

REPRESENTATIONS AND TENSOR PRODUCT GROWTH

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ABSTRACT. The deep theory of approximate subgroups establishes 3-step product growth for subsets of finite simple groups G of Lie type of bounded rank. In this paper we obtain 2-step growth results for representations of such groups G (including those of unbounded rank), where products of subsets are replaced by tensor products of representations.

Let G be a finite simple group of Lie type and χ a character of G . Let $|\chi|$ denote the sum of the squares of the degrees of all (distinct) irreducible characters of G which are constituents of χ . We show that for all $\delta > 0$ there exists $\epsilon > 0$, independent of G , such that if χ is an irreducible character of G satisfying $|\chi| \leq |G|^{1-\delta}$, then $|\chi^2| \geq |\chi|^{1+\epsilon}$. We also obtain results for reducible characters, and establish faster growth in the case where $|\chi| \leq |G|^\delta$.

In another direction, we explore covering phenomena, namely situations where every irreducible character of G occurs as a constituent of certain products of characters. For example, we prove that if $|\chi_1| \cdots |\chi_m|$ is a high enough power of $|G|$, then every irreducible character of G appears in $\chi_1 \cdots \chi_m$. Finally, we obtain growth results for compact semisimple Lie groups.

1. INTRODUCTION

In the past two decades there has been intense interest in growth phenomena in groups and in finite simple groups in particular; see [He1, Hr, BGT, He2, PS, Bre]. The celebrated Product Theorem, proved independently in [BGT] and in [PS], shows that, if G is a finite simple group of Lie type of rank r , and $A \subset G$ is a generating set, then either $A^3 = G$ or $|A^3| \geq |A|^{1+\epsilon}$ where $\epsilon > 0$ depends only on r . Here A^k denotes the set of all products $a_1 a_2 \cdots a_k$ where $a_1, \dots, a_k \in A$.

The main goal of this paper is to study analogous power growth phenomena in representation theory, with emphasis on (complex) representations of finite simple groups G of Lie type. Here products of subsets of G are replaced by tensor products of representations. Our results on tensor product growth give character-theoretic analogues of the Product Theorem which are strong in two respects: we establish 2-step (instead of 3-step) growth, as well as uniform growth when the rank of G tends to infinity.

Let G be any finite group. If $X = \{\chi_1, \dots, \chi_k\}$ is a set of (pairwise distinct) irreducible characters of G , we define

$$|X| = \sum_{i=1}^k \chi_i(1)^2.$$

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This is the Plancherel measure, normalized so that $|\text{Irr}(G)| = |G|$. If χ is any character of G , we define $|\chi| = |\text{supp}(\chi)|$, where $\text{supp}(\chi)$ denotes the set of distinct irreducible constituents of χ . Recall that G is *quasisimple* if $G = [G, G]$ and $G/\mathbf{Z}(G)$ is simple.

Our first growth results are as follows:

Theorem 1. *For all $\delta > 0$, there exists $\epsilon > 0$ such that if G is a finite quasisimple group of Lie type and χ is an irreducible character of G with $|\chi| \leq |G|^{1-\delta}$, then $|\chi^2| \geq |\chi|^{1+\epsilon}$ and $|\chi\bar{\chi}| \geq |\chi|^{1+\epsilon}$.*

By a classical result of Burnside and Brauer [Br], if χ is any faithful character of a finite group G taking exactly m distinct values on the elements G , then any irreducible character of G appears as an irreducible constituent of some χ^i with $0 \leq i < m - 1$. Thus one should expect tensor growth for arbitrary characters of finite groups, and this is studied in Theorem 2.6 (below). We also give a version of Theorem 1 for general characters in groups of high rank:

Theorem 2. *For all $\delta > 0$, there exist $\epsilon > 0$ and $R > 0$ such that if G is a finite quasisimple group of Lie type and rank $\geq R$, and χ is any (not necessarily irreducible) character of G with $|\chi| \leq |G|^{1-\delta}$, then $|\chi^2| \geq |\chi|^{1+\epsilon}$ and $|\chi\bar{\chi}| \geq |\chi|^{1+\epsilon}$.*

The proofs of these results present ϵ as an explicit function of δ , e.g. $\epsilon = \frac{c\delta}{4+2c(1-\delta)}$ in Theorem 1, where $c > 0$ is the absolute constant in [LaT, Theorem A]. Moreover, if G is sufficiently large but of bounded rank r , and χ is irreducible, then $\epsilon = \frac{\delta}{2-2\delta}$ will do; for example, any irreducible character χ of G with $|\chi| \leq |G|^{1/2}$ satisfies $|\chi^2| \geq |\chi|^{3/2}$.

As shown in Example 2.5(iv), Theorem 2 does not hold for *non-simple* quasisimple groups of bounded order. On the other hand, Theorem 2.6(iii) shows that it holds true for (non-abelian) simple groups of bounded order. The question whether Theorem 2 can hold for (large enough) quasisimple groups of Lie type of bounded rank will be studied elsewhere.

Power growth of conjugacy classes A of arbitrary finite simple groups G was studied before the Product Theorem was proved. It is shown in [Sh, 2.7] there that for any $\delta > 0$ there exists $\epsilon > 0$, depending only on δ , such that $|A| \leq |G|^{1-\delta}$ implies $|A^3| \geq |A|^{1+\epsilon}$. Furthermore, if G is of Lie type then $|A^2| \geq |A|^{1+\epsilon}$ where $\epsilon > 0$ depends only on the rank of G [Sh, 10.4].

Subsequently, growth of general normal subsets (namely, unions of conjugacy classes) was also studied. It is shown in [GPSS, 1.5] that there are absolute constants $N \in \mathbb{N}$ and $\epsilon > 0$ such that for any normal subset A of a finite simple group G , either $A^N = G$ or $|A^2| \geq |A|^{1+\epsilon}$.

In [LSS] faster growth of the form $|A^2| \geq |A|^{2-\epsilon}$ for small normal subsets A of arbitrary finite simple groups is established. Our next result gives a character-theoretic analogue of [LSS, Theorem 1.3]:

Theorem 3. *For any $\epsilon > 0$, there exists an explicit $\delta = \delta(\epsilon) > 0$ such that the following statement holds. If G is a finite quasisimple group of Lie type and χ_1, χ_2 are any (not necessarily irreducible) characters of G with $|\chi_1|, |\chi_2| \leq |G|^\delta$, then*

$$|\chi_1\chi_2| \geq (|\chi_1| \cdot |\chi_2|)^{1-\epsilon}.$$

In particular, if χ is a character of G satisfying $|\chi| \leq |G|^\delta$ then $|\chi^2| \geq |\chi|^{2-2\epsilon}$.

We note that $|\chi_1\chi_2| \leq |\chi_1| \cdot |\chi_2|$ (see Lemma 2.1 below), hence the growth established in Theorem 3 is almost best possible. As a consequence of Theorem 3, we obtain:

Corollary 4. *For any $\epsilon > 0$ and any integer $k \geq 2$, there exists an explicit $\gamma = \gamma(\epsilon, k) > 0$ such that the following statement holds. If G is a finite quasisimple group of Lie type and*

$\chi_1, \chi_2, \dots, \chi_k$ are any (not necessarily irreducible) characters of G with

$$|\chi_1|, |\chi_2|, \dots, |\chi_k| \leq |G|^\gamma,$$

then

$$|\chi_1 \chi_2 \cdots \chi_k| \geq (|\chi_1| \cdot |\chi_2| \cdots |\chi_k|)^{1-\epsilon}.$$

In particular, if χ is a character of G satisfying $|\chi| \leq |G|^\gamma$ then $|\chi^k| \geq |\chi|^{k-k\epsilon}$.

The above result shows that, for any $\epsilon > 0$ and any integer $k \geq 2$ there exists an explicit $\delta = \delta(\epsilon, k) > 0$ such that, for G as above and any (not necessarily irreducible) character χ of G satisfying $|\chi| \leq |G|^\delta$ we have $|\chi^k| \geq |\chi|^{k-\epsilon}$; indeed, define $\delta(\epsilon, k) = \gamma(\epsilon/k, k)$.

Applying Theorem 3 we deduce the following result, which is a character-theoretic analogue of [LSS, Theorem 1.1]:

Theorem 5. *For all $\epsilon > 0$, there exists an explicit positive integer N such that if G is a finite simple group of Lie type and χ is any (not necessarily irreducible) character of G , then either χ^N contains every irreducible character of G , or $|\chi^2| \geq |\chi|^{2-\epsilon}$.*

The analogy with Gowers' theorem [Go, Theorem 3.3] raises the question of whether $N = 3$ suffices in Theorem 5 when $|\chi|$ is sufficiently large. A recent theorem of Sellke [Se2, Theorem 1.2] shows that the answer to this question is affirmative for large G if χ is so large that $\frac{|\chi|}{|G|}$ is bounded away from 0. We therefore ask the following:

Question. Does there exist an absolute constant $\epsilon > 0$ such that (i), respectively (ii), holds for any finite simple group G of Lie type and an arbitrary character χ of G ?

- (i) If $|\chi| \geq |G|^{1-\epsilon}$ then $|\chi^3| = |G|$.
- (ii) More generally, $|\chi^3| \geq \min(|G|, |\chi|^{1+\epsilon})$.

Question (ii) was suggested by one of the referees. Note that both (i) and (ii) are false for *non-simple* quasisimple groups, see Example 2.5(iv).

We remark that the example of $\mathrm{PSU}_{2n+1}(q)$ [HSTZ, Theorem 1.2] shows that, in general, it would be too much to ask for $|\chi^2| = |G|$. On the other hand, for certain simple groups of Lie type, we can bring N down to 6 or 7.

Theorem 6. *If $G = \mathrm{PSL}_n(q)$ and q is sufficiently large in terms of n , then for any, not necessarily irreducible, character χ of G , $|\chi| \geq |G|^{11/12}$ implies $|\chi^6| = |G|$. If $G = \mathrm{PSU}_n(q)$ and q is sufficiently large in terms of n , then for any, not necessarily irreducible, character χ of G , $|\chi| \geq |G|^{11/12}$ implies $|\chi^7| = |G|$.*

Because our proof makes essential use of [TT], we are limited to groups of type PSL and PSU and to q sufficiently large. However, we can offer the following weaker version of Theorem 6, which works for all finite non-abelian simple groups, if one is willing to take N sufficiently large.

Theorem 7. *There exists a universal constant $C > 0$ such that the following statement holds for any finite non-abelian simple group G and any $0 < \delta \leq 1$. If χ is any, not necessarily irreducible, character of G such that $|\chi| \geq |G|^\delta$, then $|\chi^N| = |G|$ for all $N \geq C/\delta$.*

By Lemma 2.1(ii), $|\chi^N| \leq |\chi|^N$, hence Theorem 7 is optimal up to a constant. Furthermore, Theorem 7 cannot hold for non-simple quasisimple groups, as shown in Example 2.5(iv).

We also offer a character-theoretic analogue of the Rodgers-Saxl theorem on products of conjugacy classes in [RS]. In the case that the characters are irreducible, this analogue was conjectured by Gill in [Gi] and proved in [LaT, Theorem 8.5].

Theorem 8. *There exists an explicit constant $c > 0$ such that the following statement holds. If G is a finite simple group of Lie type, $m \geq 1$ any integer, and $\chi_1, \chi_2, \dots, \chi_m$ are any (not necessarily irreducible) characters of G with $\prod_{i=1}^m |\chi_i| \geq |G|^c$, then $|\chi_1 \chi_2 \dots \chi_m| = |G|$ and thus $\chi_1 \chi_2 \dots \chi_m$ contains every irreducible character of G .*

Finally, we prove an analogue of Theorem 1 for compact semisimple Lie groups.

Theorem 9. *Let G be a compact semisimple Lie group. Then there exists $\epsilon = \epsilon(G) > 0$ such that for each irreducible character χ of G , we have $|\chi^2| \geq |\chi|^{1+\epsilon}$.*

It is possible that some of the presented results also hold for alternating groups (and likewise, Theorem 2 may hold for Lie-type groups of bounded rank as well). However, the techniques developed in the paper do not seem to apply to these open cases.

Some words on the structure of this paper. In Section 2 we prove some preliminary results for finite groups. Section 3 is devoted to the proofs of Theorem 1 and Theorem 2. In Section 4 we prove Theorems 3, 5, 8, and Corollary 4, while the proofs of Theorems 6 and 7 are carried out in Section 5. In Section 6 we study tensor product growth of representations of semisimple compact Lie groups and prove Theorem 9.

2. PRELIMINARIES

We begin with general inequalities for finite groups G .

Lemma 2.1. *Let α, β be any (not necessarily irreducible) characters of G . Then*

- (a) $|\alpha + \beta| \leq |\alpha| + |\beta|$.
- (b) $|\alpha\beta| \leq |\alpha| \cdot |\beta|$.

Proof. Without any loss of generality we may assume that $\alpha = \sum_i \alpha_i$ and $\beta = \sum_j \beta_j$ are multiplicity-free, with $\alpha_i, \beta_j \in \text{Irr}(G)$. Then

$$|\alpha + \beta| = \left| \sum_i \alpha_i + \sum_j \beta_j \right| \leq \sum_i \alpha_i(1)^2 + \sum_j \beta_j(1)^2 = |\alpha| + |\beta|,$$

proving (a). Next, $|\alpha_i \beta_j| \leq (\alpha_i \beta_j(1))^2 = \alpha_i(1)^2 \beta_j(1)^2$. It follows from (a) that

$$|\alpha\beta| = \left| \sum_{i,j} \alpha_i \beta_j \right| \leq \sum_{i,j} |\alpha_i \beta_j| \leq \sum_{i,j} \alpha_i(1)^2 \beta_j(1)^2 = \left(\sum_i \alpha_i(1)^2 \right) \cdot \left(\sum_j \beta_j(1)^2 \right) = |\alpha| \cdot |\beta|,$$

proving (b). □

Proposition 2.2. *Let $\chi_1, \dots, \chi_n$ be irreducible characters of G , $n \geq 2$.*

- (i) *In general, we have*

$$|\chi_1 \dots \chi_n| \geq \max_{1 \leq i \leq n} |\chi_i|.$$

- (ii) *If G is quasisimple and $\chi_i \neq 1_G$ for $1 \leq i \leq n$, then*

$$|\chi_1 \dots \chi_n| > \max_{1 \leq i \leq n} |\chi_i|.$$

- (iii) *If G is perfect, $\chi_1 \neq 1_G$, and $\chi_i \in \{\chi_1, \bar{\chi}_1\}$ for all i , then $|\chi_1 \dots \chi_n| > |\chi_1| = \dots = |\chi_n|$.*

Proof. (i) Without any loss of generality, assume $\chi_1(1) = \max_i \chi_i(1)$. Note that

$$\begin{aligned} \langle \chi_1 \cdots \chi_n, \chi_1 \cdots \chi_n \rangle &= \frac{1}{|G|} \sum_{g \in G} |\chi_1(g) \cdots \chi_n(g)|^2 \\ &\leq \frac{1}{|G|} \max_{h \in G} |\chi_2(h) \cdots \chi_n(h)|^2 \sum_{g \in G} |\chi_1(g)|^2 \\ &= \max_{h \in G} |\chi_2(h) \cdots \chi_n(h)|^2 \langle \chi_1, \chi_1 \rangle \\ &= \chi_2(1)^2 \cdots \chi_n(1)^2, \end{aligned}$$

with equality only if, for every $g \in G$,

$$(2.1) \quad \text{either } \chi_1(g) = 0 \text{ or } |\chi_2(g) \cdots \chi_n(g)| = \chi_2(1) \cdots \chi_n(1).$$

If $d_1, d_2, \dots, d_k$ are the degrees of the distinct irreducible constituents of $\chi_1 \cdots \chi_n$ and $m_1, \dots, m_k$ their multiplicities in $\chi_1 \cdots \chi_n$, then

$$\sum_i m_i^2 = \langle \chi_1 \cdots \chi_n, \chi_1 \cdots \chi_n \rangle \leq \chi_2(1)^2 \cdots \chi_n(1)^2$$

and

$$\sum_i d_i m_i = \chi_1(1) \cdots \chi_n(1).$$

So by the Cauchy-Schwarz inequality,

$$(2.2) \quad |\chi_1 \cdots \chi_n| = \sum_{i=1}^k d_i^2 \geq \frac{\chi_1(1)^2 \cdots \chi_n(1)^2}{\sum_{i=1}^k m_i^2} = \frac{\chi_1(1)^2 \cdots \chi_n(1)^2}{\langle \chi_1 \cdots \chi_n, \chi_1 \cdots \chi_n \rangle} \geq \chi_1(1)^2 = |\chi_1|,$$

as stated.

(ii) Suppose G is quasisimple, $\chi_i \neq 1_G$ for all i , but $|\chi_1 \cdots \chi_n| = |\chi_1| = \max_i |\chi_i|$. Since $\chi_i(1) > 1$, we see that $K_i := \{g \in G \mid |\chi_i(g)| = \chi_i(1)\}$ is equal to $\mathbf{Z}(G)$. Hence $\cap_{i=2}^n K_i = \mathbf{Z}(G)$, and using (2.1), we see that $\chi_1(g) = 0$ for all $g \notin \mathbf{Z}(G)$. It follows that

$$|G| = \sum_{g \in G} |\chi_1(g)|^2 = |\mathbf{Z}(G)| \chi_1(1)^2,$$

and thus $\chi_1(1) = |G/\mathbf{Z}(G)|^{1/2}$, i.e. χ_1 is of central type character for $G/\text{Ker}(\chi)$. By the Howlett-Isaacs theorem [HI], $G/\text{Ker}(\chi_1)$ is solvable, which is impossible, since it maps onto the simple group $G/\mathbf{Z}(G)$.

(iii) Suppose G is perfect and $\chi_1 \neq 1$, but $|\chi_1^{n-j} \bar{\chi}_1^j| = |\chi_1|$ for some $0 \leq j \leq n$. We again have that $K := \{g \in G \mid |\chi_1(g)| = \chi_1(1)\}$ is a normal subgroup of G , and $\chi_1(g) = 0$ for all $g \notin K$ by (2.1). It follows that

$$|G| = \sum_{g \in G} |\chi_1(g)|^2 = |K| \chi_1(1)^2,$$

and thus $\chi_1(1) = |G/K|^{1/2}$, i.e. χ_1 is a character of central type for $G/\text{Ker}(\chi_1)$. By the Howlett-Isaacs theorem, $G/\text{Ker}(\chi_1)$ is solvable, so $\text{Ker}(\chi_1) \geq [G, G] = G$ and thus $\chi_1 = 1_G$, a contradiction. $\square$

The following example shows the absence of growth in G , even for irreducible characters, in a class of groups G which are far from being simple. See also Example 2.5(iii) for examples of absence of growth for arbitrary characters (α with $|\alpha| < |G|$) in general.

Example 2.3. Let p be any prime, $n \geq 1$ any integer, and G be an extraspecial p -group of order p^{1+2n} . Then G has p^{2n} linear characters and $p - 1$ irreducible characters of degree p^n ; let χ be one of the latter (note that $\chi(1) = |G/\mathbf{Z}(G)|^{1/2}$). Now, if $k \geq 1$ is any integer, then χ^k is a multiple of the sum of p^{2n} linear characters if $p|k$, and a multiple of a single irreducible character of degree p^n if $p \nmid k$. Thus we always have $|\chi^k| = p^{2n} = |\chi| = |G|^{1-1/(2n+1)}$.

To study tensor product growth of arbitrary, not necessarily irreducible, characters of finite groups G , it is convenient to denote

$$\alpha \equiv \beta$$

for characters α, β of a finite group G whenever $\text{supp}(\alpha) = \text{supp}(\beta)$, i.e. α and β share the same irreducible constituents. Clearly, if $\alpha \equiv \beta$ and $\gamma \equiv \delta$, then

$$|\alpha| = |\beta|, (\alpha + \gamma) \equiv (\beta + \delta), \alpha\gamma \equiv \beta\delta.$$

Moreover, if G is a subgroup of H , then $\alpha \equiv \beta$ implies that $\text{Ind}_G^H(\alpha) \equiv \text{Ind}_G^H(\beta)$ as H -characters.

Example 2.4. Let A be a finite abelian group, and α, β be any two characters of A , say with $|\alpha| \geq |\beta|$. View $\text{Irr}(A)$ as an abelian group under character product. Then

$$|\alpha\beta| \geq \max(\alpha, \beta),$$

with equality attained precisely when there is a subgroup J of $\text{Irr}(A)$ such that $\text{supp}(\alpha)$ is a union of J -cosets and $\text{supp}(\beta)$ is contained in a J -coset.

Indeed, since A is abelian, $|\alpha| = m$ if $\text{supp}(\alpha) = \{\alpha_1, \dots, \alpha_m\}$. Let $\text{supp}(\beta) = \{\beta_1, \dots, \beta_n\}$. Then $\alpha\beta$ contains the m pairwise distinct irreducible characters $\alpha_i\beta_1$, so $|\alpha\beta| \geq m = |\alpha|$. Similarly, $|\alpha\beta| \geq n = |\beta|$.

Assume now that $|\alpha\beta| = m \geq n$. Note that $J := \{\lambda \in \text{Irr}(A) \mid \lambda\text{supp}(\alpha) = \text{supp}(\alpha)\}$ is a subgroup of $\text{Irr}(A)$, and $\text{supp}(\alpha)$ is then a union of J -cosets. Next, the equality $|\alpha\beta| = m$ implies that the n sets $\{\alpha_i\beta_j \mid 1 \leq i \leq m\}$ for $1 \leq j \leq n$ are all equal. It follows that the multiplication by each $\beta_j\beta_1^{-1}$ stabilizes the set $\text{supp}(\alpha)$, i.e. $\beta_j \in J\beta_1$, and thus $\text{supp}(\beta)$ is contained in the J -coset $J\beta_1$.

Conversely, suppose there is a subgroup J of $\text{Irr}(A)$ such that $\text{supp}(\alpha) = \{\alpha_1, \dots, \alpha_m\}$ is a union of J -cosets and $\text{supp}(\beta) = \{\beta_1, \dots, \beta_n\}$ is contained in a J -coset. Then $m \geq |J| \geq n$, $\lambda\text{supp}(\alpha) = \text{supp}(\alpha)$ for any $\lambda \in J$, and $\beta_j \in J\beta_1$ for any $1 \leq j \leq n$. It follows that the n sets $\{\alpha_i\beta_j \mid 1 \leq i \leq m\}$ for $1 \leq j \leq n$ are all equal, and hence $|\alpha\beta| = |\text{supp}(\alpha\beta)| = m = |\alpha|$.

Example 2.5. Let G be any finite group.

(i) If α is a character of G and λ a linear character (i.e. of degree 1) of G , then clearly $|\alpha\lambda| = |\alpha|$.

(ii) More generally, let $Z \leq \mathbf{Z}(G)$ be any central subgroup of G , γ any character of Z , and let α be any character of G such that $\alpha \equiv \text{Ind}_Z^G(\gamma)$ (i.e. $\chi \in \text{Irr}(G)$ is an irreducible constituent of α if and only if $\langle \chi|_Z, \gamma \rangle_Z > 0$). Let δ be any linear character of Z and let β be any character of G such that $\beta|_Z = \beta(1)\delta$. Then

$$(2.3) \quad |\alpha\beta| = |\alpha| = |G/Z| \cdot |\gamma|.$$

Indeed, without any loss we may assume $\gamma = \sum_{i=1}^m \gamma_i$ is a sum of m distinct linear characters, and $\alpha = \text{Ind}_Z^G(\gamma)$. For any $\chi \in \text{Irr}(G)$, since $Z \leq \mathbf{Z}(G)$ we have $\chi|_Z = \chi(1)\lambda$ for some $\lambda \in \text{Irr}(Z)$, and

$$\langle \alpha, \chi \rangle = \langle \gamma, \chi|_Z \rangle = \chi(1) \langle \gamma, \lambda \rangle_Z.$$

It follows that χ is a constituent of α if and only if $\lambda = \gamma_i$ for some i , in which case the multiplicity of χ in α is $\chi(1)$. For each γ_i , let $\text{Irr}(G|\gamma_i)$ denote the set of irreducible characters

of G that lie above γ_i ; note that these sets are disjoint when $i \neq j$. We have therefore shown that

$$\alpha = \sum_{i=1}^m \sum_{\chi \in \text{Irr}(G|\gamma_i)} \chi(1)\chi.$$

Taking the degree we obtain

$$|G/Z| \cdot |\gamma| = |G/Z|\gamma(1) = \alpha(1) = \sum_{i=1}^m \sum_{\chi \in \text{Irr}(G|\gamma_i)} \chi(1)^2 = |\alpha|,$$

establishing the second equality in (2.3).

Next, we have

$$\alpha\beta = \text{Ind}_Z^G(\gamma)\beta = \text{Ind}_Z^G(\gamma\beta|_Z) \equiv \text{Ind}_Z^G(\gamma\delta).$$

Applying the second equality in (2.3) to $\alpha\beta$, we obtain

$$|\alpha\beta| = |\text{Ind}_Z^G(\gamma\delta)| = |G/Z| \cdot |\gamma\delta| = |G/Z| \cdot |\gamma| = |\alpha|,$$

completing the proof of (2.3).

(iii) Consider any finite group G with a nontrivial central subgroup Z . Let λ and μ be any two linear characters of Z and take $\alpha := \text{Ind}_Z^G(\lambda)$. Let β be any sum of irreducible characters in $\text{Irr}(G|\mu)$ (with arbitrary multiplicities), so that $\beta|_Z \equiv \delta := \beta(1)\mu$. Then $|\alpha\beta| = |\alpha| = |G/Z|$ by (2.3). In particular, $|\alpha^k| = |\alpha| = |G|/|Z|$ for any $k \in \mathbb{Z}_{\geq 1}$ (since when $k \geq 2$ we have $\alpha^{k-1} \equiv \text{Ind}_Z^G(\lambda^{k-1})$ and so the equality follows by induction on k using $\beta := \alpha^{k-1}$).

(iv) Suppose G is a finite non-simple quasisimple group, so that $|\mathbf{Z}(G)| > 1$. First, we can find $\delta > 0$ such that $|G|^\delta \leq |\mathbf{Z}(G)|$. In this case, the character α constructed in (iii) has $|\alpha| = |G|/|\mathbf{Z}(G)| \leq |G|^{1-\delta}$, but yet $|\alpha^2| = |\alpha\bar{\alpha}| = |\alpha|$.

Secondly, given any $\delta' > 0$, we can find a finite quasisimple group G with $\mathbf{Z}(G) \cong C_2$ and $|G|^{\delta'} \geq 2$. In this case, the character α constructed in (iii) has $|\alpha| = |G|/2 \geq |G|^{1-\delta'}$, but yet $|\alpha^k| = |\alpha| < |G|$ for all $k \in \mathbb{Z}_{\geq 1}$.

Now we can prove the next result concerning tensor product growth of arbitrary characters:

Theorem 2.6. *Let α, β be any two, not necessarily irreducible, characters of a finite group G . Then the following statements hold.*

- (i) $|\alpha\beta| \geq \max(|\alpha|, |\beta|)$.
- (ii) *Suppose that $|\alpha\beta| = \max(|\alpha|, |\beta|) = |\alpha|$. Then there exist a normal subgroup M of G containing $\text{Ker}(\beta)$, a character γ of M , and a linear character δ of M , such that $\alpha \equiv \text{Ind}_M^G(\gamma)$ and $\beta|_M = \beta(1)\delta$.*
- (iii) *Suppose that G is quasisimple and that $\text{Ker}(\alpha) \neq G$, $\text{Ker}(\beta) \neq G$. Then*

$$|\alpha\beta| = \max(|\alpha|, |\beta|) = |\alpha|$$

if and only if the pair (α, β) is as constructed in Example 2.5(ii). In particular, if in addition G is simple, then $|\alpha\beta| > \max(|\alpha|, |\beta|)$ unless $|\alpha| = |G|$ or $|\beta| = G$.

Proof. (i) Consider the space $\mathcal{C}$ of class functions on G , with the usual Hermitian product $\langle \lambda, \mu \rangle = (1/|G|) \sum_{x \in G} \lambda(x) \overline{\mu(x)}$, and write $\|\lambda\|^2 := \langle \lambda, \lambda \rangle$. Also let $\text{supp}(\alpha) = \{\alpha_1, \dots, \alpha_m\}$, $\mathcal{C}_1 := \langle \alpha_1, \dots, \alpha_m \rangle_{\mathbb{C}}$, and consider the regular character ρ of G . Then $\rho_1 := \sum_{i=1}^m \alpha_i(1)\alpha_i$ is the orthogonal projection of ρ onto $\mathcal{C}_1$, and moreover

$$(2.4) \quad |G| - |\alpha| = \sum_{\chi \in \text{Irr}(G), \chi \neq \alpha_1, \dots, \alpha_m} \chi(1)^2 = \|\rho_1 - \rho\|^2.$$

Observe that $\theta := \rho_1\beta/\beta(1)$ is a linear combination of the irreducible constituents $\gamma_1, \dots, \gamma_n$ of $\alpha\beta$, and hence belongs to $\mathcal{C}_2 := \langle \gamma_1, \dots, \gamma_n \rangle_{\mathbb{C}}$. Letting ρ_2 denote the orthogonal projection of ρ onto $\mathcal{C}_2$ and applying (2.4) to $\alpha\beta$, we obtain

$$(2.5) \quad |G| - |\alpha\beta| = \|\rho_2 - \rho\|^2 \leq \|\theta - \rho\|^2.$$

Note that

$$(2.6) \quad |\theta(g)| = |\rho_1(g)| \cdot |\beta(g)|/\beta(1) \leq |\rho_1(g)|$$

for all $g \in G$. It follows that

$$\|\theta(g)\|^2 = (1/|G|) \sum_{g \in G} |\theta(g)|^2 \leq (1/|G|) \sum_{g \in G} |\rho_1(g)|^2 = \|\rho_1\|^2.$$

Since ρ is the regular character, we also have

$$\langle \theta, \rho \rangle = \theta(1) = \rho_1(1) = \langle \rho_1, \rho \rangle.$$

Hence,

$$\|\theta - \rho\|^2 = \|\theta\|^2 - 2\langle \theta, \rho \rangle + \|\rho\|^2 \leq \|\rho_1\|^2 - 2\langle \rho_1, \rho \rangle + \|\rho\|^2 = \|\rho_1 - \rho\|^2.$$

Putting it together with (2.4) and (2.5), we obtain

$$|G| - |\alpha\beta| \leq \|\theta - \rho\|^2 \leq \|\rho_1 - \rho\|^2 = |G| - |\alpha|,$$

and thus $|\alpha| \leq |\alpha\beta|$. Similarly, $|\beta| \leq |\alpha\beta|$, and the statement follows.

(ii) The arguments in (i) show that $|\alpha\beta| = |\alpha|$ implies that we must have equality in (2.6) for all $g \in G$, that is, either $\rho_1(g) = 0$ or $|\beta(g)| = \beta(1)$. If $\text{supp}(\beta) = \{\beta_1, \dots, \beta_l\}$, then $|\beta(g)| = \beta(1)$ if and only if

$$\beta_1(g)/\beta_1(1) = \beta_2(g)/\beta_2(1) = \dots = \beta_l(g)/\beta_l(1) =: \delta(g)$$

is a root of unity. For each i , the generalized kernel $K_i := \{g \in G \mid |\beta_i(g)| = \beta_i(1)\}$ is a normal subgroup of G containing $\text{Ker}(\beta)$, hence $M := \cap_{i=1}^m K_i$ is a normal subgroup of G , and the above δ yields a linear character on M , and $\beta|_M = \beta(1)\delta$.

We have shown that $\rho_1(g) = 0$ for all $g \in G \setminus M$. Setting $\gamma := \rho_1|_M$, we then have

$$|G| \cdot \langle \rho_1, \chi \rangle = \sum_{x \in M} \rho_1(x) \overline{\chi(x)} = |M| \langle \gamma, \chi|_M \rangle_M = |M| \langle \text{Ind}_M^G(\gamma), \chi \rangle$$

for any $\chi \in \text{Irr}(G)$. Thus $\chi \in \text{Irr}(G)$ belongs to $\text{supp}(\rho_1) = \text{supp}(\alpha)$ if and only if χ is an irreducible constituent of $\text{Ind}_M^G(\gamma)$. In other words, $\alpha \equiv \text{Ind}_M^G(\gamma)$.

(iii) Suppose $|\alpha\beta| = |\alpha|$, and consider the triple (M, γ, δ) as constructed in (ii). If $M = G$, then since G is perfect, $\delta = 1_G$, whence $\text{Ker}(\beta) = G$, contrary to the assumption. So $M < G$, which implies that $M \leq \mathbf{Z}(G)$ since G is quasisimple. Again without any loss of generality we may assume $\alpha = \text{Ind}_Z^G(\gamma)$, and observe that $\alpha\beta = \text{Ind}_M^G(\gamma\beta|_Z)$. Now (2.3) shows that $|\gamma\beta|_Z = |\gamma|$, so the pair (α, β) is as constructed in Example 2.5(ii). Conversely, any pair in Example 2.5(ii) certainly gives rise to an example with $|\alpha\beta| = |\alpha|$. $\square$

Remark 2.7. Let G be a finite simple group, and let α be any nontrivial, possibly reducible, character of G . Theorem 2.6(iii) shows that the sequence $|\alpha^n|$, $n = 1, 2, \dots$, is strictly increasing until it reaches $|G|$. Hence there is an integer $n = n(\alpha)$ such that any irreducible character of G occurs as an irreducible constituent of the single power α^n ; equivalently, $|\alpha^{n(\alpha)}| = |G|$.

This remark, as well as the ideas of the proofs of Theorem 2.6(i) and of Theorem 2.6(iii) in the case of simple G , was kindly pointed out to us by one of the referees. See also [LiSh1],

[LiST1], [LiST2], [LaT] for related results on the diameter of Cayley graphs for finite simple groups.

We conclude this section with an effective version of Remark 2.7. For any finite group G , let $k(G) := |\text{Irr}(G)|$ denote its class number.

Lemma 2.8. *Let G be a finite simple group and let χ be a faithful character of G .*

- (i) *Suppose χ takes exactly m distinct values on G . Then there is some integer $2 \leq j \leq m$ such that $|\chi^{j(m-1)}| = |G|$.*
- (ii) *There is some integer $1 \leq N_1 \leq k(G)(k(G) - 1)$ such that $|\chi^N| = |G|$ for all $N \geq N_1$.*

Proof. (i) As χ is faithful, χ admits a nontrivial irreducible constituent $\alpha \in \text{Irr}(G)$. By the Burnside-Brauer theorem [Br], $\text{supp}(\sum_{i=0}^{m-1} \chi^i) = \text{Irr}(G)$. Hence $\bar{\alpha}$ is a constituent of χ^{j-1} for some $2 \leq j \leq m$. As 1_G is a constituent of $\alpha\bar{\alpha}$ and α is a constituent of χ , it follows that 1_G is a constituent of χ^j . Now, χ^j also takes at most m distinct values on G , so again by the Burnside-Brauer theorem, $\text{supp}(\sum_{i=0}^{m-1} \chi^{ij}) = \text{Irr}(G)$. But note that since 1_G is a constituent of χ^j , every irreducible constituent of χ^{ij} also occurs in $\chi^{ij}\chi^j = \chi^{(i+1)j}$. Hence $\text{supp}(\sum_{i=0}^{m-1} \chi^{ij}) = \text{supp}(\chi^{j(m-1)})$, and the statement follows.

(ii) Certainly, the number m of distinct values of χ on G does not exceed the class number $k(G)$ of G . Hence, (i) implies that $|\chi^{N_1}| = |G|$ for some $1 \leq N_1 \leq k(G)(k(G) - 1)$. Now if $N \geq N_1$, then $|G| \geq |\chi^N| \geq |\chi^{N_1}|$ by Theorem 2.6(i), whence $|\chi^N| = |G|$. $\square$

3. PROOFS OF THEOREMS 1 AND 2

We begin this section by proving Theorem 2 in the case that χ is irreducible, which is also the high rank case of Theorem 1.

Proposition 3.1. *For all $\delta > 0$, there exist $\epsilon > 0$ and $R > 0$ such that if G is a finite quasisimple group of Lie type and rank $\geq R$, and $\chi \in \text{Irr}(G)$ satisfies $|\chi| \leq |G|^{1-\delta}$, then $|\chi^2| \geq |\chi|^{1+\epsilon}$ and $|\chi\bar{\chi}| \geq |\chi|^{1+\epsilon}$.*

Proof. Let $r \geq R$ be the rank of G and q the cardinality of the field of definition (for Suzuki and Ree groups, q is defined, as usual, so that q^2 is an odd power of 2 or 3.) By the main theorem of [LaT], there exists $c > 0$ such that

$$|\chi(g)| \leq \chi(1)^{1-c \frac{\log |g^G|}{\log |G|}}.$$

Given $\alpha > 0$, let $S = S_\alpha := \{g \in G \mid |g^G| \leq |G|^{\alpha/2}\}$. Then, by the well-known result of Fulman and Guralnick [FG, Theorem 1.1] on $k(G) := |\text{Irr}(G)|$, we have

$$|S| \leq k(G)|G|^{\alpha/2} \leq 27.2q^r |G|^{\alpha/2} \leq |G|^\alpha,$$

if R is taken sufficiently large in terms of α . Thus,

$$\begin{aligned}
\langle \chi^2, \chi^2 \rangle &= \langle \chi \bar{\chi}, \chi \bar{\chi} \rangle = \frac{1}{|G|} \sum_{g \in G} |\chi(g)|^4 \\
&= \frac{1}{|G|} \sum_{g \in S} |\chi(g)|^4 + \frac{1}{|G|} \sum_{g \in G \setminus S} |\chi(g)|^4 \\
&\leq \frac{|S|}{|G|} \chi(1)^4 + \frac{1}{|G|} \max_{g \in G \setminus S} |\chi(g)|^2 \sum_{g \in G \setminus S} |\chi(g)|^2 \\
&\leq |G|^{\alpha-1} \chi(1)^4 + \frac{1}{|G|} \chi(1)^{2(1-c\alpha/2)} \sum_{g \in G \setminus S} |\chi(g)|^2 \\
&\leq |G|^{\alpha-1} \chi(1)^4 + \chi(1)^{2-c\alpha}.
\end{aligned}$$

By hypothesis, $|\chi| \leq |G|^{1-\delta}$, so $\chi(1) \leq |G|^{1/2-\delta/2}$, which implies

$$\langle \chi^2, \chi^2 \rangle \leq \chi(1)^{4-\frac{2-2\alpha}{1-\delta}} + \chi(1)^{2-c\alpha} = (\chi(1)^{-\frac{2\delta-2\alpha}{1-\delta}} + \chi(1)^{-c\alpha}) \chi(1)^2.$$

The proposition is trivial for $\delta \geq 1$, so we assume $\delta < 1$, so that choosing

$$\alpha := \frac{2\delta}{2+c(1-\delta)}, \quad \epsilon := \frac{c\alpha}{4},$$

we have $\alpha > 0$. We get

$$\langle \chi^2, \chi^2 \rangle \leq 2\chi(1)^{-c\alpha} \chi(1)^2 \leq \chi(1)^{2-2\epsilon}$$

if R is taken sufficiently large. Applying the Cauchy-Schwarz inequality as in (2.2), we get

$$|\chi^2| \geq \frac{\chi(1)^4}{\langle \chi^2, \chi^2 \rangle} \geq \chi(1)^{2+2\epsilon} = |\chi|^{1+\epsilon}.$$

Likewise,

$$|\chi \bar{\chi}| \geq \frac{\chi(1)^4}{\langle \chi \bar{\chi}, \chi \bar{\chi} \rangle} \geq \chi(1)^{2+2\epsilon} = |\chi|^{1+\epsilon}.$$

□

To complete the proof of Theorem 1, we need only treat the bounded rank case. Using Proposition 2.2(iii), we may take $|G|$ to be as large as we wish.

Proposition 3.2. *Let R be fixed. For all $\delta > 0$, there exists $\epsilon > 0$ such that if G is a quasisimple group of Lie type of rank $r < R$ and χ is an irreducible character of G with $|\chi| \leq |G|^{1-\delta}$, then $|\chi^2|$ and $|\chi \bar{\chi}|$ are both at least $|\chi|^{1+\epsilon}$.*

Proof. We have already remarked that G can be assumed arbitrarily large. Since the rank r is bounded, this means we may take q arbitrarily large. According to Gluck's bound [Gl], for $g \in G \setminus \mathbf{Z}(G)$ we have $|\chi(g)/\chi(1)| \leq C/\sqrt{q}$, where C is an absolute constant. On the other hand, our upper bound on the rank gives an upper bound on $|G|$ of the form q^N for some N depending on R . Taking q sufficiently large, we therefore obtain that

$$\frac{|\chi(g)|}{\chi(1)} \leq Cq^{-1/2} \leq q^{-1/3} \leq |G|^{-\frac{1}{3N}} \leq \chi(1)^{-\frac{1}{3N}}$$

for all $g \in G \setminus \mathbf{Z}(G)$, and that $|\mathbf{Z}(G)| \leq R+1 \leq |G|^{\delta/3}$.

Setting $S := \mathbf{Z}(G)$, we proceed as in the proof of Proposition 3.1, obtaining

$$\begin{aligned} \langle \chi^2, \chi^2 \rangle &= \langle \chi \bar{\chi}, \chi \bar{\chi} \rangle = \frac{1}{|G|} \sum_{g \in G} |\chi(g)|^4 \\ &= \frac{\chi(1)^4 \cdot |\mathbf{Z}(G)|}{|G|} + \frac{1}{|G|} \sum_{g \notin \mathbf{Z}(G)} |\chi(g)|^4 \\ &\leq \frac{\chi(1)^4 \cdot |\mathbf{Z}(G)|}{|G|} + \frac{1}{|G|} \max_{g \notin \mathbf{Z}(G)} |\chi(g)|^2 \sum_{g \notin \mathbf{Z}(G)} |\chi(g)|^2 \\ &\leq \chi(1)^{4 - \frac{2(1-\delta/3)}{1-\delta}} + \chi(1)^{2(1-\frac{1}{3N})}. \end{aligned}$$

Assuming $\delta \leq \frac{1}{3N+1}$, we have $2(1 - 1/3N) \leq 2 - 2\delta/(1 - \delta)$, and so

$$\langle \chi^2, \chi^2 \rangle < 2\chi(1)^{2 - \frac{4\delta/3}{1-\delta}} \leq \chi(1)^{2 - \frac{\delta}{1-\delta}}$$

if $|G|$, and therefore $\chi(1)$, is sufficiently large. Taking $\epsilon := \frac{\delta}{2-2\delta}$ and using Cauchy-Schwarz as in (2.2), the proposition follows. $\square$

Now we prove Theorem 2.

Proof. First we apply Proposition 3.1 to the given δ to obtain ϵ and R so that whenever G has rank $\geq R$ and $\psi \in \text{Irr}(G)$ satisfies $|\psi| \leq |G|^{1-\delta}$ we have $|\psi^2|, |\psi \bar{\psi}| \geq |\psi|^{1+\epsilon}$. Now choose $N := 1/2 + 1/\epsilon$, for which

$$(3.1) \quad (1 + \epsilon) \frac{2N}{2N + 1} = 1 + \epsilon/2.$$

According to [LiSh3, Theorem 1.2], by choosing $R \geq \max(9, 2N)$ sufficiently large, whenever G has rank $r \geq R$, then

$$(3.2) \quad \sum_{\chi \in \text{Irr}(G)} \chi(1)^{-1/N} < \frac{3}{2}.$$

Since G is quasisimple, it has only one irreducible character of degree 1, and by [LaSe] all other characters have degree at least $2^{r/2} \geq 2^N$. Now for any $D \geq 2^N$, we have

$$1 + (D^{1/N} - 1)/D^{1/N} = 2 - D^{-1/N} \geq 3/2,$$

whence (3.2) implies that the number of irreducible characters of G of degree $\leq D$ is at most $D^{1/N}$.

Now consider any character χ of G with distinct irreducible constituents $\chi_1, \dots, \chi_k$ of degrees $d_1 \leq d_2 \leq \dots \leq d_k$ with $|\chi| \leq |G|^{1-\delta}$. If $d_k = 1$, then $\chi = 1_G$ and $|\chi^2| = |\chi \bar{\chi}| = |\chi|^2$. Assuming $d_k > 1$, we then have $d_k \geq 2^N$, whence $k \leq d_k^{1/N}$ by the preceding remark, and so

$$|\chi| = \sum_{i=1}^k d_i^2 \leq d_k^{1/N} d_k^2.$$

This implies that

$$|\chi_k| = d_k^2 \geq |\chi|^{\frac{2N}{2N+1}}.$$

On the other hand, $|\chi^2| \geq |\chi_k^2|$ and $|\chi \bar{\chi}| \geq |\chi_k \bar{\chi}_k|$. Applying Proposition 3.1 to $\psi := \chi_k$ and using (3.1), we obtain $|\chi^2|, |\chi \bar{\chi}| \geq |\chi_k|^{1+\epsilon} \geq |\chi|^{1+\epsilon/2}$. $\square$

It remains open whether Theorem 2 would hold true for simple groups of Lie type but without the high rank assumption. Following a referee's suggestion, we settle the PSL_2 case in the next statement.

Proposition 3.3. *There exists an absolute constant $C > 0$ such that if $G = \mathrm{SL}_2(q)$ and χ is any arbitrary nontrivial character of G then $|\chi^2|$ and $|\chi\bar{\chi}|$ are both at least $C|G|$. In particular, for all $\delta > 0$, there exists $\epsilon > 0$ such that if χ is an arbitrary character of $S := \mathrm{PSL}_2(q)$ with $|\chi| \leq |S|^{1-\delta}$, then $|\chi^2|$ and $|\chi\bar{\chi}|$ are both at least $|\chi|^{1+\epsilon}$.*

Proof. It suffices to prove the statement in the case $\chi = \sum_{i=1}^s \chi_i$ is a sum of s distinct irreducible characters $\chi_i \in \mathrm{Irr}(G)$, and we may assume that $s \geq 2$ by Proposition 3.2 and that $q \geq 4$ is as large as we wish. Now

$$C_1 s q \geq s(q+1) \geq \chi(1) \geq 1 + (s-1)(q-1)/2 \geq C_2 s q, \quad |\chi| \leq s(q+1)^2 \leq C_3 s q^2$$

for some absolute constants $C_1, C_2, C_3 > 0$. Next, $|\chi_i(g)| \leq q^{1/2}$ for all $g \in G \setminus \mathbf{Z}(G)$, $|\chi_i(g)| \leq 2$ for all non-central semisimple $g \in G$, and the total number of non-semisimple elements of G is less than $2q^2$. It follows that

$$\langle \chi^2, \chi^2 \rangle < \frac{1}{|G|} (2C_1^4 s^4 q^4 + 16s^4 |G| + 2q^2 s^4 q^2) \leq C_4 s^4 q$$

for some absolute constant $C_4 > 0$. Applying the Cauchy-Schwarz inequality as in (2.2), we get

$$|\chi^2| \geq \frac{\chi(1)^4}{\langle \chi^2, \chi^2 \rangle} \geq (C_2/C_4) q^3 > C|G|$$

with $C := C_2/C_4$.

Next we look at $S = \mathrm{PSL}_2(q)$, viewing it as $G/\mathbf{Z}(G)$ and viewing any character χ of S with $|\chi| \leq |S|^{1-\delta}$ as a character of G . By Theorem 2.6(iii), by choosing $\epsilon > 0$ small enough, we may assume q is as large as we wish, in particular, $q^{\delta^2} \geq 1/C$. In this case we have

$$|\chi^2| > C|G| \geq q^{-\delta^2} |G| \geq |G|^{1-\delta^2} \geq |\chi|^{1+\delta}.$$

The same argument applies to $\chi\bar{\chi}$. □

4. PROOF OF THEOREMS 3, 5 AND 8

The notion of the *level* $\mathbf{l}(\chi)$ of an irreducible character $\chi \in \mathrm{Irr}(G)$ of a finite classical group G was introduced in [GLT1] for groups of type A , and [GLT2, Definition 3.2] for other classical types. We briefly recall this notion for type A ; the definition for other types will be recalled in the corresponding parts of the proof of Proposition 4.1.

Consider the reducible *Weil character* $\tau = \tau_{n,q}$ of $G = \mathrm{SL}_n(q)$, which is just the permutation character of G acting on the point set of the natural module $V := \mathbb{F}_q^n$. By [GLT1, Definition 2 and Lemma 7.1], $j = \mathbf{l}(\chi)$ is the smallest non-negative integer such that χ is a constituent of τ^j .

In the case of $\mathrm{SU}_n(q)$, consider the reducible *Weil character* $\zeta = \zeta_{n,q}$ of $G = \mathrm{SU}_n(q)$:

$$\zeta(g) = (-1)^n (-q)^{\dim_{\mathbb{F}_{q^2}} \mathrm{Ker}(g-1_V)},$$

where $V := \mathbb{F}_{q^2}^n$ is the natural G -module, see [GLT1, (1.1)]. By [GLT1, Definition 2 and Lemma 7.1], $j = \mathbf{l}(\chi)$ is the smallest non-negative integer such that χ is a constituent of ζ^j .

For brevity, we use the notation SL^ϵ to denote SL when $\epsilon = +$ and SU when $\epsilon = -$. Using the results of [GLT1, GLT2], we can prove the following bound (which is useful only when $L < \sqrt{n}$).

Proposition 4.1. *Let $G = \mathrm{SL}_n^\epsilon(q)$ with $n \geq 7$, or $\mathrm{Sp}_{2n}(q)$, $\Omega_{2n+1}(q)$, $\Omega_{2n}^\pm(q)$ with $n \geq 6$, and let $L \geq 1$ be any real number.*

- (a) *Suppose $G = \mathrm{SL}_n^\epsilon(q)$ and $L \leq n/6$. Then the number $N(L)$ of $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \leq q^{nL}$ is at most q^{12L^2} .*
- (b) *Suppose $G = \mathrm{Sp}_{2n}(q)$ with $2 \nmid q$. Then the number $N(L)$ of $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \leq q^{nL}$ is at most q^{18L^2} .*
- (c) *Suppose $G = \mathrm{Sp}_{2n}(q)$ with $2|q$, or $\Omega_{2n+1}(q)$ with $2 \nmid q$, or $\Omega_{2n}^+(q)$ for any q . Then the number $N(L)$ of $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \leq q^{nL}$ is at most q^{50L^2} .*
- (d) *Suppose $G = \Omega_{2n}^-(q)$. Then the number $N(L)$ of $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \leq q^{(n-1)L}$ is at most q^{41L^2} .*

Proof. (a) First we consider any $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \leq q^{nL}$ and bound $j := \mathfrak{l}(\chi)$ in terms of L . If $j > n/2$, then by [GLT1, Theorem 1.3(ii)] we have

$$\chi(1) \geq (2/3)q^{n^2/4-3} > q^{n^2/6}$$

since $n \geq 7$, contradicting the condition $L \leq n/6$. Hence $j \leq n/2$. Now using [GLT1, Theorem 1.3(i)] we get

$$q^{nL} \geq \chi(1) \geq \frac{1}{2(q+1)}q^{j(n-j)} > q^{j(n-j)-2.6} \geq q^{jn/2-2.6},$$

which implies

$$(4.1) \quad j \leq \frac{nL + 2.6}{n/2} = 2L + \frac{5.2}{n} < \frac{11}{4}L.$$

Let $\epsilon = +$ and consider the reducible *Weil character* $\tau = \tau_{n,q}$ of $\mathrm{SL}_n(q)$, see [GLT1, (1.1)]. As τ is just the permutation character of $G = \mathrm{SL}_n(q)$ acting on the point set of $V := \mathbb{F}_q^n$, it contains 1_G , and so τ^j contains all irreducible constituents of $\sum_{i=0}^j \tau^i$. By definition, $j = \mathfrak{l}(\chi)$ is the smallest non-negative integer such that χ is a constituent of τ^j . Hence, (4.1) implies that $N(L) \leq N^*(l)$ with $l := \lfloor 11L/4 \rfloor$, where $N^*(l)$ is the number of distinct irreducible constituents of τ^l . Next,

$$N^*(l) \leq \langle \tau^l, \tau^l \rangle = \langle \tau^{2l}, 1_G \rangle,$$

the number of G -orbits on V^{2l} . By [GLT1, Lemma 2.4], the number of $\mathrm{GL}_n(q)$ -orbits on V^{2l} is at most $8q^{l^2}$. Since G has index $q-1$ in $\mathrm{GL}_n(q)$, we have

$$N(L) \leq 8(q-1)q^{l^2} \leq q^{l^2+3} \leq q^{(11L/4)^2+3} < q^{11L^2}.$$

Now suppose $\epsilon = -$ and consider the reducible *Weil character* $\zeta = \zeta_{n,q}$ of $\mathrm{SU}_n(q)$, see [GLT1, (1.1)]. Note that $\zeta^2 = (\tau_{n,q^2})|_G$ is just the permutation character of $G = \mathrm{SU}_n(q)$ acting on the point set of $U := \mathbb{F}_{q^2}^n$; in particular ζ^2 contains 1_G , and so $\zeta^j + \zeta^{j-1}$ contains all irreducible constituents of $\sum_{i=0}^j \zeta^i$. Hence, (4.1) implies that $N(L) \leq N^*(l) + N^*(l-1)$ with $l := \lfloor 11L/4 \rfloor$, where $N^*(l)$ is the number of distinct irreducible constituents of ζ^l . Next,

$$N^*(l) \leq \langle \zeta^l, \zeta^l \rangle = \langle \zeta^{2l}, 1_G \rangle = \langle (\tau_{n,q^2})|_G, 1_G \rangle,$$

the number of G -orbits on U^{2l} . By [GLT1, Lemma 2.4], the number of $\mathrm{GU}_n(q)$ -orbits on U^{2l} is at most $2q^{l^2}$. Since G has index $q+1$ in $\mathrm{GU}_n(q)$, we have

$$N(L) \leq 4(q+1)q^{l^2} < q^{l^2+4} \leq q^{(11L/4)^2+4} < q^{12L^2}.$$

- (b) In the remaining cases, we set $k := \lfloor (j+2)/3 \rfloor$ for $j := \mathfrak{l}(\chi)$, so that

$$j/3 \leq k \leq (j+2)/3.$$

Consider the case of $G = \mathrm{Sp}_{2n}(q)$ with $2 \nmid q$. By [GLT2, Lemma 3.4], $j \leq 2n + 1$, so

$$(k+1)/2 \leq (2n+6)/6 \leq n/2$$

when $n \geq 6$. Now, applying [GLT2, Theorem 1.5] we have

$$q^{nL} \geq \chi(1) \geq q^{k(n-(k+1)/2)} \geq q^{kn/2},$$

whence $k \leq 2L$ and so

$$(4.2) \quad j \leq 3k \leq 6L.$$

By definition, $\mathbf{l}(\chi) = j$ means that χ is an irreducible constituent of $(\omega + \omega^*)^j$, where ω and ω^* are the two reducible *Weil characters* of $\mathrm{Sp}_{2n}(q)$, see [GLT2, §3]. Next, $(\omega + \omega^*)^2$ always contains $(\tau_{2n,q})|_G$, and hence 1_G as well, by [GLT2, Proposition 3.1]. It follows that $(\omega + \omega^*)^j + (\omega + \omega^*)^{j-1}$ contains all irreducible constituents of $\sum_{i=0}^j (\omega + \omega^*)^i$. Hence, (4.2) implies that $N(L) \leq N^*(l) + N^*(l-1)$ with $l := \lfloor 6L \rfloor$, where $N^*(l)$ is the number of distinct irreducible constituents of $(\omega + \omega^*)^l$. By [GLT2, Proposition 3.2], $\omega^2 = (\omega^*)^2$, so in fact any irreducible constituent of $(\omega + \omega^*)^l$ is an irreducible constituent of ω^l or of $\omega^{l-1}\omega^*$. If $q \equiv 3 \pmod{4}$, then by [GLT2, Proposition 3.2] for $0 \leq i \leq l$ we have

$$\langle \omega^{l-i}(\omega^*)^i, \omega^{l-i}(\omega^*)^i \rangle = \langle \omega^{l-i}\bar{\omega}^i, \omega^{l-i}\bar{\omega}^i \rangle = \langle (\omega\bar{\omega})^{2l}, 1_G \rangle = \langle (\tau_{2n,q}^l)|_G, 1_G \rangle,$$

the number of G -orbits on V^l with $V := \mathbb{F}_q^{2n}$, which is at most $6q^{l(l-1)/2}$ by [GLT2, Lemma 2.6]. It follows that $N^*(l) \leq 12q^{l(l-1)/2}$.

If $q \equiv 1 \pmod{4}$, then by [GLT2, Proposition 3.2] for $0 \leq i \leq l$ we have

$$\langle \omega^{l-i}(\omega^*)^i, \omega^{l-i}(\omega^*)^i \rangle = \langle (\omega)^{2l-2i}(\omega^*)^{2i}, 1_G \rangle = \langle (\tau_{2n,q}^l)|_G, 1_G \rangle,$$

which is again the number of G -orbits on V^l with $V := \mathbb{F}_q^{2n}$ and so at most $6q^{l(l-1)/2}$ by [GLT2, Lemma 2.6]. Thus we also have $N^*(l) \leq 12q^{l(l-1)/2}$ in this case.

Thus $N(L) \leq 12q^{l(l-1)/2} + 12q^{(l-1)(l-2)/2} < q^{6L(6L-1)/2+3} < q^{18L^2}$.

(c) Now consider the case of $G = \mathrm{Sp}_{2n}(q)$ with $2|q$, or $\Omega_{2n+1}(q)$ with $2 \nmid q$, or $\Omega_{2n}^+(q)$. By [GLT2, Lemma 3.4], $j \leq n + 1$, so

$$k+1 \leq (n+6)/3 \leq 2n/3$$

when $n \geq 6$. Applying [GLT2, Theorem 1.5] we have

$$q^{nL} \geq \chi(1) \geq q^{2k(n-(k+1))} \geq q^{2kn/3},$$

whence $k \leq 3L/2$ and so

$$(4.3) \quad j \leq 3k \leq 9L/2.$$

By definition, $\mathbf{l}(\chi) = j$ means that χ is an irreducible constituent of $((\tau + \zeta)|_G)^j$, where τ and ζ are the reducible Weil characters of $\mathrm{SL}_D(q)$ and $\mathrm{SU}_D(q)$, with $D = 2n$ or $2n + 1$, the dimension of the natural module $V := \mathbb{F}_q^D$ of G . In particular, τ_G contains 1_G . Hence, (4.3) implies that $N(L) \leq N^*(l)$ with $l := \lfloor 9L/2 \rfloor$, where $N^*(l)$ is the number of distinct irreducible constituents of $(\tau|_G + \zeta|_G)^l$. Since $(\tau|_G)^2 = (\zeta|_G)^2$, any irreducible constituent of $(\tau|_G + \zeta|_G)^l$ is an irreducible constituent of $(\tau|_G)^l$ or of $(\tau|_G)^{l-1}(\zeta|_G)$. Now, for $0 \leq i \leq l$ we have

$$\langle (\tau|_G)^{l-i}(\zeta|_G)^i, (\tau|_G)^{l-i}(\zeta|_G)^i \rangle = \langle (\tau|_G)^{2l-2i}(\zeta|_G)^{2i}, 1_G \rangle = \langle (\tau^{2l})|_G, 1_G \rangle,$$

the number of G -orbits on V^{2l} . By [GLT2, Lemma 2.6], the number of $\tilde{G}$ -orbits on V^{2l} is at most $6q^{l(2l+1)}$, for $\tilde{G} = \mathrm{Sp}_D(q)$ or $\mathrm{GO}_D(q)$, respectively. Note that $[\tilde{G} : G]$ is 1 in the Sp -case, 2 in the Ω -case with $2 \nmid D$ or with $2 \mid q$, and 4 otherwise. It follows that

$$N^*(l) \leq 12q^{l(2l+1)/2} \cdot [\tilde{G} : G] < q^{l(2l+1)+5} < q^{50L^2}.$$

(d) Finally, assume that $G = \Omega_{2n}^-(q)$. In this case, $j \leq n+1$, so

$$k+1 \leq (n+3)/3 \leq 3(n-1)/5$$

when $n \geq 6$. Applying [GLT2, Theorem 1.5] we have

$$q^{(n-1)L} \geq \chi(1) \geq q^{2k(n-1-k)} \geq q^{4k(n-1)/5},$$

whence $k \leq 5L/4$ and so $j \leq 3k \leq 15L/4$. Now we can repeat the arguments in (c) to get $N(l) \leq N^*(l) \leq q^{l(2l+1)+5} < q^{41L^2}$. $\square$

We can now prove Theorem 3 and Corollary 4.

Proof of Theorem 3. Let $\epsilon > 0$ be given. We claim there exists $\delta > 0$ such that if G is a finite quasisimple group of Lie type, and χ_1 and χ_2 are (possibly reducible) characters of G with $|\chi_1|, |\chi_2| \leq |G|^\delta$, then

$$|\chi_1 \chi_2| \geq (|\chi_1| \cdot |\chi_2|)^{1-\epsilon}.$$

It suffices to prove the statement in the case $|\chi_1|, |\chi_2| > 1$; in particular, the χ_i contain nontrivial irreducible constituents α_1 and α_2 , respectively. We may also assume that each χ_i is multiplicity-free. If G is an exceptional group of Lie type, or a finite classical group of rank ≤ 7 , then [LaSe] implies that $\alpha_i(1) > |G|^{1/20}$. Taking $\delta \leq 1/10$, we may assume G is a classical group of rank ≥ 8 . By [GLT2, Proposition 6.3], if G is of orthogonal type, the bounds on $|\chi_i|$ imply that they arise from characters of $\Omega_{2n+1}^+(q)$, $\Omega_{2n}^+(q)$, or $\Omega_{2n}^-(q)$ by composition with a central quotient map.

We will view χ_1, χ_2 as characters of $\hat{G} = \mathrm{SL}_n^\pm(q)$, $\mathrm{Sp}_{2n}(q)$, $\Omega_{2n+1}^+(q)$, $\Omega_{2n}^+(q)$, or $\Omega_{2n}^-(q)$, respectively, where $n \geq 8$. In all cases, $\hat{G}$ contains a subgroup $H \cong \mathrm{SL}_m^\pm(q)$ with $m := n$ in the first case, and $H \cong \mathrm{SL}_m(q)$ (inside a split Levi subgroup) with $m := n, n, n$, or $n-1$, respectively, in the other four cases. One checks that $q^{m^2/2} < |G| < q^{4m^2}$ in all cases; furthermore, $\alpha_i(1) > q^{m/2}$. Writing

$$(4.4) \quad q^{mL_i/2} \leq \chi_i(1) = q^{mD_i} \leq q^{mL_i}$$

for some (real) numbers $L_i \geq 1$ and $L_i/2 \leq D_i \leq L_i$, we will choose δ so that

$$q^{mL_i} \leq |\chi_i| \leq |G|^\delta \leq q^{4m^2\delta},$$

in particular,

$$(4.5) \quad L_i \leq 4m\delta.$$

Writing

$$\chi_1 \chi_2 = \sum_{\gamma \in \mathrm{Irr}(G)} m_\gamma \gamma, \quad \chi_1^2 = \sum_{\gamma \in \mathrm{Irr}(G)} a_\gamma \gamma, \quad \chi_2^2 = \sum_{\gamma \in \mathrm{Irr}(G)} b_\gamma \gamma,$$

we have by the Cauchy-Schwarz inequality

$$\begin{aligned}
\left(\sum_{\gamma} m_{\gamma}^2\right)^2 &= \langle \chi_1 \chi_2, \chi_1 \chi_2 \rangle^2 \\
&= \langle \chi_1 \bar{\chi}_1, \chi_2 \bar{\chi}_2 \rangle^2 \\
&\leq \langle \chi_1 \bar{\chi}_1, \chi_1 \bar{\chi}_1 \rangle \cdot \langle \chi_2 \bar{\chi}_2, \chi_2 \bar{\chi}_2 \rangle \\
&= \langle \chi_1^2, \chi_1^2 \rangle \cdot \langle \chi_2^2, \chi_2^2 \rangle \\
&= \sum_{\gamma} a_{\gamma}^2 \cdot \sum_{\gamma} b_{\gamma}^2 \\
&\leq \left(\sum_{\gamma} a_{\gamma}\right)^2 \cdot \left(\sum_{\gamma} b_{\gamma}\right)^2,
\end{aligned}$$

and thus

$$(4.6) \quad \sum_{\gamma} m_{\gamma}^2 \leq \sum_{\gamma} a_{\gamma} \cdot \sum_{\gamma} b_{\gamma} = \sigma(\chi_1^2, G) \cdot \sigma(\chi_2^2, G),$$

where, $\sigma(\cdot, \cdot)$, as defined in [GLT2, (2.1)], denotes the sum of the multiplicities of all the irreducible constituents of the first argument, regarded as a character on the group specified in the second argument.

First we consider the case $\hat{G} = \mathrm{SL}_n^{\pm}(q)$. Then each irreducible constituent θ of χ_i lies under some $\tilde{\theta} \in \mathrm{Irr}(\mathrm{GL}_n^{\pm}(q))$ of degree less than $(q \mp 1)\theta(1)$. Hence χ_i lies under some character $\tilde{\chi}_i$ of $\mathrm{GL}_n^{\pm}(q)$ of degree at most $q^{mL_i+2} \leq q^{5mL_i/4}$ since $m \geq 8$. Applying [GLT2, Corollary 5.2(i)] to $\tilde{\chi}_i$ and restricting further down to $\mathrm{SL}_n^{\pm}(q)$, we have

$$\sigma(\chi_i^2, G) \leq q^{(125/4)L_i^2+2} \sigma(\chi_i, G)^2 < q^{56L_i^2}.$$

Here, we use the fact that χ_i is multiplicity-free, and so $\sigma(\chi_i, G)$ is at most the number of irreducible characters of G of degree $\leq q^{nL_i}$, which is at most $q^{12L_i^2}$ by Proposition 4.1(a).

Next suppose we are in the symplectic-orthogonal case. Taking $\delta > 0$ small enough, we may assume that $L_i \leq n/9$. Applying Theorem 5.8 and Corollary 5.9 of [GLT2] to χ and restricting further down from a normal subgroup of $\mathrm{GL}_m(q)$ containing $H = \mathrm{SL}_m(q)$ to H if necessary, we have

$$\sigma(\chi_i^2, G) \leq q^{A\sqrt{mL_i^3}+60L_i^2} \sigma(\chi_i, G)^2 \leq q^{A\sqrt{mL_i^3}+160L_i^2}$$

for some explicit A (which can be taken to be 70). Here, we again use the fact that χ_i is multiplicity-free, and so $\sigma(\chi_i, G)$ is at most the number of irreducible characters of G of degree $\leq q^{mL_i}$, which is at most $q^{50L_i^2}$ by Proposition 4.1(b)–(d).

Using (4.6), in either case we have

$$\sum_{\gamma} m_{\gamma}^2 \leq q^{A\sqrt{mL_1^3}+A\sqrt{mL_2^3}+BL_1^2+BL_2^2}$$

with $B = 160$. It follows that

$$|\chi_1 \chi_2| = \sum_{m_{\gamma} > 0} \gamma(1)^2 \geq \frac{(\sum_{\gamma} m_{\gamma} \gamma(1))^2}{\sum_{\gamma} m_{\gamma}^2} = \frac{\chi_1(1)^2 \chi_2(1)^2}{\sum_{\gamma} m_{\gamma}^2} \geq q^{2m(D_1+D_2)-A\sqrt{mL_1^3}-A\sqrt{mL_2^3}-BL_1^2-BL_2^2}.$$

Recalling (4.5), we observe

$$\frac{\sqrt{mL_1^3} + \sqrt{mL_2^3}}{m(D_1 + D_2)} \leq \frac{2mL_1\sqrt{\delta} + 2mL_2\sqrt{\delta}}{m(L_1 + L_2)/2} = 4\sqrt{\delta},$$

and

$$\frac{L_1^2 + L_2^2}{m(D_1 + D_2)} \leq \frac{2mL_1\delta + 2mL_2\delta}{m(L_1 + L_2)/2} = 4\delta.$$

Hence

$$|\chi_1\chi_2| \geq q^{2m(D_1+D_2)(1-2A\sqrt{\delta}-2B\delta)}.$$

Note that $|\chi_1| \cdot |\chi_2| \leq \chi_1(1)^2\chi_2(1)^2 = q^{2m(D_1+D_2)}$. Hence, taking $\delta > 0$ so that

$$2A\sqrt{\delta} + 2B\delta \leq \epsilon,$$

we have $|\chi_1\chi_2| \geq (|\chi_1| \cdot |\chi_2|)^{1-\epsilon}$, as desired. $\square$

Proof of Corollary 4. We prove by induction on $k \geq 2$ the following statement:

Fix any $\alpha > 0$. For any $k \geq 2$, there exists an explicit $\gamma = \gamma(\alpha, k) > 0$ such that the following statement holds. If G is a finite quasisimple group of Lie type and $\chi_1, \chi_2, \dots, \chi_k$ are any characters of G with $|\chi_1|, |\chi_2|, \dots, |\chi_k| \leq |G|^\gamma$, then

$$|\chi_1\chi_2 \cdots \chi_k| \geq (|\chi_1| \cdot |\chi_2| \cdots |\chi_k|)^{1-k\alpha}.$$

With ϵ, k given, applying this statement with $\alpha := \epsilon/k$ we obtain Corollary 4.

We will show that the above statement holds with $\gamma(\alpha, k) := \delta/(k-1)$, where $\delta = \delta(\alpha)$ is the constant in Theorem 3. The case $k = 2$ was already established in Theorem 3. For the inductive step $k \geq 3$, note by Lemma 2.1 and the induction hypothesis that

$$(|\chi_2| \cdots |\chi_k|)^{1-(k-1)\alpha} \leq |\chi_2 \cdots \chi_k| \leq \prod_{i=2}^k |\chi_i| \leq |G|^{\gamma(k-1)} = |G|^\delta,$$

since $|\chi_2|, \dots, |\chi_k| \leq |G|^{\delta/(k-1)} < |G|^{\gamma(\alpha, k-1)}$. Now, since $|\chi_1| \leq |G|^{\delta/(k-1)} \leq |G|^\delta$, by Theorem 3 we have

$$\begin{aligned} |\chi_1\chi_2 \cdots \chi_k| &\geq (|\chi_1| \cdot |\chi_2 \cdots \chi_k|)^{1-\alpha} \\ &\geq (|\chi_1| \cdot (|\chi_2| \cdots |\chi_k|)^{1-(k-1)\alpha})^{1-\alpha} \\ &\geq (|\chi_1| \cdot |\chi_2| \cdots |\chi_k|)^{1-k\alpha}. \end{aligned}$$

$\square$

The next result is required in our proof of Theorem 5.

Theorem 4.2. For any $\delta > 0$, there exists an explicit integer N such that the following statement holds. If G is a finite simple group of Lie type, and $\chi_1, \dots, \chi_N$ are any (not necessarily irreducible) characters of G with $|\chi_i| \geq |G|^\delta$ for all i , then $|\chi_1\chi_2 \cdots \chi_N| = |G|$ and thus $\chi_1\chi_2 \cdots \chi_N$ contains every irreducible character of G .

Proof. For any character χ of G , let χ^* denote (some) irreducible constituent of largest degree of χ .

First we consider the case $k(G) \geq |G|^{\delta/3}$. If G is of rank r over $\mathbb{F}_q$, then by [FG] we have $k(G) \leq (27.2)q^r$, whereas $|G| \geq q^{r^2}$. It follows that $r \leq r_0$, where r_0 depends only on δ . On the other hand, $|\chi_i| > 1$ implies that $\chi_i^* \neq 1_G$, and so $\chi_i^*(1) > q^r/3$ by [LaSe]. As $r \leq r_0$, $\chi_i^*(1) \geq |G|^{\delta_0}$ for some δ_0 depending on δ . Applying [LaT, Theorem 8.5] and taking $N \geq \epsilon/\delta_0$ (with ϵ the constant in [LaT, Theorem 8.5]), we see that $\chi_1^* \cdots \chi_N^*$ contains $\text{Irr}(G)$.

Now suppose that $k(G) \leq |G|^{\delta/3}$. Then $\chi_i^*(1) \geq (|\chi_i|/k(G))^{1/2} \geq |G|^{\delta/3}$. Applying [LaT, Theorem 8.5] and taking $N \geq 3\epsilon/\delta$, we again see that $\chi_1^* \cdots \chi_N^*$ contains $\text{Irr}(G)$. $\square$

Proof of Theorem 5. Fix $\epsilon > 0$. It follows from Theorem 3, applied for $\chi_1 = \chi_2 = \chi$ with ϵ replaced by $\epsilon/2$, that there exists some $\delta > 0$ depending only on ϵ such that if $|\chi| \leq |G|^\delta$, then $|\chi^2| \geq |\chi|^{2-\epsilon}$. By Theorem 4.2, there is some integer $N > 0$ depending only on ϵ , such that if $|\chi| \geq |G|^\delta$ then χ^N contains $\text{Irr}(G)$. The result follows. $\square$

Proof of Theorem 8. For any character χ of G , again let χ^* denote (some) irreducible constituent of largest degree of χ . It suffices to prove the statement in the case $|\chi_i| > 1$, and thus $\chi_i^* \neq 1_G$. By the results of [FG] and [LaSe], the degree of any non-trivial irreducible character of G is at least $k(G)^{1/6}$; in particular,

$$\chi_i^*(1) \geq k(G)^{1/6} \geq m_i^{1/6},$$

if m_i denotes the number of distinct irreducible constituents of χ_i . It follows that

$$|\chi_i| \leq m_i \chi_i^*(1)^2 \leq \chi_i^*(1)^8,$$

and so $\chi_i^*(1) \geq |\chi_i|^{1/8}$. Now applying [LaT, Theorem 8.5] and taking $c = 8\delta$ (with δ the constant in [LaT, Theorem 8.5]), we have that $\prod_{i=1}^m \chi_i^*(1) \geq |G|^\delta$, and so $\chi_1^* \dots \chi_N^*$ contains $\text{Irr}(G)$. $\square$

5. PROOFS OF THEOREMS 6 AND 7

Proof of Theorem 6. First we prove the theorem when $G = \text{PSL}_2(q)$. Let χ_1 and χ_2 denote any irreducible characters of G . Then $|\chi_i(g)| \leq q^{1/2}$ for all $g \in G$, $|\chi_i(g)| \leq 2$ for all non-trivial semisimple $1 \neq g \in G$, and the total number of non-semisimple elements of G is less than q^2 . Therefore

$$\sum_{g \neq 1} |\chi_1^6(g) \chi_2(g)| \leq \chi_2(1) q^5 + 64 q^3.$$

If χ_1 is non-trivial, it has degree at least $\frac{q-1}{2}$, so for q sufficiently large,

$$\sum_{g \in G} \chi_1^6(g) \chi_2(g) \neq 0,$$

and applying this to any non-trivial constituent χ_1 or χ_2 of χ , we get the desired result.

As before, we use the notation PSL^ϵ to denote PSL when $\epsilon = +$ and PSU when $\epsilon = -$. Next we consider the case $G = \text{PSL}_3^\epsilon(q)$. Again, the generic character table has been computed explicitly [SF]. There are $O(q)$ characters of degree $\leq q^2 + q + 1$, and all other characters have degree at least $q^3/4$ (when q is not too small). Therefore, the sum of the squares of the degrees of irreducible characters of G of degree $< q^3/4$ is smaller than $|G|^{11/12}$ for large values of q . We therefore assume χ has an irreducible constituent χ_1 of degree $\geq q^3/4$. We have $|\chi_i(g)| = O(q)$ for all irreducible characters and all non-trivial elements, so

$$\sum_{g \neq 1} |\chi_1^6(g) \bar{\theta}(g)| = O(q^6 |G| \theta(1)) = O(q^{14} \theta(1)) = o(\chi_1(1)^6 \theta(1)).$$

Therefore, if $\chi_1(1) \geq q^3/4$ and q is sufficiently large, any $\theta \in \text{Irr}(G)$ is a constituent of χ_1^6 .

We may therefore assume $G = \text{PSL}_n^\epsilon(q)$ with $n \geq 4$. We claim that for q sufficiently large in terms of n , $|\chi| \geq |G|^{11/12}$ implies that χ has an irreducible constituent χ_1 with

$$(5.1) \quad \chi_1(1) \geq q^{\frac{44n^2}{131}}.$$

Indeed, the number of characters of G which do not satisfy this inequality is less than $k(G) < 27.2q^{n-1}$ and $|G| < q^{n^2-1}$, so it suffices to note that

$$\frac{88n^2}{131} + (n-1) < \frac{11(n^2-1)}{12}$$

for $n \geq 4$.

It suffices to prove that the Steinberg character St is a constituent of χ_1^3 . Indeed, if $\epsilon = +$ or if $\epsilon = -$ but $2|n$, then every irreducible character of G appears in St^2 [HSTZ, Theorem 1.2], and so it follows that $|\chi_1^6| = |G|$. Consider the case $\epsilon = -$ and $2 \nmid n$. Then by [HSTZ, Theorem 1.2], St^2 contains all but one irreducible character α of $\text{PSU}_n(q)$, which has smallest degree $(q^n - q)/(q + 1)$ among all nontrivial irreducible characters of G . Let $\theta \in \text{Irr}(G)$. By Proposition 2.2(i) and (5.1),

$$|\theta\bar{\chi}_1| \geq |\chi_1| > |\alpha|,$$

and so $\theta\bar{\chi}_1 \neq m\alpha$ for any $m \in \mathbb{N}$ by (5.1). Hence $\theta\bar{\chi}_1$ contains some irreducible character $\beta \neq \alpha$, whence $\langle \text{St}^2, \beta \rangle > 0$ and so

$$\langle \chi_1^7, \theta \rangle = \langle \chi_1^6, \theta\bar{\chi}_1 \rangle \geq \langle \text{St}^2, \beta \rangle > 0,$$

and thus χ_1^7 contains $\text{Irr}(G)$, as desired.

The rest of the proof is to show that if q is sufficiently large in terms of n , (5.1) implies that the Steinberg character St is a constituent of χ_1^3 , whence we are done as explained above. We need to show for large enough q that

$$(5.2) \quad \sum_{g \neq 1} \frac{|\text{St}(g)\chi_1(g)^3|}{|\text{St}(1)\chi_1(1)^3|} < 1.$$

As St vanishes at all non-semisimple elements, we need only consider semisimple elements g in (5.2). Let X_s denote the set of conjugacy classes of semisimple elements in G of support s , i.e., for which the largest dimension of any $\mathbb{F}_q$ -eigenspace is $n - s$. We can rewrite (5.2) as

$$(5.3) \quad \sum_{s=1}^{n-1} \sum_{C \in X_s} |C| \frac{|\text{St}(C)|}{|\text{St}(1)|} \left(\frac{|\chi_1(C)|}{\chi_1(1)} \right)^3 < 1.$$

We consider the factors of the summand in the left hand side of (5.3), one by one. If $C \in X_s$ has an element g represented by a matrix M with eigenvalue multiplicities $m_1, \dots, m_k$, then the centralizer of M in $\text{GL}_n^\epsilon(q)$ is the group of $\mathbb{F}_q$ points (possibly twisted) of a connected algebraic group of dimension $\sum_i m_i^2$, so its order is $(1 + o(1))q^{\sum_i m_i^2}$. Therefore, C , which is contained in the $\text{PSL}_n^\epsilon(q)$ -conjugacy class of g , has cardinality

$$O(\gcd(n, q - \epsilon)q^{n^2 - \sum_i m_i^2}) = O(1)q^{n^2 - \sum_i m_i^2},$$

where the implicit constant $O(1)$ depends only on n .

The absolute value of $\text{St}(C) = \text{St}(g)$ is the order of a p -Sylow subgroup of the centralizer of M , which is $q^{\sum_i \binom{m_i}{2}}$, and $\text{St}(1) = q^{\binom{n}{2}}$. Therefore,

$$\frac{|\text{St}(C)|}{|\text{St}(1)|} = q^{-\frac{n^2 - \sum_i m_i^2}{2}},$$

and

$$|C| \frac{|\text{St}(C)|}{|\text{St}(1)|} = O(1)q^{\frac{n^2 - \sum_i m_i^2}{2}}.$$

As some m_i equals $n - s$, we have

$$\sum_i m_i^2 \geq (n - s)^2 + s \cdot 1^2,$$

so

$$|C| \frac{|\text{St}(C)|}{|\text{St}(1)|} = O(1)q^{\frac{2sn-s^2-s}{2}}.$$

By [TT, Theorem 1.13] and [LiST1, Theorem 3.1], for q large enough compared to n we have

$$\frac{|\chi_1(g)|}{\chi_1(1)} = O(1)\chi_1(1)^{-s/n} = O(1)q^{-\frac{44sn}{131}},$$

where the implicit constant $O(1)$ depends only on n . Thus, the summand as a whole is $O(1)q^{-\frac{sn}{131} - \frac{s^2+s}{2}}$.

Finally, we claim that $|X_s| = O(1)q^s$, where $O(1)$ again depends only on n . Since each conjugacy class of $\text{PGL}_n^\epsilon(q)$ decomposes into at most $\gcd(n, q - \epsilon)$ conjugacy classes of G , it suffices to prove the same thing for $\text{PGL}_n^\epsilon(q)$ conjugacy classes containing semisimple elements of $\text{PSL}_n^\epsilon(q)$ or, indeed, $\text{GL}_n^\epsilon(\mathbb{F}_q)$ conjugacy classes containing semisimple elements of $\text{SL}_n^\epsilon(\mathbb{F}_q)$. Since any such element has connected centralizer in $\text{GL}_n(\mathbb{F}_q)$, any such class is uniquely determined by the spectrum of its representatives or, equivalently, by their common characteristic polynomial.

Let $Y_{s,i}$ denote the set of conjugacy classes in $\text{GL}_n^\epsilon(\mathbb{F}_q)$ of semisimple elements M in $\text{SL}_n^\epsilon(\mathbb{F}_q)$ which have exactly i eigenvalues of multiplicity $n - s$ (and all other eigenvalues of lower multiplicity), where $1 \leq i \leq t := \lfloor n/(n - s) \rfloor$. Let $P(x)$ be the monic polynomial of degree i whose roots are the i eigenvalues of M of multiplicity $n - s$, and $Q(x)$ be the monic polynomial of degree $n - i(n - s)$, such that $P(x)^{n-s}Q(x)$ is the characteristic polynomial of M . Then

$$P(0)^{n-s}Q(0) = (-1)^n \det(M) = (-1)^n.$$

So the constant term of $P(x)$ is determined up to at most $i \leq n$ possibilities by the constant term of $Q(x)$, and so there are at most nq^{i-1} possibilities for $P(x)$ for any fixed (constant term of) $Q(x)$. Since there are at most $q^{n-i(n-s)}$ possibilities for $Q(x)$, we see that the total number of possibilities for $P(x)^{n-s}Q(x)$, which, as explained above, gives an upper bound for $|Y_{s,i}|$, is at most

$$nq^{i-1}q^{n-i(n-s)} = nq^{s+(1-i)(n-s-1)} = O(1)q^s.$$

Since q is large compared to n , it follows that $|X_s| = \sum_{i=1}^t |Y_{s,i}| = O(1)q^s$, as claimed.

We conclude that the contribution of each $1 \leq s \leq n - 1$ to the left hand side of (5.3) is $O(1)q^{-sn/131}$. Hence the left hand side of (5.3) is $O(1)q^{-n/131}$, where $O(1)$ depends only on n . Taking q large enough compared to n , the statement now follows. $\square$

Proof of Theorem 7. First let G be any of the 26 sporadic simple groups. Then $k(G) \leq 194$, whence the statement follows from Lemma 2.8(ii) by taking $C \geq 37442$. (Note by Theorem 2.6(i) that if $|\chi^k| = |G|$ then $|\chi^l| = |G|$ for all $l \geq k$.)

In the remaining cases, let φ denote the irreducible constituent of χ of largest degree.

Suppose G is of Lie type. By the results of [FG] and [LaSe] we have $\varphi(1) \geq k(G)^{1/6}$. Hence $|G|^\delta \leq |\chi| \leq \varphi(1)^8$, and so $\varphi(1) \geq |G|^{\delta/8}$. If G has rank at most 8, then (the proof of) [LiST1, Theorem 2] shows that $|\varphi^N| = |G|$ whenever $N \geq 8^2 \cdot 489$, so we are done in this case by taking $C \geq 31296$. Assume now that G has rank at least 9. Then [LaT, Theorem 8.3] shows that there is an absolute constant $\gamma > 0$ such that whenever $N \geq 8\gamma/\delta$ we have $|\varphi^N| = |G|$. Hence we are done in this case by taking $C \geq 8\gamma$.

Assume now that $G = A_n$ with $n \geq 5$. By [LiSh2, Corollary 2.7], there is some integer $n_1 \geq 20$ such that, when $n > n_1$ for any $1_G \neq \psi \in \text{Irr}(G)$, there exists at most $\psi(1)^2$ irreducible characters of G of degree at most $\psi(1)$. Taking $C \geq k(A_{n_1})^2$ and applying Lemma 2.8(ii), we may therefore assume that $n > n_1$. In this case, χ has at most $\varphi(1)$ distinct irreducible constituents. Also note that

$$\log |G| > \log((n/e)^n/e) = n \log n - (n+1) \geq n \log(n)/2$$

as $n \geq 21$. It follows that

$$e^{n \log(n)\delta/2} \leq |G|^\delta \leq |\chi| \leq \varphi(1)^3,$$

and so $\log(\varphi(1)) > n \log(n)\delta/6$. By [LiST2, Theorem 2], there exists an integer $n_2 \geq 5$ such that $|\varphi^N| = |G|$ whenever $N \geq 30C_1^2/\delta$ and $n \geq n_2$, where C_1 is the constant in [LiST2, (4.2)] (whose existence is established in [Se1, Theorem 1.4]). Taking $C \geq k(A_{n_2})^2$ and applying Lemma 2.8(ii), we may assume $n \geq n_2$, in which case we are done by taking $C \geq 30C_1^2$. $\square$

Remark 5.1. Theorem 7 does not hold for *abelian* finite simple groups (even if we replace C/δ by any $C = C(\delta)$), because sums of subsets in $\mathbb{F}_p$ can have very slow growth. Indeed, given any $0 < \delta < 1$ and any $C > 0$, we can find a prime number p such that $p^{1-\delta} > 2(C+1)$. Since $p^\delta > 1$, $[p^\delta, 2p^\delta]$ contains an integer k . For such p and k , we fix a faithful linear character λ of the cyclic group G of order p and take $\chi = \sum_{i=1}^k \lambda^i$. Then $|\chi| = k \geq |G|^\delta$, whereas for $N = \lceil C \rceil$ we have $C \leq N < C+1$, $\chi^N \equiv \sum_{i=1}^{Nk} \lambda^i$, and so $|\chi^N| = kN < 2p^\delta(C+1) < |G|$.

6. SEMISIMPLE COMPACT LIE GROUPS

In this section, we briefly consider the situation when G is a compact (connected) semisimple Lie group instead of a finite simple group of Lie type. For χ a character of G , the definition of $|\chi|$ works as before. We have the following analogue of Theorem 2.6.

Proposition 6.1. *Let G be a semisimple compact Lie group, $n \geq 2$, and $\chi_1, \chi_2, \dots, \chi_n$ non-trivial characters. Then $|\chi_1 \chi_2 \cdots \chi_n| > \max(|\chi_1|, |\chi_2|, \dots, |\chi_n|)$.*

Proof. It suffices to treat the case $n = 2$ and to prove in this case that if χ_1 is non-trivial, then $|\chi_1 \chi_2| > |\chi_2|$. It is enough to treat the case that χ_1 is irreducible. Let λ_1 denote the highest weight of χ_1 . Let $\{\varphi_1, \dots, \varphi_k\}$ denote the irreducible constituents of χ_2 , and let μ_i denote the highest weight of φ_i .

Then $\lambda_1 + \mu_i$ is the highest weight in $\chi_1 \varphi_i$, so there is an irreducible constituent ψ_i of $\chi_1 \chi_2$ with highest weight $\lambda_1 + \mu_i$. By the Weyl dimension formula,

$$(6.1) \quad |\chi_1 \chi_2| \geq \sum_{i=1}^k |\psi_i|^2 = \sum_{i=1}^k \prod_{\alpha > 0} \frac{(\delta + \lambda_1 + \mu_i, \alpha)^2}{(\delta, \alpha)^2} = \sum_{i=1}^k \prod_{\alpha > 0} \frac{((\delta + \mu_i, \alpha) + (\lambda_1, \alpha))^2}{(\delta, \alpha)^2},$$

where δ denotes half the sum of the positive roots. As λ_1 is a non-zero dominant weight, we have (λ_1, α) non-negative for all positive roots α and strictly positive for at least one of them. Therefore, the right hand side of (6.1) is strictly greater than

$$\sum_{i=1}^k \prod_{\alpha > 0} \frac{(\delta + \mu_i, \alpha)^2}{(\delta, \alpha)^2} = \sum_{i=1}^k |\varphi_i|^2 = |\chi_2|.$$

$\square$

When G is of positive dimension, it no longer makes sense to compare $|\chi|$ to $|G|$, so we do not have an analogue of Theorem 3 or any of the subsequent results proved above for groups of Lie type. We can still ask about the power growth of $|\chi|$. The case of $SU(2)$ illustrates the

situation: we get uniform power growth when χ is irreducible, but there is no such growth for general characters.

Proposition 6.2. *Let $G = \mathrm{SU}(2)$.*

(i) *If χ ranges over the irreducible characters of G , then*

$$\lim_{\chi} \frac{\log |\chi^2|}{\log |\chi|} = \frac{3}{2}.$$

(ii) *If χ ranges over all characters of G , then*

$$\liminf_{\chi} \frac{\log |\chi^2|}{\log |\chi|} = 1.$$

Proof. If χ_n denotes the unique irreducible character of G of degree $n+1$, then by the Clebsch-Gordan formula ([Hu, §22, Ex. 7]),

$$\chi_n^2 = \sum_{i=0}^n \chi_{2i}.$$

Thus,

$$|\chi_n^2| = \sum_{i=0}^n (2i+1)^2 = \binom{2n+3}{3},$$

while $|\chi_n| = (n+1)^2$. This gives (i).

Again by Clebsch-Gordan, χ_n^4 is a linear combination of $\chi_0, \chi_2, \chi_4, \dots, \chi_{4n}$ with all coefficients positive, so

$$|\chi_n^4| = \binom{4n+3}{3}.$$

Therefore, as χ ranges over $\{\chi_n^2, | n \in \mathbb{N}\}$, the limit of $\frac{|\chi^2|}{|\chi|}$ is 8, and the limit of $\frac{\log |\chi^2|}{\log |\chi|}$ is 1. $\square$

Next, we prove Theorem 9 establishing uniform power growth for irreducible characters of any fixed compact semisimple Lie group.

Proof of Theorem 9. Let $\varpi_1, \dots, \varpi_r$ denote the fundamental weights of G . Let χ_λ be the irreducible character with highest weight $\lambda = a_1\varpi_1 + \dots + a_r\varpi_r$. We define

$$\langle \lambda, \alpha \rangle := \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}.$$

By [Hu, §22, Ex. 1], $\lambda - k\alpha_i$ is a weight of χ_λ for $0 \leq k \leq \langle \lambda, \alpha_i \rangle = a_i$.

We claim that $\delta + 2\lambda - k\alpha_i$ is dominant for $0 \leq k \leq a_i$. It suffices to check the non-negativity of $\langle \delta + 2\lambda - k\alpha_i, \alpha_j \rangle$ for $1 \leq j \leq r$. For $j \neq i$, we have

$$\langle \delta + 2\lambda - k\alpha_i, \alpha_j \rangle = 1 + 2a_j - \langle \alpha_i, \alpha_j \rangle \geq 1,$$

and for $j = i$, we have

$$\langle \delta + 2\lambda - k\alpha_i, \alpha_i \rangle = 1 + 2a_i - 2k \geq 1.$$

Therefore, by a theorem of Brauer [Hu, §24, Ex. 9], for each k in this range, χ_λ^2 contains the irreducible character with highest weight $2\lambda - k\alpha_i$.

By the Weyl dimension formula [Hu, §24.3],

$$\chi_\lambda(1) = \prod_{\alpha > 0} \frac{(\lambda + \delta, \alpha)}{(\delta, \alpha)},$$

where the product is taken over the set Φ^+ of positive roots α . Now, for $1 \leq i \leq r$, $(\varpi_i, \alpha) \geq 0$, so regarded as a function in λ , $\chi_\lambda(1)$ is a product of affine linear functions in λ which take positive values in the cone of dominant weights. Thus, if $a_i \geq 1$ and $0 \leq k \leq a_i$, then

$$(6.2) \quad \chi_{2\lambda - k\alpha_i}(1) \geq (1 - k/a_i)^{|\Phi^+|} \chi_{2\lambda}(1) \geq (1 - k/a_i)^{|\Phi^+|} \chi_\lambda(1).$$

We fix i such that $a_i = \max(a_1, \dots, a_r)$. Thus,

$$|\chi_\lambda| = \chi_\lambda(1)^2 = O(a_i^{2|\Phi^+|}) = O(a_i^{|\Phi|}).$$

On the other hand, by (6.2),

$$|\chi_\lambda^2| \geq \sum_{k=0}^{a_i} |\chi_{2\lambda - k\alpha_i}| \geq |\chi_\lambda| \sum_{k=0}^{a_i} (1 - k/a_i)^{|\Phi|}.$$

As $a_i \rightarrow \infty$, the sum $\sum_k (1 - k/a_i)^{|\Phi|}$ can be bounded below by a positive constant multiple of a_i and therefore by a positive constant multiple of $|\chi_\lambda|^{1/|\Phi|}$. If $\epsilon < 1/|\Phi|$, therefore, $|\chi_\lambda^2| \geq |\chi_\lambda|^{1+\epsilon}$ with finitely many exceptions λ . The theorem is trivial for $\lambda = 0$, and for each $\lambda \neq 0$, it holds when $\epsilon > 0$ is small enough by Proposition 6.1. $\square$

Note that, unlike Theorem 1, Theorem 9 does not guarantee that the power growth is uniform in G : the constant $\epsilon = \epsilon(G)$ depends on G . This is unavoidable, as the following proposition shows.

Proposition 6.3. *There exists a sequence $\chi_2, \chi_3, \dots$ of non-trivial irreducible characters of the Lie groups $\mathrm{SU}(2), \mathrm{SU}(3), \dots$ respectively, such that*

$$\liminf_n \frac{\log |\chi_n^2|}{\log |\chi_n|} = 1.$$

Proof. We choose for each $n \geq 2$ a positive integer k_n and define χ_n to be the character of the irreducible representation of $\mathrm{SU}(n)$ with highest weight $\lambda_n := k_n \delta_n$, where δ_n is half the sum of the set Φ_n^+ of positive roots of $\mathrm{SU}(n)$. By the Weyl dimension formula,

$$\chi_n(1) = \prod_{\alpha \in \Phi_n^+} \frac{(\lambda_n + \delta_n, \alpha)}{(\delta_n, \alpha)} = (k_n + 1)^{|\Phi_n^+|} = (k_n + 1)^{n(n-1)/2}.$$

On the other hand, for each fixed n , the weights of χ_n are contained in the convex hull of $\{w(\lambda_n) \mid w \in \mathcal{S}_n\}$. This can be expressed as $k_n X_n$, where X_n denotes the convex hull of $\{w(\delta_n) \mid w \in \mathcal{S}_n\}$. Thus, the weights of χ_n^2 are contained in $2k_n X_n$. Since the highest weight of every irreducible constituent of χ_n^2 is a lattice point in the fixed polytope $2X_n \subset \mathbb{R}^{n-1}$, scaled by k_n , the number of such highest weights is $O(k_n^{n-1})$, where the implicit constant depends only on n .

On the other hand, by the Weyl dimension formula, the degree of each irreducible factor of χ_n^2 is $O(k_n^{n(n-1)/2})$. Therefore, assuming that each k_n is sufficiently large, we can guarantee

$$\frac{\log |\chi_n^2|}{\log |\chi_n|} < \frac{n+1}{n},$$

which implies the proposition. $\square$

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