TWO RESULTS ON END SPACES OF INFINITE TYPE SURFACES

KATHRYN MANN AND KASRA RAFI

Abstract. We answer two questions about the topology of end spaces of infinite type surfaces and the action of the mapping class group that have appeared in the literature. First, we give examples of infinite type surfaces with end spaces that are not self-similar, but a unique maximal type of end, either a singleton or Cantor set. Secondly, we use an argument of Tsankov to show that the "local complexity" relation $\preccurlyeq$ on end types gives an equivalence relation that agrees with the notion of being locally homeomorphic.

1. Introduction

The paper [9] introduced the notion of *self-similar* end spaces for infinite type surfaces, and proved that a self-similar end space necessarily contains a unique *maximal* type of end (with respect to the partial order defined below), with the set of ends of this type either a singleton or a Cantor set. This notion has turned out to be a useful one and it has appeared many times in the literature, see for instance [1, 3, 4, 10]. A partial converse to this statement was proved in [9, Prop. 4.8]. Namely, under the additional hypothesis that an infinite type surface Σ contains no nondisplaceable subsurfaces, it was shown that the end space of Σ is self-similar if and only if the set of maximal ends is either a singleton or a Cantor set of points of the same type. However, the necessity of this extra hypothesis (no nondisplaceable subsurfaces) was not discussed there, raising the question of whether it could be eliminated. This appeared as Question 1.4 in [7], and Remark 6.2 in [8]. Here we answer the question, showing the strict converse (without extra assumptions) to [9, Prop. 4.8] is false:

Theorem 1.1. There exist examples of surfaces that have non self-similar end spaces with a unique maximal type of end and set of maximal ends homeomorphic to a singleton or to a Cantor set.

The partial order defined in [9, Section 4] is as follows. For $x, y \in E$, we say $y \preccurlyeq x$ if every neighborhood of x contains a point locally homeomorphic to y, and we say x and y are of the same type if $x \preccurlyeq y$ and $y \preccurlyeq x$, so that $\preccurlyeq$ descends to a partial order on types. Informally, we think of $\preccurlyeq$ as describing the local complexity of an end. Of course, if $h(x) = y$ for some homeomorphism h of E, then x and y are of the same type. Here we prove the converse to this statement, following an argument of T. Tsankov. This answers another question from [9].

Theorem 1.2. If $x \preccurlyeq y$ and $y \preccurlyeq x$, then there is a mapping class (equivalently, a homeomorphism) h of Σ taking x to y.

The outline of the paper is as follows. First, we provide details on a local construction of non-comparable points (an idea sketched loosely in [9]) as a tool towards the proof of Theorem 1.1. We then prove Theorem 1.1 building examples first in the case where the

maximal end is a singleton, followed by the Cantor set case. Finally, the proof of Theorem 1.2 is given in Section 4.

2. Toolkit for Theorem 1.1: Non-comparable points

For a surface Σ, we denote the space of ends of Σ by $E(\Sigma)$ or simply E. We define an equivalence relation on the end space by saying points $x, x' \in E$ are *locally homeomorphic* if there exists some clopen neighborhood of x in E that is homeomorphic to some clopen neighborhood of x' via a homeomorphism taking x to x'. This is equivalent to saying that there is a homeomorphism of Σ such the the induced map on the end space sends x to x'. For an end x, we let $\mathrm{Accu}(x)$ denote the set of accumulation points of the set of all ends locally homeomorphic to x.

We work with the relation $\preccurlyeq$ on points of E as given above in the introduction. One may equivalently define $y \preccurlyeq x$ if $x \in \mathrm{Accu}(y)$. We say x and y are *of the same type* if $x \preccurlyeq y$ and $y \preccurlyeq x$, so that $\preccurlyeq$ descends to a partial order on types. We say x is a *maximal type* if $x \preccurlyeq y$ implies $y \preccurlyeq x$ and denote the set of maximal points in E by $\mathcal{M}(E)$. We say $x, y \in E$ are non-comparable if neither $x \preccurlyeq y$ nor $y \preccurlyeq x$ holds. See [9] for more details and discussion.

The first building block in our construction is a sequence of surfaces D_n indexed by $n \in \mathbb{N}$, each with one boundary component, such that D_n contains a unique maximal end z_n and for all $i \neq j$ the ends z_i and z_j are non-comparable (the reader should picture D_i and D_j as disjoint subsurfaces of Σ).

Note that a construction such as this is not possible when the surface is planar and has a countable number of ends. A classical result of Mazurkiewicz and Sierpinski [11] states that, for any surface with a countable set of ends, there exists a countable ordinal α such that the end space E is homeomorphic to the ordinal $\omega^\alpha \cdot m + 1$ where m is a positive integer. The assumption that E has one maximal point implies that $m = 1$. Now assume D, D' are two genus zero surfaces with one boundary and a countable end space (E and E') such that that each end space has one maximal end ($x \in E$ and $x' \in E'$). Then their end spaces are respectively homeomorphic to $\omega^\alpha + 1$ and $\omega^{\alpha'} + 1$ for some countable ordinals α and α'. Now if $\alpha \leq \alpha$ then $x \preccurlyeq x'$, which means x and x' are comparable.

We carry out the construction in both remaining cases, namely, when the set of ends is uncountable and the surface is planar (the proof easily generalizes to non-planar surfaces), and when the set of ends is countable and the surface is non-planar.

Uncountable planar case. Let D be a disc, let $C_n = Q_n \cup C \subset D$ be the union of a countable set Q_n and a Cantor set C, with Cantor-Bendixson rank n such that, for each derived set of C_n that has isolated points, the accumulation set of the isolated points contains the Cantor set. For example, one may take the C to be the standard middle-thirds Cantor set, and insert in each missing interval a set homeomorphic to $\omega^n + 1$ to form Q_n. Now for each C_n, select a single point z_n and let C'_n be another Cantor set contained in D so that $C_n \cap C'_n = \{z_n\}$. Puncturing D along $C_n \cup C'_n$ gives a surface D_n with one boundary component such that z_n is the unique maximal end (see Figure 1). By construction, z_i and z_j are non-comparable when $i \neq j$.

Countable non-planar case. Let D be a disk and let α and β be two countable ordinals with $\beta < \alpha$. Let E_α be a subset of D homeomorphic to $\omega^\alpha + 1$ and denote its (unique) maximal point by $z_{\alpha,\beta}$. Now, consider a closed subset $E_\beta \subset E_\alpha$ homeomorphic to $\omega^\beta + 1$

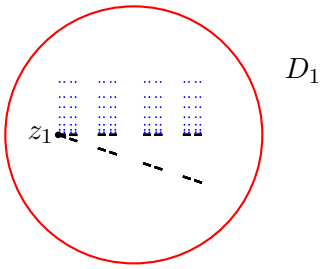

FIGURE 1. The point z_1 is in the intersection of C_1 and C_1'. There is a countable set Q_1 of punctures (the blue points) that accumulate to C_1. Points in C_1' are not accumulated by punctures except for z_1.

where $z_{\alpha,\beta}$ is again the maximal point of E_β. For every isolated point y of E_β remove a disk around y (keeping these disks pairwise disjoint) and glue back in a one-ended, infinite genus surface with one boundary component. We also puncture D along the remaining points of E_α to obtain a surface $D_{\alpha,\beta}$. The point $z_{\alpha,\beta}$ is the unique maximal end of this surface (see Figure 2). Moreover, for two pairs or countable ordinals (α, β) and (α', β') (satisfying $\beta < \alpha$ and $\beta' < \alpha'$) if $\alpha \geq \alpha'$ and $\beta < \beta'$ then $D_{\alpha,\beta}$ and $D_{\alpha',\beta'}$ are non-comparable. In fact, no end of $D_{\alpha,\beta}$ is of the same type as $z_{\alpha',\beta'}$ and vice versa. Hence we can, for example, fix α and vary β to get a countable family of surfaces with one boundary where the maximal points are non-comparable.

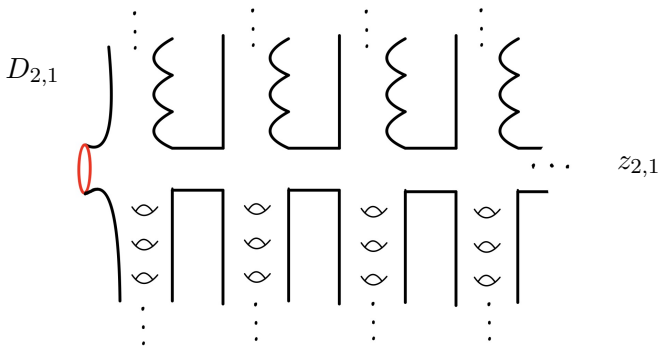

FIGURE 2. The end $z_{2,1}$ is accumulated by a set of non-planar ends homeomorphic to ω^1 and a set of planar ends homeomorphic to ω^2. That is, the space of non-planar ends is homeomorphic to $\omega + 1$ and the total end space is homeomorphic to $\omega^2 + 1$.

Uncountably many non-comparable points. It is also possible for a surface to contain uncountably many non-comparable points. For example, let Σ be a sphere minus a Cantor set. Visualize Σ as a union of pairs of pants. Enumerate the pairs of pants, remove a disk from each pair of pants, and glue back in a copy of D_n to the n-th pair of pants. Call the resulting surface Σ'. Then all the ends of Σ' coming from Σ are non-comparable since small enough neighborhoods of any two such ends contain non-comparable points.

3. Proof of Theorem 1.1

Now we construct the surface that will furnish the examples needed for Theorem 1.1. We give the construction first for the case where $\mathcal{M}(E)$ is a singleton. We then modify the construction to produce examples where $\mathcal{M}(E)$ is a Cantor set.

Start with a flute surface, meaning a sphere punctured along a sequence of points $p_1, p_2, ...$ accumulating at an end p_∞. For each $i \neq \infty$, replace a neighborhood of the puncture p_i with a Cantor tree T_i. We think of T_i as a union of pants surfaces, indexed by finite binary strings, so that the pants indexed by a string $s_1...s_n$ has cuffs glued to the pants indexed by $s_1...s_{n-1}$, $s_1...s_n0$, and $s_1...s_n1$, and the first pair of pants $P_\emptyset$ is glued on where the original puncture was removed. Now for each i, we will replace a countable set of discs in T_i with discs homeomorphic to copies of the previously constructed discs D_n (from either construction in the previous section), according to the following recipe.

For tree T_i, place copies of $D_1, D_2, \dots D_i$ on the first pants surface, $P_\emptyset$, and place a copy of D_k on each pants indexed by a word of length $k - i$. Thus, for each $k \geq i$ there are 2^{k-i} copies of D_k on T_i. Call the resulting punctured surface S. An illustration is given in Figure 3.

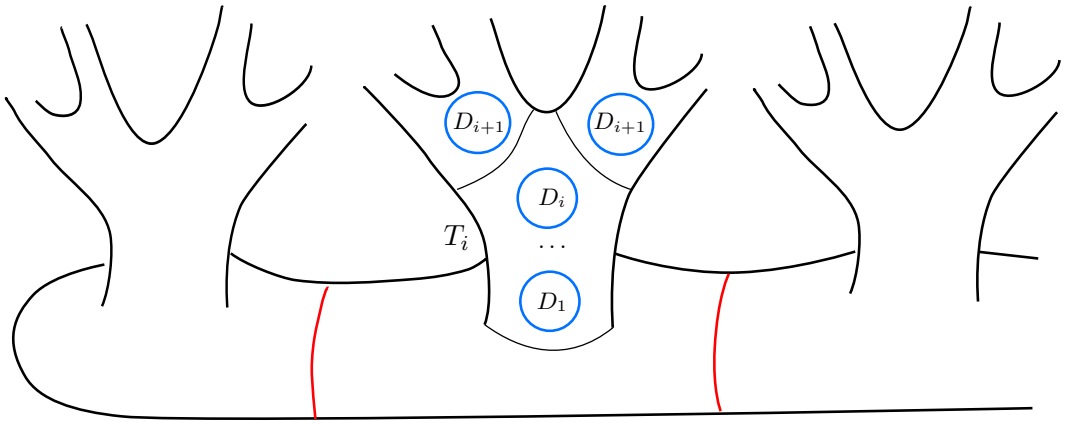

Figure 3. Construction of the surface with unique maximal end.

Note that all of the ends of each of the trees T_i are pairwise locally homeomorphic. The end p_∞ of our surface S is the unique accumulation point of these tree ends, so it is the unique maximal end. We will now show that the end space of S is not self-similar. Let E_i denote the end space of the tree T_i.

Consider the decomposition of the end space $E_1 \sqcup (E - E_1)$. Since E_1 does not contain a maximal end, to show the end space of the surface is not self-similar, it suffices to show that its complement contains no homeomorphic copy of E_1. Suppose for contradiction that we could find such. Note the the sets $U_i := \bigcup_{n=i}^{\infty} E_n$ form a neighborhood basis of p_∞ in the end space. Since p_∞ is the unique maximal end and E_1 is closed, any homeomorphic copy of E_1 must avoid some neighborhood of p_∞ so is contained in a finite union $E_2 \cup E_3 \cup \ldots \cup E_N$.

By construction, E_1 contains 2^N locally homeomorphic copies of the end z_{N+1}. But $E_2 \cup E_3 \cup \ldots \cup E_N$ contains $\sum_{i=1}^{N-1} 2^i < 2^N$ copies of z_{N+1}. A contradiction. Thus, E_1 cannot be mapped into its complement, so the end space is not self-similar.

Cantor set case. A variation on the construction above can be used to produce a non self-similar surface with a unique maximal type and a Cantor set of maximal ends. First, following a similar procedure to the construction of the punctured trees T_i, for each $i \in \mathbb{N}$ we can build a Cantor tree T_i' with a single boundary component that contains one copy of each of the discs D_1, D_2, ... D_i, and for each $k > i$ contains $2^{(2^{k-i})}$ copies of D_k, with each end of the tree locally homeomorphic.

Now instead of starting with the flute, start with a Cantor tree constructed of pairs of pants indexed by binary strings, with the first pair of pants $P_\emptyset$ capped off by a disc on one of its boundary components. From each pair of pants indexed by a string of length i, remove a disc and glue in a copy of T_{i+1}' to it along its boundary. In particular, T_1' is glued to the first pair of pants indexed by the empty string. In the resulting surface, the ends of the original Cantor tree are precisely the maximal ends, forming a Cantor set of maximal ends of a single type. We claim again that this is not self-similar. To see this, let E_1 denote the end space of T_1' and consider the decomposition of its end space into $E_1 \sqcup (E - E_1)$. Suppose for contradiction that $E - E_1$ contained a homeomorphic copy of E_1. As before, since E_1 and the set of maximal ends are both closed, the homeomorphic image of E_1 avoids some neighborhood of the maximal ends, so is contained in a union of end spaces of trees homeomorphic to T_i' for a *bounded* set of indices i. We consider the maximal such index N, and again count copies of ends of type z_{N+1}. Without loss of generality, we may take $N \geq 4$. The set E_1 contains $2^{(2^N)}$ copies of z_{N+1}. Since our surface is constructed using 2^{k-1} copies of each tree T_k', the number of copies of z_{N+1} in the union of all trees T_k' for $2 \leq k \leq N$ is equal to

$$2 \cdot 2^{(2^{N-1})} + 2^2 \cdot 2^{(2^{N-2})} + \ldots + 2^{N-1} \cdot 2^{(2^{N-N+1})}$$

Set $j = 2^{N-1} + 1$. Then this sum is bounded above by

$$2^j + 2^{j-1} + \ldots + 2^{j-N+2} < 2^{(2^N)}$$

which gives the desired contradiction.

4. Proof of Theorem 1.2

We now give the proof of Theorem 1.2, following an argument of T. Tsankov. The key ingredient is a zero-one law for Baire sets invariant under certain actions of Polish groups. We recall the following standard definition and results from descriptive set theory.

Definition 4.1. A subset A of a topological space has the *Baire property* if it differs from an open set (in the sense of symmetric difference) by a meager set. (It is also sometimes said that such a set A *is Baire measurable*, to emphasize the distinction between "sets with the Baire property" and "Baire spaces", which are not equivalent notions.)

Theorem 4.2. (See [5, Theorem 8.46].) Let G be a group of homeomorphisms of a Baire space X, and assume that for all open $U, V \subset X$ there exists $g \in G$ with $gU \cap V \neq \emptyset$. Suppose that $A \subset X$ is a G-invariant set with the Baire property. Then A is either meager or has meager complement in X.

Theorem 4.3. (Lusin-Sierpinski, see e.g. [2, Corollary 1159].) Let X and Y be Polish spaces, and $f : X \to Y$ a continuous map. Then $f(X)$ has the Baire property in Y. (Such a continuous image of a Polish space is called an *analytic set*.)

Proof of Theorem 1.2. Suppose $x \preccurlyeq y$ and $y \preccurlyeq x$. Let H_x denote the ends that are locally homeomorphic to x and H_y the ends locally homeomorphic to y. If H_x is finite, then it is easy to see that $H_x = H_y$. Otherwise, we have $\overline{H_x} = \overline{H_y}$. Moreover, since every point of $\overline{H_x}$ is an accumulation point of H_y, we have that $\overline{H_x} \subset E$ is homeomorphic to a Cantor set. Denote this cantor set by C. This set C is preserved by the action of $\mathrm{Map}(\Sigma)$ on the end space.

We claim the following: if $z \in C$ has a dense orbit in C under $\mathrm{Map}(\Sigma)$, then this orbit is comeager in C. By the Baire Category Theorem, there cannot be two disjoint comeager subsets (equivalently, two meager sets whose union is C), thus, provided the claim holds, there is at most one dense orbit. This implies that x is in the same orbit as y, or equivalently it is locally homeomorphic to y. In short, the claim proves the Theorem.

To prove the claim, let $G \subset \mathrm{Homeo}(C)$ denote the quotient of $\mathrm{Map}(\Sigma)$ defined by restricting the action to C. Since $\mathrm{Map}(\Sigma)$ is a Polish group and the kernel of the restriction map is closed, so is its quotient G. Furthermore, the action of G on C has the topological transitivity property that for every open U and U' in C, there exists some $g \in G$ with $gU \cap U' \neq \emptyset$, since both U and U' intersect H_x.

Finally, any orbit of G is a set with the Baire property by Theorem 4.3, being the continuous image of a Polish space, so one may thus apply the topological zero-one law (Theorem 4.2) and conclude that a non-meager orbit is necessarily comeager in C.

Thus, it suffices to show that any point z with a dense orbit has a non-meager orbit. By a condition of Kechris-Rosendal [6, Prop 3.2], for this it suffices to show that for any open subgroup V of G, the orbit Vz is somewhere dense in C. Let V be an open subgroup. Without loss of generality, we may take V to be the subgroup consisting of homeomorphisms preserving a fixed decomposition of C into finitely many clopen sets $C = X_1 \sqcup \ldots \sqcup X_n$ (possibly permuting the clopen sets) since such open subgroups form a basis for the topology of the homeomorphisms of the Cantor set. Again, without loss of generality, assume $z \in X_1$. We will show that Vz is dense in X_1. Given some open set $U \subset X_1$, there exists $g \in G$ such that $gz \in U$. We assume also $gz \neq z$. Let W be a clopen neighborhood of z in E small enough $W \cap C \subset X_1$, $gW \cap C \subset X_1$ and $gW \cap W = \emptyset$. Now define a homeomorphism h of E to agree with g on W, agree with g^{-1} on gW, and restrict to the identity elsewhere. Then the restriction of h to C (i.e. the image of h in G) lies in V, and $h(z) = g(z) \in U$. This is what we needed to show. □

Remark 4.4. This proof came about as a response to the question: *does a homeomorphism h of $\{0,1\}^{\mathbb{Z}}$ that set-wise preserves each periodic orbit of the full shift necessarily preserve all orbits of the shift?* Tsankov [12] answered this question in the negative, showing that, as in the proof above, there is only a single dense orbit under the group of periodic-orbit preserving maps of $\{0,1\}^{\mathbb{Z}}$. The argument above is a direct translation to this setting.

Acknowledgements. K.M. was partially supported by NSF grant DMS-1844516. K.R was partially supported by NSERC Discovery grant RGPIN 06486. We would like to thank Jing Tao, Sanghoon Kwak, and Nicholas Vlamis for comments on an earlier version of this paper, and Todor Tsankov for furnishing the proof of Theorem 1.2.

References

[1] A. Bar-Natan and Y. Verberne. The grand arc graph arXiv:2110.14761 [math.GT]
[2] A. Dasgupta. Set Theory With an Introduction to Real Point Sets Springer Birkhauser, 1st ed. 2014

[3] C. Grant and K. Rafi and Y. Verberne Asymptotic Dimension of Big Mapping Class Groups arXiv:2110.03087 [math.GT]
[4] C. Horbez and Y. Qing and K. Rafi. Big mapping class groups with hyperbolic actions: Classification and applications. Journal of the Institute of Mathematics of Jussieu, page 1–32, 2021.
[5] A. S. Kechris. Classical descriptive set theory. Springer GTM 156, 1995.
[6] A. S. Kechris and C. Rosendal. Turbulence, amalgamation and generic automorphisms of homogeneous structures. Proceedings of the London Mathematical Society. 94(2), 302-350, 2007.
[7] J. Lanier and M. Loving. Curve graphs of surfaces with finite-invariance index 1. arXiv:2106.14916 [math.GT].
[8] J. Lanier and N. Vlamis. Mapping class groups with the Rokhlin property arXiv:2105.11280 [math.GT].
[9] K. Mann and K. Rafi. Large scale geometry of big mapping class groups. arXiv:1912.10914 [math.GT].
[10] J. Malestein and J. Tao Self-similar surfaces: involutions and perfection arXiv:2106.03681 [math.GT]
[11] S. Mazurkiewicz and W. Sierpinski. Contribution á la topologie des ensembles dénombrables. Fundamenta Mathematicae, 1, 17–27, 1920.
[12] T. Tsankov. Personal communication.