RECONSTRUCTING MAPS OUT OF GROUPS

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Abstract. We show that, in many situations, a homeomorphism f of a manifold M may be recovered from the (marked) isomorphism class of a finitely generated group of homeomorphisms containing f. As an application, we relate the notions of *critical regularity* and of *differentiable rigidity*, give examples of groups of diffeomorphisms of 1-manifolds with strong differential rigidity, and in so doing give an independent, short proof of a recent result of Kim and Koberda that there exist finitely generated groups of C^α diffeomorphisms of a 1-manifold M, not embeddable into $\mathrm{Diff}^\beta(M)$ for any $\beta > \alpha \geqslant 1$.

1. Introduction

1.1. Motivation.

It is a classical and fundamental problem to describe to what extent the algebraic structure of a group determines the topological spaces on which the group can act, or constrains the regularity of those actions. For example, Whittaker [25] showed that connected, compact topological manifolds can be completely recovered from the algebraic structure of their groups of homeomorphisms: the existence of an isomorphism between $\mathrm{Homeo}(M)$ and $\mathrm{Homeo}(N)$ implies that $M = N$ and the isomorphism is an inner automorphism. This was generalized by Rubin to homeomorphism groups of other topological spaces, and Filipkiewicz [8] improved this to the groups of C^r diffeomorphisms of manifolds, showing that the algebraic structure of $\mathrm{Diff}^r(M)$ can even detect the regularity r.

All of these could be considered *recognition* or *reconstruction* theorems, showing that spaces can be recognized by their transformation groups. Another approach to this family of problems is to relate the complexity of a topological space to the algebraic complexity of (finitely generated) subgroups of its homeomorphism or diffeomorphism groups. This is, in some sense the "generalized Zimmer program," Zimmer's conjecture being that groups of high algebraic complexity, namely lattices of higher rank, cannot act by smooth or volume-preserving diffeomorphisms on low-dimensional manifolds.

This broad line of investigation has been particularly successful in dimension one. Here we know several purely algebraic conditions that prevent finitely generated groups from acting on one-manifolds with a given regularity. In the C^0 setting, this is the presence of left- or circular-orderability. In class C^1, many obstructions come from the *Thurston stability theorem*, while in higher regularity this program can be traced back all the way to Denjoy's work on rotations of the circle. To give some more recent examples, Navas [17] showed that Kazhdan's property T is an algebraic obstruction to acting on the circle with C^α regularity for $\alpha > 3/2$, and in [18] he showed that having intermediate growth is an obstruction to acting on the interval with regularity $\alpha > 1$. Castro–Jorquera–Navas [5] gave examples of nilpotent groups with sharp bounds on the Hölder regularity of their actions on the closed interval I; see also Jorquera–Navas–Rivas [13]. More recently, Kim and Koberda [14] gave examples of finitely generated subgroups of "critical regularity $\alpha \geqslant 1$," embeddable in $\mathrm{Diff}^\alpha(M)$ but not in $\mathrm{Diff}^\beta(M)$ for any $\beta > \alpha$ when $M = S^1$ or I.

1.2. **Results.** Our aim here is to contribute both to the general program of recognition and reconstruction, and to the problem of restricting regularity, with a specific application to the one-dimensional case.

We give general criteria for a group $\Gamma \subset \mathrm{Homeo}(X)$ of homeomorphisms of a space X to "reconstruct" or "recognize" other homeomorphisms of X purely through algebraic relations (Theorem 1.1). We also construct groups acting on 1-manifolds with a strong *differentiable rigidity* property (Theorem 1.5 and following), by using recent work of Bonatti–Monteverde–Navas–Rivas [1] and a precise version of the Sternberg linearization theorem. Building on all this, we deduce the existence of groups with *critical regularity* (Theorem 1.4). This gives an alternative short proof (and some generalization) of the critical regularity result of Kim and Koberda mentioned above. However, their techniques go further in a different direction than ours: they also give groups whose critical regularity passes to finite index subgroups, simple groups of given regularity, and define dynamical notions "δ-fast" and "λ-expansive" that are useful for explicitly constructing groups of specified regularity.

The remainder of this introductory section is devoted to giving precise statements of our results.

First result: map recognition. In general, if G is a group, $\Gamma \subset G$ is a subgroup and $g \in G$, we will say that Γ *recognizes* g, if for every element $h \in G$, the existence of a group isomorphism

$$\phi\colon \langle \Gamma, g\rangle \to \langle \Gamma, h\rangle$$

with $\phi_{|\Gamma} = \mathrm{id}_\Gamma$ and $\phi(g) = h$ implies that $h = g$. Essentially, Γ recognizes g if some equalities involving g and the elements of Γ characterize g among G. Note that if $\Gamma \subset \Gamma'$ then every element recognized by Γ is still recognized by Γ', and that every subgroup Γ recognizes at least its own elements. In this article we will consider the case when $G = \mathrm{Homeo}(X)$, where X is a topological space. If $\mathcal{C}$ is a subset of $\mathrm{Homeo}(X)$ we will say that Γ *recognizes maps in* $\mathcal{C}$, if it recognizes every element of $\mathcal{C}$. Typically, we will be interested in finitely generated groups Γ recognizing large classes of elements in $\mathrm{Homeo}(X)$.

The following theorem, proved in Section 2, shows that examples of such groups abound. We introduce some terminology needed for the statement. Recall that, for a group $\Gamma \subset \mathrm{Homeo}(X)$ and $\gamma \in \Gamma$, the *support* of γ is the closure of the set $\{x \in X \mid f(x) \neq x\}$. *Non-total support* means $\mathrm{Supp}(\gamma) \neq X$. We say that Γ has *small supports everywhere*[1] if, for every nonempty open set $U \subset X$, there exists $\gamma \in \Gamma \smallsetminus \{\mathrm{id}\}$ with $\mathrm{Supp}(\gamma) \subset U$, and that Γ *has the contraction property* if, for any nonempty open set $U \subset X$, there exists $\gamma \in \Gamma$ such that $\gamma(X \smallsetminus U) \subset U$.

Theorem 1.1 (Map recognition). *Let X be a Hausdorff topological space, and* $\Gamma \subset \mathrm{Homeo}(X)$.

(1) If Γ has maps with small supports everywhere*, then Γ recognizes all maps in* $\mathrm{Homeo}(X)$.
(2) If Γ acts on X with the contraction property*, then Γ recognizes homeomorphisms of X with non-total support.*

Similar conditions have been used elsewhere in the literature. The reconstruction theorems of Whittaker, Epstein, and Rubin [7, 25, 20] all use variations on the idea of small supports. To our knowledge, the contraction property was first used (under

[1] this is called *micro-supported* in [3, 4] and closely related to Rubin's notion of *locally moving* in [21]

the more cumbersome name of "minimality and strong expansivity") in the proof by Margulis of the Tits' alternative in $\mathrm{Homeo}_+(S^1)$; see [15, 10].

We also show that Baumslag-Solitar groups give additional examples of groups with map recognition. These are needed for our applications and do not fall in the domain of Theorem 1.1.

Theorem 1.2. *The affine Baumslag-Solitar subgroup* $\mathrm{BS}(1,n) \subset \mathrm{Homeo}(\mathbb{R})$ *recognizes maps with compact support.*

Here $\mathrm{BS}(1,n)$ denotes the group generated by the maps $x \mapsto x+1$ and $x \mapsto nx$. Theorem 1.2 is proved in Section 3, where we actually prove something stronger – see Proposition 3.1. It would be interesting to find a simple and general condition that would simultaneously imply both the statements of Theorem 1.1 and Theorem 1.2.

Application: differential rigidity from critical regularity. Following Kim and Koberda [14], we will say that a group $\Gamma \subset \mathrm{Homeo}(M)$ has *critical regularity* α if it is embeddable in $\mathrm{Diff}^\delta(M)$ for all $\delta < \alpha$ but not in $\mathrm{Diff}^\beta(M)$ for any $\beta > \alpha$. We will say that a group $\Gamma \subset \mathrm{Diff}^\infty(M)$ is C^α*-rigid* if for all $\beta \geqslant \alpha$, any injective morphism $\Gamma \to \mathrm{Diff}^\beta(M)$ comes from conjugation by some element of $\mathrm{Diff}^\beta(M)$. This definition is motivated by the work of Ghys [9], whose main result gives examples of C^3-rigid subgroups of $\mathrm{Diff}^\infty(S^1)$, see Section 5.1.

In the definitions above, α is assumed to take real values, with the convention that a map $f\colon M \to M$ is of class C^α if it is $C^{\lfloor\alpha\rfloor}$ and if it is $\lfloor\alpha\rfloor$-derivatives are $(\alpha - \lfloor\alpha\rfloor)$-Hölder. However, most of our work in the 1-dimensional case actually applies to maps whose regularity is given by more general moduli of continuity. We assume Hölder regularity here only for simplicity of the statement. See Remark 4.4 below.

The following proposition illustrates that *critical regularity follows from differentiable rigidity* in a general sense: this is the guiding principle and original motivation of our work.

Proposition 1.3 (Critical regularity from differential rigidity)**.** *Let M be a manifold and $\alpha \geqslant 1$. Let $\Gamma \subset \mathrm{Diff}^\infty(M)$ be C^α-rigid, and suppose that for some nonempty open set U (possibly equal to M), Γ recognizes maps with support in U. Then for any map $f \in \mathrm{Homeo}(M)$ with support in U, and any $\beta \geqslant \alpha$, the group $\langle \Gamma, f\rangle$ admits an injective morphism to $\mathrm{Diff}^\beta(M)$ if and only if $f \in \mathrm{Diff}^\beta(M)$.*

The proof is a quick consequence of the definitions, we give it at the beginning of Section 5.

Examples of groups with differential rigidity and critical regularity. Proposition 1.3 motivates the construction of differentiably rigid groups that have the map recognition property. We will give several examples, described below, when $\dim(M) = 1$. Combined with Proposition 1.3 and variations on it, these constructions give a short proof of the following result, due to Kim and Koberda for S^1 and $[0,1]$.

Theorem 1.4 (Compare Kim–Koberda [14])**.** *For $M = S^1$, $\mathbb{R}$, or $[0,1]$, and for all $\alpha \geqslant 1$ there exist finitely generated subgroups of $\mathrm{Diff}^\infty(M)$ of critical regularity C^α.*

In fact, as we mentioned above, the statement we obtain here is valid in much more generality than regularities C^α with $\alpha \in \mathbb{R}_+$, but with more general moduli of continuity; we prove that these finer regularities are detected by the algebraic structures of the groups. This implies, in particular, that there exist uncountably many non-isomorphic finitely generated groups in each class of critical regularity in Theorem 1.4, and responds to Question 7.1(1) of [14] which asks if similar results hold in the setting of k+Lipschitz regularity. We do not seek to state Theorem 1.4 in maximal degree of generality here – see the discussions following Remark 4.4 and the proof of Proposition 1.3 in Section 5.

The groups we construct for the use of Proposition 1.3 are actually quite easy to describe. The simplest case is $M = S^1$. Let $\Gamma_T \subset \mathrm{PSL}(2,\mathbb{R})$ be a Fuchsian triangle group $(2,3,7)$, with presentation

$$\Gamma_T = \langle s, r, t \mid s^2 = r^3 = t^7 = trs = 1\rangle,$$

choose an integer $n \geqslant 2$, consider a proper interval $I \subset S^1$, and let Γ_A be a copy of an affine subgroup containing $\mathrm{BS}(1,n)$ and an extra irrational homothety $x \mapsto \mu x$, acting smoothly on S^1 by a conjugate of the affine action on I, and by the identity on $S^1 \setminus I$. Let Γ denote the group generated by Γ_T and Γ_A. While there are many choices involved in this construction, we show all resulting groups are rigid:

Theorem 1.5 (Differential rigidity on S^1)**.** *Any group Γ obtained by the construction above is C^α-rigid, for all $\alpha > 1$.*

For other 1-manifolds, the situation is more delicate. For example, no groups can act in a differentiably rigid way on the closed interval $I = [0,1]$, as one can always conjugate an action to make it infinitely tangent to a linear action at 0, "double" the interval at 0 and glue two copies of the action side by side. However, our strategy can be adapted to prove critical regularity using weaker forms of map recognition for group actions on $\mathbb{R}$ and I. On the line, one can arrive at this by lifting maps from S^1. We show:

Proposition 1.6. *Let $\widetilde{\Gamma}$ be the group of lifts to $\mathrm{Diff}(\mathbb{R})$ of elements of a group $\Gamma \subset \mathrm{Diff}^\infty(S^1)$ defined above. Then $\widetilde{\Gamma}$ is C^α-rigid, for all $\alpha > 1$. Moreover, if $\mathrm{Homeo}_+^{\mathbb{Z}}(\mathbb{R})$ denotes the set of homeomorphisms of $\mathbb{R}$ which commute with integer translations, then $\widetilde{\Gamma}$ recognizes maps of $\mathrm{Homeo}_+^{\mathbb{Z}}(\mathbb{R})$ up to integer translations.*

On the closed interval, more work is needed. Let Γ_f denote the group generated by Γ_A as above, acting smoothly on $I = [0,1]$ and conjugate on $(0,1)$ to the standard affine action, together with a non-trivial homeomorphism f with support in $(0,1)$. Then we have the following.

Theorem 1.7 (Differential rigidity on the interval)**.** *Let $\alpha > 1$ and let $\phi\colon \Gamma_f \to \mathrm{Diff}^\alpha([0,1])$ be an injective morphism. Then there exists an interval $(a,b) \subset [0,1]$, invariant under $\phi(\Gamma_f)$, and a C^α-diffeomorphism $h\colon (0,1) \to (a,b)$ conjugating $\phi(\Gamma)$ to the original action on $[0,1]$. In particular, this implies $f \in \mathrm{Diff}^\alpha([0,1])$.*

Regularity of conjugacies. A major ingredient the examples above is a result on *regularity of conjugacies* (Proposition 4.1), reminiscent of a theorem of Takens, which may be of independent interest. We show that the group Γ_A described above has the property that, if $\alpha > 1$ and $\phi\colon \Gamma_A \to \mathrm{Diff}^\alpha([0,1])$ is conjugate to the standard affine action by a homeomorphism f, then f is in fact of class C^α. This is the main content of Section 4.

Higher dimension. We hope that this application to problems of critical regularity (*via* Proposition 1.3) provides motivation to construct and study groups of diffeomorphisms of higher dimensional manifolds with differential rigidity, or that exhibit the regularity of conjugacies property of Proposition 4.1. This seems to be a challenging problem, and the situation there may be quite different. Note, for example, that Harrison [12, 11] constructed C^r diffeomorphisms of manifolds (in all dimensions $\geqslant 2$) that are not topologically conjugate to any C^s diffeomorphisms for any $s > r$.

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2. Proof of Theorem 1.1: map recognition

Statements (1) and (2) of this theorem follow the same general strategy of proof, so we treat them in parallel. Throughout this section, X denotes a Hausdorff topological space. We will suppose furthermore that X has cardinal $\geqslant 3$. Provided that there exists a group acting on it with small supports everywhere or with the contraction property this implies that X is infinite, and has no isolated points. In the case $\mathrm{Card}(X) \leqslant 2$, Theorem 1.1 is immediate.

Lemma 2.1. *Let $\Gamma \subset \mathrm{Homeo}(X)$ be a group with maps with small supports everywhere. Let $f\colon X \to X$ be a continuous map, and let $x \in X$. Then the following holds.*

(1) If $f(x) \neq x$, then for any sufficiently small neighborhood U_x of x and $\gamma \in \Gamma \smallsetminus \{\mathrm{id}\}$ with $\mathrm{Supp}(\gamma) \subset U_x$, the maps γ and f do not commute.

(2) If $x \notin \mathrm{Supp}(f)$, then for any sufficiently small neighborhood U_x of x and $\gamma \in \Gamma$ with $\mathrm{Supp}(\gamma) \subset U_x$, the maps γ and f commute.

The proof is a straightforward exercise, which we omit. The version for groups with the contraction property is more interesting:

Lemma 2.2. *Let $\Gamma \subset \mathrm{Homeo}(X)$ have the contraction property, let $f\colon X \to X$ be a continuous map, and let $x \in X$. Then the following holds.*

(1) If $f(x) \neq x$, then for any sufficiently small neighborhood U_x of x and every $\gamma \in \Gamma$ mapping $X \smallsetminus U_x$ into U_x, the maps f and $\gamma f \gamma^{-1}$ do not commute.

(2) If $x \notin \mathrm{Supp}(f)$, then for any sufficiently small neighborhood U_x of x, and every $\gamma \in \Gamma$ mapping $X \smallsetminus U_x$ into U_x, the maps f and $\gamma f \gamma^{-1}$ commute.

Proof. The second item is nearly immediate. Simply take U_x contained in $X \smallsetminus \mathrm{Supp}(f)$. Then the support of $\gamma f \gamma^{-1}$ lies in U_x, so f and $\gamma f \gamma^{-1}$ have disjoint supports, hence commute.

For the first item, suppose $f(x) \neq x$. Since X has no isolated points, there exists some other point $z \in X$ such that the set $\{x, f(x), z, f(z)\}$ has cardinality 4. Let U_x be a neighborhood of x such that $\{z, f(z)\} \cap U_x = \emptyset$ and $U_x \cap f(U_x) = \emptyset$. (Such a neighborhood exists since X is Hausdorff). Let $\gamma \in \Gamma$ satisfy $\gamma(X \smallsetminus U_x) \subset U_x$. Note that this also implies that $\gamma^{-1}(X \smallsetminus U_x) \subset U_x$. Thus $f(z) = f\gamma^{-1}(\gamma z) \notin U_x$ so $\gamma f(z) \in U_x$, from which it follows that $f \circ (\gamma f \gamma^{-1})(\gamma z) \in f(U_x)$. On the other hand, we have $\gamma z \in U_x$, and thus $f(\gamma z) \notin U_x$, so $\gamma^{-1} f(\gamma z) \in U_x$, thus $f\gamma^{-1} f(\gamma z) \notin U_x$ and finally $(\gamma f \gamma^{-1}) \circ f(\gamma z) \in U_x$. Since $U_x \cap f(U_x) = \emptyset$ the computation above shows that the maps $f \circ (\gamma f \gamma^{-1})$ and $(\gamma f \gamma^{-1}) \circ f$ differ; this proves the first item. □

The lemma above will allow us to reconstruct maps, first by recovering their support.

Lemma 2.3. *Let* $\Gamma \subset \mathrm{Homeo}(X)$ *be a subgroup, either with small supports everywhere, or with the contracting property. Let* $f, h \in \mathrm{Homeo}(M)$ *be any homeomorphisms. Suppose that there exists a group isomorphism*

$$\phi\colon \langle \Gamma, f\rangle \to \langle \Gamma, h\rangle$$

such that $\phi_{|\Gamma} = \mathrm{id}_\Gamma$ *and* $\phi(f) = h$. *Then* $\mathrm{Supp}(f) = \mathrm{Supp}(h)$.

Proof. Note that if $f = \mathrm{id}_X$, or more generally if $f \in \Gamma$, then the condition $\phi_{|\Gamma} = \mathrm{id}_\Gamma$ implies $f = h$. So now we will suppose $f \neq \mathrm{id}_X$. Let $x \in X$ be such that $f(x) \neq x$. Suppose for contradiction that $x \notin \mathrm{Supp}(h)$.

Suppose first that Γ has maps with small supports everywhere. We use Lemma 2.1. Let U_x be a small neighborhood of x and let $\gamma \in \Gamma \smallsetminus \{\mathrm{id}_X\}$ with support in U_x. Then f does not commute with γ, while h commutes with γ: this contradicts that ϕ is an isomorphism. If instead Γ is contracting, we use Lemma 2.2: let U_x be a small enough neighborhood of x and let $\gamma \in \Gamma$ be an element mapping $X \smallsetminus U_x$ inside U_x. Then f and $\gamma f \gamma^{-1}$ do not commute, while h and $\gamma h \gamma^{-1}$ commute: this again contradicts the existence of ϕ.

Hence, we have proved the inclusion $\{x \in X \mid f(x) \neq x\} \subset \mathrm{Supp}(h)$. Taking closures, this implies $\mathrm{Supp}(f) \subset \mathrm{Supp}(h)$. The reverse inclusion follows since the roles of f and h are symmetric. □

We now conclude the proof of Theorem 1.1. Let X be a Hausdorff topological space, and, as a first case, assume $\Gamma \subset \mathrm{Homeo}(X)$ is a group with small supports everywhere. Let $f, h \in \mathrm{Homeo}(X)$ be any two homeomorphisms, and suppose there exists a group isomorphism ϕ as in the statement of Theorem 1.1. Suppose for contradiction that there exists $x \in X$ such that $h(x) \notin \{x, f(x)\}$. Let U_x be a neighborhood of x such that $h(x)$ is not in $U_x \cup f(U_x)$, and let $\gamma \neq \mathrm{id}_X$ be an element of Γ with support in U_x. Up to replacing x with another point in U_x, we may further suppose that $x \in \mathrm{Supp}(\gamma)$. Then the commutator map $[f^{-1}, \gamma] = f \circ \gamma^{-1} \circ f^{-1} \circ \gamma$ has support in $U_x \cup f(U_x)$, while $h(x) \in \mathrm{Supp}([h^{-1}, \gamma])$. Hence, the maps $f_2 = [f^{-1}, \gamma]$ and $h_2 = [h^{-1}, \gamma]$ have distinct supports. On the other hand, the isomorphism ϕ restricts to an isomorphism $\phi_2\colon \langle \Gamma, f_2\rangle \to \langle \Gamma, h_2\rangle$ with the same properties, and this gives a contradiction with Lemma 2.3. Thus, for every x we have $h(x) \in \{x, f(x)\}$, and symmetrically we have $f(x) \in \{x, h(x)\}$. This implies $h = f$.

Finally, suppose instead that Γ is contracting, let $f \in \mathrm{Homeo}(X)$ be a map with non-total support, let h be any map, and suppose there exists a group isomorphism ϕ as above. By Lemma 2.3 we have $\mathrm{Supp}(f) = \mathrm{Supp}(h)$, in particular h also has non-total support and f and h again play symmetric roles. For contradiction suppose there exists $x \in X$ such that $h(x) \notin \{x, f(x)\}$. Let $y \in X$ be a point which is not in $\mathrm{Supp}(f)$, and distinct from $x, f(x), h(x)$. As X is Hausdorff, there exist a neighborhood U_x of x, and a neighborhood U_y of y, such that the sets $h(U_x)$, $U_x \cup f(U_x)$ and U_y are pairwise disjoint, and such that $U_y \cap \mathrm{Supp}(f) = \emptyset$. Let $\gamma_x, \gamma_y \in \Gamma$ be such that $\gamma_x(X \smallsetminus U_x) \subset U_x$ and $\gamma_y(X \smallsetminus U_y) \subset U_y$. The map $\gamma_y^{-1} \circ f \circ \gamma_y$ has support in U_y, so the map $f_3 = \gamma_x^{-1}\gamma_y^{-1} f \gamma_y \gamma_x$ has support in U_x. The same holds for the map $h_3 = \gamma_x^{-1}\gamma_y^{-1} h \gamma_y \gamma_x$. It follows that the map $f_4 = [f^{-1}, f_3]$ has support in $U_x \cup f(U_x)$, while the map $h_4 = [h^{-1}, h_3]$ has support in $U_x \cup h(U_x)$. Also, h_4 is not the identity in $h(U_x)$, simply because h_3 is non-trivial. On the other hand, f_4 acts trivially on $h(U_x)$, so h_4 and f_4 have different supports. However, ϕ restricts to a group isomorphism $\langle \Gamma, f_4\rangle \to \langle \Gamma, h_4\rangle$, thus Lemma 2.3 yields a contradiction, as in the preceding case. □

3. Proof of Theorem 1.2: maps recognized by $\mathrm{BS}(1,n)$

For $2 \leqslant n \in \mathbb{N}$, let $\mathrm{BS}(1,n) = \langle a, b \mid aba^{-1} = b^n \rangle$ denote the Baumslag–Solitar group, with its standard affine action on the real line, defined by $a\colon x \mapsto nx$ and $b\colon x \mapsto x+1$.

This section is devoted to the proof of the following stronger version of Theorem 1.2. Here we require only a morphism, not an isomorphism between groups. We will need to use this weaker hypothesis in our discussion of an analog of differential rigidity on the closed interval in Section 6. Theorem 1.2 follows immediately from the statement below by taking ϕ to be an isomorphism, and applying the result also to ϕ^{-1}.

Proposition 3.1. *Let $f, h \in \mathrm{Homeo}_+(\mathbb{R})$, and suppose f has compact support. Suppose there is a (not necessarily injective) morphism $\phi\colon \langle \mathrm{BS}(1,n), f\rangle \to \langle \mathrm{BS}(1,n), h\rangle$ restricting to the identity on $\mathrm{BS}(1,n)$ and mapping f to h. Then either h is a translation, or $h = f$.*

In the statement, $\langle \mathrm{BS}(1,n), f\rangle \subset \mathrm{Homeo}(\mathbb{R})$ denotes the group generated by $\mathrm{BS}(1,n)$ and f; the same applies to h. The proof of Proposition 3.1, like that of Theorem 1.1, is through a careful study of the supports of (non)-commuting elements, and in particular, the study of which conjugates of f commute. With this idea we cannot rule out the possibility that h is a translation, for translations commute with all their conjugates in $\mathrm{Aff}(\mathbb{R})$. Our main technical tool is the following.

Proposition 3.2. *Let $h \in \mathrm{Homeo}_+(\mathbb{R})$ be different from a translation. Then the following statements are equivalent:*

1. *h has compact support.*
2. *There exists a compact set $K \subset \mathbb{R}$ such that for every element $u \in \mathrm{BS}(1,n)$ with $u(K) \cap K = \emptyset$, we have $[h, uhu^{-1}] = 1$; and for every element $\gamma \in \mathrm{BS}(1,n)$, the commutator $[h,\gamma]$ also satisfies the same hypothesis: there exists a compact K' such that for all $u \in \mathrm{BS}(1,n)$ with $u(K') \cap K' = \emptyset$, we have $[[h,\gamma], u[h,\gamma]u^{-1}] = 1$.*

In the second point of the statement above the set K' depends on γ.

Before embarking on the proof, we begin with an easy and useful lemma. If $u, v \in \mathrm{Homeo}_+(\mathbb{R})$ we write $u < v$ if for all $x \in \mathbb{R}$, $u(x) < v(x)$. We write $u \leqslant v$ if for all $x \in \mathbb{R}$, $u(x) \leqslant v(x)$.

Lemma 3.3. *Let $u, v \in \mathrm{Homeo}_+(\mathbb{R})$ be commuting maps, each without fixed points. Suppose $u < v$ and there exists $w \in \mathrm{Homeo}_+(\mathbb{R})$ with $wuw^{-1} = v$. Then w has fixed points.*

Proof. Up to replacing u, v and w with their inverses and switching the role of u and v, we may assume without loss of generality that $\mathrm{id} < u < v$. Since v is fixed point free, up to conjugacy we may further assume $v(x) = x+1$. Since u commutes with v, it is determined by its restriction to the compact set $[0,1]$, hence there exists $\varepsilon > 0$ such that for all $x \in \mathbb{R}$, we have $u(x) \leqslant x + 1 - \varepsilon$. By a simple induction this implies that, for any integer $N \geqslant 1$, we have $u^N(x) \leqslant x + N - N\varepsilon$ and $u^{-N}(x) \geqslant x - N + N\varepsilon$.

Now let $x \in \mathbb{R}$ be any point. If $w(x) < x$ then take $N \geqslant 1$ such that $w(x) - x + N\varepsilon > 0$, and set $y = u^N(x)$. Then we have

$$w(y) = v^N w(x) = w(x) + N \geqslant w(x) - x + N\varepsilon + u^N(x) > y.$$

Hence, $w(x) < x$ and $w(y) > y$: this implies that w has at least one fixed point. If instead $w(x) > x$, take $N \geqslant 1$ with $x - w(x) - N\varepsilon > 0$; the same reasoning shows that $w(u^{-N})(x) < u^{-N}(x)$, also implying w has a fixed point. □

Going forward, we denote by A the abelian subgroup $\mathbb{Z}[1/n] \subset \mathrm{BS}(1,n)$ consisting of translations, *i.e.*, the normal subgroup generated by b. For $t \in \mathbb{R}$, let $\tau_t \colon \mathbb{R} \to \mathbb{R}$ denote the translation $x \mapsto x + t$. We note the following standard fact.

Observation 3.4. *The centralizer of A in $\mathrm{Homeo}_+(\mathbb{R})$ is the translation subgroup. Indeed, this is true when A is replaced with any dense subgroup of translations.*

Proof of Proposition 3.2. The implication (1) ⇒ (2) is immediate, simply take $K = \mathrm{Supp}(h)$, and, for any $\gamma \in \mathrm{BS}(1,n)$, take $K' = \mathrm{Supp}([h,\gamma])$. So we need only prove the converse. The proof has three preliminary steps. We state these as Lemmas since we will later apply them to a commutator involving h, rather than h.

Lemma 3.5. *Suppose $h \in \mathrm{Homeo}_+(\mathbb{R})$, and $K \subset \mathbb{R}$ is a compact set such that $[h, uhu^{-1}] = 1$ for each u where $u(K) \cap K = \emptyset$. If the germ of h at either $+\infty$ or $-\infty$ is a nontrivial translation, then h is a translation.*

Proof. Suppose that the germ at $+\infty$ agrees with that of translation by some real number $t_0 \neq 0$. (The case for the germ at $-\infty$ is exactly the same.) By Observation 3.4, to show that h is a translation it suffices to show that h commutes with arbitrarily small translations.

Let $x \in \mathbb{R}$ and $k \geqslant 0$. By hypothesis, h commutes with $a^{-k}b^{-N}hb^Na^k$ provided N is large enough. Also, for $N > 0$ large enough, we have

$$a^{-k}b^{-N}hb^Na^k(x) = x + \frac{t_0}{n^k}$$

and

$$a^{-k}b^{-N}hb^Na^k(h(x)) = h(x) + \frac{t_0}{n^k}.$$

This yields $h(x) + \frac{t_0}{n^k} = h(x + \frac{t_0}{n^k})$ and we are done. □

Lemma 3.6. *Suppose $h \in \mathrm{Homeo}_+(\mathbb{R})$ is as in the previous lemma, namely, there is a compact $K \subset \mathbb{R}$ such that $[h, uhu^{-1}] = 1$ for each u where $u(K) \cap K = \emptyset$. Suppose also that h does not have compact support. Then $\mathrm{Fix}(h) = \emptyset$.*

Proof. From Lemma 3.5 above, either h is a translation (in which case we are done) or its germ at $+\infty$ is non-trivial and not equal to that of a translation. Equivalently, the displacement map $x \mapsto h(x) - x$ is not constant in any neighborhood of $+\infty$.

Suppose for contradiction that h does have a fixed point, $x_0 \in \mathbb{R}$. We will show that h has a dense subset of fixed points, *i.e.* $h = \mathrm{id}$, contradicting that h was assumed to have noncompact support.

Let $x_1 \in \mathbb{R}$ and $\varepsilon > 0$. We want to prove that h has a fixed point in the interval $(x_1 - \varepsilon, x_1 + \varepsilon)$. Let K be the compact set given by condition (2), and let $C > 0$ be such that $K \subset (-C, C)$. Set $m = \max(x_0, x_1)$, and let y_1 be a point in $(m + 2C, +\infty)$ where the displacement of h is not locally constant. By continuity, we can find a point $y_2 \in (m + 2C, +\infty)$ such that the displacements $d_1 = h(y_1) - y_1$ and $d_2 = h(y_2) - y_2$ are independent over $\mathbb{Q}$. Now we make the following claim.

Claim. Let $x, y \in (-\infty, m)$ differ by d_1 or d_2. Then x is a fixed point of h if and only if y is.

Let us prove this claim. We treat the case where $y = x + d_1$ and x is a fixed point of h, the other cases are symmetric. Let $\delta > 0$. Since $A = \mathbb{Z}[1/n]$ acts minimally on $\mathbb{R}$, and since h is continuous at y_1, we can find a point $y_1' \in A \cdot x$ such that

$|y_1 - y_1'| < \delta$, and such that $|(h(y_1) - y_1) - (h(y_1') - y_1')| < \delta$. Provided δ is small enough, we also have $|x - y_1'| > 2C$. Let $t = x - y_1' \in A$. By hypothesis, h commutes with $\tau_t h \tau_t^{-1}$, and since x is a fixed point of h, this implies that $\tau_t h \tau_t^{-1}(x)$ is also a fixed point of h. Now, $\tau_t h \tau_t^{-1}(x) = x + (h(y_1') - y_1')$ is within distance δ from y, hence y admits fixed points of h in all its neighborhoods, and the claim is proved.

Now we can finish the proof of the lemma. Since d_1 and d_2 are independent over $\mathbb{Q}$, there exist $p, q \in \mathbb{Z}$ such that $|(x_1 - x_0) - (pd_1 + qd_2)| < \varepsilon$. Taking p and q to be large, we can also suppose that the vectors pd_1 and qd_2 have opposite sign. So up to exchanging the two, suppose $pd_1 < 0$. The claim above implies that $x_0 + d_1 \in \mathrm{Fix}(h) \cap (-\infty, m)$ and hence, can be applied iteratively, showing that $x_0 + pd_1 \in \mathrm{Fix}(h) \cap (-\infty, m)$. A similar inductive argument with $x_0 + pd_1$ playing the role of x_0 shows that $x_0 + pd_1 + qd_2 \in \mathrm{Fix}(h)$. This proves the lemma. □

Lemma 3.7. *Let $h \in \mathrm{Homeo}_+(\mathbb{R})$ be a homeomorphism satisfying statement (2) of Proposition 3.2. Suppose $\tau_{t_0} \in A$ is such that $[h, \tau_{t_0} h \tau_{t_0}^{-1}] = 1$ and $[h, \tau_{t_0}] \neq 1$. Then $[h, \tau_{t_0}]$ has compact support.*

Proof. Suppose that the hypotheses of the lemma hold, and suppose for contradiction that the map $g = [h, \tau_{t_0}]$ does not have compact support. Note that this implies that h also does not have compact support. Then we can apply Lemma 3.6 to g, and deduce that g has no fixed points in $\mathbb{R}$. Hence, we have $h > \tau_{t_0} h \tau_{t_0}^{-1}$ or $h < \tau_{t_0} h \tau_{t_0}^{-1}$. In either case, Lemma 3.3 immediately gives a contradiction: indeed, since the conjugator τ_{t_0} has no fixed points, Lemma 3.3 implies that h must have fixed points, while Lemma 3.6 applied to h implies that h has no fixed points. □

Now we can finish the proof of Proposition 3.2. Suppose $h \in \mathrm{Homeo}_+(\mathbb{R})$ satisfies (2) and is not a translation. By Observation 3.4 the set of $t \in \mathbb{R}$ such that h commutes with τ_t is nowhere dense, hence the set $A_0 = \{\tau_t \in A \mid [h, \tau_t] \neq 1\}$ is dense in the set of translations. Also, for t large enough and $\tau_t \in A_0$, the maps h and $\tau_t h \tau_t^{-1}$ commute, hence, by Lemma 3.7, both germs of $[h, \tau_t]$ are trivial. Thus, both germs of h are t-periodic, for a set of real numbers t which has accumulation points. This implies that both germs of h have constant displacement, and by Lemma 3.5 this displacement is zero. Hence, h has compact support. □

Using this, we prove the main result of this section.

Proof of Proposition 3.1. Let $f \in \mathrm{Homeo}_+(\mathbb{R})$ have compact support, and suppose that $\phi : \langle \mathrm{BS}(1,n), f \rangle \to \langle \mathrm{BS}(1,n), h \rangle$ is a morphism restricting to the identity on $\mathrm{BS}(1,n)$, and with $\phi(f) = h$. Suppose also that h is not a translation. (In particular, $h \neq \mathrm{id}_{\mathbb{R}}$.) Using Proposition 3.2 and commutation relations among f and elements of $\mathrm{BS}(1,n)$ we can conclude that h has compact support. We will now show that $h = f$.

Suppose for contradiction that there is some point x with $h(x) \neq f(x)$. Let U_1 be a neighborhood of x, chosen small enough so that $f(U_1)$ and $h(U_1)$ are disjoint, and let $\gamma_1 \in \mathrm{BS}(1,n)$ be a dilatation with fixed point in U_1, and derivative large enough so that $\gamma_1^{-1}(\mathrm{Supp}(h) \cup \mathrm{Supp}(f)) \subset U_1$. Then $\gamma_1^{-1} h \gamma_1$ has support in U_1, hence $h_2 = h\gamma_1^{-1} h \gamma_1 h^{-1}$ has support in $h(U_1)$, while $f_2 = f\gamma_1^{-1} f \gamma_1 f^{-1}$ has support in $f(U_1)$, hence acts trivially on $h(U_1)$.

Now choose $y \in h(U_1)$ such that $h_2(y) \neq y$, and let $U_2 \subset h(U_1)$ be a neighborhood of y small enough so that U_2 and $h_2(U_2)$ are disjoint. Let $\gamma_2 \in \mathrm{BS}(1,n)$ be a dilatation with fixed point in U_2, and derivative large enough so that $\gamma_2^{-1}(\mathrm{Supp}(h) \cup \mathrm{Supp}(f)) \subset U_2$. Then $\gamma_2^{-1} f \gamma_2$ has support in U_2, where f_2 acts trivially, so that $f_3 = [f_2, \gamma_2^{-1} f \gamma_2] = 1$, while $h_3 = [h_2, \gamma_2^{-1} h \gamma_2]$ is the composition of the maps $\gamma_2^{-1} h \gamma_2$

and $h_2^{-1}\gamma_2^{-1}h^{-1}\gamma_2 h_2$, both non-trivial and with supports contained respectively in U_2 and $h_2^{-1}(U_2)$. These supports are disjoint, hence $h_3 \neq 1$, but $\phi(f_3) = h_3$, a contradiction. □

4. Regularity of conjugacies

One of our ingredients for differential rigidity will be the following analogue of a theorem of Takens [23]. Takens' theorem states that a homeomorphism between two smooth manifolds M and N, which conjugates $\mathrm{Diff}^r(M)$ to $\mathrm{Diff}^r(N)$, is necessarily a diffeomorphism of class C^r. Here we specialize to $M = N = [0,1]$, but need only a conjugacy between a finitely generated affine subgroup.

As in the introduction, let $n \geqslant 2$ and consider the Baumslag-Solitar group $\mathrm{BS}(1,n)$, with its affine action, together with an extra homothety $x \mapsto \mu x$, with $\mu \notin \mathbb{Q}$, and let Γ_A denote this subgroup of $\mathrm{Aff}(\mathbb{R})$. There is a conjugate of this action to an action by diffeomorphisms on $(0,1)$; which may even taken to be C^∞-tangent to the identity at 0 and 1.

Proposition 4.1. *Let $\alpha > 1$, and let $\phi\colon \Gamma_A \to \mathrm{Diff}^\alpha([0,1])$ be an action, C^0-conjugate to the standard affine action, so there exists a homeomorphism $f\colon (0,1) \to \mathbb{R}$ such that for every $\gamma \in \Gamma_A$, we have $f \circ \phi(\gamma) = \gamma \circ f$. Then f is of class C^α. The same holds if α is replaced with any modulus of continuity $r + \omega$ satisfying Sternberg linearization, as discussed below.*

The proof has two main ingredients. The first is a recent result of Bonatti–Monteverde–Navas–Rivas [1].

Theorem 4.2 (Theorems 1.3 and 1.7 in [1])**.** *If $\mathrm{BS}(1,n)$ acts by C^1 diffeomorphisms of $[0,1]$ with no fixed point in $(0,1)$ and non-Abelian image, then the action is C^0 conjugate to the standard action, and the derivative of a at its (unique) interior fixed point is $\pm n$.*

The second ingredient is the Sternberg linearization theorem, or more precisley, Yoccoz's proof of this theorem in [26], which applies to a more general setting than C^α regularity, for $\alpha > 1$. Using this Proposition 4.1 can be seen to hold when C^α is replaced by any modulus of continuity to which this proof applies. We now describe the context of interest to us.

Recall that if $\omega\colon [0,+\infty) \to [0,+\infty)$ is a homeomorphism, a map $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *ω-continuous* if for some $C > 0$ we have $|f(x) - f(y)| \leqslant C\omega(|x-y|)$ for all $x, y \in \mathbb{R}$. For $\omega(t) = t$, or $\omega(t) = t^\alpha$, this is the notion of Lipschitz, or Hölder functions, respectively. A map f is said to be of class $C^{r+\omega}$ if it is C^r and $f^{(r)}$ is ω-continuous. We will say ω satisfies the *Sternberg linearization condition* if there exists an increasing map $\nu\colon [0,+\infty) \to [0,+\infty)$ which sends $(0,1)$ into $(0,1)$, such that for all $x \in [0,1]$ and $t \in [0,+\infty)$ we have $\omega(tx) \leqslant \nu(t)\omega(x)$.

Theorem 4.3 (Sternberg linearization)**.** *Let $\omega\colon [0,+\infty) \to [0,+\infty)$ be a homeomorphism satisfying the Sternberg linearization condition above. Let f be a germ of a diffeomorphism of $\mathbb{R}$, with $f(0) = 0$ and $f'(0) = a < 1$, of class $C^{r+\omega}$, with $r \geqslant 1$, or of class C^r, $r \geqslant 2$. Then there exists a unique germ h of diffeomorphism of $\mathbb{R}$, with $h(0) = 0$ and $h'(0) = 1$, with same regularity as f, and such that h conjugates f into the multiplication by a.*

Remark 4.4. In the Sternberg linearization condition above, the existence of ν is sufficient to ensure that for all $r \geqslant 1$, the set $C^{r+\omega}$ is stable under composition, while the condition $\nu(0,1) \subset (0,1)$ comes into play in the proof in regularity $C^{1+\omega}$.

Examples for ω include maps equal to $x \mapsto x^\alpha \ln(1/x)^\beta$ for small x, for $\alpha \in (0,1)$ and $\beta \in \mathbb{R}$.

By choosing appropriate regularities, *e.g.* using a function agreeing near 0 with $x \mapsto x^\alpha \ln(1/x)^\beta$ and varying α and β, one may obtain uncountably many finitely generated groups with critical regularity α. Depending on the sign of β one may take these groups to be embeddable or not in $\mathrm{Diff}^\alpha(M)$, for each 1-manifold considered in Theorem 1.4. One also obtains critical regularity more broadly in the sense of different moduli of continuity.

We do not give the proof here, as it is classical, but refer the reader to the proof appearing in Yoccoz [26, Appendice 4]. (See also Navas [19, Theorem 3.6.2].) While our statement of Theorem 4.3 is more general than that of Yoccoz, his proof works in this setting as well: one applies the Picard-Banach fixed point theorem to an operator on a Banach space of functions of a given regularity, a fixed point of this operator gives the map h. It follows that there is no loss of regularity between the map f and the conjugating map h, contrarily to Sternberg's original proof. The condition on ν in our statement is easily verified to be a sufficient condition for the operator used in the proof to be a contraction when $r = 1$. (However, the reader should keep in mind that Theorem 4.3 is false in regularity C^1; a counterexample was given by Sternberg himself [22], see also [19, Example 3.6.4].) S. Kim and T. Koberda inform us that this condition has a natural equivalent formulation, called *sub-tameness* of ω in [6]. It is also shown in [6] that one may equally well work only with *concave* moduli of continuity; however we find the ν condition most straightforward to use in Yoccoz's proof.

Now we give the proof of Proposition 4.1.

Proof of Proposition 4.1. We assume for simplicity that f is orientation preserving, this does not affect the argument. From Theorem 4.2, $\phi(a)$ has derivative n at $f^{-1}(0)$. By Sternberg linearization, there exists a unique germ $[h]$ of a C^α-diffeomorphism from $0 \in \mathbb{R}$ to $f^{-1}(0) \in (0,1)$ conjugating $\phi(a)$ to multiplication by n, and such that $h'(0) = 1$. In other words, there exists a neighborhood $(-\delta, \delta)$ of 0 and a map $h\colon (-\delta,\delta) \to (0,1)$ sending 0 to $f^{-1}(0)$ such that $h'(0) = 1$, and $\phi(a)(h(x)) = h(nx)$ for all x small enough. Note that the map $x \mapsto \phi(\mu)(h(\frac{x}{\phi(\mu)'(f^{-1}(0))}))$ satisfies the same conditions, hence has the same germ at $f^{-1}(0)$ at h, by uniqueness. Thus, h conjugates $\phi(a)$ to multiplication by n, and, simultaneously, conjugates $\phi(\mu)$ to multiplication by some scalar $\phi(\mu)'(f^{-1}(0))$. Considering the action of $\phi(\mu)$ on the translation subgroup of $\phi(\Gamma_A)$, we conclude that $\phi(\mu)'(f^{-1}(0)) = \mu$.

Hence, the map $f \circ h$, which is defined on $(-\delta, \delta)$, commutes with a dense group of dilatations and so is itself a multiplication by a scalar. In particular, f is of class C^α on some neighborhood U of 0. This is enough to deduce that f has C^α regularity everywhere, since for any compact set K, there exists $\gamma \in \Gamma_A$ with $\gamma(K) \subset U$ and $\phi(\gamma)(f(K)) \subset f(U)$, so we may write f on K as a composition of locally C^α maps. □

5. Differential rigidity and critical regularity

This and the following section are devoted to giving examples of groups with differential rigidity and critical regularity. Our guiding principle is Proposition 1.3, which we prove now.

Proof of Proposition 1.3. Let $\Gamma \subset \mathrm{Diff}^\infty(M)$ be a C^α rigid group, and $\beta \geqslant \alpha$. If $f \in \mathrm{Diff}^\beta(M)$ then of course, the inclusion maps the group $\langle \Gamma, f \rangle$ into $\mathrm{Diff}^\beta(M)$.

Conversely, suppose that $\varphi\colon \langle \Gamma, f\rangle \to \mathrm{Diff}^\beta(M)$ is an injective morphism. Since $\beta \geqslant \alpha$, the restriction of φ to Γ coincides with the conjugation by some element $g^{-1} \in \mathrm{Diff}^\beta(M)$; denote by c_g the inverse of this conjugation. Hence $c_g \circ \varphi\colon \langle \Gamma, f\rangle \to \mathrm{Diff}^\beta(M)$ is an injective morphism, restricting to the identity on Γ and mapping f to $c_g(\varphi(f))$. By recognition and since f has support in U, it follows that $f = c_g(\varphi(f))$, hence f is of class C^β. $\square$

The simplicity of this proof allows quite a bit of flexibility in the construction of groups of critical regularity. For example, if $\Gamma \subset \mathrm{Diff}^\infty(M)$ is C^α-rigid for all $\alpha > 1$, and if $f \in \mathrm{Diff}^1(M)$ is not in $\mathrm{Diff}^\beta(M)$ for any $\beta > 1$, then the group $\langle \Gamma, f\rangle$ is a subgroup of $\mathrm{Diff}^1(M)$ and is not embeddable in $\mathrm{Diff}^\beta(M)$ for any $\beta > 1$. Thus, our techniques give groups of critical regularity α for all $\alpha \geqslant 1$, including 1. Similarly, if $f \in \mathrm{Diff}^{k+\mathrm{lip}}(M) \smallsetminus \mathrm{Diff}^{k+1}(M)$, then $\langle \Gamma, f\rangle$ cannot be embedded in $\mathrm{Diff}^{k+1}(M)$, which answers [14, Question 7.1(1)].

5.1. **Proof of Theorem 1.5.** The remainder of this section is devoted to the proof of Theorem 1.5, describing examples of C^α-rigid groups of diffeomorphisms of S^1, for all $\alpha > 1$. We note that examples of C^3-rigid such groups have actually been known for some time: the notion of differential rigidity essentially appeared in work of Ghys [9], where he proved that representations of surface groups with maximal Euler class into $\mathrm{Diff}^r(S^1)$, $r \geqslant 3$, are C^r-conjugate to representations in $\mathrm{PSL}(2,\mathbb{R})$. Together with an observation of Calegari [2], this implies that, for example the Fuchsian $(2,3,7)$–triangle group in $\mathrm{PSL}(2,\mathbb{R}) \subset \mathrm{Diff}^\infty(S^1)$ is C^3-differentiably rigid. For the proof of Theorem 1.5, it will be convenient to work with the following consequence (essentially a restatement) of the theorem of Bonatti, Monteverde, Navas and Rivas given above at Theorem 4.2.

Corollary 5.1. *Let $\phi\colon \mathrm{BS}(1,n) \to \mathrm{Diff}^1([0,1])$ be an injective morphism. Then there exists an integer $m \geqslant 1$, and m open intervals $I_1, \dots, I_m \subset [0,1]$, each invariant under the action by ϕ, and on which the ϕ-action of $\mathrm{BS}(1,n)$ is C^0-conjugate (possibly by an orientation reversing homeomorphism) to the standard action of $\mathrm{BS}(1,n)$ on $\mathbb{R}$. Moreover, $\phi(b)$ restricts to the identity on $[0,1] \smallsetminus \cup_j I_j$, and $\phi(a)$ has derivative $\pm n$ at its (unique) fixed point in each I_j.*

Proof. Let $I_1, I_2, \dots$ be the connected components of $[0,1] \smallsetminus \mathrm{Fix}(\phi(\mathrm{BS}(1,n)))$. Apply Theorem 4.2 to the restriction of the action to each I_i. If the action is faithful on some I_i, then $\phi(a)$ has derivative $\pm n$; since $\phi(a)$ is C^1 there can only be finitely many such. It remains only to show that every non-faithful action of $\mathrm{BS}(1,n)$ on the line has b in its kernel. This easily follows from the observation that every nontrivial element of $\mathrm{BS}(1,n)$ has a normal form $a^{-i} b^j a^k$ for some $i, k \geqslant 0$ and $j \in \mathbb{Z}$. Thus, the only nontrivial, proper, torsion free quotient of $\mathrm{BS}(1,n)$ is $\mathbb{Z} \simeq \mathrm{BS}(1,n)/\langle b\rangle$. $\square$

Proof of Theorem 1.5. Let Γ be the group generated by Γ_A et Γ_T as described in the Introduction, let $\alpha > 1$ and let $\phi\colon \Gamma \to \mathrm{Diff}_+^\alpha(S^1)$ be an injective morphism. The bulk of the proof is devoted to showing that the action of $\phi(\Gamma_T)$ is minimal, which we do now. Calegari [2] showed that, for any nontrivial action of Γ_T on S^1 by homeomorphisms, the Euler number of the action must be maximal. It then follows from work of Matsumoto [16] that the action is *semi-conjugate* to the standard one. Supposing, for contradiction, that $\phi(\Gamma_T)$ is not minimal, this means that there is an invariant closed set $K \subset S^1$, homeomorphic to a Cantor set, and a surjective, monotone, degree one map $s\colon S^1 \to S^1$ collapsing each complimentary region of K to a point, which intertwines the action of $\phi(\Gamma_T)$ with the standard action. Both circles (at the source and target of the map s) are endowed with an action of the group Γ: the target has the standard action and the source the action *via* ϕ. Since s intertwines the actions, we know that for all $x \in S^1$, we have $\gamma s(x) = s(\phi(\gamma)(x))$ for all $\gamma \in \Gamma_T$. We will now use the relations in the group Γ to recover some similar information about $\phi(\gamma)$ for *all* $\gamma \in \Gamma$, in the same spirit as the reconstruction lemmas in Section 2.

Let $\mathring{K}$ denote the set of two-sided accumulation points of K. This is also a $\phi(\Gamma_T)$–invariant set, and the restriction of s to $\mathring{K}$ is a homeomorphism conjugating $\phi(\Gamma_T)$ to the standard action of Γ_T on $s(\mathring{K})$, which is a dense subset of S^1. Thus, the action of $\phi(\Gamma_T)$ on $\mathring{K}$ has the contraction property. It follows that for all $x \in \mathring{K}$ and all $h \in \Gamma$,

(1) if $s(x) \notin \mathrm{Supp}(h)$, then $\phi(h)(x) = x$, and
(2) if $s(x) \notin \mathrm{Fix}(h)$, then $x \in \mathrm{Supp}(\phi(h))$.

This is a straightforward adaptation of Lemma 2.2, and follows from considering whether h and $\gamma h \gamma^{-1}$ commute, when $\gamma \in \Gamma_T$ maps the complement of a small neighborhood U_x of x into U_x. Now as in Section 3 we denote by $a\colon x \mapsto nx$ and $b\colon x \mapsto x+1$ the two standard generators of $\mathrm{BS}(1,n) \subset \Gamma_A$. From (1) applied to a and b, we know that $\phi(a)$ and $\phi(b)$ share a fixed point in S^1, so we may regard them as acting on the interval. Corollary 5.1 then asserts that there is a finite collection of open intervals $I_1, \dots, I_m \subset S^1$ on which the action of $\langle a, b\rangle$ is topologically conjugate to the standard action, with $\mathrm{Fix}(\phi(b)) = S^1 \smallsetminus \cup_j I_j$. Since $s(\mathring{K})$ is dense, its intersection with $\mathrm{Fix}(b)$ has infinite cardinality, so (1) implies that the compliment of $\cup_j I_j$ has infinite cardinality. In particular, this complement contains some open interval J which intersects $\mathring{K}$. Similarly, (2) implies that some nonempty subcollection of the intervals $I_1, \dots, I_m$ have nontrivial intersection with $\mathring{K}$. Reindexing if needed, we suppose $I_1 \cap \mathring{K} \neq \emptyset$.

Since J and I_1 each contain an open subset of $\mathring{K}$, there exists $\gamma \in \Gamma_T$ such that $\phi(\gamma)(S^1 \smallsetminus J) \subset I_1$, hence $\phi(\gamma b \gamma^{-1})$ has support inside I_1. Now take $U \subset I_1$ to be a connected component of $S^1 \smallsetminus K$. Since the action of $\mathrm{BS}(1,n)$ on I_1 is standard, there is some $w \in \mathrm{BS}(1,n)$ mapping the support of $\phi(\gamma b \gamma^{-1})$ into U, and so

$$\mathrm{Supp}(\phi(w\gamma b\gamma^{-1}w^{-1})) \subset U.$$

Set $g = w\gamma b \gamma^{-1} w^{-1} \in \mathrm{Diff}^\infty_+(S^1)$. It follows that for all $x \in \mathring{K}$, we have $x \notin \mathrm{Supp}(\phi(g))$. Hence, by point (2) above applied to g, we have $s(x) \in \mathrm{Fix}(g)$ for all $x \in \mathring{K}$. But $s(\mathring{K})$ is dense in S^1, hence $g = \mathrm{id}_{S^1}$, while g is conjugate to b. From this contradiction, we derive that $\phi(\Gamma_T)$ acts minimally on S^1.

Since the action of $\phi(\Gamma_T)$ is minimal, it is topologically conjugate to the standard action of Γ_T. Thus, after conjugation by some $f \in \mathrm{Homeo}(S^1)$, we may assume that ϕ restricts to the identity morphism on Γ_T. Now Γ_T has the contracting property, so by Theorem 1.1 it recognizes a, b and μ, which have non-total support. It follows that ϕ is obtained by conjugation by f. Finally, Proposition 4.1 asserts that the map f is C^α on the interval I where a, b and μ are supported. By minimality of the action of Γ, we conclude that f is C^α everywhere, and the Theorem is proved. $\square$

As a consequence, we have the following.

Proof of Theorem 1.4 for $M = S^1$. Since the map Γ constructed above acts on S^1 with the contraction property, and contains maps with non-total support, Γ also contains maps with small supports everywhere. By Theorem 1.1, it thus follows that Γ recognizes all of $\mathrm{Homeo}(S^1)$. Proposition 1.3 now proves Theorem 1.4 in the case $M = S^1$. $\square$

6. Rigidity and critical regularity for actions on $\mathbb{R}$ and $[0,1]$

We proceed with the proofs of Proposition 1.6 and Theorem 1.7. Recall that, as noted in the introduction, we will be forced to work with slight modifications of the notion of map recognition rather than directly applying Proposition 1.3.

6.1. **Groups acting on the line.** We have an exact sequence

$$1 \to \mathbb{Z} \to \mathrm{Homeo}_+^{\mathbb{Z}}(\mathbb{R}) \to \mathrm{Homeo}_+(S^1) \to 1,$$

where $\mathrm{Homeo}_+^{\mathbb{Z}}(\mathbb{R})$ is the group of all homeomorphisms of the real line which commute with the map $z\colon x \mapsto x+1$. Let Γ be the group from the previous section, and let $\widetilde{\Gamma}$ denote its preimage in $\mathrm{Homeo}_+^{\mathbb{Z}}(\mathbb{R})$. Thus, the group $\widetilde{\Gamma}$ is a subgroup of $\mathrm{Diff}_+^{\infty}(\mathbb{R})$, and it is not hard to check that it is also generated by 6 elements.

Proposition 6.1. *The group $\widetilde{\Gamma}$ is C^α-rigid, for any $\alpha > 1$.*

Proof. Much of this proof is an adaptation of an argument by Calegari, see [2].

Consider an injective morphism $\phi\colon \widetilde{\Gamma} \to \mathrm{Diff}_+^{\alpha}(\mathbb{R})$, for some $\alpha > 1$. The elements $s, r, t \in \Gamma_T \subset \Gamma$ admit lifts $\widetilde{s}$, $\widetilde{r}$ and $\widetilde{t}$ satisfying $\widetilde{s}^2 = \widetilde{r}^3 = \widetilde{t}^7 = z$. Suppose that $\phi(z)$ admits a fixed point in $\mathbb{R}$. Then $\phi(\widetilde{s})$, $\phi(\widetilde{r})$ and $\phi(\widetilde{t})$ each fixes pointwise the fixed point set of $\phi(z)$, simply because the dynamics of any map on $\mathbb{R}$ is monotone on its orbits. Hence $\widetilde{\Gamma_T}$ has a global fixed point in $\mathbb{R}$, but this violates the Thurston stability theorem of [24].

Thus $\phi(z)$ has no fixed point in $\mathbb{R}$, and so $\phi(z)$ is topologically conjugate to the map $z\colon x \mapsto x+1$ itself. As $\phi(z)$ is central in $\phi(\widetilde{\Gamma})$, the map ϕ descends to the quotient, defining an injective morphism $\overline{\phi}\colon \Gamma \to \mathrm{Diff}_+^{\alpha}(S^1)$, which is a C^α-conjugation by Theorem 1.5 above. The conjugating map then lifts to a C^α diffeomorphism of $\mathbb{R}$, realizing ϕ by conjugation. □

Using this, we complete the proof of Proposition 1.6, showing that $\widetilde{\Gamma}$ recognizes maps in $\mathrm{Homeo}_+^{\mathbb{Z}}(\mathbb{R})$ up to integer translation.

Proof of Proposition 1.6. We have already shown that $\widetilde{\Gamma}$ is C^α-rigid. So let $h \in \mathrm{Homeo}_+^{\mathbb{Z}}(\mathbb{R})$ and let $f \in \mathrm{Homeo}_+(\mathbb{R})$ be any map, and suppose that there is a group isomorphism $\phi\colon \langle \widetilde{\Gamma}, h\rangle \to \langle \widetilde{\Gamma}, f\rangle$ which restricts to the identity on $\widetilde{\Gamma}$ and with $\phi(h) = f$. We want to prove that $f = h \circ z^k$ for some k.

First, as h commutes with z, so does f, and $f \in \mathrm{Homeo}_+^{\mathbb{Z}}(\mathbb{R})$. Hence, f descends to a homeomorphism $\overline{f}$ of the circle $\mathbb{R}/\mathbb{Z}$. Since the cyclic group generated by z is central in both $\langle \widetilde{\Gamma}, h\rangle$ and $\langle \widetilde{\Gamma}, f\rangle$, the map ϕ descends to a group isomorphism between the quotients $\langle \widetilde{\Gamma}, h\rangle/\langle z\rangle$ and $\langle \widetilde{\Gamma}, f\rangle/\langle z\rangle$. These groups are naturally isomorphic to $\langle \Gamma, \overline{h}\rangle$ and $\langle \Gamma, \overline{f}\rangle$, where $\overline{h}$ is the homeomorphism of the circle defined by h. Now, the group Γ has maps with small supports everywhere so it follows from Theorem 1.1 that the maps $\overline{h}$ and $\overline{f}$ agree. Hence, h and f may differ only by an integer translation. □

Combining Propositions 1.3 and 1.6 gives the following, which proves Theorem 1.4 for $M = \mathbb{R}$.

Corollary 6.2 (Critical regularity on the line)**.** *Let $f \in \mathrm{Homeo}_+(S^1)$ be a map with non-total support, and suppose $f \notin \mathrm{Diff}^{\beta}(S^1)$ for some $\beta > 1$. Let $\tilde{f}$ be a lift of f. Then $\langle \widetilde{\Gamma}, f\rangle$ is not isomorphic to any subgroup of $\mathrm{Diff}^{\beta}(\mathbb{R})$.*

6.2. **The closed interval.** We recall the set-up of Theorem 1.7. Fix $1 < n \in \mathbb{N}$ and consider the affine group generated by $\mathrm{BS}(1,n)$ and an irrational dilatation μ, as in Section 5. Let $f \in \mathrm{Homeo}(\mathbb{R})$ have compact support, and suppose $f \neq \mathrm{id}_{\mathbb{R}}$. Let Γ denote the group generated by $\mathrm{BS}(1,n)$, μ, and f, and suppose $\phi\colon \Gamma \to \mathrm{Diff}^{\alpha}([0,1])$ is an injective morphism for some $\alpha > 1$. We will show that there exists an interval $(x,y) \subset [0,1]$, invariant under $\phi(\Gamma)$, and a C^{α}-diffeomorphism $h\colon \mathbb{R} \to (x,y)$ conjugating $\phi(\Gamma)$ with the standard action on $\mathbb{R}$.

Note that this will also immediately imply the remaining case of the critical regularity statement given in Theorem 1.4 in the introduction.

Proof of Theorem 1.7. Let $\phi\colon \Gamma \to \mathrm{Diff}^{\alpha}([0,1])$ be as above. Corollary 5.1 states that the complement of $\mathrm{Fix}(\phi(b))$ is a union of disjoint intervals $I_1 = (x_1, y_1), \ldots, I_m = (x_m, y_m)$, each of which admits a homeomorphism $\psi_j\colon \mathbb{R} \to I_j$ conjugating the standard action of $\mathrm{BS}(1,n)$ on $\mathbb{R}$ with its action *via* ϕ on I_j. The proof has three steps, which we separate into short lemmas.

Lemma 6.3. *$\phi(f)$ preserves each interval I_j.*

Proof. For this, it suffices to show that $\phi(f)(x_1) = x_1$, and $\phi(f)(y_1) = y_1$, as the remaining intervals can then be shown invariant by applying this argument iteratively to the restriction of the action to $[y_1, 1]$ and so on. As before, we will denote by $\tau_t \in \mathrm{BS}(1,n)$ the translation by t. Up to changing signs we will suppose below that for $t > 0$, τ_t is increasing on the interval I_1 (*i.e.*, ψ_1 is orientation-preserving).

Suppose for contradiction that $\phi(f)(x_1) \neq x_1$, up to replacing f with its inverse we may assume that $\phi(f)(x_1) < x_1$. Then we also have $\phi(f)(x_1+\varepsilon) < x_1$ for some $\varepsilon > 0$. Let $x_0 = \phi(f)(x_1+\varepsilon)$. Then, for all $N \in \mathbb{Z}$, we have $\phi(b^N f^{-1} b^{-N}) \circ \phi(f)(x_0) = x_0$, while $\phi(f) \circ \phi(b^N f^{-1} b^{-N})(x_0) \neq x_0$ for all N. This contradicts that f and $b^N f b^{-N}$ commute for N large enough, hence $\phi(f)(x_1) = x_1$.

Now suppose for contradiction that $\phi(f)(y_1) > y_1$. Let $x_0 = \phi(f)^{-1}(y_1) \in (x_1, y_1)$. For all $t > 0$ sufficiently large, $\tau_t f \tau_t^{-1}$ and f commute, hence $\phi(\tau_t f \tau_t^{-1}) \circ \phi(f)(x_0) = \phi(f) \circ \phi(\tau_t f \tau_t^{-1})(x_0)$. Since the set $\phi(\tau_t^{-1})(x_0)$ accumulates to x_1, we get that $\phi(f)(x) > x$ for a dense and open set of points near x_1.

Now, consider a dilatation $d \in \mathrm{BS}(1,n)$ whose fixed point x_d, in $\mathbb{R}$, is (strictly) to the left of $\mathrm{Supp}(f) \cup \{\psi_1^{-1}(x_0)\}$. Thus, f and $d^{-N} f d^N$ commute, for all N large. Since $\phi(f)(x) > x$ for an open dense set of points near x_1, we may in fact choose the dilatation d so that $p = \psi_1(x_d)$ satisfies $\phi(f)(p) > p$.

As N approaches $+\infty$, we have $\phi(d^N)(x_0) \to y_1$, hence $\phi(f^{-1} d^N)(x_0) \to x_0$, so $\phi(d^{-N} f^{-1} d^N)(x_0) \to p$ and finally $\phi(f) \circ \phi(d^{-N} f^{-1} d^N)(x_0) \to \phi(f)(p)$. On the other hand, $\phi(f)(x_0) = y_1$ is fixed by $\phi(d)$, hence $\phi(f^{-1} d^N f)(x_0) = x_0$ for all N, hence $\phi(d^{-N} f^{-1} d^N) \circ \phi(f)(x_0) = \phi(d^{-N})(x_0)$ approaches the fixed point p as N goes to $+\infty$. But $p \neq \phi(f)(p)$. It follows that, for N large enough, $\phi(f)$ and $\phi(d^{-N} f^{-1} d^N)$ cannot commute, a contradiction. □

Lemma 6.4. *The commutator $\phi([f,b])$ acts nontrivially on some I_j, and on any such interval, ψ_j conjugates the action of $\langle \mathrm{BS}(1,n), f\rangle$ to the standard action.*

Proof. Since ϕ is injective, the commutator $[f,b]$ acts nontrivially and so f acts nontrivially on at least one of the (f-invariant) intervals I_j in the complement of b. Identifying I_j with $\mathbb{R}$ *via* ψ_j, we obtain a morphism from $\langle \mathrm{BS}(1,n), f\rangle$ to $\langle \mathrm{BS}(1,n), \phi(f)\rangle$ that is the identity on $\mathrm{BS}(1,n)$ and sends f to $\phi(f)$. By Proposition 3.1, we conclude that, on any such interval, $\phi(f)$ either acts as a translation (which does not occur if $\phi([f,b]) \neq \text{id}$ on I_j), or we have $\phi(f) = \psi_j f \psi_j^{-1}$, hence the lemma is proved. □

Lemma 6.5. *For some j, the map ψ_j conjugates the action of Γ on $\mathbb{R}$ to that of $\phi(\Gamma)$ on I_j.*

Proof. Let $g = \mu f \mu^{-1}$. Then $[f,b]$ and $[g,b]$ are both nontrivial and compactly supported. We claim that for some $\tau_t \in \mathrm{BS}(1,n)$, the commutator $\gamma = [[f,b], \tau_t[g,b]\tau_t^{-1}]$ is a nontrivial element of Γ. To see this, take some τ_t that moves one of the two extremal points of $\mathrm{Supp}([g,b])$ to a non-fixed point of $[f,b]$. Now $[f,b]$ does not preserve the support of $\tau_t[g,b]\tau_t^{-1}$, so the maps $[f,b]$ and $\tau_t[g,b]\tau_t^{-1}$ cannot commute.

Since ϕ is injective and γ is nontrivial, there exists an interval I_j where $\phi([f,b])$ and $\phi([g,b])$ are simultaneously nontrivial. Applying Lemma 6.4 to the actions of f and g on this interval, we get that ψ_j conjugates $\mathrm{BS}(1,n)$, f and g to the standard action. Now we will see that ψ_j conjugates μ to the standard action as well, and hence conjugates all of Γ.

To see this, for any $x \in I_j$, take a nested sequence $U_{k,x}$ of intervals with $\bigcap_k U_{k,x} = \{x\}$. Since the action of $\langle \mathrm{BS}(1,n), f\rangle$ on I_j has small supports everywhere, we may take $\gamma_k \in \langle \mathrm{BS}(1,n), f\rangle$ with $\phi(\gamma_k)$ supported on $U_{k,x}$. Thus, γ_k is supported on $\psi_j^{-1}(U_{k,x})$, and its conjugate by μ is supported on $\mu(\psi_j^{-1}U_{k,x})$. Applying Proposition 3.1 to $\langle \mathrm{BS}(1,n), \mu\gamma_k\mu^{-1}\rangle$, it follows that $\phi(\mu\gamma_k\mu^{-1})$ is supported on $\psi_j(\mu)\psi_j^{-1}(U_{k,x})$. We conclude that, $\phi(\mu)(x) = \psi_j \mu \psi_j^{-1}(x)$, as desired. □

Conclusion of proof. It remains only to remark that, by Proposition 4.1, the map ψ_j obtained from Lemma 6.5 is of class C^α. □

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