Min-max and stat game representations for nonlinear optimal control problems

Peter M. Dower[†] William M. McEneaney[‡] Yifei Zheng[♯]

***Abstract*— A finite horizon nonlinear optimal control problem is considered for which the associated Hamiltonian satisfies a uniform semiconcavity property with respect to its state and costate variables. It is shown that the value function for this optimal control problem is equivalent to the value of a min-max game, provided the time horizon considered is sufficiently short. This further reduces to maximization of a linear functional over a convex set. It is further proposed that the min-max game can be relaxed to a type of stat (stationary) game, in which no time horizon constraint is involved.**

I. Introduction

Computational solution of nonlinear deterministic optimal control problems continues to be a particularly challenging area within the controls field. A classical but surprisingly related area of research regards solution of n-body problems in astrodynamics, c.f. [1], [2]. In [1], using an action-functional approach, it was demonstrated that on sufficiently short time horizons, gravitational n-body problems can be represented as zero-sum games. Extension to indefinitely long time horizons motivated the development of "staticization" and "stat" duality [3], [4], [5], [6].

The game-based representation provided in [1] generated solutions of n-body problems as suprema of linear functionals over convex sets of solutions of differential Riccati equations (DREs). A corresponding representation was provided in [7] for solutions of certain state constrained control problems with linear dynamics. In the current work, it is shown that a class of nonlinear optimal control problems may be similarly converted into min-max games. These are of a form such that the value function may again be obtained through supremization of linear functionals over convex sets of solutions of DREs. The main result is Corollary 1.

The development commences with the optimal control problem of interest in Section II, and the relevant notions of duality for semiconcave functions in Section III. This duality is used in Section IV to relax the optimal control problem to a min-max game, for a sufficiently short time horizon. An extension to longer horizons is discussed in Section V, with stat duality used in place of semiconcave duality, to propose solutions via staticization over sets of solutions of DREs. Efficient numerical implementation is yet to be considered.

Research partially supported by AFOSR grants †FA2386-22-1-4074 and ‡FA9550-22-1-0015, and ARC grant †LP190100104.

†Department of Electrical & Electronic Engineering, University of Melbourne, Victoria 3010, Australia. †pdower@unimelb.edu.au

‡,♯Department of Mechanical & Aerospace Engineering, University of California at San Diego, Gillman Drive, La Jolla CA 92093, USA. ‡wmceneaney@eng.ucsd.edu ♯yiz152@ucsd.edu

Throughout, the natural numbers, the real numbers, and the extended reals are denoted by $\mathbb{N}$, $\mathbb{R}$ and $\overline{\mathbb{R}}$ respectively. Euclidean space is denoted by $\mathbb{R}^n$, $n \in \mathbb{N}$, and the corresponding space of $n \times n$ matrices by $\mathbb{R}^{n\times n}$. The subsets of self-adjoint and self-adjoint positive (semi-) definite matrices are denoted by $\mathbb{S}^n$ and ($\mathbb{S}^n_{\geq 0}$) $\mathbb{S}^n_{>0}$. Given $c \in \mathbb{R}$, $C \in \mathbb{S}^n$, the inequality $c\, I_n > C$ is equivalent to $c\, I_n - C \in \mathbb{S}^n_{>0}$, where I_n is the identity in $\mathbb{R}^{n\times n}$. The space of bounded linear operators mapping between Banach spaces $\mathscr{Y}$ and $\mathscr{Z}$ is denoted by $\mathcal{L}(\mathscr{Y};\mathscr{Z})$. Spaces of continuous and k-times Fréchet differentiable mappings between $\mathscr{Y}$ and $\mathscr{Z}$ are denoted by $\mathscr{C}(\mathscr{Y};\mathscr{Z})$ and $\mathscr{C}^k(\mathscr{Y};\mathscr{Z})$, $k \in \mathbb{N}$. Where $\mathscr{Y}$ is a Hilbert space, the Riesz representation of the Fréchet derivative $D_y\Theta$ of $\Theta \in \mathscr{C}^1(\mathscr{Y};\mathbb{R})$ evaluated at $y \in \mathscr{Y}$ is denoted by $\nabla_y\Theta(y) \in \mathscr{Y}$. Where $\Theta \in \mathscr{C}^2(\mathscr{Y};\mathbb{R})$, $\nabla_{yy}\Theta(y) \in \mathcal{L}(\mathscr{Y}) \doteq \mathcal{L}(\mathscr{Y};\mathscr{Y})$ denotes the second derivative given by $\nabla_{yy}\Theta(y)\, h \doteq D_y\nabla_y\Theta(y)\, h$ for all $h \in \mathscr{Y}$. With $\mathscr{Y} \doteq \mathbb{R}^n$, this second derivative is identified with a self-adjoint matrix, i.e. $\nabla_{yy}\Theta(y) \in \mathbb{S}^n$, $y \in \mathbb{R}^n$. Given an interval $I \subset \mathbb{R}$, the space of Lebesgue square-integrable functions mapping I to $\mathscr{Z}$ is denoted by $\mathscr{L}^2(I;\mathscr{Y})$. Norms are denoted by $\|\cdot\| \equiv \|\cdot\|_{\mathscr{Y}}$, with the subscript omitted where the space $\mathscr{Y}$ involved is contextually clear.

II. Optimal control problem

Attention is restricted to a class of finite dimensional input affine nonlinear systems, evolving in continuous time on the finite interval $[t, T]$ according to the dynamics

$$\dot{\xi}_s = A\,\xi_s + f(\xi_s) + B\,\mu_s, \quad s \in [t,T], \tag{1}$$

subject to the initial state $\xi_t = x \in \mathbb{R}^n$ and control input $\mu \in \mathscr{U}[t,T] \doteq \mathscr{L}^2([t,T];\mathbb{R}^m)$, given nonlinear $f \in \mathscr{C}^2(\mathbb{R}^n;\mathbb{R}^n)$, and matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. The optimal control problem of interest is defined subject to (1) via the value and cost functions

$$\overline{W}_t(x) \doteq \inf_{\mu\in\mathscr{U}[t,T]} \bar{J}_t(x,\mu), \tag{2}$$

$$\bar{J}_t(x,\mu) \doteq \int_t^T \ell(\xi_s) + \tfrac{1}{2}\langle \xi_s,\, C\,\xi_s\rangle + \tfrac{1}{2}\|\mu_s\|^2\, ds,$$

for all $x \in \mathbb{R}^n$, $\mu \in \mathscr{U}[t,T]$, given non-quadratic running cost term $\ell \in \mathscr{C}^2(\mathbb{R}^n;\mathbb{R}_{\geq 0})$, and $C \in \mathbb{S}^n_{\geq 0}$. For simplicity, the terminal cost is set to zero. The attendant Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE) is

$$\begin{cases} 0 = -\dfrac{\partial U_s}{\partial s}(x) + \bar{H}(x, \nabla_x U_s(x)), & s \in (t,T), \\ U_T(x) = 0, & x \in \mathbb{R}^n\,, \end{cases} \tag{3}$$

in which the Hamiltonian $\bar{H}:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ is defined with respect to its quadratic and non-quadratic dependencies by

$$\begin{aligned}\bar{H}(x,p) &\doteq \bar{Q}(x,p)+\bar{N}(x,p),\\ \bar{Q}(x,p) &\doteq -\tfrac{1}{2}\langle x,\,C\,x\rangle - \langle p,\,A\,x\rangle + \tfrac{1}{2}\langle p,\,B\,B'\,p\rangle,\\ \bar{N}(x,p) &\doteq -\ell(x) - \langle p,\,f(x)\rangle,\end{aligned} \tag{4}$$

for all $x,p\in\mathbb{R}^n$. Under reasonable conditions [8], [9], the value function $\overline{W}_t$ of (2) may be characterized as the unique viscosity solution of (3).

III. Semiconcavity and stationarity

A. Semiconcavity

Given a Hilbert space $\mathscr{X}$, (uniform) semiconcavity of extended real valued functions is defined with respect to a quadratic function $\varphi:\mathscr{X}\times\mathscr{X}\to\mathbb{R}$ given by

$$\varphi(y,z)\doteq\tfrac{c}{2}\,\|y-z\|_{\mathscr{X}}^2,\quad y,z\in\mathscr{X}, \tag{5}$$

with $c\in\mathbb{R}$ fixed. Spaces of (uniformly) semiconvex and semiconcave functions on $\mathscr{X}$ are defined respectively by

$$\begin{aligned}\mathcal{S}_\varphi^+ &= \mathcal{S}_\varphi^+(\mathscr{X})\doteq\left\{\Theta:\mathscr{X}\to\overline{\mathbb{R}}\,\middle|\,\begin{array}{l}y\mapsto\varphi(y,0)+\Theta(y)\\ \text{convex, lower closed}\end{array}\right\},\\ \mathcal{S}_\varphi^- &= \mathcal{S}_\varphi^-(\mathscr{X})\doteq\left\{N:\mathscr{X}\to\overline{\mathbb{R}}\,\middle|\,-N\in\mathcal{S}_\varphi^+\right\}.\end{aligned} \tag{6}$$

These spaces are in duality, via the semiconcave transform $\mathcal{D}_\varphi^-$ and its inverse, the semiconvex transform $\mathcal{D}_\varphi^+$.

$$\mathcal{S}_\varphi^+ \underset{\mathcal{D}_\varphi^-\;\equiv\;[\mathcal{D}_\varphi^+]^{-1}}{\overset{\mathcal{D}_\varphi^+\;\equiv\;[\mathcal{D}_\varphi^-]^{-1}}{\rightleftarrows}} \mathcal{S}_\varphi^-$$

The semiconcave transform and its inverse are given by

$$\begin{aligned}\mathcal{D}_\varphi^-\,N &= \sup_{z\in\mathscr{X}}\{N(z)-\varphi(\cdot,z)\},\quad N\in\mathcal{S}_\varphi^-,\\ [\mathcal{D}_\varphi^-]^{-1}\,\Theta &= \inf_{y\in\mathscr{X}}\{\Theta(y)+\varphi(y,\cdot)\},\quad \Theta\in\mathcal{S}_\varphi^+,\end{aligned} \tag{7}$$

for all $y,z\in\mathscr{X}$, see for example [7], [10]. Note that

$$\begin{aligned}\mathcal{D}_\varphi^-\,N &= -\mathcal{D}_\varphi^+\,(-N) = [\mathcal{D}_\varphi^+]^{-1}N,\\ [\mathcal{D}_\varphi^-]^{-1}\,\Theta &= -[\mathcal{D}_\varphi^+]^{-1}\,(-\Theta) = \mathcal{D}_\varphi^+\,\Theta.\end{aligned}$$

B. Stationarity

Functions that are simultaneously semiconvex and semiconcave are of particular interest. Such functions are continuously differentiable.

Lemma 1: $\mathcal{S}_\varphi^+(\mathscr{X})\cap\mathcal{S}_\varphi^-(\mathscr{X})\subset\mathscr{C}^1(\mathscr{X};\mathbb{R})$.

Proof: Fix $N\in\mathcal{S}_\varphi^+(\mathscr{X})\cap\mathcal{S}_\varphi^-(\mathscr{X})$, and let $D^-N(z)$ and $D^+N(z)$ denote the sub- and superdifferential of N at $z\in\mathscr{X}$. Note that $D^-N(z)\neq\emptyset$ as $N\in\mathcal{S}_\varphi^+(\mathscr{X})$, and $D^+N(z)\neq\emptyset$ as $N\in\mathcal{S}_\varphi^-(\mathscr{X})$. With both differentials non-empty, existence of $\nabla_z N(z)$ is guaranteed, with $D^-N(z)=\{\nabla_z N(z)\}=D^+N(z)$. Continuity of $z\mapsto\nabla_z N(z)$ follows, see [11, Theorem 3.3.7, p.60]. ■

Given $N\in\mathcal{S}_\varphi^+(\mathscr{X})\cap\mathcal{S}_\varphi^-(\mathscr{X})$, Lemma 1 motivates the introduction of a pair of stationarity operations that generalize sup (inf) and arg max (arg min), as given by

$$\begin{aligned}\operatorname*{stat}_{z\in\mathscr{O}} N(z) &\doteq \left\{N(\bar{z})\,\middle|\,\bar{z}\in\operatorname*{arg\,stat}_{z\in\mathscr{O}} N(z)\right\},\\ \operatorname*{arg\,stat}_{z\in\mathscr{O}} N(z) &\doteq \left\{\bar{z}\in\mathscr{O}\,\middle|\,0=\nabla_z N(\bar{z})\right\},\end{aligned} \tag{8}$$

in which $\mathscr{O}\subset\mathscr{X}$ is an open set. A further restriction of N yields the following [4], [12].

Lemma 2: Given $\mathscr{O}\subset\mathscr{X}$, suppose that $N\in\mathscr{C}^2(\mathscr{O};\mathbb{R})$ satisfies $\sup_{z\in\mathscr{O}}\|\nabla_{zz}N(z)\|\doteq\bar{c}<\infty$. Then, the following properties hold:

(i) there exists $c<\infty$ sufficiently large such that the map $z\mapsto z-\frac{1}{c}\nabla_z N(z)$ defines a bijection between $\mathscr{O}$ and its image $\tilde{\mathscr{O}}\doteq\{z-\frac{1}{c}\nabla_z N(z)\,|\,z\in\mathscr{O}\}$ on $\mathscr{O}$;

(ii) given $c<\infty$ from *(i)*, the *stat-quad transform* [4] of N, denoted $\mathcal{D}_\psi\,N$, is uniquely defined via (5) by

$$\begin{aligned}\Theta(y) &\doteq (\mathcal{D}_\varphi N)(y)\doteq\operatorname*{stat}_{z\in\mathscr{O}}\{N(z)-\varphi(y,z)\},\quad y\in\tilde{\mathscr{O}},\\ N(z) &= (\mathcal{D}_\varphi^{-1}\Theta)(z)\doteq\operatorname*{stat}_{y\in\tilde{\mathscr{O}}}\{\Theta(y)+\varphi(y,z)\},\quad z\in\mathscr{O};\end{aligned}$$

(iii) the stat-quad transform $\Theta\doteq\mathcal{D}_\varphi N$ of *(ii)* satisfies $\Theta\in\mathscr{C}^2(\tilde{\mathscr{O}};\mathbb{R})$ with $\sup_{y\in\tilde{\mathscr{O}}}\|\nabla_{yy}\Theta(y)\|\leq 2\,\bar{c}$.

Remark 1: Under the restrictions of Lemma 2, the semiconcave transform and stat-quad transform coincide, allowing the former to be restricted to the open sets $\mathscr{O}$ and $\tilde{\mathscr{O}}$.

IV. Min-max and stat game representations

Motivated by Remark 1, a special case of problem (2) is considered in which the non-quadratic dependence $\bar{N}$ of the Hamiltonian (4) is restricted to some open set $\mathscr{O}\subset\mathbb{R}^{2n}$.

Assumption 1: $\bar{N}\in\mathscr{C}^2(\mathscr{O};\mathbb{R})$ and $c\in\mathbb{R}$ in (5) satisfy

$$\sup_{z\in\mathscr{O}}\|\nabla_{zz}\bar{N}(z)\|\doteq\bar{c}\leq\tfrac{c}{4}<\infty. \tag{9}$$

Assumption 1 and Remark 1 imply that $\bar{N}$ has the form

$$\begin{aligned}\bar{N}(x,p) &= \inf_{y\in\tilde{\mathscr{O}}}\left\{\bar{\Theta}(y)+\tfrac{c}{2}\,\|(x,p)-y\|^2\right\},\\ \tilde{\mathscr{O}} &\doteq \left\{z-\tfrac{1}{c}\,\nabla_z\bar{N}(z)\,\middle|\,z\in\mathscr{O}\right\}.\end{aligned} \tag{10}$$

A. Game representations

Motivated by [1], [2], [7], relaxation of the optimizing variable y appearing in (10) suggests a game representation for the optimal control problem (2), involving two players. With $\tilde{\mathscr{O}}\subset\mathbb{R}^{2n}$ as per (10), the relevant spaces of actions are

$$\mathscr{V}[t,T]\doteq\mathscr{L}^2([t,T];\mathbb{R}^n),\ \ \mathscr{W}[t,T]\doteq\mathscr{L}^2([t,T];\tilde{\mathscr{O}}), \tag{11}$$

noting that $\mathscr{V}[t,T]$ in (11) is different from $\mathscr{U}[t,T]$ in (1)-(2), see Remark 2. A strategy for the first player is a map $\bar{\nu}:\mathscr{W}[t,T]\to\mathscr{V}[t,T]$, while a strategy for the second player is similarly a map $\bar{\omega}:\mathscr{V}[t,T]\to\mathscr{W}[t,T]$. The corresponding spaces of non-anticipating strategies are defined by

$$\mathcal{V}\doteq\left\{\bar{\nu}:\mathscr{W}[t,T]\to\mathscr{V}[t,T]\,\middle|\,\begin{array}{l}\bar{\nu}[\tilde{\omega}]_r=\bar{\nu}[\breve{\omega}]_r\ \forall\,r\in[t,\tau],\\ \text{given any }\tilde{\omega},\breve{\omega}\in\mathscr{W}[t,T]\\ \text{and }\tau\in(t,T]\text{ satisfying}\\ \tilde{\omega}_r=\breve{\omega}_r\ \forall\,r\in[t,\tau]\end{array}\right\},$$

$$\mathcal{W}\doteq\left\{\bar{\omega}:\mathscr{V}[t,T]\to\mathscr{W}[t,T]\,\middle|\,\begin{array}{l}\bar{\omega}[\tilde{\nu}]_r=\bar{\omega}[\breve{\nu}]_r\ \forall\,r\in[t,\tau],\\ \text{given any }\tilde{\nu},\breve{\nu}\in\mathscr{V}[t,T]\\ \text{and }\tau\in(t,T]\text{ satisfying}\\ \tilde{\nu}_r=\breve{\nu}_r\ \forall\,r\in[t,\tau]\end{array}\right\}.$$

Define the upper and lower values $W^\pm$ of a differential game, the lower value V^- of a static game, and its generalization

$\widetilde{V}^-$ obtained using the stat operation (8), all in terms of the same cost J, by (respectively)

$$W_t^+(x) \doteq \sup_{\bar\omega\in\mathcal{W}} \inf_{\nu\in\mathscr{V}[t,T]} J_t(x,\nu,\bar\omega[\nu]), \quad (12)$$

$$W_t^-(x) \doteq \inf_{\bar\nu\in\mathcal{V}} \sup_{\omega\in\mathscr{W}[t,T]} J_t(x,\bar\nu[\omega],\omega), \quad (13)$$

$$V_t^-(x) \doteq \sup_{\omega\in\mathscr{W}[t,T]} \inf_{\nu\in\mathscr{V}[t,T]} J_t(x,\nu,\omega), \quad (14)$$

$$\widetilde{V}_t^-(x) \doteq \operatorname*{stat}_{\omega\in\mathscr{W}[t,T]} \operatorname*{stat}_{\nu\in\mathscr{V}[t,T]} J_t(x,\nu,\omega), \quad (15)$$

$$\begin{aligned} J_t(x,\nu,\omega) &\doteq \int_t^T L(\zeta_s,\nu_s,\omega_s)\,ds, \quad (16)\\ L(x,v,w) &\doteq \tfrac{1}{2}\langle v,\,\Sigma\,v\rangle - \tfrac12\langle x,\,(c\,I_n - C)\,x\rangle \\ &\qquad + \langle x,\,c\,E_1\,w\rangle - \tfrac{c}{2}\|w\|^2 - \bar\Theta(w)\,,\\ \Gamma &\doteq c\,I_n + B\,B', \quad \Sigma \doteq \Gamma^{-1},\\ E_1 &\doteq \left(\ I_n \ \middle|\ 0_{n\times n}\ \right), \quad E_2 \doteq \left(\ 0_{n\times n}\ \middle|\ I_n\ \right), \end{aligned}$$

for all $x\in\mathbb{R}^n$, $\nu\in\mathscr{V}[t,T]$, $\omega\in\mathscr{W}[t,T]$, $v\in\mathbb{R}^n$, $w\in\tilde{\mathscr{O}}$, in which $\Gamma,\Sigma\in\mathbb{S}^n_{>0}$, $E_1,E_2\in\mathbb{R}^{n\times 2n}$, and $\bar\Theta$ is as per (10), while $s\mapsto\zeta_s$ satisfies the linear dynamics

$$\dot\zeta_s = A\,\zeta_s + \nu_s + c\,E_2\,\omega_s\,, \quad s\in[t,T], \quad (17)$$

subject to the initial state $\zeta_t = x\in\mathbb{R}^n$, and player actions $\nu\in\mathscr{V}[t,T]$ and $\omega\in\mathscr{W}[t,T]$. Under suitable conditions, it will be shown that the games defined by (12)-(15) coincide with the optimal control problem (2).

Remark 2: It is important to observe that neither player in (12)-(15) corresponds to the (single) control player in (2). This is a departure from [1], [7], where the first player was inherited directly from the optimal control problem, while the second player dealt with the non-quadratic terms in the running cost. The key difference here is that, unlike [1], [7], the costate variable features in the relaxation (10), thereby affecting the underlying dynamics, and hence both players.

B. Relevant properties of the cost (16)

Some useful properties of the cost (16) relevant to existence of the values (12)-(15) are summarized as follows.

Lemma 3: The cost J_t of (16) has the semi-quadratic form

$$J_t(x,\nu,\omega) = f_1(\omega) + \langle f_2(\omega),\,\nu\rangle_{\mathscr{V}[t,T]} + \tfrac12\langle \bar B_3\,\nu,\,\nu\rangle_{\mathscr{V}[t,T]}, \quad (18)$$

for all $x\in\mathbb{R}^n$, $\nu\in\mathscr{V}[t,T]$, $\omega\in\mathscr{W}[t,T]$, with $f_1:\mathscr{W}[t,T]\mapsto\mathbb{R}$, $f_2:\mathscr{W}[t,T]\mapsto\mathscr{V}[t,T]$, and self-adjoint and invertible $\bar B_3\in\mathcal{L}(\mathscr{V}[t,T])$ given respectively by

$$\begin{aligned} f_1(\omega) &\doteq f_1(\omega;x) \doteq -\tfrac12\langle x, \mathcal{F}'\,(c\,I_n - C)\,\mathcal{F}\,x\rangle_{\mathbb{R}^n}\\ &+ \left\langle \omega, \begin{pmatrix} c\,\mathcal{F}\,x \\ -c\,\mathcal{G}'\,(c\,I_n - C)\,\mathcal{F}\,x\end{pmatrix}\right\rangle_{\mathscr{W}[t,T]} - \int_t^T \bar\Theta(\omega_s)\,ds\\ &- \tfrac{c}{2}\left\langle \omega, \begin{pmatrix} I_n & -c\,\mathcal{G} \\ -c\,\mathcal{G}' & I_n + c\,\mathcal{G}'\,(c\,I_n - C)\,\mathcal{G}\end{pmatrix}\omega\right\rangle_{\mathscr{W}[t,T]} \end{aligned} \quad (19)$$

$$\begin{aligned} f_2(\omega) &\doteq f_2(\omega;x) \doteq -\mathcal{G}'\,(c\,I_n - C)\,\mathcal{F}\,x\\ &\qquad + \left(\ c\,\mathcal{G}' \ \middle|\ -c\,\mathcal{G}'\,(c\,I_n - C)\,\mathcal{G}\ \right)\omega, \quad (20)\\ \bar B_3 &\doteq \Sigma - \mathcal{G}'\,(c\,I_n - C)\,\mathcal{G}, \quad (21) \end{aligned}$$

in which $\mathcal{F}\in\mathcal{L}(\mathbb{R}^n;\mathscr{V}[t,T])$, $\mathcal{G}\in\mathcal{L}(\mathscr{V}[t,T])$ are defined with respect to elements $U_{s,\tau}\doteq\exp(A\,(s-\tau))\in\mathcal{L}(\mathbb{R}^n)$, $s,\tau\in[t,T]$, of the generated semigroup by

$$[\mathcal{F}\,x]_s \doteq U_{s,t}\,x, \quad [\mathcal{G}\,v]_s \doteq \int_t^s U_{s,\sigma}\,v_\sigma\,d\sigma, \quad (22)$$

for all $s\in[t,T]$, $x\in\mathbb{R}^n$, $v\in\mathscr{V}[t,T]$, and

$$\begin{aligned} \|\mathcal{F}\| &\le K_1\,(T-t)^{\frac12}, \quad \|\mathcal{G}\| \le K_1\,(T-t),\\ K_1 &\doteq \sup_{s,\sigma\in[t,T]} \|U_{s,\sigma}\|. \end{aligned} \quad (23)$$

Proof: Fix $x\in\mathbb{R}^n$, $\nu\in\mathscr{V}[t,T]$, and $\omega\in\mathscr{W}[t,T]$. Applying (22), the unique solution $s\mapsto\zeta_s$ of (17) is $\zeta = (\mathcal{F}\,x + c\,\mathcal{G}\,E_2\,\omega) + \mathcal{G}\,\nu = (\mathcal{F}\,x + \mathcal{E}_2\,\omega) + \mathcal{G}\,\nu$, with $\mathcal{E}_2 \doteq c\,\mathcal{G}\,E_2$. Substituting ζ in (16),

$$\begin{aligned} &J_t(x,\nu,\omega)\\ &= \tfrac12\langle\nu,\,\Sigma\,\nu\rangle_{\mathscr{V}[t,T]} - \tfrac12\langle(\mathcal{F}\,x + \mathcal{E}_2\,\omega) + \mathcal{G}\,\nu,\\ &\qquad\qquad (c\,I_n - C)\,[(\mathcal{F}\,x + \mathcal{E}_2\,\omega) + \mathcal{G}\,\nu]\rangle_{\mathscr{V}[t,T]}\\ &\quad + \langle(\mathcal{F}\,x + \mathcal{E}_2\,\omega) + \mathcal{G}\,\nu, c\,E_1\,\omega\rangle_{\mathscr{V}[t,T]}\\ &\quad - \tfrac{c}{2}\|\omega\|^2_{\mathscr{W}[t,T]} - \int_t^T \bar\Theta(\omega_s)\,ds\\ &= -\tfrac12\langle\mathcal{F}\,x + \mathcal{E}_2\,\omega,\,(c\,I_n - C)\,(\mathcal{F}\,x + \mathcal{E}_2\,\omega)\rangle_{\mathscr{V}[t,T]}\\ &\quad - \langle\nu,\,\mathcal{G}'\,(c\,I_n - C)\,(\mathcal{F}\,x + \mathcal{E}_2\,\omega)\rangle_{\mathscr{V}[t,T]}\\ &\quad + \tfrac12\langle\nu,\,[\Sigma - \mathcal{G}'\,(c\,I_n - C)\,\mathcal{G}]\,\nu\rangle_{\mathscr{V}[t,T]}\\ &\quad + \langle\nu,\,c\,\mathcal{G}'\,E_1\,\omega\rangle_{\mathscr{V}[t,T]}\\ &\quad + \langle\omega,\,c\,E_1'(\mathcal{F}\,x + \mathcal{E}_2\,\omega)\rangle_{\mathscr{W}[t,T]}\\ &\quad - \tfrac{c}{2}\|\omega\|^2_{\mathscr{W}[t,T]} - \int_t^T \bar\Theta(\omega_s)\,ds, \end{aligned}$$

in which $\mathcal{G}'$ is the Hilbert adjoint of $\mathcal{G}$ on $\mathscr{V}[t,T]$, i.e.

$$[\mathcal{G}'\,\tilde\nu]_s = \int_s^T U'_{\sigma,s}\,\tilde\nu_\sigma\,d\sigma, \quad s\in[t,T], \quad \tilde\nu\in\mathscr{V}[t,T].$$

Collecting terms in $\nu\in\mathscr{V}[t,T]$,

$$\begin{aligned} &J_t(x,\nu,\omega)\\ &= -\tfrac12\langle\mathcal{F}\,x + \mathcal{E}_2\,\omega,\,(c\,I_n - C)\,(\mathcal{F}\,x + \mathcal{E}_2\,\omega)\rangle_{\mathscr{V}[t,T]}\\ &\quad + \langle\omega,\,c\,E_1'(\mathcal{F}\,x + \mathcal{E}_2\,\omega)\rangle_{\mathscr{W}[t,T]}\\ &\quad - \tfrac{c}{2}\|\omega\|^2_{\mathscr{W}[t,T]} - \int_t^T \bar\Theta(\omega_s)\,ds\\ &\quad + \langle\nu,\,c\,\mathcal{G}'\,E_1\,\omega\rangle_{\mathscr{V}[t,T]}\\ &\quad - \langle\nu,\,\mathcal{G}'\,(c\,I_n - C)\,(\mathcal{F}\,x + \mathcal{E}_2\,\omega)\rangle_{\mathscr{V}[t,T]}\\ &\quad + \tfrac12\langle\nu,\,[\Sigma - \mathcal{G}'\,(c\,I_n - C)\,\mathcal{G}]\,\nu\rangle_{\mathscr{V}[t,T]}\\ &= -\tfrac12\langle x,\,\mathcal{F}'\,(c\,I_n - C)\,\mathcal{F}\,x\rangle_{\mathscr{V}[t,T]}\\ &\quad + \langle\omega,\,[c\,E_1' - \mathcal{E}_2{}'\,(c\,I_n - C)]\,\mathcal{F}\,x\rangle_{\mathscr{W}[t,T]} - \int_t^T \bar\Theta(\omega_s)\,ds\\ &\quad - \tfrac12\langle\omega,\,[c\,I_n + \mathcal{E}_2{}'\,(c\,I_n - C)\,\mathcal{E}_2\\ &\qquad\qquad - c\,E_1'\,\mathcal{E}_2 - c\,\mathcal{E}_2{}'E_1]\,\omega\rangle_{\mathscr{W}[t,T]}\\ &\quad + \langle\nu,\,[c\,\mathcal{G}'\,E_1 - \mathcal{G}'\,(c\,I_n - C)\,\mathcal{E}_2]\,\omega\\ &\qquad\qquad - \mathcal{G}'\,(c\,I_n - C)\,\mathcal{F}\,x\rangle_{\mathscr{V}[t,T]}\\ &\quad + \tfrac12\langle\nu,\,[\Sigma - \mathcal{G}'\,(c\,I_n - C)\,\mathcal{G}]\,\nu\rangle_{\mathscr{V}[t,T]} \end{aligned}$$

$$= f_1(\omega) + \langle f_2(\omega),\, \nu\rangle_{\mathscr{V}[t,T]} + \tfrac{1}{2}\,\langle \bar B_3\,\nu,\, \nu\rangle_{\mathscr{V}[t,T]},$$

in which f_1, f_2, and $\bar B_3$ are given by (19)-(21). Note further that $\bar B_3$ is self-adjoint on $\mathscr{V}[t,T]$ by definition (21), given $\Sigma \in \mathbb{S}^n_{>0}$ of (16). Bounds (23) follow from (22), via

$$|[\mathcal{F}\,x]_s| \le K_1\,\|x\|,$$
$$|[\mathcal{G}\,\nu]_s| \le \int_t^s \|U_{s,\sigma}\|_{\mathcal{L}(\mathbb{R}^n)}\,\|\nu_\sigma\|\,d\sigma \le K_1\,\sqrt{T-t}\,\|\nu\|_{\mathscr{V}[t,T]},$$

which hold for all $s \in [t,T]$, $x \in \mathbb{R}^n$, $\nu \in \mathscr{V}[t,T]$, with the latter implying boundness of $\bar B_3$. Invertibility of $\bar B_3$ also follows, and the details are omitted. ■

Lemma 4: Under Assumption 1, the following properties of (16) hold for all $x \in \mathbb{R}^n$, $\nu \in \mathscr{V}[t,T]$, $\omega \in \mathscr{W}[t,T]$:

(i) $\omega \mapsto J_t(x,\nu,\omega)$ is concave if $c\,I_n \ge C$ and

$$T - t \le \bar T_0 \doteq \frac{1}{2\,K_1\,c}\,; \tag{24}$$

(ii) $\nu \mapsto J_t(x,\nu,\omega)$ is convex if $c\,I_n \le C$, or if

$$T - t \le \bar T_1 \doteq \frac{1}{K_1\sqrt{\|c\,I_n - C\|\,\|c\,I_n + B\,B'\|}}. \tag{25}$$

Proof: Fix $x \in \mathbb{R}^n$, $\nu \in \mathscr{V}[t,T]$, $\omega \in \mathscr{W}[t,T]$.

(i) Recalling Lemma 3 and (18)-(21), note that the map $\omega \mapsto J_t(x,\nu,\omega)$ is nonlinear. As affine terms do not affect the desired concavity, and given $(\alpha,\beta) \doteq \omega$, the relevant remaining terms are (with an abuse of notation)

$$\begin{aligned}
J_t^0(\omega) &\doteq -\int_t^T \bar\Theta(\omega_s)\,ds \\
&\quad - \tfrac{c}{2}\left\langle \omega, \begin{pmatrix} I_n & -c\,\mathcal{G} \\ -c\,\mathcal{G}' & I_n + c\,\mathcal{G}'\,(c\,I_n - C)\,\mathcal{G} \end{pmatrix} \omega \right\rangle_{\mathscr{W}[t,T]} \\
&= -\tfrac{c^2}{2}\,\langle \beta,\, \mathcal{G}'\,(c\,I_n - C)\,\mathcal{G}\,\beta\rangle_{\mathscr{V}[t,T]} \\
&\quad - \tfrac{c}{2}\,\|\omega\|^2_{\mathscr{W}[t,T]} + c^2\,\langle \alpha,\, \mathcal{G}\,\beta\rangle_{\mathscr{V}[t,T]} - \int_t^T \bar\Theta(\omega_s)\,ds.
\end{aligned}$$

By Assumption 1, $\bar N \in \mathscr{C}^2(\mathscr{O};\mathbb{R})$, so that by Lemma 2, Cauchy-Schwarz, (23), and with $T - t \le \bar T_0$,

$$\begin{aligned}
&\tfrac{1}{2}\,\langle \tilde\omega,\, \nabla_{\omega\omega} J_t^0(\omega)\,\tilde\omega\rangle_{\mathscr{W}[t,T]} + \tfrac{c^2}{2}\,\langle \tilde\beta,\, \mathcal{G}'\,(c\,I_n - C)\,\mathcal{G}\,\tilde\beta\rangle_{\mathscr{V}[t,T]} \\
&\le -\tfrac{c}{2}\,\|\tilde\omega\|^2_{\mathscr{W}[t,T]} + c^2\langle \tilde\alpha,\, \mathcal{G}\,\tilde\beta\rangle_{\mathscr{V}[t,T]} \\
&\qquad + \int_t^T \tfrac{1}{2}\,\|\nabla_{ww}\bar\Theta(\alpha_s,\beta_s)\|\,\|\tilde\omega_s\|^2\,ds \\
&\le c^2\,\|\mathcal{G}\|\,\|\tilde\alpha\|_{\mathscr{V}[t,T]}\,\|\tilde\beta\|_{\mathscr{V}[t,T]} - (\tfrac{c}{2} - \bar c)\,\|\tilde\omega\|^2_{\mathscr{W}[t,T]} \\
&\le c^2\,K_1\,(T-t)\,\|\tilde\alpha\|_{\mathscr{V}[t,T]}\,\|\tilde\beta\|_{\mathscr{V}[t,T]} - \tfrac{c}{4}\,\|\tilde\omega\|^2_{\mathscr{W}[t,T]} \\
&\le -\tfrac{c}{4}\,\|\tilde\alpha\|^2_{\mathscr{V}[t,T]} + \tfrac{c}{2}\,\|\tilde\alpha\|_{\mathscr{V}[t,T]}\,\|\tilde\beta\|_{\mathscr{V}[t,T]} - \tfrac{c}{4}\,\|\tilde\beta\|^2_{\mathscr{V}[t,T]} \\
&= -\tfrac{c}{4}\,\left(\|\tilde\alpha\|_{\mathscr{V}[t,T]} - \|\tilde\beta\|_{\mathscr{V}[t,T]}\right)^2 \le 0,
\end{aligned}$$

for all $\tilde\omega \doteq (\tilde\alpha, \tilde\beta) \in \mathscr{W}[t,T]$. That is, $\omega \mapsto J_t^0(\omega)$ is concave. With $c\,I_n \ge C$, it then follows that $\omega \mapsto J_t(\omega)$ is also concave. (Note that the asserted requirement $c\,I_n \ge C$ may be relaxed by further restricting the horizon (24), via (23).)

(ii) As the map $\nu \mapsto J_t(x,\nu,\omega)$ is quadratic by Lemma 3, the relevant term in (18)-(21) is (again abusing notation)

$$J_t^1(\nu) \doteq \tfrac{1}{2}\,\langle \bar B_3\,\nu,\, \nu\rangle_{\mathscr{V}[t,T]}.$$

The map $\nu \mapsto J_t^1(\nu)$ is convex if $\bar B_3$ is positive semidefinite. With $c\,I_n \le C$, and noting that $\Sigma \in \mathbb{S}^n_{>0}$, the first assertion in *(ii)* is then immediate by (21). Otherwise, again recalling (21), given $T - t \le \bar T_1$,

$$\begin{aligned}
&\langle \tilde\nu,\, \nabla_{\nu\nu} J_t^1(\nu)\,\tilde\nu\rangle_{\mathscr{V}[t,T]} = \langle \bar B_3\,\tilde\nu,\, \tilde\nu\rangle_{\mathscr{V}[t,T]} \\
&\ge \left(\|\Gamma\|^{-1} - \|\mathcal{G}\|^2\,\|c\,I_n - C\|\right)\|\tilde\nu\|^2_{\mathscr{V}[t,T]} \\
&\ge \left(\|\Gamma\|^{-1} - K_1^2\,(T-t)^2\,\|c\,I_n - C\|\right)\|\tilde\nu\|^2_{\mathscr{V}[t,T]} \ge 0,
\end{aligned}$$

for all $\tilde\nu \in \mathscr{V}[t,T]$, yielding the second assertion in *(ii)*. ■

In view of the time horizons required for application of Lemma 4, the following is assumed for the remainder.

Assumption 2: Constant c in (5) and horizon $T-t$ satisfy

$$c \ge \|C\|, \qquad T - t \in (0, \bar T), \tag{26}$$

in which $\bar T \doteq \min(\bar T_0,\, \bar T_1)$ is defined via (24)-(25).

C. Equivalence of value functions (2) and (12)-(15)

The upper and lower values (12) and (13) can be characterized as the unique viscosity solutions U^+ and U^- of the respective Hamilton-Jacobi-Isaacs (HJI) PDEs [8, p.379]

$$\begin{cases} 0 = -\dfrac{\partial U_s^\pm}{\partial s}(x) + H^\pm(x, \nabla_x U_s^\pm(x)), & s \in (t,T), \\ U_T^\pm(x) = 0, & x \in \mathbb{R}^n, \end{cases} \tag{27}$$

in which the corresponding Hamiltonians $H^\pm$ are given by

$$H^+(x,p) \doteq \sup_{v\in\mathbb{R}^n}\ \inf_{w\in\tilde{\mathscr{O}}} G(x,p,v,w), \tag{28}$$
$$H^-(x,p) \doteq \inf_{w\in\tilde{\mathscr{O}}}\ \sup_{v\in\mathbb{R}^n} G(x,p,v,w), \tag{29}$$

and G is defined with respect to L of (16) by

$$\begin{aligned}
&G(x,p,v,w) \doteq -L(x,v,w) - \langle p,\, A\,x + v + c\,E_2\,w\rangle \\
&= \tfrac{1}{2}\,\langle x,\, (c\,I_n - C)\,x\rangle - \langle p,\, A\,x + v + c\,E_2\,w\rangle - \tfrac{1}{2}\,\langle v,\, \Sigma\,v\rangle \\
&\quad - \langle x,\, c\,E_1\,w\rangle + \tfrac{c}{2}\,\|w\|^2 + \bar\Theta(w),
\end{aligned} \tag{30}$$

for all $x,p,v \in \mathbb{R}^n$, $w \in \tilde{\mathscr{O}}$.

Lemma 5: The Hamiltonians $\bar H$ and $H^\pm$ of (4) and (28)-(29) for the optimal control problem (2), and for the upper and lower values (12)-(13) of the game are equivalent, i.e.

$$\bar H(x,p) = H^\pm(x,p), \tag{31}$$

for all $x,p \in \mathbb{R}^n$, so that Isaacs' condition holds.

Proof: Fix $x,p \in \mathbb{R}^n$. Recalling (28)-(29),

$$\begin{aligned}
H^+(x,p) &= \sup_{v\in\mathbb{R}^n}\ \inf_{w\in\tilde{\mathscr{O}}} G(x,p,v,w) \\
&= \sup_{v\in\mathbb{R}^n}\ \inf_{w\in\tilde{\mathscr{O}}} \{-L(x,v,w) - \langle p,\, A\,x + v + c\,E_2\,w\rangle\} \\
&= \tfrac{1}{2}\,\langle x,\, (c\,I_n - C)\,x\rangle - \langle p,\, A\,x\rangle \\
&\quad + \sup_{v\in\mathbb{R}^n}\{-\tfrac{1}{2}\,\langle v,\, \Sigma\,v\rangle - \langle p,\, v\rangle\} \\
&\quad + \inf_{w\in\tilde{\mathscr{O}}}\{\tfrac{c}{2}\,\|w\|^2 - \langle x,\, c\,E_1\,w\rangle - \langle p,\, c\,E_2\,w\rangle + \bar\Theta(w)\} \\
&= \inf_{w\in\tilde{\mathscr{O}}}\ \sup_{v\in\mathbb{R}^n} \{-L(x,v,w) - \langle p,\, A\,x + v + c\,E_2\,w\rangle\} \\
&= H^-(x,p),
\end{aligned}$$

so that Isaacs' condition holds. By completion of squares with respect to v, and recalling (4), (10),

$$\begin{aligned}
&H^+(x,p) = H^-(x,p) \\
&= -\tfrac{1}{2}\langle x,\, C\,x\rangle - \langle p,\, A\,x\rangle + \tfrac{1}{2}\langle p,\, B\,B'\,p\rangle \\
&\ + \inf_{w\in\tilde{\mathscr{O}}}\{\bar\Theta(w) + \tfrac{c}{2}\|w\|^2 - c\,\langle (x,p),\, w\rangle + \tfrac{c}{2}\|(x,p)\|^2\} \\
&= \bar{Q}(x,p) + \bar{N}(x,p) = \bar{H}(x,p).
\end{aligned}$$

as required. ■

Motivated by the lower value (14) of the static game, it is useful to define an additional optimal control problem with respect to the same cost. In particular, given any fixed $\omega \in \mathscr{W}[t,T]$, define the value V^ω of this additional problem by

$$V_t^\omega(x) \doteq \inf_{\nu\in\mathscr{V}[t,T]} J_t(x,\nu,\omega), \tag{32}$$

for all $x \in \mathbb{R}^n$, in which J is as per (16). Note that V^ω is the unique viscosity solution of the HJB PDE

$$\begin{cases} 0 = -\dfrac{\partial U_s^\omega}{\partial s}(x) + H_s^\omega(x, \nabla_x U_s^\omega(x)), & s\in(t,T), \\ U_T^\omega(x) = 0, & x\in\mathbb{R}^n, \end{cases} \tag{33}$$

in which the Hamiltonian $H_s^\omega : \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ is

$$\begin{aligned}
&H_s^\omega(x,p) \doteq \sup_{v\in\mathbb{R}^n}\left\{ -L(x,v,\omega_s) - \langle p,\, A\,x + v + c\,E_2\,\omega_s\rangle \right\} \\
&= \tfrac{1}{2}\langle x,\, (c\,I_n - C)\,x\rangle - \langle p,\, A\,x + c\,E_2\,\omega_s\rangle + \tfrac{1}{2}\langle p,\, \Gamma\,p\rangle \\
&\quad - \langle x,\, c\,E_1\,\omega_s\rangle + \tfrac{c}{2}\|\omega_s\|^2 + \bar\Theta(\omega_s),
\end{aligned} \tag{34}$$

for all $s\in[t,T]$.

Theorem 1: Under Assumptions 1-2, the value (2) of the optimal control problem, the upper and lower values (12)-(13) of the differential game, the lower value (14) of the static game, and its stat generalization (15) are equal, with

$$\begin{aligned}
\overline{W}_t(x) &= W_t^+(x) = W_t^-(x) \\
&= \widetilde{V}_t^-(x) = V_t^-(x) = \sup_{\omega\in\mathscr{W}[t,T]} V_t^\omega(x),
\end{aligned} \tag{35}$$

for all $x\in\mathbb{R}^n$, $\omega\in\mathscr{W}[t,T]$.

Proof: It is well known, see for example [8], [9], that

$$V_t^-(x) \le W_t^-(x) \le W_t^+(x) \le V_t^+(x),$$

for all $x\in\mathbb{R}^n$, in which V_t^+ is the upper value corresponding to (14) of the static game. By Lemma 5, Isaacs' condition holds for the upper and lower Hamiltonians $H^\pm$ of (28)-(29), i.e. $H^+ \equiv H^-$. Hence, by [8, Corollary 6.1, p.387], the upper and lower values (12)-(13) of the differential game are equivalent. Moreover, again by Lemma 5, as $\bar{H} \equiv H^\pm$, HJB PDE (3) and HJI PDEs (27) are identical, and so their respective unique viscosity solutions (2) and (12)-(13) must be equal. That is, the first two equalities in (35) hold, with

$$V_t^-(x) \le W_t^-(x) = \overline{W}_t(x) = W_t^+(x),$$

for all $x\in\mathbb{R}^n$.

In order to conclude the third and fourth equalities in (35), recent results on stat dynamic programming and verification [3], [12], [13], [14] are applied. In particular, the value $\widetilde{V}^-$ is the unique solution of a corresponding HJB PDE given by

$$\begin{cases} 0 = -\dfrac{\partial \tilde{U}_s}{\partial s}(x) + \tilde{H}(x, \nabla_x \tilde{U}_s(x)), & s\in(t,T), \\ \tilde{U}_T(x) = 0, & x\in\mathbb{R}^n\,, \end{cases} \tag{36}$$

in which the Hamiltonian $\tilde{H} : \mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ is

$$\tilde{H}(x,p) \doteq \operatorname*{stat}_{w\in\tilde{\mathscr{O}}}\ \operatorname*{stat}_{v\in\mathbb{R}^n} G(x,p,v,w), \tag{37}$$

for all $x,p\in\mathbb{R}^n$, and G is as per (30). Note that $v\mapsto G(x,p,v,w)$ is concave as $\Sigma\in\mathbb{S}_{>0}^n$, and $w\mapsto G(x,p,v,w)$ is convex as $\bar\Theta\in\mathcal{S}_\varphi^-$, so that $\bar{H}\equiv H^\pm\equiv\tilde{H}$ by (28)-(29), (37). Hence, the unique solution (15) of HJB PDE (36) is identical to the unique solutions (12)-(14) of HJB PDE (3) and HJI PDEs (27). Moreover, by Lemma 4, $\nu\mapsto J_t(x,\nu,\omega)$ is convex, while $\omega\mapsto J_t(x,\nu,\omega)$ is concave, so that

$$\begin{aligned}
&\overline{W}_t(x) = W_t^\pm(x) = \widetilde{V}_t^-(x) = \operatorname*{stat}_{\omega\in\mathscr{W}[t,T]}\ \operatorname*{stat}_{\nu\in\mathscr{V}[t,T]} J_t(x,\nu,\omega) \\
&= \operatorname*{stat}_{\omega\in\mathscr{W}[t,T]}\ \inf_{\nu\in\mathscr{V}[t,T]} J_t(x,\nu,\omega) = \sup_{\omega\in\mathscr{W}[t,T]}\ \inf_{\nu\in\mathscr{V}[t,T]} J_t(x,\nu,\omega) \\
&= V_t^-(x),
\end{aligned}$$

for all $x\in\mathbb{R}^n$, as per the third and fourth equalities in (35). The fifth equality in (35) follows by (14) and (32). ■

D. Interpretation of the value (2) via DREs

By inspection of the cost (16), the value V^ω of (32) defines an LQR problem parameterized by $\omega\in\mathscr{W}[t,T]$. Consequently, V_t^ω is described by the solution of a DRE.

Theorem 2: Under Assumptions 1-2, and given any $\omega\in\mathscr{W}[t,T]$, the value V_t^ω of (32) has the explicit quadratic form

$$V_t^\omega(x) = \tfrac{1}{2}\left\langle \begin{pmatrix} x \\ 1 \end{pmatrix},\, \Pi_t^\omega \begin{pmatrix} x \\ 1 \end{pmatrix} \right\rangle \tag{38}$$

for all $x\in\mathbb{R}^n$, in which $s\mapsto\Pi_s^\omega$ satisfies the DRE

$$\begin{cases} -\dot\Pi_s = \mathcal{R}_s^\omega(\Pi_s), & s\in(t,T), \\ \Pi_T = 0 \in \mathbb{S}^{(n+1)\times(n+1)}, \end{cases} \tag{39}$$

subject to the problem data

$$\begin{aligned}
&\mathcal{R}_s^\omega(\Pi) \doteq (A_s^\omega)'\,\Pi + \Pi\,A_s^\omega - \Pi\,\Lambda\,\Pi + V_s^\omega, \qquad (40) \\
&A_s^\omega \doteq \begin{pmatrix} A & c\,E_2\,\omega_s \\ 0_{n\times 1}' & 0 \end{pmatrix},\ \Lambda \doteq \begin{pmatrix} c\,I_n + B\,B' & 0_{n\times 1} \\ 0_{n\times 1}' & 0 \end{pmatrix}, \\
&V_s^\omega \doteq \begin{pmatrix} -(c\,I_n - C) & c\,E_1\,\omega_s \\ c\,(E_1\,\omega_s)' & -c\,\|\omega_s\|^2 - 2\,\bar\Theta(\omega_s) \end{pmatrix}.
\end{aligned}$$

Proof: Fix $\omega\in\mathscr{W}[t,T]$. Recall that V^ω of (32) is the unique viscosity solution of the HJB PDE (33). In view of its quadratic structure, propose the candidate solution

$$U_s^\omega(x) \doteq \tfrac{1}{2}\left\langle \begin{pmatrix} x \\ 1 \end{pmatrix},\, \Pi_s^\omega \begin{pmatrix} x \\ 1 \end{pmatrix} \right\rangle,\ \Pi_s^\omega \doteq \begin{pmatrix} P_s^\omega & q_s^\omega \\ (q_s^\omega)' & r_s^\omega \end{pmatrix},$$

in which $s\mapsto\Pi_s^\omega$ satisfies (39). Note that $\frac{\partial}{\partial s}U_s^\omega(x) = \frac{1}{2}\langle x,\, \dot{P}_s^\omega\,x\rangle + \langle x,\, \dot{q}_s^\omega\rangle + \frac{1}{2}\dot{r}_s^\omega$ and $\nabla_x U_s^\omega(x) = P_s^\omega\,x + q_s^\omega$. By substitution in HJB PDE (33),

$$-\frac{\partial U_s^\omega}{\partial s}(x) + H_s^\omega(x, \nabla_x U_s^\omega(x))$$

$$
\begin{aligned}
&= -\tfrac{1}{2}\,\langle x,\, \dot P_s^\omega\, x\rangle - \langle x,\, \dot q_s^\omega\rangle - \tfrac{1}{2}\,\dot r_s^\omega + \tfrac{1}{2}\,\langle x,\, (c\, I_n - C)\, x\rangle \\
&\quad - \langle x,\, c\, E_1\, \omega_s\rangle + \tfrac{c}{2}\, \|\omega_s\|^2 + \bar\Theta(\omega_s) \\
&\quad - \langle P_s^\omega\, x + q_s^\omega,\, A\, x + c\, E_2\, \omega_s\rangle \\
&\quad + \tfrac{1}{2}\, \langle P_s^\omega\, x + q_s^\omega,\, \Gamma\, (P_s^\omega\, x + q_s^\omega)\rangle,
\end{aligned}
$$

which is of the form $\frac{1}{2}\,\langle x,\, X_s^\omega\, x\rangle + \langle x, y_s^\omega\rangle + \frac{1}{2}\, z_s^\omega$, with

$$
\begin{aligned}
X_s^\omega &= -\dot P_s^\omega + (c\, I_n - C) - A'\, P_s^\omega - P_s^\omega\, A + P_s^\omega\, \Gamma\, P_s^\omega, \\
y_s^\omega &= -\dot q_s^\omega - c\, E_1\, \omega_s - A'\, q_s^\omega - c\, P_s^\omega\, E_2\, \omega_s + P_s^\omega\, \Gamma\, q_s^\omega, \\
z_s^\omega &= -\dot r_s^\omega + c\, \|\omega_s\|^2 + 2\, \bar\Theta(\omega_s) \\
&\qquad - 2\, c\, \langle q_s^\omega,\, E_2\, \omega_s\rangle + \langle q_s^\omega,\, \Gamma\, q_s^\omega\rangle.
\end{aligned}
$$

However, by substitution of $s \mapsto \Pi_s^\omega$ in (39), $X_s^\omega = 0$, $y_s^\omega = 0$, $z_s^\omega = 0$ for all $s \in (t, T)$. Hence, the proposed candidate solution $(s, x) \mapsto U_s^\omega(x)$ is indeed the unique solution of the HJB PDE (33), so that V^ω of (32) satisfies $V_s^\omega(x) = U_s^\omega(x)$ for all $s \in [t, T]$, $x \in \mathbb{R}^n$. That is, (38) holds. ■

Theorems 1 and 2 together imply that the value function $\overline{W}_t$ of the nonlinear optimal control problem (2) is equivalent to a supremum of an affine function over a family $\mathscr{D}$ of forced DRE solutions, where

$$
\mathscr{D} \doteq \left\{ \Pi_t \in \mathbb{S}^{(n+1)\times(n+1)} \;\middle|\; \begin{array}{c} \text{(39) holds, given any} \\ \omega \in \mathscr{W}[t, T] \end{array} \right\}.
$$

Corollary 1: The value function $\overline{W}_t$ of (2) satisfies

$$
\overline{W}_t(x) = \sup_{\Pi \in \operatorname{co} \mathscr{D}} \tfrac{1}{2} \left\langle \begin{pmatrix} x \\ 1 \end{pmatrix}, \Pi \begin{pmatrix} x \\ 1 \end{pmatrix} \right\rangle,
$$

for all $x \in \mathbb{R}^n$.

The maximization representation provided by Corollary 1 has been applied successfully in an astrodynamics problem [2], and in a state constrained optimal control problem with linear dynamics [7]. Similar results are anticipated here.

V. Limitations and extension

Assumptions 1 and 2 are restrictive. Assumption 1 invokes a uniform bound on the second derivative $\nabla_{zz}\bar N(z) \in \mathbb{S}^{2n}$, which is evaluated for $z = (x, p) \in \mathbb{R}^{2n}$ via bounds on

$$
\begin{aligned}
\nabla_{xx}\bar N(x, p) &= -\nabla_{xx} l(x) - \sum_{i=1}^n p_i\, \nabla_{xx} f_i(x), \\
\nabla_{xp}\bar N(x, p) &= \left(-\nabla_x f_1(x) \;|\; \cdots \;|\; -\nabla_x f_n(x)\right), \\
\nabla_{pp}\bar N(x, p) &= 0_{n\times n}.
\end{aligned}
$$

Uniform bounds on $\nabla_{xx} l$ and $\nabla_{xx} f_i$, for $i \in \mathbb{N}_{\le n}$, and on p are thus required. The latter corresponds to a global Lipschitz requirement for the value function (2), via an open set selection $\mathscr{O} \doteq \mathbb{R}^n \times \{p \in \mathbb{R}^n \,|\, \|p\| \le K\}$ for some Lipschitz constant $K < \infty$. Meanwhile, Assumption 2 invokes a bound on the admissible time horizon (26), which is further limited by the choice of c in (5) needed to satisfy Assumption 1.

Assumptions 1 and 2 are both used in the proof of Lemma 4 to infer convexity and concavity properties of the cost (16), which are fundamental to the min-max game formulation of (12)-(14). However, by relaxing min-max to *iterated stat* operations at the expense of some regularity requirements [6], these properties are no longer necessary (but are sufficient) in an ensuing stat game representation (15). This type of relaxation has similarly been employed in related work [1], [3], [4], [13], [15], [16], wherein the order of stat operations can be interchanged. In the context of (2), this stat order interchange amounts to being able to equate the lower value (14) of the static game with its corresponding upper value, thereby providing a direct connection between (2) and an appropriate extension of the DRE representation of Corollary 1. This extension takes the analogous form

$$
\overline{W}_t(x) = \operatorname*{stat}_{\Pi \in \operatorname{co} \mathscr{D}} \tfrac{1}{2} \left\langle \begin{pmatrix} x \\ 1 \end{pmatrix}, \Pi \begin{pmatrix} x \\ 1 \end{pmatrix} \right\rangle,
$$

for all $x \in \mathbb{R}^n$, and the details are omitted.

References

[1] W. McEneaney and P. Dower, "The principle of least action and fundamental solutions of mass-spring and n-body two-point boundary value problems," *SIAM J. Control & Optimization*, vol. 53, no. 5, pp. 2898–2933, 2015.

[2] S. Han and W. McEneaney, "The principle of least action and a two-point boundary value problem in orbital mechanics," *Applied Math. and Optim.*, vol. 77, pp. 129–172, 2018.

[3] W. McEneaney and P. Dower, "Staticization, its dynamic program, and solution propagation," *Automatica [DOI:10.1016/j.automatica.2017.03.004]*, vol. 81, pp. 56–67, 2017.

[4] ——, "Static duality and a stationary action application," *J. Differential Eq. [DOI:10.1016/j.jde.2017.08.054]*, vol. 264, no. 2, pp. 525–549, 2018.

[5] P. Dower and W. McEneaney, "Verification of stationary action trajectories via optimal control," in *proc. American Control Conference (Denver CO) [DOI:10.23919/ACC45564.2020.9147332]*, 2020, pp. 1779–1784.

[6] W. McEneaney and R. Zhao, "Staticization and iterated staticization," *SIAM J Control and Optim.*, vol. 60, no. 3, pp. 1840–1862, 2022.

[7] P. Dower and W. McEneaney, "A fundamental solution concept for a family of state constrained regulator problems," in *Proc. 25^{th} International Symposium on Mathematical Theory of Networks and Systems (Bayreuth, Germany)*. IFAC, 2022.

[8] W. H. Fleming and H. Soner, *Controlled Markov processes and viscosity solutions*, 2nd ed. Springer, 2006.

[9] M. Bardi and I. Capuzzo-Dolcetta, *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, ser. Systems & Control: Foundations & Application. Birkhauser, 1997.

[10] W. Fleming and W. McEneaney, "A max-plus-based algorithm for a Hamilton-Jacobi-Bellman equation of nonlinear filtering," *SIAM J. Control & Optimization*, vol. 38, no. 3, pp. 683–710, 2000.

[11] P. Cannarsa and C. Sinestrari, *Semiconcave functions, Hamilton-Jacobi equations, and optimal control*. Birkhaüser, 2004.

[12] W. McEneaney, P. Dower, and T. Wang, "Approximation of second-order Hamilton-Jacobi PDE problems and certain related first-order problems," *In review, preprint (29 pages)*, 2023.

[13] W. McEneaney, "A stochastic control verification theorem for the dequantized Schrödinger equation not requiring a duration restriction," *Applied Mathematics & Optimization [DOI:0.1007/s00245-017-9442-0]*, vol. 79, no. 2, pp. 427–452, 2019.

[14] W. McEneaney and P. Dower, "Conversion of a class of stochastic control problems to fundamental-solution deterministic control problems," in *proc. American Control Conference (Denver CO) [DOI:10.23919/ACC45564.2020.9147368]*, 2020, pp. 2814–2819.

[15] P. Dower and W. McEneaney, "Solving two-point boundary value problems for a wave equation via the principle of stationary action and optimal control," *SIAM J. Control & Optimization [DOI:10.1137/151003167]*, vol. 55, no. 4, pp. 2151–2205, 2017.

[16] ——, "Verifying fundamental solution groups for lossless wave equations via stationary action and optimal control," *Applied Mathematics and Optimization [36 pages, 10.1007/s00245-020-09700-4]*, 2020.