P>0$. ∎

We now prove Theorem 3.2.

Proof of Theorem 3.2. Let $\Theta_{\scriptscriptstyle T}(t)$ and $\widehat{\Theta}_{\scriptscriptstyle T}(t)$ be as in (5.5) and (5.6), respectively, and let

$$\Theta=-\mathcal{R}(P)^{-1}\mathcal{S}(P),\quad \widehat{\Theta}=-\widehat{\mathcal{R}}(P)^{-1}\widehat{\mathcal{S}}(P,\Pi).$$

For notational simplicity, we write

$$\begin{aligned}&\mathcal{A}=A+B\Theta,\quad \mathcal{C}=C+D\Theta,\quad \mathcal{A}_{\scriptscriptstyle T}(t)=A+B\Theta_{\scriptscriptstyle T}(t),\quad \mathcal{C}_{\scriptscriptstyle T}(t)=C+D\Theta_{\scriptscriptstyle T}(t),\\ &\widehat{\mathcal{A}}=\widehat{A}+\widehat{B}\widehat{\Theta},\quad \widehat{\mathcal{C}}=\widehat{C}+\widehat{D}\widehat{\Theta},\quad \widehat{\mathcal{A}}_{\scriptscriptstyle T}(t)=\widehat{A}+\widehat{B}\widehat{\Theta}_{\scriptscriptstyle T}(t),\quad \widehat{\mathcal{C}}_{\scriptscriptstyle T}(t)=\widehat{C}+\widehat{D}\widehat{\Theta}_{\scriptscriptstyle T}(t).\end{aligned}$$

Then the process

$$H_{\scriptscriptstyle T}(t)\triangleq \widetilde{X}_{\scriptscriptstyle T}(t)-\mathbb{E}[\widetilde{X}_{\scriptscriptstyle T}(t)]-X^*(t),\quad t\in[0,T],$$

satisfies $H_{\scriptscriptstyle T}(0)=0$ and

$$\begin{aligned}dH_{\scriptscriptstyle T}(t)=&\big\{\mathcal{A}_{\scriptscriptstyle T}(t)H_{\scriptscriptstyle T}(t)+[\mathcal{A}_{\scriptscriptstyle T}(t)-\mathcal{A}]X^*(t)\big\}dt\\ &+\big\{\mathcal{C}_{\scriptscriptstyle T}(t)H_{\scriptscriptstyle T}(t)+[\mathcal{C}_{\scriptscriptstyle T}(t)-\mathcal{C}]X^*(t)+\widehat{\mathcal{C}}_{\scriptscriptstyle T}(t)\mathbb{E}[\widetilde{X}_{\scriptscriptstyle T}(t)]+\widehat{D}\widehat{\theta}_{\scriptscriptstyle T}(t)\big\}dW(t).\end{aligned}$$

Let $P>0$ be the solution to the ARE (2.7). Then Itô's rule implies that

$$\begin{aligned}\mathbb{E}\langle PH_{\scriptscriptstyle T}(t),H_{\scriptscriptstyle T}(t)\rangle=\mathbb{E}\int_0^t\Big\{&2\langle PH_{\scriptscriptstyle T},\mathcal{A}_{\scriptscriptstyle T}H_{\scriptscriptstyle T}+(\mathcal{A}_{\scriptscriptstyle T}-\mathcal{A})X^*\rangle\\ &+\langle P[\mathcal{C}_{\scriptscriptstyle T}H_{\scriptscriptstyle T}+(\mathcal{C}_{\scriptscriptstyle T}-\mathcal{C})X^*+h_{\scriptscriptstyle T}],\mathcal{C}_{\scriptscriptstyle T}H_{\scriptscriptstyle T}+(\mathcal{C}_{\scriptscriptstyle T}-\mathcal{C})X^*+h_{\scriptscriptstyle T}\rangle\Big\}ds,\end{aligned}\tag{5.15}$$

where we have suppressed s in the integrand, and

$$h_{\scriptscriptstyle T}(s)=\widehat{\mathcal{C}}_{\scriptscriptstyle T}(s)\mathbb{E}[\widetilde{X}_{\scriptscriptstyle T}(s)]+\widehat{D}\widehat{\theta}_{\scriptscriptstyle T}(s).$$

Note that

$$\mathbb{E}[X^*(t)]=0,\quad \mathbb{E}[H_{\scriptscriptstyle T}(t)]=0\quad \forall t\in[0,T].$$

Thus,

$$\begin{aligned}&\mathbb{E}\langle P[\mathcal{C}_{\scriptscriptstyle T}H_{\scriptscriptstyle T}+(\mathcal{C}_{\scriptscriptstyle T}-\mathcal{C})X^*+h_{\scriptscriptstyle T}],\mathcal{C}_{\scriptscriptstyle T}H_{\scriptscriptstyle T}+(\mathcal{C}_{\scriptscriptstyle T}-\mathcal{C})X^*+h_{\scriptscriptstyle T}\rangle\\ &\quad=\mathbb{E}\langle P[\mathcal{C}_{\scriptscriptstyle T}H_{\scriptscriptstyle T}+(\mathcal{C}_{\scriptscriptstyle T}-\mathcal{C})X^*],\mathcal{C}_{\scriptscriptstyle T}H_{\scriptscriptstyle T}+(\mathcal{C}_{\scriptscriptstyle T}-\mathcal{C})X^*\rangle+\langle Ph_{\scriptscriptstyle T},h_{\scriptscriptstyle T}\rangle.\end{aligned}$$

Substituting the above into (5.15) and expanding the integrand yields

$$\begin{aligned}\mathbb{E}\langle PH_{\scriptscriptstyle T}(t),H_{\scriptscriptstyle T}(t)\rangle=\int_0^t\mathbb{E}\Big\{&\big\langle(P\mathcal{A}_{\scriptscriptstyle T}+\mathcal{A}_{\scriptscriptstyle T}^\top P+\mathcal{C}_{\scriptscriptstyle T}^\top P\mathcal{C}_{\scriptscriptstyle T})H_{\scriptscriptstyle T},H_{\scriptscriptstyle T}\big\rangle\\ &+\big\langle\widehat{\mathcal{C}}_{\scriptscriptstyle T}^\top P\widehat{\mathcal{C}}_{\scriptscriptstyle T}\mathbb{E}[\widetilde{X}_{\scriptscriptstyle T}],\mathbb{E}[\widetilde{X}_{\scriptscriptstyle T}]\big\rangle\\ &+2\big\langle[P(\mathcal{A}_{\scriptscriptstyle T}-\mathcal{A})+\mathcal{C}_{\scriptscriptstyle T}^\top P(\mathcal{C}_{\scriptscriptstyle T}-\mathcal{C})]X^*,H_{\scriptscriptstyle T}\big\rangle+k_{\scriptscriptstyle T}\Big\}ds,\end{aligned}\tag{5.16}$$

where

$$k_T(s) = \big\langle P[\mathcal{C}_T(s) - \mathcal{C}]X^*(s), [\mathcal{C}_T(s) - \mathcal{C}]X^*(s)\big\rangle + \big\langle 2P\widehat{\mathcal{C}}_T(s)\mathbb{E}[\widetilde{X}_T(s)] + P\widehat{D}\widehat{\theta}_T(s), \widehat{D}\widehat{\theta}_T(s)\big\rangle.$$

Next, let $\Pi > 0$ be the solution to the ARE (2.8). Then integration by parts gives

$$\begin{aligned}&\big\langle \Pi\mathbb{E}[\widetilde{X}_T(t)], \mathbb{E}[\widetilde{X}_T(t)]\big\rangle - \langle \Pi\tilde{x}, \tilde{x}\rangle \\ &\quad = \int_0^t \Big\{\big\langle (\Pi\widehat{\mathcal{A}}_T + \widehat{\mathcal{A}}_T^\top\Pi)\mathbb{E}[\widetilde{X}_T], \mathbb{E}[\widetilde{X}_T]\big\rangle + 2\big\langle \Pi\mathbb{E}[\widetilde{X}_T], \widehat{B}\widehat{\theta}_T\big\rangle\Big\}ds. \end{aligned} \tag{5.17}$$

Adding (5.16) and (5.17), we obtain

$$\begin{aligned}&\mathbb{E}\langle PH_T(t), H_T(t)\rangle + \big\langle \Pi\mathbb{E}[\widetilde{X}_T(t)], \mathbb{E}[\widetilde{X}_T(t)]\big\rangle - \langle \Pi\tilde{x}, \tilde{x}\rangle \\ &\quad = \int_0^t \mathbb{E}\Big\{\big\langle (P\mathcal{A}_T + \mathcal{A}_T^\top P + \mathcal{C}_T^\top P\mathcal{C}_T)H_T, H_T\big\rangle \\ &\qquad + \big\langle (\Pi\widehat{\mathcal{A}}_T + \widehat{\mathcal{A}}_T^\top\Pi + \widehat{\mathcal{C}}_T^\top P\widehat{\mathcal{C}}_T)\mathbb{E}[\widetilde{X}_T], \mathbb{E}[\widetilde{X}_T]\big\rangle \\ &\qquad + 2\big\langle [P(\mathcal{A}_T - \mathcal{A}) + \mathcal{C}_T^\top P(\mathcal{C}_T - \mathcal{C})]X^*, H_T\big\rangle + \phi_T\Big\}ds,\end{aligned}$$

or equivalently,

$$\begin{aligned}&\frac{d}{dt}\Big\{\mathbb{E}\langle PH_T(t), H_T(t)\rangle + \big\langle \Pi\mathbb{E}[\widetilde{X}_T(t)], \mathbb{E}[\widetilde{X}_T(t)]\big\rangle\Big\} \\ &\quad = \mathbb{E}\Big\{\big\langle (P\mathcal{A}_T + \mathcal{A}_T^\top P + \mathcal{C}_T^\top P\mathcal{C}_T)H_T, H_T\big\rangle \\ &\qquad + \big\langle (\Pi\widehat{\mathcal{A}}_T + \widehat{\mathcal{A}}_T^\top\Pi + \widehat{\mathcal{C}}_T^\top P\widehat{\mathcal{C}}_T)\mathbb{E}[\widetilde{X}_T], \mathbb{E}[\widetilde{X}_T]\big\rangle \\ &\qquad + 2\big\langle [P(\mathcal{A}_T - \mathcal{A}) + \mathcal{C}_T^\top P(\mathcal{C}_T - \mathcal{C})]X^*, H_T\big\rangle + \phi_T\Big\},\end{aligned} \tag{5.18}$$

where

$$\phi_T(t) = k_T(t) + 2\big\langle \Pi\mathbb{E}[\widetilde{X}_T(t)], \widehat{B}\widehat{\theta}_T(t)\big\rangle.$$

We observe the following facts:

$$\begin{aligned}&P\mathcal{A}_T(s) + \mathcal{A}_T(s)^\top P + \mathcal{C}_T(s)^\top P\mathcal{C}_T(s) \\ &\quad = P\mathcal{A} + \mathcal{A}^\top P + \mathcal{C}^\top P\mathcal{C} + P[\mathcal{A}_T(s) - \mathcal{A}] + [\mathcal{A}_T(s) - \mathcal{A}]^\top P \\ &\qquad + [\mathcal{C}_T(s) - \mathcal{C}]^\top P\mathcal{C} + \mathcal{C}_T(s)^\top P[\mathcal{C}_T(s) - \mathcal{C}],\end{aligned}$$

$$P\mathcal{A} + \mathcal{A}^\top P + \mathcal{C}^\top P\mathcal{C} = -(Q + \Theta^\top R\Theta + S^\top\Theta + \Theta^\top S) < 0,$$

$$\begin{aligned}&\Pi\widehat{\mathcal{A}}_T(s) + \widehat{\mathcal{A}}_T(s)^\top\Pi + \widehat{\mathcal{C}}_T(s)^\top P\widehat{\mathcal{C}}_T(s) \\ &\quad = \Pi\widehat{\mathcal{A}} + \widehat{\mathcal{A}}^\top\Pi + \widehat{\mathcal{C}}^\top P\widehat{\mathcal{C}} + \Pi[\widehat{\mathcal{A}}_T(s) - \widehat{\mathcal{A}}] + [\widehat{\mathcal{A}}_T(s) - \widehat{\mathcal{A}}]^\top\Pi \\ &\qquad + [\widehat{\mathcal{C}}_T(s) - \widehat{\mathcal{C}}]^\top P\widehat{\mathcal{C}} + \widehat{\mathcal{C}}_T(s)^\top P[\widehat{\mathcal{C}}_T(s) - \widehat{\mathcal{C}}],\end{aligned}$$

$$\Pi\widehat{\mathcal{A}} + \widehat{\mathcal{A}}^\top\Pi + \widehat{\mathcal{C}}^\top P\widehat{\mathcal{C}} = -(\widehat{Q} + \widehat{\Theta}^\top\widehat{R}\widehat{\Theta} + \widehat{S}^\top\widehat{\Theta} + \widehat{\Theta}^\top\widehat{S}) < 0.$$

Also, observe that for some constant $K > 0$,

$$|\mathcal{A}_T(t)| + |\widehat{\mathcal{A}}_T(t)| + |\mathcal{C}_T(t)| + |\widehat{\mathcal{C}}_T(t)| \leqslant K \quad \forall\, 0 \leqslant t \leqslant T < \infty,$$

and similar to (5.13), we have

$$|\mathcal{A}_T(t)-\mathcal{A}|+|\widehat{\mathcal{A}}_T(t)-\widehat{\mathcal{A}}|+|\mathcal{C}_T(t)-\mathcal{C}|+|\widehat{\mathcal{C}}_T(t)-\widehat{\mathcal{C}}|\leqslant Ke^{-\lambda(T-t)}\quad\forall 0\leqslant t\leqslant T<\infty$$

for some constants $K,\lambda>0$. Then it follows that for some constant $\alpha>0$,

$$\begin{aligned}&\mathbb{E}\big\langle[P\mathcal{A}_T(s)+\mathcal{A}_T(s)^\top P+\mathcal{C}_T(s)^\top P\mathcal{C}_T(s)]H_T(s),H_T(s)\big\rangle\\&\quad\leqslant K\Big[-2\alpha+e^{-\lambda(T-s)}\Big]\mathbb{E}|H_T(s)|^2,\end{aligned}\tag{5.19}$$

$$\begin{aligned}&\big\langle[\Pi\widehat{\mathcal{A}}_T(s)+\widehat{\mathcal{A}}_T(s)^\top\Pi+\widehat{\mathcal{C}}_T(s)^\top P\widehat{\mathcal{C}}_T(s)]\mathbb{E}[\widetilde{X}_T(s)],\mathbb{E}[\widetilde{X}_T(s)]\big\rangle\\&\quad\leqslant K\Big[-\alpha+e^{-\lambda(T-s)}\Big]|\mathbb{E}[\widetilde{X}_T(s)]|^2.\end{aligned}\tag{5.20}$$

Moreover, by the Cauchy–Schwarz inequality and (5.14),

$$\begin{aligned}&2\mathbb{E}\big\langle[P(\mathcal{A}_T(s)-\mathcal{A})+\mathcal{C}_T(s)^\top P(\mathcal{C}_T(s)-\mathcal{C})]X^*(s),H_T(s)\big\rangle\\&\quad\leqslant K\Big[\alpha\mathbb{E}|H_T(s)|^2+e^{-\lambda(T-s)}\Big],\end{aligned}\tag{5.21}$$

and by Lemma 5.2 and Propositions 5.3 and 5.4,

$$\mathbb{E}|\phi_T(s)|\leqslant Ke^{-\lambda(T-s)}.\tag{5.22}$$

Substituting of (5.19)–(5.22) into (5.18), we obtain

$$\begin{aligned}&\frac{d}{dt}\Big\{\mathbb{E}\langle PH_T(t),H_T(t)\rangle+\big\langle\Pi\mathbb{E}[\widetilde{X}_T(t)],\mathbb{E}[\widetilde{X}_T(t)]\big\rangle\Big\}\\&\leqslant K\Big[-\alpha+e^{-\lambda(T-t)}\Big]\Big\{\mathbb{E}|H_T(t)|^2+|\mathbb{E}[\widetilde{X}_T(t)]|^2\Big\}+Ke^{-\lambda(T-t)}\\&\leqslant K\Big[-\alpha+e^{-\lambda(T-t)}\Big]\Big\{\mathbb{E}\langle PH_T(t),H_T(t)\rangle+\big\langle\Pi\mathbb{E}[\widetilde{X}_T(t)],\mathbb{E}[\widetilde{X}_T(t)]\big\rangle\Big\}+Ke^{-\lambda(T-t)},\end{aligned}$$

with possibly a different constant K. Then, by Gronwall's inequality, we have

$$\mathbb{E}\langle PH_T(t),H_T(t)\rangle+\big\langle\Pi\mathbb{E}[\widetilde{X}_T(t)],\mathbb{E}[\widetilde{X}_T(t)]\big\rangle\leqslant K\Big[e^{-\lambda t}+e^{-\lambda(T-t)}\Big]\quad\forall t\in[0,T],$$

with possibly different constants $K,\lambda>0$. Since $P,\Pi>0$, the above implies that

$$\mathbb{E}|H_T(t)|^2+|\mathbb{E}[\widetilde{X}_T(t)]|^2\leqslant K\Big[e^{-\lambda t}+e^{-\lambda(T-t)}\Big]\quad\forall t\in[0,T].$$

The desired (3.7) now follows from

$$\begin{aligned}X_T^*(t)-\boldsymbol{X}^*(t)&=H_T(t)+\mathbb{E}[\widetilde{X}_T(t)],\\u_T^*(t)-\boldsymbol{u}^*(t)&=\Theta_T(t)[X_T^*(t)-\boldsymbol{X}^*(t)]+[\Theta_T(t)-\Theta]X^*(t)\\&\quad+[\widehat{\Theta}_T(t)-\Theta_T(t)]\mathbb{E}[\widetilde{X}_T(t)]+\widehat{\theta}_T(t).\end{aligned}$$

The proof is complete. ∎

Next, we prove Corollary 3.4.

Proof of Corollary 3.4. By Proposition 5.4,

$$\mathbb{E}|\boldsymbol{X}^*(t)|^2+\mathbb{E}|\boldsymbol{u}^*(t)|^2\leqslant K\quad\forall t\geqslant 0$$

for some constant $K>0$, and hence by Theorem 3.2,

$$\mathbb{E}|X_T^*(t)|^2+\mathbb{E}|u_T^*(t)|^2\leqslant K\quad\forall\, 0\leqslant t\leqslant T<\infty$$

for some possibly different constant $K>0$. The above, together with Corollary 3.3, implies that

$$\begin{aligned}
\lim_{T\to\infty}\frac{1}{T}V_T(x)&=\lim_{T\to\infty}\frac{1}{T}J_T(x;u_T^*(\cdot))\\
&=\lim_{T\to\infty}\frac{1}{T}\int_0^T\Big\{\mathbb{E}\big[\langle QX_T^*(t),X_T^*(t)\rangle+2\langle SX_T^*(t),u_T^*(t)\rangle+\langle Ru_T^*(t),u_T^*(t)\rangle\big]\\
&\quad+\langle\bar{Q}\mathbb{E}[X_T^*(t)],\mathbb{E}[X_T^*(t)]\rangle+2\langle\bar{S}\mathbb{E}[X_T^*(t)],\mathbb{E}[u_T^*(t)]\rangle+\langle\bar{R}\mathbb{E}[u_T^*(t)],\mathbb{E}[u_T^*(t)]\rangle\\
&\quad+2\langle q,\mathbb{E}[X_T^*(t)]\rangle+2\langle r,\mathbb{E}[u_T^*(t)]\rangle\Big\}dt\\
&=\lim_{T\to\infty}\frac{1}{T}\int_0^T\Big\{\mathbb{E}\big[\langle Q\boldsymbol{X}^*(t),\boldsymbol{X}^*(t)\rangle+2\langle S\boldsymbol{X}^*(t),\boldsymbol{u}^*(t)\rangle+\langle R\boldsymbol{u}^*(t),\boldsymbol{u}^*(t)\rangle\big]\\
&\quad+\langle\bar{Q}\mathbb{E}[\boldsymbol{X}^*(t)],\mathbb{E}[\boldsymbol{X}^*(t)]\rangle+2\langle\bar{S}\mathbb{E}[\boldsymbol{X}^*(t)],\mathbb{E}[\boldsymbol{u}^*(t)]\rangle+\langle\bar{R}\mathbb{E}[\boldsymbol{u}^*(t)],\mathbb{E}[\boldsymbol{u}^*(t)]\rangle\\
&\quad+2\langle q,\mathbb{E}[\boldsymbol{X}^*(t)]\rangle+2\langle r,\mathbb{E}[\boldsymbol{u}^*(t)]\rangle\Big\}dt.
\end{aligned}$$

Noting that

$$\mathbb{E}[\boldsymbol{X}^*(t)]=\mathbb{E}[X^*(t)+x^*]=x^*,\quad \mathbb{E}[\boldsymbol{u}^*(t)]=\mathbb{E}[\Theta X^*(t)+u^*]=u^*,$$

we further have

$$\begin{aligned}
\lim_{T\to\infty}\frac{1}{T}V_T(x)&=\lim_{T\to\infty}\frac{1}{T}\int_0^T\Big\{\mathbb{E}\big[\langle QX^*(t),X^*(t)\rangle+\langle Qx^*,x^*\rangle\\
&\qquad+2\langle SX^*(t),\Theta X^*(t)\rangle+2\langle Sx^*,u^*\rangle\\
&\qquad+\langle R\Theta X^*(t),\Theta X^*(t)\rangle+\langle Ru^*,u^*\rangle\big]\\
&\qquad+\langle\bar{Q}x^*,x^*\rangle+2\langle\bar{S}x^*,u^*\rangle+\langle\bar{R}u^*,u^*\rangle+2\langle q,x^*\rangle+2\langle r,u^*\rangle\Big\}dt\\
&=\lim_{T\to\infty}\frac{1}{T}\int_0^T\mathbb{E}\langle(Q+S^\top\Theta+\Theta^\top S+\Theta^\top R\Theta)X^*(t),X^*(t)\rangle dt\\
&\qquad+\langle\widehat{Q}x^*,x^*\rangle+2\langle\widehat{S}x^*,u^*\rangle+\langle\widehat{R}u^*,u^*\rangle+2\langle q,x^*\rangle+2\langle r,u^*\rangle.
\end{aligned}\tag{5.23}$$

On the other hand, with $P>0$ being the solution to the ARE (2.7) and noting that $\mathbb{E}[X^*(t)]=0$, we have

$$\begin{aligned}
&\mathbb{E}\langle PX^*(T),X^*(T)\rangle\\
&\quad=\mathbb{E}\int_0^T\Big\{2\langle PX^*(t),(A+B\Theta)X^*(t)\rangle\\
&\qquad+\langle P[(C+D\Theta)X^*(t)+\sigma^*],(C+D\Theta)X^*(t)+\sigma^*\rangle\Big\}dt\\
&\quad=\mathbb{E}\int_0^T\Big\{\langle[P(A+B\Theta)+(A+B\Theta)^\top P\\
&\qquad+(C+D\Theta)^\top P(C+D\Theta)]X^*(t),X^*(t)\rangle+\langle P\sigma^*,\sigma^*\rangle\Big\}dt\\
&\quad=\mathbb{E}\int_0^T\Big[-\langle(Q+S^\top\Theta+\Theta^\top S+\Theta^\top R\Theta)X^*(t),X^*(t)\rangle+\langle P\sigma^*,\sigma^*\rangle\Big]dt.
\end{aligned}$$

Since $\mathbb{E}|X^*(T)|^2$ is bounded in T (see Proposition 5.4), we obtain

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \mathbb{E}\langle (Q+S^\top\Theta+\Theta^\top S+\Theta^\top R\Theta)X^*(t),X^*(t)\rangle dt$$
$$=\lim_{T\to\infty}\frac{1}{T}\left[-\mathbb{E}\langle PX^*(T),X^*(T)\rangle+\int_0^T\langle P\sigma^*,\sigma^*\rangle dt\right]=\langle P\sigma^*,\sigma^*\rangle. \tag{5.24}$$

Combining (5.23) and (5.24), we get the desired result. ∎

Finally, we prove Corollary 3.5.

Proof of Corollary 3.5. By a tedious but straightforward calculation we can verify that in (5.3), the adapted solution $(\widetilde{Y}_T(\cdot),Z_T^*(\cdot))$ to the backward SDE has the following relation with $(\widetilde{X}_T(\cdot),\tilde{u}_T(\cdot))$:

$$\widetilde{Y}_T(t)=P_T(t)\Big\{\widetilde{X}_T(t)-\mathbb{E}[\widetilde{X}_T(t)]\Big\}+\Pi_T(t)\mathbb{E}[\widetilde{X}_T(t)]+\widehat{\varphi}_T(t),$$
$$Z_T^*(t)=P_T(t)\Big\{C\widetilde{X}_T(t)+D\tilde{u}_T(t)+\bar{C}\mathbb{E}[\widetilde{X}_T(t)]+\bar{D}\mathbb{E}[\tilde{u}_T(t)]+\sigma^*\Big\}.$$

Using the relation (5.8), we further obtain

$$Z_T^*(t)=P_T(t)\Big\{[C+D\Theta_T(t)]\widetilde{X}_T(t)+[\bar{C}-D\Theta_T(t)]\mathbb{E}[\widetilde{X}_T(t)]+\widehat{D}\mathbb{E}[\tilde{u}_T(t)]+\sigma^*\Big\}.$$

For $\mathbb{E}|Y_T^*(t)-\boldsymbol{Y}^*(t)|^2$, we first observe that

$$\begin{aligned}Y_T^*(t)-\boldsymbol{Y}^*(t)&=\widetilde{Y}_T(t)-PX^*(t)\\&=P_T(t)\widetilde{X}_T(t)-PX^*(t)+[\Pi_T(t)-P_T(t)]\mathbb{E}[\widetilde{X}_T(t)]+\widehat{\varphi}_T(t)\\&=P_T(t)[X_T^*(t)-\boldsymbol{X}^*(t)]+[P_T(t)-P]X^*(t)\\&\quad+[\Pi_T(t)-P_T(t)]\mathbb{E}[\widetilde{X}_T(t)]+\widehat{\varphi}_T(t).\end{aligned}$$

By Lemma 2.3 and Theorem 4.1, $P_T(\cdot)$ and $\Pi_T(\cdot)$ are bounded uniformly in T, and by Proposition 5.4, $\mathbb{E}|X^*(\cdot)|$ is also bounded. The above then implies that

$$\mathbb{E}|Y_T^*(t)-\boldsymbol{Y}^*(t)|^2\leqslant K\Big\{\mathbb{E}|X_T^*(t)-\boldsymbol{X}^*(t)|^2+|P_T(t)-P|^2+|\mathbb{E}[\widetilde{X}_T(t)]|^2+|\widehat{\varphi}_T(t)|^2\Big\}$$

for some constant $K>0$. Recalling Lemma 2.3, Theorem 3.2, Lemma 5.2, and Proposition 5.3, we obtain that for some constants $K,\lambda>0$ independent of T,

$$\mathbb{E}|Y_T^*(t)-\boldsymbol{Y}^*(t)|^2\leqslant K\Big[e^{-\lambda t}+e^{-\lambda(T-t)}\Big]\quad\forall t\in[0,T].$$

In a similar manner, we can show that

$$\mathbb{E}|Z_T^*(t)-\boldsymbol{Z}^*(t)|^2\leqslant K\Big[e^{-\lambda t}+e^{-\lambda(T-t)}\Big]\quad\forall t\in[0,T].$$

The proof is complete. ∎

REFERENCES

[1] M. Basei and H. Pham, *A weak martingale approach to linear-quadratic McKean-Vlasov stochastic control problems*, J. Optim. Theory Appl., 181 (2019), pp. 347–382.

[2] T. Breiten and L. Pfeiffer, *On the turnpike property and the receding-horizon method for linear-quadratic optimal control problems*, SIAM J. Control Optim., 58 (2020), pp. 1077–1102, https://doi.org/10.1137/18M1225811.

[3] R. Buckdahn, B. Djehiche, J. Li, and S. Peng, *Mean-field backward stochastic differential equations: A limit approach*, Ann. Probab., 37 (2009), pp. 1524–1565.
[4] R. Buckdahn, J. Li, and S. Peng, *Mean-field backward stochastic differential equations and related partial differential equations*, Stochastic Process. Appl., 119 (2009), pp. 3133–3154.
[5] D. A. Carlson, A. B. Haurie, and A. Leizarowitz, *Infinite Horizon Optimal Control—Deterministic and Stochastic Systems*, 2nd ed., Springer-Verlag, Berlin, 1991.
[6] T. Damm, L. Grüne, M. Stieler, and K. Worthmann, *An exponential turnpike theorem for dissipative discrete time optimal control problems*, SIAM J. Control Optim., 52 (2014), pp. 1935–1957, https://doi.org/10.1137/120888934.
[7] R. Dorfman, P. A. Samuelson, and R. M. Solow, *Linear Programming and Economics Analysis*, McGraw-Hill, New York, 1958.
[8] T. Faulwasser and L. Grüne, *Turnpike properties in optimal control: An overview of discrete-time and continuous-time results*, in Numerical Control: Part A, Handb. Numer. Anal. 23, North-Holland, Amsterdam, 2022, pp. 367–400.
[9] L. Grüne and R. Guglielmi, *Turnpike properties and strict dissipativity for discrete time linear quadratic optimal control problems*, SIAM J. Control Optim., 56 (2018), pp. 1282–1302, https://doi.org/10.1137/17M112350X.
[10] L. Grüne and R. Guglielmi, *On the relation between turnpike properties and dissipativity for continuous time linear quadratic optimal control problems*, Math. Control Relat. Fields, 11 (2021), pp. 169–188.
[11] J. Huang, X. Li, and J. Yong, *A linear-quadratic optimal control problem for mean-field stochastic differential equations in infinite horizon*, Math. Control Relat. Fields, 5 (2015), pp. 97–139.
[12] N. Li, X. Li, and Z. Yu, *Indefinite mean-field type linear-quadratic stochastic optimal control problems*, Automatica, 122 (2020), 109267.
[13] Q. Lü, *Stochastic linear quadratic optimal control problems for mean-field stochastic evolution equation*, ESAIM Control Optim. Calc. Var., 26 (2020), 127.
[14] H. Lou and W. Wang, *Turnpike properties of optimal relaxed control problems*, ESAIM Control Optim. Calc. Var., 25 (2019), 74.
[15] J. von Neumann, *A model of general economic equilibrium*, Rev. Econ. Stud., 13 (1945), pp. 1–9.
[16] A. Porretta and E. Zuazua, *Long time versus steady state optimal control*, SIAM J. Control Optim., 51 (2013), pp. 4242–4273, https://doi.org/10.1137/130907239.
[17] N. Sakamoto and E. Zuazua, *The turnpike property in nonlinear optimal control—a geometric approach*, Automatica, 134 (2021), 109939.
[18] J. Sun, *Mean-field stochastic linear quadratic optimal control problems: Open-loop solvabilities*, ESAIM Control Optim. Calc. Var., 23 (2017), pp. 1099–1127.
[19] J. Sun and J. Yong, *Stochastic Linear-Quadratic Optimal Control Theory: Differential Games and Mean-Field Problems*, Springer Briefs Math., Springer, Cham, 2020.
[20] J. Sun, H. Wang, and J. Yong, *Turnpike properties for stochastic linear-quadratic optimal control problems*, Chinese Ann. Math. Ser. B, 43 (2022), pp. 999–1022.
[21] E. Trélat and E. Zuazua, *The turnpike property in finite-dimensional nonlinear optimal control*, J. Differential Equations, 258 (2015), pp. 81–114.
[22] J. Yong, *Linear-quadratic optimal control problems for mean-field stochastic differential equations*, SIAM J. Control Optim., 51 (2013), pp. 2809–2838, https://doi.org/10.1137/120892477.
[23] A. J. Zaslavski, *Turnpike Properties in the Calculus of Variations and Optimal Control*, Nonconvex Optim. Appl. 80, Springer, New York, 2006.
[24] A. J. Zaslavski, *Turnpike Theory of Continuous-Time Linear Optimal Control Problems*, Springer Optim. Appl. 148, Springer, Cham, 2019.
[25] E. Zuazua, *Large time control and turnpike properties for wave equations*, Annu. Rev. Control, 44 (2017), pp. 199–210.