Polynomial inclusions: definitions, applications, and open problems

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Abstract

Predictive modelling in physical science and engineering is mostly based on solving certain partial differential equations where the complexity of solutions is dictated by the geometry of the domain. Motivated by the broad applications of explicit solutions for spherical and ellipsoidal domains, in particular, the Eshelby's solution in elasticity, we propose a generalization of ellipsoidal shapes called polynomial inclusions. A polynomial inclusion (or p-inclusion for brevity) of degree k is defined as a smooth, connected and bounded body whose Newtonian potential is a polynomial of degree k inside the body. From this viewpoint, ellipsoids are identified as the only p-inclusions of degree two; many fundamental problems in various physical settings admit simple closed-form solutions for general p-inclusions as for ellipsoids. Therefore, we anticipate that p-inclusions will be useful for applications including predictive materials models, optimal designs, and inverse problems. However, the existence of p-inclusions beyond degree two is not obvious, not to mention their explicit algebraic parameterizations.

In this work, we explore alternative definitions and properties of p-inclusions in the context of potential theory. Based on the theory of variational inequalities, we show that p-inclusions do exist for certain polynomials, though a complete characterization remains open. We reformulate the determination of surfaces of p-inclusions as nonlocal geometric flows which are convenient for numerical simulations and studying geometric properties of p-inclusions. In two dimensions, by the method of conformal mapping we find an explicit algebraic parameterization of p-inclusions. We also propose a few open problems whose solution will deepen our understanding of relations between domain geometry, Newtonian potentials, and solutions to general partial differential equations. We conclude by presenting examples of applications of p-inclusions in the context of Eshelby inclusion problems and magnet designs.

Keywords: polynomial inclusion, potential theory, Eshelby inclusion problem

1. Introduction

From the viewpoint of field theory, most physical laws, if not all, can be formulated as partial differential equations, e.g., the gravitation theory of Newton, the electromagnetic theory of Maxwell (Maxwell, 1873; Jackson, 1999), and the general relativity of Einstein (1916). Predictions by these theories are achieved by solving *boundary value problems*. In two and higher dimensions, the complexity of solutions for typical boundary conditions and source terms is dictated by the geometry of the domain, and for some geometries, the solutions can be exceptionally simple. [1] For example, the Newtonian potentials are simple quadratic functions of positions inside homogeneous spherical and ellipsoidal bodies and the Einstein field equations admit the closed-form Schwarzschild solution outside a spherical mass in the theory of general relativity (Schwarzschild, 1916).

To be precise, below we restrict our discussions to the Newtonian potential problem described by the Poisson equation, though our motivations arise from and results apply to problems in more general physical settings (Eshelby,

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[1] In the context of nonlinear elasticity, there are important exceptions: families of exact or universal solutions are discovered in seminal works of Rivlin (1948, 1949a,b) and Ericksen (1954, 1955), and extended in recent works of Goodbrake et al. (2020); Yavari (2021); Yavari and Goriely (2022).

1957, 1961; Milton, 2002; Cherkaev, 2000). The simplicity of solutions for spherical bodies can be understood from symmetry; the simplicity of solutions for ellipsoidal bodies is related with the quadratic equations for ellipsoids (Poisson, 1826; Maxwell, 1873). Examples of closed-form nontrivial solutions of partial differential equations are rare for domains other than spheres, ellipsoids, polygons and polyhedras (Wu and Yin, 2021; Wu et al., 2021). On the other hand, the known explicit solutions for these bodies may be insufficient for practical shape design problems, e.g., designing geometries to realize sextupole or octupole magnetic fields in syncrotrons (Halbach, 1980; Mallinso, 1973), or for finding shapes of inhomogeneities with minimum stress concentration or maximum effective conductivity (Cherkaev and Gibiansky, 1996; Allaire, 2002; Lipton, 2004, 2005; Vigdergauz, 2006, 2008). We are therefore motivated to ponder on the possible generalizations of ellipsoidal shapes such that the Newtonian potentials of these special geometries can again be expressed in terms of elementary functions and definitive physical predictions can be made without an involved numerical procedure of solving partial differential equations.

One kind of generalizations have been proposed by Liu et al. (2007, 2008), which are named as *E-inclusions* for their association with ellipsoids, Eshelby's solutions, and extremal properties of such geometric shapes. Restricted to two-dimensional simply-connected bodies in a periodic unit cell, they were first discovered by Vigdergauz and referred to as Vigdergauz microstructures (Vigdergauz, 1986). The E-inclusions retain the properties of *piecewise* quadratic Newtonian potential in each of the connected components of the body in spite of the mutual interactions between individual components. Following the celebrated works of Eshelby (1957), we have found a number of intriguing applications of (periodic) E-inclusions in the modeling and optimal designs of composite materials (Liu et al., 2008; Liu, 2010a, 2011).

In this work, we propose a second kind of generalization of ellipsoidal bodies, i.e., a bounded body whose Newtonian potential is a polynomial function of degree k inside the body (cf., Definition 1). As is well-known, the Newtonian potential problem for a homogeneous bounded body is unique within an additive constant; requiring the potential being a polynomial inside the body places strong restriction on the geometry of the body and the existence of such geometries is not obvious at all. Thanks to the theory of *variational inequalities*, we can show such geometric shapes indeed exist for some polynomial p of degree $k > 2$. We call such geometric shapes **polynomial inclusions** (or **p-inclusions** for brevity) associated with the polynomial p. By this definition, ellipsoids can be regarded as polynomial inclusions of degree two.

Below we will present the precise definitions of p-inclusions and our preliminary results about these geometric shapes. Two related nonlocal geometric flow or free boundary problems are formulated which can be used to prove the existence and local uniqueness of p-inclusions and to numerically compute p-inclusions for given classes of polynomials. In two dimensions, the powerful method of conformal mapping enables us to construct an explicit parameterization of general p-inclusions and relate the parameterization with the Newtonian potential. However, in three and higher dimensions, explicit parameterization of general p-inclusions remains open. In addition, we propose a conjecture pertaining to the properties of p-inclusions for k-harmonic potential problems ($k > 1$). Solutions or methods toward these problems will deepen our understanding of these novel geometric shapes, and *most importantly*, inspire us to achieve simple, closed-form solutions to fundamental problems in broad physical setting that will be critical for developing novel materials models, verifying numerical algorithms, and validating experimental measurements (Eshelby, 1957; Brown, 1962; Mura, 1987).

The paper will be organized as follows. In Section 2, we introduce a few equivalent definitions of p-inclusions in terms of Newtonian potential or single-layer potential. We then show the existence of p-inclusions by the theory of variational inequalities in Section 3. In Section 4, we reformulate the determination of surfaces of p-inclusions as nonlocal geometric flow problems and present a few examples of p-inclusions obtained by numerical simulations. In Section 5, we restrict ourselves to two-dimensional space, and by the method of conformal mapping, construct an algebraic parametrization of p-inclusions. Further, in Section 6 we present applications of p-inclusions. In particular, we obtain explicit solutions to the Eshelby inclusion problem and polarization/magnetization problem for two-dimensional p-inclusions. Hopefully, these solutions could motivate advancement to optimal or inverse design problems that are of engineering importance, e.g., minimizing the stress concentration in load-bearing structures and generating specific and large multi-pole magnetic fields in electric motors, MRI, or nuclear fusion devices.

Notation. For an n-tuple nonnegative integer index $(\alpha_1, \cdots, \alpha_n)$ and a vector $\mathbf{x} = (x_1, \cdots, x_n)$, we denote by

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \qquad \mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \qquad \nabla^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

For integer $k \geq 1$, $\nabla^k \phi$ denotes the collection of all partial derivatives of ϕ of order-k. By the usual notation, we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ by $z = x_1 + ix_2$. Let $\bar{z} = x_1 - ix_2$ be the complex conjugate of z. The chain rule implies

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right), \tag{1.1}$$

and hence the Laplace operator can be written as

$$\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}}. \tag{1.2}$$

2. Definitions of polynomial inclusions

Let $G(\mathbf{x})$ be the fundamental solution of the Laplacian on $\mathbb{R}^n$ (i.e., $-\Delta G(\mathbf{x}) = \delta(\mathbf{x})$):

$$G(\mathbf{x}) = \begin{cases} -\frac{1}{2\pi}\log(|\mathbf{x}|) & \text{if } n = 2, \\ \frac{1}{(n-2)\omega_n |\mathbf{x}|^{n-2}} & \text{if } n \geq 3, \end{cases} \tag{2.1}$$

where ω_n is the surface area of the unit sphere S^{n-1} in $\mathbb{R}^n$. From the definition, it is clear that the Green's function $G(\mathbf{x}-\mathbf{y})$ admits the following properties:

$$\partial_{x_j} G(\mathbf{x}-\mathbf{y}) = -\partial_{y_j} G(\mathbf{x}-\mathbf{y}) \qquad \forall\, j = 1, \cdots, n. \tag{2.2}$$

For a bounded source term $f : \mathbb{R}^n \to \mathbb{R}$, we refer to

$$u(\mathbf{x}) = \int_{\mathbb{R}^n} G(\mathbf{x}-\mathbf{y}) f(\mathbf{y})\, \mathrm{d}\mathbf{y} \tag{2.3}$$

as the Newtonian potential induced by f (Gilbarg and Trudinger, 1983). Subsequently, by a body Ω we mean a connected open bounded domain $\Omega \subset \mathbb{R}^n$ whose boundary $\partial\Omega$ is at least continuously differentiable. The Newtonian potential of this body Ω is given by (2.3) with $f = \chi_\Omega$, where χ_Ω, equal to one on Ω and zero otherwise, is the characteristic function of Ω. Clearly, the Newtonian potential u of the body Ω satisfies the Poisson equation

$$-\Delta u = \chi_\Omega \qquad \text{on } \mathbb{R}^n. \tag{2.4}$$

It is well-known (Gilbarg and Trudinger, 1983) that the Newtonian potential u induced by a body Ω is continuously differentiable, and the second gradient $\nabla\nabla u$ (the Hessian) is uniformly bounded almost everywhere (i.e. $u \in C^{1,1}(\mathbb{R}^n)$) and satisfies

$$[\![\nabla\nabla u]\!] = \mathbf{n} \otimes \mathbf{n} \qquad \text{on } \partial\Omega. \tag{2.5}$$

Here and subsequently, $[\![\nabla\nabla u]\!] = |_{\partial\Omega^+} - |_{\partial\Omega^-}$ denotes the difference of the boundary values between the exterior of Ω ($+$ side) and the interior of Ω ($-$ side), and $\mathbf{n}$ is the unit outward normal on $\partial\Omega$.

Physically, the Newtonian potential induced by a body Ω can also be interpreted as the electric potential induced by uniformly distributed charges on Ω. In this context, the electric potential induced by a surface charge distribution $\sigma : \partial\Omega \to \mathbb{R}$ is referred to as a single-layer potential and given by

$$v(\mathbf{x}) = \int_{\partial\Omega} \sigma(\mathbf{y}) G(\mathbf{x}-\mathbf{y}) \mathrm{d}S(\mathbf{y}). \tag{2.6}$$

The single-layer potential (2.6) is continuous on $\mathbb{R}^n$ (more precisely, $\in C^{0,1}$) and satisfies that

$$\begin{cases} -\Delta v = 0 & \text{on } \mathbb{R}^n \setminus \partial\Omega, \\ -[\![\nabla v]\!] = \sigma \mathbf{n} & \text{on } \partial\Omega, \\ |\nabla v| \to 0 & \text{as } |\mathbf{x}| \to +\infty. \end{cases} \tag{2.7}$$

We remark that relations in parallel to (2.6)-(2.7) exist in the context of elasticity and Eshelby inclusion problem. For instance, a uniform eigenstrain in an inclusion is equivalent to certain prescribed surface tractions.

To verify that (2.6) implies (2.7), in particular, the jump condition $(2.7)_2$, we extend the definition of the unit normal $\mathbf{n}(\mathbf{y})$, surface charge density $\sigma(\mathbf{y})$, and characteristic function χ_Ω smoothly and trivially to the entire space. With an abuse of notion, (2.6) can be rewritten as

$$\begin{aligned} \int_{\partial\Omega} \sigma(\mathbf{y}) G(\mathbf{x}-\mathbf{y}) \mathrm{d}S(\mathbf{y}) &= \int_{\partial\Omega} n_k n_k \sigma(\mathbf{y}) G(\mathbf{x}-\mathbf{y}) \mathrm{d}S(\mathbf{y}) \\ &= \int_{\mathbb{R}^n} [n_k \sigma(\mathbf{y}) G(\mathbf{x}-\mathbf{y})]_{,y_k} \chi_\Omega d\mathbf{y} \\ &= \int_{\mathbb{R}^n} \Big\{ [n_k \sigma(\mathbf{y}) G(\mathbf{x}-\mathbf{y}) \chi_\Omega]_{,y_k} - n_k \sigma(\mathbf{y}) G(\mathbf{x}-\mathbf{y}) \partial_{y_k} \chi_\Omega \Big\} d\mathbf{y} \\ &= \int_{\mathbb{R}^n} \underbrace{-(\sigma(\mathbf{y}) \mathbf{n} \cdot \nabla \chi_\Omega)}_{\text{charge density}} G(\mathbf{x}-\mathbf{y}) d\mathbf{y}, \end{aligned} \tag{2.8}$$

where the second and last equalities follow from the divergence theorem. Therefore, for any regular domain $U \subset \mathbb{R}^n$, the total charges contained in U is given by

$$-\int_U (\sigma(\mathbf{y}) \mathbf{n} \cdot \nabla \chi_\Omega) d\mathbf{y} = \int_U [\sigma(\mathbf{y}) n_k]_{,y_k} \chi_\Omega d\mathbf{y} = \int_{U \cap \Omega} [\sigma(\mathbf{y}) n_k]_{,y_k} d\mathbf{y} = \int_{U \cap \partial\Omega} \sigma(\mathbf{y}) dS(\mathbf{y}). \tag{2.9}$$

In particular, for an infinitesimal cylinder $U_\varepsilon = \{\mathbf{y} + t\mathbf{n} : \mathbf{y} \in A_\varepsilon \subset \partial\Omega, t \in (-\varepsilon, \varepsilon)\}$ ($|A_\varepsilon|$ denotes the area of the infinitesimal surface element $A_\varepsilon \subset \partial\Omega$), the Gauss's Law implies that

$$|A_\varepsilon| \sigma(\mathbf{y}) \approx \int_{A_\varepsilon} \sigma(\mathbf{y}) dS(\mathbf{y}) = \int_{U_\varepsilon} -\Delta v d\mathbf{y} = -\int_{\partial U_\varepsilon} \nabla v \cdot \mathbf{n} dS(\mathbf{y}) \approx -([\![\nabla v]\!] \cdot \mathbf{n}) |A_\varepsilon|, \tag{2.10}$$

i.e., the second equation of (2.7).

Polynomial inclusions, as a generalization of ellipsoids, are defined as special shapes that induce polynomial interior Newtonian potential.

Definition 1. *Let $\Omega \subset \mathbb{R}^n$ be a connected bounded domain with smooth boundary and $p : \mathbb{R}^n \to \mathbb{R}$ be a polynomial. The body Ω is a polynomial inclusion or, for brevity, p-inclusion associated with the polynomial p if the Newtonian potential u of the body satisfies*

$$u = p \qquad \text{on } \Omega. \tag{2.11}$$

The degree of the p-inclusion Ω is the degree of the polynomial p.

From the symmetries (2.1)-(2.2) of the Green's function, it is clear that a translation, rotation, reflection and inversion of a p-inclusion remains as a p-inclusion of the same degree. Upon scaling $\Omega \to \Omega_\lambda := \{\lambda \mathbf{z} : \mathbf{z} \in \Omega\}$ for $\lambda > 0$, we find the Newtonian potential of Ω_λ is given by

$$u_\lambda(\mathbf{x}) = \int_{\Omega_\lambda} G(\mathbf{x}-\mathbf{y}) d\mathbf{y} = \lambda^n \int_\Omega G(\mathbf{x} - \lambda \mathbf{z}) d\mathbf{z} = \lambda^2 \int_\Omega G(\frac{\mathbf{x}}{\lambda} - \mathbf{z}) d\mathbf{z} = \lambda^2 u(\frac{\mathbf{x}}{\lambda}). \tag{2.12}$$

Therefore, if Ω is a p-inclusion associated with $p(\mathbf{x})$, Ω_λ is a p-inclusion associated with $\lambda^2 p(\frac{\mathbf{x}}{\lambda})$.

Polynomial inclusions can also be characterized in terms of magnetization (or polarization) problems. By (2.3)-(2.2), we find that the gradient of the Newtonain potential induced by a body Ω is given by

$$\nabla u(\mathbf{x}) = -\int_{\mathbb{R}^n} \nabla_{\mathbf{y}} G(\mathbf{x}-\mathbf{y})\chi_\Omega(\mathbf{y})\mathrm{d}\mathbf{y} = -\int_{\partial\Omega} \mathbf{n}G(\mathbf{x}-\mathbf{y})\mathrm{d}S(\mathbf{y}), \tag{2.13}$$

where the last equality follows from the divergence theorem. For any constant vector $\mathbf{m} \in \mathbb{R}^n$, let $u_{\mathbf{m}} := -\mathbf{m}\cdot\nabla u$. By (2.5) and (2.13) we verify that

$$\begin{cases} \Delta u_{\mathbf{m}} = 0 & \text{on } \mathbb{R}^n \setminus \partial\Omega, \\ [\![\nabla u_{\mathbf{m}}]\!] = -(\mathbf{m}\cdot\mathbf{n})\mathbf{n} & \text{on } \partial\Omega, \\ |\nabla u_{\mathbf{m}}| \to 0 & \text{as } |\mathbf{x}| \to +\infty, \end{cases} \tag{2.14}$$

which can also be written as

$$\begin{cases} \mathrm{div}(-\nabla u_{\mathbf{m}} + \mathbf{m}\chi_\Omega) = 0 & \text{in } \mathbb{R}^n, \\ \qquad |\nabla u_{\mathbf{m}}| \to 0 & \text{as } |\mathbf{x}| \to +\infty. \end{cases} \tag{2.15}$$

Physically, the scalar function $u_{\mathbf{m}}$ is recognized as the magnetic potential induced by the uniformly magnetized body Ω with magnetization $\mathbf{m}$ or surface magnetic charge of density $\sigma = \mathbf{m}\cdot\mathbf{n}$. Therefore, p-inclusions can be equivalently defined as follows.

Definition 1′. Let $\Omega \subset \mathbb{R}^n$ be a connected bounded domain with smooth boundary. The body Ω is a p-inclusion of degree k if the magnetic field induced by the uniformly magnetized body Ω is a polynomial of degree $k-2$ inside the body for any directions of magnetization.

The calculations from (2.5) to (2.15) indicate intimate relations between Newtonian potentials and single-layer potentials. More generally, we consider the following potential for some constant $\alpha \in \mathbb{R}$, $\mathbf{m} \in \mathbb{R}^n$, and skew-symmetric matrix $\mathbf{W} \in \mathbb{R}^{n\times n}$:

$$v(\mathbf{x}) = 2\alpha u(\mathbf{x}) - (\mathbf{m} + \alpha\mathbf{x} + \mathbf{W}\mathbf{x})\cdot\nabla u(\mathbf{x}). \tag{2.16}$$

By index notation (summation over repeated indices), we have

$$\begin{aligned} v_{,k} &= \alpha u_{,k} - W_{ik}u_{,i} - (m_i + \alpha x_i + W_{ij}x_j)u_{,ik}, \\ v_{,kk} &= -(m_i + \alpha x_i + W_{ij}x_j)u_{,ikk}. \end{aligned}$$

Therefore, if $u \in C^{1,1}$ is the Newtonian potential of the body Ω, then $v : \mathbb{R}^n \to \mathbb{R}$ defined in (2.16) is continuous and satisfies

$$\begin{cases} \Delta v = 0 & \text{on } \mathbb{R}^n \setminus \partial\Omega, \\ [\![\nabla v]\!] = -(\mathbf{m}\cdot\mathbf{n} + \alpha\mathbf{x}\cdot\mathbf{n} + \mathbf{n}\cdot\mathbf{W}\mathbf{x})\mathbf{n} & \text{on } \partial\Omega, \\ |\nabla v| \to 0 & \text{as } |\mathbf{x}| \to +\infty. \end{cases} \tag{2.17}$$

In other words, the potential v defined by (2.16) can be alternatively regarded as a single layer potential with charge density $\sigma(\mathbf{y}) = (\mathbf{m} + \alpha\mathbf{y} + \mathbf{W}\mathbf{y})\cdot\mathbf{n}(\mathbf{y})$ on $\partial\Omega$:

$$v(\mathbf{x}) = \int_{\partial\Omega} (\mathbf{m} + \alpha\mathbf{y} + \mathbf{W}\mathbf{y})\cdot\mathbf{n}G(\mathbf{x}-\mathbf{y})dS(\mathbf{y}). \tag{2.18}$$

The following theorem summarizes properties of p-inclusions in terms of Newtonian potentials and single-layer potentials.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded connected domain with smooth boundary. The following statements are equivalent.*

1. *The body Ω is a polynomial inclusion of degree k.*
2. *The single-layer potential*

$$v(\mathbf{x};\mathbf{m}) = \int_{\partial\Omega} \mathbf{n}\cdot\mathbf{m}G(\mathbf{x}-\mathbf{y})\mathrm{d}S(\mathbf{y}) \tag{2.19}$$

is a polynomial of degree $k-1$ on Ω for any $\mathbf{m}\in\mathbb{R}^n$.

3. *Independent of the position of the body Ω in space, the single-layer potential*

$$v(\mathbf{x};\alpha) = \int_{\partial\Omega} \alpha\mathbf{n}\cdot\mathbf{y}G(\mathbf{x}-\mathbf{y})\mathrm{d}S(\mathbf{y}) \tag{2.20}$$

is a polynomial of degree k on Ω for any $\alpha\in\mathbb{R}$.

4. *Independent of the position of the body Ω in space, the single-layer potential*

$$v(\mathbf{x};\mathbf{W}) = \int_{\partial\Omega} (\mathbf{n}\cdot\mathbf{W}\mathbf{y})G(\mathbf{x}-\mathbf{y})\mathrm{d}S(\mathbf{y}) \tag{2.21}$$

is a polynomial of degree k on Ω for any skew-symmetric matrix $\mathbf{W}\in\mathbb{R}^{n\times n}$.

Proof: $1\Leftrightarrow 2, 1\Rightarrow 3,4$ follow immediately from (2.13), and (2.16)-(2.18).
$3\Rightarrow 2$. Consider a translation by $\mathbf{m}\in\mathbb{R}^n$ of the body Ω: $\Omega\to\Omega' = \Omega+\mathbf{m} = \{\mathbf{x}+\mathbf{m} : \mathbf{x}\in\Omega\}$. Then the single-layer potential for $\partial\Omega'$ defined by (2.20) is given by

$$\begin{aligned} v'(\mathbf{x};\alpha) &= \alpha\int_{\partial\Omega'} \mathbf{n}\cdot(\mathbf{y}'-\mathbf{m}+\mathbf{m})G((\mathbf{x}-\mathbf{m})-(\mathbf{y}'-\mathbf{m}))\mathrm{d}S(\mathbf{y}') \\ &= \underbrace{\alpha\int_{\partial\Omega} \mathbf{n}\cdot\mathbf{y}G((\mathbf{x}-\mathbf{m})-\mathbf{y})\mathrm{d}S(\mathbf{y})}_{=v(\mathbf{x}-\mathbf{m};\alpha)\text{ defined in (2.20)}} + \alpha\int_{\partial\Omega} \mathbf{n}\cdot\mathbf{m}G((\mathbf{x}-\mathbf{m})-\mathbf{y})\mathrm{d}S(\mathbf{y}), \end{aligned} \tag{2.22}$$

which implies that the single-layer potential (2.19) must be a polynomial of degree $k-1$ on Ω for any $\mathbf{m}\in\mathbb{R}^n$. Similar calculations work for $4\Rightarrow 2$.

■

Let u_i be the Newtonian potential induced by $f(\mathbf{x}) = x_i\chi_\Omega(\mathbf{x})$, which are collected to form a vectorial Newtonian potential:

$$\mathbf{u}(\mathbf{x}) = \int_{\mathbb{R}^n} \mathbf{y}\chi_\Omega G(\mathbf{x}-\mathbf{y})d\mathbf{y}. \tag{2.23}$$

By the divergence theorem, we rewrite (2.20) and (2.21) as

$$\begin{aligned} v(\mathbf{x};\alpha) &= \alpha\int_{\Omega} nG(\mathbf{x}-\mathbf{y}) + \mathbf{y}\cdot\nabla_{\mathbf{y}}G(\mathbf{x}-\mathbf{y})\mathrm{d}\mathbf{y} = \alpha(nu(\mathbf{x}) - \mathrm{div}\mathbf{u}(\mathbf{x})), \\ v(\mathbf{x};\mathbf{W}) &= \int_{\Omega} (\mathbf{W}\mathbf{y})\cdot\nabla_{\mathbf{y}}G(\mathbf{x}-\mathbf{y})\mathrm{d}S(\mathbf{y}) = \mathbf{W}\cdot\nabla_{\mathbf{x}}\int_{\Omega}\mathbf{y}G(\mathbf{x}-\mathbf{y})\mathrm{d}S(\mathbf{y}) = \mathbf{W}\cdot\nabla\mathbf{u}(\mathbf{x}). \end{aligned} \tag{2.24}$$

Recognizing that independent components of $\mathbf{W}\cdot\nabla\mathbf{u}(\mathbf{x})$ for skew-symmetric matrix $\mathbf{W}$ are equivalent to $\nabla\times\mathbf{u}$ or curl$\mathbf{u}$, we see that the body Ω being a p-inclusion of degree k implies the divergence and curl of the vectorial potential $\mathbf{u}$ defined by (2.23) must be polynomials of degree k on the body Ω. In other words, if the vectorial potential $\mathbf{u}$ defined by (2.23) is a polynomial of degree $k+1$ on Ω independent of the position of the body Ω in space, the body Ω must be a p-inclusion of degree k. We speculate that the converse holds, i.e., Ω being p-inclusion of degree k implies the vectorial potential $\mathbf{u}$ defined by (2.23) is a polynomial of degree $k+1$ on Ω (the case $k=2$ is proved in, e.g. Liu (2013) and the following Theorem 3.1).

3. Existence and examples of polynomial inclusions

3.1. Necessary conditions for the existence of p-inclusions

We notice that equation (2.11) is an overdetermined condition for the Poisson equation (2.4) and places strong restrictions on the geometry of Ω. For the existence of corresponding p-inclusion, the polynomial p necessarily satisfies the following conditions:

1. If the spatial dimension $n \geq 3$, p is positive on the p-inclusion Ω.
2. The polynomial p admits a local maximum in the interior of p-inclusion Ω.
3. For any vector $\mathbf{m} \in \mathbb{R}^n$, the polynomial p and associated p-inclusion Ω satisfies

$$-\int_{\Omega} \mathbf{m} \cdot (\nabla\nabla p)\mathbf{m} = \int_{\mathbb{R}^n} |\nabla u_{\mathbf{m}}|^2 \geq 0. \tag{3.1}$$

The first and second conditions can be seen from the maximum principle: u being bounded from above must attain its global maximum inside Ω since $-\Delta u = \chi_\Omega \geq 0$ on $\mathbb{R}^n$. From (2.15), for any $\Omega \subset \mathbb{R}^n$ we have

$$\begin{aligned} 0 = \int_{\mathbb{R}^n} u_{\mathbf{m}} \mathrm{div}(-\nabla u_{\mathbf{m}} + \mathbf{m}\chi_\Omega) = \int_{\mathbb{R}^n} (\nabla u_{\mathbf{m}}) \cdot (-\nabla u_{\mathbf{m}} + \mathbf{m}\chi_\Omega) \quad \textit{i.e.}, \\ \int_{\Omega} (\nabla u_{\mathbf{m}}) \cdot \mathbf{m} = \int_{\mathbb{R}^n} |\nabla u_{\mathbf{m}}|^2 \geq 0 \qquad \forall\, \mathbf{m} \in \mathbb{R}^n. \end{aligned} \tag{3.2}$$

Therefore, if Ω is a p-inclusion, from Definition $1'$ we see (3.1) necessarily holds.

If the p-inclusion is of degree two, by (3.1) we see the quadratic polynomial $p = -\frac{1}{2}\mathbf{x} \cdot \mathbf{Q}\mathbf{x} + \cdots$ must be concave. Below we show the existence of polynomial inclusions for special polynomials of various kinds. A complete characterization of polynomials p for the existence of associated p-inclusions is not known.

3.2. Ellipsoids: p-inclusions of degree two

It is well-known that the Newtonian potential of an ellipsoid is quadratic inside the ellipsoid (Kellogg, 1929), and hence a p-inclusion of degree two in our terminology. Following Eshelby (1957), we reproduce the argument to show this fact which will motivate some interesting open problems for more general p-inclusions.

For any body $\Omega \subset \mathbb{R}^n$ and an interior point $\mathbf{x} \in \Omega$, by direct integration we find that

$$\begin{aligned} \nabla u(\mathbf{x}) &= \int_{\mathbb{R}^n} \nabla_{\mathbf{x}} G(\mathbf{x}-\mathbf{y})\chi_\Omega(\mathbf{y})\mathrm{d}\mathbf{y} \\ &= -\int_{\Omega} \frac{1}{\omega_n |\mathbf{x}-\mathbf{y}|^{n-1}} \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \mathrm{d}\mathbf{y} = \int_{S^{n-1}} \int_0^{r(\mathbf{x},\boldsymbol{l})} \frac{1}{\omega_n \rho^{n-1}} \boldsymbol{l} \rho^{n-1} d\rho dS(\boldsymbol{l}) \\ &= \frac{1}{\omega_n} \int_{S^{n-1}} \boldsymbol{l} r(\mathbf{x},\boldsymbol{l}) \mathrm{d}S(\boldsymbol{l}), \end{aligned} \tag{3.3}$$

where $\boldsymbol{l} = \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}$ is the unit directional vector starting from $\mathbf{x} \in \Omega$ and pointing to a boundary point $\mathbf{y} \in \partial\Omega$, S^{n-1} is the unit spherical surface on $\mathbb{R}^n$, and $r(\mathbf{x},\boldsymbol{l}) = |\mathbf{y}-\mathbf{x}|$ is the length of this vector.

Now, suppose that $\Omega \subset \mathbb{R}^n$ is an ellipsoid with semi-axis lengths $a_1, \cdots, a_n$. Since $\mathbf{y} \in \partial\Omega$, we have

$$\sum_{i=1}^{n} \frac{(x_i + r l_i)^2}{a_i^2} = 1 \qquad \Rightarrow \qquad \alpha(\boldsymbol{l})r^2 + 2\beta(\mathbf{x},\boldsymbol{l})r + \gamma(\mathbf{x}) = 0, \tag{3.4}$$

where

$$\alpha(\boldsymbol{l}) = \sum_{j=1}^{n} \frac{l_j^2}{a_j^2}, \qquad \beta(\mathbf{x},\boldsymbol{l}) = \sum_{i=1}^{n} \frac{x_i l_i}{a_i^2}, \qquad \gamma(\boldsymbol{l}) = \sum_{j=1}^{n} \frac{x_j^2}{a_j^2} - 1.$$

Solving (3.4) for r leads to

$$r(\mathbf{x},\boldsymbol{l}) = -\frac{\beta(\mathbf{x},\boldsymbol{l})}{\alpha(\boldsymbol{l})} + \Theta(\mathbf{x},\boldsymbol{l}), \qquad \Theta(\mathbf{x},\boldsymbol{l}) = \frac{\sqrt{\beta^2 - \alpha\gamma}}{\alpha}. \tag{3.5}$$

Note that $\Theta(\mathbf{x},\boldsymbol{l})$ is an even function of $\boldsymbol{l}$. Inserting the above equation into (3.3) we immediately find that $\nabla u(\mathbf{x})$ depends on $\mathbf{x}$ linearly in Ω, and hence $u(\mathbf{x})$ itself is a polynomial of degree two in Ω. Therefore, ellipsoids are p-inclusions of degree two by Definition 1. Further, it can be shown that a p-inclusion of degree two, if exists, is uniquely determined by the polynomial p and must be an ellipsoid (Kang and Milton, 2008; Liu, 2008).

We observe that the above argument relies on the simple *algebraic* parametrization of an ellipsoidal surface (3.4) and the explicit solution (3.5) to the single variable quadratic equation (3.4). Similar argument can be used to show other extraordinary properties of ellipsoids.

Theorem 3.1. *Consider the Poisson equation with a degree-k polynomial source term $P(\mathbf{x})$ on Ω for $\phi:\mathbb{R}^n\to\mathbb{R}$:*

$$\begin{cases} -\Delta\phi = P(\mathbf{x})\chi_\Omega & \text{on } \mathbb{R}^n, \\ |\nabla\phi|\to 0 & \text{as } |\mathbf{x}|\to+\infty, \end{cases} \tag{3.6}$$

and the k-harmonic equation ($k\geq 1$ is a positive integer) for $\psi:\mathbb{R}^n\to\mathbb{R}$:

$$\begin{cases} -\Delta^k\psi = \chi_\Omega & \text{on } \mathbb{R}^n, \\ |\nabla^{2k-1}\psi|\to 0 & \text{as } |\mathbf{x}|\to+\infty. \end{cases} \tag{3.7}$$

If $\Omega\subset\mathbb{R}^n$ is an ellipsoid, then a solution ϕ to (3.6) *must be a polynomial of degree $k+2$ on Ω, and a solution ψ to* (3.7) *must be a polynomial of degree $2k$ on Ω.*

Proof: For the k-harmonic problem (3.7), by successively differentiating the Green's function we find that for any multi-index $\alpha=(\alpha_1,\cdots,\alpha_n)$ with $|\alpha|=\alpha_1+\cdots+\alpha_n=2k-1$,

$$\begin{aligned} \nabla^\alpha\psi(\mathbf{x}) &\propto \int_\Omega \frac{Q_\alpha(\boldsymbol{l})}{\rho^{n-2k+|\alpha|}}\mathrm{d}\mathbf{y} = \int_{S^{n-1}}\int_0^{r(\mathbf{x},\boldsymbol{l})}\frac{Q_\alpha(\boldsymbol{l})}{\rho^{n-1}}\rho^{n-1}d\rho dS(\boldsymbol{l}) \\ &\propto \int_{S^{n-1}} Q_\alpha(\boldsymbol{l})r(\mathbf{x},\boldsymbol{l})\mathrm{d}S(\boldsymbol{l}), \end{aligned} \tag{3.8}$$

where $\rho=|\mathbf{y}-\mathbf{x}|$, $\boldsymbol{l}=|\mathbf{y}-\mathbf{x}|/\rho$, and $Q_\alpha(\boldsymbol{l})$ is a homogeneous polynomial of degree $|\alpha|=2k-1$, and hence odd with respect to $\boldsymbol{l}$. By (3.5), we conclude that $\nabla^\alpha\psi(\mathbf{x})$ is a linear function on Ω, and $\psi(\mathbf{x})$ is a polynomial of degree $2k$ on Ω.

For the Poisson problem (3.6) and a multi-index $\alpha=(\alpha_1,\cdots,\alpha_n)$ with $|\alpha|=\alpha_1+\cdots+\alpha_n=k+1$, direct differentiating gives rise to

$$\nabla^\alpha\phi(\mathbf{x}) \propto \int_\Omega \frac{P(\mathbf{y})\tilde{Q}_\alpha(\boldsymbol{l})}{\rho^{n-2+|\alpha|}}d\mathbf{y}, \tag{3.9}$$

where $\tilde{Q}_\alpha(\boldsymbol{l})$ is a homogeneous polynomial of degree $|\alpha|=k+1$. Without loss of generality, suppose the polynomial source is a monomial of degree k: $P(\mathbf{y})=\mathbf{y}^\beta$ for some multi-index β with $|\beta|=k$. By multinomial theorem we rewrite the source polynomial as

$$P(\mathbf{y})=P(\mathbf{y}-\mathbf{x}+\mathbf{x})=\sum_{\gamma\leq\beta}C_\gamma\rho^{|\gamma|}\boldsymbol{l}^\gamma\mathbf{x}^{\beta-\gamma}=\rho^k\boldsymbol{l}^\beta+\rho^{k-1}\sum_{|\gamma|=k-1}C_\gamma\boldsymbol{l}^\gamma\mathbf{x}^{\beta-\gamma}+\cdots,$$

where C_γ are multinomial coefficients. Inserting the above equation into (3.9), we see that the leading term $\rho^k\boldsymbol{l}^\beta$ will give rise to a linear term of $\mathbf{x}$ on Ω, and hence a solution to (3.6) is a polynomial of degree $k+2$ on Ω if the source polynomial $P(\mathbf{x})$ is of degree $k=1$. The result for general $k\geq 1$ follows by induction.

■

3.3. Existence of general p-inclusions

General p-inclusions of degree $k\geq 3$ can be constructed by the theory of variational inequalities (Kinderlehrer and Stampacchia, 1980; Friedman, 1982). For simplicity, in this section we restrict ourselves to a three or higher dimensional space ($n \geq 3$), though the argument can be applied to two dimensions as long as the boundary condition at infinity is appropriately controlled.

From the definition, the polynomial p necessarily satisfies $-\Delta p = 1$. Without loss of generality, we consider polynomials of form

$$p(\mathbf{x};d,t) = d - \frac{1}{2}\mathbf{x}\cdot\mathbf{Q}\mathbf{x} + t\tilde{p}(\mathbf{x}), \tag{3.10}$$

where $d > 0$, $t \in \mathbb{R}$, $\mathbf{Q} \in \mathbb{R}^{n\times n}_{\text{sym}}$ with $\text{Tr}\mathbf{Q} = 1$, $\tilde{p}(\mathbf{x}) = \sum_{m=3}^{k} h_m(\mathbf{x})$, and $h_m(\mathbf{x})$ ($m = 3, \cdots, k$) are homogeneous harmonic polynomials of degree m. Let

$$D = \{\mathbf{x} \in \mathbb{R}^n : p(\mathbf{x};d,t) > 0\} \supset D_0, \tag{3.11}$$

where D_0 is the connected component of D that contains a neighborhood of the origin. From the definition, we see that ∂D_0 is analytic and

$$p = 0 \qquad \text{on } \partial D_0.$$

If D_0 is bounded, we introduce the so-called "obstacle" function

$$\phi = \begin{cases} p & \text{on } D_0, \\ 0 & \text{on } \mathbb{R}^n \setminus D_0, \end{cases} \tag{3.12}$$

and consider the following obstacle/free boundary problem:

$$\min\Big\{G[w] = \int_{\mathbb{R}^n} \frac{1}{2}|\nabla w|^2 : w \geq \phi, w \in \mathbb{W}\Big\} =: G[u], \tag{3.13}$$

where

$$\mathbb{W} := \{w : \int_{\mathbb{R}^n} |\nabla w|^2 < +\infty \ \text{ and } \ w(\mathbf{x}) \to 0 \ \text{ as } \ |\mathbf{x}| \to +\infty\}.$$

The existence, uniqueness and regularity of the solution $u \in \mathbb{W}$ to the above obstacle problem (3.13) has been addressed in Caffarelli and Kinderlehrer (1980); Caffarelli (1998); Caffarelli and Salsa (2005) and references therein. In particular, it has been established that the solution u to the variational inequality (3.13) is differentiable whose derivatives are Lipshitz continuous ($u \in C^{1,1}(\mathbb{R}^n)$), and the coincident set

$$\bar{\Omega} = \{\mathbf{x} \in \mathbb{R}^n : u(\mathbf{x}) = \phi(\mathbf{x})\} \subset D_0 \tag{3.14}$$

is nontrivial with analytical boundary $\partial\Omega$. In other words, the solution u is precisely the Newtonian potential induced by Ω and satisfies that

$$\begin{cases} -\Delta u = \chi_\Omega & \text{on } \mathbb{R}^n, \\ u = p & \text{on } \Omega, \\ u(\mathbf{x}) \to 0 & \text{as } |\mathbf{x}| \to \infty. \end{cases} \tag{3.15}$$

Therefore, once the polynomial p in (3.10) is given, by numerically solving the obstacle/free boundary problem (3.13) we obtain the shape of the corresponding p-inclusion Ω from the coincident set (3.14).

When $t = 0$, we know that the coincident set Ω is an ellipsoid (Liu, 2008). From Definition 1 and the continuous dependence of the polynomial $p(\mathbf{x};d,t)$, domain D_0, and the solution u on t, we conclude the following existence theorem of p-inclusions.

Theorem 3.2. *For a given $d > 0$, there exists a $T > 0$ such that the domain D_0 defined in* (3.11) *is bounded, and the coincident set Ω defined by* (3.14) *is a p-inclusion associated with the polynomial $p(\mathbf{x};d,t)$ for any $t \in [0,T)$.*

Thereom 3.2 asserts the local existence of p-inclusions for infinitesimal t, i.e., we are only sure about the existence of p-inclusions that are quasi-ellipsoidal. The main difficulty for a stronger existence theorem lies in that the coincident set Ω could be complicated with multiple disconnected components for a general obstacle function. Nevertheless, we may numerically solve the obstacle problem (3.13) for some finite t and achieve explicit graphs of coincident sets that are connected, regular, significantly different from ellipsoids, and presumably p-inclusions. For instance, let

$$p(x_1,x_2,x_3) = \frac{1}{16} - \frac{1}{12}\left[\frac{11}{10}\left(\frac{4}{5}x_1 + \frac{3}{5}x_3\right)^2 + \frac{3}{10}x_2^2 + \frac{3}{5}\left(-\frac{3}{5}x_1 + \frac{4}{5}x_3\right)^2\right] - \frac{1}{36}\left(x_1^4 + x_2^4 - 6x_1^2x_2^2\right). \qquad (3.16)$$

Based on the method of quadratic programming (Liu, 2008), we solve the obstacle problem (3.13) and obtain the coincident set Ω, i.e., a p-inclusion in three dimensions as illustrated in Fig. 1.

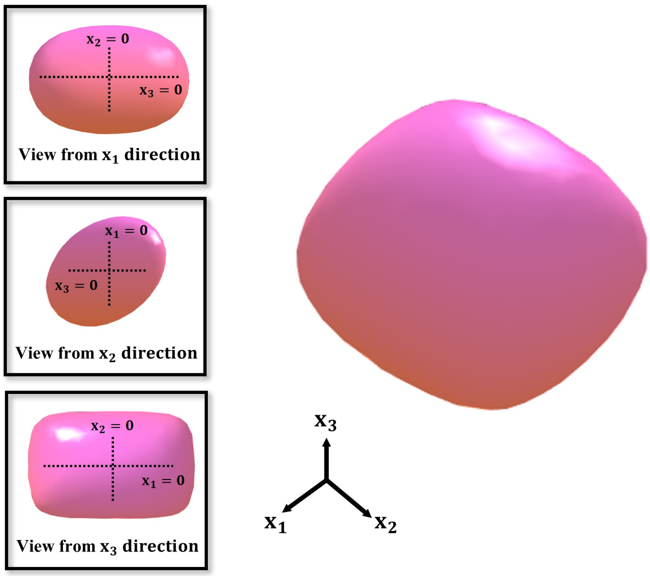


Figure 1: A p-inclusion associated with the polynomial (3.16).

Remark 3.3. *In general, the set $D = \{\mathbf{x} \in \mathbb{R}^n : p(\mathbf{x};d,t) > 0\}$ could contain multiple bounded components. Restricted to each of the components, we can apply the procedure from* (3.12) *to* (3.15) *to construct a p-inclusion associated with the polynomial $p(\mathbf{x};d,t)$. Therefore, we do not expect that a p-inclusion is uniquely determined by the polynomial $p(\mathbf{x};d,t)$ for general polynomials.*

4. Nonlocal geometric flows for p-inclusions

In this section, we reformulate the determination of p-inclusions as a nonlocal geometric flow or a Hamilton-Jacobi-type problem (Evans, 1998; Lewy, 1979). These alternative formulations are useful for analyzing properties of p-inclusions and numerical simulations. Again, we focus on the polynomial of the form (3.10):

$$p(\mathbf{x};d,t) = d - \frac{1}{2}\mathbf{x}\cdot\mathbf{Q}\mathbf{x} + t\tilde{p}(\mathbf{x}). \qquad (4.1)$$

The family of p-inclusions will be either denoted by Ω_t if $d > 0$ is fixed or Ω_d if $t > 0$ is fixed. We restrict ourselves to differentiable single-parameter families of p-inclusions in the sense that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathrm{dist}(\partial\Omega_d, \partial\Omega_{d+\varepsilon}) < +\infty,$$

where the distance between two surfaces is defined as

$$\mathrm{dist}(\partial\Omega_{d_1},\partial\Omega_{d_2}) = \min\{|\mathbf{x}_1-\mathbf{x}_2| : \mathbf{x}_i \in \partial\Omega_{d_i}\ (i=1,2)\}.$$

4.1. Nonlocal geometric flow for p-inclusion family Ω_t

Suppose the differentiable family of p-inclusions Ω_t for $t\in[0,T)$ exist and the boundary of $\partial\Omega_t$ is locally parametrized by

$$\partial\Omega_t \ni \mathbf{y} = \mathbf{f}(u^1,\cdots,u^{n-1};t) \qquad \text{where} \quad (u^1,\cdots,u^{n-1})\in U\subset\mathbb{R}^{n-1}. \tag{4.2}$$

Let $\mathbf{n}(u^1,\cdots,u^{n-1};t)$ be the unit outward normal on $\partial\Omega_t$ (or Gauss map), and

$$\mathbf{v}(u^1,\cdots,u^{n-1};t) = \partial_t\mathbf{f}(u^1,\cdots,u^{n-1};t). \tag{4.3}$$

The vector field $\mathbf{v}(\cdot,t):\partial\Omega_t\to\mathbb{R}^n$ can be interpreted as the velocity of surface points. Since tangential velocity amounts to a reparametrization of the surface, without loss of generality we restrict ourselves to motions with normal velocity:

$$\mathbf{v} = \partial_t\mathbf{f} = \sigma\mathbf{n}. \tag{4.4}$$

The normal speed $\sigma(\cdot,t):\partial\Omega_t\to\mathbb{R}$ can be determined from the condition that Ω_t is the p-inclusion associated with polynomial (4.1). From the definition of Newtonian potential (2.3) we find that for any $\mathbf{x}\in\Omega_t\cap\Omega_{t+\varepsilon}$,

$$\frac{1}{\varepsilon}[p(\mathbf{x};d,t+\varepsilon)-p(\mathbf{x};d,t)] = \frac{1}{\varepsilon}\int_{\mathbb{R}^n} G(\mathbf{x}-\mathbf{y})(\chi_{\Omega_{t+\varepsilon}}(\mathbf{y})-\chi_{\Omega_t}(\mathbf{y}))d\mathbf{y}. \tag{4.5}$$

As $\varepsilon\to 0$, we obtain that

$$\tilde{p}(\mathbf{x}) = \int_{\partial\Omega_t} G(\mathbf{x}-\mathbf{y})\sigma(\mathbf{y},t)d\mathbf{y} \qquad \forall\,\mathbf{x}\in\Omega_t. \tag{4.6}$$

Therefore, the single-layer potential $v(\cdot,t):\mathbb{R}^n\to\mathbb{R}$

$$v(\mathbf{x},t) = \int_{\partial\Omega_t} G(\mathbf{x}-\mathbf{y})\sigma(\mathbf{y},t)d\mathbf{y} \tag{4.7}$$

satisfies the following boundary value problem:[2]

$$\begin{cases} \Delta v(\mathbf{x},t) = 0 & \text{on } \mathbb{R}^n\setminus\partial\Omega_t, \\ v(\mathbf{x},t) = \tilde{p}(\mathbf{x}) & \text{on } \Omega_t, \\ |\nabla v(\mathbf{x},t)|\to 0 & \text{as } |\mathbf{x}|\to+\infty. \end{cases} \tag{4.8}$$

Upon solving the above problem on the exterior domain $\mathbb{R}^n\setminus\Omega_t$, we can uniquely determine the surface charge density $\sigma(\cdot,t)$ on $\partial\Omega_t$:

$$\sigma[\mathbf{f}](\mathbf{y},t) = -[\![\nabla v(\mathbf{y},t)]\!]\cdot\mathbf{n} \qquad \forall\,\mathbf{y}\in\partial\Omega_t, \tag{4.9}$$

where the notation $\sigma[\mathbf{f}]$ signifies that the surface charge density (4.9) depends on the overall surface $\partial\Omega_t$ (and hence the parametriztion $\mathbf{f}$). Inserting the above equation into (4.4), we obtain an evolution equation for $\mathbf{f}(\cdot,t):U\to\mathbb{R}^n$ which may be regarded as a nonlocal geometric flow.

More explicitly, let $\{\mathbf{e}_i : i=1,\cdots,n\}$ be the canonical orthonormal basis for $\mathbb{R}^n$,

$$f^i(u^1,\cdots,u^{n-1};t) = \mathbf{e}_i\cdot\mathbf{f}(u^1,\cdots,u^{n-1};t) \qquad (i=1,\cdots,n)$$

[2] In 2D, we shall require $\nabla v = \frac{\mathbf{x}}{r^2} + o(\frac{1}{r})$ as $r=|\mathbf{x}|\to+\infty$.

the components of parametrization $\mathbf{f}(u^1,\cdots,u^{n-1};t)$, and $(f^i_{,j}=\partial_{u^j}f^i)$

$$N^i=(-1)^{i-1}\det\underbrace{\begin{bmatrix} f^1_{,1} & f^2_{,1} & \cdots & f^n_{,1}\\ f^1_{,2} & f^2_{,2} & \cdots & f^n_{,2}\\ \cdots & \cdots & \cdots & \cdots\\ f^1_{,n-1} & f^2_{,n-1} & \cdots & f^n_{,n-1}\end{bmatrix}}_{i^{th}\text{ column is removed}}.$$

Then the unit outward normal vector $\mathbf{n}=n^i\mathbf{e}_i$ is identified as

$$n^i(u^1,\cdots,u^{n-1};t)=\frac{N^i}{|\mathbf{N}|}.$$

Consequently, the evolution equations for $f^i(\cdot,t):U\to\mathbb{R}$ $(i=1,\cdots,n)$ can be written as

$$\begin{cases}\partial_t f^i(u^1,\cdots,u^{n-1};t)=\sigma[\mathbf{f}]n^i(u^1,\cdots,u^{n-1};t),\\ f^i(u^1,\cdots,u^{n-1};t)|_{t=0}\quad\text{parametrizes the ellipsolid }\Omega_0.\end{cases}\tag{4.10}$$

In particular, in two dimensional space $\mathbb{R}^2$ the parametrization $\mathbf{y}=\mathbf{f}(u;t)$ for the curve $\partial\Omega_t$ should satisfy

$$\begin{cases}\partial_t f^1=\sigma[\mathbf{f}]\dfrac{f^2_{,u}}{\sqrt{(f^1_{,u})^2+(f^2_{,u})^2}},\\ \partial_t f^2=\sigma[\mathbf{f}]\dfrac{-f^1_{,u}}{\sqrt{(f^1_{,u})^2+(f^2_{,u})^2}}.\end{cases}\tag{4.11}$$

For numerical computation of surfaces $\partial\Omega_t$, it is sometimes more convenient to characterize $\partial\Omega_t$ by a level set function $F(\mathbf{x},t)$ in the sense that

$$F(\mathbf{x},t)<0\text{ if }\mathbf{x}\in\Omega_t,\qquad F(\mathbf{x},t)=0\text{ if }\mathbf{x}\in\partial\Omega_t,\qquad F(\mathbf{x},t)>0\text{ if }\mathbf{x}\in\mathbb{R}^n\setminus\bar{\Omega}_t.$$

Suppose $|\nabla F(\mathbf{x},t)|\neq 0$ on $\partial\Omega_t$. We identify

$$\mathbf{n}(\mathbf{y},t)=\left.\frac{\nabla F(\mathbf{x},t)}{|\nabla F(\mathbf{x},t)|}\right|_{\mathbf{x}=\mathbf{y}\in\partial\Omega_t}$$

as the unit outward normal on $\partial\Omega_t$. For $\varepsilon\ll 1$ and a fixed $t>0$, the normal velocity $\sigma:\partial\Omega_t\to\mathbb{R}$ introduced in (4.4) should satisfy

$$F(\mathbf{y}+\varepsilon\sigma\mathbf{n},t+\varepsilon)=o(\varepsilon)\qquad\forall\,\mathbf{y}\in\partial\Omega_t.\tag{4.12}$$

Dividing (4.12) by ε and sending $\varepsilon\to 0$, we find that

$$\sigma(\mathbf{y},t)=-\frac{\partial_t F(\mathbf{y},t)}{|\nabla_{\mathbf{y}}F(\mathbf{y},t)|}.\tag{4.13}$$

Further, we can extend the normal velocity $\sigma(\mathbf{y},t)$ continuously and trivially to the entire space $\mathbb{R}^n$. Then equation (4.13) can be rewritten as a Hamilton-Jacobi-type problem for $F(\cdot,t):\mathbb{R}^n\to\mathbb{R}$:

$$\begin{cases}\partial_t F(\mathbf{x},t)+\sigma(\mathbf{x},t)|\nabla F(\mathbf{x},t)|=0 & \text{if } t>0,\\ F(\mathbf{x},t)=F_0(\mathbf{x}) & \text{if } t=0,\end{cases}\tag{4.14}$$

where $F_0(\mathbf{y})=0$ characterizes the boundary of Ω_0, i.e., an ellipsoid associated with $p(\mathbf{x};d,t=0)$ (cf., (4.1)).

Based on (4.14), we can numerically compute p-inclusions associated with $p(\mathbf{x};d,t)$ for a finite t. For instance, starting from a sphere and for $\tilde{p} = \left(x_1^4 + x_2^4 - 6x_1^2x_2^2\right)$, we numerically solve (4.14) until $t = 1/18$ by finite difference. The level set $F(\mathbf{x},t) = 0$ is shown in Fig. 2 which is the corresponding p-inclusion.

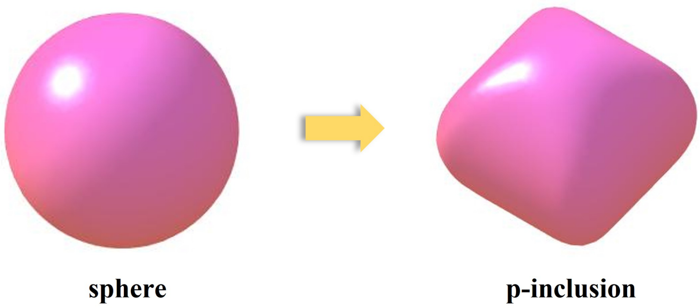


Figure 2: A p-inclusion computed by solving the Hamilton-Jacobi-type problem (4.14). The polynomial for the p-inclusion is given by $p(\mathbf{x};d,t) = \frac{1}{2} - \frac{1}{6}|\mathbf{x}|^2 + t\left(x_1^4 + x_2^4 - 6x_1^2x_2^2\right)$ with $t = \frac{1}{18}$.

4.2. *Nonlocal geometric flow for p-inclusion family Ω_d*

A similar argument can be applied to the family of p-inclusion Ω_d for a fixed $t > 0$. The key difference lies in the definition of normal speed or surface charge density. Instead of the boundary value problem (4.8), we should consider the equipotential problem:

$$\begin{cases} \Delta v(\mathbf{x},d) = 0 & \text{on } \mathbb{R}^n \setminus \partial\Omega_d, \\ v(\mathbf{x},d) = 1 & \text{on } \Omega_d, \\ |\nabla v(\mathbf{x},d)| \to 0 & \text{as } |\mathbf{x}| \to +\infty, \end{cases} \tag{4.15}$$

since $\partial_d p(\mathbf{x};d,t) = 1$. The unique solution to the above problem determines the surface charge density $\sigma(\cdot,d)$ on $\partial\Omega_d$:

$$\sigma[\mathbf{f}](\mathbf{y},d) = -[\![\nabla v(\mathbf{y},d)]\!]\cdot\mathbf{n} \qquad \forall\, \mathbf{y} \in \partial\Omega_d. \tag{4.16}$$

Consequently, the geometric flow for determining $\mathbf{y} = \mathbf{f}(u^1,\cdots,u^{n-1};d)$ remains to be the same form as $(4.10)_1$ with $\sigma[\mathbf{f}]$ replaced by (4.16). So is the Hamilton-Jacobi-type equation $(4.14)_1$ for the evolution of the level-set function $F(\cdot,t)$. The initial condition for the evolutions can be chosen at $d = 1$.

From (4.15), we see that $\partial\Omega_d$ attains the maximum of harmonic potential $v(\mathbf{x},d)$ on the exterior domain $\mathbb{R}^n \setminus \Omega_d$. From the Hopf lemma (Evans, 1998), we see $\sigma[\mathbf{f}](\mathbf{y},d) > 0$ on $\partial\Omega_d$. In other words, the flow $(4.10)_1$ is strictly outward, implying the following monotonicity theorem.

Theorem 4.1. *Suppose that the family of $p(\mathbf{x};d,t)$-inclusions Ω_d exists for some fixed $t > 0$ and $d \in (a,b)$ (cf., (4.1)). Then the family of p-inclusions Ω_d strictly increases as d increases:*

$$\Omega_{d_1} \subset \Omega_{d_2} \quad \text{and} \quad \operatorname{dist}(\partial\Omega_{d_1},\partial\Omega_{d_2}) > 0 \qquad \text{if } d_1 < d_2.$$

5. Algebraic parametrization of p-inclusions in two dimensions

In two dimensions, p-inclusions can be explicitly constructed by the method of conformal mapping, which also gives rise to explicit parametrizations of p-inclusions.

Theorem 5.1. *Let $S^1 := \{t \in \mathbb{C} : |t| = 1\}$ be the unit circle on the complex t-plane, $\mathbb{D}$ the interior domain of the unit circle, and $\mathbb{D}^c = \mathbb{C} \setminus \mathbb{D}$ the complement domain. Consider a mapping $\omega : \mathbb{C} \to \mathbb{C}$*

$$z = \omega(t) := \rho t + \frac{c_1}{t} + \cdots + \frac{c_k}{t^k} = \rho t + \sum_{m=1}^{k} \frac{c_m}{t^m}, \tag{5.1}$$

for some $\rho > 0$ and $c_m \in \mathbb{C}$ $(m = 1, \cdots, k)$. Denote by $\Omega = \omega(\mathbb{D})$ and $\Omega^c = \omega(\mathbb{D}^c)$ the image domains.

If the map $\omega : \mathbb{D}^c \to \Omega^c$ is bijective and

$$\frac{d}{dt}\omega(t) \neq 0 \qquad \forall\, t \in \mathbb{D}^c, \tag{5.2}$$

then the image curve $\Gamma = \{z = \omega(t) : t \in S^1\}$ is simple and closed, and the enclosed domain Ω is a p-inclusion of degree $k+1$.

Proof: The argument consists of two main ingredients: (i) the introduction of an auxiliary function satisfying $\bar{z} = D(z)$ on the image curve (Ru, 1999), and (ii) the *Plemelj formulas* (Muskhelishvili, 1963) for single-layer potential problems in two dimensions.

Step 1. By (5.2), we see that the map $\omega : \mathbb{D}^c \to \Omega^c$ is bijective and conformal, and the inverse map $\omega^{-1} : \Omega^c \to \mathbb{D}^c$ can be written as

$$t = \omega^{-1}(z) = \frac{z}{\rho} + \varsigma(z), \tag{5.3}$$

where $\varsigma(z)$ is analytic on $\Omega^c \cup \{\infty\}$ (satisfying $\varsigma(z) \to 0$ as $|z| \to +\infty$), and hence admits Laurent series representation for some $R > 0$:

$$\varsigma(z) = \sum_{n=1}^{\infty} \frac{b_n}{z^n} \qquad \text{if } |z| > R. \tag{5.4}$$

On the unit circle S^1, $\bar{t} = 1/t$. Taking the conjugate of (5.1) we obtain

$$\bar{z} = \rho \bar{t} + \sum_{m=1}^{k} \frac{\bar{c}_m}{\bar{t}^m} = \frac{\rho}{t} + \sum_{m=1}^{k} \bar{c}_m t^m \qquad \forall z \in \Gamma, \tag{5.5}$$

which, together with (5.3), motivates the introduction of the *auxiliary function* $D : \Omega^c \to \mathbb{C}$:

$$D(z) = \frac{\rho}{\omega^{-1}(z)} + \sum_{m=1}^{k} \bar{c}_m [\omega^{-1}(z)]^m = \frac{\rho^2}{z + \rho\varsigma(z)} + \sum_{m=1}^{k} \bar{c}_m [\frac{z}{\rho} + \varsigma(z)]^m. \tag{5.6}$$

It is clear that $D(z)$ is analytic on Ω^c since $\omega^{-1}(z) \neq 0$. In addition, the right-hand-side of (5.5) implies that $D(z) = O(z^k)$ as $z \to \infty$, and hence

$$D(z) = P_k(z) + \Theta(z) \qquad \text{on } \Omega^c \tag{5.7}$$

for some degree-k polynomial $P_k(z)$ and an analytic function $\Theta(z)$ on $\Omega^c \cup \{\infty\}$.

Step 2. Recall the *Plemelj formulas* that for any smooth $f : \Gamma \to \mathbb{C}$, the Cauchy integral

$$F(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta$$

is analytic on $(\{\infty\} \cup \mathbb{C}) \setminus \Gamma$ and satisfies that

$$\left[\left[\frac{d^k}{dz^k} F(\zeta)\right]\right] = -f^{(k)}(\zeta) \qquad \forall\, \zeta \in \Gamma. \tag{5.8}$$

Here and subsequently, the k^{th}-order derivative for a smooth function $f : \mathit{\Gamma} \to \mathbb{C}$ is recursively defined as

$$f^{(k)}(\zeta) = \lim_{\zeta_1 \to \zeta,\, \zeta_1 \in \mathit{\Gamma}} \frac{f^{(k-1)}(\zeta_1) - f^{(k)}(\zeta)}{\zeta_1 - \zeta}; \qquad f^{(0)}(\zeta) = f(\zeta). \tag{5.9}$$

We remark that the derivatives defined above are restricted to the curve $\mathit{\Gamma}$, equal to the regular complex derivatives if $f(z)$ is analytic on a neighborhood containing $\mathit{\Gamma}$, and well-defined for non-analytic smooth functions. For instance, if $f(z) = \bar{z}$, we have

$$f^{(1)}(\zeta) = \lim_{\zeta_1 \to \zeta,\, \zeta_1 \in \mathit{\Gamma}} \frac{\overline{\zeta_1} - \overline{\zeta}}{\zeta_1 - \zeta} = e^{-2i\gamma}, \tag{5.10}$$

where γ is the angle between the positive tangent on the counterclockwise contour $\mathit{\Gamma}$ and the positive real axis.

We will focus on the magnetization problem (2.14) and express the single-layer potential $u_{\mathbf{m}}$ in (2.14) in terms of Cauchy integral. For any $\mathbf{m} \in \mathbb{R}^2$, let $z_{\mathbf{m}} = m_1 - im_2$,

$$f(\zeta) = \frac{1}{2}(\overline{\zeta}\bar{z}_{\mathbf{m}} - \zeta z_{\mathbf{m}}) \qquad \zeta \in \mathit{\Gamma}, \tag{5.11}$$

and

$$\xi_{10}(z) = \frac{1}{2\pi i} \oint_{\mathit{\Gamma}} \frac{f(\zeta)\mathrm{d}\zeta}{\zeta - z} = \frac{1}{2\pi i * 2} \oint_{\mathit{\Gamma}} \frac{(\bar{z}_{\mathbf{m}} D(\zeta) - z_{\mathbf{m}}\zeta)}{\zeta - z}\mathrm{d}\zeta, \tag{5.12}$$

where the second equality follows from $\bar{z} = D(z)$ on $\mathit{\Gamma}$ (cf., (5.5)-(5.6)). From the *Plemelj formulas* (5.8) we see that $\mathrm{Re}\,[\![\xi_{10}(\zeta)]\!] = 0$,

$$\left[\!\!\left[\frac{d}{dz}\xi_{10}(z)\right]\!\!\right] = \left[\!\!\left[\Big(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\Big)\mathrm{Re}[\xi_{10}(z)]\right]\!\!\right] = -f^{(1)}(\zeta) = -\frac{1}{2}(\bar{z}_{\mathbf{m}} e^{-2i\gamma} - z_{\mathbf{m}}),$$

and hence

$$\begin{aligned}
[\![\nabla \mathrm{Re}[\xi_{10}(z)]]\!] &= [\partial_{x_1}\mathrm{Re}[\xi_{10}(z)],\ \partial_{x_2}\mathrm{Re}[\xi_{10}(z)]] \\
&= -\frac{1}{2}\left[(m_1\cos 2\gamma + m_2 \sin 2\gamma - m_1),\ (m_1 \sin 2\gamma - m_2\cos 2\gamma - m_2)\right] \\
&= -(-m_1 \sin\gamma + m_2\cos\gamma)[\sin\gamma,\ -\cos\gamma] \\
&= -(\mathbf{m}\cdot\mathbf{n})\mathbf{n},
\end{aligned}$$

where $\mathbf{n} = [\sin\gamma,\ -\cos\gamma]$ is the outward unit normal to $\mathit{\Gamma}$. Therefore, the solution to (2.14) is given by (Liu, 2009)

$$u_{\mathbf{m}}(x_1, x_2) = \mathrm{Re}[\xi_{10}(z)] = \frac{1}{2}[\xi_{10}(z) + \overline{\xi_{10}(z)}]. \tag{5.13}$$

Since the auxiliary function $D : \mathit{\Omega}^c \to \mathbb{C}$ is analytic, by (5.7) and the Cauchy theorem (Ahlfors, 1979), we conclude that

$$\xi_{10}(z) = \begin{cases} \frac{\bar{z}_{\mathbf{m}}}{2}\mathit{\Theta}(z) & \text{if } z \in \mathit{\Omega}^c, \\ -\frac{z_{\mathbf{m}} z}{2} + \frac{\bar{z}_{\mathbf{m}}}{2}P_k(z) & \text{if } z \in \mathit{\Omega}. \end{cases} \tag{5.14}$$

By (5.13) we conclude that the magnetic potential $u_{\mathbf{m}}$ is a polynomial of degree k for any $\mathbf{m} \in \mathbb{R}^2$, and hence the body $\mathit{\Omega}$ is a p-inclusion of degree $k+1$ (cf., Definition 1′).

■

Based on Theorem 5.1, the boundary of a two-dimensional p-inclusion can be parametrized as

$$x_1 + ix_2 = \rho(\cos\theta + i\sin\theta) + \sum_{m=1}^{k} c_m(\cos\theta - i\sin\theta)^m, \qquad \theta \in [0, 2\pi). \tag{5.15}$$

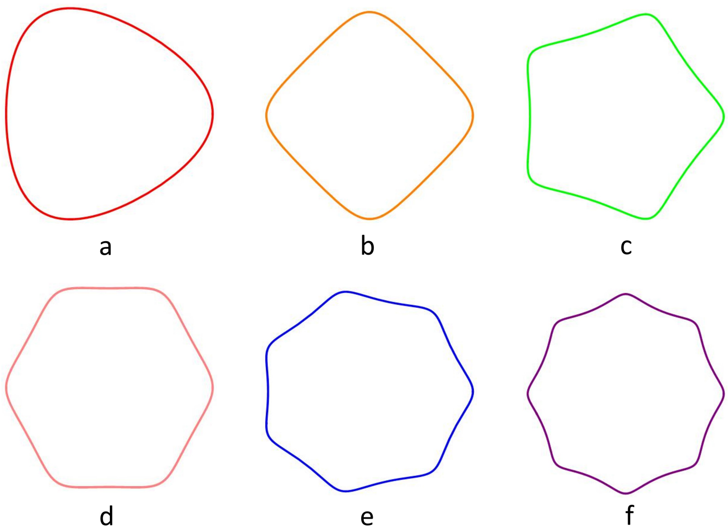


Figure 3: Two-dimensional p-inclusions with $(k+1)$-fold symmetry parametrized by $x_1 + ix_2 = (\cos\theta + i\sin\theta) + 0.1(\cos\theta - i\sin\theta)^k$, $k = 2, \cdots, 6$ (also called hypocycloid).

In Fig. 3, we plot some 2D p-inclusions with $(k+1)$-fold symmetries by choosing $\rho = 1$, $c_k = 0.1$ and vanishing c_m for $m < k$. Moreover, for any simply connected domain $\Omega \subset \mathbb{C}$, it is known form Ahlfors (1979); Muskhelishvili (1963); Markushevich (1977) that there exists a conformal mapping of the form

$$z = \omega(t) = \rho t + \sum_{m=0}^{\infty} \frac{c_m}{t^m},$$

such that the exterior of the unit circle in the t-plane is conformally mapped onto the exterior of Ω on the z-plane. Upon truncating the above infinite series, we conjecture that the sequence of $\Gamma_k := \{\rho t + \sum_{m=0}^{k} \frac{c_m}{t^m} : t \in S^1\}$ are boundaries of p-inclusions in regard of (5.1) and the sequence Γ_k converges to $\partial\Omega$.

In regard of (5.1), algebraic parameterizations and Newtonian potentials of general two-dimensional p-inclusions are, by and large, solved, though the relation between the algebraic parameterization and the polynomial p is more complicated than ellipses. The holy grail in this topic, in our opinion, is the following open problem:

Open problem. *In three and higher dimensions, find explicit parameterizations of p-inclusions in terms of elementary functions (if they exist), relate the parameterizations with the polynomial p, and characterize the collection of domains that can be approximated by p-inclusions.*

6. Applications

6.1. Explicit solutions to the Eshelby inclusion problem

The first application concerns solutions to the Eshelby inclusion problem for p-inclusions in linear elasticity which will be useful for generalizing the Eshelby's analysis in material models such as cracks, composites and solid-to-solid phase transition (Mura, 1987). Mathematically, the Eshelby inclusion problem aims to solve the mechanical equilibrium equation for the displacement $\mathbf{u}^e : \mathbb{R}^n \to \mathbb{R}^n$:

$$\mathrm{div}(\mathbf{C}\nabla\mathbf{u}^e + \sigma^* \chi_\Omega) = \mathbf{0}, \tag{6.1}$$

where $\mathbf{C} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ is the isotropic stiffness tensor with Lamé constants μ, λ, and $\sigma^* \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ is called eigenstress.

The importance of the above problems cannot be overstated, as evident in the two most cited papers (Eshelby, 1957; Mori and Tanaka, 1973) in the area of solid mechanics. The closed-form solutions obtained by Eshelby (1957) for ellipsoidal inclusions have played a significant role in the development of many material models for composites, fractures, phase transition, etc. Based on the Eshelby's solution and the Eshelby's equivalent inclusion method (EIM), physically important quantities such as energy, strain and stress fields can be explicitly calculated for ellipsoidal inclusions by solving a system of linear algebraic equations instead of the original partial differential equations.

However, generic microstructures (i.e., inclusions) in real-world materials are non-ellipsoidal; the analysis based on the Eshelby's solution, strictly speaking, should be regarded as some kind of approximations to the actual problem. To improve the accuracy of such analysis, we may solve the Eshelby inclusion problem (6.1) for more general inclusions that preserve the simplicity of the Eshelby's solution to certain extent. From this viewpoint, the p-inclusions, interpolating between ellipsoidal inclusions and general shapes and inducing polynomial interior Newtonian potentials, are precisely the generalization that we are looking for (Markenscoff, 1998a,b; Lubarda and Markenscoff, 1998; Ru, 1999). Specifically, one can use a "best-fitting" p-inclusion to approximate any inclusion with smooth boundary, in analogy with using the truncated Taylor series to approximate a smooth function. The closed-form solution of p-inclusions will greatly facilitate the subsequent analysis on the physical and mechanical property of the inclusion.

We now consider the Eshelby inclusion problem (6.1) for p-inclusions. First of all, since the stiffness tensor $\mathbf{C}$ is isotropic, by the method of Green's function or Fourier analysis, a solution to (6.1) for general inclusion $\Omega \subset \mathbb{R}^n$ can be expressed in terms of Newtonian potential and biharmonic potential as (Eshelby, 1957):

$$\mathbf{u}^e = \frac{1}{\mu}\sigma^*\nabla u - \frac{\lambda+\mu}{\mu(\lambda+2\mu)}(\nabla\nabla\nabla h)\sigma^*, \tag{6.2}$$

where $u : \mathbb{R}^n \to \mathbb{R}$ is the Newtonian potential induced by Ω (cf., (2.4)) and $h : \mathbb{R}^n \to \mathbb{R}$ is a biharmonic potential satisfying

$$\begin{cases} -\Delta^2 h = -\Delta\Delta h = -\Delta u = \chi_\Omega & \text{in } \mathbb{R}^n, \\ \nabla\nabla\nabla h \to 0 & \text{as } |\mathbf{x}| \to +\infty. \end{cases} \tag{6.3}$$

From (6.2) and (6.3), we immediately see that if the eigenstress $\sigma^* = \sigma^0\mathbf{I}$ with $\mathbf{I}$ being the identity matrix,

$$\mathbf{u}^e = \frac{\sigma^0}{\lambda+2\mu}\nabla u \qquad \text{in } \mathbb{R}^n. \tag{6.4}$$

Therefore, if Ω is a p-inclusion of degree k, *the interior strain induced by a dilatational eigenstress on Ω is precisely a polynomial of degree $k-2$.* Note that the relation between $\mathbf{u}^e$ and ∇u in (6.4) is also applicable to transversely isotropic media if the eigenstress is transversely isotropic (Yuan et al., 2022), which indicates the same property holds for transversely isotropic media with transversely isotropic eigenstress.

For a p-inclusion of degree $k = 2$, i.e., an ellipsoid, the property of degree-$(k-2)$ polynomial, i.e., uniform, interior strain holds for all uniform eigenstress, which is known as the Eshelby uniformity property of ellipsoids. This motivates us to pose a question: Does a p-inclusion of degree k always induce polynomial interior strains for any eigenstress, and if so, what is the degree of the polynomials and the relation between the interior strains and the polynomial p of the p-inclusion? To answer this question, we need to solve the Eshelby's problem (6.1) for non-dilatational eigenstress, i.e., the biharmonic potential problem in (6.3).

We are able to achieve an affirmative answer to the question in two dimensions by the method of conformal mapping. The question remains open in higher dimensions.

Theorem 6.1. *Consider the Eshelby inclusion problem* (6.1) *for an isotropic stiffness tensor* $\mathbf{C}$ *and uniform eigenstress* σ^* *in the inclusion* Ω*. If the body* Ω *is a p-inclusion of degree* $k+1$ *defined by the conformal map* (5.1)*, then*

(i) *a dilatational eigenstress in* Ω *induces polynomial interior strain* $\nabla\mathbf{u}^e$ *of degree* $k-1$ *in* Ω*, and*
(ii) *a non-dilatational eigenstress* $\sigma^* \in \mathbb{R}^{2\times 2}_{\text{sym}}$ *eigenstress in* Ω *induces a polynomial interior strain* $\nabla\mathbf{u}^e$ *of degree at most* $2(k-1)$ *in the p-inclusion* Ω.

Proof: Part (i) of the theorem follows from the definition of p-inclusion and (6.4). For part (ii), we need to construct a solution to the biharmonic problem (6.3), which will be divided into a few steps.

Step 1. Following Liu (2009), we focus on the single-layer potential $u_{\mathbf{m}}$ in (2.14) and gradient of biharmonic potential $v_{\mathbf{m}}$ for some $\mathbf{m} \in \mathbb{R}^n$:

$$u_{\mathbf{m}} = -\mathbf{m} \cdot \nabla u \qquad \text{and} \qquad v_{\mathbf{m}} = -\mathbf{m} \cdot \nabla h. \tag{6.5}$$

From (6.3), it is clear that

$$\begin{cases} \Delta v_{\mathbf{m}} = u_{\mathbf{m}} & \text{in } \mathbb{R}^n, \\ \nabla\nabla v_{\mathbf{m}} \to 0 & \text{as } |\mathbf{x}| \to +\infty. \end{cases} \tag{6.6}$$

To utilize methods from complex analysis in 2D, we rewrite $v_{\mathbf{m}} : \mathbb{R}^2 \to \mathbb{R}$ as a function of z and $\bar{z}$ in $\mathbb{C}$, i.e., $v_{\mathbf{m}} = v_{\mathbf{m}}(z, \bar{z})$, and $(6.6)_1$ as (cf., (1.2) and (5.13))

$$4 \frac{\partial^2 v_{\mathbf{m}}(z,\bar{z})}{\partial z \partial \bar{z}} = u_{\mathbf{m}}(z,\bar{z}) = \mathrm{Re}[\xi_{10}]. \tag{6.7}$$

The above equation motivates the following form of solution (Muskhelishvili, 1963):

$$v_{\mathbf{m}}(z,\bar{z}) = \mathrm{Re}[\xi_{20}(z) + \bar{z}\xi_{21}(z) + d_1 z + d_2], \tag{6.8}$$

where $d_1, d_2 \in \mathbb{C}$ are constants, and $\xi_{20}, \xi_{21} : \mathbb{C} \to \mathbb{C}$ are analytic on $\mathbb{C} \setminus \mathit{\Gamma}$. To find $\xi_{21}(z)$, inserting (6.8) into (6.7) we are led to

$$\frac{\mathrm{d}\xi_{21}(z)}{\mathrm{d}z} = \frac{1}{4}\xi_{10}(z) \qquad \text{on } \mathbb{C} \setminus \mathit{\Gamma}, \tag{6.9}$$

which, by direct integration, determines $\xi_{21}(z)$ in $\mathit{\Omega}$ and $\mathit{\Omega}^c$ within additive constants. Across the interface $\mathit{\Gamma}$, by *Plemelj formulas* (5.8) we find that

$$\left[\!\left[\bar{z}\frac{\mathrm{d}^2\xi_{21}(z)}{\mathrm{d}z^2} \right]\!\right]\Big|_{z=\zeta} = \frac{\overline{\zeta}}{4} \left[\!\left[\frac{\mathrm{d}\xi_{10}(z)}{\mathrm{d}z} \right]\!\right]\Big|_{z=\zeta} = -\frac{i}{4}\overline{\zeta} f^{(1)}(\zeta) \qquad \zeta \in \mathit{\Gamma}. \tag{6.10}$$

Step 2. Next, we construct the other unknown function ξ_{20} in (6.8). Since $u_{\mathbf{m}}$ is continuous on $\mathbb{C}$, $v_{\mathbf{m}}$ as determined by (6.7) necessarily admits continuous second-order derivatives, meaning that for any $\zeta \in \mathit{\Gamma}$,

$$\left[\!\left[\frac{\partial^2 v_{\mathbf{m}}(z,\bar{z})}{\partial z^2} \right]\!\right]\Big|_{z=\zeta} = \frac{1}{2} \left[\!\left[\bar{z}\frac{\mathrm{d}^2\xi_{21}(z)}{\mathrm{d}z^2} + \frac{\mathrm{d}^2\xi_{20}(z)}{\mathrm{d}z^2} \right]\!\right]\Big|_{z=\zeta} = 0. \tag{6.11}$$

By (6.10), we arrive at

$$\left[\!\left[\frac{\mathrm{d}^2\xi_{20}(z)}{\mathrm{d}z^2} \right]\!\right]\Big|_{z=\zeta} = \frac{i}{4}\overline{\zeta} f^{(1)}(\zeta) \qquad \zeta \in \mathit{\Gamma}, \tag{6.12}$$

which, by Plemelj formulas, indicates

$$\frac{\mathrm{d}^2\xi_{20}(z)}{\mathrm{d}z^2} = -\frac{1}{8\pi} \int_{\mathit{\Gamma}} \frac{\overline{\zeta} f^{(1)}(\zeta)}{\zeta - z} \mathrm{d}\zeta. \tag{6.13}$$

Step 3. In the last step, we recall that the auxiliary function $D(z)$ in (5.5)-(5.7) satisfies $\bar{z} = D(z)$ on $\mathit{\Gamma}$, and hence

$$f^{(1)}(\zeta) = \frac{1}{2i}\left[\frac{\mathrm{d}D(\zeta)}{\mathrm{d}\zeta}\bar{z}_{\mathbf{m}} - z_{\mathbf{m}} \right] = \frac{1}{2i}\left[\left(\frac{\mathrm{d}P_k(\zeta)}{\mathrm{d}\zeta} + \frac{\mathrm{d}\Theta(\zeta)}{\mathrm{d}\zeta} \right)\bar{z}_{\mathbf{m}} - z_{\mathbf{m}} \right]. \tag{6.14}$$

Inserting (5.7), (6.14) into (6.13), we find that

$$\frac{\mathrm{d}^2\xi_{20}(z)}{\mathrm{d}z^2} = -\frac{1}{16\pi i} \int_{\mathit{\Gamma}} [P_k(\zeta) + \Theta(\zeta)] \left[\left(\frac{\mathrm{d}P_k(\zeta)}{\mathrm{d}\zeta} + \frac{\mathrm{d}\Theta(\zeta)}{\mathrm{d}\zeta} \right)\bar{z}_{\mathbf{m}} - z_{\mathbf{m}} \right] \frac{\mathrm{d}\zeta}{\zeta - z} \qquad \forall\, z \in \mathit{\Omega}. \tag{6.15}$$

Since $P_k(z)$ is a polynomial of degree k and $\Theta(z)$ is analytic on $\Omega^c \cup \{\infty\}$, by Cauchy integral Theorem we see that $\frac{d^2\xi_{20}(z)}{dz^2}$ is necessarily a polynomial of degree $2k-1$ on Ω. By (6.8) we conclude that $\frac{\partial^3 v_{\mathbf{m}}}{\partial z^3}$ is a polynomial of degree at most $2k-2$ inside Ω. Since (i) all third-order derivatives of $v_{\mathbf{m}}$ can be expressed as linear combinations of $\frac{\partial^3 v_{\mathbf{m}}}{\partial z^3}$, $\frac{\partial^3 v_{\mathbf{m}}}{\partial z^2 \partial \bar{z}}$, and their conjugates, and (ii) $\frac{\partial^3 v_{\mathbf{m}}}{\partial z^2 \partial \bar{z}}$ is a polynomial of degree $k-1$, we see the induced elastic strain $\nabla \mathbf{u}^e$ (cf., (6.2) and (6.5)) must be a polynomial of degree at most $2k-2$.

■

In analogy with Theorem 3.1, we observe that not only the harmonic potential problems but also the biharmonic problems exhibit polynomial solutions for p-inclusions. This observation motivates us to propose the following conjecture:

Conjecture. *Suppose that $\Omega \subset \mathbb{R}^n$ is a p-inclusion of degree k ($k \geq 2$). Then the q-harmonic potential ψ for a positive integer $q \geq 1$ induced by the p-inclusion Ω, i.e., a solution to*

$$\begin{cases} -\Delta^q \psi = \chi_\Omega & \text{on } \mathbb{R}^n, \\ |\nabla^{2q-1}\psi| \to 0 & \text{as } |\mathbf{x}| \to +\infty, \end{cases} \tag{6.16}$$

coincides with a polynomial of degree qk inside Ω.

An explicit example. Below we present explicit expressions of the polynomial interior strain induced by uniform eigenstresses on two-dimensional p-inclusions. For simplicity, we focus on p-inclusions of degree $k+1$ that are parameterized by

$$x_1 + ix_2 = (\cos\theta + i\sin\theta) + e(\cos\theta - i\sin\theta)^k, \qquad \theta \in [0, 2\pi), \tag{6.17}$$

where $0 < e < 1$ is "eccentricity" of the p-inclusion. We remark that such shapes are also called hypotrochoids (Ru, 1999; Zou et al., 2010), and enjoy $(k+1)$-fold symmetry as can be seen in Fig. 3. From (5.14), we see that that

$$u_{\mathbf{m}} = -\mathbf{m}\cdot\nabla u = \mathrm{Re}(\xi_{10}(z)) = \mathrm{Re}(-\frac{z_{\mathbf{m}} z}{2} + \frac{\bar{z}_{\mathbf{m}}}{2} P_k(z)) \quad \text{if } z \in \Omega, \tag{6.18}$$

where the polynomial $P_k(z)$ of degree k is given by (cf., (5.12))

$$P_k(z) = \frac{1}{2\pi i}\oint_{\partial\Omega} \frac{\bar{\zeta}\,d\zeta}{\zeta - z} \qquad \forall z \in \Omega. \tag{6.19}$$

For the p-inclusion parameterized by (6.17), i.e., $S^1 \ni t \mapsto \zeta = t + e/t^k \in \Gamma$, upon a change of variable $\zeta \to t$ we obtain

$$\begin{aligned} P_k(z) &= \frac{1}{2\pi i}\oint_{S^1} \frac{(\frac{1}{t} + et^k)(1 - ekt^{-(k+1)})}{t + \frac{e}{t^k} - z}\,dt \\ &= \frac{1}{2\pi i}\oint_{S^1} (1 + et^{k+1})(1 - ekt^{-(k+1)}) \left[1 + \sum_{n=1}^{\infty} \frac{(z - \frac{e}{t^k})^n}{t^n}\right] \frac{1}{t^2}\,dt \\ &= ez^k, \end{aligned} \tag{6.20}$$

where the last equality follows from integration term by term and the fact $\oint_{S^1} t^m dt = 2\pi i$ if $m = -1$ and $= 0$ if $m \neq -1$. For more general p-inclusions, the polynomial $P_k(z)$ is calculated in Appendix A, (A2). Then by (6.18) the interior Newtonian potential of the p-inclusion is given by

$$p(x_1, x_2) = d - \frac{1}{4}(x_1^2 + x_2^2) + \frac{e}{4(k+1)}\left[(x_1 + ix_2)^{k+1} + (x_1 - ix_2)^{k+1}\right],$$

where $d > 0$ is a constant that accounts for the size of the p-inclusion. In particular, the interior Newtonian potential of the ellipse ($k = 1$) is

$$p(x_1, x_2) = d - \frac{1}{4}\left[(1-e)x_1^2 + (1+e)x_2^2\right].$$

Inserting (6.20) into (6.18) we obtain

$$\frac{\partial u_{\mathbf{m}}}{\partial z} = -\frac{z_{\mathbf{m}}}{4} + \frac{\bar{z}_{\mathbf{m}}}{4} kez^{k-1} \qquad z \in \Omega. \tag{6.21}$$

In addition, by (6.8)-(6.15) we see that for any $z \in \Omega$,

$$\begin{aligned} \frac{\partial^3 v_{\mathbf{m}}}{\partial z^2 \partial \bar{z}} &= -\frac{z_{\mathbf{m}}}{16} + \frac{\bar{z}_{\mathbf{m}}}{16} \frac{\mathrm{d}P_k(z)}{\mathrm{d}z}, \\ \frac{\partial^3 v_{\mathbf{m}}}{\partial z^3} &= \frac{z_{\mathbf{m}}}{16} \frac{\mathrm{d}P_k(z)}{\mathrm{d}z} + \frac{\bar{z}_{\mathbf{m}} \bar{z}}{16} \frac{\mathrm{d}^2 P_k(z)}{\mathrm{d}z^2} - \frac{\bar{z}_{\mathbf{m}}}{32\pi i} \frac{\mathrm{d}}{\mathrm{d}z} \left(\int_\Gamma \frac{\overline{\zeta}}{\zeta - z} \frac{\mathrm{d}D(\zeta)}{\mathrm{d}\zeta} \mathrm{d}\zeta \right), \end{aligned} \tag{6.22}$$

where $D(\zeta)$ is the auxiliary function introduced in (5.5). For the p-inclusions parameterized by (6.17), by similar calculations (see details in (A3), Appendix A) we find that

$$\int_\Gamma \frac{\overline{\zeta}}{\zeta - z} \frac{\mathrm{d}D(\zeta)}{\mathrm{d}\zeta} \mathrm{d}\zeta = 2\pi i \left[ke^2 z^{2k-1} + (k-1)(-ke^2+1)ez^{k-2}\right] \qquad \forall z \in \Omega, \tag{6.23}$$

and consequently,

$$\begin{aligned} \frac{\partial^3 v_{\mathbf{m}}}{\partial z^3} =& \frac{z_{\mathbf{m}}}{16} kez^{k-1} + \frac{\bar{z}_{\mathbf{m}}}{16} k(k-1)ez^{k-2}\bar{z} \\ &- \frac{\bar{z}_{\mathbf{m}}}{16}\left[(2k-1)ke^2 z^{2k-1} + (k-1)(k-2)(-ke^2+1)ez^{k-3}\right] \qquad z \in \Omega. \end{aligned} \tag{6.24}$$

From (6.21) and (6.22), we can find the expressions for $\nabla u_{\mathbf{m}}$ and $\nabla\nabla\nabla v_{\mathbf{m}}$ on the p-inclusion Ω by the following relations:

$$\frac{\partial u_{\mathbf{m}}}{\partial x_1} = 2\mathrm{Re}[\frac{\partial u_{\mathbf{m}}}{\partial z}], \quad \frac{\partial u_{\mathbf{m}}}{\partial x_2} = -2\mathrm{Im}[\frac{\partial u_{\mathbf{m}}}{\partial z}] \tag{6.25}$$

and

$$\begin{aligned} \frac{\partial^3 v_{\mathbf{m}}}{\partial x_1^3} &= 2\mathrm{Re}[\frac{\partial^3 v_{\mathbf{m}}}{\partial z^3} + 3\frac{\partial^3 v_{\mathbf{m}}}{\partial z^2 \partial \bar{z}}], \qquad & \frac{\partial^3 v_{\mathbf{m}}}{\partial x_1^2 \partial x_2} &= 2\mathrm{Im}[-\frac{\partial^3 v_{\mathbf{m}}}{\partial z^3} - \frac{\partial^3 v_{\mathbf{m}}}{\partial z^2 \partial \bar{z}}], \\ \frac{\partial^3 v_{\mathbf{m}}}{\partial x_2^3} &= 2\mathrm{Im}[\frac{\partial^3 v_{\mathbf{m}}}{\partial z^3} - 3\frac{\partial^3 v_{\mathbf{m}}}{\partial z^2 \partial \bar{z}}], \qquad & \frac{\partial^3 v_{\mathbf{m}}}{\partial x_2^2 \partial x_1} &= 2\mathrm{Re}[-\frac{\partial^3 v_{\mathbf{m}}}{\partial z^3} + \frac{\partial^3 v_{\mathbf{m}}}{\partial z^2 \partial \bar{z}}]. \end{aligned} \tag{6.26}$$

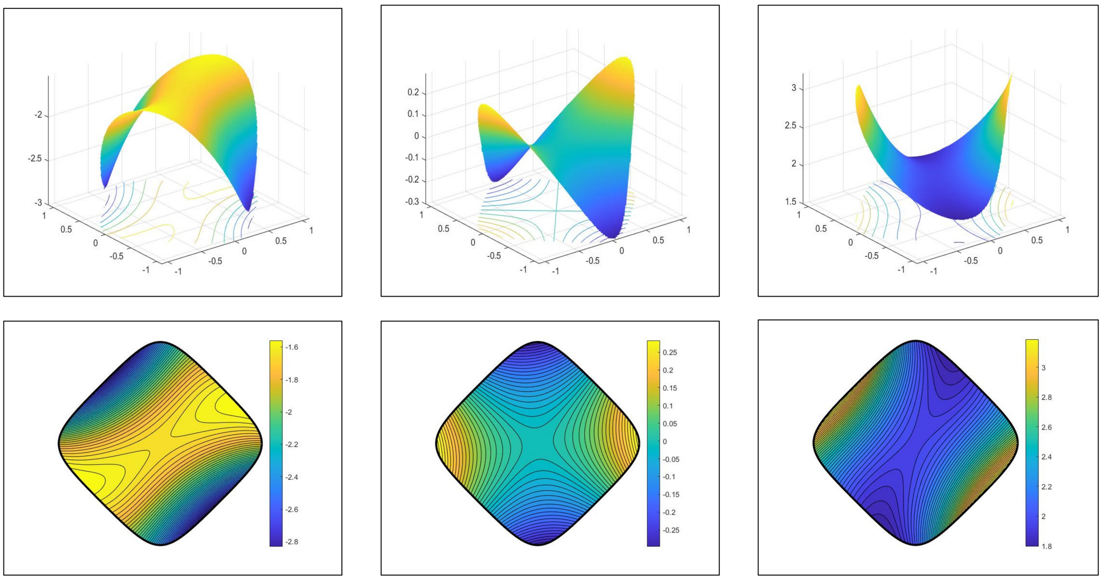
It is noteworthy that the above procedure for explicit expressions of $\nabla u_{\mathbf{m}}$ and $\nabla\nabla\nabla v_{\mathbf{m}}$ can be extended to arbitrary p-inclusions of form (5.1), which will be detailed in Appendix A.

To find the explicit expression of the interior strain as determined by the Eshelby inclusion problem (6.1), we denote the column vector of eigenstress σ^* by

$$\mathbf{m}^{(j)} := [\sigma^*_{j1}, \sigma^*_{j2}]^T \quad \text{so that} \quad z_{\mathbf{m}}^{(j)} \equiv m_1^{(j)} - i m_2^{(j)} = \sigma^*_{j1} - i\sigma^*_{j2} \qquad (j = 1, 2), \tag{6.27}$$

and

$$u_{\mathbf{m}}^{(j)} := \mathbf{m}^{(j)} \cdot \nabla u \qquad \text{and} \qquad v_{\mathbf{m}}^{(j)} := \mathbf{m}^{(j)} \cdot \nabla h. \tag{6.28}$$



ε^e_{11} ε^e_{12} ε^e_{22}

Figure 4: The contour of the quartic strain field inside the p-inclusion parameterized by (6.17) with $k = 3$ and $e = 0.1$, when the p-inclusion is subjected to uniform eigenstress with $\sigma^*_{11} = 1$, $\sigma^*_{12} = 1$, and $\sigma^*_{22} = 2$. The Lamé constants λ and μ for the isotropic elastic medium are both set to be one.

Combining (6.28) with (6.2), the induced strain field ε^e_{pq} can be expressed as

$$\varepsilon^e_{pq} \equiv \frac{1}{2}\left(\frac{\partial u^e_p}{\partial x_q} + \frac{\partial u^e_q}{\partial x_p}\right) = \frac{1}{2\mu}\left(\frac{\partial u^{(p)}_{\mathbf{m}}}{\partial x_q} + \frac{\partial u^{(q)}_{\mathbf{m}}}{\partial x_p}\right) + \sum_{j=1}^{2} \frac{\lambda+\mu}{\mu(\lambda+2\mu)} \frac{\partial^3 v^{(j)}_{\mathbf{m}}}{\partial x_j \partial x_p \partial x_q} \quad (p,q = 1,2). \tag{6.29}$$

By (6.21)-(6.29), we can obtain an explicit expression for the induced strain ε^e_{pq}. The detailed expression will be presented in Appendix B.

Furthermore, we can select specific values for the parameters: $k = 3$, $e = 0.1$, $\sigma^*_{11} = 1$, $\sigma^*_{12} = 1$, $\sigma^*_{22} = 2$, $\lambda = 1$, and $\mu = 1$. These choices correspond to a specifically chosen p-inclusion, a uniform eigenstress, and an isotropic medium, respectively. For this particular case, we plot the distribution of the strain field ε^e_{pq} inside the p-inclusion, as shown in Fig. 4. The theoretical results depicted in Fig. 4 exhibit a relative error of approximately 5% when compared to the numerical results.

6.2. *Designing or optimizing a field using p-inclusions*

A second application of p-inclusions involves the design and optimization of various fields, including magnetic/electric fields, stress/strain fields, wave fields, etc. For example, minimizing stress or strain concentration in load-bearing structures is essential for enhancing the safety and reliability of the structures (Lipton, 2005; Wheeler, 2004). Also, designing a passive structure or a shield so that wave fields are negligibly small in a subregion facilitates the development of cloaking devices (Pendry et al., 2006; Leonhardt, 2006; Milton et al., 2006; Norris, 2008; Liu, 2010b; Yavari and Golgoon, 2019; Golgoon and Yavari, 2021). In the context of elastic cloaks, the reader is referred to works of Chen et al. (2021); Wang et al. (2022); Sozio et al. (2023) and reference therein for recent development on the design. Conversely, amplifying remote fields in subregions is also valuable. In the case of high-precision measurements of magnetic or electric fields, the sensitivity of the measuring devices can be improved as much as an auxiliary passive structure can amplify the measured fields (Griffith et al., 2009).

The challenging problem in these applications lies in how to design the geometries or material distributions of the structure to realize a particular field or to optimize the strength of the field. The concept of p-inclusions provides a flexible approach to directly realizing certain prescribed field in space. For the design of magnets, we recall that (2.15) determines the magnetic potential and field induced by a uniformly magnetized body Ω. Since the problem (2.15) is linear, we can consider a superposition of multiple magnetized bodies:

$$\begin{cases} \operatorname{div}(-\nabla\phi + \sum_{i=1}^{m} \mathbf{m}^{(i)} \chi_{\Omega^i}) = 0 & \text{in } \mathbb{R}^n, \\ |\nabla\phi| \to 0 & \text{as } |\mathbf{x}| \to +\infty. \end{cases} \tag{6.30}$$

If each of the bodies Ω^i is a p-inclusion associated with the polynomial p_i, from the linearity we see that the magnetic field in the intersection of all p-inclusions is given by

$$-\nabla\phi(\mathbf{x}) = \sum_{i=1}^{m} \big(\nabla\nabla p_i(\mathbf{x})\big)\mathbf{m}^{(i)} \qquad \forall\, \mathbf{x} \in \cap_{i=1}^{m} \Omega^i. \tag{6.31}$$

Moreover, we can make use of the property that p-inclusions remain as p-inclusions upon translations, rotations and reflections. For instance, to create a multi-pole magnetic field, e.g., a 10-pole magnetic field that can be used as a magnetic lens to control the trajectories of electrons in synchrotrons, we can simply superimpose uniformly magnetized p-inclusions of degree six with a 30° rotation of itself with opposite magnetization (see Fig. 5). Similarly, general $2N$-multipole fields can be constructed using p-inclusions of degree $N+1$. From this viewpoint, the concept of p-inclusions can be used to systematically improve the quality of magnetic fields and reducing the weight of materials, particularly in nuclear fusion devices such as Tokamak and Stellarator where specific and gigantic magnetic fields are required to confine high-temperature plasma (Wilson, 1983; Oku et al., 2008; Machida and Fenning, 2010).

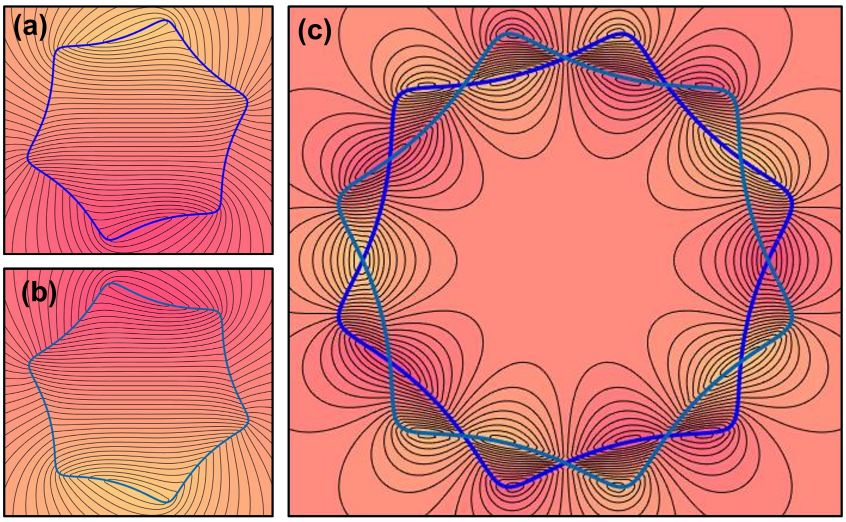


Figure 5: Designing a perfect 10-pole field by p-inclusions. (a) Contour plot of magnetic potential induced by a p-inclusion of degree six with uniform magnetization $\mathbf{m} = \mathbf{e}_2$; (b) a 30^o rotation of (a) with uniform magnetization $\mathbf{m} = -\mathbf{e}_2$; (c) a superposition of (a) and (b). The intersection of (a) and (b) now has zero magnetization, i.e., the bore region of magnetic lens.

For a three-dimensional example, we consider a superposition of the p-inclusion Ω_1 associated with $p_1 = \frac{1}{2} - \frac{1}{6}|\mathbf{x}|^2 + \frac{1}{18}\left(x_1^4 + x_2^4 - 6x_1^2 x_2^2\right)$ in Fig. 2 with uniform magnetization $\mathbf{m} \in \mathbb{R}^n$ and an ellipsoid Ω_2 associated with $p_2 = \frac{4}{5} - \frac{1}{16}(3x_1^2 + 3x_2^2 + 2x_3^2)$ with opposite magnetization $-\mathbf{m}$. By (6.31), the magnetic field in the bore region $\Omega_1 \cap \Omega_2$ is approximately given by

$$\begin{aligned} -\nabla\phi(\mathbf{x}) = {} & \left[\frac{2}{3}(x_1^2 - x_2^2)m_1 + \frac{1}{24}m_1 - \frac{4}{3}x_1 x_2 m_2\right]\mathbf{e}_1 \\ & + \left[-\frac{4}{3}x_1 x_2 m_1 + \frac{2}{3}(x_2^2 - x_1^2)m_2 + \frac{1}{24}m_2\right]\mathbf{e}_2 - \frac{1}{12}m_3\mathbf{e}_3. \end{aligned} \tag{6.32}$$

From (6.32), we see if the uniform magnetization $\mathbf{m}$ is vertical (along $\mathbf{e}_3$), the superposition of Ω_1 and Ω_2 will induce vertical uniform magnetic field in the bore region. And if the uniform magnetization $\mathbf{m}$ is horizontal (parallel to the plane spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$), the superposition of Ω_1 and Ω_2 will induce horizontal quadratic magnetic field in the bore region. In particular, we take $\mathbf{m} = \mathbf{e}_1$. Then the quadratic magnetic field induced by the superposition of Ω_1 and Ω_2 in the bore region is illustrated in Fig. 6.

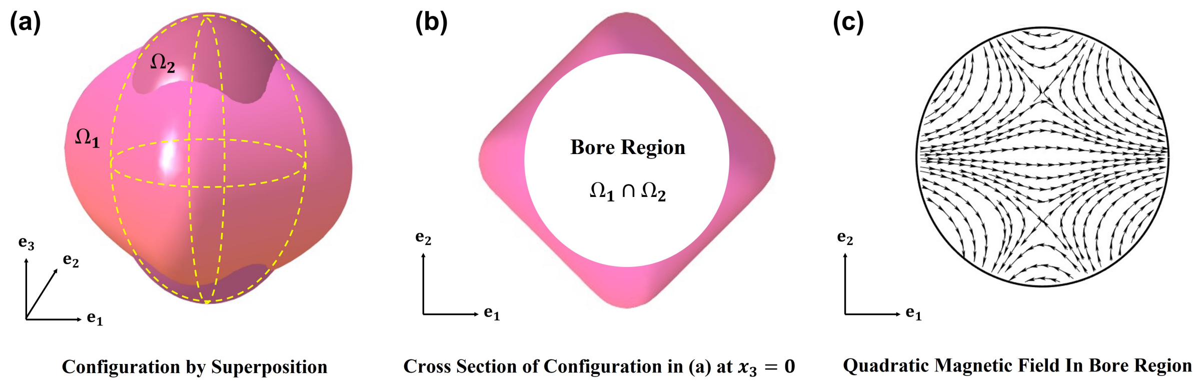


Figure 6: The quadratic magnetic field realized by the superposition of the p-inclusion Ω_1 associated with $p_1 = \frac{1}{2} - \frac{1}{6}|\mathbf{x}|^2 + \frac{1}{18}\left(x_1^4 + x_2^4 - 6x_1^2x_2^2\right)$ in Fig. 2 with uniform magnetization $\mathbf{e}_1$ and an ellipsoid Ω_2 associated with $p_2 = \frac{4}{5} - \frac{1}{16}(3x_1^2 + 3x_2^2 + 2x_3^2)$ with opposite magnetization $-\mathbf{e}_1$. (a) The resulting configuration by the superposition, where $\Omega_1 \cap \Omega_2$ is the bore region; (b) the cross section of the configuration in (a) at $x_3 = 0$; and (c) the quadratic magnetic field induced in the bore region $\Omega_1 \cap \Omega_2$, which is parallel to the plane spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$.

More generally, for many shape design or optimization problems it can be advantageous to restrict the admissible shapes to p-inclusions of degrees less than N. The choice of N depends on the desired precision for the problem. The set of all p-inclusions of degrees less than N forms a finite-dimensional space. Then the infinite-dimensional optimization problem over all possible shapes is converted into an optimization problem over a finite-dimensional space. This finite-dimensional formulation leverages a more tractable and efficient approach for solving the optimization problem.

7. Concluding Remarks

In summary, we have introduced the concept of p-inclusions and explored the properties of p-inclusions in the context of potential theory. Based on the theory of variational inequalities, we have shown the existence of p-inclusion in all dimensions. The shape of p-inclusions in two dimensions is algebraically parameterized by the method of conformal mapping, whereas in higher dimensions, examples of p-inclusions are numerically computed by solving an obstacle problem or Hamilton-Jacobi problem. Based on the properties of p-inclusions, we present two applications: (i) explicit solutions to the Eshelby inclusion problems for p-inclusions with arbitrary eigenstress, and (ii) designing magnets with some prescribed magnetic field.

Some open problems on p-inclusions are proposed. As evident in the ubiquitousness of models based on ellipsoids in applied science and engineering, we believe further study of explicit parameterizations and properties of p-inclusion can unveil more hidden connections between partial differential equations and algebraic geometry, and more importantly, may have significant impacts on many applications in optimal designs and inverse problems.

Declaration of Competing Interest

The authors declare no competing interests.

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Appendix A. The explicit expression of $\nabla u_{\mathbf{m}}$ and $\nabla\nabla\nabla v_{\mathbf{m}}$ inside general p-inclusions

We introduce two notations to simplify the upcoming expressions:

$$A_s^{p,q} = \binom{s}{q} \sum_{|\alpha|=p} (-1)^q (c_{\alpha_1} \cdots c_{\alpha_q}) \quad \text{with } A_s^{0,0} = 1,$$
$$B_s^{p,q} = \binom{s}{q} \sum_{|\alpha|=p-s} (-1)^s c_1^{s-q} \times (c_{\alpha_1+1} \cdots c_{\alpha_q+1}),$$

where $\binom{s}{q} \equiv \frac{s!}{(s-q)!q!}$ are binomial coefficients, "$\cdots$" denotes the multiplication of scalars, and c_{α_j} $(\alpha_j = 1, \cdots, k)$ are the parameters for the p-inclusions of form (5.15).

For a general p-inclusion parameterized by (5.15), one can calculate the integral expression (6.19) of $P_k(z)$ in the same way as that proposed in (6.20), which leads to

$$\begin{aligned} P_k(z) &= \frac{1}{2\pi i} \oint_{S^1} \left[\frac{\rho}{t} + \sum_{m=1}^{k} \overline{c}_m t^m\right] \left[\rho - \sum_{m=1}^{k} \frac{m c_m}{t^{m+1}}\right] \frac{1}{\rho t} \left[1 - \frac{(z - \sum_{m=1}^{k} \frac{c_m}{t^m})}{\rho t}\right]^{-1} \mathrm{d}t \\ &= \frac{1}{2\pi i} \oint_{S^1} \left[\frac{\rho}{t} + \sum_{m=1}^{k} \overline{c}_m t^m\right] \left[\rho - \sum_{m=1}^{k} \frac{m c_m}{t^{m+1}}\right] \left[\frac{1}{\rho t} + \sum_{n=1}^{\infty} \frac{(z - \sum_{m=1}^{k} \frac{c_m}{t^m})^n}{\rho^{n+1} t^{n+1}}\right] \mathrm{d}t \qquad \forall z \in \Omega. \end{aligned} \tag{A1}$$

Applying the Cauchy residue theorem to (A1) yields

$$\begin{aligned} P_k(z) = {} & \sum_{m=1}^{k} \overline{c}_m \rho \left[\sum_{s=\lfloor \frac{m+1}{2} \rfloor}^{m} \left(\sum_{q=\min\{m-s,1\}}^{m-s} A_s^{m-s,\,q} \left(\frac{z^{s-q}}{\rho^{s+1}}\right)\right)\right] + \sum_{m=3}^{k} \overline{c}_m \rho \left[\sum_{s=1}^{\lfloor \frac{m-1}{2} \rfloor} \left(\sum_{q=1}^{\min\{m-2s,s\}} \frac{B_s^{m-s,\,q}}{\rho^{s+1}}\right)\right] \\ & - \sum_{m=2}^{k} \overline{c}_m \left\{\sum_{n=1}^{m-1} n c_n \left[\sum_{s=\lfloor \frac{m-n-1}{2} \rfloor}^{m-n-1} \left(\sum_{q=\min\{m-n-s-1,1\}}^{m-n-s-1} A_s^{m-n-s-1,\,q} \left(\frac{z^{s-q}}{\rho^{s+1}}\right)\right)\right]\right\} \\ & - \sum_{m=5}^{k} \overline{c}_m \left\{\sum_{n=1}^{m-4} n c_n \left[\sum_{s=1}^{\lfloor \frac{m-n-2}{2} \rfloor} \left(\sum_{q=1}^{\min\{m-n-2s-1,s\}} \frac{B_s^{m-n-s-1,\,q}}{\rho^{s+1}}\right)\right]\right\}, \end{aligned} \tag{A2}$$

where "$\lfloor\ \rfloor$" denotes the integer part. It is straightforward to verify that (A2) will degenerate into (6.20) if the p-inclusions degenerate into hypotrochoids with the parameterization (6.17). Then, the explicit expression of $\nabla u_{\mathbf{m}}$ can be directly obtained by substituting (A2) into (6.21) and using the relations (6.25).

Next, we turn to derive the explicit expression of $\nabla\nabla\nabla v_{\mathbf{m}}$. To achieve this, we need to show the expression of the integral $\int_\Gamma \frac{\overline{\zeta}}{\zeta - z} \frac{\mathrm{d}D(\zeta)}{\mathrm{d}\zeta} \mathrm{d}\zeta$ in (6.22). We can reformulate such integral in a similar manner to the reformulation of $P_k(z)$ in (A1), which leads to

$$\int_\Gamma \frac{\overline{\zeta}}{\zeta - z} \frac{\mathrm{d}D(\zeta)}{\mathrm{d}\zeta} \mathrm{d}\zeta = \oint_{S^1} \left[\frac{\rho}{t} + \sum_{m=1}^{k} \overline{c}_m t^m\right] \left[-\frac{\rho}{t^2} + \sum_{m=1}^{k} m \overline{c}_m t^{m-1}\right] \left[\frac{1}{\rho t} + \sum_{n=1}^{\infty} \frac{(z - \sum_{m=1}^{k} \frac{c_m}{t^m})^n}{\rho^{n+1} t^{n+1}}\right] \mathrm{d}t \quad \forall z \in \Omega. \tag{A3}$$

By the Cauchy residue theorem, we observe that

$$\begin{aligned}\int_{\Gamma}\frac{\overline{\zeta}}{\zeta-z}\frac{\mathrm{d}D(\zeta)}{\mathrm{d}\zeta}\mathrm{d}\zeta=&\sum_{m=1}^{k}\overline{c}_m\left\{\sum_{n=1}^{k}n\overline{c}_n\left[\sum_{s=\lfloor\frac{m+n-1}{2}\rfloor}^{m+n-1}\left(\sum_{q=\min\{m+n-s-1,1\}}^{m+n-s-1}A_s^{m+n-s-1,\,q}\left(\frac{z^{s-q}}{\rho^{s+1}}\right)\right)\right]\right\}\\&+\sum_{m=1}^{k}\overline{c}_m\left\{\sum_{n=\max\{4-m,\,1\}}^{k}nc_n\left[\sum_{s=1}^{\lfloor\frac{m+n-2}{2}\rfloor}\left(\sum_{q=1}^{\min\{m+n-2s-1,s\}}\frac{B_s^{m+n-s-1,\,q}}{\rho^{s+1}}\right)\right]\right\}\\&+\sum_{m=2}^{k}(m-1)\overline{c}_m\left[\sum_{s=\lfloor\frac{m-2}{2}\rfloor}^{m-2}\left(\sum_{q=\min\{m-s-2,\,1\}}^{m-s-2}A_s^{m-s-2,\,q}\left(\frac{z^{s-q}}{\rho^{s+1}}\right)\right)\right]\\&+\sum_{m=5}^{k}(m-1)\overline{c}_m\left[\sum_{s=1}^{\lfloor\frac{m-3}{2}\rfloor}\left(\sum_{q=1}^{\min\{m-2s-2,s\}}\frac{B_s^{m-s-2,\,q}}{\rho^{s+1}}\right)\right].\end{aligned}\tag{A4}$$

It is easy to verify that for the p-inclusions parameterized by (6.17), (A4) degenerates into (6.23). Likewise, the explicit expression of $\nabla\nabla\nabla v_{\mathbf{m}}$ can be found by substituting (A4) into (6.22) and using the relations (6.26) in the main text.

Once the explicit expressions of $\nabla u_{\mathbf{m}}$ and $\nabla\nabla\nabla v_{\mathbf{m}}$ are gained as described above, by the same procedure from (6.27) to (6.29) we can ultimately derive the explicit expression of the strain field inside arbitrary p-inclusion.

Appendix B. The explicit expression of the strain field inside p-inclusion parameterized by (6.17)

For brevity, let σ^1, σ^2, and σ^3, i.e.,

$$\sigma^1=\begin{bmatrix}1&0\\0&1\end{bmatrix},\ \sigma^2=\begin{bmatrix}0&1\\1&0\end{bmatrix},\ \sigma^3=\begin{bmatrix}1&0\\0&-1\end{bmatrix},$$

to the bases of $\mathbb{R}^{2\times2}_{\mathrm{sym}}$ such that a symmetric eigenstresses $\sigma^*=[\sigma^*_{ij}]$ can be expressed as

$$\sigma^*=\frac{1}{2}(\sigma^*_{11}+\sigma^*_{22})\sigma^1+\sigma^*_{12}\sigma^1+\frac{1}{2}(\sigma^*_{11}-\sigma^*_{22})\sigma^3.$$

Let ε^q $(q=1,2,3)$ be the elastic strain induced by the basis eigenstress σ^q $(q=1,2,3)$. From the linearity of the Eshelby inclusion problem (6.1), the elastic strain due to eigenstress σ^* can be written as

$$\varepsilon^e=\frac{1}{2}(\sigma^*_{11}+\sigma^*_{22})\varepsilon^1+\sigma^*_{12}\varepsilon^2+\frac{1}{2}(\sigma^*_{11}-\sigma^*_{22})\varepsilon^3.$$

We now calculate the explicit expression of ε^q $(q=1,2,3)$. For future convenience, we introduce the vectorial functions $\mathbf{P}(x_1,x_2,k)$ and $\mathbf{P}^*(x_1,x_2)$, defined as follows:

$$\mathbf{P}(x_1,x_2,k):=\begin{bmatrix}\sum_{q=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{2q}(-1)^q x_1^{k-2q}x_2^{2q}\\ \sum_{q=0}^{\lfloor\frac{k-1}{2}\rfloor}\binom{k}{2q+1}(-1)^q x_1^{k-1-2q}x_2^{2q+1}\end{bmatrix}\quad\text{and}\quad\mathbf{P}^*(x_1,x_2):=\begin{bmatrix}\mathbf{P}(x_1,x_2,k-1)\\\mathbf{P}(x_1,x_2,k-2)\\\mathbf{P}(x_1,x_2,k-3)\\\mathbf{P}(x_1,x_2,2k-2)\end{bmatrix}.$$

In addition, we introduce parameters related to the "eccentricity" e in (6.17):

$$\begin{aligned}&h_1:=\frac{2\mu ke}{\lambda+\mu},\quad h_2:=(k-1)kex_1,\quad h_3:=(k-1)kex_2,\\&h_4:=-(k-2)(k-1)(ke^2-1)e,\quad h_5:=(2k-1)ke^2,\end{aligned}$$

and three mutually orthogonal vectors in $\mathbb{R}^3$:

$$\mathbf{E}^1 := \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \ \mathbf{E}^2 := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{E}^3 := \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

Denote by

$$\varepsilon^q_{vec}(x_1,x_2) := \begin{bmatrix} \varepsilon^q_{11}(x_1,x_2) \\ \varepsilon^q_{12}(x_1,x_2) \\ \varepsilon^q_{22}(x_1,x_2) \end{bmatrix} \qquad (q = 1,2,3)$$

the vectorized elastic strain for two-dimensional problems. Upon substituting (6.21), (6.22), (6.25) and (6.26) into (6.29), by tedious but straightforward calculations we obtain the vectorized strain $\varepsilon^q_{vec}(x_1,x_2)$ $(q = 1,2,3)$ induced by each basis eigenstress as

$$\begin{aligned} \varepsilon^1_{vec} &= \frac{1}{2(\lambda+2\mu)}\left(\mathbf{E}^1 + \frac{\lambda+\mu}{2\mu}\mathbf{H}^1\mathbf{P}^*\right), \\ \varepsilon^2_{vec} &= \frac{1}{2(\lambda+2\mu)}\left(\frac{\lambda+3\mu}{2\mu}\mathbf{E}^2 + \frac{\lambda+\mu}{2\mu}\mathbf{H}^2\mathbf{P}^*\right), \\ \varepsilon^3_{vec} &= \frac{1}{2(\lambda+2\mu)}\left(\frac{\lambda+3\mu}{2\mu}\mathbf{E}^3 + \frac{\lambda+\mu}{2\mu}\mathbf{H}^3\mathbf{P}^*\right), \end{aligned} \tag{B1}$$

where $\mathbf{H}^i$ $(i = 1,2,3)$ are 3×8 matrices composed of h_i, i.e.,

$$\begin{aligned} \mathbf{H}^1 &= \begin{bmatrix} h_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -h_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -h_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ \mathbf{H}^2 &= \begin{bmatrix} 0 & -h_1 & -h_3 & h_2 & 0 & -h_4 & 0 & -h_5 \\ 0 & 0 & h_2 & h_3 & -h_4 & 0 & -h_5 & 0 \\ 0 & -h_1 & h_3 & -h_2 & 0 & h_4 & 0 & h_5 \end{bmatrix}, \\ \mathbf{H}^3 &= \begin{bmatrix} h_1 & 0 & -h_2 & -h_3 & h_4 & 0 & h_5 & 0 \\ 0 & 0 & -h_3 & h_2 & 0 & -h_4 & 0 & -h_5 \\ -h_1 & 0 & -h_2 & h_3 & -h_4 & 0 & -h_5 & 0 \end{bmatrix}. \end{aligned} \tag{B2}$$

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