DEGREES AND FIELDS OF VALUES OF IRREDUCIBLE CHARACTERS

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Abstract. We completely describe all the possible fields of values of irreducible characters of degree up to 3 of finite groups. The obtained result points toward a rather surprising connection between the field of values and the degree of an arbitrary irreducible character.

To the memory of Irina Suprunenko

1. Introduction

Let χ be an irreducible (complex) character of a finite group G. It is clear that each value of χ is a sum of $|G|$-th roots of unity, and thus the field of values of χ, denoted by $\mathbb{Q}(\chi)$, is an abelian extension of the field $\mathbb{Q}$ of rational numbers. It is even true that, by Brauer's theorem, χ can always be afforded by a representation with entries in the $|G|$-th cyclotomic field. If χ is linear, i.e., χ is a homomorphism from G to the multiplicative group of nonzero complex numbers $\mathbb{C}^\times$, all the values $\{\chi(g) : g \in G\}$ of χ are roots of unity, and therefore $\mathbb{Q}(\chi)$ is a full cyclotomic field. A naive question emerges: *what abelian extension of $\mathbb{Q}$ could be the field of values of an irreducible character of degree 2*? While we see several examples of degree-2 irreducible characters with fields of values $\mathbb{Q}(\sqrt{5})$ (in $SL_2(5)$ for instance) and $\mathbb{Q}(\sqrt{k})$ with $|k| \leqslant 3$ (in $C_3 \times \mathsf{S}_3, C_3 \rtimes Q_8, D_{16}, \ldots$), it seems that other quadratic fields only show up as fields of values of characters of degree at least 3.

In this paper we resolve the above question for characters of degree 2 and 3, see Section 2. In what follows, the *conductor* $c(F)$ of an abelian extension F of $\mathbb{Q}$ is the smallest $n \in \mathbb{Z}^+$ such that $F \subseteq \mathbb{Q}_n := \mathbb{Q}(e^{2i\pi/n})$.

Theorem 1.1. *Let F be an abelian extension of the rational numbers.*

(i) *F is the field of values of an irreducible character of degree 2 of some finite group if and only if $[\mathbb{Q}_{c(F)} : F] \leqslant 2$.*
(ii) *F is the field of values of an irreducible character of degree 3 of some finite group if and only if $[\mathbb{Q}_{c(F)} : F] \in \{1, 3\}$ or $F = \mathbb{Q}_k(\sqrt{5})$ for some $k \in \mathbb{Z}^+$ not divisible by 5.*

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More than thirty years ago, Cram [Cra88] already proved, using Gajendragadkar-Isaacs's theory of p-special characters of p-solvable groups, that if $\chi \in \mathrm{Irr}(G)$ where G is *solvable*, then $[\mathbb{Q}_{c(\chi)} : \mathbb{Q}(\chi)]$ divides $\chi(1)$. This, unfortunately, no longer holds in arbitrary non-solvable groups, as shown in the case $\chi(1) = 3$ and $\mathbb{Q}(\chi) = \mathbb{Q}_k(\sqrt{5})$ of Theorem 1.1. Theorem 1.1, however, seems to suggest the following. Here, the conductor of a character χ is defined as $c(\chi) := c(\mathbb{Q}(\chi))$.

Conjecture 1.2. *Let G be a finite group and χ an irreducible character of G. Then*

$$[\mathbb{Q}_{c(\chi)} : \mathbb{Q}(\chi)] \leqslant \chi(1).$$

It is pointed out to us by the referee that the extension-field degree $[\mathbb{Q}_{c(\chi)} : \mathbb{Q}(\chi)]$ has been called the *cyclotomic deficiency* of χ, a term coined by I. M. Isaacs.

Two remarks are in order. First, Conjecture 1.2 explains our initial observation on quadratic fields of values. Assume that $\chi \in \mathrm{Irr}(G)$ with $\mathbb{Q}(\chi) = \mathbb{Q}(\sqrt{k})$, where k is a square-free integer. Then it is expected that $\chi(1) \geqslant \varphi(|k|)/2$, where φ is Euler's phi function. (In fact, if $k \not\equiv 1 \pmod 4$, then $\chi(1) \geqslant \varphi(4|k|)/2$.) Secondly, by [FG72, Theorem 8] (see also [NT21, Theorem 2.2]), given any abelian extension F, there exists a (solvable) group G of order $c(F)[\mathbb{Q}_{c(F)} : F]$ and $\chi \in \mathrm{Irr}(G)$ of degree $\chi(1) = [\mathbb{Q}_{c(F)} : F]$ such that $\mathbb{Q}(\chi) = F$. Therefore, the inequality is as tight as possible.

Some evidence in support of Conjecture 1.2 is provided in Sections 3 and 4, where we verify it for the alternating groups and, respectively, the general linear and unitary groups.

When χ has odd degree, Conjecture 1.2 follows from a strengthened version of the celebrated McKay conjecture that includes the degree of the field extension $\mathbb{Q}_{c(\chi)}/\mathbb{Q}(\chi)$. This and some other related problems will be discussed in the final Section 5.

2. Characters of small degree

Given an abelian extension F of $\mathbb{Q}$ and a positive integer d, we are interested in finding a solution χ (and the corresponding group) to the following system:

$$\begin{cases} \mathbb{Q}(\chi) = F, \\ \chi(1) = d. \end{cases}$$

Our notation is fairly standard and follows [Isa76, Nav18]. Recall that, for a character χ, the field of values $\mathbb{Q}(\chi)$ of χ is the extension of $\mathbb{Q}$ generated by the set $\{\chi(g) : g \in G\}$. The conductor $c(\chi)$ of χ is the smallest positive integer such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_{c(\chi)}$ - that is, $\mathbb{Q}_{c(\chi)}$ is the *cyclotomic closure* of $\mathbb{Q}(\chi)$. For notational simplicity, sometimes we write

$$\mathbf{f}(\chi) := [\mathbb{Q}_{c(\chi)} : \mathbb{Q}(\chi)]$$

to be the degree of the field extension $\mathbb{Q}_{c(\chi)}/\mathbb{Q}(\chi)$. Therefore, $f(\chi)$ measures how far the field of values of χ is from its cyclotomic closure. When F is an abelian extension of $\mathbb{Q}$, we also use

$$\mathbf{f}(F) := [\mathbb{Q}_{c(F)} : F],$$

where $c(F)$ is the conductor of F.

Lemma 2.1. *Let $d \in \mathbb{Z}^+$. Then there exists a solvable group G and a rational-valued character $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) = d$.*

Proof. The lemma is trivial if $d = 1$. Using direct products of groups and (outer) tensor products of characters if necessary, we may assume that d is a prime. As the symmetric group S_3 has a desired character of degree 2, we further assume that d is an odd prime. Then the cyclic group $A := C_d$ of order d, viewed as a subgroup of the linear group $\mathrm{GL}_{d-1}(2)$, naturally acts on the $(d-1)$-dimensional vector space V over $\mathbb{F}_2$. The resulting semidirect product $V \rtimes A$ then has a rational-valued irreducible character of degree d, namely any character induced from a (linear) character of V that is not fixed by A. (Note that if G is not required to be solvable, one can consider the Steinberg character of the linear group $\mathrm{PSL}_2(d)$, which has degree d and is rational-valued for every prime $d \geqslant 5$.) □

Theorem 2.2. *Let F be an abelian extension of $\mathbb{Q}$ and $d \in \mathbb{Z}^+$. Then there exists a solvable group G and $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) = d$ and $\mathbb{Q}(\chi) = F$ if and only if $[\mathbb{Q}_{c(F)} : F]$ divides d.*

Proof. The only if implication is Cram's theorem [Cra88]. For the reverse implication, write $e := [\mathbb{Q}_{c(F)} : F]$, and assume that e divides d. By Fein-Gordon's result [FG72], there exists a solvable group H of order $e \cdot c(F)$ and $\psi \in \mathrm{Irr}(H)$ of degree $\psi(1) = e$ such that $\mathbb{Q}(\psi) = F$. By Lemma 2.1, there is another solvable group K and $\varphi \in \mathrm{Irr}(K)$ such that $\varphi(1) = d/e$ and $\mathbb{Q}(\varphi) = \mathbb{Q}$. Now the direct product $G := H \times K$ and $\chi := \psi \times \varphi \in \mathrm{Irr}(G)$ fulfill the requirement. □

Given two positive integers e and d such that $e \leqslant d$, it is an interesting problem to determine whether there is an irreducible character χ (of a certain group) such that $\chi(1) = d$ and $\mathbf{f}(\chi) = e$. When e is not a divisor of d, the relevant groups must be non-solvable, and it is hard to find a canonical construction. The answer is probably not 'yes' all the time. For instance, we do not have an example of an irreducible character χ with $\chi(1) = 5$ and $[\mathbb{Q}_{c(\chi)} : \mathbb{Q}(\chi)] = 4$.

Another problem we find interesting is to determine the distribution of the values of the function

$$f(\chi) := \frac{[\mathbb{Q}_{c(\chi)} : \mathbb{Q}(\chi)]}{\chi(1)}$$

on all the irreducible characters of finite groups. Note that the linear group $\mathrm{PSL}_2(p)$ with $p \equiv 1 \bmod 4$ has an irreducible character χ of degree $(p+1)/2$ and fields of values $\mathbb{Q}(\sqrt{p})$, and so $f(\chi) = (p-1)/(p+1)$. Since there are infinitely many primes of the form $4k+1$, the values of f are *dense* on the interval $(0, 1]$. Again, we do not know if $4/5$ is a value of f.

The following consequence of Theorem 2.2 generalizes [NT21, Theorem 2.8] from odd-degree characters to p'-degree characters.

Corollary 2.3. *Let F be an abelian extension of $\mathbb{Q}$. Then there exists a solvable group G and $\chi \in \mathrm{Irr}_{p'}(G)$ such that $\mathbb{Q}(\chi) = F$ if and only if $[\mathbb{Q}_{c(F)} : F]$ is not divisible by p.*

Theorem 2.4. *Let F be an abelian extension of the rational numbers.*

(i) *If $[\mathbb{Q}_{c(F)} : F] \leqslant 2$ then there exists a finite group G and $\chi \in \mathrm{Irr}(G)$ such that $\mathbb{Q}(\chi) = F$ and $\chi(1) = 2$.*
(ii) *If $[\mathbb{Q}_{c(F)} : F] \in \{1, 3\}$ or $F = \mathbb{Q}_k(\sqrt{5})$ for some $k \in \mathbb{Z}^+$ not divisible by 5, then there exists a finite group G and $\chi \in \mathrm{Irr}(G)$ such that $\mathbb{Q}(\chi) = F$ and $\chi(1) = 3$.*

Proof. Let $\mathbf{f}(F) := [\mathbb{Q}_{c(F)} : F]$. We already mentioned a result of Fein and Gordon that, given any abelian extension F, there exists a (solvable) group G of order $c(F)\mathbf{f}(F)$ and $\chi \in \mathrm{Irr}(G)$ of degree $\chi(1) = \mathbf{f}(F)$ such that $\mathbb{Q}(\chi) = F$. This and Lemma 2.1 prove assertion (i) and the part $\mathbf{f}(F) \in \{1, 3\}$ of (ii).

So we suppose that $F = \mathbb{Q}_k(\sqrt{5})$ for some $k \in \mathbb{Z}^+$ not divisible by 5, in which case $\mathbf{f}(F) = 2$. Note then that, by Cram's result, there is no solvable group G having an irreducible character χ such that $\mathbb{Q}(\chi) = F$ and $\chi(1) = 3$, and therefore we have to search among non-solvable groups.

In fact, the alternating group A_5 has an irreducible character, say α, such that $\alpha(1) = 3$ and $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{5})$. Now let $G := \mathsf{A}_5 \times C_k$ – the direct product of A_5 and the cyclic group of order k. Let $\beta \in \mathrm{Irr}(C_k)$ of order k and consider $\chi := \alpha \times \beta \in \mathrm{Irr}(G)$. Then $\chi(1) = \alpha(1) = 3$ and $\mathbb{Q}(\chi) = \mathbb{Q}(\alpha, \beta) = \mathbb{Q}_k(\sqrt{5}) = F$, as desired. □

Lemma 2.5. *Let $H < G$, $\chi \in \mathrm{Irr}(G)$ and $\psi \in \mathrm{Irr}(H)$ such that $\chi = \psi^G$. Then $\mathbb{Q}(\psi) \subseteq \mathbb{Q}(\chi)$ and $[\mathbb{Q}(\psi) : \mathbb{Q}(\chi)] \leqslant |G : H|$.*

Proof. The fact that $\mathbb{Q}(\psi) \subseteq \mathbb{Q}(\chi)$ follows from the induction character formula. Let $\sigma \in \mathrm{Gal}(\mathbb{Q}(\psi)/\mathbb{Q}(\chi))$. Then σ fixes χ_H and $[\chi_H, \psi^\sigma] = [(\chi_H)^\sigma, \psi^\sigma] = [\chi_H, \psi]^\sigma = [\chi_H, \psi]$. Thus the Galois $\mathrm{Gal}(\mathbb{Q}(\psi)/\mathbb{Q}(\chi))$-conjugates of ψ are irreducible constituents of χ_H. Note that no nontrivial automorphisms in $\mathrm{Gal}(\mathbb{Q}(\psi)/\mathbb{Q}(\chi))$ fix ψ. Hence there are precisely $|\mathrm{Gal}(\mathbb{Q}(\psi)/\mathbb{Q}(\chi))| = [\mathbb{Q}(\psi) : \mathbb{Q}(\chi)]$ different $\mathrm{Gal}(\mathbb{Q}(\psi)/\mathbb{Q}(\chi))$-conjugates of ψ among those constituents. These conjugates all have the same degree as ψ. We deduce that

$$\chi(1) \geqslant [\mathbb{Q}(\psi) : \mathbb{Q}(\chi)]\psi(1).$$

Since $\chi(1) = |G : H|\psi(1)$, the lemma follows. □

Lemma 2.6. *Let G be a finite group and χ an irreducible monomial character of G. Then $[\mathbb{Q}_{c(\chi)} : \mathbb{Q}(\chi)] \leqslant \chi(1)$. In particular, Conjecture 1.2 holds for non-primitive characters of prime degree.*

Proof. We have $\chi = \lambda^G$, where λ is a linear character of some (not necessarily proper) subgroup H of G. Since $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(\lambda)$ and $\mathbb{Q}(\lambda)$ is a full cyclotomic field, we have $\mathbb{Q}_{c(\chi)} \subseteq \mathbb{Q}(\lambda)$. It follows that $[\mathbb{Q}_{c(\chi)} : \mathbb{Q}(\chi)]$ divides $[\mathbb{Q}(\lambda) : \mathbb{Q}(\chi)]$, and thus $[\mathbb{Q}_{c(\chi)} : \mathbb{Q}(\chi)] \leqslant |G : H|$ by Lemma 2.5. The first statement of the lemma follows immediately.

For the second part, note that a non-primitive character of G of prime degree must be induced from a linear character of some subgroup of G, and therefore is automatically monomial. □

The following easy observation might be useful in the future.

Theorem 2.7. *Conjecture 1.2 is true if it holds whenever the character χ is primitive.*

Proof. Assume that χ is a non-primitive irreducible character of a finite group G. Then $\chi = \psi^G$ for some primitive character $\psi \in \mathrm{Irr}(H)$, where H is a proper subgroup of G (see [Isa76, Theorem 5.8]). Clearly $c(\chi) \leqslant c(\psi)$. Since the conjecture holds for primitive characters, we have $[\mathbb{Q}_{c(\psi)} : \mathbb{Q}(\psi)] \leqslant \psi(1)$. By Lemma 2.5, $[\mathbb{Q}(\psi) : \mathbb{Q}(\chi)] \leqslant |G : H|$. It then follows that

$$[\mathbb{Q}_{c(\chi)} : \mathbb{Q}(\chi)] \leqslant [\mathbb{Q}_{c(\psi)} : \mathbb{Q}(\psi)][\mathbb{Q}(\psi) : \mathbb{Q}(\chi)] \leqslant \psi(1)|G : H| = \chi(1),$$

as desired. □

Theorem 2.8. *Let G be a finite group and let $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \leqslant 3$. Then we have $[\mathbb{Q}_{c(\chi)} : \mathbb{Q}(\chi)] \leqslant \chi(1)$. Furthermore, if $\chi(1) = 3$ and $[\mathbb{Q}_{c(\chi)} : \mathbb{Q}(\chi)] = 2$, then $\mathbb{Q}(\chi) = \mathbb{Q}_k(\sqrt{5})$ for some $k \in \mathbb{Z}^+$ not divisible by 5.*

Proof. As before, we write $\mathbf{f}(\chi) := [\mathbb{Q}_{c(\chi)} : \mathbb{Q}(\chi)]$. First we observe that the degree, the field of values, and the conductor of χ are all unchanged if χ is viewed as a character $G/\mathrm{Ker}(\chi)$. Therefore without loss we may assume that χ is faithful. We also may assume that $\chi(1) \in \{2, 3\}$. In particular, χ has prime degree and the first part of the theorem is done if χ is non-primitive, by Lemma 2.6.

We will prove that the hypothesis of the second part does not occur in the case χ being non-primitive. Assume so. Then $\chi = \lambda^G$, where λ is a nontrivial linear character of a subgroup $H \leqslant G$ of index 3. Also, $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(\lambda)$, $\mathbb{Q}(\lambda)$ is a full cyclotomic field, and $[\mathbb{Q}(\lambda) : \mathbb{Q}(\chi)] \leqslant |G : H| = 3$ by Lemma 2.5. On the other hand, $\mathbf{f}(\chi) = [\mathbb{Q}_{c(\chi)} : \mathbb{Q}(\chi)] = 2$ by the hypothesis, and so we must have $\mathbb{Q}(\lambda) = \mathbb{Q}_{c(\chi)}$. Therefore,

$$\chi_H = \lambda + \lambda^\sigma + \mu,$$

where σ is the nontrivial automorphism of $\mathrm{Gal}(\mathbb{Q}(\lambda)/\mathbb{Q}(\chi))$ and $\mu \in \mathrm{Irr}(H)$ is of necessarily degree 1. Moreover, $\mu \notin \{\lambda, \lambda^\sigma\}$ because otherwise $\chi(1) \geqslant 4$ (note that λ and λ^σ appears in χ_H with the same multiplicity). But then $\chi = \mu^G$, and by the same reasoning, we would have $\mathbb{Q}(\mu) = \mathbb{Q}_{c(\chi)} = \mathbb{Q}(\lambda)$, and thus the order-2 group $\mathrm{Gal}(\mathbb{Q}(\lambda)/\mathbb{Q}(\chi))$ permutes the three constituents of χ_H with no fixed points. This is a contradiction.

We may now assume that χ is primitive. Note that we are also done if G is solvable by Theorem 2.2. In summary, it is sufficient to assume that G is a non-solvable primitive linear group of degree 2 or 3. By the classification of the primitive linear groups of these degrees (see [Bli17, Chapter V, Section 81]), we have

$$S := G/\mathbf{Z}(G) \cong \mathsf{A}_5, \mathsf{A}_6, \text{ or } \mathrm{PSL}_2(7).$$

Let M be a minimal member (in terms of inclusion) among all the non-solvable normal subgroups of G. We note that M is perfect and is contained in the last term of the derived series of G. Write $Z := \mathbf{Z}(G)$. We claim that $G = MZ$ is a central product with a central amalgamated subgroup $\mathbf{Z}(M) = M \cap Z$. First, as G/Z is simple, M is non-solvable, and Z is central in G, it is clear that $G = MZ$ is a central product. We then have $M/(M \cap Z) \cong MZ/Z = G/Z$ is simple. It follows that $M \cap Z = \mathbf{Z}(M)$ and indeed M is a perfect central cover of S.

Write $A := \mathbf{Z}(M) = M \cap Z$. For each $\lambda \in \mathrm{Irr}(A)$, there exists a bijective correspondence

$$\mathrm{Irr}(M \mid \lambda) \times \mathrm{Irr}(Z \mid \lambda) \to \mathrm{Irr}(G \mid \lambda)$$

such that if (α, β) corresponds to χ, then $\chi(1) = \alpha(1)\beta(1) = \alpha(1)$ (see, for instance, [Nav18, Theorem 10.7]). Furthermore, $\chi(xz) = \alpha(x)\beta(z)$ for every $x \in M$ and $z \in Z$, so that $\mathbb{Q}(\chi) = \mathbb{Q}(\alpha, \beta)$. In particular, $c(\chi) = \mathrm{lcm}(c(\alpha), c(\beta))$. Note that $\beta \in \mathrm{Irr}(Z)$ is linear and $\mathbb{Q}(\beta) = \mathbb{Q}_{c(\beta)}$. All together, we have

$$\begin{aligned}[\mathbb{Q}_{c(\chi)} : \mathbb{Q}(\chi)] &= [\mathbb{Q}_{c(\chi)} : \mathbb{Q}(\alpha, \beta)] \\ &= [\mathbb{Q}_{c(\alpha)}\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\alpha, \beta)] \\ &= [\mathbb{Q}_{c(\alpha)} : (\mathbb{Q}_{c(\alpha)} \cap \mathbb{Q}(\alpha, \beta))] \\ &\leqslant [\mathbb{Q}_{c(\alpha)} : \mathbb{Q}(\alpha)],\end{aligned}$$

where the third equality is due to the natural irrationality. (The last inequality is indeed a divisibility.) For the first statement of the theorem, we in fact have reduced from $\chi \in \mathrm{Irr}(G)$ to $\alpha \in \mathrm{Irr}(M)$, where M is a perfect central cover of $\mathsf{A}_5, \mathsf{A}_6$ or $\mathrm{PSL}_2(7)$. The required inequality for α then can be verified by direct inspection of the character tables of the relevant groups available in [Atl]. In fact, if α is a non-rational character in consideration, then either $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{5})$ or $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{7})$, where the latter case occurs only when α is one of the two irreducible characters of degree 3 of $\mathrm{PSL}_2(7)$.

Now assume that $\chi(1) = 3$ and $\mathbf{f}(\chi) = 2$ (and χ is still primitive as in the preceding paragraph). Then $\alpha(1) = 3$ and $\mathbf{f}(\alpha)$ is divisible by 2. This happens only when M is a cover of A_5, so that $M \in \{\mathsf{A}_5, \mathrm{SL}_2(5)\}$, or M is the triple cover of A_6. In both instances, we have $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{5})$ and so $c(\alpha) = 5$. If $c(\beta)$ is divisible by 5 then $\mathbb{Q}(\chi)$ would be the full cyclotomic field $\mathbb{Q}(\beta)$, which is not the case. Hence $k := c(\beta)$ is coprime to 5, and so $\mathbb{Q}(\chi) = \mathbb{Q}(\alpha, \beta) = \mathbb{Q}_k(\sqrt{5})$. The proof is complete. □

Theorem 1.1 readily follows from Theorems 2.4 and 2.8.

3. Alternating groups

In this section we establish Conjecture 1.2 for alternating groups A_n, see Theorem 3.2. Note that, since characters of symmetric groups S_n are all rational-valued, the conjecture for S_n is a triviality.

We recall some needed background of the representation theory of S_n, as well as A_n, and we refer the reader to [JK81] for further details. There is a one-to-one correspondence between the irreducible characters of S_n and the partitions of n. Let λ be a partition of n. The Young diagram corresponding to λ, denoted by Y_λ, is the finite subset of $\mathbb{N} \times \mathbb{N}$ such that

$$(i, j) \in Y_\lambda \text{ if and only if } i \leqslant \lambda_j.$$

The conjugate partition of λ, denoted by $\overline{\lambda}$, is the partition whose associated Young diagram is obtained from Y_λ by reflecting it about the line $y = x$. So $\lambda = \overline{\lambda}$ if and only if Y_λ is symmetric and in that case we say that λ is self-conjugate.

For each node $(i, j) \in Y_\lambda$, the *hook length* $h_\lambda(i, j)$ is the number of nodes that are directly above it, directly to the right of it, or equal to it:

$$h_\lambda(i, j) := 1 + \lambda_j + \overline{\lambda}_i - i - j.$$

Let χ_λ denote the irreducible character of S_n corresponding to λ. The irreducible characters of A_n can be obtained by restricting those of S_n to A_n. More specifically, the restrictions of both χ_λ and $\chi_{\overline{\lambda}}$ are irreducible if λ is not self-conjugate, and such restriction χ_λ splits into two different irreducible characters of A_n, say χ_λ^+ and χ_λ^-, of the same degree if λ

is self-conjugate. Recall that irreducible characters of S_n are rational-valued. Therefore members of $\mathrm{Irr}(\mathsf{A}_n)$ that are restrictions of those in $\mathrm{Irr}(\mathsf{S}_n)$ trivially satisfy Conjecture 1.2.

We therefore assume that λ is a self-conjugate partition of n and focus on characters of the form $\chi_\lambda^\pm$.

Write $h_\lambda(i) := h_\lambda(i,i)$ - the hook length at the position (i,i) in the main diagonal of Y_λ, and consider the partition

$$h(\lambda) := (h_\lambda(1), h_\lambda(2), ..., h_\lambda(k)),$$

of n, where k is the length of the main diagonal of the Young diagram Y_λ. Let $C_{h(\lambda)}$ denotes the conjugacy class of S_n whose cycle partition is exactly $h(\lambda)$. Note that all the parts of $h(\lambda)$ are pairwise different and odd, and thus $C_{h(\lambda)}$ splits into two A_n-classes of equal size, say $C^+_{h(\lambda)}$ and $C^-_{h(\lambda)}$.

Lemma 3.1. *Let λ be a self-conjugate partition of $n \in \mathbb{Z}^{\geqslant 3}$ and k the length of the main diagonal of the Young diagram Y_λ. Then*

$$\mathbb{Q}(\chi_\lambda^\pm) = \mathbb{Q}\left(\sqrt{\prod_{1\leqslant i\leqslant k; 3\leqslant h_\lambda(i)} \epsilon_{h_\lambda(i)} h_\lambda(i)}\right),$$

where, for an odd integer $a > 1$, $\epsilon_a = (-1)^{(a-1)/2}$. In particular, if $\chi := \chi_\lambda^\pm$ then

$$[\mathbb{Q}_{c(\chi)} : \mathbb{Q}(\chi)] \leqslant \frac{1}{2} \prod_{1\leqslant i\leqslant k; 3\leqslant h_\lambda(i)} (h_\lambda(i) - 1).$$

Proof. The values of $\chi_\lambda^\pm$ are well-known, see [JK81, Lemmas 2.5.12 and 2.5.13] for instance, as follows.

(i) $\chi_\lambda(C_{h(\lambda)}) = (-1)^{(n-k)/2}$,

(ii) $\chi_\lambda^\pm(C^+_{h(\lambda)}) = \frac{1}{2}\left(\chi_\lambda(C_{h(\lambda)}) \pm \sqrt{\chi_\lambda(C_{h(\lambda)}) \prod_{i=1}^k h_\lambda(i)}\right)$ and
$\chi_\lambda^\pm(C^-_{h(\lambda)}) = \frac{1}{2}\left(\chi_\lambda(C_{h(\lambda)}) \mp \sqrt{\chi_\lambda(C_{h(\lambda)}) \prod_{i=1}^k h_\lambda(i)}\right)$ for a suitable labeling of χ_λ^+ and χ_λ^-,

(iii) $\chi_\lambda^\pm(C) = \chi_\lambda(C)/2$ for any class C different from $C^\pm_{h(\lambda)}$.

(Here, we write $\chi(C)$ for the value of χ at any element in the conjugacy class C.) Note that $n - k = \sum_{i=1}^k (h_\lambda(i) - 1)$, and hence $\chi_\lambda(C_{h(\lambda)}) = (-1)^{(n-k)/2} = \prod_{i=1}^k \epsilon_{h_\lambda(i)}$. The first statement of the lemma immediately follows.

To prove the second part, let $p_1, p_2, ..., p_t$ be the primes that occur with odd exponent in the prime factorization of $\prod_{1\leqslant i\leqslant k; 3\leqslant h_\lambda(i)} \epsilon_{h_\lambda(i)} h_\lambda(i)$. Recall that all of $h_\lambda(i)$s are odd, and so are the p_is. Now

$$\mathbb{Q}\left(\sqrt{\prod_{1\leqslant i\leqslant k; 3\leqslant h_\lambda(i)} \epsilon_{h_\lambda(i)} h_\lambda(i)}\right) = \mathbb{Q}\left(\sqrt{\prod_{1\leqslant i\leqslant t} \epsilon_{p_i} p_i}\right).$$

It is well-known from the quadratic Gauss sum that $c(\sqrt{\epsilon_{p_i} p_i}) = p_i$. We therefore deduce that

$$c(\chi) = \prod_{1 \leqslant i \leqslant t} p_i,$$

which yields

$$[\mathbb{Q}_{c(\chi)} : \mathbb{Q}] \leqslant \prod_{1 \leqslant i \leqslant t} (p_i - 1).$$

It is clear that $\prod_{1 \leqslant i \leqslant t}(p_i - 1) \leqslant \prod_{1 \leqslant i \leqslant k; 3 \leqslant h_\lambda(i)}(h_\lambda(i) - 1)$, and the proof is complete. □

Theorem 3.2. *For every irreducible character* $\chi \in \mathrm{Irr}(\mathsf{A}_n)$ *with* $n \in \mathbb{Z}^+$, *we have* $[\mathbb{Q}_{c(\chi)} : \mathbb{Q}(\chi)] \leqslant \chi(1)$.

Proof. As already mentioned, it suffices to prove the statement for those characters of the form $\chi_\lambda^\pm$, where λ is a self-conjugate partition of n. In view of Lemma 3.1, we wish to show that

$$\chi_\lambda(1) \geqslant \prod_{1 \leqslant i \leqslant k; 3 \leqslant h_\lambda(i)} (h_\lambda(i) - 1),$$

where χ_λ is the character of S_n corresponding to the partition λ, of degree $\chi_\lambda = 2\chi_\lambda^\pm(1)$. For notational convenience, we write $h_i := h_\lambda(i)$ from now on.

First consider the case that $k = 1$ (which implies that n must be odd), i.e., $\lambda = (\lambda_1, 1, 1, ...)$. Then

$$\chi_\lambda(1) = \frac{n!}{n \cdot (((n-1)/2)!)^2} = \frac{(n-1) \cdots ((n+1)/2)}{(n-1)/2)!},$$

which can be easily shown to be at least $n - 1 = h_1 - 1$, as desired.

So suppose that $k > 1$. Consider

$$\mu := (\lambda_2 - 1, \lambda_3 - 1, ...),$$

so that μ is still self-conjugate and the Young diagram Y_μ corresponding to μ can be obtained from that of λ by removing the entire (largest) hook at the position $(1, 1)$. By the hook length formula [FRT54],

$$\chi_\lambda(1) = \frac{n(n-1) \cdots (n - h_1 + 1)}{h_1 \cdot \prod_{2 \leqslant i \leqslant \lambda_1} (h_\lambda(i, 1))^2} \cdot \chi_\mu(1),$$

where $\chi_\mu \in \mathrm{Irr}(\mathsf{S}_{n-h_1})$. (Here, we note that $h_1 = 2\lambda_1 - 1$ and $\prod_{2 \leqslant i \leqslant \lambda_1} h_\lambda(i, 1)$ is simply the product of all the hook lengths of λ on the horizontal arm of the largest hook, not counting the position $(1, 1)$. Note also that the hook lengths on the vertical arm are exactly the same as those on the horizontal arm, due to the symmetrical feature of the Young diagram Y_λ.) By induction on n, we just need to prove that

$$\frac{n(n-1) \cdots (n - h_1 + 1)}{h_1 \cdot \prod_{2 \leqslant i \leqslant \lambda_1} (h_\lambda(i, 1))^2} \geqslant h_1 - 1.$$

If $\lambda_2 < \lambda_1$ then $h_\lambda(\lambda_1, 1) = 1$, and one easily sees that the left side is at least n, and the inequality follows. Thus we may assume that $\lambda_2 = \lambda_1$. Then $n \geqslant h_1 + h_2 = 2h_1 - 2$, and so $n - h_1 + 1 \geqslant h_1 - 1$. The desired inequality is now reduced to

$$n(n-1)\cdots(n-h_1+2) \geqslant h_1 \cdot \prod_{2\leqslant i\leqslant \lambda_1} (h_\lambda(i,1))^2.$$

Observe that the Young diagram Y_λ contains all the nodes (i,j) with $1 \leqslant i,j \leqslant h_\lambda(\lambda_1,1)$, we have

$$n \geqslant (h_\lambda(\lambda_1,1))^2$$

(with equality when Y_λ is a square). Therefore, we wish to establish

$$(n-1)\cdots(n-h_1+2) \geqslant h_1 \cdot \prod_{2\leqslant i\leqslant \lambda_1-1} (h_\lambda(i,1))^2,$$

where the product $\prod_{2\leqslant i\leqslant\lambda_1-1}(h_\lambda(i,1))^2$ is assumes to be 1 if $\lambda_1 \leqslant 2$. This turns out to be clear since the number of terms on both sides are the same ($h_1 - 2 = 2\lambda_1 - 3$) and the smallest term, namely $n - h_1 + 2$, on the left-hand side is at least the largest term, namely h_1, on the right-hand side. □

4. Linear and unitary groups

In this section we verify Conjecture 1.2 for general linear and unitary groups.

We will use the notation $G = G_n = \mathrm{GL}_n^\epsilon(q)$, with $\epsilon = \pm$ and q any prime power, where GL^+ stands for GL and GL^- stands for GU. We can identify the dual group G^* with $G = \mathrm{GL}_n^\epsilon(q)$ and use Lusztig's classification of complex characters of G, see [Car85], [DM91]. If $s \in G$ is a semisimple element, then $\mathcal{E}(G,(s))$ denotes the rational series of irreducible characters of G labeled by the G-conjugacy class of s. For any semisimple $s \in G$, we can decompose $V = V^0 \oplus V^1$ as direct (orthogonal if $\epsilon = -$) sum of s-invariant subspaces, where $V^0 = \oplus_{\delta\in\mu_{q-\epsilon 1}} V_\delta$, s acts on V_δ as $\delta \cdot 1_{V_\delta}$, and no eigenvalue of $s^1 := s|_{V^1}$ belongs to

$$\mu_{q-\epsilon 1} := \{x \in \mathbb{F}_{q^2}^\times \mid x^{q-\epsilon 1} = 1\}.$$

Then

$$\mathbf{C}_G(s) = \prod_{\delta\in\mu_{q-\epsilon 1}} \mathrm{GL}^\epsilon(V_\delta) \times \mathbf{C}_{\mathrm{GL}^\epsilon(V^1)}(s^1).$$

Correspondingly, any unipotent character ψ of $\mathbf{C}_G(s)$ can be written in the form

$$\psi = \boxtimes_{\delta\in\mu_{q-\epsilon 1}} \psi^{\boldsymbol{\gamma}_\delta} \boxtimes \psi_1, \tag{4.1}$$

where $\psi^{\boldsymbol{\gamma}_\delta}$ is the unipotent character of $\mathrm{GL}^\epsilon(V_\delta)$ labeled by a partition $\boldsymbol{\gamma}_\delta$ of $\dim_{\mathbb{F}_Q} V_\delta$, and ψ_1 is a unipotent character of $\mathbf{C}_{\mathrm{GL}^\epsilon(V^1)}(s^1)$. If $V_\delta = 0$, then we view $\boldsymbol{\gamma}_\delta$ as the partition (0) of 0.

Fix an embedding of $\bar{\mathbb{F}}^\times$ into $\mathbb{C}^\times$. Then one can identify $\mathbf{Z}(\mathbf{C}_G(s))$ with

$$\mathrm{Hom}(\mathbf{C}_G(s)/[\mathbf{C}_G(s),\mathbf{C}_G(s)],\mathbb{C}^\times)$$

as in [FS82, (1.16)], and the linear character of $\mathbf{C}_G(s)$ corresponding to s will be denoted by $\hat{s}$. In particular, the element s and the linear character $\hat{s}$ have the same order, and hence

$$\mathbb{Q}(\hat{s}) = \mathbb{Q}_{|s|}. \tag{4.2}$$

Now, the irreducible character χ of G labeled by s and the unipotent character ψ is

$$\chi = \pm R^G_{\mathbf{C}_G(s)}(\hat{s}\psi), \tag{4.3}$$

see [FS82, p. 116].

Let $\mathcal{G} = \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ and let F denote the Frobenius endomorphism

$$X = (x_{ij}) \mapsto X^{(q)} := (x_{ij}^q)$$

or $F : X \mapsto \left({}^t X^{(q)}\right)^{-1}$, so that $\mathcal{G}^F \cong \mathrm{GL}_n(q)$, respectively $\mathrm{GU}_n(q)$. Following [FS82, §1], we will *always* fix an F-stable maximal torus $\mathcal{T}_1$ consisting of *diagonal* matrices, so that $|\mathcal{T}_1^F| = (q-1)^n$, respectively $(q+1)^n$. Then the $\mathcal{G}^F$-conjugacy classes of maximal tori in $\mathcal{G}$ are parametrized by conjugacy classes in the Weyl group $W = \mathbf{N}_{\mathcal{G}}(\mathcal{T}_1)/\mathcal{T}_1 \cong \mathbb{S}_n$. Furthermore, the unipotent characters of $\mathcal{G}^F$ are parametrized by the irreducible characters λ of W, which in turn are parametrized by partitions $\boldsymbol{\lambda} \vdash n$. For $w \in W$, let $\mathcal{T}_w$ denote an F-stable maximal torus of $\mathcal{G}$ corresponding to the W-conjugacy class of w. Then, for any $\lambda \in \mathrm{Irr}(W)$ labeled by $\boldsymbol{\lambda} \vdash n$, the corresponding unipotent character $\psi^\lambda = \psi^{\boldsymbol{\lambda}}$ of $\mathcal{G}^F$ is given by

$$\psi^\lambda = \frac{a_\lambda}{|W|} \sum_{w \in W} \lambda(w) R^{\mathcal{G}}_{\mathcal{T}_w}(1_{\mathcal{T}_w^F}) \tag{4.4}$$

for some $a_\lambda = \pm 1$, see [FS82, (1.13)]. The same construction extends to direct products of groups of type GL, equipped with Frobenius endomorphisms stabilizing each factor, in particular to F-stable Levi subgroups of GL_n.

As before, we can identity the dual group G^* of $G = \mathcal{G}^F$ with G. For any semisimple element $s \in G$,

$$\mathcal{L} := \mathbf{C}_{\mathcal{G}}(s) = \mathcal{G}_1 \times \mathcal{G}_2 \times \ldots \times \mathcal{G}_m,$$

and likewise the Weyl group

$$W_{\mathcal{L}} \cong \mathbb{S}_{n_1} \times \mathbb{S}_{n_2} \times \ldots \times \mathbb{S}_{n_m}$$

of $\mathcal{L}$ is a direct product of symmetric groups. Hence, any unipotent character ψ^μ of $\mathbf{C}_G(s)$ is labeled by an irreducible character $\mu \in \mathrm{Irr}(W_{\mathcal{L}})$, as described in (4.4). Recall that any such μ is rational-valued. For $w \in W_{\mathcal{L}}$, let $\mathcal{T}_w$ denote an F-stable maximal torus of $\mathcal{L} = \mathbf{C}_{\mathcal{G}}(s)$ corresponding to the $W_{\mathcal{L}}$-conjugacy class of w. Then, according to [FS82, (1.18)] we have that

$$\chi^{s,\mu} = \frac{a_{s,\mu}}{|W_{\mathcal{L}}|} \sum_{w \in W_{\mathcal{L}}} \mu(w) R^{\mathcal{G}}_{\mathcal{T}_w}(\hat{s}|_{\mathcal{T}_w^F}) \tag{4.5}$$

for some $a_{s,\mu} = \pm 1$, where the linear character $\hat{s}$ of $\mathbf{C}_G(s)$ is introduced before (4.3). Now, (4.2), (4.5), and the formula for Lusztig induction [DM91, Proposition 12.2] shows that

$$\mathbb{Q}(\chi^{s,\mu}) \subseteq \mathbb{Q}_{|s|}.$$

In particular, for $\chi = \chi^{s,\mu}$ we have $c(\chi)$ divides $|s|$.

Now we can prove

Theorem 4.1. *Conjecture 1.2 holds for $G = \mathrm{GL}_n(q)$ and $G = \mathrm{GU}_n(q)$ with $n \geqslant 5$.*

Proof. In the notation of the above discussion, we have shown that $c(\chi)$ divides $|s|$ for $\chi = \chi^{s,\mu} \in \mathrm{Irr}(G)$. Decomposing the natural module for G into a direct sum of (pairwise orthogonal and non-degenerate, in the case $G = \mathrm{GU}_n(q)$, g-invariant nonzero subspaces, with as many summands as possible, one can see that there are integers $n_1, \ldots, n_m \geqslant 1$ and signs $\epsilon_1, \ldots, \epsilon_m = \pm 1$ (and $\epsilon_i = 1$ for all i if $G = \mathrm{GL}_n(q)$, such that $|s|$ divides $\mathrm{lcm}_{i=1}^m (q^{n_i} - \epsilon_i)$. Now applying [GMPS15, Lemma 2.9] when $\epsilon = -$, we obtain

$$|s| \leqslant D := \frac{q^{n+1}}{(q-1)\gcd(2, q-1)},$$

whereas

$$|s| \leqslant D := q^n - 1$$

when $\epsilon = +$. Hence, to complete the verification of Conjecture 1.2 for G, it remains to check it for χ with

$$\chi(1) < D. \tag{4.6}$$

As the result can be checked directly using [GAP] for $G = \mathrm{GU}_6(2)$, we assume in addition that $G \not\cong \mathrm{GU}_6(2)$. Let θ be an irreducible constituent of $\chi|_{[G,G]}$ (note that $S := [G, G] \cong \mathrm{SL}_n^\epsilon(q)$). Applying Theorems 3.1 and 4.1 of [TZ96] to θ, we see that, under the assumption $n \geqslant 5$, θ is either an irreducible Weil character, or 1_S. In the former case, θ extends to a Weil character of G. As $G/S \cong C_{q-\epsilon}$ is cyclic, χ itself is also a Weil character. The character values of Weil characters of G are well-known, see e.g. [Tie15] and [TZ97], and one readily checks that $c(\chi)$ divides $q - \epsilon$, implying the conjecture for χ. In the latter case, χ is a linear character of G/S, so $\mathbb{Q}_{c(\chi)} = \mathbb{Q}(\chi)$. ☐

Theorem 4.2. *Conjecture 1.2 holds for $G = \mathrm{GL}_n(q)$ and $G = \mathrm{GU}_n(q)$ with $n \leqslant 4$.*

Proof. For smaller-rank groups, the rough bound for the order $|s|$ of the semisimple element s presented in the proof of Theorem 4.1 is not enough for our purpose. Instead, we examine the detailed structure of the centralizer $\mathbf{C}_G(s)$ and directly compare $|s|$ with the degree $\chi(1)$ of the character $\chi = \chi^{s,\mu}$ defined in (4.5). Recall the degree formula in Lusztig's parametrization of complex irreducible characters that

$$\chi^{s,\mu}(1) = |G : \mathbf{C}_G(s)|_{p'} \psi^\mu(1), \tag{4.7}$$

where, as before, ψ^μ is the unipotent character of $\mathbf{C}_G(s)$ labeled by μ and p is the defining characteristic of G, see [DM91, Remark 13.24].

If s is central in G then $\mathbf{C}_G(s) = G$ and $\chi^{s,\mu}$ is simply the product of a unipotent character (which is rational-valued) and a linear character of $G/[G, G]$ (whose field of values is a full cyclotomic field), in which case $\mathbb{Q}(\chi^{s,\mu})$ is cyclotomic. We therefore may assume that s is not central, so that $\mathbf{C}_G(s)$ is properly contained in G.

Let $G = \mathrm{GL}_n(q)$. Then

$$\mathbf{C}_G(s) \cong \mathrm{GL}_{k_1}(q^{n_1}) \times \mathrm{GL}_{k_2}(q^{n_2}) \times \cdots,$$

where $n_i, k_i \in \mathbb{Z}^+$ and $n = \sum_i n_i k_i$ (see, for instance, [Car81]). Clearly s belongs to $\mathbf{Z}(\mathbf{C}_G(s))$, which is a direct product of cyclic groups of order $q^{n_i} - 1$. It follows that

$$|s| \leqslant \mathrm{lcm}(q^{n_1} - 1, q^{n_2} - 1, \ldots).$$

Since $n \leqslant 4$, there are only a few possibilities for n_is and k_is, and in each case, it is straightforward to check that

$$\mathrm{lcm}(q^{n_1}-1, q^{n_2}-1, ...) \leqslant |G : \mathbf{C}_G(s)|_{p'}.$$

As noted earlier, $c(\chi^{s,\mu})$ divides $|s|$. The formula 4.7 then implies that $c(\chi^{s,\mu}) \leqslant \chi^{s,\mu}(1)$, and the result follows.

Now let $G = \mathrm{GU}_n(q)$. Then

$$\mathbf{C}_G(s) \cong \prod_i \mathrm{GL}_{k_i}(q^{2n_i}) \times \prod_j \mathrm{GU}_{l_j}(q^{2m_j-1}),$$

where $n_i, k_i, m_j, l_j \in \mathbb{Z}^+$ and $n = \sum_i 2n_ik_i + \sum_j l_j(2m_j-1)$, and thus

$$|s| \leqslant \mathrm{lcm}(q^{2n_1}-1, q^{2n_2}-1, ..., q^{2m_1-1}+1, q^{2m_2-1}+1, ...).$$

Recall the assumption that s is not central in S, so $(l_1, m_1) \neq (4, 1)$. A similar case-by-case check as in the linear-group case then shows that $|s| \leqslant |G : \mathbf{C}_G(s)|_{p'}$, and the result again follows. □

5. Odd-degree characters and further discussion

We have seen the relevance of the invariant $\mathbf{f}(\chi) := [\mathbb{Q}_{c(\chi)} : \mathbb{Q}(\chi)]$ in the connection between degrees and fields of values of characters. Let $\mathbf{f}(G, e) := |\{\chi \in \mathrm{Irr}(G) : \mathbf{f}(\chi) = e\}|$ and $\mathbf{f}(G) := \max_e\{\mathbf{f}(G, e)\}$. A result of Moretó [Mor21] implies that there exists a real-valued function h such that $|G| \leqslant h(\mathbf{f}(G))$. We find it interesting to study the class of finite groups G with $\mathbf{f}(G) = |\mathrm{Irr}(G)|$; that is, $\mathbf{f}(\chi) = 1$ for all $\chi \in \mathrm{Irr}(G)$ or, equivalently, all the irreducible characters of G have cyclotomic fields of values.

Problem 5.1. *Describe the groups whose irreducible complex characters are all cyclotomic.*

This class includes three important subclasses. The first is the obvious abelian groups. The second is the odd-order p-groups (see [NT21, Theorem 2.3]). And the third is the well-known rational groups - those having irreducible characters that are all rational-valued. Though rational groups have been studied extensively in the literature (see [Gow76, FS89, Tho08]), a complete understanding is still far from reach.

One may wonder if there is an *element-level* version of Conjecture 1.2.

Question 5.2. *Let G be a finite group. Is it true that*

$$[\mathbb{Q}_{c(\mathbb{Q}(\chi(g)))} : \mathbb{Q}(\chi(g))] \leqslant \chi(1)$$

for every $\chi \in \mathrm{Irr}(G)$ and every $g \in G$.

Question 5.2 is related to a problem in algebraic number theory that, unfortunately, we do not have an answer at this time: for $z = \zeta_1 + \zeta_2 + ... + \zeta_k$ a sum of k (primitive) roots of unity (of possibly different orders), is it true that $[\mathbb{Q}_{c(z)} : \mathbb{Q}(z)] \leqslant k$, where $c(z)$ is the smallest positive integer such that $z \in \mathbb{Q}_{c(z)}$.

As promised, we now offer a connection between Conjecture 1.2 and the well-known McKay conjecture ([Nav18, Conjecture 9.1]) in the case of odd-degree characters.

As usual let $\mathrm{Irr}_{2'}(G)$ denote the set of all odd-degree irreducible characters of G. Let $P \in \mathrm{Syl}_2(G)$. The McKay conjecture asserts that the two sets $\mathrm{Irr}_{2'}(G)$ and $\mathrm{Irr}_{2'}(\mathbf{N}_G(P))$

have the same cardinality. Recall our earlier notation $\mathbf{f}(\chi) := [\mathbb{Q}_{c(\chi)} : \mathbb{Q}(\chi)]$. We have not found a counter example to the following:

Question 5.3. *Let G be a finite group and $P \in \mathrm{Syl}_2(G)$. Is it true that there always exists a bijection*

$$^* : \mathrm{Irr}_{2'}(G) \to \mathrm{Irr}_{2'}(\mathbf{N}_G(P))$$

such that

$$\frac{\mathbf{f}(\chi)}{\chi(1)} \leqslant \frac{\mathbf{f}(\chi^*)}{\chi^*(1)} \tag{5.1}$$

for every $\chi \in \mathrm{Irr}_{2'}(G)$.

Of course one can ask the same question for primes other than 2. We focus on only $p = 2$ because of the following.

Theorem 5.4. *An affirmative answer to Question 5.3 implies Conjecture 1.2 for odd-degree characters.*

Proof. Note that $\mathbf{N}_G(P)$ is solvable by Feit-Thompson's odd-order theorem. Therefore, by Theorem 2.2, $\mathbf{f}(\chi^*)$ divides $\chi^*(1)$, and the result follows. □

The additional condition (5.1) to the McKay bijection turns out to be satisfied in several cases where the bijection is known to be natural/canonical.

If G is solvable, in [Isa73, Theorem 10.9], Isaacs constructed a natural one-to-one correspondence $^* : \mathrm{Irr}_{2'}(G) \to \mathrm{Irr}_{2'}(\mathbf{N}_G(P))$. This bijection can be shown to commute with the the Galois automorphisms (in $\mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$) and hence preserve the field of values of corresponding characters. Moreover, $\chi^*(1)$ divides $\chi(1)$ (see [Tur07, Riz19] for more discussion on the degree divisibility in character correspondence). The extra condition is therefore satisfied.

Question 5.3 also has an affirmative answer for alternating and symmetric groups. Let $G = \mathsf{A}_n$ or S_n with $n \geqslant 5$. The existence of a natural McKay bijection is also well-known in this case (see [Ols76]). It is also well-known that a 2-Sylow subgroup $P \in \mathrm{Syl}_2(G)$ is self-normalizing, i.e., $P = \mathbf{N}_G(P)$. Thus all the members of $\mathrm{Irr}_{2'}(\mathbf{N}_G(P)) = \mathrm{Irr}_{2'}(P)$ are linear whose fields of values are cyclotomic. (Indeed, these linear characters are rational-values, because P/P' is elementary abelian.) The condition $\mathbf{f}(\chi)/\chi(1) \leqslant \mathbf{f}(\chi^*)/\chi^*(1)$ is then reduced to $\mathbf{f}(\chi) \leqslant \chi(1)$, which we have already established in Section 3.

Our work in Section 4 also confirms Question 5.3 in the cases $G = \mathrm{GL}_n(q)$ or $\mathrm{GU}_n(q)$ where $2 \nmid q$. As before we write $G = \mathrm{GL}_n^\epsilon(q)$ for $\epsilon = \pm$ suitably. Then a canonical McKay correspondence was constructed in [GKNT17, Theorem E]. More concretely, let

$$n = 2^{t_1} + 2^{t_2} + \cdots 2^{t_k}$$

be the 2-adic expansion of n, and $P_i \in \mathrm{Syl}_2(\mathrm{GL}^\epsilon_{2^{t_i}}(q))$, so that

$$P := P_1 \times P_2 \times \cdots \times P_k \in \mathrm{Syl}_2(G).$$

As each P_i is an irreducible subgroup of $\mathrm{GL}^\epsilon_{2^{t_i}}(q)$, we have

$$\mathbf{N}_G(P) = \mathbf{N}_{\mathrm{GL}^\epsilon_{2^{t_1}}(q)}(P_1) \times \mathbf{N}_{\mathrm{GL}^\epsilon_{2^{t_2}}(q)}(P_2) \times \cdots \times \mathbf{N}_{\mathrm{GL}^\epsilon_{2^{t_k}}(q)}(P_k).$$

It is well-known that $\mathbf{N}_{\mathrm{GL}^{\epsilon}_{2^{t_i}}(q)}(P_i) = \mathbf{O}_{2'}(\mathrm{GL}^{\epsilon}_{2^{t_i}}(q)) \times P_i$, and therefore all the odd-degree irreducible characters of $\mathbf{N}_G(P)$ are linear. Inequality (5.1) is now reduced to $\mathbf{f}(\chi) \leqslant \chi(1)$, which follows from Theorems 4.1 and 4.2.

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