Unsteady Incompressible Thermal Laminar Boundary Layers Subject to Streamwise Pressure Gradients

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Abstract. This paper focuses on the laminar boundary layer startup process (momentum and thermal) in incompressible flows. The unsteady boundary layer equations can be solved via similarity analysis by normalizing the stream-wise (x), wall-normal (y) and time (t) coordinates by a variable η and τ, respectively. The resulting ODEs are solved by a finite difference explicit algorithm. This can be done for two cases: flat plate flow where the change in pressure are zero (Blasius solution) and wedge or Falkner-Skan flow where the changes in pressure can be favorable (FPG) or adverse (APG). In addition, transient passive scalar transport is examined by setting several Prandtl numbers in the governing equation at two different wall thermal conditions: isothermal and isoflux. Numerical solutions for the transient evolution of the momentum and thermal boundary layer profiles are compared with analytical approximations for both small times (unsteady flow) and large (steady-state flow) times.

BACKGROUND

The Navier-Stokes (NS) equations are the basis of the fluid mechanics theory. This set of partial differential equations (PDE) consists of a continuity and momentum equations which study the mass transport and the flow dynamics, respectively. An additional transport equation can be solved in incompressible flows, to describe the temporal/spatial evolution of thermal energy. If the temperature difference in the thermal boundary layer can be assumed sufficiently small, buoyancy forces and temperature dependence of fluid properties are negligible. Therefore, the temperature may be treated as a passive scalar, and the momentum and the heat transfer equations are uncoupled [1]. According to the Clay Mathematics Institute [2], there is not yet an analytical solution to the Navier-Stokes equations; however, there are several solutions to this equation based on some assumptions to reduce its complexity. The Prandtl or boundary layer theory is the solution of external flow for the Navier-Stokes equation. This approximated solution is achieved by applying the following assumptions to NS [3]: (i) wall-normal gradients are much larger than srtreamwise gradients, and (ii) pressure gradients may vary, or not, in the direction parallel to the wall (streamwise) but always remain approximately constant in the direction perpendicular to the wall. Once these assumptions are applied to the NS equations, the Prandtl boundary layer equations are obtained. In the present scenario, it is assumed an unsteady problem where the effects of time are taken into account. The energy equation for external flow is obtained in a similar manner to the external flow momentum equations by assuming that the variation of temperature in the direction perpendicular to the wall (gradient) is higher than the variation parallel to the wall [4]. As previously described, flow buoyancy is neglected due to the fact that the variation of the flow properties as the pressure and temperature changes is very small and remains quasi-constant, so, the energy equation can be decoupled form the momentum equation. This turns the temperature in the energy equation in to a passive scalar that does not affect the flow dynamics in the momentum equation [5]. According to Schlichting [6], unsteady boundary layers can be start-up or shutdown processes (flow that start from rest or that die away in time) and periodic processes. The boundary layers assumptions previously stated are also applied to unsteady flow with the addition of the dimensionless time reference quantity, τ, used in the small time solution different from the large time, where the flow approaches the steady state solution. In the case of start up processes the fluid surrounding the body starts at rest in the form of an irrotational potential flow. As time progresses, the boundary layer evolves and start to increase its thickness, eventually, a very strong APG may induce boundary layer detachment and the boundary layer equations can no longer be applied. In this manuscript, the Prandtl boundary layer equations are solved numerically (or semi-analytically) by applying finite difference. A similarity analysis is applied by normalizing the stream-wise (x), wall-normal (y) and time (t) coordinates by new variable η and τ, respectively. The family of the Falkner-Skan [7] flows is considered where it is assumed that the freestream velocity varies as a power law of the streamwise coordinate, i.e. $U_\infty \sim x^m$. Consequently, flow acceleration or FPG is related to positive values of m, while flow deceleration or APG is connected to negative values of m. As a particular case for $m = 0$ or ZPG, the Blasius solution arises. The resulting unsteady third order differential equation is solved by an explicit finite difference scheme, second order accurate.

Unsteady Boundary Layer Governing Equations

The unsteady laminar boundary layer equation for incompressible flows with a passive scalar are:

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

Momentum:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e\frac{\partial u_e}{\partial x} + \frac{\partial^2 u}{\partial y^2} \tag{2}$$

Energy:

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{Pr}\frac{\partial^2 T}{\partial y^2} \tag{3}$$

According to Harris et al. [8, 9], a similarity analysis can be applied to reduce the momentum equation 2 to a third order ODE:

$$\frac{\partial^3 f}{\partial \eta^3} + \left[\frac{m+1}{2}f + (m-1)\tau\frac{\partial f}{\partial \tau}\right]\frac{\partial^2 f}{\partial \eta^2} + m\left[1 - (\frac{\partial f}{\partial \eta})^2\right] = \left[1 + (m-1)\tau\frac{\partial f}{\partial \eta}\right]\frac{\partial^2 f}{\partial \eta \partial \tau}, \tag{4}$$

where m is the power law parameter related to the pressure gradient, $f(\eta, \tau)$ is associated with the stream function, $f'(\eta, \tau)$ is associated with the streamwise velocity and $f''(\eta, \tau)$ is associated with the shear stress. The corresponding η $\left(= x^{\frac{m-1}{2}} y\right)$ and τ $\left(= x^{m-1} t\right)$ terms are related to space and time, respectively. Similarly, the transformed energy equations for isothermal and isoflux wall conditions read, respectively:

$$\frac{1}{Pr}\frac{\partial^2 g}{\partial \eta^2} + \left[\frac{1+m}{2}f + (m-1)\tau\frac{\partial f}{\partial \tau}\right]\frac{\partial g}{\partial \eta} = \left[1 + (m-1)\tau\frac{\partial f}{\partial \eta}\right]\frac{\partial g}{\partial \tau} \tag{5}$$

$$\frac{1}{Pr}\frac{\partial^2 g}{\partial \eta^2} + \left[\frac{m+1}{2}f + (m-1)\tau\frac{\partial f}{\partial \tau}\right]\frac{\partial g}{\partial \eta} - \frac{1-m}{2}g\frac{\partial f}{\partial \eta} = \left[1 + (m-1)\tau\frac{\partial f}{\partial \eta}\right]\frac{\partial g}{\partial \tau}, \tag{6}$$

here, Pr is the molecular Prandtl number and $g(\eta, \tau)$ $(= (T - T_w)/(T_\infty - T_w))$ is the dimensionless temperature. The boundary conditions for $\tau \geq 0$ are: $f(0, \tau) = 0$, $\frac{\partial f}{\partial \eta}(0, \tau) = 0$, $\frac{\partial f}{\partial \eta}(\infty, \tau) = 1$, $\frac{\partial g}{\partial \eta}(0, \tau) = -1$ (constant heat flux), $f(0, \tau) = 0$ (constant wall temperature), $g(\infty, \tau) = 0$. The following initial conditions ($\tau < 0$) are considered: $f = g = 0$ for all η. Harris *et al.* [8, 9] stated that for small time solution where $\tau << 1$ or all impulsive changes in temperature or heat flux, there is a short period in which the effects are confined to a thin boundary layer adjacent to the surface. Therefore, for small time solution the diffusion scale can be set to $\sqrt{\tau}$ with the following dimensionless variables: For the momentum conservation eq.: $f = 2\sqrt{\tau}F(\zeta, \tau)$ and $\zeta = \frac{\eta}{2\sqrt{\tau}}$.

For the energy conservation eq. with wall constant temperature: $g = G(\zeta, \tau)$.

For the energy conservation eq. with wall constant heat flux: $g = 2\sqrt{\tau}^{\frac{1}{2}}G(\zeta, \tau)$.

Applying the above non-dimensional variables, the following unsteady transformed equations are obtained,

$$\frac{\partial^3 F}{\partial \zeta^3} + \left[2\zeta + 4m\tau F + 4(m-1)\tau^2\frac{\partial F}{\partial \tau}\right]\frac{\partial^2 F}{\partial \zeta^2} + 4m\tau\left[1 - (\frac{\partial F}{\partial \zeta})^2\right] = 4\tau\left[1 + (m-1)\tau\frac{\partial^2 F}{\partial \zeta}\right]\frac{\partial^2 F}{\partial \zeta \partial \tau} \tag{7}$$

$$\frac{1}{Pr}\frac{\partial^2 G}{\partial \zeta^2} + \left[2\zeta + 4m\tau F + 4(m-1)\tau^2\frac{\partial F}{\partial \tau}\right]\frac{\partial G}{\partial \zeta} = 4\tau\left[1 + (m-1)\tau\frac{\partial F}{\partial \tau}\right]\frac{\partial G}{\partial \zeta} \tag{8}$$

$$\frac{1}{Pr}\frac{\partial^2 G}{\partial \zeta^2} + \left[2\zeta + 4m\tau F + 4(m-1)\tau^2\frac{\partial F}{\partial \tau}\right]\frac{\partial G}{\partial \zeta} - 2G = 4\tau\left[1 + (m-1)\tau\frac{\partial F}{\partial \tau}\right]\frac{\partial G}{\partial \zeta} \tag{9}$$

for momentum (7), constant wall temperature (8) and constant heat flux (9) transient equations. It is also known that for small time, the solution of eq. 7 have the form $F(\zeta, \tau) = F_0(\zeta) + F_1(\zeta)\tau + ...$; and, $G(\zeta, \tau) = G_0(\zeta) + G_1(\zeta)\tau + ...$ for eqns. 8 and 9, respectively. Applying the proposed solutions to eqns. 7, 8 and 9, the new set of ODE's is solved by applying a Finite Difference algorithm with the corresponding boundary conditions. In addition, the skin friction coefficient, C_f, can be computed as $C_f\sqrt{Re_x} = \frac{\partial^2 f}{\partial \eta^2}(0, \tau)$. Whereas, the analytical expression for C_f is as follows:

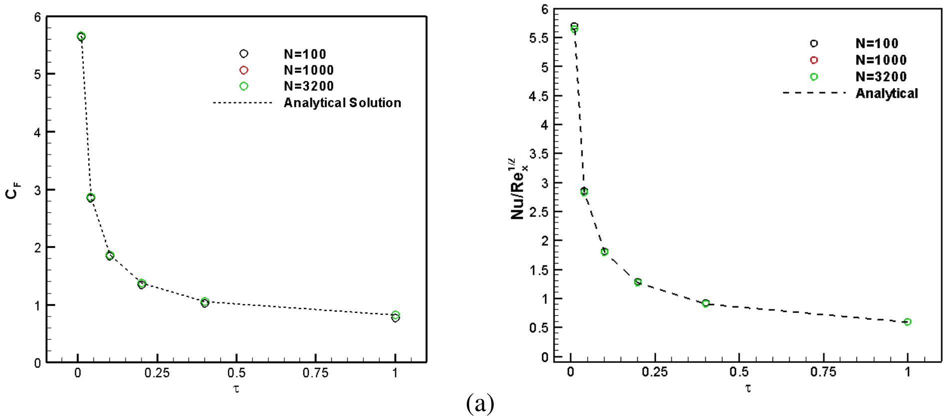

FIGURE 1. Grid independence test of the FD algorithm employed vs. analytical solution at β = 0.5: (a) Skin friction coefficient, (b) Nusselt number for a constant wall temperature condition.

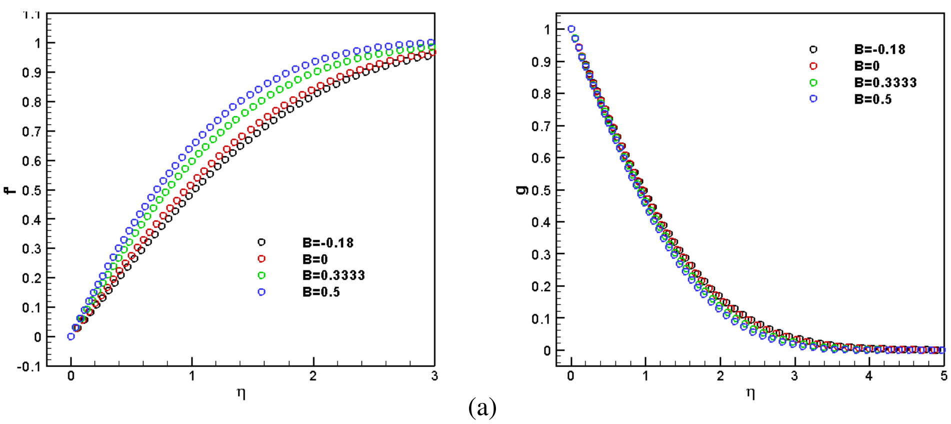

FIGURE 2. (a) Normalized streamwise velocity, f', and (b) temperature (constant wall temperature condition) for β =-0.18, 0, 0.3333 and 0.5 at τ= 1.

$$C_f \sqrt{Re_x} = \frac{1}{\sqrt{\pi}} \sqrt{\tau} + \frac{m}{\sqrt{\pi}}(1 + \frac{4}{3\pi}) \sqrt{\tau} \tag{10}$$

Similarly, the Nusselt number, N_u, for a constant wall temperature condition is defined as can be obtained as $\frac{Nu}{Re_x^{\frac{1}{2}}} = -\frac{\partial g}{\partial \eta}(0, \tau)$. The corresponding analytical expression is:

$$\frac{Nu}{Re_x^{\frac{1}{2}}} = \sqrt{\frac{Pr}{\pi}}\tau^{-\frac{1}{2}} + \frac{m}{2}\sqrt{\frac{Pr}{\pi}}[(1 + Pr)^2 tan^{-1}(\frac{1}{\sqrt{Pr}}) - \frac{1}{3}\sqrt{Pr}(5 + 3Pr)]\tau^{\frac{1}{2}} \tag{11}$$

Figures 1 exhibits a grid independence test for the unsteady and normalized skin friction ($C_f \sqrt{Re_x}$) and Nusselt number ($Nu/\sqrt{Re_x}$) in a favorable pressure gradient flow (β =0.5 and $m = 1/3$). Here, β is a constant defining the wedge angle ($\pi\beta$) and $m = \beta/(2 - \beta)$ is the acceleration or deceleration exponent in Falkner-Skan flows. Furthermore, the agreement with the corresponding analytical approximations (10 and 11) is excellent. Fig. 2(a) shows the normalized streamwise velocity, f', at different streamwise pressure gradients, from very strong APG (β = -0.18) to very strong FPG (β = 0.5) at a dimensionless time of τ = 1. It can be observed that as the flow decelerates (i.e., β = -0.18), the transient boundary layer thickness grows. The β-0.18 profile does not depict the typical "S" shape (inflexion point) as in the steady solution, indicating that the flow is far from the detachment stage at this τ value. On the contrary, the streamwise pressure gradient seems not to significantly affect the thermal field at Pr = 1, as seen in fig. 2(b).

CONCLUSIONS

A transient study is performed in momentum and thermal laminar boundary layers subject to streamwise pressure gradients. An efficent FD scheme is employed in the solution of the unsteady ODE of Falkner-Skan flows with passive scalar transport. A sharp decrease of C_f and N_u is seen within the initial unitary dimensionless time ($\tau = 1$), in the order of 6X, with values very close to the steady state solution. However, the streamwise velocity (f') inside the boundary layer is far from being developed at $\tau = 1$. Clearly, the laminar flow exhibits a development time scale much smaller in the near wall region. Future study will involve the analysis of molecular Prandtl number effect on the transient evolution of thermal boundary layers subject to isothermal and isoflux wall conditions.

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