Well-posedness of solutions to stochastic fluid-structure interaction

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Abstract

In this paper we introduce a constructive approach to study well-posedness of solutions to stochastic fluid-structure interaction with stochastic noise. We focus on a benchmark problem in stochastic fluid-structure interaction, and prove the existence of a unique weak solution in the probabilistically strong sense. The benchmark problem consists of the 2D time-dependent Stokes equations describing the flow of an incompressible, viscous fluid interacting with a linearly elastic membrane modeled by the 1D linear wave equation. The membrane is stochastically forced by the time-dependent white noise. The fluid and the structure are linearly coupled. The constructive existence proof is based on a time-discretization via an operator splitting approach. This introduces a sequence of approximate solutions, which are random variables. We show the existence of a subsequence of approximate solutions which converges, almost surely, to a weak solution in the probabilistically strong sense. The proof is based on uniform energy estimates in terms of the *expectation* of the energy norms, which are the backbone for a weak compactness argument giving rise to a weakly convergent subsequence of *probability measures* associated with the approximate solutions. Probabilistic techniques based on the Skorohod representation theorem and the Gyöngy-Krylov lemma are then employed to obtain almost sure convergence of a subsequence of the random approximate solutions to a weak solution in the probabilistically strong sense. The result shows that the deterministic benchmark FSI model is robust to stochastic noise, even in the presence of rough white noise in time. To the best of our knowledge, this is the first well-posedness result for stochastic fluid-structure interaction.

1 Introduction

In this paper, we introduce a constructive approach to study solutions of stochastic fluid-structure interaction (SFSI) with stochastic noise. Problems of this type arise in many applications. One example is the flow of blood in human coronary arteries, which sit of the surface of the heart. In particular, it is now well-known that the flow of blood through the heart has a strong stochastic component. The stochastic fluctuations of the single ion channels and the sub-cellular dynamics in tissue and organ scale get reflected in the macroscopic random cardiac events [52]. Another example is sperm swimming, which exhibits stochastic fluctuations in the sperm swimming paths [23]. More generally, studying SFSI is important because well-posedness of SFSI models provides confidence that the deterministic FSI models are, indeed, robust to stochastic noise that occurs naturally in real-life problems.

From the mathematic point of view, this manuscript is written as an introduction to the use of stochastic techniques to study SFSI, and is aimed at audiences that have experience with deterministic FSI, but may be new to stochastic analysis. We focus on a benchmark problem in which a stochastically forced linearly elastic membrane interacts with the flow of a viscous incompressible Newtonian fluid in two spatial dimensions. The membrane is modeled by the linear wave equation, while the fluid is modeled by the 2D time-dependent Stokes equations. The problem is forced by a "rough" stochastic forcing given by a time-dependent white noise $\dot{W}(t)$, where W is a given one-dimensional Brownian motion with respect to a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with complete filtration $\{\mathcal{F}_t\}_{t\geq 0}$. The fluid and the membrane are coupled via a two-way coupling describing continuity of fluid and structure velocities at the fluid-structure interface, and continuity of contact forces at the interface. The coupling is calculated at the linearized, fixed interface, rendering this problem a linear stochastic fluid-structure interaction problem. The goal is to show that despite the rough white noise, the resulting problem is well-posed, showing that the underlying deterministic fluid-structure interaction problem is robust to noise. Indeed, we prove the existence of a unique weak solution in the probabilistically strong sense (see Definition 4.2 in Section 4) to this stochastic fluid-structure interaction problem. This means that there exist unique random variables (stochastic processes), describing the fluid velocity $\boldsymbol{u}$, the structure velocity v, and the structure displacement η, such that those stochastic processes are adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, i.e., they only depend on the past history of the processes up to time t and not on the future, which satisfy the weak formulation of the original problem almost surely. This is the main result of this manuscript, stated in Theorem 4.1.

To prove the existence of a unique weak solution in the probabilistically strong sense, we design a constructive existence proof. The constructive existence proof is based on semi-discretizing the problem in time by dividing the time interval $(0, T)$ into N subintervals of width $\Delta t = T/N$, and using a time-splitting scheme, introduced in [5], to construct approximate solutions. The goal is to show that the approximate solutions converge almost surely with respect to a certain topology, to the unique weak solution as Δt goes to zero. In contrast to the deterministic case, see the works of Muha and Čanić in [43–45], where a time-discretization via operator splitting approach was used to study existence of weak solutions, our proposed splitting scheme for the current stochastic FSI problem involves an additional subproblem that separates the stochastic contribution to the FSI dynamics from the fluid/structure subproblems. This gives rise to a *three-way splitting scheme* which separates each of the individual components of this multiphysical stochastic FSI problem from each other: the random noise, the structure elastodynamics, and the fluid. More precisely, along each time sub-interval $(t^n_N, t^{n+1}_N), n = 0, \dots, N-1$, the following three sub-problems are solved to obtain approximate solutions consisting of the fluid and structure velocities, and the structure displacement, $(\boldsymbol{u}, v, \eta)$. First, in **Step 1**, the structure displacement and structure velocity are updated using only the structure displacement and structure velocity from the previous time step. The resulting random variables are measurable with respect to the sigma algebra $\mathcal{F}_{t^n_N}$. Then, in **Step 2**, which is the stochastic step, the structure velocity is updated by adding to the structure velocity calculated in Step 1 the stochastic noise increment from time step t^n_N to time step t^{n+1}_N. Since the structure velocity obtained in Step 1 is a random variable that is measurable with respect to the sigma algebra $\mathcal{F}_{t^n_N}$, and the stochastic increment from t^n_N to t^{n+1}_N is independent of it, we will be able to obtain boundedness of the stochastic integral involving these two quantities by using their independence. This will lead to stability. The resulting updated structure velocity is a random variable that is measurable with respect to the sigma algebra $\mathcal{F}_{t^{n+1}_N}$. Finally, in **Step 3**, the fluid and structure velocities are updated by using the information from the just calculated structure velocity in Step 2. This gives rise to random variables that are measurable with respect to the sigma algebra $\mathcal{F}_{t^{n+1}_N}$. We would like to show that the sequence or a subsequence of random variables constructed this way converges in a certain topology to a weak solution in the probabilistically strong sense of the coupled SFSI problem.

Based on this splitting scheme, uniform energy estimates in terms of expectation can be derived. In addition to estimating the expectation of the kinetic and elastic energy of the problem, it is important to get a uniform bound on the expectation of the numerical dissipation, to show that the numerical dissipation is bounded and that it in fact, approaches zero as the time step Δt goes to zero, which is crucial in the convergence proof. This is provided in Proposition 6.7. We further remark that the numerical dissipation terms in the energy estimate describe dissipation that arises as a result of the time discretization, and is given explicitly by the sum of the squares of the differences of the approximate solutions on adjacent time steps measured in appropriate energy-level spatial function spaces. Hence, having a uniform bound in expectation of the numerical dissipation terms will be essential for estimates on time shifts of the random approximate solutions (defined as functions of both space and time), which will be used in later compactness arguments, see for example Lemma 8.2. Furthermore, another interesting observation is that the energy estimates will have an extra term on the right-hand side which accounts for the energy pumped into the problem by the stochastic noise. This is in addition to the energy/work contributions by the initial and boundary data. These energy estimates define an energy function space for the unknown functions $(\boldsymbol{u}, v, \eta)$. A separable subspace of the energy space, specified in (34) in Section 8.1 is called a *phase space*, and is denoted by $\mathcal{X}$.

Uniform estimates are the backbone for weak compactness, giving rise to convergent subsequences whose limits potentially satisfy the original problem in a certain sense. In the deterministic case, the uniform energy estimates typically imply existence of weakly- and weakly*-convergent subsequences in the appropriate topologies, which is usually sufficient to pass to the limit in the linear problem and recover the weak solution. Similarly, having uniform boundedness, even just in expectation, of the random approximate solutions constructed in the splitting scheme is sufficient for passing to the limit in the random approximate solutions on the given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, as one can use the weak convergence of the random approximate solutions weakly and weakly star in function spaces involving both the probability space and the natural spacetime function spaces in order to pass to the limit in the semidiscrete weak formulations. See the discussion in the Appendix in Section 11. However, while such an argument works well for the specific case of this fully linear stochastic system of PDEs, this is an approach that does not generalize to more complex cases of interest. This includes the case of linearly coupled FSI with nonlinearities, for example in the intensity of the noise [35], and the case of *nonlinearly coupled* stochastic FSI systems, in which the fluid domain is a moving domain whose top boundary is determined by the time-dependent configuration of an elastic structure that is stochastically forced [57]. This gives rise to a complex stochastic system of PDEs in which additional geometric nonlinearities and probabilistic difficulties arise from the fact that the Navier-Stokes equations for the fluid are posed on an a priori unknown, random, time-dependent fluid domain. Thus, the goal of the current manuscript is to develop a general methodology for studying these complex stochastic systems. We hence develop a framework for existence using compactness arguments and probabilistic methods from stochastic PDEs that generalize to more complex nonlinear stochastic FSI systems. The goal of the compactness arguments that we use is to obtain the existence of weakly convergent subsequences of *probability measures*, or *laws*, describing the *distributions* of the random approximate solutions. Once weak convergence of probability measures is established, one can work on getting almost sure convergence of the random approximate solutions, which can be used to recover a weak solution. We emphasize that even in the fully linear case, this approach provides stronger convergence results in terms of convergence in probability and hence convergence almost surely along a subsequence of the (random) approximate solutions that cannot be deduced just from uniform boundedness results alone. The additional compactness arguments that must be done to show stronger convergence in probability of the (random) approximate solutions, in contrast to what can be deduced directly from uniform boundedness of approximate solutions in expectation, is important for numerical method development. The fact that the random approximate solutions generated via the splitting scheme converge in probability implies that numerical solutions to a splitting scheme based on our constructive proof would converge in probability to the unique true solution to the fully coupled stochastic FSI problem.

To establish weak convergence of probability measures, one must show that the probability measures are **tight**. More precisely, one must show that for each $\epsilon > 0$, there exists a **compact set** in the phase space $\mathcal{X}$ of displacements and fluid and structure velocities, such that the probability that our approximate solutions $(\boldsymbol{u}_N, v_N, \eta_N)$ live in that compact set is greater than $1 - \epsilon$. See Definition 8.1 for tightness of measures. The proof of tightness of the sequence of probability measures μ_N corresponding to the laws of the approximate solutions $(\boldsymbol{u}_N, v_N, \eta_N)$ will follow from a *deterministic* compactness argument alla Aubin-Lions. The compactness argument will establish the existence of a compact subset of the phase space $\mathcal{X}$ that contains the approximate solutions $(\boldsymbol{u}_N, v_N, \eta_N)$ with probability greater than $1 - \epsilon$, thus verifying the tightness property.

Once we have established the existence of a subsequence of probability measures μ_N that converges weakly to some probability measure μ as $N \to \infty$, or equivalently, as $\Delta t \to 0$, we would like to show that on a further subsequence, the *random variables* $(\boldsymbol{u}_N, v_N, \eta_N)$ will converge almost surely to a random variable with the law μ, with respect to the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For our current fully linear benchmark stochastic FSI model, showing this almost sure convergence with respect to the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, will be done in two parts. In the first part, we get a hold of a subsequence of approximate solutions that converge almost surely **but** on another probability space, and then use this information in the second part to construct a convergent subsequence of approximate solutions that converge on the *original* probability space. The following is a more detailed albeit succinct description of the two parts.

Part 1. We use the Skorohod representation theorem to deduce that there exists a sequence of random variables $(\tilde{\boldsymbol{u}}, \tilde{v}, \tilde{\eta})_N$, defined on **a probability space** $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, which is *not necessarily the same as* the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that the laws of $(\tilde{\boldsymbol{u}}_N, \tilde{v}_N, \tilde{\eta}_N)$ are μ_N, and $(\tilde{\boldsymbol{u}}_N, \tilde{v}_N, \tilde{\eta}_N)$ converge almost surely to a random variable $(\tilde{\boldsymbol{u}}, \tilde{v}, \tilde{\eta})$ with the law μ, on the "tilde" probability space. On this "tilde" probability space we also show that the almost sure limit $(\tilde{\boldsymbol{u}}, \tilde{v}, \tilde{\eta})$ satisfies the weak formulation of the original problem almost surely, **but** with respect to the "tilde" probability space. This means that this limit is a **weak solution to the original problem in the probabilistically *weak* sense**, see Definition 4.1. This result will be useful in showing the existence of a unique weak solution in the probabilistically *strong* sense on the *original* probability space $(\Omega, \mathcal{F}, \mathbb{P})$, discussed in the second part. We remark that although transferring all of the probabilistic information to a new probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with equivalence of laws may at first seem to be an abstract construction, we emphasize that this is a mathematically natural construction, as it is well-known that the new probability space can be taken to be one of the simplest probability spaces, in which $\tilde{\Omega} = [0, 1)$, $\mathcal{F}$ is the set of Borel measurable subsets on $[0, 1)$ and $\mathbb{P}$ is the Lebesgue measure, see Theorem 2.4 and its proof in [51].

Part 2. We would like to be able to prove that our sequence of approximate solutions $(\boldsymbol{u}_N, v_N, \eta_N)$, obtained using our time-discretization via operator splitting approach described above, converges almost surely to a random variable $(\boldsymbol{u}, v, \eta)$ on the original probability space, and satisfies the weak formulation almost surely on the original probability space. Namely, we would like to prove that the limit is a **weak solution to the original problem in the probabilistically *strong* sense**. If we could obtain that the sequence $(\boldsymbol{u}_N, v_N, \eta_N)$ converges **in probability** to a random variable on the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$, namely $(\boldsymbol{u}_N, v_N, \eta_N) \xrightarrow{p} (\boldsymbol{u}, v, \eta)$, then the almost sure convergence along a subsequence will follow immediately. To obtain convergence *in probability* of $(\boldsymbol{u}, v, \eta)_N$, we will invoke a standard Gyöngy-Krylov argument [28].

More precisely, to prove that $X_N = (\boldsymbol{u}, v, \eta)_N$ converge *in probability* to some random variable $X^* = (\boldsymbol{u}, v, \eta)$ on $(\Omega, \mathcal{F}, \mathbb{P})$, $X_N \xrightarrow{p} X^*$, based on the Gyöngy-Krylov lemma [28], we need to show that for every two subsequences X_l and X_m, there exists a subsequence $x_k = (X_{l_k}, X_{m_k})$ such that the following two properties hold:

1. The joint laws $\nu_{X_{l_k}, X_{m_k}}$ of the subsequence x_k converge to some probability measure ν as $k \to \infty$;
2. The limiting law is supported on the diagonal: $\nu(\{(X, Y) : X = Y\}) = 1$.

The first property will follow from the tightness of measures μ_l and μ_m, which are the laws associated with the random variables $X_l = (\boldsymbol{u}_l, v_l, \eta_l)$ and $X_m = (\boldsymbol{u}_m, v_m, \eta_m)$. The tightness of the measures μ_l and μ_m implies tightness of the joint measures ν_{X_l, X_m} as well. To show that the second property holds, we will use the result of Part 1 above, combined with a *deterministic uniqueness* argument. Namely, Part 1 gives us the existence of the almost surely convergent subsequences $\tilde{X}_l = (\tilde{\boldsymbol{u}}_l, \tilde{v}_l, \tilde{\eta}_l)$ and $\tilde{X}_m = (\tilde{\boldsymbol{u}}_m, \tilde{v}_m, \tilde{\eta}_m)$ on the "tilde" probability space that have the same laws μ_l and μ_m as $X_l = (\boldsymbol{u}_l, v_l, \eta_l)$ and $X_m = (\boldsymbol{u}_m, v_m, \eta_m)$. Those two "tilde" subsequences of random variables converge to the limits $\tilde{X}^1$ and $\tilde{X}^2$, respectively, each of which has the law μ, and a joint law of $(\tilde{X}^1, \tilde{X}^2)$ equal to ν from Property 1 above. Recall, from Step 1, that both $\tilde{X}^1$ and $\tilde{X}^2$ are weak solutions in the probabilistically weak sense. To show that this joint law ν is supported on the diagonal, namely, to show Property 2 above, it is sufficient to show that $\tilde{X}^1$ is equal to $\tilde{X}^2$ almost surely, namely it will be sufficient to show that $\tilde{\mathbb{P}}(\tilde{X}^1 = \tilde{X}^2) = 1$. Indeed, proving the diagonal condition from the Gyöngy-Krylov lemma is associated with proving pathwise uniqueness of weak solutions, which we present in Section 9.1.

Once the properties from the Gyöngy-Krylov lemma have been verified, we can conclude that there exists a subsequence of $(\boldsymbol{u}_N, v_N, \eta_N)$, which we continue to denote by N, such that $(\boldsymbol{u}_N, v_N, \eta_N) \xrightarrow{p} (\boldsymbol{u}, v, \eta)$, which implies almost sure convergence along a subsequence on the original probability space. This is presented in Section 9.2.

Finally, the proof that the limiting function $(\boldsymbol{u}, v, \eta)$ recovered above is a weak solution in the probabilistically strong sense is presented in Section 9.3.

To the best of our knowledge, this is the first well-posedness result in the context of stochastic fluid-structure interaction. The result shows that our deterministic benchmark FSI model is robust to stochastic noise, even in the presence of rough white noise in time. This proof combines stochastic PDE analysis tools with deterministic FSI approaches. Additionally, the constructive proof lays out a framework for the development of a numerical scheme for this class of SFSI problems, and provides a framework for the analysis of a broader class of more complex stochastic FSI problems, involving nonlinear dependence on the solution and involving geometric nonlinearities due to the consideration of fluid equations posed on (random) moving fluid domains.

In the next section, we provide a brief review of the related literature.

2 Literature review

The mathematical analysis of deterministic fluid-structure interaction began around twenty years ago by focusing on rigorous well-posedness for linearly coupled fluid-structure interaction models. Linearly coupled FSI models are models where the fluid and structure coupling conditions are evaluated along a fixed fluid-structure interface, and the fluid equations are posed on a fixed fluid domain, even though the structure is assumed to be elastic and displaces from its reference configuration. The results concerning these linearly coupled models typically deal with establishing existence/uniqueness of weak or strong solutions. The existence and uniqueness of a weak solution to a linearly coupled model involving an interaction between the linear Stokes equations and the equations of linear elasticity was established in [19] using a Galerkin method. The Navier-Stokes equations for an incompressible, viscous fluid linearly coupled to immersed elastic solids were considered in [2, 3, 37]. In particular, the work in [2] deals with showing the existence of energy-level weak solutions, by a careful examination of the trace regularity of the hyperbolic structure dynamics in terms of the normal stress at the fluid-structure interface. The results in [3, 37] deal with establishing sufficient regularity of initial data that provides existence of strong solutions of the corresponding linearly coupled systems.

The well-posedness analysis of deterministic FSI models was extended later to nonlinearly coupled models, where the fluid domain changes in time according to the structure displacement, and hence the problem is a moving boundary problem where the fluid domain is not known a priori. There is by now an extensive mathematical literature dealing with the well-posedness of such models, see e.g., [4, 8–10, 13, 14, 24–26, 30, 31, 36, 39, 40, 43–47, 53] and the references therein. Of these references, we note that the approach outlined in [27, 43–47] is closely related to the approach used in the current manuscript. In particular, the approach is based on using a splitting scheme, known as the Lie operator splitting scheme, that discretizes the nonlinearly coupled problem in time by a time step Δt, and separates the coupled problem into fluid and structure subproblems. Then, compactness arguments of Aubin-Lions type (see [1, 42, 48]) are used to pass to the limit as $\Delta t \to 0$ in the approximate weak formulations satisfied by the approximate solutions, in order to obtain a *constructive existence proof* for weak solutions to nonlinearly coupled fluid-structure interaction problems. This approach proved to be quite robust for *deterministic* fluid-structure interaction problems, since it provided existence of weak solutions for several different scenarios involving thin, thick, and multi-layered structures coupled to the flow of an incompressible, viscous fluid via the no-slip or Navier slip boundary conditions, see [43], [44], [45], [46], [47].

In the present work, a version of this approach is extended to deal with *stochastic* fluid-structure interaction problems, by combining stochastic calculus with stochastic operator splitting approaches introduced in [5] and analyzed in [29]. More precisely, we design a time-discretized, operator splitting method in just the right way so that all the stochastic integrals are well-defined, and the resulting time-discretized scheme is stable, allowing us to show, using stochastic calculus, an almost sure convergence of approximate solutions to a weak solution in the probabilistically strong sense of the coupled fluid-structure interaction problem. To the best of our knowledge, this is the first well-posedness result on fully coupled *stochastic fluid-structure interaction.* Our result builds on recent developments in the area of *stochastic partial differential equations* (SPDEs).

Stochastic partial differential equations are PDEs that feature some sort of random noise forcing, such as white noise forcing in either time, or both time and space, or spatially homogeneous Gaussian noise that is independent at every time but potentially correlated in space. They are motivated by the fact that many real-life systems modeled by PDEs exhibit some type of random noise, which can significantly impact the resulting dynamics of the system. The current manuscript considers a stochastic linearly coupled fluid-structure interaction model involving the interaction between a fluid modeled by the linear Stokes equations and an elastic membrane modeled by the wave equation. Although the coupled stochastic FSI model has not been previously considered in the stochastic PDE literature, there are many works that study either stochastic fluid dynamics or stochastic wave equations separately, as we summarize below.

In terms of stochastic fluid equations, the consideration of stochastic Navier-Stokes equations is an active area of research, see e.g., [6, 7, 22, 38]. The study of stochastic Navier-Stokes equations was initiated in the work of [6], which considered an abstract stochastic equation of Navier-Stokes type, with an additive random noise forcing in time, and a random initial condition. It was shown that there exists a solution that satisfies the problem almost surely in a distributional sense. The approach in the work [6] is semi-deterministic in the sense that the solution map Γ for the deterministic problem, which is potentially multi-valued since the uniqueness of a solution to the deterministic problem is not known, is studied, and then the solution to the equation with stochastic forcing is given by a composition of a mapping from the probability space to the random quantities of the problem with a measurable section of the solution map Γ. In contrast, the works of [7, 22] feature genuinely probabilistic methods of studying the stochastic equations of Navier-Stokes type, and in these works, this abstract equation of Navier-Stokes type is extended to more general settings where there is nonlinear dependence of the intensity of the random noise forcing on the actual solution itself. These two works [7, 22] consider different abstract conditions on this nonlinear dependence and prove existence of martingale, or probabilistically weak, solutions to the resulting stochastic equations. Both of these works use a Galerkin scheme to construct solutions and obtain existence by establishing uniform bounds on the sequence of random functions satisfying the finite-dimensional Galerkin problems. We note that passing to the limit in the Galerkin solutions in [7, 22] was done by using standard probabilistic methods, such as establishing tightness of laws, showing weak convergence in law, and invoking the Skorohod representation theorem, which are standard techniques that we will employ for our current problem as well. While there are many works on stochastic fluid dynamics, we mention in particular a recent work [41], which establishes the existence of local martingale solutions, which are martingale solutions up to some stopping time, for a system of one layer shallow water equations for fluid velocity and water depth in two spatial dimensions, driven by random noise forcing described by cylindrical Wiener processes. We remark that [41] employs similar probabilistic methods in passing to the limit in a sequence of random approximate solutions (obtained by a Galerkin method) that motivated many of the probabilistic arguments in this manuscript, though the methods used in this current manuscript for constructing approximate solutions are different, as they are based on time discretization using an operator splitting approach, and not spatial discretization using a Galerkin method. One reason for the use of time-discretization via operator splitting, versus a Galerkin approach, is a possible extension to the moving boundary case. In the Galerkin case, the basis functions for the moving boundary case will depend on the *random* solution itself, which is difficult to deal with.

In terms of stochastic wave equations, there is extensive work on well-posedness and properties of solutions. It is classically well-known that the stochastic wave equation with spacetime white noise has a mild solution only in dimension one, but not in dimensions two and higher (see for example [16]), where spacetime white noise is a type of random noise defined for $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$ with a random intensity that is independent at every point in space and time. Spacetime white noise, due to its independence properties in *both* space and time, is very rough, and hence defining a mild solution via integration against spacetime white noise requires sufficient regularity of the integrand. The stochastic wave equation with spacetime white noise does not have a mild solution in spatial dimensions two and higher because the fundamental solution of the linear wave equation for spatial dimensions two and higher does not possess enough regularity to be integrated against spacetime white noise, as it is not square integrable in spacetime in dimension two, and in higher dimensions, it is not even function-valued. Hence, work on the stochastic wave equation in dimensions two and higher, focuses on considering stochastic wave equations with a more general type of noise, such as spatially homogeneous Gaussian noise (see for example [51]) which is independent in time but is correlated in space. In particular, the authors of [15, 17, 32] consider conditions for this spatially homogeneous Gaussian noise, such that the resulting stochastic wave equation has a solution that is function-valued (rather than just a distribution) in dimensions two and higher. Existence results for such stochastic wave equations in higher dimensions are also considered in [12], and the Hölder continuity and regularity properties of stochastic wave equations in higher dimensions are considered in [12,18]. We remark that for our current benchmark stochastic FSI model, in the case of a two-dimensional fluid domain, our problem similarly requires the use of a random noise forcing that is independent at different times, but is correlated (and is in fact constant) in space, as the analysis of such a stochastic FSI system with spacetime white noise forcing does not yield a bounded energy estimate.

We conclude this literature review by mentioning a recent work [34] by the current authors, where a *stochastic viscous wave equation* was derived as a model for a stochastic linearly coupled fluid-structure interaction problem in a geometry that allowed the entire fluid-structure system to be modeled by a single stochastic viscous wave equation, describing the random displacement of the structure from its reference configuration. This model describes the interaction between a two-dimensional infinite plate, modeled by the 2D wave equation, and a 3D fluid in the lower half space, modeled by the stationary Stokes equations, under the additional influence of spacetime white noise (random noise that is formally independent at every point in space and time). The work in [34] considers well-posedness for the stochastic viscous wave equation and establishes existence and uniqueness of a mild solution in spatial dimensions one and two, in addition to improved Hölder regularity properties. This result is interesting because the classical heat and wave equations driven by spacetime white noise in dimension two, do not possess a mild solution. The main reason why the stochastic viscous wave equation studied in [34] admits a Hölder continuous mild solution in dimension two (which is the physical dimension) is the "right" scaling and the regularity properties of the fractional derivative operator (Dirichlet-to-Neumann operator), which models the effects of viscous fluid regularization on the elastodynamics of a stochastically perturbed 2D membrane.

While the results in [34] provide an insight into the behavior of solutions to stochastic FSI, they are restricted by the fact that the stochastic viscous wave equation is not a fully coupled model, it is defined in a special geometry on the entire $\mathbb{R}^2$, and it does not include the fluid inertia effects. This allowed the use of mathematical techniques that are not available in the fully coupled case of stochastic FSI. The goal of the current manuscript is to develop techniques for studying *fully coupled* stochastic fluid-structure interaction systems, defined on physically relevant geometries, including fluid inertia effects described by the time-dependent Stokes equations.

3 Description of the model

The model problem considered here is defined on a fixed fluid domain, which is a rectangle $\Omega_f = [0, L] \times [0, R]$. The boundary $\partial\Omega_f$ of the fluid domain consists of four parts: the moving boundary part denoted by Γ (it is the reference configuration of the moving boundary), the bottom of the "channel" denoted by Γ_b, and the inlet and outlet parts of the boundary Γ_{in} and Γ_{out} where the pressure data is prescribed. The flow in the fluid domain Ω_f is driven by the inlet and outlet pressure data, and by the motion of the moving boundary. See Fig. 1. We will use $\boldsymbol{x} = (z, r)$ to denote the coordinates of points in the fluid domain.

The **fluid flow** in Ω_f will be modeled by the time-dependent Stokes equations for an incompressible, viscous fluid:

$$\left.\begin{array}{rcl} \partial_t \boldsymbol{u} & = & \nabla \cdot \boldsymbol{\sigma}, \\ \nabla \cdot \boldsymbol{u} & = & 0, \end{array}\right\} \quad \text{in } \Omega_f, \tag{1}$$

where $\boldsymbol{u}(t, \boldsymbol{x}) = (u_z(t, \boldsymbol{x}), u_r(t, \boldsymbol{x}))$ is the fluid velocity, $\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\mu\boldsymbol{D}(\boldsymbol{u})$ is the Cauchy stress tensor describing a Newtonian fluid, and p is the fluid pressure. This gives rise to the following system:

$$\left.\begin{array}{rcl} \partial_t \boldsymbol{u} - \mu\Delta\boldsymbol{u} + \nabla p & = & 0, \\ \nabla \cdot \boldsymbol{u} & = & 0, \end{array}\right\} \quad \text{in } \Omega_f. \tag{2}$$

At the top boundary Γ of the fluid domain, an elastic membrane interacts with the fluid flow. We assume that this elastic structure experiences displacement only in the vertical direction from its reference configuration Γ, and we denote the magnitude of this displacement by $\eta(t, z)$. The **elastodynamics of the structure** will be modeled by the wave equation:

$$\eta_{tt} - \Delta\eta = f, \qquad \text{on } \Gamma, \tag{3}$$

where f is an external forcing term.

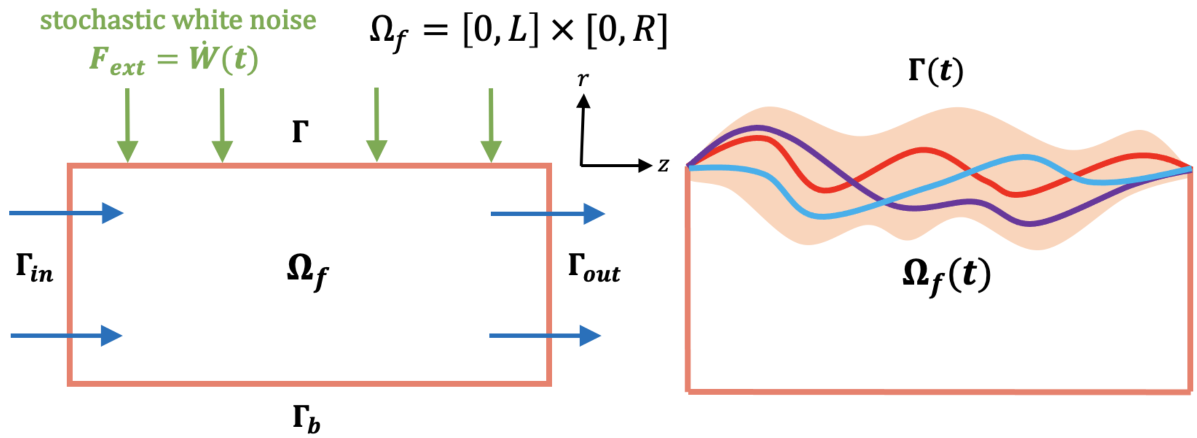

Figure 1: *Left: A sketch of the linearly coupled stochastic FSI problem, with Ω_f denoting the reference fluid domain, Γ denoting the reference configuration of the structure, and $\dot{W}(t)$ denoting stochastic white noise forcing on the structure. Right: The different colors represent different possible outcomes for the random configuration $\Gamma(t)$ of the structure at some time t. The lightly shaded region represents a confidence interval of where the structure is likely to be.*

The fluid and structure are coupled via two sets of coupling conditions, the kinematic and dynamic coupling conditions, which are evaluated along the *fixed interface*. This is known as **linear coupling**. The **kinematic coupling condition** considered in this work describes the continuity of velocities at the fluid-structure interface

$$\boldsymbol{u} = \eta_t \boldsymbol{e}_r, \qquad \text{on } \Gamma, \tag{4}$$

also known as the *no-slip* condition. The **dynamic coupling condition** describes balance of forces at the interface. Namely, it states that the elastodynamics of the thin elastic structure is driven by the jump in the force acting on the structure, coming from the normal component of the normal fluid stress $\boldsymbol{\sigma e_r} \cdot \boldsymbol{e_r}$ on one side, and the external forcing F_{ext} on the other:

$$\eta_{tt} - \Delta\eta = -\boldsymbol{\sigma e_r} \cdot \boldsymbol{e_r} + F_{ext}, \qquad \text{on } \Gamma,$$

where $\boldsymbol{e_r}$ is the unit outer normal to the fixed fluid-structure interface Γ.

In this manuscript, we consider the external force F_{ext} to be a stochastic force. In particular, as a start, we consider

$$F_{ext} = \dot{W}(t),$$

where W is a one-dimensional Brownian motion in time. Note that the stochastic force is constant on the whole structure at each time. As a result, the stochastic noise is rough temporally but is constant spatially. We remark that although this is a simplified model, we use it to demonstrate the difficulties present in the stochastic case in the simplest possible setting.

More precisely, we let W denote a one-dimensional Brownian motion with respect to an underlying probability space with filtration, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, in which case dW is formally the derivative of this Brownian motion. This is a purely formal notation that we will give precise meaning to later, as Brownian motion is almost surely nowhere differentiable.

In addition, we will assume that the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is a **complete filtration**, which means that $\mathcal{F}_t$ contains all null sets of $(\Omega, \mathcal{F}, \mathbb{P})$ for every $t \geq 0$, where a **null set** is defined to be any measurable set in $\mathcal{F}$ that has probability zero. This technical assumption will be useful to pass to the limit in our analysis of the stochastic problem above, as it allows us to bypass technicalities regarding null sets when considering almost sure limits of stochastic processes. *In particular, the almost sure limit of $\mathcal{F}_t$ measurable random variables for any arbitrary $t \geq 0$ is still $\mathcal{F}_t$ measurable under the assumption of a complete filtration.* This is not a restrictive assumption, as one can complete a filtration by simply adding all null sets to $\mathcal{F}_t$ for all $t \geq 0$, and

W will still be a Brownian motion with respect to the completed filtration. See Section 1.4 in Revuz and Yor [55] for more information about complete filtrations.

In summary, the coupled stochastic fluid-structure interaction problem studied in this manuscript, supplemented with initial and boundary data, is given by the following: *Find* $(\boldsymbol{u}, \eta)$ *such that*

$$\left.\begin{array}{rcl} \eta_{tt} - \Delta\eta & = & -\boldsymbol{\sigma}\boldsymbol{e_r}\cdot\boldsymbol{e_r} + dW(t), \\ \boldsymbol{u} & = & \eta_t\boldsymbol{e_r}, \end{array}\right\} \quad \text{on } \Gamma, \qquad \left.\begin{array}{rcl} \partial_t\boldsymbol{u} & = & \nabla\cdot\boldsymbol{\sigma}, \\ \nabla\cdot\boldsymbol{u} & = & 0, \end{array}\right\} \quad \text{in } \Omega_f, \tag{5}$$

with boundary data:

$$\left.\begin{array}{rcl} u_r & = & 0, \\ p & = & P_{in/out}(t), \end{array}\right\} \quad \text{on } \Gamma_{in/out}, \qquad u_r = \partial_r u_z = 0, \qquad \text{on } \Gamma_b, \tag{6}$$

and the following deterministic initial data:

$$\boldsymbol{u}(0,z,r) = \boldsymbol{u}_0(z,r), \qquad \eta(0,z,R) = \eta_0(z), \qquad \partial_t\eta(0,z,R) = v_0(z), \tag{7}$$

where $\boldsymbol{u}_0 \in L^2(\Omega_f)$*,* $\eta_0 \in H_0^1(\Gamma)$*, and* $v_0 \in L^2(\Gamma)$*, and* W *is a given one-dimensional Brownian motion with respect to the complete probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with complete filtration* $\{\mathcal{F}_t\}_{t\geq 0}$*.*

Thus, the problem is driven by deterministic inlet and outlet pressure data $P_{in/out}(t)$ prescribed on $\Gamma_{in/out}$, with the flow symmetry condition imposed at the bottom boundary Γ_b. Notice that throughout this manuscript, we will be using Ω to denote the underlying probability space, while Ω_f denotes the fluid domain.

4 Definition of a weak solution and main result

To define the space of weak solutions to the above problem, we first introduce the function space for the fluid velocity:

$$\mathcal{V}_F = \{\boldsymbol{u} = (u_z, u_r) \in H^1(\Omega_f)^2 : \nabla\cdot\boldsymbol{u} = 0,\ u_z = 0 \text{ on } \Gamma,\ u_r = 0 \text{ on } \partial\Omega_f\backslash\Gamma\}. \tag{8}$$

Since the structure subproblem is given by the wave equation with clamped ends, the natural space of functions for the structure is

$$\mathcal{V}_S = H_0^1(\Gamma). \tag{9}$$

Motivated by the energy inequality presented in Sec. 5, we introduce the following solution spaces in time for the fluid and structure subproblems:

$$\mathcal{W}_F(0,T) = L^2(\Omega; L^\infty(0,T; L^2(\Omega_f))) \cap L^2(\Omega; L^2(0,T; \mathcal{V}_F)). \tag{10}$$

$$\mathcal{W}_S(0,T) = L^2(\Omega; W^{1,\infty}(0,T; L^2(\Gamma))) \cap L^2(\Omega; L^\infty(0,T; \mathcal{V}_S)). \tag{11}$$

We emphasize that $\boldsymbol{u}$ and η are random variables, and that the $L^2(\Omega)$ part of the solution spaces reflects the fact that the energy estimate will hold in expectation.

Finally, we introduce the solution space for the *stochastic coupled FSI problem*:

$$\mathcal{W}(0,T) = \{(\boldsymbol{u}, \eta) \in \mathcal{W}_F(0,T) \times \mathcal{W}_S(0,T) : \boldsymbol{u}|_\Gamma = \eta_t\boldsymbol{e_r} \text{ for almost every } t \in [0,T], \text{ a.s.}\}. \tag{12}$$

Notice that in this solution space, the kinematic coupling condition is enforced strongly.

As in the deterministic case, we define weak solutions by integrating in space and time against an appropriate space of test functions, which we define to be:

$$\mathcal{Q}(0,T) = \{(\boldsymbol{q}, \psi) \in C_c^1([0,T); \mathcal{V}_F \times \mathcal{V}_S) : \boldsymbol{q}(t,z,R) = \psi(t,z)\boldsymbol{e_r}\}. \tag{13}$$

These test functions are deterministic functions. Because the fluid domain does not change in time with the assumption of linear coupling, we can define

$$\mathcal{Q} = \{(\boldsymbol{q}, \psi) \in \mathcal{V}_F \times \mathcal{V}_S : \boldsymbol{q}|_\Gamma = \psi\boldsymbol{e}_r\}, \tag{14}$$

and hence view the test functions as differentiable, compactly supported functions on $[0, T)$ that take values in the fixed function space $\mathcal{Q}$.

To motivate the definition of a weak solution, we will proceed as in [43]. For the purposes of the derivation of the weak solution, we consider, for the moment, the case of a general deterministic external force $F_{ext}(t)$ in place of $\dot{W}(t)$, so that the first equation for the structure becomes

$$\eta_{tt} - \Delta\eta = -\boldsymbol{\sigma}\boldsymbol{e_r}\cdot\boldsymbol{e_r} + F_{ext}(t).$$

We will derive the standard deterministic partial differential equation definition of a weak solution, assuming that $F_{ext}(t)$ is a purely deterministic function in time, and then generalize this to the stochastic case.

We start by taking a test function $(\boldsymbol{q}, \psi) \in \mathcal{Q}(0, T)$, and multiplying the linear Stokes equation by $\boldsymbol{q}$ and integrating in space and time. We obtain

$$\int_0^T \int_{\Omega_f} \partial_t \boldsymbol{u}\cdot\boldsymbol{q} d\boldsymbol{x}dt = \int_0^T \int_{\Omega_f} (\nabla\cdot\boldsymbol{\sigma})\cdot\boldsymbol{q} d\boldsymbol{x}dt.$$

By integrating the first term by parts in time, we obtain:

$$\int_0^T \int_{\Omega_f} \partial_t \boldsymbol{u}\cdot\boldsymbol{q} d\boldsymbol{x}dt = \int_{\Omega_f} \boldsymbol{u}\cdot\boldsymbol{q} d\boldsymbol{x}\Big|_{t=0}^{t=T} - \int_0^T \int_{\Omega_f} \boldsymbol{u}\cdot\partial_t\boldsymbol{q} d\boldsymbol{x}dt = -\int_{\Omega_f} \boldsymbol{u_0}\cdot\boldsymbol{q}(0)d\boldsymbol{x} - \int_0^T \int_{\Omega_f} \boldsymbol{u}\cdot\partial_t\boldsymbol{q} d\boldsymbol{x}dt.$$

By integrating the second term by parts in space and using the divergence free condition on $\boldsymbol{q}$, we obtain:

$$\int_{\Omega_f} (\nabla\cdot\boldsymbol{\sigma})\cdot\boldsymbol{q} d\boldsymbol{x} = \int_{\partial\Omega_f} (\boldsymbol{\sigma}\boldsymbol{n})\cdot\boldsymbol{q} dS - 2\mu\int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{q})d\boldsymbol{x},$$

where $\boldsymbol{D}(\boldsymbol{u})$ and $\boldsymbol{D}(\boldsymbol{q})$ represent the symmetrized gradient. Using the definition of the Cauchy stress tensor, $\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\mu\boldsymbol{D}(\boldsymbol{u})$, and integrating in time, we obtain

$$\begin{aligned}\int_0^T \int_{\Omega_f} (\nabla\cdot\boldsymbol{\sigma})\cdot\boldsymbol{q} d\boldsymbol{x}dt &= \int_0^T \int_{\Gamma_{in}} pq_z drdt - \int_0^T \int_{\Gamma_{out}} pq_z drdt - 2\mu\int_0^T \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{q})d\boldsymbol{x}dt \\ - \int_0^T \int_\Gamma \nabla\eta\cdot\nabla\psi dzdt &+ \int_0^T \int_\Gamma \partial_t\eta\partial_t\psi dzdt + \int_\Gamma v_0\psi(0)dz + \int_0^T \left(\int_\Gamma \psi dz\right) F_{ext}(t)dt.\end{aligned}$$

Putting this all together, we get that

$$\begin{aligned}&-\int_0^T \int_{\Omega_f} \boldsymbol{u}\cdot\partial_t\boldsymbol{q} d\boldsymbol{x}dt + 2\mu\int_0^T \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{q})d\boldsymbol{x}dt - \int_0^T \int_\Gamma \partial_t\eta\partial_t\psi dzdt + \int_0^T \int_\Gamma \nabla\eta\cdot\nabla\psi dzdt \\ &= \int_0^T P_{in}(t)\left(\int_{\Gamma_{in}} q_z dr\right)dt - \int_0^T P_{out}(t)\left(\int_{\Gamma_{out}} q_z dr\right)dt + \int_{\Omega_f} \boldsymbol{u_0}\cdot\boldsymbol{q}(0)d\boldsymbol{x} + \int_\Gamma v_0\psi(0)dz + \int_0^T \left(\int_\Gamma \psi dz\right) F_{ext}(t)dt,\end{aligned}$$

where we used the fact that $P_{in/out}(t) = p$ on $\Gamma_{in/out}$.

Now, we formally substitute $F_{ext}(t) = \dot{W}(t)$, into the definition of the deterministic weak solution, to get that the term containing $F_{ext}(t)$ can be interpreted in the stochastic case as:

$$\int_0^T \left(\int_\Gamma \psi dz\right) dW(t).$$

Since W is a one dimensional Brownian motion and since $\int_\Gamma \psi dz$ is a deterministic function in time, we can interpret this term as a stochastic integral.

Before we give the definition of a weak solution to the stochastic FSI problem above, we recall the definition of a **stochastic basis**. A **stochastic basis** $\mathcal{S}$ is an ordered quintuple (see [41] for the notation)

$$\mathcal{S} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W),$$

where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $\{\mathcal{F}_t\}_{t\geq 0}$ is a complete filtration with respect to this probability space, and W is a one-dimensional Brownian motion on the probability space with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, meaning that: (1) W has continuous paths, almost surely, (2) W is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, and (3) $W(t) - W(s)$ is independent of $\mathcal{F}_s$ for all $t \geq s$ and $W(t) - W(s) \sim N(0, t-s)$ for all $0 \leq s \leq t$, where N denotes the normal distribution.

We will define two notions of solution: (1) a weak solution in a probabilistically weak sense, and (2) a weak solution in a probabilistically strong sense. The second one is stronger than the first, but we will need the first to be able to prove the existence of a weak solution in a probabilistically strong sense.

Definition 4.1. An ordered triple $(\tilde{\mathcal{S}}, \tilde{\boldsymbol{u}}, \tilde{\eta})$ is a *weak solution in a probabilistically* ***weak*** *sense* if there exists a stochastic basis

$$\tilde{\mathcal{S}} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t\geq 0}, \tilde{\mathbb{P}}, \tilde{W})$$

and $(\tilde{\boldsymbol{u}}, \tilde{\eta}) \in \mathcal{W}(0,T)$ with paths almost surely in $C(0,T;\mathcal{Q}')$, which satisfies:

- $(\tilde{\boldsymbol{u}}, \tilde{\eta})$ is adapted to the filtration $\{\tilde{\mathcal{F}}_t\}_{t\geq 0}$,
- $\tilde{\eta}(0) = \eta_0$ almost surely, and
- for all $(\boldsymbol{q}, \psi) \in \mathcal{Q}(0,T)$,

$$-\int_0^T \int_{\Omega_f} \tilde{\boldsymbol{u}} \cdot \partial_t \boldsymbol{q} d\boldsymbol{x} dt + 2\mu \int_0^T \int_{\Omega_f} \boldsymbol{D}(\tilde{\boldsymbol{u}}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} dt - \int_0^T \int_\Gamma \partial_t \tilde{\eta} \partial_t \psi dz dt + \int_0^T \int_\Gamma \nabla \tilde{\eta} \cdot \nabla \psi dz dt$$
$$= \int_0^T P_{in}(t) \left(\int_{\Gamma_{in}} q_z dr \right) dt - \int_0^T P_{out}(t) \left(\int_{\Gamma_{out}} q_z dr \right) dt + \int_{\Omega_f} \boldsymbol{u_0} \cdot \boldsymbol{q}(0) d\boldsymbol{x} + \int_\Gamma v_0 \psi(0) dz + \int_0^T \left(\int_\Gamma \psi dz \right) d\tilde{W},$$

  almost surely.

Definition 4.2. Let $\mathcal{S} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W)$ be a stochastic basis with a complete filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and a one-dimensional Brownian motion $\{W_t\}_{t\geq 0}$ on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. An ordered pair $(\boldsymbol{u}, \eta)$ is a *weak solution in a probabilistically* ***strong*** *sense* if $(\boldsymbol{u}, \eta) \in \mathcal{W}(0,T)$ with paths almost surely in $C(0,T;\mathcal{Q}')$, satisfies:

- $(\boldsymbol{u}, \eta)$ is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$
- $\eta(0) = \eta_0$ almost surely, and
- for all $(\boldsymbol{q}, \psi) \in \mathcal{Q}(0,T)$,

$$-\int_0^T \int_{\Omega_f} \boldsymbol{u} \cdot \partial_t \boldsymbol{q} d\boldsymbol{x} dt + 2\mu \int_0^T \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} dt - \int_0^T \int_\Gamma \partial_t \eta \partial_t \psi dz dt + \int_0^T \int_\Gamma \nabla \eta \cdot \nabla \psi dz dt$$
$$= \int_0^T P_{in}(t) \left(\int_{\Gamma_{in}} q_z dr \right) dt - \int_0^T P_{out}(t) \left(\int_{\Gamma_{out}} q_z dr \right) dt + \int_{\Omega_f} \boldsymbol{u_0} \cdot \boldsymbol{q}(0) d\boldsymbol{x} + \int_\Gamma v_0 \psi(0) dz + \int_0^T \left(\int_\Gamma \psi dz \right) dW.$$

  almost surely.

In a probabilistically strong solution as in the second definition above, we have a random solution satisfying the initial conditions on the originally given (arbitrary) probability space with a one dimensional Brownian motion with respect to a complete filtration. In a probabilistically weak solution, we have a weaker requirement that the random solution exists on a *particular (not arbitrary) probability space*, where the initial conditions are satisfied "in law". We will show the existence of a weak solution in the probabilistically strong sense. To get to that solution, we will introduce a general methodology for passing to the limit almost surely in the (random) approximate solutions, that will generalize to more complex stochastic FSI systems. This will involve first showing existence of a convergent subsequence of probability measures corresponding to the laws of the approximate solutions. Then we will construct a weak solution in the probabilistically weak sense using the Skorohod representation theorem, and we will conclude by using the Gyöngy-Krylov argument [28] to get to a weak solution in the probabilistically strong sense.

The main result of this work is stated in the following theorem.

Theorem 4.1 (Main Result). Let $\boldsymbol{u}_0 \in L^2(\Omega_f)$, $v_0 \in L^2(\Gamma)$, and $\eta_0 \in H_0^1(\Gamma)$. Let $P_{in/out} \in L^2_{loc}(0, \infty)$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a Brownian motion W with respect to a given complete filtration $\{\mathcal{F}_t\}_{t \geq 0}$. Then, for any $T > 0$, there exists a unique weak solution in a probabilistically strong sense to the given stochastic fluid-structure interaction problem (5)–(7).

5 A priori energy estimate

We derive a formal energy estimate by assuming that the solution is pathwise regular enough to justify the integration by parts. We use $\|\cdot\|_{L^2(\Gamma)}$ and $(\cdot, \cdot)$ to denote the norm and inner product on $L^2(\Gamma)$, and $\|\cdot\|_{L^2(\Omega_f)}$ and $\langle \cdot, \cdot \rangle$ to denote the norm and inner product on $L^2(\Omega_f)$.

We define the total energy at time T by

$$E(T) := \frac{1}{2}\int_\Gamma |\nabla \eta|^2 dz + \frac{1}{2}\int_\Gamma |v|^2 dz + \frac{1}{2}\int_{\Omega_f} |\boldsymbol{u}|^2 d\boldsymbol{x} = \frac{1}{2}\left(\|\nabla \eta\|^2_{L^2(\Gamma)} + \|v\|^2_{L^2(\Gamma)} + \|\boldsymbol{u}\|^2_{L^2(\Omega_f)}\right),$$

and the total dissipation by time T by

$$D(T) = \int_0^T \int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u})|^2 d\boldsymbol{x}.$$

To estimate the total energy and dissipation for the stochastic processes $\boldsymbol{u}$, v and η, we rewrite the stochastic fluid-structure interaction problem in the following stochastic differential formulation:

$$\begin{aligned} d\eta &= v dt, \\ dv &= (\Delta \eta - \boldsymbol{\sigma}\boldsymbol{e_r} \cdot \boldsymbol{e_r})dt + dW, \\ d\boldsymbol{u} &= (\nabla \cdot \boldsymbol{\sigma})dt. \end{aligned}$$

Notice that the first equation implies $d(\nabla \eta) = (\nabla v)dt$. To obtain an energy estimate, we first apply Itö's formula to express the differentials of the L^2-norms of the stochastic processes that define the total energy of the problem:

$$d(\|\nabla \eta\|^2_{L^2(\Gamma)}) = 2(\nabla \eta, \nabla v)dt,$$

$$d(\|v\|^2_{L^2(\Gamma)}) = [2(\Delta \eta, v) - 2(\boldsymbol{\sigma}\boldsymbol{e_r} \cdot \boldsymbol{e_r}, v) + L]dt + 2(1, v)dW,$$

$$d(\|\boldsymbol{u}\|^2_{L^2(\Omega_f)}) = 2\langle \nabla \cdot \boldsymbol{\sigma}, \boldsymbol{u} \rangle dt.$$

In the preceding calculation, we emphasize an important distinction from the corresponding energy estimates in the deterministic case. In particular, because of the influence of the random noise forcing, there is an extra Ldt term in the differential of $||v||^2_{L^2(\Gamma)}$, which is a result of an extra "correction term" in Itô's formula, which intuitively can be regarded as a stochastic analogue of the chain rule, see Theorem 4.1.2 in [50], pg. 70-71 of [21], and Theorem 3.3 in Chapter IV of [55] for example. This additional term in Itô's formula arises from the fact that the quadratic variation of Brownian motion on a given closed interval is given by the change in the time parameter, see the discussion in Theorem 2.4 in Chapter I of [55], which can be easily seen in the current discussion of squaring the velocity v to obtain the kinetic energy of the membrane, in the computation above. By adding these equations together, we obtain that the differential of the total energy satisfies:

$$d(\|\nabla \eta\|^2_{L^2(\Gamma)} + \|v\|^2_{L^2(\Gamma)} + \|\boldsymbol{u}\|^2_{L^2(\Omega_f)}) = [2\langle \nabla \cdot \boldsymbol{\sigma}, \boldsymbol{u} \rangle - 2(\boldsymbol{\sigma}\boldsymbol{e_r} \cdot \boldsymbol{e_r}, v) + L]dt + 2(1, v)dW,$$

where we have used that $(\Delta \eta, v) = -(\nabla \eta, \nabla v)$ under the assumption that η and v are smooth and vanish at the endpoints of Γ. Recalling the kinematic coupling condition $\boldsymbol{u}|_\Gamma = v$, we obtain that

$$\int_{\Omega_f} (\nabla \cdot \boldsymbol{\sigma}) \cdot \boldsymbol{u} d\boldsymbol{x} = \int_{\Gamma_{in}} p u_z dr - \int_{\Gamma_{out}} p u_z dr + \int_\Gamma (\boldsymbol{\sigma}\boldsymbol{e_r} \cdot \boldsymbol{e_r}) v dz - 2\mu \int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u})|^2 d\boldsymbol{x},$$

which implies

$$d\left(\frac{1}{2}\int_\Gamma |\nabla\eta|^2 dz + \frac{1}{2}\int_\Gamma |v|^2 dz + \frac{1}{2}\int_{\Omega_f} |\boldsymbol{u}|^2 d\boldsymbol{x}\right)$$

$$= \left(\frac{L}{2} - 2\mu\int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u})|^2 d\boldsymbol{x} + \int_{\Gamma_{in}} p u_z dr - \int_{\Gamma_{out}} p u_z dr\right) dt + \left(\int_\Gamma v dz\right) dW.$$

Therefore, after integration, for all $T \geq 0$, we have

$$E(T) + 2\mu\int_0^T\int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u})|^2 d\boldsymbol{x}dt = E_0 + \frac{LT}{2} + \int_0^T\int_{\Gamma_{in}} P_{in}(t) u_z dr dt - \int_0^T\int_{\Gamma_{out}} P_{out}(t) u_z dr dt + \int_0^T\left(\int_\Gamma v dz\right) dW. \tag{15}$$

We estimate the terms on the right hand side of (15) as follows. For the pressure term we use Hölder's inequality, the trace inequality, Poincaré's inequality, and Korn's inequality [33] to get

$$\left|\int_0^T\left(\int_{\Gamma_{in}} u_z dr\right) P_{in}(t) dt\right| \leq C\left|\int_0^T\left(\int_{\Gamma_{in}} |u_z|^2 dr\right)^{1/2} P_{in}(t) dt\right| \leq C\left|\int_0^T ||\nabla\boldsymbol{u}||_{L^2(\Omega_f)} P_{in}(t) dt\right|$$
$$\leq C\left|\int_0^T ||\boldsymbol{D}(\boldsymbol{u})||_{L^2(\Omega_f)} P_{in}(t) dt\right| \leq C(D(T))^{1/2}||P_{in}(t)||_{L^2(0,T)} \leq \epsilon D(T) + C(\epsilon)||P_{in}(t)||^2_{L^2(0,T)}. \tag{16}$$

We note that the constant $C(\epsilon)$ depends only on ϵ and the parameters of the problem. The same computation holds for the outlet pressure.

For the stochastic integral, we will bound the expectation $\mathbb{E}\left(\max_{0\leq\tau\leq T}\left|\int_0^\tau\left(\int_\Gamma \partial_t\eta dz\right) dW\right|\right)$ since the final energy estimate will be given in terms of expectation of the total energy and dissipation at time T. To bound this quantity, we use the Burkholder-Davis-Gundy (BDG) inequality under the assumption that the process $\partial_t\eta$ is a predictable stochastic process with respect to the given filtration $\{\mathcal{F}_t\}_{t\geq 0}$:

$$\mathbb{E}\left(\max_{0\leq s\leq T}\left|\int_0^s\left(\int_\Gamma \partial_t\eta dz\right) dW\right|\right) \leq \mathbb{E}\left(\left|\int_0^T\left(\int_\Gamma \partial_t\eta dz\right)^2 dt\right|^{1/2}\right) \leq C\mathbb{E}\left(\left|\int_0^T ||\partial_t\eta||^2_{L^2(\Gamma)} dt\right|^{1/2}\right)$$
$$\leq C\left(\mathbb{E}\left|\int_0^T ||\partial_t\eta||^2_{L^2(\Gamma)} dt\right|\right)^{1/2} \leq CT^{1/2}\cdot\left[\mathbb{E}\left(\max_{0\leq t\leq T} ||\partial_t\eta(t,\cdot)||^2_{L^2(\Gamma)}\right)\right]^{1/2}$$
$$\leq C(\epsilon)T + \epsilon\mathbb{E}\left(\max_{0\leq t\leq T} ||\partial_t\eta(t,\cdot)||^2_{L^2(\Gamma)}\right) \leq C(\epsilon)T + \epsilon\mathbb{E}\left(\max_{0\leq t\leq T} E(t)\right). \tag{17}$$

Now, we first use (16) in (15) to obtain

$$E(T) + 2\mu D(T) \leq E(0) + \frac{LT}{2} + 2\epsilon D(T) + C(\epsilon)\left(||P_{in}(t)||^2_{L^2(0,T)} + ||P_{out}(t)||^2_{L^2(0,T)}\right) + \int_0^T\left(\int_\Gamma v dz\right) dW,$$

and then choose $\epsilon < \frac{\mu}{2}$ and $\epsilon < \frac{1}{2}$ to get

$$E(T) + \mu D(T) \leq E(0) + \frac{LT}{2} + C(\epsilon)\left(||P_{in}(t)||^2_{L^2(0,T)} + ||P_{out}(t)||^2_{L^2(0,T)}\right) + \int_0^T\left(\int_\Gamma v dz\right) dW.$$

Taking a maximum over times $t \in [0, T]$, taking an expectation, and then using the estimate in (17), we obtain the following *a priori* energy estimate for the coupled problem (5)–(7):

$$\mathbb{E}\left(\max_{0\leq t\leq T} E(t) + \mu\int_0^t\int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u})|^2 d\boldsymbol{x}ds\right) \leq C\left(T + E(0) + ||P_{in}(t)||^2_{L^2(0,T)} + ||P_{out}(t)||^2_{L^2(0,T)}\right),$$

where C is independent of T, depending only on the parameters of the problem.

Remark 5.1. The right hand side of the energy estimate shows the four sources of energy input into the system: $E(0)$ represents the initial kinetic and potential energy, the two final terms represent the energy input from the inlet and outlet pressure, and CT represents the energy input from the stochastic forcing on the structure.

6 The splitting scheme

To prove the existence of a weak solution to the given stochastic FSI problem we adapt a Lie operator splitting scheme that was first designed in the context of nonlinear fluid-structure interaction by Muha and Čanić in [43]. See also [27]. In this section, we introduce a new splitting scheme that splits each of the three different contributions to the fully coupled dynamics of the problem (the stochastic noise, the structure, and the fluid) apart from each other. To do this, we use a stochastic splitting introduced in [5], which has been used in stochastic differential equations to split stochastic effects from all other deterministic effects. We design a three part splitting scheme that involves a structure subproblem, a stochastic subproblem, and a fluid subproblem, **which gives rise to a stable and convergent scheme**, as we show below. While it is possible to design a scheme that combines the stochastic and structure subproblems (see [57]), the advantage of using a three-way scheme is that all of the different multiphysical components of the problem are fully split from each other, which allows one to isolate the influence of the stochasticity, structure elastodynamics, and the fluid on the fully coupled dynamics numerically.

Given a fixed time $T > 0$, for each positive integer N, let $\Delta t = \frac{T}{N}$ denote the associated time step, and let $t_N^n = n\Delta t$ denote the discrete times for $n = 0, 1, ..., N-1, N$. At each time step, we update the following vector using a three step method described below:

$$\boldsymbol{X}_N^{n+\frac{i}{3}} = \left(\boldsymbol{u}_N^{n+\frac{i}{3}}, v_N^{n+\frac{i}{3}}, \eta_N^{n+\frac{i}{3}}\right)^T, \quad n = 0, 1, ..., N-1, \quad i = 1, 2, 3,$$

where $i = 1$ corresponds to the result after updating the structure subproblem, $i = 2$ corresponds to the stochastic subproblem, and $i = 3$ corresponds to the fluid subproblem, with the initial data $\boldsymbol{X}_N^0 = (\boldsymbol{u}_0, v_0, \eta_0)^T$ for each N.

6.1 The structure subproblem

In this subproblem, we keep the fluid velocity fixed, so that

$$\boldsymbol{u}_N^{n+\frac{1}{3}} = \boldsymbol{u}_N^n,$$

and update the structure displacement and the structure velocity by requiring that $(\eta_N^{n+\frac{1}{3}}, v_N^{n+\frac{1}{3}})$ satisfy the following first order system in weak variational form:

$$\int_\Gamma \frac{\eta_N^{n+\frac{1}{3}} - \eta_N^n}{\Delta t}\phi dz = \int_\Gamma v_N^{n+\frac{1}{3}}\phi dz, \qquad \text{for all } \phi \in L^2(\Gamma),$$

$$\int_\Gamma \frac{v_N^{n+\frac{1}{3}} - v_N^n}{\Delta t}\psi dz + \int_\Gamma \nabla\eta_N^{n+\frac{1}{3}} \cdot \nabla\psi dz = 0, \qquad \text{for all } \psi \in H_0^1(\Gamma), \tag{18}$$

where this system is solved pathwise for each $\omega \in \Omega$ separately. We note that $(\eta_N^{n+\frac{1}{3}}, v_N^{n+\frac{1}{3}})$ is a random variable taking values in $H_0^1(\Gamma) \times H_0^1(\Gamma)$. To verify this, we must check that it is a measurable function of the probability space.

Proposition 6.1. Suppose that η_N^n and v_N^n are $\mathcal{F}_{t_N^n}$ measurable random variables taking values in $H_0^1(\Gamma)$ and $L^2(\Gamma)$ respectively. Then, the structure problem (18) has a unique solution $(\eta_N^{n+\frac{1}{3}}, v_N^{n+\frac{1}{3}})$, which is a random variable taking values in $H_0^1(\Gamma) \times H_0^1(\Gamma)$ that is measurable with respect to $\mathcal{F}_{t_N^n}$.

Proof. Let $F_N^n : (\eta_N^n, v_N^n) \to (\eta_N^{n+\frac{1}{3}}, v_N^{n+\frac{1}{3}})$ be the deterministic linear map that sends deterministic data $(\eta_N^n, v_N^n) \in H_0^1(\Gamma) \times L^2(\Gamma)$ to the unique solution $(\eta_N^{n+\frac{1}{3}}, v_N^{n+\frac{1}{3}}) \in H_0^1(\Gamma) \times H_0^1(\Gamma)$ satisfying the weak formulation (18) as a deterministic problem. We claim that this deterministic linear map $F_N^n : (\eta_N^n, v_N^n) \to (\eta_N^{n+\frac{1}{3}}, v_N^{n+\frac{1}{3}})$ is a continuous (or equivalently, bounded) linear map from $H_0^1(\Gamma) \times L^2(\Gamma)$ to $H_0^1(\Gamma) \times H_0^1(\Gamma)$.

By plugging the first equation in (18) into the second equation, $\eta_N^{n+\frac{1}{3}}$ satisfies the weak formulation:

$$\int_\Gamma \eta_N^{n+\frac{1}{3}} \psi dz + (\Delta t)^2 \int_\Gamma \nabla \eta_N^{n+\frac{1}{3}} \cdot \nabla \psi dz = (\Delta t) \int_\Gamma v_N^n \psi dz + \int_\Gamma \eta_N^n \psi dz, \qquad \text{for all } \psi \in H_0^1(\Gamma). \tag{19}$$

The existence of a unique $\eta_N^{n+\frac{1}{3}} \in H_0^1(\Gamma)$ satisfying (19) is given by the Lax-Milgram lemma, and $v_N^{n+\frac{1}{3}} = \dfrac{\eta_N^{n+\frac{1}{3}} - \eta_N^n}{\Delta t} \in H_0^1(\Gamma)$. The linear map $F^n : (\eta_N^n, v_N^n) \to (\eta_N^{n+\frac{1}{3}}, v_N^{n+\frac{1}{3}})$ is a bounded linear map from $H_0^1(\Gamma) \times L^2(\Gamma)$ to $H_0^1(\Gamma) \times H_0^1(\Gamma)$, which can be seen by substituting $\psi = \eta_N^{n+\frac{1}{3}}$ in (19). Thus, $F_N^n : (\eta_N^n, v_N^n) \to (\eta_N^{n+\frac{1}{3}}, v_N^{n+\frac{1}{3}})$ is a bounded linear map from $H_0^1(\Gamma) \times L^2(\Gamma)$ to $H_0^1(\Gamma) \times H_0^1(\Gamma)$, and so the result of the structure subproblem, which consists of the random functions $(\eta_N^{n+\frac{1}{3}}, v_N^{n+\frac{1}{3}}) = F_N^n \circ (\eta_N^n, v_N^n)$, is a pair of $\mathcal{F}_{t_N^n}$ measurable random variables, taking values in $H_0^1(\Gamma) \times H_0^1(\Gamma)$. $\square$

To show that the approximate solutions defined by the subproblems converge to the weak solution of the continuous problem as $\Delta t \to 0$, we will need uniform bounds on the approximating sequences, which will follow from the uniform bounds on the discrete energy of the problem. For this purpose, we define the discrete energy at time t_n by

$$E_N^{n+\frac{i}{3}} = \frac{1}{2} \left(\int_{\Omega_f} |\boldsymbol{u}_N^{n+\frac{i}{3}}|^2 d\boldsymbol{x} + ||v_N^{n+\frac{i}{3}}||^2_{L^2(\Gamma)} + ||\nabla \eta_N^{n+\frac{i}{3}}||^2_{L^2(\Gamma)} \right), \tag{20}$$

and we define the fluid dissipation at time t_n by

$$D_N^n = (\Delta t)\mu \int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u}_N^n)|^2 d\boldsymbol{x}. \tag{21}$$

We emphasize that these are random variables.

Proposition 6.2. The following discrete energy equality is satisfied pathwise:

$$E_N^{n+\frac{1}{3}} + \frac{1}{2}\left(||v_N^{n+\frac{1}{3}} - v_N^n||^2_{L^2(\Gamma)} \right) + \frac{1}{2}\left(||\nabla \eta_N^{n+\frac{1}{3}} - \nabla \eta_N^n||^2_{L^2(\Gamma)} \right) = E_N^n.$$

Proof. Because $v_N^{n+\frac{1}{3}} \in H_0^1(\Gamma)$, we can substitute $\psi = v_N^{n+\frac{1}{3}}$ in the weak formulation to obtain that pathwise,

$$\int_\Gamma (v_N^{n+\frac{1}{3}} - v_N^n) \cdot v_N^{n+\frac{1}{3}} dz + (\Delta t) \int_\Gamma \nabla \eta_N^{n+\frac{1}{3}} \cdot \nabla v_N^{n+\frac{1}{3}} dz = 0.$$

By using the identity $(a-b)\cdot a = \frac{1}{2}(|a|^2 + |a-b|^2 - |b|^2)$, along with the fact that $v_N^{n+\frac{1}{3}} = \frac{\eta_N^{n+\frac{1}{3}} - \eta_N^n}{\Delta t}$, we obtain that the following identity holds pathwise:

$$\frac{1}{2}||v_N^{n+\frac{1}{3}}||^2_{L^2(\Gamma)} + \frac{1}{2}||\nabla \eta_N^{n+\frac{1}{3}}||^2_{L^2(\Gamma)} + \frac{1}{2}||v_N^{n+\frac{1}{3}} - v_N^n||^2_{L^2(\Gamma)} + \frac{1}{2}||\nabla \eta_N^{n+\frac{1}{3}} - \nabla \eta_N^n||^2_{L^2(\Gamma)} = \frac{1}{2}||v_N^n||^2_{L^2(\Gamma)} + \frac{1}{2}||\nabla \eta_N^n||^2_{L^2(\Gamma)}.$$

The result follows once we note that $\boldsymbol{u}_N^{n+\frac{1}{3}} = \boldsymbol{u}_N^n$. $\square$

6.2 The stochastic subproblem

In this subproblem, we incorporate only the effects of the stochastic forcing, which appears only in the structure equation. In this step, we keep the structure displacement and fluid velocity fixed

$$\eta_N^{n+\frac{2}{3}} = \eta_N^{n+\frac{1}{3}}, \qquad \boldsymbol{u}_N^{n+\frac{2}{3}} = \boldsymbol{u}_N^{n+\frac{1}{3}},$$

and only update the structure velocity as

$$v_N^{n+\frac{2}{3}} = v_N^{n+\frac{1}{3}} + [W((n+1)\Delta t) - W(n\Delta t)]. \tag{22}$$

In particular, we are splitting the stochastic part of the structure problem from the deterministic part. This is necessary to obtain a stable scheme. We state the following simple proposition.

Proposition 6.3. Suppose that $v_N^{n+\frac{1}{3}}$ is an $\mathcal{F}_{t_N^n}$ measurable random variable taking values in $H_0^1(\Gamma)$. Then, $v_N^{n+\frac{2}{3}}$ is an $\mathcal{F}_{t_N^{n+1}}$ measurable random variable taking values in $H^1(\Gamma)$.

Notice that the solution $v_N^{n+\frac{2}{3}}$ to the stochastic subproblem taking values in $H^1(\Gamma)$, satisfies pathwise the following integral equality, which will be useful later:

$$\int_\Gamma \frac{v_N^{n+\frac{2}{3}} - v_N^{n+\frac{1}{3}}}{\Delta t}\psi dz = \int_\Gamma \frac{W((n+1)\Delta t) - W(n\Delta t)}{\Delta t}\psi dz, \qquad \text{for all } \psi \in H_0^1(\Gamma). \tag{23}$$

Proposition 6.4. The following discrete energy identity holds pathwise:

$$E_N^{n+\frac{2}{3}} = E_N^{n+\frac{1}{3}} + [W((n+1)\Delta t) - W(n\Delta t)]\int_\Gamma v_N^{n+\frac{1}{3}} dz + \frac{L}{2}[W((n+1)\Delta t) - W(n\Delta t)]^2.$$

Proof. From $v_N^{n+\frac{2}{3}} = v_N^{n+\frac{1}{3}} + [W((n+1)\Delta t) - W(n\Delta t)]$, we get that

$$\frac{1}{2}|v_N^{n+\frac{2}{3}}|^2 = \frac{1}{2}|v_N^{n+\frac{1}{3}}|^2 + v_N^{n+\frac{1}{3}} \cdot [W((n+1)\Delta t) - W(n\Delta t)] + \frac{1}{2}[W((n+1)\Delta t) - W(n\Delta t)]^2.$$

Therefore, after integrating over Γ, one gets the desired energy equality, after recalling that η and $\boldsymbol{u}$ do not change in this subproblem. $\square$

6.3 The fluid subproblem

In this subproblem, we keep the structure displacement fixed

$$\eta_N^{n+1} = \eta_N^{n+\frac{2}{3}},$$

and update the fluid and structure velocities. To define the problem satisfied by the fluid and structure velocities, we introduce the following notation for the corresponding fixed time function spaces:

$$\mathcal{V} = \{(\boldsymbol{u}, v) \in \mathcal{V}_F \times L^2(\Gamma) : \boldsymbol{u}|_\Gamma = v\boldsymbol{e_r}\}, \quad \mathcal{Q} = \{(\boldsymbol{q}, \psi) \in \mathcal{V}_F \times H_0^1(\Gamma) : \boldsymbol{q}|_\Gamma = \psi \boldsymbol{e}_r\},$$

where $\mathcal{V}_F$ is defined by (8). Note that the definition of $\mathcal{V}$ and $\mathcal{Q}$ does not depend on N or n.

Then, the fluid subproblem is to find $(\boldsymbol{u}_N^{n+1}, v_N^{n+1})$ taking values in $\mathcal{V}$ pathwise, such that

$$\int_{\Omega_f} \frac{\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^{n+\frac{2}{3}}}{\Delta t} \cdot \boldsymbol{q} d\boldsymbol{x} + 2\mu \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}_N^{n+1}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} + \int_\Gamma \frac{v_N^{n+1} - v_N^{n+\frac{2}{3}}}{\Delta t}\psi dz$$
$$= P_{N,in}^n \int_0^R (q_z)|_{z=0} dr - P_{N,out}^n \int_0^R (q_z)|_{z=L} dr, \quad \forall (\boldsymbol{q}, \psi) \in \mathcal{Q}, \tag{24}$$

pathwise for each outcome $\omega \in \Omega$, where $P_{N,in/out}^n = \frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} P_{in/out}(t)dt$.

Proposition 6.5. Suppose that $\boldsymbol{u}_N^{n+\frac{2}{3}}$ and $v_N^{n+\frac{2}{3}}$ are $\mathcal{F}_{t_N^{n+1}}$ measurable random variables taking values in $\mathcal{V}_F$ and $H^1(\Gamma)$ respectively. Then, the fluid subproblem (24) has a unique solution $(\boldsymbol{u}_N^{n+1}, v_N^{n+1})$ that is an $\mathcal{F}_{t_N^{n+1}}$ measurable random variable taking values in $\mathcal{V}$.

Proof. We establish this result again using the Lax Milgram lemma. We let $T_N^n : \mathcal{V}_F \times H^1(\Gamma) \times \mathbb{R} \times \mathbb{R} \to \mathcal{V}$ denote the deterministic map that sends deterministic data $(\boldsymbol{u}_N^{n+\frac{2}{3}}, v_N^{n+\frac{2}{3}}, P_{N,in}^n, P_{N,out}^n) \in \mathcal{V}_F \times H^1(\Gamma) \times \mathbb{R} \times \mathbb{R}$ to the unique solution $(\boldsymbol{u}_N^{n+1}, v_N^{n+1}) \in \mathcal{V}$ satisfying the deterministic form of the weak formulation (24). We want to show that the deterministic linear map $T_N^n : (\boldsymbol{u}_N^{n+\frac{2}{3}}, v_N^{n+\frac{2}{3}}, P_{N,in}^n, P_{N,out}^n) \to (\boldsymbol{u}_N^{n+1}, v_N^{n+1})$ is a continuous map. We start by showing that the bilinear form $B : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ given by

$$B((\boldsymbol{u}, v), (\boldsymbol{q}, \psi)) = \int_{\Omega_f} \boldsymbol{u} \cdot \boldsymbol{q} d\boldsymbol{x} + 2\mu(\Delta t) \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} + \int_\Gamma v\psi dz,$$

is coercive and continuous. Coercivity follows from the Korn equality (see for example, Lemma 6 on pg. 377 in [8]), applied to

$$B((\boldsymbol{u}, v), (\boldsymbol{u}, v)) = \int_{\Omega_f} |\boldsymbol{u}|^2 d\boldsymbol{x} + 2\mu(\Delta t) \int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u})|^2 d\boldsymbol{x} + \int_\Gamma v^2 dz,$$

to obtain $||\nabla \boldsymbol{u}||^2_{L^2(\Omega_f)} = 2||\boldsymbol{D}(\boldsymbol{u})||^2_{L^2(\Omega_f)}$. Continuity of the bilinear form B follows from an application of the Cauchy-Schwarz inequality.

Next, one can verify that the map sending

$$(\boldsymbol{q}, \psi) \to \int_{\Omega_f} \boldsymbol{u}_N^{n+\frac{2}{3}} \cdot \boldsymbol{q} d\boldsymbol{x} + \int_\Gamma v_N^{n+\frac{2}{3}} \psi dz + (\Delta t) \left(P_{N,in}^n \int_0^R (q_z)|_{z=0} dr - P_{N,out}^n \int_0^R (q_z)|_{z=L} dr \right),$$

is a continuous linear functional on $\mathcal{V}$. Thus, the existence of a unique $(\boldsymbol{u}_N^{n+1}, v_N^{n+1}) \in \mathcal{V}$ satisfying (24) with the larger space of test functions $(\boldsymbol{q}, \psi) \in \mathcal{V}$ is guaranteed by the Lax-Milgram lemma. Note that $\mathcal{V}$ is a larger space than the space $\mathcal{Q}$ required for the test functions in the fluid subproblem (24). However, we still have the desired uniqueness of the solution in $\mathcal{V}$ if we restrict the test functions to $\mathcal{Q}$ as in (24) because $\mathcal{Q}$ is dense in $\mathcal{V}$.

Then, using coercivity, the trace inequality for $\boldsymbol{u} \in H^1(\Omega_f)$, and the fact that

$$\begin{aligned} &B((\boldsymbol{u}_N^{n+1}, v_N^{n+1}), (\boldsymbol{u}_N^{n+1}, v_N^{n+1})) \\ &= \int_{\Omega_f} \boldsymbol{u}_N^{n+\frac{2}{3}} \cdot \boldsymbol{u}_N^{n+1} d\boldsymbol{x} + \int_\Gamma v_N^{n+\frac{2}{3}} \cdot v_N^{n+1} dz + (\Delta t) \left(P_{N,in}^n \int_0^R (\boldsymbol{u}_N^{n+1})_z|_{z=0} dr - P_{N,out}^n \int_0^R (\boldsymbol{u}_N^{n+1})_z|_{z=L} dr \right), \end{aligned}$$

we obtain the continuity of the map T^n.

Thus, since $\boldsymbol{u}_N^{n+\frac{2}{3}}$ and $v_N^{n+\frac{2}{3}}$ are $\mathcal{F}_{t_N^{n+1}}$ measurable by assumption, the random functions $(\boldsymbol{u}_N^{n+1}, v_N^{n+1}) = T_N^n \circ (\boldsymbol{u}_N^{n+\frac{2}{3}}, v_N^{n+\frac{2}{3}})$, which solve the fluid subproblem, are $\mathcal{F}_{t_N^{n+1}}$ measurable random variables also. □

Proposition 6.6. The following discrete energy identity holds pathwise:

$$\begin{aligned} E_N^{n+1} + 2\mu(\Delta t) \int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u}_N^{n+1})|^2 d\boldsymbol{x} + \frac{1}{2} \left(||\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^{n+\frac{2}{3}}||^2_{L^2(\Omega_f)} \right) + \frac{1}{2} \left(||v_N^{n+1} - v_N^{n+\frac{2}{3}}||^2_{L^2(\Gamma)} \right) \\ = E_N^{n+\frac{2}{3}} + (\Delta t) \left(P_{N,in}^n \int_0^R (\boldsymbol{u}_N^{n+1})_z|_{z=0} dr - P_{N,out}^n \int_0^R (\boldsymbol{u}_N^{n+1})_z|_{z=L} dr \right). \end{aligned}$$

Proof. We can substitute $(\boldsymbol{q}, \psi) = (\boldsymbol{u}_N^{n+1}, v_N^{n+1})$ into the weak formulation of the fluid subproblem since we showed in Proposition 6.5 that (24) holds more generally for test functions in $\mathcal{V}$. We obtain

$$\begin{aligned} \int_{\Omega_f} \frac{\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^{n+\frac{2}{3}}}{\Delta t} \cdot \boldsymbol{u}_N^{n+1} d\boldsymbol{x} + 2\mu \int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u}_N^{n+1})|^2 d\boldsymbol{x} + \int_\Gamma \frac{v_N^{n+1} - v_N^{n+\frac{2}{3}}}{\Delta t} \cdot v_N^{n+1} dz \\ = P_{N,in}^n \int_0^R (\boldsymbol{u}_N^{n+1})_z|_{z=0} dr - P_{N,out}^n \int_0^R (\boldsymbol{u}_N^{n+1})_z|_{z=L} dr. \end{aligned}$$

The desired equality follows after multiplication by Δt, and by using the identity $(a-b) \cdot a = \frac{1}{2}(|a|^2 + |a-b|^2 - |b|^2)$. □

6.4 The full, coupled semidiscrete problem

By adding the weak formulations of the stochastic and fluid subproblems (23) and (24), and the second equation in the structure subproblem (18), we have that the solution to the full semidiscrete problem is $(\boldsymbol{u}_N^{n+1}, v_N^{n+1}) \in \mathcal{V}$, and $(v_N^{n+\frac{1}{3}}, \eta_N^{n+\frac{1}{3}}) \in H_0^1(\Gamma) \times H_0^1(\Gamma)$, satisfying the following equality pathwise:

$$\begin{gathered}
\int_{\Omega_f} \frac{\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^n}{\Delta t} \cdot \boldsymbol{q} d\boldsymbol{x} + 2\mu \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}_N^{n+1}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} + \int_\Gamma \frac{v_N^{n+1} - v_N^n}{\Delta t} \psi dz + \int_\Gamma \nabla \eta_N^{n+1} \cdot \nabla \psi dz \\
= \int_\Gamma \frac{W((n+1)\Delta t) - W(n\Delta t)}{\Delta t} \psi dz + P_{N,in}^n \int_0^R (q_z)|_{z=0} dr - P_{N,out}^n \int_0^R (q_z)|_{z=L} dr, \ \forall (\boldsymbol{q}, \psi) \in \mathcal{Q}, \\
\int_\Gamma \frac{\eta_N^{n+1} - \eta_N^n}{\Delta t} \phi dz = \int_\Gamma v_N^{n+\frac{1}{3}} \phi dz, \quad \forall \phi \in L^2(\Gamma),
\end{gathered} \tag{25}$$

where $P_{N,in/out}^n = \dfrac{1}{\Delta t} \displaystyle\int_{n\Delta t}^{(n+1)\Delta t} P_{in/out}(t) dt$. Note that $\eta_N^{n+1} = \eta_N^{n+\frac{1}{3}}$ by the way we constructed the splitting scheme.

The following proposition provides uniform estimates on the expectation of the kinetic and elastic energy for the full, semidiscrete coupled problem (uniform in the number of time steps N, or equivalently, uniform in Δt), as well as uniform estimates on the expectation of the numerical dissipation.

Proposition 6.7. Let $N > 0$ and let $\Delta t = \frac{T}{N}$. There exists a constant C independent of N and depending only on the initial data, the parameters of the problem, and $||P_{in/out}||^2_{L^2(0,T)}$, such that the following uniform energy estimates hold:

1. ***Uniform semidiscrete kinetic energy and elastic energy estimates:***

$$\mathbb{E}\left(\max_{n=0,1,\dots,N-1} E_N^{n+\frac{1}{3}}\right) \le C, \ \ \mathbb{E}\left(\max_{n=0,1,\dots,N-1} E_N^{n+\frac{2}{3}}\right) \le C, \ \text{and} \ \ \mathbb{E}\left(\max_{n=0,1,\dots,N-1} E_N^{n+1}\right) \le C.$$

2. ***Uniform semidiscrete viscous fluid dissipation estimate:***

$$\sum_{j=1}^N \mathbb{E}(D_N^j) \le C.$$

3. ***Uniform numerical dissipation estimates:***

$$\sum_{n=0}^{N-1} \left(\mathbb{E}\left(||v_N^{n+\frac{1}{3}} - v_N^n||^2_{L^2(\Gamma)} \right) + \mathbb{E}\left(||\nabla \eta_N^{n+\frac{1}{3}} - \nabla \eta_N^n||^2_{L^2(\Gamma)} \right) \right) \le C.$$

$$\sum_{n=0}^{N-1} \mathbb{E}\left(||v_N^{n+\frac{2}{3}} - v_N^{n+\frac{1}{3}}||^2_{L^2(\Gamma)} \right) \le C.$$

$$\sum_{n=0}^{N-1} \left(\mathbb{E}\left(||\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^{n+\frac{2}{3}}||^2_{L^2(\Omega_f)} \right) + \mathbb{E}\left(||v_N^{n+1} - v_N^{n+\frac{2}{3}}||^2_{L^2(\Gamma)} \right) \right) \le C.$$

Proof. First, recall the definitions of the discrete energy $E_N^{n+\frac{i}{3}}$ and the discrete fluid dissipation D_N^n from (20) and (21). We start with the second uniform numerical dissipation estimate. This estimate follows directly from the stochastic subproblem (22) after integration

$$\int_\Gamma |v_N^{n+\frac{2}{3}} - v_N^{n+\frac{1}{3}}|^2 dz = L \cdot [W((n+1)\Delta t) - W(n\Delta t)]^2,$$

and summation of the expectations of both sides:

$$\sum_{n=0}^{N-1} \mathbb{E}\left(||v_N^{n+\frac{2}{3}} - v_N^{n+\frac{1}{3}}||^2_{L^2(\Gamma)}\right) = \sum_{n=0}^{N-1} \mathbb{E}\left(L \cdot [W((n+1)\Delta t) - W(n\Delta t)]^2\right) = LT.$$

We now verify the remaining uniform energy estimates. By summing the structure, stochastic, and fluid discrete energy identities, we obtain

$$\begin{aligned} E_N^{n+1} &+ \sum_{k=0}^{n}\left(2\mu(\Delta t)\int_{\Omega_f}|\boldsymbol{D}(\boldsymbol{u}_N^{k+1})|^2 d\boldsymbol{x} + \frac{1}{2}\left(||\boldsymbol{u}_N^{k+1} - \boldsymbol{u}_N^{k+\frac{2}{3}}||^2_{L^2(\Omega_f)}\right) + \frac{1}{2}\left(||v_N^{k+1} - v_N^{k+\frac{2}{3}}||^2_{L^2(\Gamma)}\right)\right) \\ &+ \sum_{k=0}^{n}\left(\frac{1}{2}\left(||v_N^{k+\frac{1}{3}} - v_N^{k}||^2_{L^2(\Gamma)}\right) + \frac{1}{2}\left(||\nabla\eta_N^{k+\frac{1}{3}} - \nabla\eta_N^{k}||^2_{L^2(\Gamma)}\right)\right) \\ &= E_0 + (\Delta t)\sum_{k=0}^{n}\left(P_{N,in}^k\int_0^R (\boldsymbol{u}_N^{k+1})_z|_{z=0}dr - P_{N,out}^k\int_0^R(\boldsymbol{u}_N^{k+1})_z|_{z=L}dr\right) \\ &+ \sum_{k=0}^{n}\left([W((k+1)\Delta t) - W(k\Delta t)]\int_\Gamma v_N^{k+\frac{1}{3}}dz + \frac{L}{2}[W((k+1)\Delta t) - W(k\Delta t)]^2\right), \end{aligned} \tag{26}$$

$$\begin{aligned} E_N^{n+\frac{2}{3}} &+ \sum_{k=0}^{n-1}\left(2\mu(\Delta t)\int_{\Omega_f}|\boldsymbol{D}(\boldsymbol{u}_N^{k+1})|^2 d\boldsymbol{x} + \frac{1}{2}\left(||\boldsymbol{u}_N^{k+1} - \boldsymbol{u}_N^{k+\frac{2}{3}}||^2_{L^2(\Omega_f)}\right) + \frac{1}{2}\left(||v_N^{k+1} - v_N^{k+\frac{2}{3}}||^2_{L^2(\Gamma)}\right)\right) \\ &+ \sum_{k=0}^{n}\left(\frac{1}{2}\left(||v_N^{k+\frac{1}{3}} - v_N^{k}||^2_{L^2(\Gamma)}\right) + \frac{1}{2}\left(||\nabla\eta_N^{k+\frac{1}{3}} - \nabla\eta_N^{k}||^2_{L^2(\Gamma)}\right)\right) \\ &= E_0 + (\Delta t)\sum_{k=0}^{n-1}\left(P_{N,in}^k\int_0^R (\boldsymbol{u}_N^{k+1})_z|_{z=0}dr - P_{N,out}^k\int_0^R(\boldsymbol{u}_N^{k+1})_z|_{z=L}dr\right) \\ &+ \sum_{k=0}^{n}\left([W((k+1)\Delta t) - W(k\Delta t)]\int_\Gamma v_N^{k+\frac{1}{3}}dz + \frac{L}{2}[W((k+1)\Delta t) - W(k\Delta t)]^2\right), \end{aligned}$$

and

$$\begin{aligned} E_N^{n+\frac{1}{3}} &+ \sum_{k=0}^{n-1}\left(2\mu(\Delta t)\int_{\Omega_f}|\boldsymbol{D}(\boldsymbol{u}_N^{k+1})|^2 d\boldsymbol{x} + \frac{1}{2}\left(||\boldsymbol{u}_N^{k+1} - \boldsymbol{u}_N^{k+\frac{2}{3}}||^2_{L^2(\Omega_f)}\right) + \frac{1}{2}\left(||v_N^{k+1} - v_N^{k+\frac{2}{3}}||^2_{L^2(\Gamma)}\right)\right) \\ &+ \sum_{k=0}^{n}\left(\frac{1}{2}\left(||v_N^{k+\frac{1}{3}} - v_N^{k}||^2_{L^2(\Gamma)}\right) + \frac{1}{2}\left(||\nabla\eta_N^{k+\frac{1}{3}} - \nabla\eta_N^{k}||^2_{L^2(\Gamma)}\right)\right) \\ &= E_0 + (\Delta t)\sum_{k=0}^{n-1}\left(P_{N,in}^k\int_0^R (\boldsymbol{u}_N^{k+1})_z|_{z=0}dr - P_{N,out}^k\int_0^R(\boldsymbol{u}_N^{k+1})_z|_{z=L}dr\right) \\ &+ \sum_{k=0}^{n-1}\left([W((k+1)\Delta t) - W(k\Delta t)]\int_\Gamma v_N^{k+\frac{1}{3}}dz + \frac{L}{2}[W((k+1)\Delta t) - W(k\Delta t)]^2\right), \end{aligned}$$

for $n = 0, 1, ..., N-1$. Therefore,

$$\begin{aligned} \mathbb{E}&\left(\max_{i=1,2,3}\left[\max_{n=0,1,...,N-1} E_N^{n+\frac{i}{3}}\right]\right) + \sum_{k=0}^{N-1}\left[\mathbb{E}\left(2\mu(\Delta t)\int_{\Omega_f}|\boldsymbol{D}(\boldsymbol{u}_N^{k+1})|^2 d\boldsymbol{x}\right) + \frac{1}{2}\mathbb{E}\left(||\boldsymbol{u}_N^{k+1} - \boldsymbol{u}_N^{k+\frac{2}{3}}||^2_{L^2(\Omega_f)}\right)\right. \\ &\left.+ \frac{1}{2}\mathbb{E}\left(||v_N^{k+1} - v_N^{k+\frac{2}{3}}||^2_{L^2(\Gamma)}\right) + \left(\frac{1}{2}\mathbb{E}\left(||v_N^{k+\frac{1}{3}} - v_N^{k}||^2_{L^2(\Gamma)}\right) + \frac{1}{2}\mathbb{E}\left(||\nabla\eta_N^{k+\frac{1}{3}} - \nabla\eta_N^{k}||^2_{L^2(\Gamma)}\right)\right)\right] \\ &\leq 2E_0 + 2\mathbb{E}\left[\max_{n=0,1,...,N-1}(\Delta t)\sum_{k=0}^{n}\left(P_{N,in}^k\int_0^R (\boldsymbol{u}_N^{k+1})_z|_{z=0}dr - P_{N,out}^k\int_0^R(\boldsymbol{u}_N^{k+1})_z|_{z=L}dr\right)\right] \\ &+ 2\mathbb{E}\left[\max_{n=0,1,...,N-1}\sum_{k=0}^{n}\left([W((k+1)\Delta t) - W(k\Delta t)]\int_\Gamma v_N^{k+\frac{1}{3}}dz + \frac{L}{2}[W((k+1)\Delta t) - W(k\Delta t)]^2 dr\right)\right]. \end{aligned}$$

What is left is to bound the quantities

$$I_1 := \mathbb{E}\left[\max_{n=0,1,\dots,N-1}(\Delta t)\sum_{k=0}^{n}\left(P^k_{N,in}\int_0^R (\boldsymbol{u}_N^{k+1})_z|_{z=0}dr - P^k_{N,out}\int_0^R(\boldsymbol{u}_N^{k+1})_z|_{z=L}dr\right)\right],$$

and

$$\mathbb{E}\left[\max_{n=0,1,\dots,N-1}\sum_{k=0}^{n}\left([W((k+1)\Delta t)-W(k\Delta t)]\int_\Gamma v_N^{k+\frac{1}{3}}dz + \frac{L}{2}[W((k+1)\Delta t)-W(k\Delta t)]^2 dr\right)\right]$$
$$\leq \mathbb{E}\left[\max_{n=0,1,\dots,N-1}\sum_{k=0}^{n}\left([W((k+1)\Delta t)-W(k\Delta t)]\int_\Gamma v_N^{k+\frac{1}{3}}dz\right)\right] + \frac{L}{2}\mathbb{E}\left(\sum_{k=0}^{N-1}[W((k+1)\Delta t)-W(k\Delta t)]^2\right) := I_2 + I_3.$$

Bound for I_1: The same argument will work for $P^k_{N,in}$ and $P^k_{N,out}$ so without loss of generality, we perform the bounds below for $P^k_{N,in}$. We recall that $P^k_{N,in} = \frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t}P_{in}(t)dt$, where $P^k_{N,in}$ is deterministic. Therefore, we have the following bound, for the term in I_1 that involves $P^k_{N,in}$:

$$\mathbb{E}\left[\max_{n=0,1,\dots,N-1}(\Delta t)\sum_{k=0}^{n}\left(P^k_{N,in}\int_0^R(\boldsymbol{u}_N^{k+1})_z|_{z=0}dr\right)\right] \leq \mathbb{E}\left(\sum_{k=0}^{N-1}(\Delta t)|P^k_{N,in}|\left|\int_0^R(\boldsymbol{u}_N^{k+1})_z|_{z=0}dr\right|\right)$$
$$\leq \sum_{k=0}^{N-1}\mathbb{E}\left[(\Delta t)\frac{1}{4\epsilon}|P^k_{N,in}|^2 + \epsilon(\Delta t)\left(\int_0^R(\boldsymbol{u}_N^{k+1})_z|_{z=0}dr\right)^2\right]$$
$$\leq \sum_{k=0}^{N-1}\mathbb{E}\left[\frac{1}{4\epsilon}\cdot\frac{1}{\Delta t}\left(\int_{k\Delta t}^{(k+1)\Delta t}P_{in}(t)dt\right)^2 + C\epsilon(\Delta t)\int_0^R(\boldsymbol{u}_N^{k+1})_z^2|_{z=0}dr\right]$$
$$\leq \sum_{k=0}^{N-1}\mathbb{E}\left[\frac{1}{4\epsilon}||P_{in}||^2_{L^2(k\Delta t,(k+1)\Delta t)} + C\epsilon(\Delta t)\int_{\Omega_f}|\boldsymbol{D}(\boldsymbol{u}_N^{k+1})|^2 d\boldsymbol{x}\right] = \frac{1}{4\epsilon}||P_{in}||^2_{L^2(0,T)} + \sum_{k=0}^{N-1}\mathbb{E}\left(C\epsilon(\Delta t)\int_{\Omega_f}|\boldsymbol{D}(\boldsymbol{u}_N^{k+1})|^2 d\boldsymbol{x}\right),$$

where we used Korn's inequality in the last line. Therefore,

$$I_1 \leq \frac{1}{4\epsilon}||P_{in}||^2_{L^2(0,T)} + \frac{1}{4\epsilon}||P_{out}||^2_{L^2(0,T)} + \sum_{k=0}^{N-1}\mathbb{E}\left(2C\epsilon(\Delta t)\int_{\Omega_f}|\boldsymbol{D}(\boldsymbol{u}_N^{k+1})|^2 d\boldsymbol{x}\right).$$

Note that the constant C is independent of Δt and N. It is the geometric constant arising from the application of the Poincaré inequality on the fluid domain Ω_f.

Bound for I_2: Next, we examine I_2 and start with an estimate involving the absolute values:

$$I_2 \leq \mathbb{E}\left(\max_{n=0,1,\dots,N-1}\left|\sum_{k=0}^{n}\left(\int_0^L v_N^{k+\frac{1}{3}}dz\right)\cdot[W((k+1)\Delta t)-W(k\Delta t)]\right|\right).$$

Next, we consider the expression under the absolute value sign, and consider it as the following *stochastic integral*:

$$\sum_{k=0}^{n}\left(\int_0^L v_N^{k+\frac{1}{3}}dz\right)\cdot[W((k+1)\Delta t)-W(k\Delta t)] = \int_0^{(n+1)\Delta t}f(t)dW(t),$$

where $f(t)$ is the random function on $[0,T]$ defined by:

$$f(t) = \sum_{k=0}^{N-1}\left(\int_0^L v_N^{k+\frac{1}{3}}dz\right)\cdot 1_{(k\Delta t,(k+1)\Delta t]}(t). \tag{27}$$

Because $v_N^{k+\frac{1}{3}}$ is $\mathcal{F}_{t_N^k}$ measurable, this integrand is predictable. *This is a direct consequence of how we split the stochastic part of the problem from the structure subproblem.* Without such a splitting, we would not be able to make the same conclusion. Hence, since the stochastic integral is a continuous process in time, we have

$$I_2 \leq \mathbb{E}\left(\max_{0\leq s\leq T}\left|\int_0^s f(t)dW\right|\right).$$

Using the BDG inequality, we obtain that

$$I_2 \leq \mathbb{E}\left[\left(\int_0^T |f(t)|^2 dt\right)^{1/2}\right] = \mathbb{E}\left[(\Delta t)^{1/2}\left(\sum_{k=0}^{N-1}\left(\int_0^L v_N^{k+\frac{1}{3}}dz\right)^2\right)^{1/2}\right] \leq \epsilon(\Delta t)\mathbb{E}\sum_{k=0}^{N-1}\left(\int_0^L v_N^{k+\frac{1}{3}}dz\right)^2 + \frac{1}{4\epsilon}$$

$$\leq \epsilon L(\Delta t)\mathbb{E}\sum_{k=0}^{N-1}||v_N^{k+\frac{1}{3}}||^2_{L^2(\Gamma)} + \frac{1}{4\epsilon} \leq \epsilon LN(\Delta t)\mathbb{E}\left(\max_{k=0,1,\dots,N-1}||v_N^{k+\frac{1}{3}}||^2_{L^2(\Gamma)}\right) + \frac{1}{4\epsilon} \leq 2\epsilon LN(\Delta t)\mathbb{E}\left(\max_{n=0,1,\dots,N-1}E_N^{n+\frac{1}{3}}\right) + \frac{1}{4\epsilon}.$$

Bound for I_3: Finally, by using the properties of Brownian motion, we immediately deduce that

$$I_3 := \frac{L}{2}\mathbb{E}\left(\sum_{k=0}^{N-1}[W((k+1)\Delta t) - W(k\Delta t)]^2\right) = \frac{LT}{2}.$$

Conclusion: From the above estimates, we conclude that

$$\mathbb{E}\left(\max_{i=1,2,3}\left[\max_{n=0,1,\dots,N-1}E_N^{n+\frac{i}{3}}\right]\right) + \sum_{k=0}^{N-1}\left[\mathbb{E}\left(2\mu(\Delta t)\int_{\Omega_f}|\boldsymbol{D}(\boldsymbol{u}_N^{k+1})|^2 d\boldsymbol{x}\right) + \frac{1}{2}\mathbb{E}\left(||\boldsymbol{u}_N^{k+1} - \boldsymbol{u}_N^{k+\frac{2}{3}}||^2_{L^2(\Omega_f)}\right)\right.$$
$$\left. + \frac{1}{2}\mathbb{E}\left(||v_N^{k+1} - v_N^{k+\frac{2}{3}}||^2_{L^2(\Gamma)}\right) + \left(\frac{1}{2}\mathbb{E}\left(||v_N^{k+\frac{1}{3}} - v_N^k||^2_{L^2(\Gamma)}\right) + \frac{1}{2}\mathbb{E}\left(||\nabla\eta_N^{k+\frac{1}{3}} - \nabla\eta_N^k||^2_{L^2(\Gamma)}\right)\right)\right]$$
$$\leq 2E_0 + \frac{1}{2\epsilon}||P_{in}||^2_{L^2(0,T)} + \frac{1}{2\epsilon}||P_{out}||^2_{L^2(0,T)} + \sum_{k=0}^{N-1}\mathbb{E}\left(4C\epsilon(\Delta t)\int_{\Omega_f}|\boldsymbol{D}(\boldsymbol{u}_N^{k+1})|^2 d\boldsymbol{x}\right) + 4\epsilon LT\cdot\mathbb{E}\left(\max_{n=0,1,\dots,N-1}E_N^{n+\frac{1}{3}}\right) + \frac{1}{2\epsilon} + LT.$$

We note that the constant C depends only on the fluid domain Ω_f and not on Δt or N. The result follows once we fix $\epsilon > 0$, independent of Δt, such that $4C\epsilon < \mu$ and $4\epsilon LT < \frac{1}{2}$, and move the associated terms from the right hand side to the left hand side. We emphasize that this gives a uniform energy estimate because the choice of ϵ is independent of Δt and hence N.

$\square$

Remark 6.1. We remark that the specific form of the splitting scheme enabled us to directly estimate the terms involving the white noise as stochastic integrals, such as the second to last term in estimate (26), as a result of measurability considerations. Because $v_N^{k+\frac{1}{3}}$ is $\mathcal{F}_{t_N^k}$ measurable, the stochastic increment $[W((k+1)\Delta t) - W(k\Delta t)]$ is independent of the integral of $v_N^{k+\frac{1}{3}}$, and hence, we were able to rewrite this term as a stochastic integral, see (27).

7 Approximate solutions

We use the solutions at fixed times of our semidiscrete scheme, $\boldsymbol{u}_N^{n+\frac{i}{3}}$, $\eta_N^{n+\frac{i}{3}}$, and $v_N^{n+\frac{i}{3}}$ for $i = 1, 2, 3$, to create approximate solutions for the given stochastic FSI problem in time on the time interval $[0, T]$, for each N, which we will need to pass to the limit as $\Delta t \to 0$. The approximate solutions will be defined as piecewise functions in time. However, we must be careful in this construction of approximate solutions to make sure that they are adapted to the given filtration $\{\mathcal{F}_t\}_{t\geq 0}$ associated to the given Brownian motion.

7.1 Definition of approximate solutions

We start with the fluid. We define the approximate random function $\boldsymbol{u}_N$ on $[0,T] \times \Omega_f$ to be the piecewise constant function

$$\boldsymbol{u}_N(t,\cdot) = \boldsymbol{u}_N^{n-1}, \qquad \text{for } t \in ((n-1)\Delta t, n\Delta t].$$

Note that because $\boldsymbol{u}_N^n$ is $\mathcal{F}_{t_N^n}$ measurable, the choice of $\boldsymbol{u}_N^{n-1}$ instead of $\boldsymbol{u}_N^n$ above is used so that the resulting process $\boldsymbol{u}_N$ is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

Next, we consider the functions associated with the structure. Note that η_N^n, $\eta_N^{n+\frac{1}{3}}$, and $v_N^{n+\frac{1}{3}}$ are $\mathcal{F}_{t_N^n}$ measurable while $v_N^{n+\frac{2}{3}}$ is $\mathcal{F}_{t_N^{n+1}}$ measurable. It turns out that we will not need to keep track of $v_N^{n+\frac{2}{3}}$ when passing to the limit, since it does not appear in (25). So it suffices to define

$$\eta_N(t,\cdot) = \eta_N^{n-1}, \qquad v_N(t,\cdot) = v_N^{n-1}, \qquad v_N^*(t,\cdot) = v_N^{n-\frac{2}{3}}, \qquad \text{for } t \in ((n-1)\Delta t, n\Delta t],$$

and these are all adapted to the given filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Note that v_N defined on $[0,T] \times \Gamma$ is pathwise the trace of the fluid velocity $\boldsymbol{u}_N$ defined on $[0,T] \times \Omega_f$ for all $t \in [0,T]$, but this is not true for v_N^*, since v_N^* is the structure velocity obtained after the structure subproblem in the semidiscrete scheme, which does not update the fluid velocity directly.

We also introduce a piecewise linear interpolation $\overline{\eta}_N$ of η_N to add additional regularity to the structure displacement, since we will want the structure displacement to be in $W^{1,\infty}(0,T;L^2(\Gamma))$ almost surely in the limit as $\Delta t \to 0$. Thus, $\overline{\eta}_N$ is piecewise linear such that

$$\overline{\eta}_N(n\Delta t) = \eta_N^n, \qquad \text{for } n = 0,1,...,N. \tag{28}$$

Note that $\overline{\eta}_N$ has Lipschitz continuous paths in time, and furthermore,

$$\partial_t \overline{\eta}_N = v_N^*. \tag{29}$$

Because both η_N^n and η_N^{n+1} are $\mathcal{F}_{t_N^n}$ adapted, $\overline{\eta}_N$ is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. We will also introduce a piecewise constant function $\eta_N^{\Delta t}$ for the structure displacement, given by

$$\eta_N^{\Delta t}(t,\cdot) = \eta_N^n, \qquad \text{for } t \in ((n-1)\Delta t, n\Delta t]. \tag{30}$$

Note that $\eta_N^{\Delta t}$ is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and is a time-shifted version of η_N, which is emphasized in the notation by the superscript of Δt. This time-shifted structure displacement will be useful for passing to the limit in Section 8.4.

We will also consider the corresponding piecewise linear interpolations for the fluid velocity and structure velocity, which satisfy

$$\overline{\boldsymbol{u}}_N(n\Delta t) = \boldsymbol{u}_N^n, \qquad \overline{v}_N(n\Delta t) = v_N^n, \qquad \text{for } n = 0,1,...,N. \tag{31}$$

We will need to consider $\overline{\boldsymbol{u}}_N$ and $\overline{v}_N$ because we will express the discrete time derivatives $\frac{\boldsymbol{u}_N^{n+1}-\boldsymbol{u}_N^n}{\Delta t}$ and $\frac{v_N^{n+1}-v_N^n}{\Delta t}$ in the semidiscrete formulation (25) in terms of the time derivatives of $\overline{\boldsymbol{u}}_N$ and $\overline{v}_N$. We will also need to consider piecewise constant time-shifted functions $\boldsymbol{u}_N^{\Delta t}$ and $v_N^{\Delta t}$ for the fluid velocity and the structure velocity, defined by

$$\boldsymbol{u}_N^{\Delta t}(t,\cdot) = \boldsymbol{u}_N^n, \qquad v_N^{\Delta t}(t,\cdot) = v_N^n, \qquad \text{for } t \in ((n-1)\Delta t, n\Delta t]. \tag{32}$$

We note that $\boldsymbol{u}_N^{\Delta t}$ and $v_N^{\Delta t}$ are time-shifted versions of $\boldsymbol{u}_N$ and v_N. We will need these time-shifted functions because the fluid dissipation estimate in Proposition 6.7 implies that $\boldsymbol{u}_N^{\Delta t}$, rather than $\boldsymbol{u}_N$, is uniformly bounded in $L^2(\Omega; L^2(0,T;H^1(\Omega_f)))$. See Proposition 7.2.

We make the following important observation. Unlike $\overline{\eta}_N$, we note that $\overline{\boldsymbol{u}}_N$ and $\overline{v}_N$ are *not* necessarily adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, even though they can still be considered as random variables taking values in their appropriate path spaces. Similarly, $\boldsymbol{u}_N^{\Delta t}$ and $v_N^{\Delta t}$, unlike $\boldsymbol{u}_N$ and v_N, are *not* necessarily adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. However, this will not be an issue, because we will see later in Lemma 8.3 that $\overline{\boldsymbol{u}}_N$, $\boldsymbol{u}_N^{\Delta t}$, $\overline{v}_N$, and $v_N^{\Delta t}$ are almost surely "close to" the random processes $\boldsymbol{u}_N$ and $\boldsymbol{v}_N$, which are adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, as $N \to \infty$ along a subsequence.

We summarize some of the previously discussed measure theoretic properties of the stochastic approximate solutions in the following proposition, for future reference.

Proposition 7.1. Recall that W is a one dimensional Brownian motion with respect to the probability space with complete filtration, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. For all $N \in \mathbb{N}$, $\boldsymbol{u}_N$, v_N, v_N^*, η_N, and $\overline{\eta}_N$ are adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ with left continuous paths, with $\overline{\eta}_N$ having continuous paths. In addition, for some fixed $t > 0$ and for each N, define $n_0 = \lfloor \frac{t}{\Delta t} \rfloor + 1$. Then, $W_\tau - W_t$ is independent of each of the random variables in the following collection of random variables for each N and for each $\tau > t$:

$$\{\boldsymbol{u}_N^{n-1}, v_N^{n-1}, v_N^{n-\frac{2}{3}} : 1 \leq n \leq n_0\}, \{\eta_N^n : 0 \leq n \leq n_0\}, \{\overline{\eta}_N(s) : s \in [0, n_0\Delta t]\}.$$

7.2 Uniform boundedness of approximate solutions

Using the previous discrete energy estimates, we establish uniform boundedness of the approximate solutions in the following proposition. We note that in contrast to the case of deterministic FSI, the uniform boundedness of these (random) approximate solutions is only in *expectation*.

Proposition 7.2. The following uniform boundedness results hold:

- $(\eta_N)_{N\in\mathbb{N}}$ is uniformly bounded in $L^2(\Omega; L^\infty(0, T; H_0^1(\Gamma)))$.
- $(v_N)_{N\in\mathbb{N}}$ is uniformly bounded in $L^2(\Omega; L^\infty(0, T; L^2(\Gamma)))$.
- $(v_N^{\Delta t})_{N\in\mathbb{N}}$ is uniformly bounded in $L^2(\Omega; L^2(0, T; H^{1/2}(\Gamma)))$.
- $(v_N^*)_{N\in\mathbb{N}}$ is uniformly bounded in $L^2(\Omega; L^\infty(0, T; L^2(\Gamma)))$.
- $(\boldsymbol{u}_N)_{N\in\mathbb{N}}$ is uniformly bounded in $L^2(\Omega; L^\infty(0, T; L^2(\Omega_f)))$.
- $(\boldsymbol{u}_N^{\Delta t})_{N\in\mathbb{N}}$ is uniformly bounded in $L^2(\Omega; L^2(0, T; H^1(\Omega_f)))$.

Proof. The only part of this result that does not follow directly from Proposition 6.7 is to show that $(\boldsymbol{u}_N^{\Delta t})_{N\in\mathbb{N}}$ is uniformly bounded in $L^2(\Omega; L^2(0, T; H^1(\Omega_f)))$. We compute

$$||\boldsymbol{u}_N^{\Delta t}||^2_{L^2(\Omega;L^2(0,T;H^1(\Omega_f)))} = \mathbb{E}\left(\int_0^T ||\boldsymbol{u}_N^{\Delta t}||^2_{H^1(\Omega_f)} dt\right) = (\Delta t)\mathbb{E}\left(\sum_{k=1}^N ||\boldsymbol{u}_N^k||^2_{H^1(\Omega_f)}\right)$$
$$\leq C(\Delta t)\mathbb{E}\left(\sum_{k=1}^N ||\boldsymbol{u}_N^k||_{L^2(\Omega_f)} + \sum_{k=1}^N ||\boldsymbol{D}(\boldsymbol{u}_N^k)||^2_{L^2(\Omega_f)}\right),$$

where we used Korn's inequality (see [33] and Theorem 6.3-3 in [11]). The result follows from the uniform boundedness of the sum of the dissipation terms and the uniform boundedness of the kinetic energy of the fluid, stated in Proposition 6.7. By taking the trace of the r component of the fluid velocity $\boldsymbol{u}_N^n$, which is in $H^{1/2}(\Gamma)$, we get the corresponding boundedness of $(v_N^{\Delta t})_{N\in\mathbb{N}}$ in $L^2(\Omega; L^2(0, T; H^{1/2}(\Gamma)))$. $\square$

We also state the corresponding uniform boundedness property for the linear interpolations $(\overline{\eta}_N)_{N\in\mathbb{N}}$. Note that in terms of distributional derivatives, $\partial_t \overline{\eta}_N = v_N^*$ holds pathwise for $\omega \in \Omega$.

Therefore, we have:

Proposition 7.3. The sequence of linear interpolations of the structure displacements $(\overline{\eta}_N)_{N\in\mathbb{N}}$ is uniformly bounded in $L^2(\Omega; L^\infty(0, T; H_0^1(\Gamma))) \cap L^2(\Omega; W^{1,\infty}(0, T; L^2(\Gamma)))$.

Remark 7.1. To be very precise, one must check that the stochastic approximate solutions are measurable, as random variables taking values in a given path space. The measurability of these stochastic processes is easy to see by using the measurability properties of the functions $\boldsymbol{u}_N^n$, $v_N^{n+\frac{i}{3}}$, and η_N^n. For example, η_N is measurable as a map from the probability space Ω to $L^\infty(0,T;H_0^1(\Gamma))$ because η_N can be considered as the composition of a measurable map F_1 with a continuous map F_2. F_1 is the map from $\omega \in \Omega$ to the space of bounded sequences of length N with values in $H_0^1(\Gamma)$, given by $F_1 : \omega \to (\eta_N^0, \eta_N^1, ..., \eta_N^{N-1})$, which is measurable by the measurability properties of each η_N^n. F_2 is the map from the space of bounded sequences of length N with values in $H_0^1(\Gamma)$ to $L^\infty(0,T;H_0^1(\Gamma))$, given by $F_2 : (\eta_N^0, \eta_N^1, ..., \eta_N^{N-1}) \to \sum_{k=0}^{N-1} \eta_N^k \cdot 1_{(k\Delta t,(k+1)\Delta t]}(t)$, which is continuous.

8 Passage to the limit

We would like to show that our approximate solution sequences converge in a certain sense, to a weak solution of the original problem. While uniform boundedness results give weak convergence of the random approximate solutions in probabilistic function spaces, which can be used to pass to the limit in the semidiscrete formulation, as discussed in the Appendix in Section 11, our goal will be to develop a methodology that will extend more generally to stochastic FSI problems with nonlinearities, such as linearly coupled FSI models with nonlinear dependence of the random noise on the solution itself and nonlinearly coupled FSI models with geometric nonlinearities arising from the fact that the fluid equations are posed on a random moving fluid domain. For this reason, we will develop a mathematical framework for strengthening the convergence of the random approximate solutions from weak convergence in probabilistic function spaces to almost sure convergence, which is a strong enough convergence to pass to the limit in this linear problem and more generally in nonlinear stochastic FSI problems also. To do this, we will use a compactness argument, which will first imply the existence of a convergent subsequence of the *probability measures* which describe the *laws* or equivalently, the *distributions* of the approximate solutions. From here, we will eventually be able to get to *almost sure* convergence of the stochastic approximate solutions themselves.

We start by designing compactness arguments that will provide weak convergence of the probability measures describing the laws of our random approximate solutions.

8.1 Weak convergence of measures

We first show that along subsequences, the probability measures, or the laws describing the distributions of our stochastic approximate solutions constructed earlier, converge to a probability measure, as the time step $\Delta t \to 0$, or $N \to \infty$. For this purpose, we recall that we are given a probability space with complete filtration $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, with a one dimensional Brownian motion W with respect to the given filtration. For each N, we define the probability measure (or the law) μ_N:

$$\mu_N = \mu_{\eta_N} \times \mu_{\overline{\eta}_N} \times \mu_{\eta_N^{\Delta t}} \times \mu_{\boldsymbol{u}_N} \times \mu_{v_N} \times \mu_{\boldsymbol{u}_N} \times \mu_{v_N^*} \times \mu_{\overline{\boldsymbol{u}}_N} \times \mu_{\overline{v}_N} \times \mu_{\boldsymbol{u}_N^{\Delta t}} \times \mu_{v_N^{\Delta t}} \times \mu_W, \tag{33}$$

defined on the phase space $\mathcal{X}$:

$$\mathcal{X} = [L^2(0,T;L^2(\Gamma))]^3 \times [L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma))]^4 \times C(0,T;\mathbb{R}). \tag{34}$$

Here, μ_{η_N} denotes the law of η_N on $L^2(0,T;L^2(\Gamma))$, $\mu_{\boldsymbol{u}_N}$ denotes the law of $\boldsymbol{u}_N$ on $L^2(0,T;L^2(\Omega_f))$, μ_W denotes the law of W on $C(0,T;\mathbb{R})$, and so on. Thus, μ_N is the joint law of the random variables η_N, $\overline{\eta}_N$, $\eta_N^{\Delta t}$, $\boldsymbol{u}_N$, v_N, $\boldsymbol{u}_N$,v_N^*, $\overline{\boldsymbol{u}}_N$, $\overline{v}_N$, $\boldsymbol{u}_N^{\Delta t}$, $v_N^{\Delta t}$, and W. As we shall see below, it is easier to work with the fluid velocity and the structure velocity in pairs, which is the reason why in (33) above, we consider $(\mu_{\boldsymbol{u}_N}, \mu_{v_N})$, $(\mu_{\boldsymbol{u}_N}, \mu_{v_N^*})$, $(\mu_{\overline{\boldsymbol{u}}_N}, \mu_{\overline{v}_N})$, and $(\mu_{\boldsymbol{u}_N^{\Delta t}}, \mu_{v_N^{\Delta t}})$. The main result of this subsection is the following.

Theorem 8.1. Along a subsequence (which we will continue to denote by N), μ_N converges weakly as probability measures to a probability measure μ on $\mathcal{X}$.

To show weak convergence of these probability measures along a subsequence, stated in Theorem 8.1, we must show that the probability measures are **tight**.

Definition 8.1. The probability measures μ_N are **tight** if for every $\epsilon > 0$, there exists a compact set A_ϵ, compact in $\mathcal{X}$, such that

$$\mu_N(A_\epsilon) > 1 - \epsilon, \qquad \text{for all } N.$$

To get a hold of the compact subset A_ϵ, we will need the following two deterministic compactness results for the structure displacements $\{\eta_N(\omega)\}$ and for the fluid and structure velocities $\{\boldsymbol{u}_N(\omega)\}$ and $\{v_N(\omega)\}$. The two results are obtained in the following two lemmas.

The first lemma, which will be applied to the structure displacements $\{\eta_N(\omega)\}$, is a direct consequence of the classical Aubin-Lions compactness lemma [1, 42]:

Lemma 8.1. The following holds: $[W^{1,\infty}(0,T;L^2(\Gamma)) \cap L^\infty(0,T;H_0^1(\Gamma))] \subset\subset L^\infty(0,T;L^2(\Gamma))$.

The Aubin-Lions compactness lemma actually gives a stronger compact embedding of $W^{1,\infty}(0,T;L^2(\Gamma)) \cap L^\infty(0,T;H_0^1(\Gamma))$ into $C(0,T;L^2(\Gamma))$, but since we want η_N and $\overline{\eta}_N$ to take values in the same path space, we use $L^\infty(0,T;L^2(\Gamma))$ since η_N is not continuous.

To handle the compactness argument for the structure and fluid velocities, we consider the subsets $\mathcal{K}$ and $\mathcal{K}_R$ in $L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma))$, defined as follows.

Definition 8.2 (Definition of $\mathcal{K}$ and $\mathcal{K}_R$). The sets $\mathcal{K}$ and $\mathcal{K}_R$ of paths (or realizations) are defined as follows:

• For the pathwise left continuous approximate functions $\boldsymbol{u}_N(\omega), v_N(\omega)$ on $[0,T]$, we define:

$$\mathcal{K} = \{(\boldsymbol{u}, v) \in L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma)) : \boldsymbol{u} = \boldsymbol{u}_N(\omega) \text{ and } v = v_N(\omega) \text{ for some } \omega \in \Omega \text{ and } N \in \mathbb{N}\}.$$

• For any arbitrary positive constant R, define $\mathcal{K}_R$ to be the subset of paths $(\boldsymbol{u}_N(\omega), v_N(\omega)) \in \mathcal{K}$ where ω and N satisfy the following properties.

1. *Uniform boundedness:* $||(\boldsymbol{u}_N^{\Delta t}, v_N^{\Delta t})||_{L^2(0,T;H^1(\Omega_f))\times L^2(0,T;H^{1/2}(\Gamma))} \leq R$, $||\boldsymbol{u}_N||_{L^\infty(0,T;L^2(\Omega_f))} \leq R$, $||v_N||_{L^\infty(0,T;L^2(\Gamma))} \leq R$, $||\eta_N||_{L^\infty(0,T;H_0^1(\Gamma))} \leq R$.
2. *Boundedness of numerical dissipation:* $\sum_{n=0}^{N-1} ||\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^{n+\frac{2}{3}}||^2_{L^2(\Omega_f)} \leq R$, $\sum_{n=0}^{N-1} ||v_N^{n+\frac{1}{3}} - v_N^n||^2_{L^2(\Gamma)} \leq R$, $\sum_{n=0}^{N-1} ||v_N^{n+\frac{2}{3}} - v_N^{n+\frac{1}{3}}||^2_{L^2(\Gamma)} \leq R$, $\sum_{n=0}^{N-1} ||v_N^{n+1} - v_N^{n+\frac{2}{3}}||^2_{L^2(\Gamma)} \leq R$.
3. *Boundedness of fluid dissipation:* $(\Delta t)\sum_{n=1}^{N} \int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u}_N^n)|^2 d\boldsymbol{x} \leq R$.
4. *Boundedness of* $1/4$*-Hölder exponent of Brownian motion:* $\displaystyle\sup_{s,t\in[0,T], s\neq t} \frac{|W(t) - W(s)|}{|t-s|^{1/4}} \leq R$.

Remark 8.1. In the fourth condition above, any positive Hölder exponent that is strictly less than $1/2$ would suffice, since Brownian motion is "almost" $1/2$-Hölder continuous, but we have fixed $1/4$ for concreteness.

The following lemma provides the desired compactness result for $(\boldsymbol{u}_N, v_N)$.

Lemma 8.2. For any arbitrary positive constant R, the set $\mathcal{K}_R$ is precompact in $L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma))$.

Proof. We use the Simon's compactness theorem [48, 56]. According to Simon's theorem, it suffices to check two conditions.

First condition: We must first show that for any $0 < t_1 < t_2 < T$, the collection $\left\{\int_{t_1}^{t_2} f(t)dt : f \in \mathcal{K}_R\right\}$ is relatively compact in $L^2(\Omega_f) \times L^2(\Gamma)$. Consider a sequence $\{f_m(t,\cdot)\}_{m=1}^\infty$ in $\mathcal{K}_R$, where $f_m(t,\cdot) = (\boldsymbol{u}_m(t,\cdot), v_m(t,\cdot))$. We want to show that there is a subsequence $\left\{\int_{t_1}^{t_2} f_{m_k}(t)dt\right\}_{k=1}^\infty$ that converges in $L^2(\Omega_f) \times L^2(\Gamma)$.

For each m, there exists some N_m and $\omega_m \in \Omega$ (both depending on m) such that

$$\boldsymbol{u}_m(t) = \boldsymbol{u}_0 \cdot 1_{t\in[0,(\Delta t)_m]} + \sum_{n=1}^{N_m-1} \boldsymbol{u}^n_{N_m}(\omega_m) \cdot 1_{t\in(n(\Delta t)_m,(n+1)(\Delta t)_m]},$$
$$v_m(t) = v_0 \cdot 1_{t\in[0,(\Delta t)_m]} + \sum_{n=1}^{N_m-1} v^n_{N_m}(\omega_m) \cdot 1_{t\in(n(\Delta t)_m,(n+1)(\Delta t)_m]},$$

where $(\Delta t)_m = T/N_m$. Therefore, we have that

$$\int_{t_1}^{t_2} \boldsymbol{u}_m(t)dt = a_m\boldsymbol{u}_0 + \int_{\max(t_1,(\Delta t)_m)}^{\max(t_2,(\Delta t)_m)} \boldsymbol{u}_m(t)dt, \qquad \int_{t_1}^{t_2} v_m(t)dt = a_m v_0 + \int_{\max(t_1,(\Delta t)_m)}^{\max(t_2,(\Delta t)_m)} v_m(t)dt,$$

where $a_m = \max(0, (\Delta t)_m - t_1)$. Because $a_m \in [0, T]$, we can find a subsequence $\{m_k\}_{k=1}^{\infty}$ such that $a_{m_k} \to a$ as $k \to \infty$, for some $a \in [0, T]$. Because $\boldsymbol{u}_0$ and v_0 are the fixed initial data for the fluid velocity and the structure velocity, $a_{m_k}\boldsymbol{u}_0$ and $a_{m_k}v_0$ converge along this subsequence in $L^2(\Omega_f)$ and $L^2(\Gamma)$.

It remains to show that the sequences in k given by

$$\int_{\max\left(t_1,(\Delta t)_{m_k}\right)}^{\max\left(t_2,(\Delta t)_{m_k}\right)} \boldsymbol{u}_{m_k}(t)dt \qquad \text{and} \qquad \int_{\max\left(t_1,(\Delta t)_{m_k}\right)}^{\max\left(t_2,(\Delta t)_{m_k}\right)} v_{m_k}(t)dt \tag{35}$$

converge in $L^2(\Omega_f)$ and $L^2(\Gamma)$ respectively along a further subsequence. Because of the compact embedding $H^1(\Omega_f) \times H^{1/2}(\Gamma) \subset\subset L^2(\Omega_f) \times L^2(\Gamma)$, it suffices to show that the two sequences in k given in (35) are uniformly bounded in $H^1(\Omega_f)$ and $H^{1/2}(\Gamma)$. This can be easily verified by using the uniform boundedness property of functions in $\mathcal{K}_R$ in Definition 8.2:

$$\begin{aligned}
&\left|\left|\int_{\max\left(t_1,(\Delta t)_{m_k}\right)}^{\max\left(t_2,(\Delta t)_{m_k}\right)} \boldsymbol{u}_{m_k}(t)dt\right|\right|_{H^1(\Omega_f)} + \left|\left|\int_{\max\left(t_1,(\Delta t)_{m_k}\right)}^{\max\left(t_2,(\Delta t)_{m_k}\right)} v_{m_k}(t)dt\right|\right|_{H^{1/2}(\Gamma)} \\
&\le \int_{(\Delta t)_{m_k}}^{T} ||\boldsymbol{u}_{m_k}(t)||_{H^1(\Omega_f)}dt + \int_{(\Delta t)_{m_k}}^{T} ||v_{m_k}(t)||_{H^{1/2}(\Gamma)}dt \\
&\le T^{1/2}\left(\int_{(\Delta t)_{m_k}}^{T} ||\boldsymbol{u}_{m_k}(t)||^2_{H^1(\Omega_f)}dt\right)^{1/2} + T^{1/2}\left(\int_{(\Delta t)_{m_k}}^{T} ||v_{m_k}(t)||^2_{H^{1/2}(\Gamma)}dt\right)^{1/2} \le 2T^{1/2}R.
\end{aligned}$$

Thus, we can further refine the subsequence $\{m_k\}_{k=1}^{\infty}$ to obtain that $\left\{\int_{t_1}^{t_2} (\boldsymbol{u}_{m_k}(t), v_{m_k}(t))\, dt\right\}_{k=1}^{\infty}$ converges in $L^2(\Omega_f) \times L^2(\Gamma)$, where we continue to denote the refined subsequence by $\{m_k\}_{k=1}^{\infty}$.

Second condition: We must show that $||\tau_h f - f||_{L^2(h,T;L^2(\Omega_f)\times L^2(\Gamma))} \to 0$ uniformly for all $f = (\boldsymbol{u}, v) \in \mathcal{K}_R$, as $h \to 0$. Here τ_h for $h > 0$ denotes the time shift map $(\tau_h f)(t, \cdot) = f(t - h, \cdot)$. Consider an arbitrary $\epsilon > 0$. We want to find $h > 0$ sufficiently small such that

$$||\tau_h\boldsymbol{u} - \boldsymbol{u}||_{L^2(h,T;L^2(\Omega_f))} < \epsilon \quad \text{and} \quad ||\tau_h v - v||_{L^2(h,T;L^2(\Gamma))} < \epsilon \quad \forall(\boldsymbol{u}, v) \in \mathcal{K}_R.$$

To verify this, we can write $h = l(\Delta t) + s$, for each $\Delta t = \frac{T}{N}$, where $0 \le s < \Delta t$, so that

$$||\tau_h\boldsymbol{u} - \boldsymbol{u}||_{L^2(h,T;L^2(\Omega_f))} \le ||\tau_s\tau_{l\Delta t}\boldsymbol{u} - \tau_{l\Delta t}\boldsymbol{u}||_{L^2(h,T;L^2(\Omega_f))} + ||\tau_{l\Delta t}\boldsymbol{u} - \boldsymbol{u}||_{L^2(h,T;L^2(\Omega_f))},$$

$$||\tau_h v - v||_{L^2(h,T;L^2(\Gamma))} \le ||\tau_s\tau_{l\Delta t}v - \tau_{l\Delta t}v||_{L^2(h,T;L^2(\Gamma))} + ||\tau_{l\Delta t}v - v||_{L^2(h,T;L^2(\Gamma))}.$$

The above estimates require one estimate for the small s time shift, and one for the larger $l\Delta t$ time shift. We will handle the first time shift estimate using the numerical dissipation estimate holding for $\mathcal{K}_R$, specified in Definition 8.2, and we will handle the second time shift estimate using an Ehrling property.

Estimate for time shift by s: Consider arbitrary $(\boldsymbol{u}_N, v_N) \in \mathcal{K}_R$. Recalling that $0 \le s < \Delta t$, we compute

$$||\tau_s \tau_{l\Delta t} \boldsymbol{u}_N - \tau_{l\Delta t} \boldsymbol{u}_N||^2_{L^2(h,T;L^2(\Omega_f))} = s \sum_{n=0}^{N-l-2} ||\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^n||^2_{L^2(\Omega_f)} \le s \sum_{n=0}^{N-1} ||\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^n||^2_{L^2(\Omega_f)} \le sR.$$

where we used that $\boldsymbol{u}_N^n = \boldsymbol{u}_N^{n+\frac{2}{3}}$ and the numerical dissipation estimate in the last inequality. Similarly,

$$||\tau_s \tau_{l\Delta t} v_N - \tau_{l\Delta t} v_N||^2_{L^2(h,T;L^2(\Gamma))} = s \sum_{n=0}^{N-l-2} ||v_N^{n+1} - v_N^n||^2_{L^2(\Gamma)} \le s \sum_{n=0}^{N-1} ||v_N^{n+1} - v_N^n||^2_{L^2(\Gamma)}$$
$$\le 3s \left(\sum_{n=0}^{N-1} ||v_N^{n+\frac{1}{3}} - v_N^n||^2_{L^2(\Gamma)} + \sum_{n=0}^{N-1} ||v_N^{n+\frac{2}{3}} - v_N^{n+\frac{1}{3}}||^2_{L^2(\Gamma)} + \sum_{n=0}^{N-1} ||v_N^{n+1} - v_N^{n+\frac{2}{3}}||^2_{L^2(\Gamma)} \right) \le 9sR.$$

Recalling that $h = s + l\Delta t$ so that $0 < s \le h$, we can make these quantities arbitrarily small by taking h sufficiently small, since R is a fixed arbitrary positive constant.

Estimate for time shift by $l\Delta t$: Consider arbitrary $(\boldsymbol{u}_N, v_N) \in \mathcal{K}_R$. We want to estimate

$$||\tau_{l\Delta t} \boldsymbol{u}_N - \boldsymbol{u}_N||_{L^2(h,T;L^2(\Omega_f))} + ||\tau_{l\Delta t} v_N - v_N||_{L^2(h,T;L^2(\Gamma))}.$$

This is identically zero if $h < \Delta t$, so we assume for the following estimate that $h \ge \Delta t$. We use the chain of embeddings $H^1(\Omega_f) \times H^{1/2}(\Gamma) \subset\subset L^2(\Omega_f) \times L^2(\Gamma) \subset \mathcal{Q}'$, where $\mathcal{Q}$ is the test space defined in (14). Applying the uniform Ehrling property, see e.g., [48, 54], we obtain

$$\begin{aligned}
&||\tau_{l\Delta t} \boldsymbol{u}_N - \boldsymbol{u}_N||_{L^2(h,T;L^2(\Omega_f))} + ||\tau_{l\Delta t} v_N - v_N||_{L^2(h,T;L^2(\Gamma))} \\
&\le 2||\tau_{l\Delta t}(\boldsymbol{u}_N, v_N) - (\boldsymbol{u}_N, v_N)||_{L^2(h,(l+1)\Delta t;L^2(\Omega_f)\times L^2(\Gamma))} + 2||\tau_{l\Delta t}(\boldsymbol{u}_N, v_N) - (\boldsymbol{u}_N, v_N)||_{L^2((l+1)\Delta t,T;L^2(\Omega_f)\times L^2(\Gamma))} \\
&\le 2||\tau_{l\Delta t}(\boldsymbol{u}_N, v_N) - (\boldsymbol{u}_N, v_N)||_{L^2(h,(l+1)\Delta t;L^2(\Omega_f)\times L^2(\Gamma))} + \delta||\tau_{l\Delta t}(\boldsymbol{u}_N, v_N) - (\boldsymbol{u}_N, v_N)||_{L^2((l+1)\Delta t,T;H^1(\Omega_f)\times H^{1/2}(\Gamma))} \\
&+ C(\delta)||\tau_{l\Delta t}(\boldsymbol{u}_N, v_N) - (\boldsymbol{u}_N, v_N)||_{L^2((l+1)\Delta t,T;\mathcal{Q}')} := I_1 + I_2 + I_3.
\end{aligned}$$

To estimate I_1, we use the triangle inequality, the assumption that $h \ge \Delta t$, and the uniform boundedness property of $\mathcal{K}_R$ in Definition 8.2:

$$I_1 \le 2||\tau_{l\Delta t}(\boldsymbol{u}_N, v_N)||_{L^2(h,(l+1)\Delta t;L^2(\Omega_f)\times L^2(\Gamma))} + 2||(\boldsymbol{u}_N, v_N)||_{L^2(h,(l+1)\Delta t;L^2(\Omega_f)\times L^2(\Gamma))} \le 8(\Delta t)^{1/2} R \le 8h^{1/2} R.$$

To estimate I_2, we use the triangle inequality and the uniform boundedness property of $\mathcal{K}_R$ in Definition 8.2:

$$I_2 \le \delta \left(||\tau_{l\Delta t}(\boldsymbol{u}_N, v_N)||_{L^2((l+1)\Delta t,T;H^1(\Omega_f)\times H^{1/2}(\Gamma))} + ||(\boldsymbol{u}_N, v_N)||_{L^2((l+1)\Delta t,T;H^1(\Omega_f)\times H^{1/2}(\Gamma))} \right)$$
$$\le 2\delta||(\boldsymbol{u}_N, v_N)||_{L^2(\Delta t,T;H^1(\Omega_f)\times H^{1/2}(\Gamma))} \le 2\delta R.$$

To estimate I_3, we multiply the first equation in the weak formulation (25) by Δt to obtain:

$$\int_{\Omega_f} (\boldsymbol{u}_N^{n+l} - \boldsymbol{u}_N^n) \cdot \boldsymbol{q} d\boldsymbol{x} + \int_\Gamma (v_N^{n+l} - v_N^n)\psi dz = \int_\Gamma [W((n+l)\Delta t) - W(n\Delta t)]\psi dz + (\Delta t) \sum_{k=1}^{l} \left(P_{N,in}^{n+k-1} \int_0^R (q_z)|_{z=0} dr \right.$$
$$\left. - P_{N,out}^{n+k-1} \int_0^R (q_z)|_{z=L} dr - 2\mu \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}_N^{n+k}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} - \int_\Gamma \nabla \eta_N^{n+k} \cdot \nabla \psi dz \right), \qquad \forall (\boldsymbol{q}, \psi) \in \mathcal{Q}.$$

We estimate the terms on the right hand side as follows. For $(\boldsymbol{q}, \psi) \in \mathcal{Q}$, where $\mathcal{Q}$ is defined in (14), with $||(\boldsymbol{q}, \psi)||_{\mathcal{Q}} \le 1$, we have the following estimates.

- Using Cauchy-Schwarz and the boundedness of the 1/4-Holder exponent of Brownian motion in the definition of $\mathcal{K}_R$, see Definition 8.2, we obtain

$$\left| \int_\Gamma [W((n+l)\Delta t) - W(n\Delta t)]\psi dz \right| \le \left(\int_\Gamma |W((n+l)\Delta t) - W(n\Delta t)|^2 dz \right)^{1/2} \le \left(\int_\Gamma \left| R(l\Delta t)^{1/4} \right|^2 dz \right)^{1/2} \le C(l\Delta t)^{1/4}.$$

- Next, we recall the definition of the discretized pressure $P^n_{N,in/out} = \frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} P_{in/out}(t)dt$, and use the trace inequality on the integral involving q_z to obtain

$$(\Delta t)\left|\sum_{k=1}^{l} P_{N,in}^{n+k-1}\int_0^R (q_z)|_{z=0}dr\right| = (\Delta t)\left|\sum_{k=1}^{l} P_{N,in}^{n+k-1}\right| \cdot \left|\int_0^R (q_z)|_{z=0}dr\right| \leq C(\Delta t)\left|\sum_{k=1}^{l} P_{N,in}^{n+k-1}\right|$$
$$= C\left|\int_{n\Delta t}^{(n+l)\Delta t} P_{in}(t)dt\right| \leq C(l\Delta t)^{1/2}||P_{in}||_{L^2(n\Delta t,(n+l)\Delta t)} \leq C(l\Delta t)^{1/2}||P_{in}||_{L^2(0,T)} = C(l\Delta t)^{1/2}.$$

The same estimate holds for the outlet pressure term.

- Using Cauchy-Schwarz and the uniform fluid dissipation estimate in Definition 8.2 of $\mathcal{K}_R$, we get

$$(\Delta t)\left|\sum_{k=1}^{l} 2\mu\int_{\Omega_f}\boldsymbol{D}(\boldsymbol{u}_N^{n+k}) : \boldsymbol{D}(\boldsymbol{q})d\boldsymbol{x}\right| \leq C(\Delta t)\sum_{k=1}^{l}\left(\int_{\Omega_f}|\boldsymbol{D}(\boldsymbol{u}_N^{n+k})|^2 d\boldsymbol{x}\right)^{1/2}$$
$$\leq Cl^{1/2}(\Delta t)\left(\sum_{k=1}^{l}\int_{\Omega_f}|\boldsymbol{D}(\boldsymbol{u}_N^{n+k})|^2 d\boldsymbol{x}\right)^{1/2} \leq Cl^{1/2}(\Delta t)\left(\sum_{k=1}^{N}\int_{\Omega_f}|\boldsymbol{D}(\boldsymbol{u}_N^{k})|^2 d\boldsymbol{x}\right)^{1/2} \leq C(l\Delta t)^{1/2}.$$

- Using Cauchy-Schwarz and the uniform boundedness of η_N in Definition 8.2 of $\mathcal{K}_R$, we get:

$$(\Delta t)\left|\sum_{k=1}^{l}\int_\Gamma \nabla\eta_N^{n+k}\cdot\nabla\psi dz\right| \leq (\Delta t)\sum_{k=1}^{l}\int_\Gamma|\nabla\eta_N^{n+k}\cdot\nabla\psi|dz \leq C(\Delta t)\sum_{k=1}^{l}||\eta_N^{n+k}||_{H_0^1(\Gamma)} \leq Cl(\Delta t).$$

Here, all constants C are independent of n, l, and Δt and hence N, but can depend on the fixed, arbitrary constant R, and on the given parameters of the problem. Combining all of these estimates together, we obtain that

$$||(\boldsymbol{u}_N^{n+l}, v_N^{n+l}) - (\boldsymbol{u}_N^n, v_N^n)||_{\mathcal{Q}'} \leq C(l\Delta t)^{1/4}, \tag{36}$$

where we use the estimate $0 \leq l(\Delta t) \leq T$ to reduce all exponents on $(l\Delta t)$ to the smallest one, which is 1/4. Hence,

$$||\tau_{l\Delta t}(\boldsymbol{u}_N, v_N) - (\boldsymbol{u}_N, v_N)||^2_{L^2((l+1)\Delta t,T;\mathcal{Q}')}$$
$$= (\Delta t)\sum_{n=1}^{N-1-l}||(\boldsymbol{u}_N^{n+l}, v_N^{n+l}) - (u_N^n, v_N^n)||^2_{\mathcal{Q}'} \leq C(\Delta t)\sum_{n=0}^{N-1}(l\Delta t)^{1/2} \leq C(l\Delta t)^{1/2}.$$

and so $I_3 := C(\delta)||\tau_{l\Delta t}(\boldsymbol{u}_N, v_N) - (\boldsymbol{u}_N, v_N)||_{L^2((l+1)\Delta t,T;\mathcal{Q}')} \leq C(\delta)(l\Delta t)^{1/4}$.

Combining the estimates for I_1, I_2, and I_3, we obtain

$$||\tau_{l\Delta t}\boldsymbol{u}_N - \boldsymbol{u}_N||_{L^2(h,T;L^2(\Omega_f))} + ||\tau_{l\Delta t}v_N - v_N||_{L^2(h,T;L^2(\Gamma))} \leq 8h^{1/2}R + 2\delta R + C(\delta)(l\Delta t)^{1/4}.$$

We can now conclude the verification of the second condition of Simon's compactness result. Namely, we have shown that

$$||\tau_h\boldsymbol{u} - \boldsymbol{u}||_{L^2(h,T;L^2(\Omega_f))} \leq (sR)^{1/2} + 8h^{1/2}R + 2\delta R + C(\delta)(l\Delta t)^{1/4},$$
$$||\tau_h v - v||_{L^2(h,T;L^2(\Gamma))} \leq 3(sR)^{1/2} + 8h^{1/2}R + 2\delta R + C(\delta)(l\Delta t)^{1/4}.$$

Now, since $h = s + l\Delta t$ and $s, l\Delta t \in [0, h]$, we get

$$||\tau_h\boldsymbol{u} - \boldsymbol{u}||_{L^2(h,T;L^2(\Omega_f))} \leq (hR)^{1/2} + 8h^{1/2}R + 2\delta R + C(\delta)h^{1/4},$$
$$||\tau_h v - v||_{L^2(h,T;L^2(\Gamma))} \leq 3(hR)^{1/2} + 8h^{1/2}R + 2\delta R + C(\delta)h^{1/4}.$$

Therefore, given $\epsilon > 0$, we can first choose $\delta > 0$ so that $2\delta R < \frac{\epsilon}{2}$, which fixes a value for $C(\delta)$. Then, we can choose $h > 0$ sufficiently small so that

$$3(hR)^{1/2} + 8h^{1/2}R + C(\delta)h^{1/4} < \frac{\epsilon}{2}.$$

This establishes the desired equicontinuity estimate, and hence Lemma 8.2 follows from Simon's compactness theorem. □

Finally, we note that we have obtained compactness results only for the velocity approximate function v_N and not v_N^*. In addition, when passing to the limit, we will consider the linear interpolations and time-shifted versions of the fluid velocity and of the structure displacement and velocity. We recall that the linear interpolations are piecewise linear functions defined by (28), (31), and the time-shifted functions are piecewise constant functions defined by (30), (32). Hence, we will need the following result.

Lemma 8.3. For an appropriate subsequence, which we continue to denote by N,

$$\begin{aligned}
||v_N - v_N^*||_{L^2(0,T;L^2(\Gamma))} &\to 0, && \text{as } N \to \infty, \text{ almost surely,}\\
||v_N - \overline{v}_N||_{L^2(0,T;L^2(\Gamma))} &\to 0, && \text{as } N \to \infty, \text{ almost surely,}\\
||v_N - v_N^{\Delta t}||_{L^2(0,T;L^2(\Gamma))} &\to 0, && \text{as } N \to \infty, \text{ almost surely,}\\
||\boldsymbol{u}_N - \overline{\boldsymbol{u}}_N||_{L^2(0,T;L^2(\Omega_f))} &\to 0, && \text{as } N \to \infty, \text{ almost surely,}\\
||\boldsymbol{u}_N - \boldsymbol{u}_N^{\Delta t}||_{L^2(0,T;L^2(\Omega_f))} &\to 0, && \text{as } N \to \infty, \text{ almost surely,}\\
||\eta_N - \overline{\eta}_N||_{L^2(0,T;L^2(\Gamma))} &\to 0, && \text{as } N \to \infty, \text{ almost surely,}\\
||\eta_N - \eta_N^{\Delta t}||_{L^2(0,T;L^2(\Gamma))} &\to 0, && \text{as } N \to \infty, \text{ almost surely.}
\end{aligned}$$

Proof. We start by showing the first convergence result. To do that, we introduce the events

$$E_{j,N} = \left\{ ||v_N - v_N^*||_{L^2(0,T;L^2(\Gamma))} \le \frac{1}{j} \right\}, \quad j \ge 1,$$

and show that the probability that the complements of $E_{j,N}$ occur for infinitely many j, is zero. Indeed, by multiplying by Δt the uniform numerical dissipation estimate from Proposition 6.7 and keeping only the first term on the left hand side, we obtain

$$\mathbb{E}\left(\Delta t \sum_{n=0}^{N-1} ||v_N^{n+\frac{1}{3}} - v_N^n||^2_{L^2(\Gamma)}\right) = \mathbb{E}\left(||v_N - v_N^*||^2_{L^2(0,T;L^2(\Gamma))}\right) \le C(\Delta t). \tag{37}$$

By Chebychev's inequality, we get $\mathbb{P}(E_{j,N}^c) \le C(\Delta t)j^2 = CTN^{-1}j^2$. Thus, for the events $E_{j,N=j^4}$, we have $\sum_{j=1}^{\infty} \mathbb{P}(E_{j,N=j^4}^c) \le CT \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty$. Therefore, by the Borel-Cantelli lemma,

$$\mathbb{P}\left(E_{j,N=j^4}^c \text{ occurs for infinitely many } j\right) = 0.$$

This implies that for almost every $\omega \in \Omega$, there exists $j_0(\omega)$ such that $||v_{N_j} - v_{N_j}^*||_{L^2(0,T;L^2(\Gamma))} \le \frac{1}{j}$ for all $j \ge j_0(\omega)$, where $N_j := j^4$, which implies the desired result, where our subsequence N_j will continue to be denoted by N for simplicity of notation.

To show the the remaining convergence results, we use Proposition 6.7 to conclude that there exists a uniform constant C independent of N such that

$$\sum_{n=0}^{N-1} \mathbb{E}\left(||v_N^{n+1} - v_N^n||^2_{L^2(\Gamma)}\right) \le C, \quad \sum_{n=0}^{N-1} \mathbb{E}\left(||\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^n||^2_{L^2(\Omega_f)}\right) \le C, \quad \sum_{n=0}^{N-1} \mathbb{E}\left(||\nabla\eta_N^{n+1} - \nabla\eta_N^n||^2_{L^2(\Gamma)}\right) \le C,$$

where we recall that $\boldsymbol{u}_N^{n+\frac{2}{3}} = \boldsymbol{u}_N^n$ and $\eta_N^{n+\frac{1}{3}} = \eta_N^{n+1}$, and where we used the triangle inequality to obtain the first estimate. Then, the same argument as above gives the desired result, once we note that

$$\mathbb{E}\left(||\overline{v}_N - v_N||^2_{L^2(0,T;L^2(\Gamma))}\right) \leq (\Delta t)\sum_{n=0}^{N-1}\mathbb{E}\left(||v_N^{n+1} - v_N^n||^2_{L^2(\Gamma)}\right) \leq C(\Delta t) \to 0, \qquad \text{as } N \to \infty, \tag{38}$$

$$\mathbb{E}\left(||v_N^{\Delta t} - v_N||^2_{L^2(0,T;L^2(\Gamma))}\right) = (\Delta t)\sum_{n=0}^{N-1}\mathbb{E}\left(||v_N^{n+1} - v_N^n||^2_{L^2(\Gamma)}\right) \leq C(\Delta t) \to 0, \qquad \text{as } N \to \infty, \tag{39}$$

$$\mathbb{E}\left(||\overline{\boldsymbol{u}}_N - \boldsymbol{u}_N||^2_{L^2(0,T;L^2(\Omega_f))}\right) \leq (\Delta t)\sum_{n=0}^{N-1}\mathbb{E}\left(||\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^n||^2_{L^2(\Omega_f)}\right) \leq C(\Delta t) \to 0, \qquad \text{as } N \to \infty, \tag{40}$$

$$\mathbb{E}\left(||\boldsymbol{u}_N^{\Delta t} - \boldsymbol{u}_N||^2_{L^2(0,T;L^2(\Omega_f))}\right) = (\Delta t)\sum_{n=0}^{N-1}\mathbb{E}\left(||\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^n||^2_{L^2(\Omega_f)}\right) \leq C(\Delta t) \to 0, \qquad \text{as } N \to \infty, \tag{41}$$

$$\mathbb{E}\left(||\overline{\eta}_N - \eta_N||^2_{L^2(0,T;L^2(\Gamma))}\right) \leq (\Delta t)\sum_{n=0}^{N-1}\mathbb{E}\left(||\eta_N^{n+1} - \eta_N^n||^2_{L^2(\Gamma)}\right) \leq C'(\Delta t) \to 0, \qquad \text{as } N \to \infty, \tag{42}$$

$$\mathbb{E}\left(||\eta_N^{\Delta t} - \eta_N||^2_{L^2(0,T;L^2(\Gamma))}\right) = (\Delta t)\sum_{n=0}^{N-1}\mathbb{E}\left(||\eta_N^{n+1} - \eta_N^n||^2_{L^2(\Gamma)}\right) \leq C'(\Delta t) \to 0, \qquad \text{as } N \to \infty, \tag{43}$$

where we used Poincaré's inequality to deduce (42) and (43). □

Notice that this result follows from the numerical dissipation estimates in Proposition 6.7, which imply convergence to zero in expectation, of the numerical dissipation terms, shown in (38), (39), (40), (41), (42), and (43), from which we were able to deduce the almost sure convergence.

Proof of Theorem 8.1. To show weak convergence of probability measures along a subsequence, we must show that the probability measures μ_N are tight, see Definition 8.1. Here, we note that for reasons that will be clear later (see Step 2 below), we will take N to be the subsequence provided by Lemma 8.3 and begin with this indexing convention of N.

Step 1: Weak convergence of $\mu_{\overline{\eta}_N}$ and $\mu_{\boldsymbol{u}_N} \times \mu_{v_N}$ along a subsequence. We show this by showing that $\mu_{\overline{\eta}_N}$ and $\mu_{\boldsymbol{u}_N} \times \mu_{v_N}$ are tight. To show the tightness of $\mu_{\overline{\eta}_N}$, we define the set

$$A_R = \{\overline{\eta} \in W^{1,\infty}(0,T;L^2(\Gamma)) \cap L^\infty(0,T;H_0^1(\Gamma)) : ||\overline{\eta}||_{W^{1,\infty}(0,T;L^2(\Gamma))} \leq R, ||\overline{\eta}||_{L^\infty(0,T;H_0^1(\Gamma))} \leq R\}.$$

By Lemma 8.1, $\overline{A_R}$ is a compact set in $L^2(0,T;L^2(\Gamma))$ since $L^\infty(0,T;L^2(\Gamma))$ embeds continuously into $L^2(0,T;L^2(\Gamma))$, where the closure is taken in the topology of $L^2(0,T;L^2(\Gamma))$. So by Chebychev's inequality and the previous uniform boundedness results, we have that for an arbitrary $\epsilon > 0$,

$$\mu_{\overline{\eta}_N}(\overline{A_R}) > 1 - \epsilon,$$

if R is chosen sufficiently large. So there exists a subsequence, which we continue to denote by N, for which $\mu_{\overline{\eta}_N}$ converges weakly to some probability measure μ_η on $L^2(0,T;L^2(\Gamma))$.

To show the tightness of $\mu_{\boldsymbol{u}_N} \times \mu_{v_N}$, recall the definition of the set $\mathcal{K}_R$, and note that by Lemma 8.2, $\overline{\mathcal{K}_R}$ is a compact set in $L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma))$. Furthermore, using the uniform boundedness estimates from Proposition 7.2 combined with Chebychev's inequality, we have that for any $\epsilon > 0$, we can find R sufficiently large such that

$$(\mu_{\boldsymbol{u}_N} \times \mu_{v_N})(\overline{\mathcal{K}_R}) > 1 - \epsilon.$$

Hence, there exists a subsequence, which we continue to denote by N, for which the measures $\mu_{\boldsymbol{u}_N} \times \mu_{v_N}$ converge weakly to some limiting probability measure on $L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma))$, which we denote by $\mu_{\boldsymbol{u}} \times \mu_v$.

Step 2: Weak convergence of $\mu_{\boldsymbol{u}_N} \times \mu_{v_N^*}$, $\mu_{\overline{\boldsymbol{u}}_N} \times \mu_{\overline{v}_N}$, $\mu_{\boldsymbol{u}_N^{\Delta t}} \times \mu_{v_N^{\Delta t}}$, μ_{η_N}, and $\mu_{\eta_N^{\Delta t}}$ along the subsequence obtained from Step 1. Since $\mu_{\boldsymbol{u}_N} \times \mu_{v_N} \Longrightarrow \mu_{\boldsymbol{u}} \times \mu_v$, by the definition of weak convergence, we have

$$\mathbb{E}[f(\boldsymbol{u}_N, v_N)] \to \int_{L^2(0,T;L^2(\Omega_f))\times L^2(0,T;L^2(\Gamma))} f d(\mu_{\boldsymbol{u}} \times \mu_v),$$

for all bounded, Lipschitz continuous functions $f : L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma)) \to \mathbb{R}$. However, because $||v_N - v_N^*||_{L^2(0,T;L^2(\Gamma))} \to 0$ a.s. due to Lemma 8.3, we have that by the Lipschitz continuity of f,

$$|f(\boldsymbol{u}_N, v_N) - f(\boldsymbol{u}_N, v_N^*)| \leq \text{Lip}(f)||v_N - v_N^*||_{L^2(0,T;L^2(\Gamma))} \to 0, \quad \text{a.s. as } N \to \infty.$$

Hence, by the bounded convergence theorem, $\mathbb{E}[f(\boldsymbol{u}_N, v_N)] - \mathbb{E}[f(\boldsymbol{u}_N, v_N^*)] \to 0$, as $N \to \infty$, and hence,

$$\mathbb{E}[f(\boldsymbol{u}_N, v_N^*)] \to \int_{L^2(0,T;L^2(\Omega_f))\times L^2(0,T;L^2(\Gamma))} f d(\mu_{\boldsymbol{u}} \times \mu_v),$$

for all bounded, Lipschitz continuous functions $f : L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma)) \to \mathbb{R}$. Thus, along the subsequence generated from Step 1, we have that both $\mu_{\boldsymbol{u}_N} \times \mu_{v_N}$ and $\mu_{\boldsymbol{u}_N} \times \mu_{v_N^*}$ converge weakly to the same limiting probability measure $\mu_{\boldsymbol{u}} \times \mu_v$ on $L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma))$.

The same argument can be used to show that $\mu_{\overline{\boldsymbol{u}}_N} \times \mu_{\overline{v}_N}$ and $\mu_{\boldsymbol{u}_N^{\Delta t}} \times \mu_{v_N^{\Delta t}}$ also converge weakly to $\mu_{\boldsymbol{u}} \times \mu_v$. This follows from the result on a.s. convergence of $||\overline{\boldsymbol{u}}_N - \boldsymbol{u}_N||_{L^2(0,T;L^2(\Omega_f))}$, $||\overline{v}_N - v_N||_{L^2(0,T;L^2(\Gamma))}$, $||\boldsymbol{u}_N^{\Delta t} - \boldsymbol{u}_N||_{L^2(0,T;L^2(\Omega_f))}$, and $||v_N^{\Delta t} - v_N||_{L^2(0,T;L^2(\Gamma))}$ in Lemma 8.3.

Finally, we have from Step 1 that $\mu_{\overline{\eta}_N}$ converges weakly to some probability measure μ_η, as probability measures on $L^2(0,T;L^2(\Gamma))$. Then, the weak convergence of μ_{η_N} and $\mu_{\eta_N^{\Delta t}}$ to this same weak limit μ_η follows from the same argument as above, and the result from Lemma 8.3 that $||\overline{\eta}_N - \eta_N||_{L^2(0,T;L^2(\Gamma))} \to 0$ and $||\eta_N^{\Delta t} - \eta_N||_{L^2(0,T;L^2(\Gamma))} \to 0$, as $N \to \infty$, a.s.

Step 3: Tightness of full measures μ_N along the subsequence obtained from Step 1. We now consider the full probability measures μ_N specified in (33) on the phase space $\mathcal{X}$ specified in (34). We want to show that these probability measures μ_N are tight along the subsequence N constructed as a result of Step 1.

Consider $\epsilon > 0$. We want to construct a compact set in the phase space $\mathcal{X}$ for which the probability measure μ_N has probability greater than $1 - \epsilon$ on this compact set, for all N. We will construct this compact set component-wise, using $\pi_1, \ldots, \pi_{12}$ to denote the projections onto the components 1 through 12 of μ_N.

By the weak convergence of the measures $\mu_{\overline{\eta}_N}$, μ_{η_N}, and $\mu_{\eta_N^{\Delta t}}$, by Prohorov's theorem (see for example Proposition 6.1 in [41]), there exist compact sets B_1, B_2, and B_3 in $L^2(0,T;L^2(\Gamma))$ such that

$$\pi_1(\mu_N)(B_1) > 1 - \frac{\epsilon}{8}, \qquad \pi_2(\mu_N)(B_2) > 1 - \frac{\epsilon}{8}, \qquad \pi_3(\mu_N)(B_3) > 1 - \frac{\epsilon}{8}, \qquad \text{for all } N.$$

Similarly, because $(\boldsymbol{u}_N, v_N)$, $(\boldsymbol{u}_N, v_N^*)$, $(\overline{\boldsymbol{u}}_N, \overline{v}_N)$, and $(\boldsymbol{u}_N^{\Delta t}, v_N^{\Delta t})$ converge weakly along this subsequence N by Step 1 and Step 2, there exist compact sets $B_{4,5}$, $B_{6,7}$, $B_{8,9}$, and $B_{10,11}$ in $L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma))$ such that

$$\begin{aligned} &\pi_{4,5}(\mu_N)(B_{4,5}) > 1 - \frac{\epsilon}{8}, \qquad \pi_{6,7}(\mu_N)(B_{6,7}) > 1 - \frac{\epsilon}{8}, \\ &\pi_{8,9}(\mu_N)(B_{8,9}) > 1 - \frac{\epsilon}{8}, \qquad \pi_{10,11}(\mu_N)(B_{10,11}) > 1 - \frac{\epsilon}{8}, \qquad \text{for all } N. \end{aligned}$$

Finally, the last component of μ_N, which is μ_W, is constant in N. Hence, the probability measures $\pi_{12}(\mu_N)$ defined on $C(0,T;\mathbb{R})$ are trivially, weakly compact. Therefore, the collection $\pi_{12}(\mu_N)$ for all N is tight, and hence, there exists a compact set $B_{12} \subset C(0,T;\mathbb{R})$ such that

$$\pi_{12}(\mu_N)(B_{12}) > 1 - \frac{\epsilon}{8}, \qquad \text{for all } N.$$

Based on this construction, we have the set $M_\epsilon := B_1 \times B_2 \times B_3 \times B_{4,5} \times B_{6,7} \times B_{8,9} \times B_{10,11} \times B_{12}$, which is a compact subset of the phase space $\mathcal{X}$, satisfying $\mu_N(M_\epsilon) > 1 - \epsilon$, for all N. This establishes the desired tightness of the probability measures, and completes the proof of Proposition 8.1. $\square$

8.2 Continuity properties of the weak limit

In order to prove certain measure theoretic properties of the limiting solutions, we need to establish continuity of the limiting solution. This is because many measure theoretic properties are simpler for stochastic processes with continuous paths in time. This is simple to do for the structure displacements, since the approximate structure displacements $\overline{\eta}_N$ all have Lipschitz continuous paths. However, because the approximate fluid and structure velocities $\boldsymbol{u}_N$ and v_N have paths that are not continuous, we want to establish that the limiting solutions for the fluid and structure velocities have continuous paths in time, with an appropriate notion of continuity. We first introduce the following definition, which will be used throughout this section.

Definition 8.3. Let B be an arbitrary Banach space and let $f, g : [0, T] \to B$. The function $g : [0, T] \to B$ is a **version** of f if $f = g$ a.e. on $[0, T]$.

The goal is to show that the limit function $(\boldsymbol{u}, v)$ is in $C(0, T; \mathcal{Q}')$ almost surely. This continuity will later allow us to show that $(\boldsymbol{u}, v)$ is predictable as a stochastic process. To show this continuity, we will use the idea of p-variation. The notion of considering total variations of functions is a classical idea [49], [58]. We remark however that our definition below differs slightly from classical definitions of total p-variation.

Definition 8.4. For any real number $p \geq 1$ and any $\delta > 0$, we define the *p-variation of length scale δ* of a given function $(\boldsymbol{u}, v) : [0, T] \to \mathcal{Q}'$ by

$$V_p^\delta(\boldsymbol{u}, v) = \sup_{|P| \leq \delta} \sum_{i=1}^{M} ||(\boldsymbol{u}(t_i), v(t_i)) - (\boldsymbol{u}(t_{i-1}), v(t_{i-1}))||_{\mathcal{Q}'}^p,$$

where P denotes a partition $0 \leq t_0 < t_1 < ... < t_M \leq T$ for some positive integer M, and the condition $|P| \leq \delta$ means that $|t_i - t_{i-1}| \leq \delta$ for all $i = 1, 2, ..., M$.

We introduce this definition of the p-variation of length scale δ because we will invoke estimates on the time shifts, as in (36), in order to deduce continuity in $\mathcal{Q}'$. The strategy will be to show that almost surely, the limiting fluid velocity and structure velocity, denoted by the pair $(\boldsymbol{u}, v)$, has a variation that goes to zero as the length scale δ goes to zero, which would imply that the pair $(\boldsymbol{u}, v)$ cannot have any discontinuities and is hence continuous in $\mathcal{Q}'$. We hence want to define and examine the subset of functions whose p-variation of length scale δ is bounded above by a certain parameter ϵ. We do this in the following lemma.

Lemma 8.4. Let $A_{p,\delta,\epsilon}$ be the set of functions $(\boldsymbol{u}, v) : [0, T] \to \mathcal{Q}'$ in $X = L^2(0, T; L^2(\Omega_f)) \times L^2(0, T; L^2(\Gamma))$ such that the following properties hold:

1. $(\boldsymbol{u}, v)$ has a version that is left continuous on $[0, T]$ as a function of time, taking values in $\mathcal{Q}'$.
2. This version of $(\boldsymbol{u}, v)$ is also right continuous at $t = 0$ as a function taking values in $\mathcal{Q}'$.
3. For this (necessarily unique) left continuous version, $V_p^\delta(\boldsymbol{u}, v) \leq \epsilon$.

Then, for any $p \geq 1$, $\delta > 0$, and $\epsilon > 0$, $A_{p,\delta,\epsilon}$ is a closed set in X.

Proof. To show that $A_{p,\delta,\epsilon}$ is a closed set in X, we consider a sequence $\{(\boldsymbol{u}_n, v_n)\}_{n=1}^\infty$ in $A_{p,\delta,\epsilon}$ that converges to some element $(\boldsymbol{u}, v) \in X$ in the norm of X. We claim that $(\boldsymbol{u}, v) \in A_{p,\delta,\epsilon}$.

We start by showing Property 3 above, namely $V_p^\delta(\boldsymbol{u}, v) \leq \epsilon$. Because $(\boldsymbol{u}_n, v_n) \to (\boldsymbol{u}, v)$ in $L^2(0, T; L^2(\Omega_f) \times L^2(\Gamma))$ and $L^2(\Omega_f) \times L^2(\Gamma) \subset \mathcal{Q}'$, we have that along a subsequence (denoted by the same index),

$$(\boldsymbol{u}_n(t), v_n(t)) \to (\boldsymbol{u}(t), v(t)), \text{ in } \mathcal{Q}', \ \forall t \in S, \tag{44}$$

where S is some measurable subset of $[0, T]$, which consists of almost every $t \in [0, T]$.

By the convergence (44) and the fact that $V_p^\delta(\boldsymbol{u}_n, v_n) \le \epsilon$ for all n, for any partition P with $|P| \le \delta$ *consisting only of points in* S,

$$\sum_{i=1}^{M} ||(\boldsymbol{u}(t_i), v(t_i)) - (\boldsymbol{u}(t_{i-1}), v(t_{i-1}))||_{\mathcal{Q}'}^p \le \epsilon. \tag{45}$$

We use the inequality (45) to construct a version of $(\boldsymbol{u}, v)$ which satisfies Properties 1 and 2. Then, we will conclude the proof by verifying Property 3 for this new version. We start with Property 1 above, namely that $(\boldsymbol{u}, v)$ must have a version that is left continuous. We do this in the following steps.

Step 1: Using the fact that any partition consisting of points in S with $|P| \le \delta$ satisfies (45), we can conclude that the left and right limits of $(\boldsymbol{u}, v)$ along points in S exists, where the limit is considered in the norm of $\mathcal{Q}'$. This is useful, as S is dense in $[0, T]$. In addition, the density of S in $[0, T]$ means that for all $t \in [0, T]$, the notion of a left and right limit along points in S makes sense. In particular, given any point $t_0 \in [0, T]$, $\lim_{t \to t_0^-, t \in S}(\boldsymbol{u}(t), v(t))$ and $\lim_{t \to t_0^+, t \in S}(\boldsymbol{u}(t), v(t))$ both exist and are finite, where these are limits in $\mathcal{Q}'$.

Step 2: Next, we show that there can only be countably many points $t_0 \in (0, T)$ for which the limits $\lim_{t \to t_0^-, t \in S}(\boldsymbol{u}(t), v(t))$ and $\lim_{t \to t_0^+, t \in S}(\boldsymbol{u}(t), v(t))$, which take values in $\mathcal{Q}'$, do not agree. Suppose for contradiction that there are uncountably many points in $(0, T)$ for which these limits do not agree. Then, there exists $\rho > 0$ sufficiently small such that there are infinitely many points $t_0 \in (0, T)$ for which

$$|| \lim_{t \to t_0^-, t \in S}(\boldsymbol{u}(t), v(t)) - \lim_{t \to t_0^+, t \in S}(\boldsymbol{u}(t), v(t))||_{\mathcal{Q}'} \ge \rho.$$

Let M be sufficiently large such that $M\left(\frac{\rho}{2}\right)^p > \epsilon$ and select $t_1, ..., t_M$ points of discontinuity in $(0, T)$ with

$$|| \lim_{t \to t_n^-, t \in S}(\boldsymbol{u}(t), v(t)) - \lim_{t \to t_n^+, t \in S}(\boldsymbol{u}(t), v(t))||_{\mathcal{Q}'} \ge \rho, \qquad \text{for } n = 1, 2, ..., M.$$

We can order these points as $t_1 < t_2 < ... < t_M$, and select $2M$ points $\{s_{n,i}\}_{1 \le n \le M, i=1,2}$ in S, such that

1. $s_{1,1} < s_{1,2} < s_{2,1} < s_{2,2} < ... < s_{M,1} < s_{M,2}$.
2. For each $n = 1, 2, ..., M$, $t_n - \frac{\delta}{2} < s_{n,1} < t_n < s_{n,2} < t_n + \frac{\delta}{2}$.
3. For each $n = 1, 2, ..., M$, $||(\boldsymbol{u}(s_{n,1}), v(s_{n,1})) - \lim_{t \to t_n^-, t \in S}(\boldsymbol{u}(t), v(t))||_{\mathcal{Q}'} < \frac{\rho}{4}$, and $||(\boldsymbol{u}(s_{n,2}), v(s_{n,2})) - \lim_{t \to t_n^+, t \in S}(\boldsymbol{u}(t), v(t))||_{\mathcal{Q}'} < \frac{\rho}{4}$.

Then, we can form a partition of points in S that interlaces the sequence $s_{1,1} < s_{1,2} < s_{2,1} < s_{2,2} < ... < s_{M,1} < s_{M,2}$ with additional points so that the resulting partition P has $|P| < \delta$, since S is dense in $[0, T]$. We can do this in a way that keeps the points $s_{n,i}$ for $i = 1, 2$ consecutive in the partition for each $n = 1, 2, ..., M$. Since $M\left(\frac{\rho}{2}\right)^p > \epsilon$, we have that the variation for this resulting partition is greater than ϵ, which is a contradiction.

We conclude that *there are only countably many points* $t_0 \in S$ *for which* $\lim\limits_{t \to t_0^-, t \in S}(\boldsymbol{u}(t), v(t)) \neq (\boldsymbol{u}(t_0), v(t_0))$, by using a similar argument. So let S^* be the set of points $t_0 \in S$ such that $\lim\limits_{t \to t_0^-, t \in S}(\boldsymbol{u}(t), v(t)) = (\boldsymbol{u}(t_0), v(t_0))$. Since countable sets have measure zero, $[0, T] - S^*$ is of measure zero and S^* is still dense in $[0, T]$. *We emphasize that now,* $(\boldsymbol{u}(t), v(t))$ *has the useful property that it is left continuous on* S^*.

Step 3: Because $S^* \subset S$ and is still a dense set in $[0, T]$, the result from Step 1 implies that:

$\lim_{t \to t_0^-, t \in S^*}(\boldsymbol{u}(t), v(t))$ and $\lim_{t \to t_0^+, t \in S^*}(\boldsymbol{u}(t), v(t))$ exist for all $t_0 \in [0, T]$.

However, these limits are only along points in S^*. By the density of S^* in $[0, T]$ and the fact that $[0, T] - S^*$ has measure zero, we can redefine $(\boldsymbol{u}, v)$ up to a version, so that

$$(\boldsymbol{u}(t_0), v(t_0)) \text{ is unchanged if } t_0 \in S^*, \text{ and } (\boldsymbol{u}(t_0), v(t_0)) = \lim_{t \to t_0^-, t \in S^*}(\boldsymbol{u}(t), v(t)) \text{ if } t_0 \in [0, T] - S^*. \tag{46}$$

For the remainder of this proof, $(\boldsymbol{u}, v)$ will denote this newly defined version in (46). We claim that

$$\lim_{t\to t_0^-, t\in S^*} (\boldsymbol{u}(t), v(t)) = \lim_{t\to t_0^-} (\boldsymbol{u}(t), v(t)) \qquad \text{and} \qquad \lim_{t\to t_0^+, t\in S^*} (\boldsymbol{u}(t), v(t)) = \lim_{t\to t_0^+} (\boldsymbol{u}(t), v(t)), \tag{47}$$

for all $t_0 \in [0, T]$. We will just prove the first statement in (47), as the second statement is proved analogously. To see this, note that by the definition of the version and by the definition of S^* in Step 2,

$$(\boldsymbol{u}(t_0), v(t_0)) = \lim_{t\to t_0^-, t\in S^*} (\boldsymbol{u}(t), v(t)), \qquad \text{for all } t_0 \in [0, T]. \tag{48}$$

Given any $t_n \nearrow t_0$ where t_n is not necessarily in S^*, $\lim_{n\to\infty}(\boldsymbol{u}(t_n), v(t_n)) = \lim_{t\to t_0^-, t\in S^*}(\boldsymbol{u}(t), v(t))$ as a result of (48) and the density of S^* in $[0, T]$, which establishes the desired result in (47). Note that this version $(\boldsymbol{u}(t), v(t))$ is left continuous by (47) and (48), with only countably many points of discontinuity by Step 2.

Conclusion: We have constructed a left continuous version of $(\boldsymbol{u}(t), v(t))$ on $[0, T]$ taking values in $\mathcal{Q}'$ in Step 3. At the left boundary, $t = 0$, we can set the version of $(\boldsymbol{u}, v)$ so that $(\boldsymbol{u}(0), v(0)) = \lim_{t\to 0^+}(\boldsymbol{u}(t), v(t))$, so that we have right continuity at $t = 0$. This is possible since this limit exists by Step 1 and (47). For the newly defined version of $(\boldsymbol{u}(t), v(t))$, we have that $\sum_{i=1}^{N} ||(\boldsymbol{u}(x_i), v(x_i)) - (\boldsymbol{u}(x_{i-1}), v(x_{i-1}))||_{\mathcal{Q}'}^p \le \epsilon$, for all partitions P consisting of points in S^* with $|P| \le \delta$, since we did not change the original $(\boldsymbol{u}(t), v(t))$ on points of S^*, which is a subset of S. We can now show that this p-variation inequality holds more generally for all partitions P with points in $[0, T]$ with $|P| \le \delta$. To do this, we note that since S^* is dense in $[0, T]$, we can approximate any partition P of arbitrary points in $[0, T]$ with $|P| \le \delta$ by a sequence of partitions $\{P_k\}_{k\ge 1}$ of points in S^* with $|P_k| \le \delta$ containing the same number of points as P. We can do this by approaching any partition points of P in $(0, T]$ from the left by points in S^*, and approaching $t = 0$ from the right by points in S^* if $t = 0$ is a partition point in P. We then obtain the desired result by taking the limit in k as the partitions P_k approach P. This process of taking the limit uses the fact that the version of $(\boldsymbol{u}, v)$ as defined in Step 3 is left continuous on $[0, T]$ and right continuous at $t = 0$. Therefore, we conclude that $V_p^\delta(\boldsymbol{u}, v) \le \epsilon$. $\square$

The next lemma, along with the weak convergence of the laws μ_N, will allow us to use the result above to prove almost sure continuity in $\mathcal{Q}'$ of the limiting fluid and structure velocity. In particular, this next lemma will show that if the length scale δ is chosen appropriately, then eventually, for large enough N (or equivalently small enough Δt), the approximate solutions will have p-variation (for $p > 4$) with length scale δ bounded above uniformly with high probability. This is to be expected, due to the time shift estimate (36), which is *independent of* N.

For the following results, we recall the definition of μ_N on the phase space $\mathcal{X}$ from (33) and (34), and we denote by $\pi_{4,5}\mu_N$ the projection onto the fourth and fifth components of $\mathcal{X}$, which gives the law of $(\boldsymbol{u}_N, v_N)$ on $L^2(0, T; L^2(\Omega_f)) \times L^2(0, T; L^2(\Gamma))$.

Lemma 8.5. For any $p > 4$ and any $\epsilon > 0$, there exists $\delta_0 > 0$ sufficiently small and N_0 sufficiently large such that for all $0 < \delta \le \delta_0$,

$$\pi_{4,5}\mu_N(A_{p,\delta,\epsilon}) > 1 - \epsilon, \qquad \text{for all } N \ge N_0.$$

Proof. Let $\mathcal{K}_{R,N}$ be the collection of paths in $\mathcal{K}_R$ (introduced in Definition 8.2), corresponding to path realizations of the random variables $(\boldsymbol{u}_N, v_N)$ for fixed N, satisfying the properties in the definition of $\mathcal{K}_R$. In particular, $\mathcal{K}_R = \bigcup_{N=1}^{\infty} \mathcal{K}_{R,N}$. Notice that we can choose R large enough so that

$$\pi_{4,5}\mu_N(\overline{\mathcal{K}_{R,N}}) > 1 - \epsilon, \qquad \text{for all } N,$$

where the closure is taken in $L^2(0, T; L^2(\Omega_f) \times L^2(\Gamma))$. Recall the time shift estimate (36), which holds for all $(\boldsymbol{u}_N, v_N) \in \mathcal{K}_R$, where C_R depends only on R. We use this estimate to choose $\delta_0 > 0$ and N_0 such that $\overline{\mathcal{K}_{R,N}} \subset A_{p,\delta,\epsilon}$, for all $N \ge N_0$ and $0 < \delta \le \delta_0$, from which the result $\pi_{4,5}\mu_N(A_{p,\delta,\epsilon}) > 1 - \epsilon$, for all $N \ge N_0$ and $0 < \delta \le \delta_0$ will follow. Indeed, for any given partition P with $|P| \le \delta$, the following estimate holds:

$$\sum_{i=1}^{M} ||(\boldsymbol{u}(x_i), v(x_i)) - (\boldsymbol{u}(x_{i-1}), v(x_{i-1}))||_{\mathcal{Q}'}^p \le C_R \sum_{k=1}^{l} n_k (k\Delta t)^{p/4},$$

for any $(\boldsymbol{u}, v) \in \mathcal{K}_R$, where n_k is the number of increments that have indices k apart and l is the maximum integer for which $l\Delta t < \delta + \Delta t$. This is true by the fact that the paths $(\boldsymbol{u}, v)$ in $\mathcal{K}_R$ are defined as piecewise constant functions taking values $(\boldsymbol{u}_N^n, v_N^n)$, and by inequality (36). Because the partition P has $|P| \leq \delta$, we have that l must satisfy

$$l\Delta t < \delta + \Delta t = \delta + N^{-1}T \quad \text{and} \quad \sum_{k=1}^{l} n_k(k\Delta t) \leq (N-1)\Delta t = T - \Delta t. \tag{49}$$

Therefore, since $p > 4$, we have that for any partition P with $|P| \leq \delta$ and for any $(\boldsymbol{u}, v) \in \mathcal{K}_R$,

$$\begin{aligned}\sum_{i=1}^{M} ||(\boldsymbol{u}(x_i), v(x_i)) - (\boldsymbol{u}(x_{i-1}), v(x_{i-1}))||_{\mathcal{Q}'}^p &\leq C_R \sum_{k=1}^{l} n_k(k\Delta t)^{p/4} \leq C_R \left(\sum_{k=1}^{l} n_k(k\Delta t)\right)(l\Delta t)^{\frac{p}{4}-1} \\ &\leq C_R T(l\Delta t)^{\frac{p}{4}-1} \leq C_R T(\delta + N^{-1}T)^{\frac{p}{4}-1},\end{aligned}$$

where we used (49). The proof is complete once we choose $\delta_0 > 0$ sufficiently small and N_0 sufficiently large such that $C_R T(\delta_0 + N_0^{-1}T)^{\frac{p}{4}-1} < \epsilon$. Therefore, for $(\boldsymbol{u}_N, v_N)$ in $\mathcal{K}_R$ for any $N \geq N_0$ and $0 < \delta \leq \delta_0$, we have $V_p^\delta(\boldsymbol{u}_N, v_N) \leq \epsilon$. Thus,

$$\mathcal{K}_{R,N} \subset A_{p,\delta,\epsilon}, \qquad \text{for all } N \geq N_0 \text{ and } 0 < \delta \leq \delta_0.$$

Since $A_{p,\delta,\epsilon}$ is closed in $L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma))$ by Lemma 8.4, we conclude that

$$\overline{\mathcal{K}_{R,N}} \subset A_{p,\delta,\epsilon}, \qquad \text{for all } N \geq N_0 \text{ and } 0 < \delta \leq \delta_0,$$

where the closure is taken with respect to the norm of $L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma))$. Since $\pi_{4,5}\mu_N(\overline{\mathcal{K}_{R,N}}) > 1-\epsilon$ for all positive integers N by the initial choice of R, this implies the result. $\square$

Lemma 8.6. For the weak limit μ,

$$\pi_{4,5}\mu(X \cap C(0,T;\mathcal{Q}')) = 1,$$

where $X := L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma))$. Furthermore, $\pi_{4,5}\mu$ is supported on a Borel measurable subset of X such that every function has a version in $C(0,T;\mathcal{Q}')$ that is equal to $(\boldsymbol{u}_0, v_0)$ at $t = 0$.

Remark 8.2. We remark that $X \cap C(0,T;\mathcal{Q}')$ is a Borel measurable subset of X, so the statement above makes sense. To see this, note that the inclusion map $\iota : X \to L^2(0,T;\mathcal{Q}')$ is continuous since $L^2(\Omega_f) \times L^2(\Gamma)$ embeds continuously into $\mathcal{Q}'$. Furthermore, $C(0,T;\mathcal{Q}')$ is a Borel measurable subset of $L^2(0,T;\mathcal{Q}')$, so $X \cap C(0,T;\mathcal{Q}')$ is measurable in X, as it is the preimage of $C(0,T;\mathcal{Q}') \subset L^2(0,T;\mathcal{Q}')$ under ι.

Proof. Fix $p > 4$ and set $\epsilon_k = 2^{-k}$. Then, by Lemma 8.5, there exists a decreasing sequence of positive real numbers $\{\delta_k\}_{k=1}^\infty$ and an increasing sequence of positive integers $\{N_k\}_{k=1}^\infty$, such that

$$\pi_{4,5}\mu_N(A_{p,\delta_k,\epsilon_k}) > 1 - \epsilon_k, \qquad \text{for all } N \geq N_k, \text{ and } k \in \mathbb{Z}^+.$$

Note that since μ_N converges weakly to μ, we have that $\pi_{4,5}\mu_N$ converges weakly to $\pi_{4,5}\mu$. For each fixed positive integer k, since $A_{p,\delta_k,\epsilon_k}$ is a closed set in X, we have by Portmanteau's theorem that $\pi_{4,5}\mu(A_{p,\delta_k,\epsilon_k}) \geq \limsup_{N\to\infty} \pi_{4,5}\mu_N(A_{p,\delta_k,\epsilon_k}) \geq 1 - \epsilon_k$. By the Borel Cantelli lemma, we conclude that $\pi_{4,5}\mu$ takes values in the set $\{A_{p,\delta_k,\epsilon_k}$occurs for infinitely many $k\}$ almost surely. However, one can show that

$$\{A_{p,\delta_k,\epsilon_k} \text{ occurs for infinitely many } k\} \subset X \cap C(0,T;\mathcal{Q}'),$$

which then implies the result. This inclusion follows from the fact that

$$||(\boldsymbol{u}(t), v(t)) - (\boldsymbol{u}(t_0), v(t_0))||_{\mathcal{Q}'} \leq \epsilon_k^{1/p}, \qquad \text{for all } t \in (t_0 - \delta_k, t_0 + \delta_k) \cap [0,T],$$

for all $t_0 \in [0,T]$, for any k such that $(\boldsymbol{u}, v) \in A_{p,\delta_k,\epsilon_k}$, and the fact that $\epsilon_k = 2^{-k} \to 0$ as $k \to \infty$. Therefore, we have shown the first part of the lemma, that $\pi_{4,5}\mu(X \cap C(0,T;\mathcal{Q}')) = 1$.

It remains to show that $\pi_{4,5}\mu$ is supported more specifically on a Borel measurable subset of X that consists entirely of functions that have a version that is in $C(0,T;\mathcal{Q}')$ with value $(\boldsymbol{u}_0, v_0)$ at time $t=0$. Define the set B_R to be the set of functions $(\boldsymbol{u}, v) \in L^2(0,T; L^2(\Omega_f) \times L^2(\Gamma))$ such that

$$||(\boldsymbol{u}(\cdot), v(\cdot)) - (\boldsymbol{u}_0, v_0)||_{L^2(0,h;\mathcal{Q}')} \leq C_R h^{3/4}, \qquad \text{for all } 0 < h \leq T, \tag{50}$$

where C_R is the constant from the estimate (36).

One can check that for every $R > 0$, every element of $\mathcal{K}_R$ satisfies (50) and hence is in B_R. This is because by using (36), we have that for all $0 < h \leq T$ and for all $(\boldsymbol{u}, v) \in \mathcal{K}_R$,

$$||(\boldsymbol{u}(\cdot), v(\cdot)) - (\boldsymbol{u}_0, v_0)||^2_{L^2(0,h;\mathcal{Q}')} \leq \int_0^h ||(\boldsymbol{u}(s), v(s)) - (\boldsymbol{u}_0, v_0)||^2_{\mathcal{Q}'} ds \leq C_R^2 h \cdot \left(h^{1/4}\right)^2 = C_R^2 h^{3/2}.$$

Furthermore, one checks easily that B_R is closed in $L^2(0,T; L^2(\Omega_f) \times L^2(\Gamma))$ since a sequence that converges in $L^2(0,T; L^2(\Omega_f) \times L^2(\Gamma))$ also converges in $L^2(0,T;\mathcal{Q}')$, in which case one can take the limit in (50) to get the corresponding property for the limit function. Since $\mathcal{K}_R \subset B_R$ and B_R is closed in X, we obtain that

$$\overline{\mathcal{K}_R} \subset B_R \subset X, \qquad \text{for all } R > 0.$$

Consider any $\epsilon > 0$. Choose R sufficiently large so that $\pi_{4,5}\mu_N(\overline{K_R}) > 1 - \epsilon$, for all N. Then, by Portmanteau's theorem, $\pi_{4,5}\mu(B_R) \geq \limsup_{N\to\infty} \pi_{4,5}\mu_N(B_R) \geq \limsup_{n\to\infty} \pi_{4,5}\mu_N(\overline{K_R}) \geq 1 - \epsilon$. So there exists an increasing sequence $\{R_k\}_{k=1}^\infty$ such that $\pi_{4,5}\mu\left(\bigcup_{k=1}^\infty B_{R_k}\right) = 1$. Thus,

$$\pi_{4,5}\mu\left[\left(\bigcup_{k=1}^\infty B_{R_k}\right) \cap C(0,T;\mathcal{Q}')\right] = 1, \quad \text{where} \quad \left(\bigcup_{k=1}^\infty B_{R_k}\right) \cap C(0,T;\mathcal{Q}') = \left(\bigcup_{k=1}^\infty B_{R_k}\right) \cap X \cap C(0,T;\mathcal{Q}')$$

is a Borel measurable subset of X. However, we note that any function in $(\bigcup_{k=1}^\infty B_{R_k}) \cap C(0,T;\mathcal{Q}')$ must have the property that its (unique) continuous version taking values in $\mathcal{Q}'$ must be equal to $(\boldsymbol{u}_0, v_0)$ at $t = 0$. To see this, if instead, $(\boldsymbol{u}(0), v(0)) \neq (\boldsymbol{u}_0, v_0)$, let $d = ||(\boldsymbol{u}(0), v(0)) - (\boldsymbol{u}_0, v_0)||_{\mathcal{Q}'} > 0$. Then, one can show that there exists h_0 such that for all $h \in (0, h_0]$, $||(\boldsymbol{u}(\cdot), v(\cdot)) - (\boldsymbol{u}_0, v_0)||_{L^2(0,h;\mathcal{Q}')} \geq \dfrac{d}{2}h^{1/2}$. Therefore, this function cannot satisfy an estimate $||(\boldsymbol{u}(\cdot), v(\cdot)) - (\boldsymbol{u}_0, v_0)||_{L^2(0,h;\mathcal{Q}')} \leq Ch^{3/4}$, for all $h \in (0, h_0]$, for any C, and so this function cannot be in any B_R. This completes the proof. $\square$

8.3 Skorohod representation theorem

We now use the classical Skorohod representation theorem to translate weak convergence of probability measures to almost sure convergence of random variables, which will allow us to pass to the limit in the semidiscrete weak formulation. However, this will be at the expense of working on a different probability space. Namely, the Skorohod representation theorem provides the *existence* of *a probability space*, on which we will have almost sure convergence of new random variables with the same laws as the original approximate solutions, to a weak solution with the law μ from Theorem 8.1. This probability space is not necessarily the same as the original probability space on which our problem is posed. Nevertheless, we can get back to the original probability space by using another result, known as the Gyöngy-Krylov lemma, see Section 9.2, to show that along a subsequence, the original approximate solutions on the original probability space converge almost surely to a limit with the same law μ from Theorem 8.1.

More precisely, showing convergence of our approximate solutions almost surely to a weak solution on the original probability space, consists of two steps. First, we use the Skorohod representation theorem to show that *there exists* a probability space, which we denote by "tilde", on which a sequence of random variables that are equal to our approximate solutions *in law* converges almost surely in $\mathcal{X}$ as $N \to \infty$, to a weak solution on the "tilde" probability space, where the law of this weak solution is equal to μ, obtained in Theorem 8.1. Thus, in this step, we prove the existence of a weak solution in a probabilistically *weak* sense, see Definition 4.1.

Then, in step two, we show using the Gyöngy-Krylov lemma, that we can bring that weak solution back to the original probability space, implying that we will have constructed a weak solution in a probabilistically strong sense, see Definition 4.2, of the original continuous problem. This will complete the existence proof, which is the main result of this manuscript.

To achieve these goals, we first obtain almost sure convergence along a subsequence of approximate solutions on a "tilde" probability space using Skorohod's theorem. A statement of the Skorohod representation theorem, which holds for probability measures on complete separable metric spaces, can be found in Proposition 6.2 in [41].

Before we state the result, we introduce the notation "$=_d$" to denote random variables that are "equal in distribution" i.e., the random variables have the same laws as random variables taking values on the same given phase space $\mathcal{X}$. Namely, we will say that a random variable X is equal in distribution (or equal in law) to the random variable $\tilde{X}$, and denote

$$X =_d \tilde{X} \quad \text{if} \quad \mu_X = \mu_{\tilde{X}},$$

where μ_X for example is the probability measure on $\mathcal{X}$ describing the law of the random variable X on $\mathcal{X}$.

Recall again the definition of the laws corresponding to the approximate solutions (33), and the definition of the corresponding phase space (34).

Lemma 8.7. Let μ denote the probability measure obtained as a weak limit of the measures μ_N from Theorem 8.1. Then, there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and $\mathcal{X}$-valued random variables on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$:

$$(\tilde{\eta}, \tilde{\overline{\eta}}, \tilde{\eta}^{\Delta t}, \tilde{\boldsymbol{u}}, \tilde{v}, \tilde{\boldsymbol{u}}^*, \tilde{v}^*, \tilde{\overline{\boldsymbol{u}}}, \tilde{\overline{v}}, \tilde{\boldsymbol{u}}^{\Delta t}, \tilde{v}^{\Delta t}, \tilde{W}), \text{ and } (\tilde{\eta}_N, \tilde{\overline{\eta}}_N, \tilde{\eta}_N^{\Delta t}, \tilde{\boldsymbol{u}}_N, \tilde{v}_N, \tilde{\boldsymbol{u}}_N^*, \tilde{v}_N^*, \tilde{\overline{\boldsymbol{u}}}_N, \tilde{\overline{v}}_N, \tilde{\boldsymbol{u}}_N^{\Delta t}, \tilde{v}_N^{\Delta t}, \tilde{W}_N), \text{ for each } N,$$

such that

$$(\tilde{\eta}_N, \tilde{\overline{\eta}}_N, \tilde{\eta}_N^{\Delta t}, \tilde{\boldsymbol{u}}_N, \tilde{v}_N, \tilde{\boldsymbol{u}}_N^*, \tilde{v}_N^*, \tilde{\overline{\boldsymbol{u}}}_N, \tilde{\overline{v}}_N, \tilde{\boldsymbol{u}}_N^{\Delta t}, \tilde{v}_N^{\Delta t}, \tilde{W}_N) =_d (\eta_N, \overline{\eta}_N, \eta_N^{\Delta t}, \boldsymbol{u}_N, v_N, \boldsymbol{u}_N, v_N^*, \overline{\boldsymbol{u}}_N, \overline{v}_N, \boldsymbol{u}_N^{\Delta t}, v_N^{\Delta t}, W),$$

for all N, and

$$(\tilde{\eta}_N, \tilde{\overline{\eta}}_N, \tilde{\eta}_N^{\Delta t}, \tilde{\boldsymbol{u}}_N, \tilde{v}_N, \tilde{\boldsymbol{u}}_N^*, \tilde{v}_N^*, \tilde{\overline{\boldsymbol{u}}}_N, \tilde{\overline{v}}_N, \tilde{\boldsymbol{u}}_N^{\Delta t}, \tilde{v}_N^{\Delta t}, \tilde{W}_N) \to (\tilde{\eta}, \tilde{\overline{\eta}}, \tilde{\eta}^{\Delta t}, \tilde{\boldsymbol{u}}, \tilde{v}, \tilde{\boldsymbol{u}}^*, \tilde{v}^*, \tilde{\overline{\boldsymbol{u}}}, \tilde{\overline{v}}, \tilde{\boldsymbol{u}}^{\Delta t}, \tilde{v}^{\Delta t}, \tilde{W}), \tag{51}$$

a.s. in $\mathcal{X}$, as $N \to \infty$, where the law of $(\tilde{\eta}, \tilde{\overline{\eta}}, \tilde{\eta}^{\Delta t}, \tilde{\boldsymbol{u}}, \tilde{v}, \tilde{\boldsymbol{u}}^*, \tilde{v}^*, \tilde{\overline{\boldsymbol{u}}}, \tilde{\overline{v}}, \tilde{\boldsymbol{u}}^{\Delta t}, \tilde{v}^{\Delta t}, \tilde{W})$ is equal to μ.

Furthermore, the following properties hold:

1. $\tilde{\boldsymbol{u}}_N = \tilde{\boldsymbol{u}}_N^*$, $\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}}^* = \tilde{\overline{\boldsymbol{u}}} = \tilde{\boldsymbol{u}}^{\Delta t}$ almost surely, $\tilde{v} = \tilde{v}^* = \tilde{\overline{v}} = \tilde{v}^{\Delta t}$ almost surely, and $\tilde{\eta} = \tilde{\overline{\eta}} = \tilde{\eta}^{\Delta t}$ almost surely.

2. $\tilde{\eta} \in L^2(\tilde{\Omega}; W^{1,\infty}(0,T; L^2(\Gamma)) \cap L^\infty(0,T; H_0^1(\Gamma)))$, $\tilde{\boldsymbol{u}} \in L^2(\tilde{\Omega}; L^2(0,T; H^1(\Omega_f)) \cap L^\infty(0,T; L^2(\Omega_f)))$, and $\tilde{v} \in L^2(\tilde{\Omega}; L^\infty(0,T; L^2(\Gamma)))$.

3. $\tilde{\eta}(0) = \eta_0$ almost surely.

4. $\partial_t \tilde{\eta} = \tilde{v}$ almost surely.

5. $(\tilde{\boldsymbol{u}}, \tilde{v}) \in C(0,T; \mathcal{Q}')$ and $(\tilde{\boldsymbol{u}}, \tilde{\eta}) \in \mathcal{W}(0,T)$ almost surely.

6. Define the filtration
$$\tilde{\mathcal{F}}_t = \sigma(\tilde{\eta}(s), \tilde{\boldsymbol{u}}(s), \tilde{v}(s) : 0 \le s \le t). \tag{52}$$
Then $\tilde{W}$ is a Brownian motion with respect to $\tilde{\mathcal{F}}_t$.

7. $(\tilde{\boldsymbol{u}}, \tilde{\eta}, \tilde{v})$ is a predictable process with respect to the filtration $\{\tilde{\mathcal{F}}_t\}_{0 \le t \le T}$.

Proof. The existence of the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and the given random variables follows from the previous result on weak convergence in Theorem 8.1 and the Skorohod representation theorem. So it suffices to prove the given properties.

Property 1: Because $(\tilde{\boldsymbol{u}}_N, \tilde{\boldsymbol{u}}_N^*) =_d (\boldsymbol{u}_N, \boldsymbol{u}_N)$, we have that $\tilde{\boldsymbol{u}}_N - \tilde{\boldsymbol{u}}_N^* =_d 0$ as random variables taking values in $L^2(0,T;L^2(\Omega_f))$, so $\tilde{\boldsymbol{u}}_N = \tilde{\boldsymbol{u}}_N^*$ a.s. for all N. Hence, by taking the limit as $N \to \infty$, we obtain $\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}}^*$ a.s., since $\tilde{\boldsymbol{u}}_N \to \tilde{\boldsymbol{u}}$ and $\tilde{\boldsymbol{u}}_N^* \to \tilde{\boldsymbol{u}}^*$ in $L^2(0,T;L^2(\Omega_f))$ a.s.

Because $\boldsymbol{u}_N$ and $\overline{\boldsymbol{u}}_N$ actually have different laws from each other, we must use a different argument to conclude that $\tilde{\boldsymbol{u}} = \tilde{\overline{\boldsymbol{u}}}$ a.s. However, we recall the following fact (40) from the proof of Lemma 8.3,

$$\mathbb{E}\left(||\boldsymbol{u}_N - \overline{\boldsymbol{u}}_N||^2_{L^2(0,T;L^2(\Omega_f))}\right) \to 0, \qquad \text{as } N \to \infty.$$

Hence, by the equivalence of laws,

$$\tilde{\mathbb{E}}\left(||\tilde{\boldsymbol{u}}_N - \tilde{\overline{\boldsymbol{u}}}_N||^2_{L^2(0,T;L^2(\Omega_f))}\right) \to 0, \qquad \text{as } N \to \infty.$$

Therefore, along a further subsequence, $||\tilde{\boldsymbol{u}}_N - \tilde{\overline{\boldsymbol{u}}}_N||^2_{L^2(0,T;L^2(\Omega_f))} \to 0$ almost surely, by a standard Borel Cantelli lemma argument. Since $\tilde{\boldsymbol{u}}_N \to \tilde{\boldsymbol{u}}$ and $\tilde{\overline{\boldsymbol{u}}}_N \to \tilde{\overline{\boldsymbol{u}}}$ in $L^2(0,T;L^2(\Omega_f))$, we conclude that $\tilde{\boldsymbol{u}} = \tilde{\overline{\boldsymbol{u}}}$ a.s.

The remaining statements follow from the same argument as above. In particular, by using the estimates (37)–(43) from the proof of Lemma 8.3, the equivalence of laws, and the almost sure convergence of the "tilde" random variables in (51), we obtain the desired result.

Property 2: These properties will all be handled similarly. By the uniform energy estimates in Lemma 7.2 and Lemma 7.3, we have that

$$\mathbb{E}\left(||\overline{\eta}_N||^2_{W^{1,\infty}(0,T;L^2(\Gamma))}\right) \le C, \ \mathbb{E}\left(||\overline{\eta}_N||^2_{L^\infty(0,T;H_0^1(\Gamma))}\right) \le C,$$

$$\mathbb{E}\left(||\boldsymbol{u}_N^{\Delta t}||^2_{L^2(0,T;H^1(\Omega_f))}\right) \le C, \ \mathbb{E}\left(||\boldsymbol{u}_N||^2_{L^\infty(0,T;L^2(\Omega_f))}\right) \le C, \ \mathbb{E}\left(||v_N||^2_{L^\infty(0,T;L^2(\Gamma))}\right) \le C,$$

for a constant C that is independent of N. Therefore, by the equivalence of laws, we have that these uniform estimates hold for the random variables on the new probability space, so that

$$\tilde{\mathbb{E}}\left(||\tilde{\overline{\eta}}_N||^2_{W^{1,\infty}(0,T;L^2(\Gamma))}\right) \le C, \ \tilde{\mathbb{E}}\left(||\tilde{\overline{\eta}}_N||^2_{L^\infty(0,T;H_0^1(\Gamma))}\right) \le C,$$

$$\tilde{\mathbb{E}}\left(||\tilde{\boldsymbol{u}}_N^{\Delta t}||^2_{L^2(0,T;H^1(\Omega_f))}\right) \le C, \ \tilde{\mathbb{E}}\left(||\tilde{\boldsymbol{u}}_N||^2_{L^\infty(0,T;L^2(\Omega_f))}\right) \le C, \ \tilde{\mathbb{E}}\left(||\tilde{v}_N||^2_{L^\infty(0,T;L^2(\Gamma))}\right) \le C,$$

for a constant C that is independent of N. Therefore, by this uniform boundedness, we conclude for example that $\tilde{\overline{\eta}}_N$ converges weakly star in $L^2(\tilde{\Omega}; W^{1,\infty}(0,T;L^2(\Gamma)))$ and weakly star in $L^2(\tilde{\Omega}; L^\infty(0,T;H_0^1(\Gamma)))$. Since we already have that $\tilde{\overline{\eta}}_N$ converges to $\tilde{\overline{\eta}}$ almost surely in $L^2(0,T;L^2(\Gamma))$ and $\tilde{\overline{\eta}} = \tilde{\eta}$ almost surely by Property 1, by the uniqueness of this limit, we conclude that $\tilde{\overline{\eta}}_N \rightharpoonup \tilde{\eta}$, weakly star in $L^2(\tilde{\Omega}; W^{1,\infty}(0,T;L^2(\Gamma)))$ and $L^2(\tilde{\Omega}; L^\infty(0,T;H_0^1(\Gamma)))$.

Similarly, $\tilde{\boldsymbol{u}}_N^{\Delta t} \rightharpoonup \tilde{\boldsymbol{u}}^{\Delta t}$, weakly $L^2(\tilde{\Omega}; L^2(0,T;H^1(\Omega_f)))$, $\tilde{\boldsymbol{u}}_N \rightharpoonup \tilde{\boldsymbol{u}}$ weakly star in $L^2(\tilde{\Omega}; L^\infty(0,T;L^2(\Omega_f)))$, and $\tilde{v}_N \rightharpoonup \tilde{v}$ weakly star in $L^2(\tilde{\Omega}; L^\infty(0,T;L^2(\Gamma)))$. This establishes Property 2.

Property 3: Since $\tilde{\eta} = \tilde{\overline{\eta}}$ almost surely, it suffices to show that $\tilde{\overline{\eta}}(0) = \eta_0$ almost surely. To do this, we use a method similar to the method in the proof of Lemma 8.6. We define

$$D_M = \{\eta \in L^2(0,T;L^2(\Gamma)) : ||\eta(\cdot) - \eta_0||_{L^2(0,h,L^2(\Gamma))} \le Mh^{3/2}, \text{ for all } 0 < h \le T\}. \tag{53}$$

Because of the uniform bound $\mathbb{E}\left(||\overline{\eta}_N||^2_{W^{1,\infty}(0,T;L^2(\Gamma))}\right) \le C$ for all N, from Lemma 7.3, we have that

$$\mathbb{P}(\overline{\eta}_N \in D_M) \ge 1 - \frac{C}{M^2} \qquad \text{for all } M \text{ and } N,$$

by using Chebychev's inequality. This is because if $||\overline{\eta}_N||_{W^{1,\infty}(0,T;L^2(\Gamma))} \le M$, then from the fact that $\overline{\eta}_N(0) = \eta_0$ for all $\omega \in \Omega$ and N, we have that

$$||\overline{\eta}(\cdot) - \eta_0||_{L^2(0,h,L^2(\Gamma))} = \left(\int_0^h ||\overline{\eta}(s) - \eta_0||^2_{L^2(\Gamma)} ds\right)^{1/2} \le \left(\int_0^h (Ms)^2 ds\right)^{1/2} \le Mh^{3/2}.$$

Then, by equivalence of laws,

$$\tilde{\mathbb{P}}(\tilde{\overline{\eta}}_N \in D_M) \ge 1 - \frac{C}{M^2} \qquad \text{for all } M \text{ and } N.$$

Because D_M is a closed set in $L^2(0,T;L^2(\Gamma))$ and $\tilde{\overline{\eta}}_N \to \tilde{\overline{\eta}}$ in $L^2(0,T;L^2(\Gamma))$ a.s., we conclude that

$$\tilde{\mathbb{P}}(\tilde{\overline{\eta}} \in D_M) \ge \limsup_{N\to\infty} \tilde{\mathbb{P}}(\tilde{\overline{\eta}}_N \in D_M) \ge 1 - \frac{C}{M^2} \ \text{ for all } M, \text{ which implies } \tilde{\mathbb{P}}\left(\tilde{\overline{\eta}} \in \bigcup_{M=1}^{\infty} D_M\right) = 1.$$

Because $\tilde{\overline{\eta}}$ is almost surely continuous on $[0,T]$ taking values in $L^2(\Gamma)$ by Property 2, we obtain $\tilde{\overline{\eta}}(0) = \eta_0$ almost surely. This is because if a continuous function η on $[0,T]$ taking values in $L^2(\Gamma)$ has $\eta(0) \ne \eta_0$, then

$$||\eta(\cdot) - \eta_0||_{L^2(0,h;L^2(\Gamma))} \ge \frac{d}{2} h^{1/2},$$

for all h sufficiently small where $d = ||\eta(0) - \eta_0||_{L^2(\Gamma)}$, and hence η cannot belong to $\bigcup_{M=1}^{\infty} D_M$.

Property 4: To prove this property, we recall from the second equation in the semidiscrete formulation (25) that

$$\int_\Gamma \frac{\eta_N^{n+1} - \eta_N^n}{\Delta t} \phi dz = \int_\Gamma v_N^{n+\frac{1}{3}} \phi dz,$$

almost surely for all $\phi \in L^2(\Gamma)$. Integrating in time, we obtain for all N that

$$\int_0^T \int_\Gamma \partial_t \overline{\eta}_N \phi dz dt = \int_0^T \int_\Gamma v_N^* \phi dz dt, \qquad \text{for all } \phi \in C^1([0,T); L^2(\Gamma)),$$

almost surely. Because each $\overline{\eta}_N$ is almost surely a piecewise linear continuous function satisfying $\overline{\eta}(0) = \eta_0$, we obtain by integration by parts that almost surely, for all $\phi \in C^1([0,T); L^2(\Gamma))$,

$$-\eta_0 \cdot \phi(0) - \int_0^T \int_\Gamma \overline{\eta}_N \partial_t \phi dz dt = \int_0^T \int_\Gamma v_N^* \phi dz dt,$$

and hence, by equivalence of laws,

$$-\eta_0 \cdot \phi(0) - \int_0^T \int_\Gamma \tilde{\overline{\eta}}_N \partial_t \phi dz dt = \int_0^T \int_\Gamma \tilde{v}_N^* \phi dz dt.$$

Passing to the limit, we obtain

$$-\eta_0 \cdot \phi(0) - \int_0^T \int_\Gamma \tilde{\eta} \partial_t \phi dz dt = \int_0^T \int_\Gamma \tilde{v} \phi dz dt,$$

for all $\phi \in C^1([0,T); L^2(\Gamma))$, almost surely. This implies that $\partial_t \tilde{\eta} = \tilde{v}$ holds almost surely for the limiting solution, since we showed in Property 3 that $\tilde{\eta}(0) = \eta_0$ almost surely.

Property 5: The fact that $(\tilde{\boldsymbol{u}}, \tilde{v}) \in C(0,T;\mathcal{Q}')$ almost surely follows from Lemma 8.6, since the limiting random variables with the tildes have their law given by the probability measure μ. So it remains to show that $(\tilde{\boldsymbol{u}}, \tilde{v}) \in \mathcal{W}(0,T)$, where $\mathcal{W}(0,T)$ is defined in (12).

To establish this result, first notice that we already know from Property 2 that $\tilde{\boldsymbol{u}} \in L^2(\tilde{\Omega}; L^\infty(0,T; L^2(\Omega_f)))$ and $\tilde{\boldsymbol{u}} \in L^2(\tilde{\Omega}; L^2(0,T; H^1(\Omega_f)))$, and Property 2 already gives the desired result for the structure. Thus, it remains to show that $\tilde{\boldsymbol{u}} \in L^2(0,T;\mathcal{V}_F)$ almost surely, where $\mathcal{V}_F$ is defined in (8), and that the kinematic coupling condition holds. By Property 4, we must show in particular that $\tilde{\boldsymbol{u}} = \tilde{v}\boldsymbol{e}_r$ a.s. on Γ.

To do this, define the deterministic function space

$$\mathcal{H} = \{(\boldsymbol{u}, v) \in L^2(0,T;\mathcal{V}_F) \times L^2(0,T;L^2(\Gamma)) : \boldsymbol{u} = v\boldsymbol{e_r} \text{ for almost every } t \in [0,T]\}.$$

One can check that the linear subspace $\mathcal{H} \subset L^2(0,T;H^1(\Omega_f)) \times L^2(0,T;L^2(\Gamma))$ is closed in the Hilbert space $L^2(0,T;H^1(\Omega_f)) \times L^2(0,T;L^2(\Gamma))$, and hence $\mathcal{H}$ is a Hilbert space with the inner product of $L^2(0,T;H^1(\Omega_f)) \times L^2(0,T;L^2(\Gamma))$. By equivalence of laws and the uniform boundedness in Lemma 7.2, $(\tilde{\boldsymbol{u}}_N, \tilde{v}_N)$ is uniformly bounded in $L^2(\tilde{\Omega};\mathcal{H})$, and hence converges weakly to $(\tilde{\boldsymbol{u}}, \tilde{v}) \in L^2(\tilde{\Omega};\mathcal{H})$ by uniqueness of the limit, since we already have that $(\tilde{\boldsymbol{u}}_N, \tilde{v}_N)$ converges almost surely to $(\tilde{\boldsymbol{u}}, \tilde{v})$ in $L^2(0,T;L^2(\Omega_f)) \times L^2(0,T;L^2(\Gamma))$. This gives the desired result.

Property 6: First, we sketch the idea. By construction, we have that on the original probability space, $W(t) - W(s)$ is independent of $\sigma(\boldsymbol{u}_N(\tau), v_N(\tau), \eta_N(\tau),$ for $0 \le \tau \le s)$, where we recall that these processes $(\boldsymbol{u}_N(\tau), v_N(\tau), \eta_N(\tau))$ are piecewise constant on intervals of length $\Delta t = T/N$. This is because for a given time $\tau \in [0,T]$, $(\boldsymbol{u}_N(\tau), v_N(\tau), \eta_N(\tau))$ depends only on the values of the Brownian motion at time $\lfloor \frac{\tau}{\Delta t} \rfloor \Delta t$ or earlier, from which the claim follows by the independent increments property of Brownian motion. The idea will be to transfer this independence property over to the new random variables $(\tilde{\boldsymbol{u}}_N, \tilde{v}_N, \tilde{\eta}_N)$ on the new probability space $(\tilde{\Omega}, \tilde{F}, \tilde{\mathbb{P}})$ and then take a limit as $N \to \infty$ to get the desired independence in the limit.

Note that the definition of $\tilde{\mathcal{F}}_t$ as

$$\tilde{\mathcal{F}}_t = \sigma(\tilde{\boldsymbol{u}}(s), \tilde{v}(s), \tilde{\eta}(s), \text{ for } 0 \le s \le t)$$

makes sense, since by the above properties, $\tilde{\eta}$ and $(\tilde{\boldsymbol{u}}, \tilde{v})$ are continuous on $[0,T]$ in time, taking values in $L^2(\Gamma)$ and $\mathcal{Q}'$ respectively. So it makes sense to refer to values pointwise at specific times, for example as in $\tilde{\boldsymbol{u}}(\tau)$ for a given $\tau \in [0,T]$. However, it is not clear yet, for example, what $\tilde{\boldsymbol{u}}_N(\tau)$ would be, since a priori, we only know that $\tilde{\boldsymbol{u}}_N \in L^2(0,T;L^2(\Omega_f))$, and hence, each path of $\tilde{\boldsymbol{u}}_N$ is only defined up to a version for $t \in [0,T]$.

To handle this, define the set K_N of all functions in $L^2(0,T;L^2(\Omega_f))$ that have a version that is piecewise constant on the intervals of the form $[0,\Delta t]$ and $(n\Delta t, (n+1)\Delta t]$ for $1 \le n \le N-1$, where $\Delta t = T/N$. Note that K_N is a closed subset of $L^2(0,T;L^2(\Omega_f))$, so by equivalence of laws,

$$\tilde{\mathbb{P}}(\tilde{\boldsymbol{u}}_N \in K_N) = \mathbb{P}(\boldsymbol{u}_N \in K_N) = 1.$$

Therefore, $\tilde{\boldsymbol{u}}_N$ is almost surely piecewise constant on $[0,\Delta t]$ and $(n\Delta t, (n+1)\Delta t]$ for $1 \le n \le N-1$. The same argument shows that $\tilde{v}_N$ and $\tilde{\eta}_N$ also almost surely have versions that are piecewise constant on these same intervals, since v_N and η_N on the original probability space almost surely have this property too.

Therefore, for each N, up to taking a version of $\tilde{\boldsymbol{u}}_N$, $\tilde{v}_N$, and $\tilde{\eta}_N$, we can define random variables $\tilde{\boldsymbol{u}}_N^n$, $\tilde{v}_N^n$, and $\tilde{\eta}_N^n$ for $0 \le n \le N-1$, satisfying

$$\begin{aligned}
\tilde{\boldsymbol{u}}_N(t,\omega) = \tilde{\boldsymbol{u}}_N^0(\omega), \quad \text{if } 0 \le t \le \Delta t \text{ and } \tilde{\boldsymbol{u}}_N(t,\omega) = \tilde{\boldsymbol{u}}_N^n(\omega), \quad \text{if } n\Delta t < t \le (n+1)\Delta t, \\
\tilde{v}_N(t,\omega) = \tilde{v}_N^0(\omega), \quad \text{if } 0 \le t \le \Delta t \text{ and } \tilde{v}_N(t,\omega) = \tilde{v}_N^n(\omega), \quad \text{if } n\Delta t < t \le (n+1)\Delta t, \\
\tilde{\eta}_N(t,\omega) = \tilde{\eta}_N^0(\omega), \quad \text{if } 0 \le t \le \Delta t \text{ and } \tilde{\eta}_N(t,\omega) = \tilde{\eta}_N^n(\omega), \quad \text{if } n\Delta t < t \le (n+1)\Delta t.
\end{aligned}$$

Furthermore, by the equivalence of laws, the joint distribution of $\tilde{\boldsymbol{u}}_N^n, \tilde{v}_N^n, \tilde{\eta}_N^n$ for $0 \le n \le N-1$ is the same as that of $\boldsymbol{u}_N^n, v_N^n, \eta_N^n$ for $0 \le n \le N-1$. Therefore, we can now make sense of $\tilde{\boldsymbol{u}}_N(\tau)$ for example for any $\tau \in [0,T]$, by considering the piecewise constant versions of these stochastic processes as given above. When we refer to $\tilde{\boldsymbol{u}}_N$, $\tilde{v}_N$, and $\tilde{\eta}_N$, we will refer to the piecewise constant versions defined above.

We now show the desired independence. We consider $\tau_0 \in [0,s]$ and $0 \le s \le t$, and show that $\tilde{\boldsymbol{u}}(\tau_0)$ and $\tilde{W}(t) - \tilde{W}(s)$ are independent. The same argument will work for $v(\tau_0)$ and $\eta(\tau_0)$, so it suffices to show the independence of $\tilde{W}(t) - \tilde{W}(s)$ and $\tilde{\boldsymbol{u}}(\tau_0)$ for arbitrary $\tau_0 \in [0,s]$ and $0 \le s \le t$.

Recall that $\tilde{\boldsymbol{u}}_N \to \tilde{\boldsymbol{u}}$ almost surely in $L^2(0,T;L^2(\Omega_f))$. Define the set

$$E_{N,n} = \{(t,\omega) \in [0,T] \times \tilde{\Omega} : ||\tilde{\boldsymbol{u}}(t,\omega,\cdot) - \tilde{\boldsymbol{u}}_N(t,\omega,\cdot)||_{L^2(\Omega_f)} \ge 2^{-n}\}.$$

For each positive integer n, we can choose $N := N(n)$ sufficiently large such that $N(n) > N(n-1)$ for $n \geq 2$, and

$$(dt \times \tilde{\mathbb{P}})(E_{N(n),n}) \leq 2^{-n}. \tag{54}$$

To see this, one selects $N(n)$ sufficiently large so that

$$\tilde{\mathbb{P}}\left(||\tilde{u} - \tilde{u}_{N(n)}||_{L^2(0,T;L^2(\Omega_f))} \leq 2^{-2n}\right) \geq 1 - 2^{-2n},$$

and then apply Chebychev's inequality in time. Then, by applying the Borel Cantelli lemma to (54), we obtain that

$$\tilde{\boldsymbol{u}}_N(t,\omega,\cdot) \to \tilde{\boldsymbol{u}}(t,\omega,\cdot) \qquad \text{in } L^2(\Omega_f), \tag{55}$$

for all $(t,\omega) \in S \subset [0,T] \times \tilde{\Omega}$ for a set S satisfying $(dt \times \tilde{\mathbb{P}})(S) = T$, where we continue to denote the new subsequence $N(n)$ by N. Thus, $([0,T] \times \tilde{\Omega}) - S$ has measure zero with respect to the measure $(dt \times \tilde{\mathbb{P}})$.

Let $S_0 \subset [0,T]$ be the set of all $t \in [0,T]$ such that $\tilde{\mathbb{P}}((t,\omega) \in S) = 1$. By Fubini's theorem, S_0 is a measurable subset of $[0,T]$ for which $[0,T] - S_0$ has measure zero. Note that for each $t \in S_0$, $\tilde{\boldsymbol{u}}_N(t,\cdot) \to \tilde{\boldsymbol{u}}(t,\cdot)$ almost surely as random variables taking values in $L^2(\Omega_f)$.

So if $\tau_0 \in S_0$, we deduce the independence of $\tilde{\boldsymbol{u}}(\tau_0)$ and $\tilde{W}(t) - \tilde{W}(s)$ as follows. By the fact that $\boldsymbol{u}_N(\tau_0)$ and $W(t) - W(s)$ are independent, we have by equivalence of laws that

$$\tilde{\boldsymbol{u}}_N(\tau_0) \text{ and } \tilde{W}_N(t) - \tilde{W}_N(s) \text{ are independent.}$$

Here, N denotes the subsequence $N(n)$ we used to define S and S_0. However, since $\tau_0 \in S_0$, we have that $\tilde{\boldsymbol{u}}(\tau_0)$ is the almost sure limit of $\tilde{\boldsymbol{u}}_N(\tau_0)$, and furthermore, $\tilde{W}(t) - \tilde{W}(s)$ is the almost sure limit of $\tilde{W}_N(t) - \tilde{W}_N(s)$. So since the almost sure limits of independent random variables are independent, this gives the desired result.

If $\tau_0 \notin S_0$, since $[0,T] - S_0$ has measure zero in $[0,T]$, there exists a sequence $\tau_i \in S_0$ that converges to τ_0 as $i \to \infty$, where $\tau_i \in [0,s]$. Then, since $\tilde{\boldsymbol{u}}(\tau_i)$ and $\tilde{W}(t) - \tilde{W}(s)$ are independent for each i and since $\tilde{\boldsymbol{u}}(\tau_i) \to \tilde{\boldsymbol{u}}(\tau_0)$ almost surely by continuity, the result follows. (For the case of $\tau_0 = 0$, we recall from Lemma 8.6, that $(\tilde{\boldsymbol{u}}(0), \tilde{v}(0)) = (\boldsymbol{u}_0, v_0)$ almost surely.)

We use the equivalence of laws to verify the remaining properties of Brownian motion. In particular, we just need to show that $\tilde{W}(t) - \tilde{W}(s)$ is distributed as $N(0, t-s)$. By the equivalence of laws and the fact that W is originally a Brownian motion, $\tilde{W}_N(t) - \tilde{W}_N(s) =_d W(t) - W(s)$, so that $\tilde{W}_N(t) - \tilde{W}_N(s)$ is distributed as $N(0, t-s)$. Since $\tilde{W}_N \to \tilde{W}$ a.s. in $C(0,T;\mathbb{R})$, we obtain that $\tilde{W}_N(t) - \tilde{W}_N(s) \to \tilde{W}(t) - \tilde{W}(s)$ almost surely, so that $\tilde{W}(t) - \tilde{W}(s)$ is the almost sure limit of random variables distributed as $N(0, t-s)$. Thus, we conclude that $\tilde{W}(t) - \tilde{W}(s)$ must also be distributed as $N(0, t-s)$, which concludes the proof of Property 6.

Property 7: By the definition of $\tilde{\mathcal{F}}_t$, the process $(\tilde{\boldsymbol{u}}, \tilde{v}, \tilde{\eta})$ is adapted to $\tilde{\mathcal{F}}_t$. By Property 2, $\tilde{\eta}$ almost surely has continuous paths on $[0,T]$, taking values in $L^2(\Omega_f)$. By Property 5, $(\tilde{\boldsymbol{u}}, \tilde{v})$ almost surely has continuous paths on $[0,T]$, taking values in $\mathcal{Q}'$. Since a continuous adapted process is predictable (see Proposition 5.1 in Chapter IV of Revuz and Yor [55]), this establishes the desired property.

This completes the proof of Lemma 8.7. □

8.4 Passing to the limit

We now consider the approximate solutions defined as random variables on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, discussed in Lemma 8.7, and show that the almost sure limit obtained in Lemma 8.7, satisfies the weak formulation stated in Definition 4.1, almost surely on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. For this purpose, we recall the semidiscrete formulation of the problem from (25), given by

$$\int_{\Omega_f} \frac{\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^n}{\Delta t} \cdot \boldsymbol{q} d\boldsymbol{x} + 2\mu \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}_N^{n+1}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} + \int_\Gamma \frac{v_N^{n+1} - v_N^n}{\Delta t} \psi dz + \int_\Gamma \nabla \eta_N^{n+1} \cdot \nabla \psi dz$$
$$= \int_\Gamma \frac{W((n+1)\Delta t) - W(n\Delta t)}{\Delta t} \psi dz + P_{N,in}^n \int_0^R (q_z)|_{z=0} dr - P_{N,out}^n \int_0^R (q_z)|_{z=L} dr, \ \forall (\boldsymbol{q}, \psi) \in \mathcal{Q},$$

$$\int_\Gamma \frac{\eta_N^{n+1}-\eta_N^n}{\Delta t}\phi dz = \int_\Gamma v_N^{n+\frac{1}{3}}\phi dz, \qquad \forall \phi \in L^2(\Gamma),$$

where $P_{N,in/out}^n = \frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} P_{in/out}(t)dt$. Notice that as stated, this semidiscrete formulation refers to the original variables, defined on the original probability space. Given a general $(\boldsymbol{q},\psi) \in \mathcal{Q}(0,T)$, we use the semidiscrete formulation at each fixed time and integrate in time from 0 to T to obtain for all $(\boldsymbol{q},\psi) \in \mathcal{Q}(0,T)$,

$$\int_0^T \int_{\Omega_f} \partial_t \overline{\boldsymbol{u}}_N \cdot \boldsymbol{q} d\boldsymbol{x} + 2\mu \int_0^T \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}_N^{\Delta t}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x}dt + \int_0^T \int_\Gamma \partial_t \overline{v}_N \psi dzdt$$
$$+ \int_0^T \int_\Gamma \nabla \eta_N^{\Delta t} \cdot \nabla \psi dzdt = \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_\Gamma \frac{W((n+1)\Delta t) - W(n\Delta t)}{\Delta t}\psi dzdt$$
$$+ \sum_{n=0}^{N-1}\left(\int_{n\Delta t}^{(n+1)\Delta t} P_{N,in}^n \int_0^R (q_z)|_{z=0} drdt - \int_{n\Delta t}^{(n+1)\Delta t} P_{N,out}^n \int_0^R (q_z)|_{z=L} drdt\right),$$

$$\int_0^T \int_\Gamma \partial_t \overline{\eta}_N \phi dzdt = \int_0^T \int_\Gamma v_N^* \phi dzdt, \quad \forall \phi \in C^1(0,T;L^2(\Gamma)),$$

where $\overline{\boldsymbol{u}}_N, \overline{v}_N$ and $\overline{\eta}_N$ are the piecewise linear approximations, given by (28) and (31), and $\boldsymbol{u}_N^{\Delta t}$ and $\eta_N^{\Delta t}$ are the piecewise constant time shifted functions, given by (30) and (32). Now, we convert to the new probability space by noticing that the same identities hold for the new random variables defined on the "tilde" probability space since the two sets of random variables have the same law on $\mathcal{X}$. So for all $(\boldsymbol{q},\psi) \in \mathcal{Q}(0,T)$, on the new probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with the filtration $\{\tilde{\mathcal{F}}_t\}_{t\geq 0}$ defined in (52), we obtain

$$\int_0^T \int_{\Omega_f} \partial_t \tilde{\overline{\boldsymbol{u}}}_N \cdot \boldsymbol{q} d\boldsymbol{x} + 2\mu \int_0^T \int_{\Omega_f} \boldsymbol{D}(\tilde{\boldsymbol{u}}_N^{\Delta t}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x}dt + \int_0^T \int_\Gamma \partial_t \tilde{\overline{v}}_N \psi dzdt$$
$$+ \int_0^T \int_\Gamma \nabla \tilde{\eta}_N^{\Delta t} \cdot \nabla \psi dzdt = \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_\Gamma \frac{\tilde{W}_N((n+1)\Delta t) - \tilde{W}_N(n\Delta t)}{\Delta t}\psi dzdt$$
$$+ \sum_{n=0}^{N-1}\left(\int_{n\Delta t}^{(n+1)\Delta t} P_{N,in}^n \int_0^R (q_z)|_{z=0} dr - \int_{n\Delta t}^{(n+1)\Delta t} P_{N,out}^n \int_0^R (q_z)|_{z=L} drdt\right),$$

$$\int_0^T \int_\Gamma \partial_t \tilde{\overline{\eta}}_N \phi dzdt = \int_0^T \int_\Gamma \tilde{v}_N^* \phi dzdt \quad \forall \phi \in C^1(0,T;L^2(\Gamma)).$$

We can now pass to the limit in all of the integrals, and use the almost sure convergence of the "tilde" random variables as follows.

First term: For the functions on the original probability space, note that because $\boldsymbol{q}(T) = 0$, we can integrate by parts to obtain

$$\int_0^T \int_{\Omega_f} \partial_t \overline{\boldsymbol{u}}_N \cdot \boldsymbol{q} d\boldsymbol{x}dt = -\int_0^T \int_{\Omega_f} \overline{\boldsymbol{u}}_N \cdot \partial_t \boldsymbol{q} d\boldsymbol{x}dt - \int_{\Omega_f} \boldsymbol{u}_0 \cdot \boldsymbol{q}(0) d\boldsymbol{x}. \tag{56}$$

By equivalence of laws, this identity also holds with $\tilde{\overline{\boldsymbol{u}}}_N$ in place of $\overline{\boldsymbol{u}}_N$. Then, because $\tilde{\overline{\boldsymbol{u}}}_N \to \tilde{\boldsymbol{u}}$ almost surely in $L^2(0,T;L^2(\Omega_f))$, we can pass to the limit to obtain the desired almost sure convergence,

$$\int_0^T \int_{\Omega_f} \partial_t \tilde{\overline{\boldsymbol{u}}}_N \cdot \boldsymbol{q} d\boldsymbol{x} \to -\int_0^T \int_{\Omega_f} \tilde{\boldsymbol{u}} \cdot \partial_t \boldsymbol{q} d\boldsymbol{x}dt - \int_{\Omega_f} \boldsymbol{u}_0 \cdot \boldsymbol{q}(0) d\boldsymbol{x}.$$

Third term: For the third term, we use an argument similar to that for the first term. Since $\psi(T) = 0$, we can integrate by parts,

$$\int_0^T \int_\Gamma \partial_t \overline{v}_N \psi dzdt = -\int_0^T \int_\Gamma \overline{v}_N \partial_t \psi dzdt - \int_\Gamma v_0 \psi(0) dz.$$

This holds with $\tilde{\overline{v}}_N$ in place of $\overline{v}_N$ too by equivalence of laws. Because $\tilde{\overline{v}}_N \to \tilde{v}$ in $L^2(0,T;L^2(\Gamma))$ almost surely, we have the desired almost sure convergence,

$$\int_0^T \int_\Gamma \partial_t \tilde{\overline{v}}_N \psi dz dt = -\int_0^T \int_\Gamma \tilde{\overline{v}}_N \partial_t \psi dz dt - \int_\Gamma v_0 \psi(0) dz \to -\int_0^T \int_\Gamma \tilde{v} \partial_t \psi dz dt - \int_\Gamma v_0 \psi(0) dz.$$

Second and fourth term with smooth test function: For the second and fourth term, we have to use an approximation argument, since we only have estimates of convergence of $\tilde{\boldsymbol{u}}_N$ and $\tilde{\boldsymbol{u}}_N^{\Delta t}$ in $L^2(0,T;L^2(\Omega_f))$ and $\tilde{v}_N$ in $L^2(0,T;L^2(\Gamma))$.

We will first show the desired convergence under the assumption that $(\boldsymbol{q},\psi) \in \mathcal{Q}(0,T)$ is spatially smooth at each time in $[0,T]$. Then, on the original probability space,

$$2\mu \int_0^T \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}_N^{\Delta t}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} dt = \mu \int_0^T \int_{\Omega_f} \nabla \boldsymbol{u}_N^{\Delta t} : \nabla \boldsymbol{q} d\boldsymbol{x} dt = -\mu \int_0^T \int_{\Omega_f} \boldsymbol{u}_N^{\Delta t} \cdot \Delta \boldsymbol{q} d\boldsymbol{x} dt,$$

where the last integration by parts has no boundary terms due to the properties of the solution space and test space for the fluid. Then, by the uniform dissipation estimate in Proposition 6.7, $\sum_{n=0}^{N-1} \mathbb{E}\left(||\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^n||^2_{L^2(\Omega_f)}\right) \leq C$, we have that

$$\mathbb{E}\left(||\boldsymbol{u}_N^{\Delta t} - \boldsymbol{u}_N||^2_{L^2(0,T;L^2(\Omega_f))}\right) \leq C(\Delta t) \to 0, \qquad \text{as } N \to \infty.$$

By equivalence of laws, the above identities and estimates hold for $\tilde{\boldsymbol{u}}_N^{\Delta t}$ in place of $\boldsymbol{u}_N^{\Delta t}$. By the Borel-Cantelli lemma, we have that

$$||\tilde{\boldsymbol{u}}_N^{\Delta t} - \tilde{\boldsymbol{u}}_N||_{L^2(0,T;L^2(\Omega_f))} \to 0 \text{ almost surely as } N \to \infty,$$

taking a subsequence if needed. Because $\tilde{\boldsymbol{u}}_N$ converges to $\tilde{\boldsymbol{u}}$ in $L^2(0,T;L^2(\Omega_f))$ as $N \to \infty$, we also have that

$$||\tilde{\boldsymbol{u}}_N^{\Delta t} - \tilde{\boldsymbol{u}}||_{L^2(0,T;L^2(\Omega_f))} \to 0 \text{ almost surely as } N \to \infty$$

along this subsequence, which allows us to pass to the limit to obtain

$$\begin{aligned} 2\mu \int_0^T \int_{\Omega_f} \boldsymbol{D}(\tilde{\boldsymbol{u}}_N^{\Delta t}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} dt &= -\mu \int_0^T \int_{\Omega_f} \tilde{\boldsymbol{u}}_N^{\Delta t} \cdot \Delta \boldsymbol{q} d\boldsymbol{x} dt \\ \to -\mu \int_0^T \int_{\Omega_f} \tilde{\boldsymbol{u}} \cdot \Delta \boldsymbol{q} d\boldsymbol{x} dt &= 2\mu \int_0^T \int_{\Omega_f} \boldsymbol{D}(\tilde{\boldsymbol{u}}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} dt. \end{aligned} \tag{57}$$

For the fourth term, one can use a similar argument under the assumption that the test function $(\boldsymbol{q},\psi)$ is spatially smooth. On the original probability space,

$$\int_0^T \int_\Gamma \nabla \eta_N^{\Delta t} \cdot \nabla \psi dz dt = -\int_0^T \int_\Gamma \eta_N^{\Delta t} \cdot \Delta \psi dz dt.$$

By the numerical dissipation estimate from Lemma 6.7, $\sum_{n=0}^{N-1} \mathbb{E}\left(||\nabla \eta_N^{n+\frac{1}{3}} - \nabla \eta_N^n||^2_{L^2(\Gamma)}\right) \leq C$, so we obtain, by Poincaré's inequality, that

$$\mathbb{E}\left(||\eta_N^{\Delta t} - \eta_N||^2_{L^2(0,T;L^2(\Gamma))}\right) \leq C(\Delta t) \to 0, \qquad \text{as } N \to \infty.$$

These estimates hold on the new probability space with $\tilde{\eta}_N$ in place of η_N. By the Borel-Cantelli lemma and the convergence of $\tilde{\eta}_N$ to $\tilde{\eta}$ in $L^2(0,T;L^2(\Gamma))$,

$$||\tilde{\eta}_N^{\Delta t} - \tilde{\eta}||_{L^2(0,T;L^2(\Gamma))} \to 0, \qquad \text{almost surely as } N \to \infty,$$

taking a subsequence. This allows us to pass to the limit to obtain the almost sure convergence,

$$\begin{aligned} \int_0^T \int_\Gamma \nabla \tilde{\eta}_N^{\Delta t} \cdot \nabla \psi dz dt &= -\int_0^T \int_\Gamma \tilde{\eta}_N^{\Delta t} \cdot \Delta \psi dz dt \\ \to -\int_0^T \int_\Gamma \tilde{\eta} \cdot \Delta \psi dz dt &= \int_0^T \int_\Gamma \nabla \tilde{\eta} \cdot \nabla \psi dz dt, \qquad \text{as } N \to \infty. \end{aligned} \tag{58}$$

Second and fourth term with general test function: To show the almost sure convergence in the previous step, we assumed that $(\boldsymbol{q}, \psi) \in \mathcal{Q}(0,T)$ was spatially smooth. To get the general convergence, we use an approximation argument. Suppose that $(\boldsymbol{q}, \psi) \in \mathcal{Q}(0,T)$ is not smooth spatially. It suffices to show that $\int_0^T \int_{\Omega_f} \boldsymbol{D}(\tilde{\boldsymbol{u}}_N^{\Delta t}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} dt \to \int_0^T \int_{\Omega_f} \boldsymbol{D}(\tilde{\boldsymbol{u}}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} dt$ in probability, and $\int_0^T \int_\Gamma \nabla \tilde{\eta}_N^{\Delta t} \cdot \nabla \psi dz dt \to \int_0^T \int_\Gamma \nabla \tilde{\eta} \cdot \nabla \psi dz dt$ in probability (see below for the precise definition), as we would get the desired result from the fact that we then have almost sure convergence along a subsequence. So given any $\epsilon > 0$ and $\delta > 0$, we must show that there exists N_0 such that for all $N \geq N_0$,

$$\tilde{\mathbb{P}}\left(\left|\int_0^T \int_{\Omega_f} \boldsymbol{D}(\tilde{\boldsymbol{u}}_N^{\Delta t}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} dt - \int_0^T \int_{\Omega_f} \boldsymbol{D}(\tilde{\boldsymbol{u}}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} dt\right| > \delta\right) \leq \epsilon, \tag{59}$$

$$\tilde{\mathbb{P}}\left(\left|\int_0^T \int_\Gamma \nabla \tilde{\eta}_N^{\Delta t} \cdot \nabla \psi dz dt - \int_0^T \int_\Gamma \nabla \tilde{\eta} \cdot \nabla \psi dz dt\right| > \delta\right) \leq \epsilon. \tag{60}$$

To show this, observe that by the uniform dissipation estimate in Proposition 6.7, we have that

$$\mathbb{E}\left(\sum_{n=0}^{N-1} (\Delta t) \int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{u}_N^{n+1})|^2 d\boldsymbol{x}\right) = \mathbb{E}\left(||\boldsymbol{D}(\boldsymbol{u}_N^{\Delta t})||^2_{L^2(0,T;L^2(\Omega_f))}\right) \leq C.$$

and hence by equivalence of laws,

$$\tilde{\mathbb{E}}\left(||\boldsymbol{D}(\tilde{\boldsymbol{u}}_N^{\Delta t})||^2_{L^2(0,T;L^2(\Omega_f))}\right) \leq C,$$

for a uniform constant C. Since $\tilde{\boldsymbol{u}} \in L^2(\Omega; L^2(0,T; H^1(\Omega_f)))$ by Property 2 of Lemma 8.7, we conclude that there exists a sufficiently large positive constant M such that for all N,

$$\tilde{\mathbb{P}}\left(||\boldsymbol{D}(\tilde{\boldsymbol{u}}_N^{\Delta t})||_{L^2(0,T;L^2(\Omega_f))} \geq M\right) \leq \frac{\epsilon}{3}, \qquad \tilde{\mathbb{P}}\left(||\boldsymbol{D}(\tilde{\boldsymbol{u}})||_{L^2(0,T;L^2(\Omega_f))} \geq M\right) \leq \frac{\epsilon}{3}. \tag{61}$$

For the fourth term involving structure displacements, recall from Lemma 6.7 that

$$\mathbb{E}\left(||\nabla \eta_N^{\Delta t}||^2_{L^\infty(0,T;L^2(\Gamma))}\right) \leq C,$$

and by Property 2 in Lemma 8.7, $\tilde{\eta} \in L^2(\tilde{\Omega}; L^\infty(0,T; H_0^1(\Gamma)))$. So using equivalence of laws, M can also be chosen sufficiently large so that for all N,

$$\tilde{\mathbb{P}}\left(||\nabla \tilde{\eta}_N^{\Delta t}||_{L^\infty(0,T;L^2(\Gamma))} \geq M\right) \leq \frac{\epsilon}{3}, \qquad \tilde{\mathbb{P}}\left(||\nabla \tilde{\eta}||_{L^\infty(0,T;L^2(\Gamma))} \geq M\right) \leq \frac{\epsilon}{3}. \tag{62}$$

Then, choose $(\widehat{\boldsymbol{q}}, \widehat{\psi}) \in \mathcal{Q}(0,T)$ that are smooth spatially at all times in $[0,T]$, such that

$$||\boldsymbol{D}(\boldsymbol{q}) - \boldsymbol{D}(\widehat{\boldsymbol{q}})||_{L^2(0,T;L^2(\Omega_f))} \leq \frac{\delta}{3M}, \qquad ||\nabla \psi - \nabla \widehat{\psi}||_{L^1(0,T;L^2(\Gamma))} \leq \frac{\delta}{3M}. \tag{63}$$

Then, the almost sure convergences (57) and (58), which hold for this smoother $(\widehat{\boldsymbol{q}}, \widehat{\psi})$, allow us to choose N_0 sufficiently large such that for all $N \geq N_0$,

$$\tilde{\mathbb{P}}\left(\left|\int_0^T \int_{\Omega_f} \boldsymbol{D}(\tilde{\boldsymbol{u}}_N^{\Delta t}) : \boldsymbol{D}(\widehat{\boldsymbol{q}}) d\boldsymbol{x} dt - \int_0^T \int_{\Omega_f} \boldsymbol{D}(\tilde{\boldsymbol{u}}) : \boldsymbol{D}(\widehat{\boldsymbol{q}}) d\boldsymbol{x} dt\right| > \frac{\delta}{3}\right) \leq \frac{\epsilon}{3}, \tag{64}$$

$$\tilde{\mathbb{P}}\left(\left|\int_0^T \int_\Gamma \nabla \tilde{\eta}_N^{\Delta t} \cdot \nabla \widehat{\psi} dz dt - \int_0^T \int_\Gamma \nabla \tilde{\eta} \cdot \nabla \widehat{\psi} dz dt\right| > \frac{\delta}{3}\right) \leq \frac{\epsilon}{3}. \tag{65}$$

Furthermore, the choice of $(\widehat{\boldsymbol{q}}, \widehat{\psi})$ in (63) and the choice of M in (61) and (62) give that for all N,

$$\tilde{\mathbb{P}}\left(\left|\int_0^T \int_{\Omega_f} \boldsymbol{D}(\tilde{\boldsymbol{u}}_N^{\Delta t}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} dt - \int_0^T \int_{\Omega_f} \boldsymbol{D}(\tilde{\boldsymbol{u}}_N^{\Delta t}) : \boldsymbol{D}(\widehat{\boldsymbol{q}}) d\boldsymbol{x} dt\right| > \frac{\delta}{3}\right) \leq \frac{\epsilon}{3}, \tag{66}$$

$$\tilde{\mathbb{P}}\left(\left|\int_0^T\int_\Gamma \nabla\tilde{\eta}_N^{\Delta t}\cdot\nabla\psi dzdt - \int_0^T\int_\Gamma \nabla\tilde{\eta}_N^{\Delta t}\cdot\nabla\widehat{\psi}dzdt\right| > \frac{\delta}{3}\right) \le \frac{\epsilon}{3}, \tag{67}$$

and

$$\tilde{\mathbb{P}}\left(\left|\int_0^T\int_{\Omega_f} \boldsymbol{D}(\tilde{\boldsymbol{u}}) : \boldsymbol{D}(\boldsymbol{q})d\boldsymbol{x}dt - \int_0^T\int_{\Omega_f} \boldsymbol{D}(\tilde{\boldsymbol{u}}) : \boldsymbol{D}(\widehat{\boldsymbol{q}})d\boldsymbol{x}dt\right| > \frac{\delta}{3}\right) \le \frac{\epsilon}{3}, \tag{68}$$

$$\tilde{\mathbb{P}}\left(\left|\int_0^T\int_\Gamma \nabla\tilde{\eta}\cdot\nabla\psi dzdt - \int_0^T\int_\Gamma \nabla\tilde{\eta}\cdot\nabla\widehat{\psi}dzdt\right| > \frac{\delta}{3}\right) \le \frac{\epsilon}{3}. \tag{69}$$

Combining the estimates (64), (65), (66), (67), (68), and (69) establishes the desired estimates (59) and (60), and hence proves the desired convergence in probability.

Passing to the limit in the stochastic integral. We want to pass to the limit in the stochastic integral and show that for arbitrary ψ such that $(\boldsymbol{q},\psi) \in \mathcal{Q}(0,T)$,

$$\sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t}\int_\Gamma \frac{\tilde{W}_N((n+1)\Delta t) - \tilde{W}_N(n\Delta t)}{\Delta t}\psi dzdt \to \int_0^T\left(\int_\Gamma \psi dz\right)d\tilde{W}, \qquad \text{a.s. as } N\to\infty.$$

Note that because ψ is deterministic, we can express the right hand side as a stochastic integral,

$$\begin{aligned}&\sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t}\int_\Gamma \frac{\tilde{W}_N((n+1)\Delta t) - \tilde{W}_N(n\Delta t)}{\Delta t}\psi dzdt\\ &= \int_0^T\sum_{n=0}^{N-1}\left(\frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)(\Delta t)}\int_\Gamma \psi(s,z)dzds\right)1_{t\in(n\Delta t,(n+1)\Delta t]}(t)d\tilde{W}_N(t).\end{aligned}$$

Because convergence in probability implies convergence almost surely along a subsequence, it thus suffices to prove that

$$\int_0^T\sum_{n=0}^{N-1}\left(\frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)(\Delta t)}\int_\Gamma \psi(s,z)dzds\right)1_{t\in(n\Delta t,(n+1)\Delta t]}(t)d\tilde{W}_N \to \int_0^T\left(\int_\Gamma \psi dz\right)d\tilde{W},$$

as $N\to\infty$ in probability. So we must show that given any $\delta > 0$ and any $\epsilon > 0$, there exists N_0 sufficiently large such that for all $N \ge N_0$

$$\tilde{\mathbb{P}}\left(\left|\int_0^T\sum_{n=0}^{N-1}\left(\frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)(\Delta t)}\int_\Gamma \psi(s,z)dzds\right)1_{t\in(n\Delta t,(n+1)\Delta t]}(t)d\tilde{W}_N - \int_0^T\left(\int_\Gamma \psi dz\right)d\tilde{W}\right| > \delta\right) < \epsilon.$$

We accomplish this through two estimates. We claim that we can choose N_0 sufficiently large such that

$$\tilde{\mathbb{P}}\left(\left|\int_0^T\sum_{n=0}^{N-1}\left(\frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)(\Delta t)}\int_\Gamma \psi(s,z)dzds\right)1_{t\in(n\Delta t,(n+1)\Delta t]}(t)d\tilde{W}_N - \int_0^T\left(\int_\Gamma \psi dz\right)d\tilde{W}_N\right| > \frac{\delta}{2}\right) < \frac{\epsilon}{2}, \tag{70}$$

and

$$\tilde{\mathbb{P}}\left(\left|\int_0^T\left(\int_\Gamma \psi dz\right)d\tilde{W}_N - \int_0^T\left(\int_\Gamma \psi dz\right)d\tilde{W}\right| > \frac{\delta}{2}\right) < \frac{\epsilon}{2}, \tag{71}$$

for all $N \ge N_0$.

For the first estimate (70), it suffices to use the It isometry along with the fact that

$$\left\|\sum_{n=0}^{N-1}\left(\frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)(\Delta t)}\int_\Gamma \psi(s,z)dzds\right)1_{t\in(n\Delta t,(n+1)\Delta t]}(t) - \int_\Gamma \psi dz\right\|_{L^2(0,T)} \to 0, \qquad \text{as } N\to\infty,$$

to conclude that

$$\tilde{\mathbb{E}}\left(\left|\int_0^T \sum_{n=0}^{N-1}\left(\frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)(\Delta t)}\int_\Gamma \psi(s,z)dzds\right)1_{t\in(n\Delta t,(n+1)\Delta t]}(t)d\tilde{W}_N - \int_0^T\left(\int_\Gamma \psi dz\right)d\tilde{W}_N\right|^2\right)\to 0,$$

as $N\to\infty$. The first estimate (70) thus follows from taking N_0 sufficiently large to make this expectation sufficiently small, and then using Chebychev's inequality.

For the second estimate, note that we can approximate $\int_\Gamma \psi(t,z)dz := g(t)$ by a *deterministic* step function

$$g_m(t) = g\left(\frac{kT}{m}\right) \qquad \text{if } \frac{kT}{m} < t \leq \frac{(k+1)T}{m}.$$

By the continuity of $g(t)$, we can select m sufficiently large such that

$$||g(t)-g_m(t)||^2_{L^2(0,T)} < \frac{\epsilon}{6}\cdot\left(\frac{\delta}{6}\right)^2.$$

Then, by the It isometry and Chebychev's inequality,

$$\tilde{\mathbb{P}}\left(\left|\int_0^T\left(\int_\Gamma \psi dz\right)d\tilde{W}_N - \int_0^T g_m(t)d\tilde{W}_N\right| > \frac{\delta}{6}\right) < \frac{\epsilon}{6}, \tag{72}$$

for all N, and

$$\tilde{\mathbb{P}}\left(\left|\int_0^T\left(\int_\Gamma \psi dz\right)d\tilde{W} - \int_0^T g_m(t)d\tilde{W}\right| > \frac{\delta}{6}\right) < \frac{\epsilon}{6}. \tag{73}$$

So it remains to choose N_0 sufficiently large such that for all $N\geq N_0$,

$$\tilde{\mathbb{P}}\left(\left|\int_0^T g_m(t)d\tilde{W}_N - \int_0^T g_m(t)d\tilde{W}\right| > \frac{\delta}{6}\right) < \frac{\epsilon}{6}. \tag{74}$$

We note that $|g_m(t)|\leq K$ for some constant K that is deterministic, as $g_m(t)$ is a deterministic function of time. Also, note that

$$\int_0^T g_m(t)d\tilde{W}_N = \sum_{k=0}^{m-1} g\left(\frac{kT}{m}\right)\cdot\left(\tilde{W}_N\left(\frac{(k+1)T}{m}\right) - \tilde{W}_N\left(\frac{kT}{m}\right)\right),$$

with an analogous formula for the integration against $\tilde{W}$. Hence,

$$\begin{aligned}
&\left|\int_0^T g_m(t)d\tilde{W}_N - \int_0^T g_m(t)d\tilde{W}\right| \\
&\leq \sum_{k=0}^{m-1}\left|g\left(\frac{kT}{m}\right)\cdot\left[\left(\tilde{W}_N\left(\frac{(k+1)T}{m}\right)-\tilde{W}_N\left(\frac{kT}{m}\right)\right)-\left(\tilde{W}\left(\frac{(k+1)T}{m}\right)-\tilde{W}\left(\frac{kT}{m}\right)\right)\right]\right| \\
&\leq \sum_{k=0}^{m-1} 2K||\tilde{W}-\tilde{W}_N||_{C(0,T;\mathbb{R})} \leq 2Km\cdot||\tilde{W}-\tilde{W}_N||_{C(0,T;\mathbb{R})}.
\end{aligned}$$

Because $\tilde{W}_N\to\tilde{W}$ in $C(0,T;\mathbb{R})$ almost surely, there exists N_0 sufficiently large such that

$$\tilde{\mathbb{P}}\left(||\tilde{W}-\tilde{W}_N||_{C(0,T;\mathbb{R})} > \frac{\delta}{12Km}\right) < \frac{\epsilon}{6}, \qquad \text{for all } N\geq N_0.$$

Therefore,

$$\tilde{\mathbb{P}}\left(\left|\int_0^T g_m(t)d\tilde{W}_N - \int_0^T g_m(t)d\tilde{W}\right| > \frac{\delta}{6}\right) < \frac{\epsilon}{6}, \qquad \text{for all } N\geq N_0.$$

The estimates (72), (73), and (74) thus imply the desired estimate in (71).

Convergence of the pressure term. Finally, we show that

$$\sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t} P_{N,in}^n\left(\int_0^R (q_z)|_{z=0}dr\right)dt \to \int_0^T P_{in}(t)\left(\int_0^R (q_z)|_{z=0}dr\right)dt, \qquad \text{as } N\to\infty. \tag{75}$$

The same argument will work for the outlet pressure term.

Define the following piecewise approximation of the test function $\boldsymbol{q}$,

$$\boldsymbol{q}^m(t,\cdot) = \boldsymbol{q}\left(\frac{kT}{m},\cdot\right), \qquad \text{if } \frac{kT}{m} < t \le \frac{(k+1)T}{m}.$$

For any positive integer N,

$$\begin{aligned}
\int_0^T P_{in}(t)\left(\int_0^R (q_z^N)|_{z=0}dr\right)dt &- \sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t} P_{N,in}^n\left(\int_0^R (q_z^N)|_{z=0}dr\right)dt \\
&= \sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t}(P_{in}(t)-P_{N,in}^n)\left(\int_0^R (q_z^N)|_{z=0}dr\right)dt \\
&= \sum_{n=0}^{N-1}\left(\int_0^R (q_z^N)|_{z=0}dr\right)\int_{n\Delta t}^{(n+1)\Delta t}(P_{in}(t)-P_{N,in}^n)dt = 0.
\end{aligned}$$

To establish (75), it suffices to show that

$$\int_0^T P_{in}(t)\left(\int_0^R (q_z)|_{z=0}dr\right)dt - \int_0^T P_{in}(t)\left(\int_0^R (q_z^N)|_{z=0}dr\right)dt \to 0, \qquad \text{as } N\to\infty, \tag{76}$$

$$\sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t} P_{N,in}^n\left(\int_0^R (q_z)|_{z=0}dr\right)dt - \sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t} P_{N,in}^n\left(\int_0^R (q_z^N)|_{z=0}dr\right)dt \to 0, \qquad \text{as } N\to\infty. \tag{77}$$

For (76), we compute

$$\begin{aligned}
&\left|\int_0^T P_{in}(t)\left(\int_0^R (q_z)|_{z=0}dr\right)dt - \int_0^T P_{in}(t)\left(\int_0^R (q_z^N)|_{z=0}dr\right)dt\right| \\
&= \left|\int_0^T P_{in}(t)\left(\int_0^R (q_z-q_z^N)|_{z=0}dr\right)dt\right| \le ||P_{in}||_{L^2(0,T)}\left(\int_0^T\left(\int_0^R (q_z-q_z^N)|_{z=0}dr\right)^2 dt\right)^{1/2} \\
&\le C||P_{in}||_{L^2(0,T)}\left(\int_0^T ||\boldsymbol{q}-\boldsymbol{q}^N||^2_{H^1(\Omega_f)}dt\right)^{1/2}.
\end{aligned} \tag{78}$$

Because $\boldsymbol{q}$ is continuous taking values in $\mathcal{V}_F$ equipped with the norm of $H^1(\Omega_f)$, we have that $||\boldsymbol{q}-\boldsymbol{q}^N||_{H^1(\Omega_f)} \to$

0 uniformly on $[0,T]$ as $N \to \infty$, which establishes the desired limit. Similary, to estabish (77) we calculate

$$\left|\sum_{n=0}^{N-1} P_{N,in}^n \int_{n\Delta t}^{(n+1)\Delta t} \left(\int_0^R (q_z - q_z^N)|_{z=0} dr\right) dt\right| \leq \left|\sum_{n=0}^{N-1} (\Delta t)^{1/2} P_{N,in}^n \left(\int_{n\Delta t}^{(n+1)\Delta t} \left(\int_0^R (q_z - q_z^N)|_{z=0} dr\right)^2 dt\right)^{1/2}\right|$$

$$\leq C \left|\sum_{n=0}^{N-1} (\Delta t)^{1/2} P_{N,in}^n \left(\int_{n\Delta t}^{(n+1)\Delta t} ||\boldsymbol{q} - \boldsymbol{q}^N||^2_{H^1(\Omega_f)} dt\right)^{1/2}\right|$$

$$\leq C \left(\sum_{n=0}^{N-1} (\Delta t)^{1/2} |P_{N,in}^n|\right) \cdot \max_{0 \leq n \leq N-1} \left(\int_{n\Delta t}^{(n+1)\Delta t} ||\boldsymbol{q} - \boldsymbol{q}^N||^2_{H^1(\Omega_f)} dt\right)^{1/2}$$

$$\leq C \left(\sum_{n=0}^{N-1} \frac{1}{(\Delta t)^{1/2}} \int_{n\Delta t}^{(n+1)\Delta t} |P_{in}(t)| dt\right) \cdot \max_{0 \leq n \leq N-1} \left(\int_{n\Delta t}^{(n+1)\Delta t} ||\boldsymbol{q} - \boldsymbol{q}^N||^2_{H^1(\Omega_f)} dt\right)^{1/2}$$

$$\leq C ||P_{in}||_{L^2(0,T)} \cdot \max_{0 \leq n \leq N-1} \left(\int_{n\Delta t}^{(n+1)\Delta t} ||\boldsymbol{q} - \boldsymbol{q}^N||^2_{H^1(\Omega_f)} dt\right)^{1/2}.$$

Again, because $\boldsymbol{q}$ is continuous taking values in $\mathcal{V}_F$ equipped with the norm of $H^1(\Omega_f)$, we have that $||\boldsymbol{q} - \boldsymbol{q}^N||_{H^1(\Omega_f)} \to 0$ uniformly on $[0,T]$ as $N \to \infty$, which establishes the desired limit.

We have, therefore, established the existence of a weak solution to the stochastic fluid-structure interaction problem in a probabilistically weak sense, as in Definition 4.1.

9 Return to the original probability space

We have thus constructed a stochastic process $(\tilde{\boldsymbol{u}}, \tilde{\eta})$, which satisfies the weak formulation of the continuous problem almost surely on the "tilde" probability space determined by the Skorohod representation theorem. However, we want to bring the solution back to the original probability space. In particular, we must get convergence of the original approximate solutions $(\boldsymbol{u}_N, v_N, \eta_N)$ on the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the original given complete filtration $\{\mathcal{F}_t\}_{t \geq 0}$ and the original Brownian motion $W(t)$.

To do this, we will use a standard *Gyöngy-Krylov argument* based on the following lemma, see Lemma 1.1 in [28] and Proposition 6.3 in [41].

Lemma 9.1 (Gyöngy-Krylov lemma)**.** Let $\{X_n\}_{n=1}^\infty$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in a separable Banach space B. Then X_n **converges in probability** to some B-valued random variable X^* if and only if for every two subsequences X_{l_k} and X_{m_k} of X_n, there exists a further subsequence of $x_k = (X_{l_k}, X_{m_k})$ whose laws converge weakly to a probability measure ν on $B \times B$ that is supported on the diagonal $\{(x,y) \in B \times B : x = y\}$.

In other words, the statement of the Gyöngy-Krylov lemma holds if and only if for every two subsequences X_{l_k} and X_{m_k}, there exists a further subsequence such that the joint probability measures associated with $x_k = (X_{l_k}, X_{m_k})$ on $B \times B$, defined by

$$\nu_{x_k} = \nu_{X_{l_k}, X_{m_k}}(A_1 \times A_2) = \mathbb{P}(X_{l_k} \in A_1, X_{m_k} \in A_2), \quad A_1, A_2 \in \mathcal{B}(B),$$

where $\mathcal{B}(B)$ is the Borel sigma algebra on B, converge weakly along this *further subsequence* to some probability measure ν, where ν is such that

$$\nu(\{(x,y) \in B \times B : x = y\}) = 1. \tag{79}$$

Thus, the limits of any two convergent subsequences have to be "the same" with probability 1.

Once we show convergence in probability of our original sequence using the Gyöngy-Krylov lemma, we will have almost sure convergence along a subsequence of our approximate solutions on the *original probability* *space*. Then, using the fact that our approximate solutions converge almost surely along a subsequence on the original probability space, we can adapt the arguments in Section 8.4 in order to show that the limiting weak solution on the original probability space satisfies the weak form of the continuous problem almost surely, so that the limiting solution is a weak solution in a probabilistically strong sense.

Thus, what remains to be shown is that the diagonal condition in the Gyöngy-Krylov lemma holds. Since our problem is linear and the stochastic noise is additive, using the Skorohod representation theorem, one can show that the diagonal condition is equivalent to showing *deterministic uniqueness* holding pathwise. To demonstrate this, we first prove deterministic uniqueness, and then use it to show how this implies the diagonal condition.

9.1 Uniqueness of the deterministic linear problem

Lemma 9.2 (Uniqueness for the deterministic problem)**.** Suppose that $\boldsymbol{u} \in L^\infty(0,T;L^2(\Omega_f)) \cap L^2(0,T;\mathcal{V}_F)$, $\eta \in W^{1,\infty}(0,T;L^2(\Gamma)) \cap L^\infty(0,T;\mathcal{V}_S)$, and $\boldsymbol{u}|_\Gamma = \partial_t\eta\boldsymbol{e}_r$. Suppose also that $(\boldsymbol{u},\partial_t\eta) \in C(0,T;\mathcal{Q}')$, with $\eta(0) = 0$. If for all $(\boldsymbol{q},\psi) \in \mathcal{Q}(0,T)$,

$$-\int_0^T \int_{\Omega_f} \boldsymbol{u} \cdot \partial_t \boldsymbol{q} d\boldsymbol{x}dt + 2\mu \int_0^T \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x}dt - \int_0^T \int_\Gamma \partial_t\eta\partial_t\psi dzdt + \int_0^T \int_\Gamma \nabla\eta \cdot \nabla\psi dzdt = 0,$$

then $(\boldsymbol{u},\eta) = 0$.

Proof. Observe first that to get the usual energy equality, we would want to formally substitute in $(\boldsymbol{u},\partial_t\eta)$ for $(\boldsymbol{q},\psi)$. However, since $(\boldsymbol{q},\psi)$ must have $\psi(t) \in H_0^1(\Gamma)$ by the definition of the test space $\mathcal{Q}(0,T)$, we do not have enough regularity to do this. Therefore, we use a different approach of taking an antiderivative, which is an approach used for example in establishing uniqueness of weak solutions for general hyperbolic equations (see Section 7.2 in [20]).

Consider an arbitrary s such that $0 \le s \le T$. We use the following test function,

$$(\boldsymbol{q}_0(t),\psi_0(t)) = \begin{cases} \left(\int_t^s \left(\int_0^\tau \boldsymbol{u}(\sigma)d\sigma\right) d\tau, \int_t^s \eta(\tau)d\tau\right) & \text{if } 0 \le t \le s, \\ (0,0) & \text{if } s \le t \le T. \end{cases}$$

Recall that $\eta(0) = 0$ by assumption. Note that since

$$\int_0^\tau \boldsymbol{u}(\sigma)d\sigma\Big|_\Gamma = \int_0^\tau \partial_t\eta(\sigma)d\sigma = \eta(\tau)$$

for all $\tau \in [0,T]$, the function $(\boldsymbol{q}_0,\psi_0)$ satisfies the necessary kinematic coupling condition for $\mathcal{Q}(0,T)$. While this test function is only piecewise differentiable, it is easy to show by an approximation argument that the weak formulation should still hold with this test function by approximating it with differentiable functions. For notational simplicity, we define

$$\boldsymbol{U}(t) = \int_0^t \boldsymbol{u}(\sigma)d\sigma.$$

Substituting the test function into the weak formulation, we obtain for all $s \in [0,T]$,

$$\int_0^s \int_{\Omega_f} \boldsymbol{u} \cdot \boldsymbol{U} d\boldsymbol{x}dt + 2\mu \int_0^s \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{q}_0) d\boldsymbol{x}dt + \int_0^s \int_\Gamma \partial_t\eta \cdot \eta dzdt + \int_0^s \int_\Gamma \nabla\eta \cdot \nabla\psi_0 dzdt = 0,$$

where we note that $\partial_t\boldsymbol{q}_0(t) = -U(t)$ and $\partial_t\psi_0(t) = -\eta(t)$, for $t \in [0,s)$. We handle the four terms on the left hand side as follows.

- **First term:** We note that $\boldsymbol{u} = \partial_t\boldsymbol{U}$. Hence, using the fact that $\boldsymbol{U}(0) = 0$, we get

$$\int_0^s \int_{\Omega_f} \boldsymbol{u} \cdot \boldsymbol{U} d\boldsymbol{x}dt = \int_0^s \frac{d}{dt}\left(\frac{1}{2}||\boldsymbol{U}||^2_{L^2(\Omega_f)}\right) dt = \frac{1}{2}||\boldsymbol{U}(s)||^2_{L^2(\Omega_f)} - \frac{1}{2}||\boldsymbol{U}(0)||^2_{L^2(\Omega_f)} = \frac{1}{2}||\boldsymbol{U}(s)||^2_{L^2(\Omega_f)}.$$

- **Second term:** For the second term, we again use that $\boldsymbol{u} = \partial_t \boldsymbol{U}$. Therefore,

$$2\mu \int_0^s \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{q}_0) d\boldsymbol{x} dt = 2\mu \int_0^s \int_{\Omega_f} \boldsymbol{D}(\partial_t \boldsymbol{U}) : \boldsymbol{D}(\boldsymbol{q}_0) d\boldsymbol{x} dt.$$

  We integrate by parts in time. Note that $\boldsymbol{U}(0) = 0$ and $\boldsymbol{q}_0(s) = 0$, so there are no boundary terms from the integration by parts. Hence, using the fact that $\partial_t \boldsymbol{q}_0 = -\boldsymbol{U}$, we obtain

$$2\mu \int_0^s \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{q}_0) d\boldsymbol{x} dt = -2\mu \int_0^s \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{U}) : \boldsymbol{D}(\partial_t \boldsymbol{q}_0) d\boldsymbol{x} dt = 2\mu \int_0^s \int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{U})|^2 d\boldsymbol{x} dt.$$

- **Third term:** We immediately have that

$$\int_0^s \int_\Gamma \partial_t \eta \cdot \eta dz dt = \frac{1}{2} ||\eta(s)||^2_{L^2(\Gamma)} - \frac{1}{2} ||\eta(0)||^2_{L^2(\Gamma)} = \frac{1}{2} ||\eta(s)||^2_{L^2(\Gamma)}.$$

- **Fourth term:** Since $\eta = -\partial_t \psi_0$, we have that

$$\int_\Gamma \nabla \eta \cdot \nabla \psi_0 dz = -\frac{1}{2} \frac{d}{dt} \left(||\nabla \psi_0||^2_{L^2(\Gamma)} \right),$$

  and hence, using the fact that $\psi_0(s) = 0$, we get that

$$\int_0^s \int_\Gamma \nabla \eta \cdot \nabla \psi_0 dz dt = \frac{1}{2} ||\nabla \psi_0(0)||^2_{L^2(\Gamma)}.$$

Therefore, for all $0 \leq s \leq T$, the entire expression (energy) can now be written as

$$\frac{1}{2} ||\boldsymbol{U}(s)||^2_{L^2(\Omega_f)} + 2\mu \int_0^s \int_{\Omega_f} |\boldsymbol{D}(\boldsymbol{U})|^2 d\boldsymbol{x} dt + \frac{1}{2} ||\eta(s)||^2_{L^2(\Gamma)} + \frac{1}{2} ||\nabla \psi_0(0)||^2_{L^2(\Gamma)} = 0.$$

Thus, we conclude that $\boldsymbol{U}(s) = 0$ and $\eta(s) = 0$ for all $s \in [0, T]$. From the definition of $\boldsymbol{U}$, we conclude that $\boldsymbol{u}(t) = \partial_t \boldsymbol{U}(t) = 0$ for all $t \in [0, T]$ also, which completes the proof. □

9.2 Verifying the diagonal condition of the Gyöngy-Krylov lemma

Now that we have established a uniqueness result, we can construct a solution on the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by invoking a standard argument involving the Gyöngy-Krylov argument (Lemma 9.1), to show that the random variables $(\eta_N, \boldsymbol{u}_N, v_N)$ defined on the original probability space converge in probability, and hence converge almost surely along a subsequence *in the original topology.*

Because we have already shown deterministic uniqueness in Sec. 9.1, it only remains to demonstrate how the Skorohod representation theorem can be used to show that the diagonal condition (79) from the Gyöngy-Krylov lemma is equivalent to showing deterministic uniqueness.

For this purpose, denote by $\{X^1_{M_k}\}_{k=1}^\infty$ and $\{X^2_{N_k}\}_{k=1}^\infty$ any two subsequences of our random variables (approximate solutions) defined on the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$:

$$\begin{aligned} X^1_{M_k} &= (\eta_{M_k}, \overline{\eta}_{M_k}, \eta^{\Delta t}_{M_k}, \boldsymbol{u}_{M_k}, v_{M_k}, \boldsymbol{u}_{M_k}, v^*_{M_k}, \overline{\boldsymbol{u}}_{M_k}, \overline{v}_{M_k}, \boldsymbol{u}^{\Delta t}_{M_k}, v^{\Delta t}_{M_k}, W), \\ X^2_{N_k} &= (\eta_{N_k}, \overline{\eta}_{N_k}, \eta^{\Delta t}_{N_k}, \boldsymbol{u}_{N_k}, v_{N_k}, \boldsymbol{u}_{N_k}, v^*_{N_k}, \overline{\boldsymbol{u}}_{N_k}, \overline{v}_{N_k}, \boldsymbol{u}^{\Delta t}_{N_k}, v^{\Delta t}_{N_k}, W). \end{aligned}$$

Recall that the laws corresponding to each of these these two sequences of random variables *individually* converge to the law μ. However, to verify the diagonal condition in the Gyöngy-Krylov lemma, we must examine the *joint laws* of these random variables $(X^1_{M_k}, X^2_{N_k})$.

Hence, we consider the joint probability measures (or joint laws) $\{\nu_{X^1_{M_k}, X^2_{N_k}}\}_{k=1}^\infty$ on $\mathcal{X} \times \mathcal{X}$, associated with the subsequence $(X^1_{M_k}, X^2_{N_k})$. By the tightness of the original probability measures μ_N, established in the proof of Theorem 8.1, we have that the collection of joint laws $\{\nu_{X^1_{M_k}, X^2_{N_k}}\}_{k=1}^{\infty}$ is also tight, and hence converges weakly to a probability measure ν on $\mathcal{X} \times \mathcal{X}$ along a further subsequence, which we will continue to denote by the same indexing for notational simplicity. Then, by the Skorohod representation theorem, there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and random variables

$$\begin{aligned}\tilde{X}^1_{M_k} &= (\tilde{\eta}^1_{M_k}, \tilde{\overline{\eta}}^1_{M_k}, \tilde{\eta}^{\Delta t,1}_{M_k}, \tilde{\boldsymbol{u}}^1_{M_k}, \tilde{v}^1_{M_k}, \tilde{\boldsymbol{u}}^{*,1}_{M_k}, \tilde{v}^{*,1}_{M_k}, \tilde{\overline{\boldsymbol{u}}}^1_{M_k}, \tilde{\overline{v}}^1_{M_k}, \tilde{\boldsymbol{u}}^{\Delta t,1}_{M_k}, \tilde{v}^{\Delta t,1}_{M_k}, \tilde{W}^1_{M_k}),\\ \tilde{X}^2_{N_k} &= (\tilde{\eta}^2_{N_k}, \tilde{\overline{\eta}}^2_{N_k}, \tilde{\eta}^{\Delta t,2}_{N_k}, \tilde{\boldsymbol{u}}^2_{N_k}, v^2_{N_k}, \tilde{\boldsymbol{u}}^{*,2}_{N_k}, \tilde{v}^{*,2}_{N_k}, \tilde{\overline{\boldsymbol{u}}}^2_{N_k}, \tilde{\overline{v}}^2_{N_k}, \tilde{\boldsymbol{u}}^{\Delta t,2}_{N_k}, \tilde{v}^{\Delta t,2}_{N_k}, \tilde{W}^2_{N_k}),\end{aligned}$$

such that

$$(\tilde{X}^1_{M_k}, \tilde{X}^2_{N_k}) =_d (X^1_{M_k}, X^2_{N_k}), \tag{80}$$

and $(\tilde{X}^1_{M_k}, \tilde{X}^2_{N_k}) \to (\tilde{X}^1, \tilde{X}^2)$ in $\mathcal{X} \times \mathcal{X}$ almost surely as $k \to \infty$, where

$$\begin{aligned}\tilde{X}^1 &= (\tilde{\eta}^1, \tilde{\overline{\eta}}^1, \tilde{\eta}^{\Delta t,1}, \tilde{\boldsymbol{u}}^1, \tilde{v}^1, \tilde{\boldsymbol{u}}^{*,1}, \tilde{v}^{*,1}, \tilde{\overline{\boldsymbol{u}}}^1, \tilde{\overline{v}}^1, \tilde{\boldsymbol{u}}^{\Delta t,1}, \tilde{v}^{\Delta t,1}, \tilde{W}^1),\\ \tilde{X}^2 &= (\tilde{\eta}^2, \tilde{\overline{\eta}}^2, \tilde{\eta}^{\Delta t,2}, \tilde{\boldsymbol{u}}^2, \tilde{v}^2, \tilde{\boldsymbol{u}}^{*,2}, \tilde{v}^{*,2}, \tilde{\overline{\boldsymbol{u}}}^2, \tilde{\overline{v}}^2, \tilde{\boldsymbol{u}}^{\Delta t,2}, \tilde{v}^{\Delta t,2}, \tilde{W}^2),\end{aligned}$$

are random variables on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, and ν is the law of $(\tilde{X}_1, \tilde{X}_2)$.

We want to show that ν is supported on the diagonal. It suffices to show that $\tilde{\mathbb{P}}(\tilde{X}^1 = \tilde{X}^2) = 1$. We do this in three steps.

Step 1. First we notice that $\tilde{X}^1$ *is a weak solution in a probabilistically weak sense with respect to the stochastic basis* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}^1_t\}_{t \geq 0}, \tilde{\mathbb{P}}, \tilde{W}_1)$ in the sense of Definition 4.1. This follows from the results of Lemma 8.7. Namely, the results of Lemma 8.7 imply that $\tilde{\eta}^1 = \tilde{\overline{\eta}}^1 = \tilde{\eta}^{\Delta t,1}$, $\tilde{\boldsymbol{u}}^1 = \tilde{\boldsymbol{u}}^{*,1} = \tilde{\overline{\boldsymbol{u}}}^1 = \tilde{\boldsymbol{u}}^{\Delta t,1}$, $\tilde{v}^1 = \tilde{v}^{*,1} = \tilde{\overline{v}}^1 = \tilde{v}^{\Delta t,1}$, and $\partial_t \tilde{\eta}^1 = \tilde{v}^1$ almost surely. Furthermore, $(\tilde{\boldsymbol{u}}^1, \tilde{\eta}^1) \in \mathcal{W}(0,T)$ and $(\tilde{\boldsymbol{u}}^1, \tilde{v}^1) \in C(0,T; \mathcal{Q}')$, satisfying the initial condition $\tilde{\eta}^1(0) = \eta_0$ almost surely. Furthermore, $\tilde{X}^1$ is a weak solution in a probabilistically weak sense with respect to the stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}^1_t\}_{t \geq 0}, \tilde{\mathbb{P}}, \tilde{W}_1)$ in the sense of Definition 4.1. The same is true for the components of $\tilde{X}^2$, with respect to $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}^2_t\}_{t \geq 0}, \tilde{\mathbb{P}}, \tilde{W}_2)$. Here, the filtrations $\{\tilde{\mathcal{F}}^1_t\}_{t \geq 0}$ and $\{\tilde{\mathcal{F}}^2_t\}_{t \geq 0}$ are defined by (52) with the appropriate limiting random variables with superscripts "1" and "2" respectively.

Step 2. Here we notice that the limiting white noise satisfies $\tilde{W}_1 = \tilde{W}_2$. This follows directly from (80), which implies $\tilde{W}^1_{M_k} = \tilde{W}^2_{N_k}$ almost surely, since the law of $(\tilde{W}^1_{M_k}, \tilde{W}^2_{N_k})$ is the same as that of (W, W). Thus, by the convergence of $\tilde{W}^1_{M_k}$ and $\tilde{W}^2_{N_k}$ in $C(0,T;\mathbb{R})$ almost surely to $\tilde{W}^1$ and $\tilde{W}^2$, we have that $\tilde{W}^1 = \tilde{W}^2$ almost surely in $C(0,T;\mathbb{R})$. This will allow us to make sense of the difference of the stochastic integrals with respect to $\tilde{W}_1$ and $\tilde{W}_2$ in the weak formulations on the "tilde" probability space.

Step 3. Finally, we use deterministic uniqueness to obtain the diagonal condition. We consider the difference $(\tilde{\eta}^1 - \tilde{\eta}^2, \tilde{\boldsymbol{u}}^1 - \tilde{\boldsymbol{u}}^2)$. By subtracting the weak formulations defining $(\tilde{\boldsymbol{u}}^1, \tilde{\eta}^1)$ and $(\tilde{\boldsymbol{u}}^2, \tilde{\eta}^2)$ as probabilistically weak solutions, given in Definition 4.1, and by using the result of Step 2 above, we obtain that $(\tilde{\boldsymbol{u}}^1 - \tilde{\boldsymbol{u}}^2, \tilde{\eta}^1 - \tilde{\eta}^2)$ almost surely satisfies for all $(\boldsymbol{q}, \psi) \in \mathcal{Q}(0,T)$,

$$\begin{aligned}&-\int_0^T \int_{\Omega_f} (\boldsymbol{u}_1 - \boldsymbol{u}_2) \cdot \partial_t \boldsymbol{q} d\boldsymbol{x} dt + 2\mu \int_0^T \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}_1 - \boldsymbol{u}_2) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} dt\\ &- \int_0^T \int_\Gamma \partial_t (\eta_1 - \eta_2) \partial_t \psi dz dt + \int_0^T \int_\Gamma \nabla(\eta_1 - \eta_2) \cdot \nabla \psi dz dt = 0,\end{aligned}$$

with $\tilde{\eta}^1 - \tilde{\eta}^2 = 0$ almost surely. Therefore, by using the uniqueness result in Lemma 9.2, we conclude that $\tilde{\eta}^1 = \tilde{\eta}^2$ and $\tilde{\boldsymbol{u}}^1 = \tilde{\boldsymbol{u}}^2$ almost surely. Since $\tilde{v}^1 = \partial_t \tilde{\eta}^1$ and $\tilde{v}^2 = \partial_t \tilde{\eta}^2$, we also obtain that $\tilde{v}^1 = \tilde{v}^2$ almost surely. This allows us to conclude that $\tilde{\mathbb{P}}(\tilde{X}^1 = \tilde{X}^2) = 1$, which implies that the limiting joint probability measure (or law) ν is supported on the diagonal.

This completes the verification of the diagonal condition of the Gyöngy-Krylov lemma.

9.3 Existence of a weak solution in a probabilistically strong sense

The existence of a weak solution in a probabilistically strong sense, given by Definition 4.2, now follows from the Gyöngy-Krylov lemma in Lemma 9.1. More precisely, by the Gyöngy-Krylov lemma, the original sequence $(\eta_N, \overline{\eta}_N, \eta_N^{\Delta t}, \boldsymbol{u}_N, v_N, \boldsymbol{u}_N, v_N^*, \overline{\boldsymbol{u}}_N, \overline{v}_N, \boldsymbol{u}_N^{\Delta t}, v_N^{\Delta t}, W)$ converges in probability to some random variable $(\eta, \overline{\eta}, \eta^{\Delta t}, \boldsymbol{u}, v, \boldsymbol{u}^*, v^*, \overline{\boldsymbol{u}}, \overline{v}, \boldsymbol{u}^{\Delta t}, v^{\Delta t}, W)$, where the last component must be W up to a null set, since the limit in probability of any constant sequence is almost surely exactly that constant.

Since convergence in probability implies almost sure convergence along a subsequence, we conclude that along a subsequence which we continue to denote by N, we have that

$$(\eta_N, \overline{\eta}_N, \eta_N^{\Delta t}, \boldsymbol{u}_N, v_N, \boldsymbol{u}_N, v_N^*, \overline{\boldsymbol{u}}_N, \overline{v}_N, \boldsymbol{u}_N^{\Delta t}, v_N^{\Delta t}, W) \to (\eta, \overline{\eta}, \eta^{\Delta t}, \boldsymbol{u}, v, \boldsymbol{u}^*, v^*, \overline{\boldsymbol{u}}, \overline{v}, \boldsymbol{u}^{\Delta t}, v^{\Delta t}, W), \text{ a.s. in } \mathcal{X}. \tag{81}$$

To show that this limit is a weak solution in the sense of Definition 4.2, we use the same arguments as in Lemma 8.7. All of the properties from Definition 4.2 follow from Lemma 8.7, except for uniqueness and showing that $(\boldsymbol{u}, v, \eta)$ is $\mathcal{F}_t$-adapted.

Uniqueness follows from the deterministic uniqueness result of Lemma 9.2.

$\mathcal{F}_t$-adaptedness of $(\boldsymbol{u}, v, \eta)$: Note that this is not provided by Lemma 8.7, as we want to show that this solution is adapted to the *original* filtration $\{\mathcal{F}_t\}_{t\geq 0}$, while the filtration defined in (52) is not necessarily the same filtration.

To verify this, we note that by construction, $(\boldsymbol{u}_N, v_N, \eta_N)$ is adapted to the given complete filtration $\{\mathcal{F}_t\}_{t\geq 0}$. We want to pass to the limit as $N \to \infty$. By the convergence in (81),

$$\begin{aligned} \boldsymbol{u}_N &\to \boldsymbol{u}, && \text{almost surely in } L^2(0,T;L^2(\Omega_f)), \\ v_N &\to v, && \text{almost surely in } L^2(0,T;L^2(\Gamma)), \\ \eta_N &\to \eta, && \text{almost surely in } L^2(0,T;L^2(\Gamma)). \end{aligned}$$

By the same argument used to establish (55) for example, we obtain that for a measurable set $S \subset [0,T]\times\Omega$ with $(dt \times \mathbb{P})(S) = T$,

$$\boldsymbol{u}_{N_k}(t,\omega,\cdot) \to \boldsymbol{u}(t,\omega,\cdot) \text{ in } L^2(\Omega_f), \qquad v_{N_k}(t,\omega,\cdot) \to v(t,\omega,\cdot),\ \eta_{N_k}(t,\omega,\cdot) \to \eta(t,\omega,\cdot) \text{ in } L^2(\Gamma) \tag{82}$$

along a common subsequence N_k. In particular, $([0,T]\times\Omega) - S$ has measure zero with respect to the product measure $dt \times \mathbb{P}$.

Define $S_0 \subset [0,T]$ to be all times $t \in [0,T]$ for which $\mathbb{P}((t,\omega) \in S) = 1$, so that the time slice at time t has full measure in probability. S_0 is measurable in $[0,T]$ and contains almost every time in $[0,T]$ by Fubini's theorem. So for all $t \in S_0$, the convergences (82) are almost sure convergences.

Because $\{\mathcal{F}_t\}_{t\geq 0}$ is a complete filtration by assumption, the almost sure limit of $\mathcal{F}_t$-measurable random variables must also be $\mathcal{F}_t$-measurable, since $\mathcal{F}_t$ contains all null sets of $(\Omega, \mathcal{F}, \mathbb{P})$. So for all $t \in S_0$, $\boldsymbol{u}(t)$, $v(t)$, and $\eta(t)$ are $\mathcal{F}_t$-measurable since $\boldsymbol{u}_{N_k}(t)$, $v_{N_k}(t)$, and $\eta_{N_k}(t)$ are $\mathcal{F}_t$-measurable by construction.

To show $\boldsymbol{u}(t)$, $v(t)$, and $\eta(t)$ are $\mathcal{F}_t$-measurable for $t \notin S_0$, we use the fact that S_0 has full measure in $[0,T]$ and is hence dense. We can assume $t \neq 0$, since at $t = 0$, $(\boldsymbol{u}(0), v(0), \eta(0)) = (\boldsymbol{u}_0, v_0, \eta_0)$ almost surely so the result holds. So for $t \notin S_0$ and $t \neq 0$, we can construct $t_n \in S_0$ such that $t_n \nearrow t$. By the fact that $(\boldsymbol{u}, v) \in C(0,T;\mathcal{Q}')$ and η is Lipschitz continuous almost surely, we have that $(\boldsymbol{u}(t), v(t), \eta(t))$ is the almost sure limit of $(\boldsymbol{u}(t_n), v(t_n), \eta(t_n))$, which are $\mathcal{F}_t$-measurable since $\mathcal{F}_{t_n} \subset \mathcal{F}_t$, as $t_n \leq t$. This establishes the adaptedness of $(\boldsymbol{u}, v, \eta)$ to the given complete filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

In conclusion, we have now shown that $(\boldsymbol{u}, v, \eta)$ has all of the required properties needed to be a weak solution in a probabilistically strong sense to the given fluid-structure interaction problem with respect to the Brownian motion W with complete filtration $\{\mathcal{F}_t\}_{t\geq 0}$, as in Definition 4.2. This completes the proof of the main result, stated in Theorem 4.1, and restated here:

Theorem 9.1 (Main Result). Let $\boldsymbol{u}_0 \in L^2(\Omega_f)$, $v_0 \in L^2(\Gamma)$, and $\eta_0 \in H_0^1(\Gamma)$. Let $P_{in/out} \in L^2_{loc}(0,\infty)$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a Brownian motion W with respect to a given complete filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Then, for any $T > 0$, there exists a unique weak solution in a probabilistically strong sense to the given stochastic fluid-structure interaction problem (5)–(7).

10 Conclusions

In this manuscript, we presented a constructive proof of the existence of a weak solution in a probabilistically strong sense, to a benchmark stochastic fluid-structure interaction (SFSI) problem (5)–(7). An example of such a problem is the flow of blood in coronary arteries that sit on the surface of the heart and contract and expand under the outside forcing due to the heart muscle contraction and expansion. Dynamic patient images show significant stochastic effects in the heart contractions, which can be captured by an SFSI model such as the one studied in this work. Our well-posedness result indicates that stochastic FSI models are robust in the sense that a unique weak solution in the sense of Definition 4.2 will exist even when the problem is stochastically forced by a rough time-dependent white noise, as considered in this work.

In addition to the importance of this work in terms of modeling real-life fluid structure interaction phenomena with stochastic noise, to the best of our knowledge the results of this work present a first constructive existence proof of a unique weak solution in a probabilistically strong sense to a stochastically forced and fully coupled FSI problem, as defined in Definition 4.2.

In contrast to the deterministic case, the proof based on the operator splitting strategy presented in this work has several new interesting components, which we summarize below.

1. The energy estimates are given in expectation, and do not necessarily hold pathwise. Thus, one cannot immediately deduce almost sure convergence of the random approximate solutions along a subsequence immediately from uniform boundedness of the random approximate solutions in the probabilistic finite energy spaces with norms involving expectations. Having such almost sure convergence is essential for passing to the limit in semidiscrete formulation, especially for a more general class of complex stochastic FSI problems involving nonlinearities.

2. The energy estimate has an extra term that accounts for the energy pumped into the problem by the stochastic forcing in expectation.

3. A constructive splitting scheme can be developed for this multiphysical problem that separates each of the three different contributions to the fully coupled dynamics of the multiphysical problem into three separate subproblems: (1) the structure subproblem, (2) the stochastic subproblem, and (3) the fluid subproblem. This gives rise to a loosely coupled, stable constructive existence scheme for this stochastic FSI problem that is inherently modular in nature, where each different physical component of the problem is treated individually in each subproblem.

4. To establish stronger almost sure convergence of the random approximate solutions, one must first show weak convergence of probability measures, by establishing that the probability measures are **tight**. This requires the use of a *compactness result* alla Aubin-Lions, which form a general framework for constructive existence that generalizes to the well-posedness analysis of more complex nonlinear stochastic FSI systems. These compactness arguments show that the random approximate solutions generated via the splitting scheme converge in probability, which suggests that the constructive splitting scheme introduced in this manuscript can be used as a basis for developing numerical methods for stochastic FSI, which explicitly generate numerical solutions that converge in a probability as the time step goes to zero.

5. Once weak convergence of the probability measures (laws) associated with the approximate solutions is established, probabilistic techniques based on the Skorohod representation theorem and the Gyöngy-Krylov lemma have to be employed to obtain almost sure convergence along a subsequence to a weak solution.

11 Appendix: An alternative approach to the existence proof

In this appendix, we make some additional comments about an alternative approach to passing to the limit in the random approximate solutions constructed in Section 7. We would like to thank the anonymous reviewer, whose read the manuscript carefully and provided helpful suggestions which inspired the observations stated in this appendix. In particular, in the exposition presented in the current manuscript, we used compactness arguments based on showing tightness of the laws of the random approximate solutions, and stochastic PDE techniques involving the Skorohod representation theorem and the Gyöngy-Krylov lemma to pass to the limit. As emphasized throughout the manuscript, the rationale for using these compactness arguments, even in the case of a fully linear stochastic system of PDEs, was to develop a robust framework that is applicable to a wide variety of stochastic systems of PDEs. In particular, the compactness argument framework presented in this manuscript extends to the case of linearly coupled FSI with nonlinear dependence of the intensity of the random noise [35], and nonlinearly coupled stochastic FSI in which the fluid equations are posed on a time-dependent (random) moving fluid domain which is determined by the displacement of a stochastically forced elastic membrane [57].

However, we note that in the case of a genuinely fully linear stochastic system of PDEs as in the case of the current manuscript, having uniform boundedness of the random approximate solutions, *even just in expectation*, is sufficient to pass to the limit in the semidiscrete weak formulation, in order to obtain existence of a probabilistically *strong* solution directly on the original probability space, hence bypassing the need to use the Skorohod representation theorem and eliminating the need to transfer the problem to a different probability space. We illustrate this procedure below, starting from the uniform boundedness of the approximate solutions generated by the splitting scheme, stated in Proposition 7.2 and Proposition 7.3, where these approximate solutions satisfy the semidiscrete formulation (25). Though this procedure using weak convergence of the random approximate solutions in function spaces involving both the probability space and the spacetime function spaces works well for the case of this fully linear stochastic system, we emphasize that this approach *does not generalize* to more complex stochastic systems involving nonlinearities, which require compactness arguments of the type presented in this manuscript.

We start with the uniform boundedness result stated in Proposition 7.2 and Proposition 7.3, and we conclude that there exist limiting random variables η, v, and $\boldsymbol{u}$ such that

- η_N converges weakly star to η in $L^2(\Omega; L^\infty(0,T; H_0^1(\Gamma)))$.
- $\overline{\eta}_N$ converges weakly star to η in $L^2(\Omega; W^{1,\infty}(0,T; L^2(\Gamma)))$.
- v_N converges weakly star to v in $L^2(\Omega; L^\infty(0,T; L^2(\Gamma)))$.
- $v_N^{\Delta t}$ converge weakly to v in $L^2(\Omega; L^2(0,T; H^{1/2}(\Gamma)))$.
- v_N^* converges weakly star to v in $L^2(\Omega; L^\infty(0,T; L^2(\Gamma)))$.
- $\boldsymbol{u}_N$ converges weakly star to $\boldsymbol{u}$ in $L^2(\Omega; L^\infty(0,T; L^2(\Omega_f)))$.
- $\boldsymbol{u}_N^{\Delta t}$ converges weakly to $\boldsymbol{u}$ in $L^2(\Omega; L^2(0,T; H^1(\Omega_f)))$.

In addition, by combining the uniform numerical dissipation estimates stated in Proposition 6.7 with the convergences stated above, we conclude that

- $\overline{\boldsymbol{u}}_N$ converges to $\boldsymbol{u}$ weakly in $L^2(\Omega; L^2(0,T; L^2(\Omega_f)))$.
- $\overline{v}_N$ converges to v weakly in $L^2(\Omega; L^2(0,T; L^2(\Gamma)))$.

We recall that the (random) approximate solutions satisfy the semidiscrete formulation (25) almost surely for each (deterministic) test function $(\boldsymbol{q}, \psi) \in \mathcal{Q}(0,T)$:

$$\int_{\Omega_f} \frac{\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^n}{\Delta t} \cdot \boldsymbol{q} d\boldsymbol{x} + 2\mu \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}_N^{n+1}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} + \int_\Gamma \frac{v_N^{n+1} - v_N^n}{\Delta t} \psi dz + \int_\Gamma \nabla \eta_N^{n+1} \cdot \nabla \psi dz$$
$$= \int_\Gamma \frac{W((n+1)\Delta t) - W(n\Delta t)}{\Delta t} \psi dz + P_{N,in}^n \int_0^R (q_z)|_{z=0} dr - P_{N,out}^n \int_0^R (q_z)|_{z=L} dr.$$

We want to show that the limiting functions $(\eta, v, \boldsymbol{u})$ satisfy the continuous weak formulation almost surely for each $(\boldsymbol{q}, \psi) \in \mathcal{Q}(0,T)$:

$$
-\int_0^T \int_{\Omega_f} \boldsymbol{u} \cdot \partial_t \boldsymbol{q} d\boldsymbol{x} dt + 2\mu \int_0^T \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} dt - \int_0^T \int_\Gamma v \partial_t \psi dz dt + \int_0^T \int_\Gamma \nabla \eta \cdot \nabla \psi dz dt
$$
$$
= \int_0^T P_{in}(t) \left(\int_{\Gamma_{in}} q_z dr \right) dt - \int_0^T P_{out}(t) \left(\int_{\Gamma_{out}} q_z dr \right) dt + \int_{\Omega_f} \boldsymbol{u_0} \cdot \boldsymbol{q}(0) d\boldsymbol{x} + \int_\Gamma v_0 \psi(0) dz + \int_0^T \left(\int_\Gamma \psi dz \right) dW(t).
$$

We emphasize that the convergence of the approximate solutions to the limiting functions $(\eta, v, \boldsymbol{u})$ is only convergence weakly and weakly star in function spaces *involving the probability space itself* since the uniform boundedness of the approximate solutions is only in expectation.

We consider a fixed but arbitrary deterministic test function $(\boldsymbol{q}, \psi) \in \mathcal{Q}(0,T)$ where the test space $\mathcal{Q}(0,T)$ is defined in (13), and we associate to this test function the following random variable (defined with the limiting weak formulation in consideration):

$$
X_{(\boldsymbol{q},\psi)} := -\int_0^T \int_{\Omega_f} \boldsymbol{u} \cdot \partial_t \boldsymbol{q} d\boldsymbol{x} dt + 2\mu \int_0^T \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{q}) d\boldsymbol{x} dt - \int_0^T \int_\Gamma v \partial_t \psi dz dt + \int_0^T \int_\Gamma \nabla \eta \cdot \nabla \psi dz dt
$$
$$
- \int_0^T P_{in}(t) \left(\int_{\Gamma_{in}} q_z dr \right) dt + \int_0^T P_{out}(t) \left(\int_{\Gamma_{out}} q_z dr \right) dt - \int_{\Omega_f} \boldsymbol{u_0} \cdot \boldsymbol{q}(0) d\boldsymbol{x} - \int_\Gamma v_0 \psi(0) dz - \int_0^T \left(\int_\Gamma \psi dz \right) dW(t). \tag{83}
$$

Because of the function spaces that the limiting solution $(\eta, v, \boldsymbol{u})$ belong to and the regularity of the test functions in the test space $\mathcal{Q}(0,T)$ which is defined in (13), we conclude that $X_{(\boldsymbol{q},\psi)} \in L^2(\Omega)$ is a square integrable real-valued random variable that is hence finite almost surely.

Because the semidiscrete weak formulation holds pathwise as a result of how the splitting scheme constructs the approximate solutions pathwise outcome by outcome in the probability space, we more generally have that the approximate solutions satisfy the following generalized semidiscrete formulation almost surely

$$
\int_{\Omega_f} \frac{\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^n}{\Delta t} \cdot \left(X_{(\boldsymbol{q},\psi)} \boldsymbol{q} \right) d\boldsymbol{x} + 2\mu \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}_N^{n+1}) : \boldsymbol{D} \left(X_{(\boldsymbol{q},\psi)} \boldsymbol{q} \right) d\boldsymbol{x}
$$
$$
+ \int_\Gamma \frac{v_N^{n+1} - v_N^n}{\Delta t} \left(X_{(\boldsymbol{q},\psi)} \psi \right) dz + \int_\Gamma \nabla \eta_N^{n+1} \cdot \nabla \left(X_{(\boldsymbol{q},\psi)} \psi \right) dz - \int_\Gamma \frac{W((n+1)\Delta t) - W(n\Delta t)}{\Delta t} \left(X_{(\boldsymbol{q},\psi)} \psi \right) dz
$$
$$
- P_{N,in}^n \int_0^R \left(X_{(\boldsymbol{q},\psi)} q_z \right) |_{z=0} dr + P_{N,out}^n \int_0^R \left(X_{(\boldsymbol{q},\psi)} q_z \right) |_{z=L} dr = 0, \tag{84}
$$

so that the test function $\left(X_{(\boldsymbol{q},\psi)} q, X_{(\boldsymbol{q},\psi)} \psi \right)$ in the semidiscrete weak formulation is now a random test function. Because all of the weak and weak star convergences that we have involve the probability space and hence involve convergence of quantities in expectation, we integrate from $n\Delta t$ to $(n+1)\Delta t$ in time, sum from $n = 0$ to $n = N-1$, and take the expectation of both sides of (84), and then pass to the limit as $N \to \infty$. We hence want to pass to the limit as $N \to \infty$ in the left hand side of the expression:

$$
\mathbb{E} \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \left[\int_{\Omega_f} \frac{\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^n}{\Delta t} \cdot \left(X_{(\boldsymbol{q},\psi)} \boldsymbol{q} \right) d\boldsymbol{x} + 2\mu \int_{\Omega_f} \boldsymbol{D}(\boldsymbol{u}_N^{n+1}) : \boldsymbol{D} \left(X_{(\boldsymbol{q},\psi)} \boldsymbol{q} \right) d\boldsymbol{x} \right.
$$
$$
+ \int_\Gamma \frac{v_N^{n+1} - v_N^n}{\Delta t} \left(X_{(\boldsymbol{q},\psi)} \psi \right) dz + \int_\Gamma \nabla \eta_N^{n+1} \cdot \nabla \left(X_{(\boldsymbol{q},\psi)} \psi \right) dz - \int_\Gamma \frac{W((n+1)\Delta t) - W(n\Delta t)}{\Delta t} \left(X_{(\boldsymbol{q},\psi)} \psi \right) dz
$$
$$
\left. - P_{N,in}^n \int_0^R \left(X_{(\boldsymbol{q},\psi)} q_z \right) |_{z=0} dr + P_{N,out}^n \int_0^R \left(X_{(\boldsymbol{q},\psi)} q_z \right) |_{z=L} dr \right] dt = 0, \tag{85}
$$

To handle the first term, we use an integration by parts in time as in (56) and the fact that $\boldsymbol{q}$ is compactly supported in $[0,T)$ to obtain:

$$\mathbb{E}\sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t}\int_{\Omega_f}\frac{\boldsymbol{u}_N^{n+1}-\boldsymbol{u}_N^{n}}{\Delta t}\cdot\left(X_{(\boldsymbol{q},\psi)}\boldsymbol{q}\right)d\boldsymbol{x}dt=\mathbb{E}\int_0^T\int_{\Omega_f}\partial_t\overline{\boldsymbol{u}}_N\cdot\left(X_{(\boldsymbol{q},\psi)}\boldsymbol{q}\right)d\boldsymbol{x}dt$$
$$=-\mathbb{E}\int_0^T\int_{\Omega_f}\overline{\boldsymbol{u}}_N\cdot\left(X_{(\boldsymbol{q},\psi)}\partial_t\boldsymbol{q}\right)d\boldsymbol{x}dt-\mathbb{E}\int_{\Omega_f}\boldsymbol{u}_0\cdot\left(X_{(\boldsymbol{q},\psi)}\boldsymbol{q}(0)\right)$$
$$\to-\mathbb{E}\int_0^T\int_{\Omega_f}\boldsymbol{u}\cdot\left(X_{(\boldsymbol{q},\psi)}\partial_t\boldsymbol{q}\right)d\boldsymbol{x}dt-\mathbb{E}\int_{\Omega_f}\boldsymbol{u}_0\cdot\left(X_{(\boldsymbol{q},\psi)}\boldsymbol{q}(0)\right),\quad(86)$$

by the convergence of $\overline{\boldsymbol{u}}_N$ to $\boldsymbol{u}$ weakly in $L^2(\Omega;L^2(0,T;L^2(\Omega_f)))$. Similarly, we have by the weak convergence of v_N to v in $L^2(\Omega;L^2(0,T;L^2(\Gamma)))$ that

$$\mathbb{E}\sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t}\int_{\Gamma}\frac{v_N^{n+1}-v_N^{n}}{\Delta t}\left(X_{(\boldsymbol{q},\psi)}\psi\right)dzdt\to-\mathbb{E}\int_0^T\int_{\Gamma}v\left(X_{(\boldsymbol{q},\psi)}\partial_t\psi\right)dzdt-\mathbb{E}\int_{\Omega_f}v_0\left(X_{(\boldsymbol{q},\psi)}\psi(0)\right).\quad(87)$$

By the weak convergence of $\boldsymbol{u}_N^{\Delta t}$ to $\boldsymbol{u}$ in $L^2(\Omega;L^2(0,T;H^1(\Omega_f)))$, we obtain that

$$\mathbb{E}\sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t}2\mu\int_{\Omega_f}\boldsymbol{D}(\boldsymbol{u}_N^{n+1}):\boldsymbol{D}\left(X_{(\boldsymbol{q},\psi)}\boldsymbol{q}\right)d\boldsymbol{x}dt=\mathbb{E}\left(2\mu\int_0^T\int_{\Omega_f}\boldsymbol{D}\left(\boldsymbol{u}_N^{\Delta t}\right):\boldsymbol{D}\left(X_{(\boldsymbol{q},\psi)}\boldsymbol{q}\right)d\boldsymbol{x}dt\right)$$
$$\to\mathbb{E}\left(2\mu\int_0^T\int_{\Omega_f}\boldsymbol{D}\left(\boldsymbol{u}\right):\boldsymbol{D}\left(X_{(\boldsymbol{q},\psi)}\boldsymbol{q}\right)d\boldsymbol{x}dt\right).\quad(88)$$

Similarly, by the weak convergence of $\eta_N^{\Delta t}$ to η in $L^2(\Omega;L^2(0,T;H_0^1(\Gamma)))$ which follows from the uniform numerical dissipation estimates in Proposition 6.7, where $\eta_N^{\Delta t}$ is defined in (30), we obtain that

$$\mathbb{E}\sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t}\int_{\Gamma}\nabla\eta_N^{n+1}\cdot\nabla\left(X_{(\boldsymbol{q},\psi)}\psi\right)dzdt\to\mathbb{E}\left(\int_0^T\int_{\Gamma}\nabla\eta\cdot\nabla\left(X_{(\boldsymbol{q},\psi)}\psi\right)dzdt\right).\quad(89)$$

For the stochastic integral, we claim that

$$\mathbb{E}\sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t}\int_{\Gamma}\frac{W((n+1)\Delta t)-W(n\Delta t)}{\Delta t}\left(X_{(\boldsymbol{q},\psi)}\psi\right)dzdt$$
$$=\mathbb{E}\left(X_{(\boldsymbol{q},\psi)}\sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t}\int_{\Gamma}\frac{W((n+1)\Delta t)-W(n\Delta t)}{\Delta t}\psi dzdt\right)\to\mathbb{E}\left(X_{(\boldsymbol{q},\psi)}\int_0^T\int_{\Gamma}\psi(t,z)dzdW(t)\right).\quad(90)$$

In order to show this, we first observe that

$$\mathbb{E}\sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t}\int_{\Gamma}\frac{W((n+1)\Delta t)-W(n\Delta t)}{\Delta t}\psi dzdt$$
$$=\mathbb{E}\left(\int_0^T\sum_{n=0}^{N-1}\left(\frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t}\int_{\Gamma}\psi(s,z)dzds\right)1_{[n\Delta t,(n+1)\Delta t)}(t)dW(t)\right),\quad(91)$$

and note that we have the following deterministic convergence:

$$\int_0^T\left(\int_{\Gamma}\psi(t,z)dz-\frac{1}{\Delta t}\sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t}\int_{\Gamma}\psi(s,z)dzds1_{[n\Delta t,(n+1)\Delta t)}(t)\right)^2dt\to0,\qquad\text{as }N\to\infty,\quad(92)$$

by the fact that the function $\int_\Gamma \psi(t,z)dz$ is a uniformly continuous real-valued (deterministic) function on $[0,T]$. Hence, by Itô's isometry and (91),

$$\mathbb{E}\sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t}\int_\Gamma \frac{W((n+1)\Delta t)-W(n\Delta t)}{\Delta t}\psi dzdt \to \mathbb{E}\left(\int_0^T\int_\Gamma \psi(t,z)dzdW(t)\right). \tag{93}$$

Then, because $X_{(\boldsymbol{q},\psi)} \in L^2(\Omega)$ and in addition,

$$\sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t}\int_\Gamma \frac{W((n+1)\Delta t)-W(n\Delta t)}{\Delta t}\psi dzdt, \quad \int_0^T\int_\Gamma \psi(t,z)dzdW(t) \in L^2(\Omega),$$

we conclude the desired convergence (90) by using (93) and the Cauchy-Schwarz inequality with $L^2(\Omega)$.

Finally, for the pressure term, we use the deterministic convergence (75) to conclude that

$$\begin{aligned}&\mathbb{E}\sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t}\left(P^n_{N,in/out}\int_0^R \left(X_{(\boldsymbol{q},\psi)}q_z\right)|_{z=0}dr\right)\\ &= \left(\sum_{n=0}^{N-1}\int_{n\Delta t}^{(n+1)\Delta t} P^n_{N,in/out}\int_0^R (q_z)|_{z=0}drdt\right)\mathbb{E}\left(X_{(\boldsymbol{q},\psi)}\right) \to \left(\int_0^T P_{in/out}(t)\int_0^R (q_z)|_{z=0}drdt\right)\mathbb{E}\left(X_{(\boldsymbol{q},\psi)}\right).\end{aligned} \tag{94}$$

Combining the convergences (86), (87), (88), (89), (90), and (94) and applying these convergences to take the limit in (85) as $N \to \infty$, we obtain that

$$\mathbb{E}\left(\left|X_{(\boldsymbol{q},\psi)}\right|^2\right) = 0,$$

since we can take the real-valued random variable $X_{(\boldsymbol{q},\psi)}$ out of any integrals involving space or time as a multiplicative constant. Hence, $X_{(\boldsymbol{q},\psi)} = 0$ almost surely for every fixed but arbitrary deterministic test function $(\boldsymbol{q},\psi) \in \mathcal{Q}(0,T)$, which shows that the limiting functions $(\eta, v, \boldsymbol{u})$ satisfy the continuous in time weak formulation. This is a result of the fact that $X_{(\boldsymbol{q},\psi)} = 0$ almost surely, and the definition of the real-valued random variable $X_{(\boldsymbol{q},\psi)}$ in (83).

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