ON BOUNDED DEGREE GRAPHS WITH LARGE SIZE-RAMSEY NUMBERS

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Abstract. The size-Ramsey number $\hat{r}(G')$ of a graph G' is defined as the smallest integer m so that there exists a graph G with m edges such that every 2–coloring of the edges of G contains a monochromatic copy of G'. Answering a question of Beck, Rödl and Szemerédi showed that for every $n \geq 1$ there exists a graph G' on n vertices each of degree at most three, with size-Ramsey number at least $cn \log^{\frac{1}{60}} n$ for a universal constant $c > 0$. In this note we show that a modification of Rödl and Szemerédi's construction leads to a bound $\hat{r}(G') \geq cn\, \exp(c\sqrt{\log n})$.

1. Introduction

The size-Ramsey number of a graph G' can be viewed as the number of edges in a most economical "robust version" of G', a graph G such that every 2–coloring of the edges of G contains a monochromatic copy of G' [4]. In [1], Beck asked whether every bounded degree graph has size-Ramsey number linear in the number of its vertices. The question was answered negatively by Rödl and Szemerédi in [6] who constructed for every $n \geq 1$ a graph G' on n vertices with the maximum degree at most three, such that $\hat{r}(G') \geq cn \log^{\frac{1}{60}} n$. The authors of [6] further conjectured that for every $d \geq 3$ there is a number $\varepsilon = \varepsilon(d) > 0$ such that for all sufficiently large n,

$$n^{1+\varepsilon} \leq \max\left\{\hat{r}(G') : \; G' \text{ has } n \text{ vertices, each of degree at most } d\right\} \leq n^{2-\varepsilon}.$$

The conjecture was partially confirmed in [5] where it was shown that

$$\max\left\{\hat{r}(G') : \; G' \text{ has } n \text{ vertices, each of degree at most } d\right\} \leq n^{2-1/d+o(1)}.$$

We refer to papers [2, 3] for improvements of the upper bound as well as further references regarding size-Ramsey numbers. Whereas there has been substantial progress on estimating size-Ramsey numbers of sparse graphs from above, the lower bound $cn \log^{\frac{1}{60}} n$ given in [6] seems to be the best known estimate as of this writing. The purpose of this note is to show that a modification of Rödl and Szemerédi's construction leads to an improvement of the lower bound:

Theorem 1.1. *For every $n \geq 1$ there is a graph G' on n vertices of maximum degree at most three such that $\hat{r}(G') \geq cn\, \exp(c\sqrt{\log n})$, for a universal constant $c > 0$.*

2. Proof of Theorem 1.1

Definition 2.1. *Let $k \geq 2$ be an integer parameter. We define a labeled random graph $U_k = (V_k, E_k)$ as follows. Let $T = (V_k, E_T)$ be a complete rooted binary tree of depth k, and let $V_L \subset V_k$ be its set of leaves. Let $C = (V_L, E_C)$ be a spanning cycle on V_L chosen uniformly at random. Then we let U_k be the union of T and C, namely the graph with vertex set V_k and edge set $E_T \cup E_L$. We refer to Figure 1 for a realization of U_3.*

Remark 2.2. We will call the root of T the **root** of U_k.

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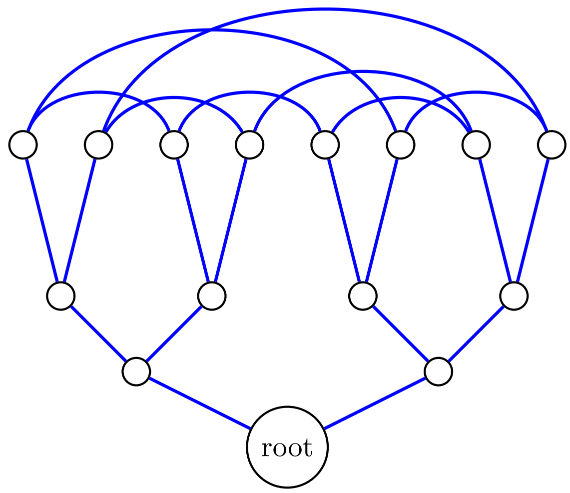
Figure 1. A realization of U_3.



Lemma 2.3. *Let $U_k=(V_k,E_k)$ be as above, and let $G=(V_G,E_G)$ be a non-random labeled graph of maximum degree d. Fix any vertex $v\in V_G$. Then*

$$\mathbb{P}\big\{\textit{There is an embedding of } U_k \textit{ into } G \textit{ mapping the root of } U_k \textit{ into } v\big\}\le \frac{d^{2^k-1}\cdot d^{2^{k+1}}}{(2^k-1)!}.$$

Proof. Let $T=(V_k,E_T)$ be the binary tree subgraph of U_k from the definition, let v_r be its root, and V_L be its set of leaves. We first claim that for any non-random embedding ϕ of T into G mapping v_r into v, ϕ can be extended to an embedding of U_k into G with probability at most

$$\prod_{\ell=1}^{2^k-1}\frac{d}{\ell}=\frac{d^{2^k-1}}{(2^k-1)!}.$$

This bound can be obtained by considering the following spanning cycle generation on V_L. Let $v_0\in V_L$ be a fixed vertex of V_L. At the first step, a vertex $v_1\in V_L\setminus\{v_0\}$ is chosen uniformly at random; at the second step, v_2 is chosen uniformly on the set $V_L\setminus\{v_0,v_1\}$, and so on. The cycle is given by the random set of edges $\{v_i,v_{(i+1)\mod |V_L|}\}$, $0\le i\le |V_L|-1$. Then, conditioned on any realization of $v_1,\dots,v_i$, the probability that $\phi(v_i)$ and $\phi(v_{i+1})$ are adjacent in G is at most $\frac{d}{2^k-1-i}$, and the claim follows.

To complete the proof of the lemma, it is sufficient to give upper bound on the number N of embeddings ϕ of T into G mapping v_r into v. Since every vertex of G has at most d neighbors, a rough bound gives

$$N\le d^{2^1+2^2+\dots+2^k}\le d^{2^{k+1}},$$

and the result follows. $\square$

As an immediate corollary, we get

Corollary 2.4. *Let $r\ge 1$, $k,d\ge 2$, and let $U^{(1)},\dots,U^{(r)}$ be i.i.d copies of U_k. Then*

$$\begin{aligned}&\mathbb{P}\big\{\textit{There is a labeled graph } G \textit{ of maximum degree } d \textit{ and a vertex } v \textit{ of } G \textit{ such that}\\ &\quad \textit{for each } i\le r \textit{ there is an embedding of } U^{(i)} \textit{ into } G \textit{ mapping the root of } U^{(i)} \textit{ into } v\big\}\\ &\quad\le\left(\frac{d^{2^k-1}\cdot d^{2^{k+1}}}{(2^k-1)!}\right)^r\cdot\left(d^{k+1}\right)^{d\cdot d^{k+1}}.\end{aligned}$$

Proof. The proof is accomplished via a union bound. We first note that the event in question coincides with the event

$$\big\{\text{There is a labeled graph } G \text{ of maximum degree } d \textit{ on } d^{k+1} \textit{ vertices} \text{ and a vertex } v \text{ of } G \text{ such that for each } i \le r \text{ there is an embedding of } U^{(i)} \text{ into } G \text{ mapping the root of } U^{(i)} \text{ into } v\big\}.$$

Indeed, the claim follows by observing that in any graph of maximum degree d, any ball of radius k contains at most $1+d+d^2+\cdots+d^k \le d^{k+1}$ vertices. We further can assume that the graphs G in the above event have a common vertex set V. The total number of such graphs G can be crudely bounded above by

$$\big(d^{k+1}\big)^{d\cdot d^{k+1}},$$

implying the result. □

Everything is ready for the proof of Theorem 1.1. Note that the result is trivial for small n by choosing the constant c in the statement of the theorem sufficiently small. From now on, we assume that n is a large integer. Let parameters $h\ge 1$, $1\le r\le h$, $k,d\ge 2$ be chosen as follows:

$$d := \big\lfloor \exp\big(\sqrt{\log n}/100\big)\big\rfloor;\quad k = \big\lfloor \sqrt{\log n}/10\big\rfloor;\quad r := d\cdot d^{k+1};\quad h := \lfloor 2^{-k-1}n\rfloor.$$

Let $U^{(1)},\dots,U^{(h)}$ be i.i.d copies of U_k (on disjoint vertex sets), and define a random graph G' as the union of $U^{(1)},\dots,U^{(h)}$. We will show that with a positive probability, $\hat r(G') \ge \exp\big(\sqrt{\log n}/1000\big)n$. For every r–subset S of $\{1,2,\dots,h\}$, let $\mathcal{E}_S$ be the event

$$\mathcal{E}_S := \big\{\text{There is a labeled graph } G \text{ of maximum degree } d \text{ and a vertex } v \text{ of } G \text{ such that for each } i\in S \text{ there is an embedding of } U^{(i)} \text{ into } G \text{ mapping the root of } U^{(i)} \text{ into } v\big\},$$

and let $\mathcal{E}$ be the intersection of the complements $\mathcal{E}_S^c$, $|S|=r$, $S\subset\{1,2,\dots,h\}$. In view of Corollary 2.4 and our choice of parameters,

$$\begin{aligned}
\mathbb{P}(\mathcal{E}) &\ge 1-\sum_{S\subset\{1,\dots,h\},\,|S|=r}\mathbb{P}(\mathcal{E}_S)\\
&\ge 1-\binom{h}{r}\left(\frac{d^{2^k-1}\cdot d^{2^{k+1}}}{(2^k-1)!}\right)^r\cdot\big(d^{k+1}\big)^{d\cdot d^{k+1}}\\
&\ge 1-\left(\frac{eh\cdot d^{2^k-1}\cdot d^{2^{k+1}}}{d(2^k-1)!}\right)^r\\
&\ge 1-\left(\frac{e\cdot n\cdot\exp\big(2^k-1+2^{k+2}\sqrt{\log n}/100\big)}{(2^k-1)^{2^k-1}}\right)^r>0.
\end{aligned}$$

Condition on any realization of G' from $\mathcal{E}$. Let $G=(V,E)$ be any graph with at most $\exp\big(\sqrt{\log n}/1000\big)\,n$ edges. We will show that there is a 2–coloring of the edges of G such that G does not contain a monochromatic copy of G'.

Denote by $E_{high}\subset E$ the collection of all edges of G incident to vertices of degree at least $d+1$, and let $\tilde G$ be the subgraph of G obtained by removing E_{high} from the edge set of G. Observe that the maximum degree of $\tilde G$ is at most $d=\lfloor\exp\big(\sqrt{\log n}/100\big)\rfloor$. By the definition of $\mathcal{E}$, for every vertex v of $\tilde G$ there are at most $r-1$ indices $i\le h$ such that $U^{(i)}$ can be embedded into $\tilde G$ with the root of $U^{(i)}$ mapped into v. Since the number of non-isolated vertices of $\tilde G$ is at most $2\exp\big(\sqrt{\log n}/1000\big)\,n$ we get that there exists an index $i_0\le h$ and a subset of vertices V_r of $\tilde G$ of size at most $r\cdot 2\exp\big(\sqrt{\log n}/1000\big)\,n/h$, such that $U^{(i_0)}$ can be embedded into $\tilde G$ only when mapping the root of $U^{(i_0)}$ into one of vertices in V_r.

At this stage, we can define a coloring of G. Color all edges from E_{high} as well as all edges incident to V_r red, and all other edges blue, and denote the corresponding sets of edges by E_{red}

and E_{blue}, respectively. Note that the blue subgraph $G_{blue} = (V, E_{blue})$ of G is also a subgraph of $\tilde{G}$, and G_{blue} cannot contain a copy of G' since, by our construction, it does not contain a copy of $U^{(i_0)}$. Assume for a moment that G' is embeddable into subgraph $G_{red} = (V, E_{red})$, and let $\phi : G' \to G_{red}$ be an embedding. Every edge from $E_{red} \setminus E_{high}$ is incident to a vertex in V_r which has degree at most d in G, and therefore

$$\begin{aligned} |E_{red} \setminus E_{high}| &\leq d \cdot r \cdot 2\exp\left(\sqrt{\log n}/1000\right) n/h \\ &\leq 4\exp\left((k+3)\sqrt{\log n}/100 + \sqrt{\log n}/1000\right) 2^{k+1} < h/2. \end{aligned}$$

Let $I \subset \{1, 2, \dots, h\}$ be the subset of all indices i such that $\phi(U^{(i)})$ contains an edge from $E_{red} \setminus E_{high}$. Then, by the above,

$$|I| < h/2.$$

For every $i \in \{1, 2, \dots, h\} \setminus I$, the edge set of the graph $\phi(U^{(i)})$ is entirely comprised by E_{high} and, in particular, more than 2^{k-1} vertices of $\phi(U^{(i)})$ have degree at least $d+1$ in G. Thus, the total number of vertices in G of degree $d+1$ or larger can be estimated from below by

$$(h - |I|) \cdot 2^{k-1} \geq \frac{h}{2} \cdot 2^{k-1}.$$

On the other hand, the number of such vertices cannot be greater than

$$\frac{2\exp\left(\sqrt{\log n}/1000\right) n}{d}.$$

We get the inequality

$$\frac{h}{2} \cdot 2^{k-1} \leq \frac{2\exp\left(\sqrt{\log n}/1000\right) n}{d},$$

which is clearly false. The contradiction shows that G' cannot be embedded into G_{red}, and the result follows.

References

[1] J. Beck, On size Ramsey number of paths, trees and circuits. II, in *Mathematics of Ramsey theory*, 34–45, Algorithms Combin., 5, Springer, Berlin. MR1083592

[2] D. Conlon, R. Nenadov and M. Trujić, The size-Ramsey number of cubic graphs, Bull. Lond. Math. Soc. **54** (2022), no. 6, 2135–2150. MR4528614

[3] N. Draganić, K. Petrova, Size-Ramsey numbers of graphs with maximum degree three, preprint, arXiv:2207.05048.

[4] P. Erdős, R. J. Faudree, C. C. Rousseau, R. H. Schelp, The size Ramsey number, Period. Math. Hungar. **9** (1978), no. 1-2, 145–161. MR0479691

[5] Y. Kohayakawa, V. Rödl, M. Schacht, E. Szemerédi, Sparse partition universal graphs for graphs of bounded degree, Adv. Math. **226** (2011), no. 6, 5041–5065. MR2775894

[6] V. Rödl and E. Szemerédi, On size Ramsey numbers of graphs with bounded degree, Combinatorica **20** (2000), no. 2, 257–262. MR1767025

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