Generic types and generic elements in divisible rigid groups [1]

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1 Introduction

Rigid solvable groups, in particular, free solvable groups, have rich, interesting and surprisingly ameanable algebraic geometry and model theory. The algebraic geometry of these groups was developed in papers [1] - [6], where, among other results, it was shown that these groups are equationally Noetherian, have finite Krull dimension, admit a robust Nullstellensatz. The fundamentals of the model theory were studied in detail in papers [7] - [10]. Among all rigid groups there are so-called divisible rigid groups that play the same part as divisible abelian groups in the class of torsion-free abelian groups, or the algebraically closed fields in the class of all fields. Namely, divisible m-rigid groups are precisely the existentially (algebraically) closed models in the class of all m-rigid groups (see [8]). Furthermore, the class $\mathcal{D}_m$ of all divisible m-rigid groups has a lot of nice model-theoretic properties: the theory of $\mathcal{D}_m$ is complete and has a natural recursive set of axioms, it admits elimination of quantifiers to boolean combinations of universal formulas, it is ω-stable, and allows an easy description of saturated models. In this paper we describe generic elements and generic types in divisible rigid groups.

Recall that a group G is m-rigid if it has a normal series (termed the rigid series)

$$G = \rho_1(G) > \rho_2(G) > \ldots > \rho_m(G) > \rho_{m+1}(G) = 1,$$

whose factors $\rho_i(G)/\rho_{i+1}(G)$ are abelian and do not have torsion, when viewed as right modules over $\mathbb{Z}[G/\rho_i(G)]$. Such a series, if it exists, is unique in the group, and the number m is equal precisely to the solvability class of G. A rigid group G is called *divisible*, if each element of the quotient $\rho_i(G)/\rho_{i+1}(G)$ is divisible by any non-zero element of the ring $\mathbb{Z}[G/\rho_i(G)]$, i.e., $\rho_i(G)/\rho_{i+1}(G)$ is a vector space over the ring (skew-field) of right fractions $Q(G/\rho_i(G))$ of the ring $\mathbb{Z}[G/\rho_i(G)]$. The definition is based on the fact that the ring $\mathbb{Z}[G/\rho_i(G)]$ is a right (left) Ore domain, so it embeds into its uniquely defined ring of the right fractions $Q(G/\rho_i(G))$ (see Section 2 for details). A divisible rigid group is determined uniquely, up to isomorphism, by the dimensions α_i of the vector spaces $\rho_i(G)/\rho_{i+1}(G)$, such a group is denoted by $M(\alpha_1, \ldots, \alpha_m)$. It was shown

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(see [1, 2]) that every m-rigid group is embedded independently into a suitable divisible rigid group $M(\alpha_1, \ldots, \alpha_m)$. The group $M(\alpha_1, \ldots, \alpha_m)$ is countable if $\alpha_i \leq \omega$, otherwise, its cardinality is equal to the maximal α_i. Every saturated divisible group of cardinality λ is isomorphic to the group $M(\lambda, \ldots, \lambda)$.

Let $G = M(\alpha_1, \ldots, \alpha_m)$ and a cardinal λ is strictly greater than ω and each of the α_i. Then G is elementarily embedded into the saturated model $\mathbb{G} = M(\lambda, \ldots, \lambda)$, which is viewed as a monster model for G. It is not hard to show that G has a unique generic type, i.e., a type $p \in S_1(G)$ whose Morley rank is equal to the Morley rank $\mathrm{RM}(G)$ of the group G. Respectively, an element $x \in \mathbb{G}$ which realizes in $\mathbb{G}$ the generic type p is called a *generic element*. In this paper we give a pure algebraic description of generic elements $x \in \mathbb{G}$ in terms of linear independence in vectors spaces $\rho_i(\mathbb{G})/\rho_{i+1}(\mathbb{G})$ over the ring of right fractions $Q(\mathbb{G}/\rho_i(\mathbb{G}))$, $i = 1, \ldots, m$. This also gives an a description in algebraic terms of the generic types $p \in S_1(G)$, which is not unlike of the description of the types of transcendental elements in fields.

2 Preliminaries

In this section we recall some notions and facts about rigid groups. For details we refer to papers [1, 2, 4, 8, 9].

Let G be a group, $\mathbb{Z}G$ the integral group ring of G, and T a right module over $\mathbb{Z}G$. If a tuple $(t_1, \ldots, t_n)$ of elements from T is linearly dependent over $\mathbb{Z}G$ then $t_1u_1 + \ldots, t_nu_n = 0$ for some elements $u_i = a_{i1}g_1 + \ldots + a_{is}g_s \in \mathbb{Z}G$, where $a_{ij} \in \mathbb{Z}, g_1, \ldots, g_s \in G$, $i = 1, \ldots, n, j = 1, \ldots, s$. We may assume that all the elements $g_1, \ldots, g_s$ are pair-wise distinct, and the $n \times s$ matrix $A = (a_{ij})$ has no zero columns. In this situation we write

$$(t_1, \ldots, t_n) \cdot A \cdot (g_1, \ldots, g_s)' = 0,$$

(here $(g_1, \ldots, g_s)'$ is the transpose of $(g_1, \ldots, g_s)$) and refer to $(t_1, \ldots, t_n)$ as *linearly dependent over $\mathbb{Z}G$ with matrix A.*

Recall that if an associative ring R is a right Ore domain (a domain where for any $0 \neq a, b \in R$ there are $0 \neq x, y \in R$ such that $ax = by$) then it has a unique, up to natural isomorphism, ring of right fractions $Q(R)$, which is a division ring (skew-field) containing R and where every element is of the form rs^{-1} for $r, s \in R, s \neq 0$ (see, for example, [11]). Similar result holds for left Ore domains. If R is a right, as well as left, Ore domain then $Q(R)$ is both the ring of right and left fractions of R. It follows from [14, 15] that the integral group ring $\mathbb{Z}G$ of a solvable torsion-free group G, in particular, a rigid group, is a right Ore domain. Note that in this case $\mathbb{Z}G$ is also a left Ore domain, so its ring of right fractions, which we denote by $Q(G)$, is also its ring of left fractions. Every finite set of elements $f_1, \ldots, f_n$ from $Q(G)$ has a right common denominator $0 \neq u \in \mathbb{Z}G$, i.e., each f_i can be presented in the form $u_i u^{-1}$ for some $u_i \in \mathbb{Z}G$, as well as a left common denominator. It follows that if in a vector space over $Q(G)$ some non-trivial linear combination $v_1 f_1 + \ldots + v_n f_n = 0$ of elements with coefficients $f_i \in Q(G)$ is equal to zero, then $v_1(f_1 u) + \ldots + v_n(f_n u) = 0$ is a non-trivial linear combination of the same elements with coefficients $f_i u \in \mathbb{Z}G$ which is also equal to zero.

Let $G \leqslant H$ be m-rigid groups. Then the rigid series of G can be obtained from the rigid series of H by intersection of each member with G, i.e., $\rho_i(G) = \rho_i(H) \cap G$. Hence the group $G/\rho_i(G)$ embeds into the group $H/\rho_i(H)$, so the ring $\mathbb{Z}[G/\rho_i(G)]$ embeds into the ring $\mathbb{Z}[H/\rho_i(H)]$, and the module $\rho_i(G)/\rho_{i+1}(G)$ over the ring $\mathbb{Z}[G/\rho_i(G)]$ embeds into the module $\rho_i(H)/\rho_{i+1}(H)$ over the ring $\mathbb{Z}[H/\rho_i(H)]$. Moreover, the field $Q(G/\rho_i(G))$ embeds into the field $Q(H/\rho_i(H))$.

The following notion is crucial in the study of model theory and algebraic geometry of rigid groups.

Definition 1. *Let $G \leqslant H$ be m-rigid groups. In the notation above, the subgroup G is termed independent in H (or the embedding $G \to H$ is independent), if any tuple of elements in $\rho_i(G)/\rho_{i+1}(G)$, linearly independent over the ring $\mathbb{Z}[G/\rho_i(G)]$, is linearly independent (as a tuple in $\rho_i(H)/\rho_{i+1}(H)$) over the larger ring $\mathbb{Z}[H/\rho_i(H)]$.*

We mentioned in the introduction that any m-rigid group embed independently into a divisible m-rigid group and any divisible rigid groups is isomorphic to the divisible group $G = M(\alpha_1, \ldots, \alpha_m)$ for some cardinals $\alpha_1, \ldots, \alpha_m$. In was shown in [8] that for divisible rigid groups an embedding is independent if and only if it is elementary.

Now we briefly describe the structure of divisible m-rigid groups $G = M(\alpha_1, \ldots, \alpha_m)$, where $\alpha_1, \ldots, \alpha_m$ are arbitrary cardinals. We build $M(\alpha_1, \ldots, \alpha_m)$ by induction on m. For $m = 1$ the group $M(\alpha_1)$ is a direct sum of α_1 copies of the additive group of rational numbers $\mathbb{Q}^+$. Assume now that the group $B = M(\alpha_1, \ldots, \alpha_{m-1})$ is already constructed. Embed the integral group ring $\mathbb{Z}B$ in its field of fractions $Q(B)$. Let T be a right vector space over $Q(B)$ of dimension α_m. Put

$$M(\alpha_1, \ldots, \alpha_m) = \begin{pmatrix} B & 0 \\ T & 1 \end{pmatrix}.$$

The group $G = M(\alpha_1, \ldots, \alpha_m)$ splits into an iterated semidirect product of its abelian subgroups $G = G_1 \ldots G_m = (\ldots (G_1 \ltimes G_2) \ltimes \ldots \ltimes G_{m-1}) \ltimes G_m$, such that G_i normalises G_j for $i < j$ and $\rho_i(G) = G_i \dots G_m$ for any $i = 1, \dots, m$, in particular, $G_i \simeq \rho_i(G)/\rho_{i+1}(G)$. In this case the subgroup G_i is equal to the centralizer of any its non-trivial element. Furthermore, any other such splitting of G can be obtained from a given one by a conjugation in G. In this paper we refer to these splittings as the *semidirect splittings* (or just *splittings*) of divisible groups. Note, that every element $x \in G$ has a unique decomposition $x = x_1 \dots x_m$ with respect to the splitting $G = G_1 \dots G_m$, where $x_i \in G_i$ for every i.

The following result is straightforward.

Lemma 1. *Let $G = G_1 \dots G_m$ be a semidirect splitting of a divisible m-rigid group. For every $i = 1, \dots, m$ fix a non-trivial element $g_i \in G_i$. Then the subgroup G_i is definable in G by the equation $[x, g_i] = 1$, and a subgroup $G_1 \dots G_i$ is definable in G by the positive existential formula*

$$\exists x_1 \dots \exists x_i(([x_1, g_1] = 1) \wedge \dots \wedge ([x_i, g_i] = 1) \wedge (x = x_1 \dots x_i)).$$

3 The main result

Let a group $G = M(\alpha_1, \dots, \alpha_m)$ be elementary (independently) embedded into a divisible m-rigid group $\mathbb{G} = M(\lambda, \dots, \lambda)$, where λ is an uncountable cardinal greater than all α_i. Note that in this case $\mathbb{G}$ is a Monster extension of the group G in the terminology of [12]. Indeed, the theory $\mathcal{T}_m$ (for a fixed m) of all divisible m-rigid groups is ω-stable (and complete), the group $\mathbb{G}$ has cardinality λ, is saturated, and every divisible group of cardinality $< \lambda$ is elementary embedded into $\mathbb{G}$ (see [9] for these results). Furthermore, any monster model of $\mathcal{T}_m$ of cardinality λ is isomorphic to $\mathbb{G}$. To the rest of this paper we fix such an embedding $G \prec \mathbb{G}$.

Note that for any semidirect spltting $G_1 \dots G_m$ of the group G there exists a unique "compatible"semidirect splitting $\mathbb{G}_1 \dots \mathbb{G}_m$ of $\mathbb{G}$, in which $\mathbb{G}_i$ is completely determined by G_i as the centralizer in $\mathbb{G}$ of the subgroup G_i, or, equivalently, the centralizer of any fixed non-trivial element $g_i \in G_i$. We refer to the splitting $\mathbb{G} = \mathbb{G}_1 \dots \mathbb{G}_m$ as *induced* by the splitting $G = G_1 \dots G_m$.

Definition 2. *Let $G \prec \mathbb{G}$.*

1) Fix a semidirect splitting

$$G = G_1 \dots G_m \tag{1}$$

of G, and consider the induced splitting $\mathbb{G} = \mathbb{G}_1 \dots \mathbb{G}_m$ of $\mathbb{G}$. An element $x \in \mathbb{G}$ is termed independent of G *relative to the splitting (1) if the induced decomposition $x = x_1 \dots x_m$ of x satisfies the following condition (Ind) for every i:*

(Ind) for any subset A of G_i which is linearly independent over $\mathbb{Z}[G_1 \dots G_{i-1}]$ the set $\{x_i\} \cup A$ in $\mathbb{G}_i$ is linearly independent over $\mathbb{Z}[\mathbb{G}_1 \dots \mathbb{G}_{i-1}]$.

2) *An element $x \in \mathbb{G}$ is termed* independent of G *if it is independent of G relative to any semidirect slitting of G.*

Recall that, in the notation of Definition 2, the following holds:

$$G_i \cong \rho_i(G)/\rho_{i+1}(G), \ \mathbb{G}_i \cong \rho_i(\mathbb{G})/\rho_{i+1}(\mathbb{G}),$$

$$\rho_i(G) = G_i \dots G_m, \ \rho_i(\mathbb{G}) = \mathbb{G}_i \dots \mathbb{G}_m,$$

$$G_1 \dots G_{i-1} \cong G/\rho_i(G), \ \mathbb{G}_1 \dots \mathbb{G}_{i-1} \cong \mathbb{G}/\rho_i(\mathbb{G}), \ x_i \in \rho_i(G).$$

Therefore, the condition (Ind) above can be stated as follows:

(Ind') Let E_i be a maximal linearly independent over $\mathbb{Z}[G/\rho_i(G)]$ set of elements from $\rho_i(G)/\rho_{i+1}(G)$. Then the set $E_i \cup \{x_i \cdot \rho_{i+1}(\mathbb{G})\}$ is linearly independent over $\mathbb{Z}[\mathbb{G}/\rho_i(\mathbb{G})]$.

Lemma 2. *For the groups $G < \mathbb{G}$ the following holds:*

1) *If an element $x \in \mathbb{G}$ is independent of G relative to some semidirect splitting of G then it is independent of G relative to any semidierct splitting of G, i.e., in this case x is independent of G.*

2) *If $x \in \mathbb{G}$ is independent of G and $g \in G$, then the elements xg and gx are also independent of G.*

3) *The set of all independent of G elements in $\mathbb{G}$ has cardinality λ, which is cardinality of $\mathbb{G}$.*

Proof. 1) Let $G = G_1 \dots G_m$ be a semidirect splitting of G and $\mathbb{G} = \mathbb{G}_1 \dots \mathbb{G}_m$ the induced splitting of $\mathbb{G}$. Suppose $x \in \mathbb{G}$ is independent of G relative to the splitting above and $x = x_1 \dots x_m$ is its corresponding decomposition. Assume now that $G = G'_1 \dots G'_m$ is another semidirect splitting of G, and $\mathbb{G} = \mathbb{G}'_1 \dots \mathbb{G}'_m$, $x = x'_1 \dots x'_m$ are the induced splittings of $\mathbb{G}$ and x. As was mentioned above there exists an element $g \in G$, such that $\mathbb{G}'_i = g^{-1}\mathbb{G}_i g$, hence there are elements $z_i \in \mathbb{G}_i$ for which $x'_i = g^{-1} z_i g$ for each i. Conjugation by g in the quotient $\rho_i(\mathbb{G})/\rho_{i+1}(\mathbb{G})$ is precisely the same as the multiplication by the element $g \cdot \rho_{i+1}(\mathbb{G})$ in the right module $\rho_i(\mathbb{G})/\rho_{i+1}(\mathbb{G})$. The element $g \cdot \rho_{i+1}(\mathbb{G})$ is invertible in $\mathbb{Z}[\mathbb{G}/\rho_i(\mathbb{G})]$, therefore for any maximal linearly independent over $\mathbb{Z}[G/\rho_i(G)]$ subset E_i of $\rho_i(G)/\rho_{i+1}(G)$ the set $E_i \cup \{x'_i \cdot \rho_{i+1}(\mathbb{G})\}$ is linearly independent over $\mathbb{Z}[\mathbb{G}/\rho_i(\mathbb{G})]$ if and only if the set $E_i \cup \{z_i \cdot \rho_{i+1}(\mathbb{G})\}$ is linearly independent over $\mathbb{Z}[\mathbb{G}/\rho_i(\mathbb{G})]$.

Clearly, $gxg^{-1} = z_1 \dots z_m$ is the induced splitting of the element gxg^{-1} relative to the splitting $\mathbb{G} = \mathbb{G}_1 \dots \mathbb{G}_m$. Let $g = g_1 \dots g_m$ be the induced splitting of g relative to the initial splitting $G = G_1 \dots G_m$, so $g_i \in G_i$. Denote $\overline{x}_i = x_1 \dots x_{i-1}$ and $\overline{g}_i = g_1 \dots g_{i-1}$ (here $x_1 = g_1 = 1$), $i = 1, \dots, m+1$. Now put

$$u_i = \overline{g}_i \overline{x}_i^{-1} g_i \overline{x}_{i+1} \overline{g}_{i+1}^{-1}, \quad i = 1, \dots, m.$$

Note, first, that $u_i \in \mathbb{G}_i$, since $\mathbb{G}_1, \dots, \mathbb{G}_i$ normalise $\mathbb{G}_i$. Secondly, after performing straightforward cancellation, one can see that

$$u_1 \dots u_m = (g_1 \dots g_m)(x_1 \dots x_m)(g_1 \dots g_m)^{-1} = gxg^{-1},$$

so $u_1 \dots u_m = gxg^{-1}$ is an induced decomposition of gxg^{-1} relative to the splitting $\mathbb{G} = \mathbb{G}_1 \dots \mathbb{G}_m$. From the uniqueness of such decompositions one gets $z_i = u_i, i = 1, \dots, m$.

In the language of modules the equality $z_i = \overline{g}_i \overline{x}_i^{-1} g_i \overline{x}_{i+1} \overline{g}_{i+1}^{-1}$ can be written as

$$z_i = g_i \overline{x}_i \overline{g}_i^{-1} + x_i \overline{g}_i^{-1} - g_i \overline{g}_i^{-1}, \quad i = 1, \dots, m. \tag{2}$$

Note, that g_i is a linear combination with coefficients in the skew-field $Q(G/\rho_i(G))$ of some finite subset of elements from E_i. It follows from 2 that z_i is a linear combination with coefficients in the skew-field $Q(\mathbb{G}/\rho_i(\mathbb{G}))$ of some finite subset of elements from $E_i \cup \{x_i \rho_{i+1}(\mathbb{G})\}$. Since the latter is linearly independent over $Q(\mathbb{G}/\rho_i(\mathbb{G}))$ the set $E_i \cup \{z_i \rho_{i+1}(\mathbb{G})\}$ is also linearly independent over $Q(\mathbb{G}/\rho_i(\mathbb{G}))$, hence over $\mathbb{Z}(\mathbb{G}/\rho_i(\mathbb{G}))$, as claimed.

2) It suffices to note that in the language of modules the i's component of the induced decomposition of xg relative to the splitting $\mathbb{G} = \mathbb{G}_1 \dots \mathbb{G}_m$ is equal to $x_i \overline{g}_i + g_i$, and the i's component of gx is equal to $g_i \overline{x}_i + x_i$.

3) Let E_i be a basis of the vector space G_i, over the skew-field $Q(G/G_i)$. The cardinality of E_i is equal to $\alpha_i < \lambda_i$. Since the embedding G into $\mathbb{G}$ is independent, the image of E_i in $\rho_i(\mathbb{G})/\rho_{i+1}(\mathbb{G})$ is linearly independent over $Q(\mathbb{G}/\mathbb{G}_i)$. The co-dimension of the subspace $\mathbb{G}_i$ over $Q(\mathbb{G}/\mathbb{G}_i)$ in $\rho_i(\mathbb{G})/\rho_{i+1}(\mathbb{G})$ is equal to λ, as well as the co-dimension of the subspace $E_i \cdot Q(\mathbb{G})$. Therefore, there are λ possibilities for any component x_i of an independent from G element $x \in \mathbb{G}$, hence λ possibilities for the element x itself. This proves the lemma. □

Lemma 3. *Let $G \prec \mathbb{G}$. If $x, y \in \mathbb{G}$ are such that x is independent of G and y is not, then there exists an $\exists$-formula $\Phi(z)$ with one free variable z and parameters from G which holds in $\mathbb{G}$ on y but does not hold in $\mathbb{G}$ on x.*

Proof. Consider a semidirect splitting $G = G_1 \dots G_m$ and the induced splitting $\mathbb{G} = \mathbb{G}_1 \dots \mathbb{G}_m$. Let $x = x_1 \dots x_m$, $y = y_1 \dots y_m$ be the induced decompositions of x and y. Since the element y is not independent of G, then for some i in G_i there exists a linear independent over $\mathbb{Z}[G_1 \dots G_{i-1}]$ tuple of elements $(e_1, \dots, e_n)$ such that for the tuple $(e_1, \dots, e_n, y_i)$ there is their linear combination over $\mathbb{Z}[\mathbb{G}_1 \dots \mathbb{G}_{i-1}]$ with matrix A of size $(n+1) \times s$. This can be expressed by a formula:

$$\exists z_1 \dots \exists z_m \exists v_1 \dots \exists v_s ((z_1 \in G_1) \wedge \dots \wedge (z_m \in G_m) \wedge (z = z_1 \dots z_m) \\ \wedge (v_1 \in G_1 \dots G_{i-1}) \wedge \dots \wedge (v_s \in G_1 \dots G_{i-1}) \bigwedge_{i \neq j} (v_i \neq v_j) \\ \wedge ((e_1, \dots, e_n, z_i) \cdot A \cdot (v_1, \dots, v_s)' = 0)),$$

where by Lemma 1 the conditions of the type $z_i \in \mathbb{G}_i$ are expressible by some $\exists$-formulas with parameters from G. This allows one to construct the required $\exists$-formula $\Phi(z)$. This proves the lemma.

□

Lemma 4. *Let $G \prec \mathbb{G}$. Any two independent from G elements in $\mathbb{G}$ are conjugated by an G-automorphism of the group $\mathbb{G}$.*

Proof. Let $x, y \in \mathbb{G}$ be independent from G elements of $\mathbb{G}$. Consider a semidirect splitting $G = G_1 \dots G_m$ and the induced splitting $\mathbb{G} = \mathbb{G}_1 \dots \mathbb{G}_m$ with the corresponding decompositions $x = x_1 \dots x_m$, $y = y_1 \dots y_m$. Recall that G_i can be viewed as a right vector space over the skew-field over $Q(G_1 \dots G_{i-1})$. Fix a basis E_i in this space. Then the sets $E_i \cup \{x_i\}$ and $E_i \cup \{y_i\}$ are linearly independent over $Q(\mathbb{G}_1 \dots \mathbb{G}_{i-1})$. There are some sets E_i' and E_i'' that extend the sets above up to some bases in $\mathbb{G}_i$. The identity map $E_i \to E_i$, an arbitrary bijection $E_i' \to E_i''$, and the map $x_i \to y_i$ give a bijection $\varphi_i : E_i \cup \{x_i\} \cup E_i' \to E_i \cup \{y_i\} \cup E_i''$. Now the tuple of maps $(\varphi_1, \dots, \varphi_m)$ gives rise to a uniquely determined G-automorphism of the group $\mathbb{G}$, which preserves the splitting $\mathbb{G} = \mathbb{G}_1 \dots \mathbb{G}_m$ and maps x to y. This proves the lemma.

□

Now we can state and prove the main result of the paper.

Theorem 1. *An element $x \in \mathbb{G}$ is generic over G if and only if x is independent from G.*

Proof. It follows from Lemma 4 that any two independent from G elements in $\mathbb{G}$ are conjugated by a G-automorphism, so they have the same types over G. Consider the action of G on the set of 1-types $S_1(G)$ in G by the right (or left) multiplication of the variable. By Lemma 2 item 2) G stabilizers $\mathrm{tp}(x/G)$, provided x is independent from G element. In this case by Lemma 7.2.3 from [12] $\mathrm{tp}(x/G)$ is a generic type and x a generic element. On the other hand if x is not indpendent of G, then by Lemma 3 the type $\mathrm{tp}(x/G)$ is not generic, so x is not a generic element. This proves the theorem.

□

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