Divergent Models with the Failure of the Continuum Hypothesis *†‡

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October 31, 2023

Abstract

We construct divergent models of AD^+ along with the failure of the Continuum Hypothesis (CH) under various assumptions. Divergent models of AD^+ plays an important role in descriptive inner model theory; all known analyses of HOD in AD^+ models (without extra iterability assumptions) are carried out in the region below the existence of divergent models of AD^+. Our results are the first step toward resolving various open questions concerning the length of definable prewellorderings of the reals and principles implying $\neg\mathsf{CH}$, like MM, that divergent models shed light on, see Question 5.1.

1 Introduction

In this paper, we identify the reals $\mathbb{R}$ with $\mathbb{N}^{\mathbb{N}}$, the set of all infinite sequences of natural numbers equipped with the Baire topology.

Definition 1.1 *Suppose M and N are transitive models of AD^+. We say that M and N are divergent models of AD^+ if there are sets of reals $A \in M$ and $B \in N$ such that $A \notin N$ and $B \notin M$.*

*2000 Mathematics Subject Classifications: 03E15, 03E45, 03E60.

†Keywords: Divergent models, inner model theory, descriptive set theory.

‡The author's research was partially supported by the NSF Career Award DMS-1945592.

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If M, N are divergent models of AD^+, then the Wadge hierarchies of M, N "diverge", or equivalently $\wp(\mathbb{R}) \cap M \not\subseteq N$ and $\wp(\mathbb{R}) \cap N \not\subseteq M$. Woodin has shown that letting $\Gamma = \wp(\mathbb{R}) \cap M \cap N$, then $\Gamma = \wp(\mathbb{R}) \cap L(\Gamma, \mathbb{R})$ and furthermore, $L(\Gamma, \mathbb{R}) \vDash \mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$. The upper-bound consistency strength of divergent models of AD^+, as shown by Woodin, is the existence of a Woodin cardinal which is a limit of Woodin cardinals. This bound is conjectured to be exact.[1] Divergent models of AD^+ plays a very important role in descriptive inner model theory; virtually, all known analyses of HOD in strong AD^+ models are carried out below this bound (see cf. [Sar14], [ST23]).

Working in a universe satisfying CH, Woodin constructed divergent models of AD^+ [Far10]. We prove that it is consistent that there are divergent models of AD^+ while CH fails.

Theorem 1.2 *Suppose* CH *holds and there are two sets of reals A, B such that*

- $(\mathbb{R}, A)^\sharp, (\mathbb{R}, B)^\sharp$ *exist and are $\aleph_1$-universally Baire,*
- $L(A, \mathbb{R}), L(B, \mathbb{R})$ *are models of* AD^+ *such that letting $H_A = HOD^{L(A,\mathbb{R})}$ and $H_B = HOD^{L(B,\mathbb{R})}$, there is some $\alpha < min\{\omega_1^{H_A}, \omega_1^{H_B}\}$ such that the α-th real in the canonical well-order of H_A is different from the α-th real in the canonical well-order of H_B.*

Let $\mathbb{P}$ be the standard ccc forcing that adds ω_2 many Cohen reals and $g \subseteq \mathbb{P}$ be V-generic. Then in $V[g]$, there are A^*, B^* and embeddings j_A, j_B such that

1. *$j_A : L(A, \mathbb{R}^V) \to L(A^*, \mathbb{R}^{V[g]})$, $j_B : L(B, \mathbb{R}^V) \to L(B^*, \mathbb{R}^{V[g]})$ fix all ordinals, and*
2. *$L(A^*, \mathbb{R}^{V[g]}), L(B^*, \mathbb{R}^{V[g]})$ are divergent models of* AD^+.

Corollary 1.3 *Con(*ZFC+ *there is a Woodin limit of Woodin cardinals) implies Con(*CH *fails and there are divergent models of* AD^+*).*

Proof. By results of Woodin's (see [Far10]), the hypothesis of Theorem 1.2 is consistent relative to the existence of a Woodin limit of Woodin cardinals. The corollary follows from Theorem 1.2. □

The following theorem is folklore. We include the proof here for self-containment. It is used in the proof of Corollary 1.5. A forcing $\mathbb{P}$ is said to be *weakly proper* if whenever $g \subset \mathbb{P}$ is V-generic, for any ordinal α, $\wp_{\omega_1}^{V[g]}(\alpha) \subset \wp_{\omega_1}^{V}(\alpha)$. Γ_∞ denotes the collection of universally Baire sets.

[1] It has come to my attention recently that G. Sargsyan (unpublished) has shown this.

Theorem 1.4 *Assume there is a proper class of Woodin cardinals and $A \subseteq \mathbb{R}$ is universally Baire. Suppose $\mathbb{P}$ is weakly proper. Then for any V-generic $g \subseteq \mathbb{P}$, there is some universally Baire set $B \in V$ such that letting B^* be the canonical interpretation of B in $V[g]$, A is Wadge reducible to B^*.*

Corollary 1.5 *Assume there is a proper class of Woodin cardinals. Suppose A, B are as in the hypothesis of Theorem 1.2. Furthermore, assume that $\Gamma_\infty \subset L(A, \mathbb{R}) \cap L(B, \mathbb{R})$. Let $\mathbb{P}$ be the forcing that adds ω_2 Cohen reals and $g \subseteq \mathbb{P}$ be V-generic. Then in $V[g]$, $\Gamma_\infty \subset L(A^*, \mathbb{R}^{V[g]}) \cap L(B^*, \mathbb{R}^{V[g]})$.*

Now we address the question of whether the hypothesis of Corollary 1.5 is consistent. We construct divergent models of AD^+ that contain the collection of universally Baire sets from a strong hypothesis. We are hopeful that with recent advancement in descriptive inner model theory, this hypothesis can be shown to be consistent.

Definition 1.6 *Let $\mathcal{M}$ be a hybrid premouse. We say that $\mathcal{M}$ is **appropriate premouse** if $\mathcal{M} = (|\mathcal{M}|, \in, \mathbb{E}, \mathbb{S})$ is an amenable J-structure that satisfies:*

1. *the predicate $\mathbb{S}$ codes $(\mathcal{P}_0, \Sigma)$, where $\mathcal{P}_0 = (\mathcal{M}|\delta_0)^{\sharp}$[2] for some Woodin cardinal δ_0 such that $\mathcal{P}_0$ is an lsa hod premouse and Σ is the short-tree strategy of $\mathcal{P}_0$;* [3]

2. *there is a proper class of Woodin cardinals and a Woodin limit of Woodin cardinals $> \delta_0$ as witnessed by a fine-extender sequence (in the sense of [Ste10]) coded by $\mathbb{E}$;*

3. *for any set generic h, Σ has a canonical interpretation Σ^h in $V[h]$; more precisely, there is a term-relation τ such that for all generic h, $\tau^h = \Sigma^h$;*

4. *in all generic extensions $V[g]$ of V for which $\mathcal{P}_0$ is countable, $\Sigma^g \notin (\Gamma_\infty)^{V[g]}$ but letting $\Gamma(\mathcal{P}_0, \Sigma^g)$ be the set of A such that there is a countable $\mathcal{T}$ according to Σ^g such that $A \leq_w \Sigma^g_{\mathcal{T},\mathcal{M}(\mathcal{T})}$, then $\Gamma(\mathcal{P}_0, \Sigma^g) = (\Gamma_\infty)^{V[g]}$. This essentially says that all lower-level strategies of Σ^g or its iterates are in $(\Gamma_\infty)^{V[g]}$.*

$(\mathcal{M}, \Psi)$ is an appropriate mouse if $\mathcal{M}$ is an appropriate premouse and Ψ is an iteration strategy for $\mathcal{M}$ such that if $i : \mathcal{M} \to \mathcal{N}$ be an iteration according to Ψ, then for any $\mathcal{N}$-generic g, $i(\tau)^g = (\Psi_N)^{sh}_{\mathcal{P}_0} \restriction \mathcal{N}[g]$, here $(\Psi_N)^{sh}_{\mathcal{P}_0}$ is the restriction of the tail strategy Ψ_N on N to short trees on $\mathcal{P}_0$.

[2] By this we mean $\mathcal{P}_0$ is the first active initial segment of $\mathcal{M}$ extending $\mathcal{M}|\delta_0$.

[3] See [ST23] for a detailed theory of lsa hod mice. Roughly, $\mathcal{P}_0$ is a hod mouse with the largest Woodin cardinal δ_0 and the least $< \delta_0$-strong cardinal is a limit of Woodin cardinals.

It is not known if the existence of an appropriate mouse is consistent; a weaker version of this is shown to be consistent in [ST19] and plays a key role in determining the exact consistency strength of Woodin's Sealing of the Universally Baire sets. Property 4, namely the assumption on Σ, is an abstraction of properties of excellent mice defined in [ST19] and is the key property that allows us to prove Theorem 1.7. The intuition giving rise to 4 comes from the construction of models of $\mathsf{LSA}-\mathsf{over}-\mathsf{UB}$ in [ST19], where the LSA model is generated by a pair $(\mathcal{P},\Sigma)$ such that Σ is a short-tree strategy for an lsa-type hod premouse $\mathcal{P}$ and $\Gamma(\mathcal{P},\Sigma)=\Gamma_\infty$. In the proof of Theorem 1.7, we use this property to show that Γ_∞ (in a generic extension of the appropriate mouse) is in both divergent models, by showing the interpretation of τ by the generic is in both models. The main difference between an appropriate mouse and an excellent mouse lies in property 2. We do not yet have a theory of layered-hod mice that reaches the level of "$\mathsf{ZFC}+$ there is a Woodin cardinal which is a limit of Woodin cardinals" (WLW), but such a theory exists for least-branch hod mice ([Ste22]), so it seems very plausible that the existence of appropriate mice is consistent.[4]

The following property abstracts out some of the features of countable substructures of models obtained by fully-backgrounded constructions (see cf. [Ste10, Nee02]). We say that V satisfies *countable self-iterability* if for any cardinal δ and any countable $X\prec V_{\delta+1}$, the transitive collapse M of X is fully iterable with δ-universally Baire strategy Λ; furthermore, letting $\tau: M\to X$ be the uncollapse map, Λ is τ-realizable, i.e. whenever $\pi: M\to N$ is an iteration map according to Λ with $|N|<\omega_1$, there is some $\sigma: N\to V_{\delta+1}$ such that $\tau=\sigma\circ\pi$.

Theorem 1.7 *Suppose $V=L[\vec{E}]$ is an extender model such that in V, there is a proper class of Woodin cardinals and countable self-iterability holds. Suppose there is an appropriate mouse $(\mathcal{M},\Psi)$ such that $\Psi\in\Gamma_\infty$. Then in some generic extension of $\mathcal{M}$, there are divergent models of AD^+ N_1,N_2 such that $\Gamma_\infty\subset N_1\cap N_2$.*

Remark 1.8 *Theorem 1.7 relates to Question 5.1(i) in light of recent development in the core model induction; in particular, one can show under MM that Γ_∞ contains very complicated mice, e.g. there are Wadge initial segments Γ such that $L(\Gamma)\vDash\mathsf{AD}_{\mathbb{R}}+$ "Θ is regular" and much more. One can hope that MM implies the existence of mice that satisfies WLW with universally Baire iteration strategies. 5.1(ii) is a weakening of 5.1(i) as MM implies $\underset{\sim}{\delta}^1_2=\omega_2$. If 5.1(ii) was true, then Γ_∞ is "large" in that $o(\Gamma_\infty)>\omega_2$. It is open whether $o(\Gamma_\infty)$ could be $>\omega_3$.*

[4]What is missing from [Ste22] is a theory of short-tree strategy mice in the least-branch hierarchy.

Acknowledgement: The author would like to thank G. Sargsyan for many illuminating discussions involving the topic of this paper. We thank the referee for comments that improve the paper. The author would also like to thank the NSF for its generous support through NSF CAREER grant DMS-1945592.

2 Preliminaries

Let Θ be the supremum of ordinals γ such that there is a surjection from $\mathbb{R}$ onto γ. A very useful extension of the Axiom of Determinacy, AD, is a theory called AD^+ isolated by Woodin. AD^+ consists of the following statements.

- $\mathsf{DC}_{\mathbb{R}}$.
- Every set of reals has an ∞-Borel code. (An ∞-Borel code is a pair (S,φ) where S is a set of ordinals and φ is a formula of set theory. Let $\mathfrak{B}_{(S,\varphi)} = \{r \in \mathbb{R} : L[S,r] \vDash \varphi(S,r)\}$. (S,φ) is an ∞-Borel code for a set $A \subseteq \mathbb{R}$ if and only if $A = \mathfrak{B}_{(S,\varphi)}$.)
- $\mathsf{Ordinal\ Determinacy}$, which is the statements that for every $\lambda < \Theta$, $X \subseteq \mathbb{R}$, and continuous function $\pi : {}^{\omega}\lambda \to \mathbb{R}$, the two player game on λ with payoff set $\pi^{-1}(X)$ is determined.

It is conjectured that under $\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}}$, AD implies AD^+. All known models of AD satisfy AD^+.

For any model M of AD^+, the ordinal Θ^M is defined to be the supremum of ordinals γ such that there is a surjection from $\mathbb{R}$ onto γ in M. For any set of reals A in M, let $w(A)$ denote the Wadge rank of A in M. A basic result due to R. Solovay, is that Θ^M is supremum of the Wadge ranks of sets of reals A in M.

We summarize basic facts about (weakly) homogeneously Suslin and universally Baire sets we need. For a more detailed discussion, the reader should consult for example [Ste09].

Given an uncountable cardinal κ, and a set Z, $meas_\kappa(Z)$ denotes the set of all κ-additive measures on $Z^{<\omega}$. If $\mu \in meas_\kappa(Z)$, then there is a unique $n < \omega$ such that $Z^n \in \mu$ by κ-additivity; we let this $n = dim(\mu)$. If $\mu, \nu \in meas_\kappa(Z)$, we say that μ *projects to* ν if $dim(\nu) = m \leq dim(\mu) = n$ and for all $A \subseteq Z^m$,

$$A \in \nu \Leftrightarrow \{u : u \restriction m \in A\} \in \mu.$$

In this case, there is a natural embedding from the ultrapower of V by ν into the ultrapower of V by μ:

$$\pi_{\nu,\mu} : Ult(V, \nu) \to Ult(V, \mu)$$

defined by $\pi_{\nu,\mu}([f]_\nu) = [f^*]_\mu$ where $f^*(u) = f(u \restriction m)$ for all $u \in Z^n$. A tower of measures on Z is a sequence $\langle \mu_n : n < k\rangle$ for some $k \leq \omega$ such that for all $m \leq n < k$, $dim(\mu_n) = n$ and μ_n projects to μ_m. A tower $\langle \mu_n : n < \omega\rangle$ is *countably complete* if the direct limit of $\{Ult(V, \mu_n), \pi_{\mu_m,\mu_n} : m \leq n < \omega\}$ is well-founded. We will also say that the tower $\langle \mu_n : n < \omega\rangle$ is well-founded.

Recall we identify the set of reals $\mathbb{R}$ with the Baire space ${}^\omega\omega$.

Definition 2.1 *Fix an uncountable cardinal κ. A function $\bar{\mu} : \omega^{<\omega} \to meas_\kappa(Z)$ is a κ-**complete homogeneity system** with support Z if for all $s, t \in \omega^{<\omega}$, writing μ_t for $\bar{\mu}(t)$:*

1. *$dom(\mu_t) = dom(t)$,*
2. *$s \subseteq t \Rightarrow \mu_t$ projects to μ_s.*

Often times, we will not specify the support Z; instead, we just say $\bar{\mu}$ is a κ-complete homogeneity system.

*A set $A \subseteq \mathbb{R}$ is κ-**homogeneous** iff there is a κ-complete homogeneity system $\bar{\mu}$ such that*

$$A = S_\mu =_{def} \{x : \bar{\mu}_x \textit{ is countably complete}\}.$$

A is homogeneous if it is κ-homogeneous for all κ. Let Hom_∞ be the collection of all homogeneous sets.

Definition 2.2 *Fix an uncountable cardinal κ. A function $\bar{\mu} : \omega^{<\omega} \to meas_\kappa(Z)$ is a κ-**complete weak homogeneity system** with support Z if it is injective and for all $t \in \omega^{<\omega}$:*

1. *$dom(\mu_t) \leq dom(t)$,*
2. *if μ_t projects to ν, then there is some $i < dom(\mu_t)$ such that $\nu = \mu_{t\restriction i}$.*

*A set $A \subseteq \mathbb{R}$ is κ-**weakly homogeneous** iff there is a κ-complete weak homogeneity system $\bar{\mu}$ such that*

$$A = W_{\bar{\mu}} =_{def} \{x : \exists (i_k : k < \omega) \in \omega^\omega \langle \mu_{x|i_k} : k < \omega\rangle \textit{ is well-founded}\}.$$

A is weakly homogeneous if it is κ-weakly homogeneous for all κ. Let $wHom_\infty$ be the collection of all weakly homogeneous sets.

Definition 2.3 *$A \subseteq \mathbb{R}$ is **κ-universally Baire** if there are trees $T, U \subseteq (\omega \times ON)^{<\omega}$ that are κ-absolutely complemented, i.e. $A = p[T] = \mathbb{R}\backslash p[U]$ and whenever $\mathbb{P}$ is a forcing such that $|\mathbb{P}| < \kappa$ and $g \subseteq \mathbb{P}$ is V-generic, in $V[g]$, $p[T] = \mathbb{R}\backslash p[U]$. In this case, we let $A_g = p[T]$ be the canonical interpretation of A in $V[g]$.*

*A is **universally Baire** if A is κ-universally Baire for all κ. Let Γ_∞ be the collection of all universally Baire sets.*

We remark that if A is κ-universally Baire as witnessed by pairs (T_1, U_1) and (T_2, U_2) and $\mathbb{P} \in V_\kappa$ and $g \subset \mathbb{P}$ is V-generic, then $A_g = p[T_1] = p[T_2]$, i.e. A_g does not depend on the choice of absolutely complemented trees that witness A is κ-universally Baire. A similar remark applies to κ-(weakly) homogeneously Suslin sets.

Suppose there is a proper class of Woodin cardinals. The following are some standard results about universally Baire sets we will use throughout our paper. The proof of these results can be found in [Ste09].

1. $\mathrm{Hom}_\infty = \mathrm{wHom}_\infty = \Gamma_\infty$.
2. For any $A \in \Gamma_\infty$, $L(A, \mathbb{R}) \vDash \mathsf{AD}^+$; furthermore, given such an A, there is a $B \in \Gamma_\infty$ such that $B \notin L(A, \mathbb{R})$ and $A \in L(B, \mathbb{R})$. In fact, $A^\sharp$ is an example of such a B.
3. Suppose $A \in \Gamma_\infty$. Let B be the code for the first order theory with real parameters of the structure $(HC, \in, A)$ (under some reasonable coding of HC by reals). Then $B \in \Gamma_\infty$ and if g is V-generic for some forcing, then in $V[g]$, $B_g \in \Gamma_\infty$ is the code for the first order theory with real parameters of $(HC^{V[g]}, \in, A_g)$.

Under the same hypothesis, the results above also imply that

- Γ_∞ is closed under Wadge reducibility,
- if $A \in \Gamma_\infty$, then $\neg A \in \Gamma_\infty$,
- if $A \in \Gamma_\infty$ and g is V-generic for some forcing, then there is an elementary embedding $j : L(A, \mathbb{R}) \to L(A_g, \mathbb{R}_g)$, where $\mathbb{R}_g = \mathbb{R}^{V[g]}$.

Finally, the reader should consult [Ste10] for the basics of inner model theory. This is the background needed to follow the proof of Theorem 1.7. Consult [ST23, ST19] for more information on the theory of short-tree strategy mice related to lsa hod mice and appropriate mice; we will not need this material in this paper, however.

In the following, we fix a natural coding of (ω_1, ω_1)-iteration strategies for countable mice by sets of reals, e.g. we fix a function $\tau : HC \to \mathbb{R}$ that codes elements of HC by reals as in [Woo10, Chapter 2] and $Code : \wp(HC) \to \wp(\mathbb{R})$ is the induced function given by: $Code(A) = \tau[A]$.

3 Divergent models of AD^+ and the failure of CH

Proof.[Proof of Theorem 1.2] Fix $A, B, \mathbb{P}, g$ as in the statement of the theorem. Let $\mathbb{R}_g = \mathbb{R}^{V[g]}$. Let α be the least such that letting x_A be the α-th real in the canonical well-order of H_A and x_B be the α-th real in the canonical well-order of H_B, then $x_A \neq x_B$.

Let (U, φ) and (W, ψ) be ∞-Borel codes for A, B respectively. Let $s \in (\wp_{\omega_1}(\omega_2))^{V[g]}$. Note that s is added by a countable suborder of $\mathbb{P}$ by the countable chain condition of $\mathbb{P}$. Let $\mathbb{R}_s = \mathbb{R}^{V[s]}$ and define A_s by: for all $x \in \mathbb{R}_s$,

$$x \in A_s \Leftrightarrow L[U, x] \vDash \varphi[x, U].$$

We define B_s using (W, ψ) in a similar fashion. Let

$$M_s = L(A_s, \mathbb{R}_s),$$

and

$$N_s = L(B_s, \mathbb{R}_s),$$

Claim 1: Suppose $t \in (\wp_{\omega_1}(\omega_2))^{V[g]}$ and $s \subseteq t$. Then the map $\pi^A_{s,t} : M_s \to M_t$ defined by: $\pi^A_{s,t} \restriction \mathbb{R}_s \cup ON = id$ and $\pi^A_{s,t}(A_s) = A_t$ is an elementary embedding. Similarly, $\pi^B_{s,t}$ is an elementary embedding.

Proof. We prove the statement for A. This follows from [Woo10, Theorem 10.63, 2.27–2.29] and [Far10, Theorem 6.3, 6.4]. The key points are:

- All sets of reals in $L(A, \mathbb{R})$ are $\aleph_1$-universally Baire, as $(\mathbb{R}, A)^\sharp$ is $\aleph_1$-universally Baire.
- The suborder of $\mathbb{P}$ adding s is weakly proper and countable, so $\pi^A_{\emptyset,s} \restriction ON = id$ and $\pi^A_{\emptyset,s}(A) = A_s$ is the canonical interpretation of A in $V[s]$.

□

Let M_∞ be the direct limit of $\mathcal{F}_A = \{M_s, \pi^A_{s,t} : s \subseteq t \in (\wp_{\omega_1}(\omega_2))^{V[g]}\}$ and N_∞ be the direct limit of $\mathcal{F}_B = \{N_s, \pi^B_{s,t} : s \subseteq t \in (\wp_{\omega_1}(\omega_2))^{V[g]}\}$.

Claim 2: M_∞, N_∞ are well-founded.

Proof. The directed systems $\mathcal{F}_A, \mathcal{F}_B$ consist of well-founded models and the directed relation ($\subseteq$) is in fact countably directed, i.e. if $(s_n : n < \omega)$ is such that for all n, $s_n \in (\wp_{\omega_1}(\omega_2))^{V[g]}$, then there is some $s \in (\wp_{\omega_1}(\omega_2))^{V[g]}$ such that $s_n \subseteq s$ for all n. Therefore, M_∞, N_∞ are well-founded as any witness that M_∞ (N_∞) is ill-founded has preimage in some M_s (N_s). □

Let

$$\pi^A : L(A, \mathbb{R}) \to M_\infty = L(A_\infty, \mathbb{R}_g)$$

and

$$\pi^B : L(B, \mathbb{R}) \to M_\infty = L(B_\infty, \mathbb{R}_g)$$[5]

be the direct limit maps. Note that $\pi^A \restriction ON = \pi^B \restriction ON = id$. Now we claim that M_∞, N_∞ are divergent models of AD^+ in $V[g]$. This finishes the proof of the theorem.

We note that $\pi^A(x_A) = x_A$ is the α-th real in the canonical well order of HOD^{M_∞}. This follows from the fact that π^A is elementary and fixes all ordinals. Similarly, $\pi^B(x_B) = x_B$ is the α-th real in the canonical well order of HOD^{M_∞}. If M_∞, N_∞ are compatible, then the α-th real in HOD^{M_∞} must be equal to the α-th real in HOD^{N_∞}. To see this, suppose without loss of generality $\wp(\mathbb{R})^{M_\infty} \subseteq \wp(\mathbb{R})^{N_\infty}$. Suppose $\beta \leq \Theta^{N_\infty}$ is such that $\wp(\mathbb{R})^{M_\infty} = \{A \in N_\infty : w(A) < \beta\}$. This easily gives HOD^{M_∞} is OD in N_∞ and that the canonical well-order of OD-reals in M_∞ is compatible with the canonical well-order of OD-reals in N_∞. So $x_A = x_B$. Contradiction. □

Proof.[Proof of Theorem 1.4] Fix $A, \mathbb{P}, g$ as in the statement of the theorem. Let κ be a measurable cardinal such that

- $\mathbb{P} \in V_\kappa$.
- A is κ-homogeneous.
- Every κ-homogeneously Suslin set in $V[g]$ is universally Baire in $V[g]$.

Let $\bar{\mu} = (\mu_s : s \in \omega^{<\omega})$ be a homogeneous system witnessing A is κ-homogeneously Suslin, i.e.

$$x \in A \Leftrightarrow (\mu_{x|i} : i < \omega) \text{ is countably complete.}$$

[5]It is clear that $\mathbb{R}^{M_\infty} = \mathbb{R}^{N_\infty} = \mathbb{R}_g$.

Since $\mathbb{P} \in V_\kappa$, for each $s \in \omega^{<\omega}$, there is $\nu \in meas_\kappa(\kappa^{|s|})$ in V such that $\nu^* = \mu_s$, where $\nu^* = \{A \in V[g] : \exists B \in \nu(B \subseteq A)\}$ is the canonical extension of ν in $V[g]$. By the weak properness of $\mathbb{P}$, there is a countable set of measures $\sigma \subset meas_\kappa(\bigcup_n \kappa^n)$ in V such that

$$\bar{\mu} \subseteq \sigma^* = \{\nu^* : \nu \in \sigma\}.$$

In V, let $\bar{\nu} = (\nu_s : s \in \omega^{<\omega})$ be an enumeration of σ such that

(i) for each $s \in \omega^{<\omega}$, ν_s concentrates on $\kappa^{|s|}$;

(ii) if ν_t projects to ν, then there is some $i < dom(\nu_t)$ such that $\nu_{t|i} = \nu$.

Now define the following set B, which is just the κ-homogeneously Suslin set given by $\bar{\nu}$: for $x \in \mathbb{R}$,

$$x \in B \Leftrightarrow (\nu_{x|k} : k < \omega) \text{ is countably complete.}$$

Let B^* be the canonical extension of B induced by $\bar{\nu^*} = (\nu_s^* : s \in \omega^{<\omega})$ in $V[g]$. Thus, B^* is κ-homogeneously Suslin and hence is universally Baire in $V[g]$. Let $f : \omega^{<\omega} \to \omega^{<\omega}$ be:

$$f(s) = t \text{ where } t \text{ is such that } \mu_s = \nu_t^*.$$

By the properties of $\bar{\nu}$ and $\bar{\mu}$, we have

(a) $f(s)$ has the same length as s for every $s \in \omega^{<\omega}$.

(b) f is order preserving, i.e. if s_0 is an initial segment of s_1 then $f(s_0)$ is an initial segment of $f(s_1)$.

Let $\hat{f} : \mathbb{R}^{V[g]} \to \mathbb{R}^{V[g]}$ be the continuous map induced by f:

$$\hat{f}(x) = \bigcup_{i<\omega} f(x|i).$$

We have for any $x \in \mathbb{R}^{V[g]}$:

$$\begin{aligned} x \in A &\Leftrightarrow (\mu_{x|i} : i < \omega) \text{ is countably complete} \\ &\Leftrightarrow (\nu^*_{f(x|i)} : i < \omega) \text{ is countably complete} \\ &\Leftrightarrow \hat{f}(x) \in B^* \end{aligned}$$

Thus $\hat{f}$ witnesses A is Wadge reducible to B^*.

□

Proof.[Proof of Corollary 1.5] First note that $\mathbb{P}$ is weakly proper so we can apply Theorem 1.4. Now note that

$$o(\Gamma_\infty)^{V[g]} = \sup[j_A \restriction o(\Gamma_\infty^V)] = \sup[j_B \restriction o(\Gamma_\infty^V)]. \tag{1}$$

Here, $o(\Gamma_\infty)$ is the length of the Wadge prewellorder on Γ_∞. To see 1, note that for each $X \in \Gamma_\infty$, $j_A(X), j_B(X) \in \Gamma_\infty^{V[g]}$[6] and is the canonical interpretation of X, so $j_A(X) = j_B(X)$. Now apply Theorem 1.4 to see that $j_A \restriction \Gamma_\infty^V = j_B \restriction \Gamma_\infty^V$ is cofinal in $\Gamma_\infty^{V[g]}$.

Finally, for each $X \in \Gamma_\infty$, X is Wadge reducible to A ($X \leq_w A$) in $L(A, \mathbb{R})$. To see this, note that $A \notin \Gamma_\infty$. Otherwise, by the facts mentioned at the end of Section 2, there is some $C \in \Gamma_\infty$ such that $A \in L(C, \mathbb{R})$; futhermore, $C^\sharp \in \Gamma_\infty$, so $C^\sharp \notin L(A, \mathbb{R})$. This contradicts $\Gamma_\infty \subset L(A, \mathbb{R})$. Since $A \notin \Gamma_\infty, \Gamma_\infty \subset L(A, \mathbb{R})$, and $L(A, \mathbb{R}) \vDash \mathsf{AD}^+$, the claim is established.

By elementarity $j_A(X) \leq_w A^*$. By (1), $\Gamma_\infty^{V[g]} \subset L(A^*, \mathbb{R}^{V[g]})$. Similarly, $\Gamma_\infty^{V[g]} \subset L(B^*, \mathbb{R}^{V[g]})$ □

4 Divergent models of AD^+ over UB

In this section, we give the proof of Theorem 1.7. The proof closely resembles Woodin's original proof of the existence of divergent models of AD^+ in [Far10, Section 6]; the reader is advised to consult that proof for details we omit here.

Let $\mathcal{M}, \Psi$ be as in the statement of the theorem and assume this is a minimal such mouse. Let $\mathcal{P}_0 = (\mathcal{M}|\delta_0)^\sharp$ be as in clause 1 of Definition 1.6. Let $\lambda = \lambda^{\mathcal{M}} > \delta_0$ be the Woodin limit of Woodin cardinals of $\mathcal{M}$. Let $c \in V$ be a Cohen real over $\mathcal{M}$ and let $A \in \Gamma_\infty$ be such that c is OD in $L(A, \mathbb{R})$.

The existence of A follows from countable self-iterability and the argument in [Far10, Section 6.2]. We sketch a proof here. A codes a pair $(P, \Lambda \restriction HC)$ where P is the transitive collapse of a countable $X \prec V_{\delta+1}$ such that $c \in X$ and δ is large enough that δ-universally Baire sets are universally Baire, and Λ is a δ-universally Baire strategy of P. P is an extender model since $V = L[\vec{E}]$ is an extender model. Therefore, A is universally Baire. So $L(A, \mathbb{R}) \vDash \mathsf{AD}^+$. By replacing P by $Hull^P(\{c\})$ we may assume P projects to ω and Λ is the unique iteration strategy for P. Since $c \in P$, P is an extender model, and $\Lambda \restriction HC$ can be extended to a unique $\omega_1 + 1$-iteration strategy for P in $L(A, \mathbb{R})$, the direct limit of all countable nondropping iterates of M via Λ is defined and is OD in $L(A, \mathbb{R})$ and hence c is OD in $L(A, \mathbb{R})$.

[6] This follows from [Woo10, Theorem 10.63]. The maps j_A, j_B maps each $X \in \Gamma_\infty^V$ to its canonical interpretation in $V[g]$.

We may and do choose A such that $Code(\Psi) <_w A$ as witnessed by a real x^*.[7] To see such an A exists, suppose $Code(\Psi) = p[T] = \mathbb{R}\backslash p[U]$, where T, U are trees witnessing $Code(\Psi)$ is δ-universally Baire for some δ. By choosing A coding the first order theory of $(HC, \in, (P, \Lambda))$ with real parameters such that

- P is the transitive collapse of some countable $X \prec V_{\gamma+1}$ and
- $(T, U) \in X$ for γ sufficiently large that Λ, the strategy for P, is universally Baire,

we can compute Ψ from A as follows. Note that Λ exists by countable self-iterability and since $\Lambda \in \Gamma_\infty$, so is A. Let $x \in Code(\Psi) = p[T]$, let $\pi : P \to N$ be the iteration map that is induced by a genericity iteration according to Λ to make x generic for the extender algebra at the first Woodin cardinal of N; we assume the first Woodin cardinal is $< \gamma$. Let (T^*, U^*) be the image of (T, U) under the transitive collapse map τ and $(\tilde{T}, \tilde{U}) = \pi(T^*, U^*)$. We claim that $N[x] \vDash x \in p[\tilde{T}]$; otherwise, since $\tilde{T}, \tilde{U}$ are absolutely complemented for forcings of size the first Woodin cardinal of N, $N[x] \vDash x \in p[\tilde{U}]$. Since Λ is a τ-realizable strategy, there is an embedding $\sigma : N \to V_{\gamma+1}$ such that $\tau = \sigma \circ \pi$. This easily gives $x \in p[U]$. Contradiction. Similarly, if $x \in p[U]$, then $N[x] \vDash x \in p[\tilde{U}]$. The above calculations show that $Code(\Psi)$ is projective in $Code(\Lambda)$: for any $x \in \mathbb{R}$, $x \in Code(\Psi)$ if and only if there is a non-dropping, countable tree $\mathcal{T}$ with last model N according to Λ such that letting $\pi : P \to N$ be the iteration map, $x \in p[\pi(T^*)]$. By the choice of A, $Code(\Psi)$ is Wadge reducible to A.

Say c is the α-th real in the canonical well-order of $HOD^{L(A,\mathbb{R})}$. Let $C = B^\sharp$, where B codes the first order theory of $(HC, \in, A)$ with real parameters; again, $C \in \Gamma_\infty$ and hence $L(C, \mathbb{R}) \vDash \mathsf{AD}^+$. Let $\pi : \mathcal{M} \to \mathcal{N}$ be the map induced by a countable iteration according to Ψ above $\mathcal{P}_0$ such that

1. letting $\lambda^* = \pi(\lambda)$, then $(C \restriction \lambda^*, \mathbb{R} \restriction \lambda^*)$ is in $\mathcal{N}[g]$, where $g \in V$ is $\mathcal{N}$-generic for $\pi(W_\lambda^{\mathcal{M}}) =_{def} W_{\lambda^*}^{\mathcal{N}}$, the λ^*-generator extender algebra of $\mathcal{N}$ at λ^*,[8]
2. $\mathbb{R} \cap L[C \restriction \lambda^*] = \mathbb{R}^{\mathcal{N}[g]}$ and $L(C \restriction \lambda^*, \mathbb{R} \restriction \lambda^*) \prec L(C, \mathbb{R})$,
3. $c, x^* \in \mathbb{R}^{\mathcal{N}[g]}$.

[7] This means x^* induces a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $a \in Code(\Psi)$ if and only if $f(a) \in A$. Recall the function $Code$ introduced in Section 2 that codes subsets of HC by sets of reals in a natural way.

[8] Since CH holds in V, we identity $(\mathbb{R}, C)$ with a subset of ω_1 that codes it in a reasonable way.

The proof of these items, making substantial use of the fact that λ is Woodin limit of Woodin cardinals, is the same as in [Far10, Section 6.3]. So in $\mathcal{N}[g]$, there is an $\aleph_1$-universally Baire set A[9] and two reals c, x such that

4. $L(A, \mathbb{R}) \vDash \mathsf{AD}^+$,

5. c is Cohen over $\mathcal{N}$ and c is the α-th real in the canonical well-order of $HOD^{L(A,\mathbb{R})}$,

6. $\pi(\tau)^g <_w A$ as witnessed by x. [10]

We note that clauses 4 and 5 follow from clause 2; clause 6 follows from clause 3 and the choice of A.

Say $p \in g$ forces (4)–(6). Note that by appropriateness of $\mathcal{N}$ (clauses 3 and 4) and (6), in $\mathcal{N}[g]$, $\Gamma_\infty \subset L(A, \mathbb{R})$. Let $g_1 \times g_2 \subset W^{\mathcal{N}}_{\lambda^*} \times W^{\mathcal{N}}_{\lambda^*}$ be $\mathcal{N}$-generic and contains (p, p). In $\mathcal{N}[g_1 \times g_2]$, for $i \in \{1, 2\}$, there is a triple (A_i, c_i, x_i) satisfying (4)–(6) for $\mathcal{N}[g_i]$. As in [Far10, Section 6.3] and the proof of Theorem 1.2, in $\mathcal{N}[g_1 \times g_2]$, there are sets A^*_1, A^*_2 and embeddings $\pi_i : L(A_i, \mathbb{R}^{\mathcal{N}[g_i]}) \to L(A^*_i, \mathbb{R}^{\mathcal{N}[g_1 \times g_2]})$ that fix the ordinals.

By (6), we have that $\pi(\tau)^{\mathcal{N}[g_1 \times g_2]} = \pi(\tau)^{\mathcal{N}[g_2 \times g_1]} \in L(A^*_i, \mathbb{R}^{\mathcal{N}[g_1 \times g_2]})$ for $i \in \{1, 2\}$. Therefore, by appropriateness,

$$\Gamma_\infty^{\mathcal{N}[g_1 \times g_2]} \subset L(A^*_1, \mathbb{R}^{\mathcal{N}[g_1 \times g_2]}) \cap L(A^*_2, \mathbb{R}^{\mathcal{N}[g_1 \times g_2]}). \tag{2}$$

As in [Far10, Section 6.3], $\pi_1(c_1) = c_1 \neq \pi_2(c_2) = c_2$ as c_1, c_2 are mutually generic over $\mathcal{N}$. So in $\mathcal{N}[g_1 \times g_2]$

$$L(A^*_1, \mathbb{R}^{\mathcal{N}[g_1 \times g_2]}), L(A^*_2, \mathbb{R}^{\mathcal{N}[g_1 \times g_2]}) \text{ are divergent models of } \mathsf{AD}^+. \tag{3}$$

By elementarity of π applied to (2) and (3), in a generic extension of $\mathcal{M}$, there are divergent models of AD^+ M_1, M_2 such that $\Gamma_\infty \subset M_1 \cap M_2$.

Remark 4.1 *We note in the construction above, letting g be a generic over $\mathcal{M}$ such that in $\mathcal{M}[g]$ there are divergent models M_1, M_2 as above, letting $\Delta = M_1 \cap M_2 \cap \wp(\mathbb{R})$, then $\Gamma_\infty^{\mathcal{M}[g]} \subsetneq \Delta$. This is because $\tau_g \in M_1 \cap M_2$. By a result of Woodin, $L(\Delta) \cap \wp(\mathbb{R}) = \Delta$ and $L(\Delta) \vDash \mathsf{AD}_{\mathbb{R}}$, therefore, there are Suslin co-Suslin sets in $M_1 \cap M_2$ that are not universally Baire.*

[9] In $\mathcal{N}[g]$, $C \upharpoonright \lambda^*$ is $\aleph_1$-universally Baire, not necessarily fully universally Baire.

[10] Recall that τ is the term relation in $\mathcal{M}$ that interprets the short-tree strategy Σ in all generic extensions of $\mathcal{M}$.

5 Open questions

We collect some open problems concerning divergent models of AD^+. First, we do not know if divergent models of AD^+ is consistent with or follows from various other strong hypotheses that imply CH fails.

Question 5.1 *1. Does* MM *imply there are divergent models of* AD^+*?*

2. Is the theory "there are divergent models of $\mathsf{AD}^+ + \delta^1_2 = \omega_2$*" consistent?*

One way to answer the following question is to show it is possible to construct appropriate mice.

Question 5.2 *Is the theory "there is a proper class of Woodin cardinals and there are divergent models of* AD^+ M *and* N *such that* $\Gamma_\infty \subset M \cap N$*" consistent?*

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