NONSOLVABLE GROUPS HAVE A LARGE PROPORTION OF VANISHING ELEMENTS

ALEXANDER MORETÓ AND PHAM HUU TIEP

Abstract. We prove that if G is a nonsolvable group, then the proportion of vanishing elements of G is at least $1067/1260$ (and this lower bound is optimal). This confirms a conjecture of Dolfi, Pacifici, and Sanus [7].

1. Introduction

Zeros of characters of finite groups have been a subject of considerable interest in the last couple of decades. We refer the reader to [7] for an exposition of the research in this area. Following [12], we say that if G if a finite group an element $g \in G$ is **nonvanishing** if $\chi(g) \neq 0$ for every irreducible character $\chi \in \mathrm{Irr}(G)$. Otherwise, we say that g is **vanishing**. Recently, it has been observed that nonsolvable groups tend to have many zeros in the character table. For instance, M. Larsen and A. Miller proved in [14] that the probability that a random entry in the character table of a simple group of Lie type is 0 goes to 1 when the rank of the group goes to infinity.

On the other hand, it was conjectured by S. Dolfi, E. Pacifici and L. Sanus in [7] that the proportion of vanishing elements in a nonsolvable group is at least $1067/1260$ (note that this is the proportion of vanishing elements in A_7). Given a finite group, we write $\mathcal{P}_v(G)$ to denote the proportion of vanishing elements of G. In other words,

$$\mathcal{P}_v(G) = \frac{|\{g \in G \mid \chi(g) = 0 \text{ for some } g \in G\}|}{|G|}.$$

It has been proven in [18] that if $\mathcal{P}_v(G) \leq 2/3$ then G is solvable. Our goal in this note is to settle the conjecture of Dolfi, Pacifici and Sanus.

Theorem A. *Let G be a finite group. If $\mathcal{P}_v(G) < \mathcal{P}_v(\mathsf{A}_7) = 1067/1260$, then G is solvable.*

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In the proof of Theorem A we use the following result, that is perhaps worth pointing out.

Theorem B. *Let G be an almost simple group with socle S. Then all the elements in $G \smallsetminus S$ are vanishing.*

Our proof of Theorem B relies on the Deligne-Lusztig theory of characters of groups of Lie type by means of Theorem 2.1 below (and Theorem 2.3 of [5]). In Section 2, we prove Theorem B assuming Theorem 2.1. Next, we prove Theorem A in Section 3 and we present the proof of Theorem 2.1 in Section 4.

2. Proof of Theorem B

In this section, we will prove Theorem B assuming the following result, which is a version for 2-elements of Theorem 2.3 of [5] and which will be proved in §4. Recall that given a group G, $\mathbf{F}(G)$ is the Fitting subgroup of G, i.e., the largest normal nilpotent subgroup of G.

Theorem 2.1. *Let S be a nonabelian simple group. Suppose that $S \leq G \leq \mathrm{Aut}(S)$. Let $xS \in \mathbf{F}(G/S)$ be an element of order 2. Then there exists $\psi \in \mathrm{Irr}(S)$ such that x does not fix any G-conjugate of ψ.*

The following is an immediate consequence of Theorem 2.1. As usual, if G is a finite group, $N \trianglelefteq G$ and $\psi \in \mathrm{Irr}(N)$ we write $\mathrm{Irr}(G|\psi)$ to denote the set of irreducible characters of G that lie over ψ.

Corollary 2.2. *Let S be a nonabelian simple group. Suppose that $S \leq G \leq \mathrm{Aut}(S)$. Let $xS \in \mathbf{F}(G/S)$ be a nontrivial 2-element. Then there exists $\psi \in \mathrm{Irr}(S)$ such that x does not fix any G-conjugate of ψ. In particular, if $\chi \in \mathrm{Irr}(G|\psi)$ then $\chi(x) = 0$.*

Proof. Apply Theorem 2.1 to $(xS)^{o(xS)/2}$. The claim that $\chi(x) = 0$ follows from Clifford's theory and the formula for the induced character. □

We will use several times the following elementary fact.

Lemma 2.3. *Let G be a finite group and let $L \leq K$ be two normal subgroups of G. If $\varphi \in \mathrm{Irr}(K)$ vanishes on $K \smallsetminus L$ and $\chi \in \mathrm{Irr}(G|\varphi)$, then χ vanishes on $K \smallsetminus L$.*

Proof. If suffices to note that by Clifford's theorem (Theorem 6.2 of [11]) χ_K is a sum of conjugates of φ and use the fact that $K \smallsetminus L$ is a normal subset. □

The following is a consequence of the well-known structure of the outer automorphism group of simple groups.

Corollary 2.4. *Let S be a nonabelian simple group. Let $O = \mathrm{Out}(S)$. Then the elements in $O \smallsetminus \mathbf{F}(O)$ are vanishing*

Proof. By [9, Theorem 2.5.12], we know that $O/\mathbf{F}(G)$ is abelian. Now the result follows from the proof of Lemma 18.1 of [16], for instance. □

Next, we prove a slightly strengthened form of Theorem B.

Theorem 2.5. *Let S be a nonabelian simple group. Suppose that $S \leq G \leq \mathrm{Aut}(S)$. Let $x \in G \smallsetminus S$. Then there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi(x) = 0$. Furthermore, if $xS \in \mathbf{F}(G/S)$ then there exists $\psi \in \mathrm{Irr}(S)$ such that xS does not fix any G-conjugate of ψ.*

Proof. Write $O = G/S$. By Corollary 2.4, we may assume that $xS \in \mathbf{F}(O)$. By Corollary 2.2, we may assume that $o(xS)$ is not a 2-power. Let p be an odd prime divisor of $o(xS)$ and let yS be the p-part of xS. By Theorem 2.3 of [5], there exists $\psi \in \mathrm{Irr}(S)$ such that yS does not fix any G-conjugate of ψ. Since yS is a power of xS, we conclude that xS does not fix any G-conjugate of ψ. Hence, if $\chi \in \mathrm{Irr}(G|\psi)$ then $\chi(x) = 0$, as wanted. □

3. Proof of Theorem A

Following [4], if Ω is a set and n is a positive integer, we write $\mathsf{P}_n(\Omega)$ to denote the set of $(n+1)$-tuples of subsets of Ω that form a partition of Ω.

Lemma 3.1. *Let G be a solvable permutation group on a set Ω. Then G has a regular orbit on $\mathsf{P}_4(\Omega)$.*

Proof. This follows from Corollary 6 of [4]. □

Suppose that H is a subgroup of a finite group G and let $x \in H$. In the next results, we will need to distinguish between being vanishing as an element of H or as an element of G. We will say that x is vanishing in G or x is vanishing as an element of G if there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi(x) = 0$.

We write

$$\begin{aligned}\mathcal{L}_2 =& \{M_{12}, M_{22}, M_{24}, J_2, HS, Suz, Ru, Co_1, Co_3, B\} \\ & \cup \{\mathsf{A}_n \mid n \neq 2m^2 + m \text{ and } n \neq 2m^2 + m + 2 \text{ for any integer } m\}\end{aligned}$$

and

$$\mathcal{L}_3 = \{Suz, Co_3\} \cup \left\{ \mathsf{A}_n \,\middle|\, \begin{array}{ll} 3n+1 = m^2 r & \text{for some } r \text{ square-free and divisible} \\ & \text{by some prime } q \equiv 2 (\mathrm{mod}\ 3) \end{array} \right\}.$$

By Corollary 1 of [10], $\mathcal{L}_p$ is the list of non-abelian simple groups that do not possess an irreducible character of p-defect zero for $p \in \{2, 3\}$ and every non-abelian simple group has some irreducible character of p-defect zero for every prime $p \geq 5$.

Lemma 3.2. *Let $N = S_1 \times \cdots \times S_n$ be a minimal normal subgroup of a finite group G, with $S_i \cong S$ non-abelian simple for every i, and let $1 \neq x \in N$. Then*

(i) *If $o(x)$ is divisible by some prime $p \geq 5$, then x is a vanishing element of G.*
(ii) *If $S \notin \mathcal{L}_2 \cup \mathcal{L}_3$, then x is a vanishing element of G.*
(iii) *If $S \neq \mathsf{A}_6$ is an alternating group, then the proportion of elements in N that are vanishing elements of G is at least $1067/1260$.*

(iv) *If $S \neq M_{24}$ is one of the sporadic groups in $\mathcal{L}_2 \cup \mathcal{L}_3$ or $S = \mathsf{A}_6$, then all the nontrivial elements in N are vanishing in G.*

(v) *If $S = M_{24}$ then the proportion of elements in N that are vanishing elements of G is at least* $1067/1260$.

Proof. By Corollary 1 of [10], we know that S has some irreducible character φ of p-defect zero. Take $\psi = \varphi \times \cdots \times \varphi \in \mathrm{Irr}(N)$. If $\chi \in \mathrm{Irr}(G)$ lies over ψ, then it follows from Clifford's theorem that χ_N is a sum of G-conjugates of ψ. Since ψ^g has p-defect zero for every $g \in G$, we deduce from Theorem 8.17 of [11] that $\chi(x) = 0$. This proves (i). The second part can be proved analogously.

Now, we want to prove (iii). Let $x = (x_1, \ldots, x_n) \in N$. Note that x is a vanishing element of N if and only if x_i is a vanishing element of S_i for some i. Since the proportion of elements in S_1 that are vanishing is at least $1067/1260$ (by Theorem 1.5 of [18]), it suffices to see that if $y \in S_1$ is vanishing in S_1, then y is a vanishing element of G. Let $\theta \in \mathrm{Irr}(S_1)$ such that $\theta(y) = 0$. Since S_1 is subnormal in G, it follows from Clifford's theorem that if $\chi \in \mathrm{Irr}(G)$ lies over $\theta \times \cdots \times \theta \in \mathrm{Irr}(N)$ then χ_{S_1} is a sum of $\mathrm{Aut}(S_1)$-conjugates of θ. By the proof of Lemma 3.1 of [18], all the $\mathrm{Aut}(S_1)$-conjugates of θ vanish at y. We deduce that $\chi(y) = 0$. Therefore, y is a vanishing element of G, as desired.

Now, assume that S is one of the sporadic groups in $\mathcal{L}_2 \cup \mathcal{L}_3$ or A_6. If $S = M_{24}$, it can be checked in the Atlas [2] that all the elements except for those in classes 1A and 2A are vanishing elements. If $S \neq M_{24}$ then we can see in [2] that any nontrivial element in S is vanishing in H for any almost simple group H with socle S. Parts (iv) and (v) follow the same reasoning as in (iii). (Note that in part (v) we have a proportion of vanishing elements that is much larger than stated.) □

Now, we can complete the proof of Theorem A.

Proof of Theorem A. Let G be a minimal counterexample. Let N be a minimal normal subgroup of G. By Lemma 2.3 of [18], $\mathcal{P}_v(G/N) \leq \mathcal{P}_v(G) < \mathcal{P}_v(\mathsf{A}_7)$. By the minimality of G, G/N is solvable. We deduce that $N = S_1 \times \cdots \times S_n$, where $S_i \cong S$ for some non-abelian simple group S and every i. Furthermore, G/N is solvable. We also have that N is the unique minimal normal subgroup of G. Hence, G is isomorphic to a subgroup of $\mathrm{Aut}(N) = \mathrm{Aut}(S) \wr \mathsf{S}_n$. Write $K = G \cap \mathrm{Aut}(S)^n$, so that $N \leq K \leq \mathrm{Aut}(S)^n$ and G/K is isomorphic to a solvable permutation group on $\Omega = \{S_1, \ldots, S_n\}$.

By Lemma 3.1, G/K has a regular orbit on $\mathsf{P}_4(\Omega)$. Hence, there exists a partition $\Gamma_0, \ldots, \Gamma_4$ of Ω such that $\bigcap_{i=0}^{4} \mathrm{stab}_{G/K}(\Gamma_1) = K/K$. Since S is simple nonabelian, S has at least 5 different irreducible characters $\gamma_0, \ldots, \gamma_4$. Let $\gamma \in \mathrm{Irr}(N)$ be the character whose factor corresponding to the direct factors in Γ_i is γ_i for every i. By the choice of γ, $I_G(\gamma) \leq K$. Hence, if $\chi \in \mathrm{Irr}(G|\gamma)$ then χ vanishes on $G \smallsetminus K$.

Put $F/N = \mathbf{F}(K/N)$. By Corollary 2.4 and Lemma 2.3, the elements in $K \smallsetminus F$ are also vanishing.

Next, using arguments from [5], we will see that the elements in $F \smallsetminus N$ are vanishing. Let $x \in F \smallsetminus N$. Since $x \in K$, x normalizes S_i for every $i = 1, \ldots, n$. In fact, K normalizes S_i for every i. By way of contradiction, suppose that x is nonvanishing. We will show first that $x \in S_i\mathbf{C}_G(S_i)$ for all $i = 1, \ldots, n$. Without loss of generality, we will show this for $i = 1$. Let $\theta \in \mathrm{Irr}(S)$, and let $\psi = \theta_1 \times \cdots \times \theta_n$, where θ_i is identified with θ for every i. By Lemma 2.3 of [12], x fixes ψ^g for some $g \in G$. Write

$$S_i^{g^{-1}} = S_{\sigma(i)} = S_1^{g_{\sigma(i)}}.$$

Hence,

$$S_1^{g_{\sigma(i)}g} = S_i.$$

Then

$$\psi^g = \theta_1^{g_{\sigma(1)}g} \times \cdots \times \theta_1^{g_{\sigma(n)}g}.$$

Since x fixes ψ^g, we have that it fixes each of the factors of ψ^g. Hence,

$$\theta_1^{g_{\sigma(1)}gx} = \theta_1^{g_{\sigma(1)}g}$$

and therefore $\theta_1^{uxu^{-1}} = \theta_1$, where $u = g_{\sigma(1)g} \in \mathbf{N}_G(S_1)$. In other words, x fixes some $\mathbf{N}_G(S_1)/\mathbf{C}_G(S_1)$-conjugate of θ_1 for every $\theta_1 \in \mathrm{Irr}(S_1\mathbf{C}_G(S_1)/\mathbf{C}_G(S_1))$. Recall that xN lies in a nilpotent normal subgroup of $K/N \leq \mathbf{N}_G(S_1)/N$. Hence, $xS_1\mathbf{C}_G(S_1)$ lies in a nilpotent normal subgroup of $\mathbf{N}_G(S_1)/S_1\mathbf{C}_G(S_1)$. Applying Theorem 2.5 with $\mathbf{N}_G(S_1)/\mathbf{C}_G(S_1)$ in the place of G and $S_1\mathbf{C}_G(S_1)/\mathbf{C}_G(S_1)$ in the place of S, we deduce that $x \in S_1\mathbf{C}_G(S_1)$, as wanted.

Now, for all $i = 1, \ldots, n$, write $x = s_ic_i$ with $s_i \in S_i$ and $c_i \in \mathbf{C}_G(S_i)$. On the other hand, we can certainly write $x = s_1s_2\cdots s_ny$ for some $y \in G$, and we work to show that $y = 1$. We get $s_1c_1 = s_1(s_2\cdots s_n)y$, whence $y = (s_2\cdots s_n)^{-1}c_1 \in \mathbf{C}_G(S_1)$. Analogously, we see that $y \in \mathbf{C}_G(S_i)$ for every i and therefore $y \in \mathbf{C}_G(N) = 1$. We conclude that $x = s_1\cdots s_n \in N$, as desired.

We have thus seen that all the elements in $G \smallsetminus N$ are vanishing. By Lemma 3.2, the proportion of elements of N that are vanishing in G is at least $1067/1260$. If follows that $\mathcal{P}_v(G) \geq 1067/1260$, which is a contradiction. This completes the proof. □

In [12], M. Isaacs, G. Navarro and T. Wolf conjectured that in a solvable group G every element in $G \smallsetminus \mathbf{F}(G)$ is vanishing. This conjecture remains open. In [5] it was proved that for any finite group G all elements of order coprime to 6 in $G \smallsetminus \mathbf{F}(G)$ are vanishing, and the coprimeness hypothesis is necessary. As mentioned in [5], it is tempting to conjecture that all the elements in $G \smallsetminus \mathbf{F}^*(G)$ are vanishing but $2^{11} \rtimes M_{24}$ is a counterexample. Our proof of Theorem A suggests that, perhaps, all the elements outside the generalized Fitting subgroup are vanishing for groups with trivial Fitting subgroup. Note that Theorem B is a particular case of this conjecture.

4. Almost simple groups

In this section we prove Theorem 2.1, which we restate.

Theorem 4.1. *Let S be a nonabelian simple group. Suppose that $S \leq G \leq \mathrm{Aut}(S)$. Let $xS \in \mathbf{F}(G/S)$ be an element of order 2. Then there exists $\psi \in \mathrm{Irr}(S)$ such that x does not fix any G-conjugate of ψ. In particular, if $\chi \in \mathrm{Irr}(G|\psi)$ then $\chi(x) = 0$.*

Note that the assumption $xS \in \mathbf{F}(G/S)$ is essential; otherwise $G = \mathrm{Aut}(\Omega_8^+(2))$ would be a counterexample.

Proof. We will assume that $x \notin S$ and aim to produce a G-orbit $\mathcal{O}$ on $\mathrm{Irr}(S)$ such that x **moves** every character in $\mathcal{O}$. Consider the subgroup $J := \langle xS \rangle$ in $G/S \leq \mathrm{Out}(S)$.

(i) First we show that the theorem holds in the case $J \lhd G/S$. Indeed, [8, Theorem C] shows that, in the action of J on the conjugacy classes of S, there is some orbit of length > 1. Since J is cyclic, this action of J is permutationally isomorphic to its action on $\mathrm{Irr}(S)$. In particular, J has an orbit $\mathcal{O}_1$ of length > 1 on $\mathrm{Irr}(S)$. Now let $\mathcal{O}$ be the G-orbit on $\mathrm{Irr}(S)$ that contains $\mathcal{O}_1$. Since $J \lhd A$, J acts semi-transitively on $\mathcal{O}$, i.e. all J-orbits on $\mathcal{O}$ have the same length. Hence we are done as $|\mathcal{O}_1| > 1$.

(ii) By the result of (i), we are done if G/S or $\mathrm{Out}(S)$ is abelian; in particular, if S is an alternating group or a sporadic simple group. The rest of the proof is to deal with simple groups of Lie type, for which the structure of $\mathrm{Out}(S)$ is described for instance in [9, Theorem 2.5.12]. Since $xS \in \mathbf{F}(G/S)$ has order 2, $\mathbf{O}_2(G/S) \neq 1$. From now on we may assume that G/S is non-abelian and that $\mathbf{O}_2(G/S)$ is non-cyclic (because otherwise $J \ \mathsf{char}\ \mathbf{O}_2(G/S) \lhd G/S$, and we are done again); in particular, the Sylow 2-subgroups of $\mathrm{Out}(S)$ are non-cyclic.

Thus we are left with the cases, where $S = \mathrm{PSL}_n^\epsilon(q)$ with $n \geq 3$ (and $2|n$ if $\epsilon = -$, since $\mathrm{Out}(\mathrm{PSU}_n(q))$ have cyclic Sylow 2-subgroups when $2 \nmid n$), $P\Omega_{2n}^\epsilon(q)$ with odd q and $n \geq 4$, or $E_6(q)$, or $P\Omega_8^+(q)$ and $2|q$. Here, $q = p^f$, and $\epsilon = +$ in the untwisted case and $\epsilon = -$ in the twisted case.

In the fourth case, $\mathrm{Out}(S) = \Phi \times \Gamma$ with $\Phi \cong \mathbf{C}_f$ and $\Gamma \cong \mathsf{S}_3$. Then $J \not\leq \Phi$ as otherwise it is central in G/S. Hence we can write $xS = ab$ with $a \in \Phi$ and $b \in \Gamma$ with $o(b) = 2$, and $\mathbf{O}_2\big(((G/S)\Phi)/\Phi\big) \neq 1$. As $\mathbf{O}_2(\mathsf{S}_3) = 1$, we have $(G/S)\Phi \neq \Gamma$, whence $G/S \leq \Phi\langle b \rangle$ and J is again central in G/S. We will now deal with the first three cases.

(iii) We can find a simple, simply connected, algebraic group $\mathcal{G}$ in characteristic p and a Frobenius endomorphism F on $\mathcal{G}$ such that $S = L/\mathbf{Z}(L)$ for $L := \mathcal{G}^F$. We will also consider the pair $(\mathcal{G}^*, F^*)$ dual to $(\mathcal{G}, F)$ and the dual group $H := (\mathcal{G}^*)^{F^*}$, cf. [1]. We will use the Deligne-Lusztig theory (cf. [13], [1], [3]).

Here we consider the case where x induces an inner-diagonal automorphism of S, i.e. $xS \in I := \mathrm{Outdiag}(S)$ in the notation of [9]. If in addition I is cyclic, then $J \ \mathsf{char}\ I \lhd \mathrm{Out}(S)$, and so we are done again. We may therefore assume that I is elementary abelian of order 4, and so $S \cong P\Omega_{2n}^+(q)$ with $2|n \geq 4$ and $2 \nmid q$. Suppose

$n = 4$. Then, as shown in the proof of [19, Proposition 5.11], $\mathrm{Aut}(S)$ has an orbit $\mathcal{O}$ of length 4 on $\mathrm{Irr}(S)$, on which I acts regularly, and hence we are done. Next assume that $n \geq 6$, and choose $\epsilon = \pm 1$ such that $q^{n/2} \equiv \epsilon (\bmod 4)$. According to [9, Table 4.5.1], $H = (\mathrm{PCO}^\circ)^+_{2n}(q)$ has a unique conjugacy class that contains an involution t (denoted by $t_{n/2}$ or $t'_{n/2}$ therein) with the properties that $t \in [H, H]$ and $|\mathbf{C}_H(t)| = 4|\mathrm{SO}^\epsilon_n(q) \times \mathrm{SO}^\epsilon_n(q)|$. Since $|\mathbf{Z}(L)| = 4 = |H/[H, H]|$, [20, Lemma 4.4] shows that the rational series corresponding to t contains four irreducible characters of L of degree

$$D := [\mathrm{SO}^+_{2n}(q) : \mathrm{SO}^\epsilon_n(q) \times \mathrm{SO}^\epsilon_n(q)]_{p'}/4.$$

If ψ denotes any of them, then ψ is trivial on $\mathbf{Z}(L)$, so can be viewed as a character of S. Moreover, this set $\mathcal{O}$ of four characters is $\mathrm{Aut}(S)$-invariant, by the uniqueness of the conjugacy class t^H. Note that $S = [H, H]$. Let $\chi \in \mathrm{Irr}(H)$ lie above ψ. It suffices to show that $\chi(1) = 4D$. (Indeed, since $|H/S| = 4$ and $x \in H$, in this case $H/S \cong \mathbf{C}_2^2$ acts regularly on the four irreducible constituents $\psi_1 = \psi, \psi_2, \psi_3, \psi_4$ of χ_S. It follows that $\{\psi_1, \ldots, \psi_4\}$ is precisely $\mathcal{O}$, the set $\mathcal{O}$ is a single $\mathrm{Aut}(S)$-orbit, and x does not fix any ψ_i.) Since $|H/S| = 4$, we have $\chi(1) = \kappa D$ with $\kappa \in \{1, 2, 4\}$; furthermore, as $H = \mathcal{G}^{*F^*}$ and $\mathcal{G}^*$ has trivial center, $\chi = \chi_s$ for some semisimple element $s \in L \cong H^*$ with $|\mathbf{C}_L(s)| = (4/\kappa)|\mathrm{SO}^\epsilon_n(q)|^2$. Furthermore, $\mathbf{C}_L(s) = \mathbf{C}_{\mathcal{G}}(s)^F$, where $\mathbf{C}_{\mathcal{G}}(s)$ is a connected reductive algebraic group (see e.g. Theorems 3.5.4 and 3.5.6 of [1]), whose simple factors are of type A or D, and in fact $|\mathbf{C}_L(s)|$ is a product of factors of the form $|\mathrm{SL}^\pm_a(q^b)|$ or $|\mathrm{SO}^\pm_{2c}(q)|$. The number e of these factors is at most the number e' of irreducible summands of the image of $\mathbf{C}_L(s)$ in $\Omega^+_{2n}(q)$, in its action on the natural module $\mathbb{F}_q^{2n}$ for $\mathrm{SO}^+_{2n}(q)$. When $\epsilon = -$, using the divisibility of $|\mathbf{C}_L(s)|$ by $(q^{n/2} + 1)^2$, one sees that $e \leq e' \leq 2$. When $\epsilon = +$, using the divisibility of $|\mathbf{C}_L(s)|$ by $(q^{n-2} - 1)^2(q^{n/2} - 1)^2$, one can also check that $e \leq 2$. Now one readily checks that $\kappa \neq 2, 4$, and thus $\chi(1) = 4D$ as desired.

Suppose now that, modulo the inner-diagonal and field automorphisms of S, x induces a graph automorphism of order 2, and moreover $S = P\Omega^+_{2n}(q)$. Then [15, Theorem 2.5] explicitly describes two unipotent characters of S such that they are permuted by a graph automorphisms of order 2 of S, but each of them is fixed by every diagonal or field automorphism of S. Now we can just choose $\mathcal{O}$ to be any such pair.

(iv) In the remaining cases, x induces an automorphism σ of order 2 which is outside of $\mathrm{Inndiag}(S)$. We aim to find a semisimple element $s \in H$ such that $\mathbf{C}_{\mathcal{G}^*}(s)$ is connected, $s \in [H, H]$, but the conjugacy class s^H of s in H is not σ-invariant. The first two conditions imply that the semisimple character $\chi = \chi_s$ of L is irreducible and trivial at $\mathbf{Z}(L)$, hence can be viewed as an irreducible character $\chi \in \mathrm{Irr}(S)$, see e.g. [20, Lemma 4.4]. Notice that, in the cases under consideration, the inner-diagonal automorphisms of S are induced by conjugation using elements in H (when we embed S in H), and so they preserve s^H; also, we may write $H = \mathrm{Inndiag}(S)$. As a result,

H fixes χ, cf. [21, §2]. Since σ moves s^H, [21, Corollary 2.4] and the disjointness of Lusztig series imply that $\chi^\sigma \neq \chi$ and so $\chi^x \neq \chi$. As shown in the proof of [6, Lemma 2.5], in the case σ is induced by a Frobenius endomorphism σ^* of $\mathcal{G}^*$, then

$$|s| \text{ does not divide } |(\mathcal{G}^*)^{\sigma^*}| \tag{4.1}$$

implies that s^H is not σ-invariant.

Now let $\mathcal{O}$ be the G-orbit of χ. Observe that, in our cases, $\mathrm{Out}(S)/H$ is either abelian, or $\mathbf{C}_f \times \mathsf{S}_3$, where the latter case occurs only when $S = P\Omega_8^+(q)$ (and $q = p^f$). Unless we are in the latter case, $\langle xH \rangle$ is a normal subgroup of $\mathrm{Out}(S)/H$, and so $\langle x(G \cap H)\rangle \lhd G/(G \cap H)$. Recall that $G \cap H$ fixes χ. Now arguing as in (i), we see that x moves every character in $\mathcal{O}$, and so we are done. Suppose we are in the latter case. Then H still fixes every member of the G-orbit $\mathcal{O}$ of χ, and $\mathrm{Out}(S)/H = \Phi \times \Gamma \cong \mathbf{C}_f \times \mathsf{S}_3$. Now, the arguments in the last paragraph of (ii), but applied to the image of GH/H in $\mathrm{Out}(S)/H$, acting on $\mathcal{O}$, and using the assumption that JH/H moves χ, yield the result.

The rest of the proof is to construct the desired element s. This construction will follow some arguments given in [17]. In what follows, once the prime ℓ is chosen, we will fix $\alpha \in \overline{\mathbb{F}}_q^\times$ of order ℓ.

(v) Let $S = \mathrm{PSL}_n(q)$ with $n \geq 3$. Then $H = \mathrm{PGL}_n(q)$. Modulo H, we may assume that σ is induced by one of the following three maps on $GL_n(q)$:

$$X := (a_{ij}) \mapsto X^{(r)} := (a_{ij}^r), \text{ or } X \mapsto {}^t\!X^{-1}, \text{ or } X \mapsto {}^t\!\big(X^{(r)}\big)^{-1}, \tag{4.2}$$

where $r := p^{f/2}$, and $2|f$ whenever r is in discussion. We may assume $q > 2$ as otherwise $\mathrm{Out}(S)$ is abelian and we are done. Hence, by [22] there is a **primitive prime divisor** (ppd for short) ℓ of $p^{nf} - 1$, that is, a prime divisor of $p^{nf} - 1$ which does not divide $\prod_{j=1}^{nf-1}(p^j - 1)$.

Assume in addition that $2 \nmid n$ if $\sigma(X) = {}^t\!X^{-1}$. Then choose $s \in GL_n(q)$ represented by the diagonal matrix $\mathrm{diag}(\alpha, \alpha^q, \ldots, \alpha^{q^{n-1}})$ over $\overline{\mathbb{F}}_q$. Abusing the notation, we will denote the image of s in H also by s (and we will do the same in subsequent parts of the proof). Notice that $\ell \geq nf + 1$, and so $\mathbf{C}_{\mathcal{G}^*}(s)$ is connected and $s \in [H, H]$ (as $o(s)$ is coprime to $|\mathbf{Z}(\mathcal{G})|$ and $|H/[H, H]|$). It remains to show that s and s^σ are not conjugate in H. According as σ is one of the three maps described in (4.2), at least one of the eigenvalues of s^σ over $\overline{\mathbb{F}_q}$ is α^r, α^{-1}, or α^{-r}. Hence it suffices to show that there is no $\lambda \in \mathbb{F}_q^\times$ and $0 \leq j \leq n - 1$ such that

$$\lambda\alpha^{q^j} = \alpha^r, \text{ or } \alpha^{-1}, \text{ or } \alpha^{-r},$$

according as σ is one of the three maps described in (4.2). Assume the contrary. Since $o(\alpha) = \ell$ is coprime to $q - 1$, we must have $\lambda = 1$. Now, if $j = 0$, then ℓ divides $p^{f/2} \mp 1$ or 2, which is a contradiction, as ℓ is a ppd of $p^{nf} - 1$. If $j > 0$, then ℓ divides $p^{jf \mp f/2} - 1$ or $p^{jf} + 1$. In the former case, the primitivity of ℓ implies that nf divides

$(j \mp 1/2)f$, i.e. $2n$ divides $2j \mp 1$, again a contradiction. In the latter case, we have $2 \nmid n$ and $nf|2jf$, whence $n|j$, which is impossible as $0 < j < n$.

Now assume that $n \geq 4$ is even and $\sigma(X) = {}^tX^{-1}$. Since $\mathrm{Out}(\mathrm{SL}_4(2))$ is abelian, we may assume that $(n,q) \neq (4,2)$, whence there exists a ppd ℓ of $p^{(n-1)f}-1$. Next, choose $s \in \mathrm{GL}_n(q)$ represented by the matrix $\mathrm{diag}(1, \alpha, \alpha^q, \alpha^{q^2}, \ldots, \alpha^{q^{n-2}})$ over $\overline{\mathbb{F}}_q$. Then $\ell \geq (n-1)f+1$, and so $\mathbf{C}_{\mathcal{G}^*}(s)$ is connected and $s \in [H,H]$. Arguing as above, we see that the eigenvalue α^{-1} of s^σ is not among the eigenvalues of s, whence s and s^σ are not conjugate in H.

(vi) Consider the case $S = \mathrm{PSU}_n(q)$ and $n \geq 3$, whence $H = \mathrm{PGU}_n(q)$. As mentioned in (ii), we may assume $n \geq 4$ is even. Since $\mathrm{Out}(\mathrm{SU}_4(2))$ is abelian, we may assume that $(n,q) \neq (4,2)$, whence there exists a ppd ℓ of $p^{2(n-1)f}-1$. Next, choose $s \in \mathrm{GU}_n(q)$ represented by the matrix $\mathrm{diag}(1, \alpha, \alpha^{-q}, \alpha^{q^2}, \ldots, \alpha^{q^{n-2}})$ over $\overline{\mathbb{F}}_q$. Then $\ell \geq (n+3)f+1$, and so $\mathbf{C}_{\mathcal{G}^*}(s)$ is connected and $s \in [H,H]$. Arguing as above, we see that s and s^σ are not conjugate in H.

(vii) Next assume that $S = P\Omega^+_{2n}(q)$ with $n \geq 4$ and $2 \nmid q$; in particular, there exists a ppd ℓ of $p^{2(n-1)f}-1$. By the considerations in (iii), we may assume that $\sigma(X) = X^{(r)}$ with $r := p^{f/2}$ as in (4.2). Next, choose $s \in \mathrm{GO}^+_{2n}(q)$ represented by the matrix $\mathrm{diag}(1, 1, \alpha, \alpha^q, \alpha^{q^2}, \ldots, \alpha^{q^{2n-3}})$ over $\overline{\mathbb{F}}_q$. Again, $\mathbf{C}_{\mathcal{G}^*}(s)$ is connected and $s \in [H,H]$. Arguing as above, we see that the eigenvalue α^r of s^σ is not among the eigenvalues of λs for any $\lambda \in \mathbb{F}_q^\times$, whence s and s^σ are not conjugate in H.

Assume $S = P\Omega^-_{2n}(q)$ and $n \geq 4$. Since $\mathrm{Out}(S)$ is non-abelian, we must have that $2 \nmid n$ and $4|(q+1)$. Here we choose ℓ to be a ppd of $p^{2nf}-1$. Next, choose $s \in \mathrm{GO}^-_{2n}(q)$ of order ℓ. Since ℓ is odd, $\mathbf{C}_{\mathcal{G}^*}(s)$ is connected and $s \in [H,H]$. Note that, by choosing a suitable matrix realization of $\mathrm{GO}^-_{2n}(q)$ over $\mathbb{F}_{q^2}$, we may assume that $\sigma(X) = X^{(r)}$ with $r := q$, and thus σ is induced by the q^{th} Frobenius endomorphism σ^*. Now, if s^H is σ-invariant, then ℓ divides $|(\mathcal{G}^*)^{\sigma^*}| = (\mathrm{PCO}^\circ)^+_{2n}(q)$ by (4.1), contrary to the choice of ℓ.

(viii) Finally, we consider the case $S = E_6(q)$. Then σ is induced by $\sigma_q\tau$, σ_r, or $\sigma_r\tau$, where τ is a graph automorphism of $\mathcal{G}$, σ_q is the q^{th} Frobenius endomorphism (which acts trivially on L), and, if $2|f$ then σ_r is the r^{th} Frobenius endomorphism, with $r := p^{f/2}$. We choose to replace τ by $\sigma_q\tau$ to make sure that the corresponding map on $\mathcal{G}$ is a Frobenius endomorphism. Accordingly, choose $s \in H$ to be of order ℓ, where ℓ is a ppd of $p^{9f}-1$, $p^{9f}-1$, or $p^{12f}-1$. This ensures that $\mathbf{C}_{\mathcal{G}^*}(s)$ is connected and $s \in [H,H]$. Now, if s^H is σ-invariant, then ℓ divides $|(\mathcal{G}^*)^{\sigma^*}| = {}^2E_6(q)_{\mathrm{ad}}$, $E_6(r)_{\mathrm{ad}}$, or ${}^2E_6(r)_{\mathrm{ad}}$ by (4.1), contrary to the choice of ℓ. □

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Departament de Matemàtiques, Universitat de València, 46100 Burjassot, València, Spain
Email address: alexander.moreto@uv.es

Department of Mathematics, Rutgers University, Piscataway, NJ 08854, USA
Email address: tiep@math.rutgers.edu