AUTOMATIC CONTINUITY FOR HOMEOMORPHISM GROUPS AND BIG MAPPING CLASS GROUPS

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Abstract. We show that, for any manifold M, compact or homeomorphic to the interior of a compact manifold, the homeomorphism group of M has automatic continuity. The same is true for the relative homeomorphism group $\mathrm{Homeo}(M, X)$ where X is homeomorphic to the union of a Cantor set and a (possibly empty) finite set, and the big *mapping class group* $\mathrm{Homeo}(M, X)/\mathrm{Homeo}_0(M, X)$. In other words, the algebraic structure of these groups is extremely rigid, and determines their topology in a very strong way.

1. Introduction

In [6] it was shown that the group of homeomorphisms of a compact manifold M (equipped with the compact-open topology) satisfies *automatic continuity*: every homomorphism from this group to any separable, topological group is necessarily continuous. The proof also applies to compact manifolds with boundary and to the subgroup $\mathrm{Homeo}(M, X)$ of homeomorphisms of any such manifold M preserving a submanifold $X \subset M$ of dimension at least one.

Here we treat the remaining cases where M is homeomorphic to the interior of a compact manifold, where X is a submanifold of dimension 0, and also where X is a Cantor set. We also discuss automatic continuity for *mapping class groups* of infinite type surfaces, answering a question posed by N. Vlamis. Our motivation for the study of the invariant Cantor set case comes from the frequent appearance of groups of Cantor set-preserving homeomorphisms in dynamics. For instance, Calegari [2] studies infinite groups acting on $\mathbb{R}^2$ with bounded orbits by collapsing each bounded component of the complement of an orbit closure to a point to produce an action of the group on a plane with an invariant finite or Cantor set, hence a homomorphism to $\mathrm{Homeo}(\mathbb{R}^2, X)$ where X is finite or Cantor; this is exactly the kind of setting we have in mind.

Our proofs use a condition formulated by Rosendal and Solecki in [10].

Definition 1.1. A topological group G is *Steinhaus* if there is some $n \in \mathbb{N}$ such that, whenever $W \subset G$ is a symmetric set such that countably many left-translates of W cover G, there exists a neighborhood of the identity of G contained in W^n.

As shown in [10, Prop. 2], the Steinhaus property implies automatic continuity (via a straightforward Baire category argument). We show the following.

Theorem 1.2. Let M be a manifold, either compact or homeomorphic to the interior of a compact manifold with boundary. Let $X \subset M$ be a set consisting of finitely many isolated points, or a Cantor set, or both. Then the group $\mathrm{Homeo}(M, X)$ of homeomorphisms preserving X (setwise) is Steinhaus, hence has automatic continuity.

By contrast, when M is noncompact (but is homeomorphic to the interior of a compact manifold $\bar{M}$), the group $\mathrm{Homeo}_c(M)$ of compactly supported diffeomorphisms (equipped with the compact-open topology) does not have automatic continuity, as the obvious inclusion of $\mathrm{Homeo}_c(M)$ into $\mathrm{Homeo}(\bar{M})$ is not continuous.

Before embarking on the proof of Theorem 1.2, we give an application to automatic continuity for some mapping class groups of infinite type surfaces, answering Vlamis' question and raising several new problems.

2. Mapping class groups

The *mapping class group* of a surface Σ is the group of homeomorphisms up to isotopy, equivalently the group $\pi_0(\mathrm{Homeo}(\Sigma))$. When Σ is *finite type* (compact or homeomorphic to the interior of a compact surface), these are discrete groups, and have been known since the work of Dehn to be finitely generated. By contrast, mapping class groups of *infinite type* surfaces provide interesting examples of totally disconnected topological groups.

We are interested more generally in understanding the algebraic and topological properties of π_0 of homeomorphism groups, and of relative homeomorphism groups, of noncompact manifolds. Such "generalized mapping class groups" are Polish groups: indeed, for any manifold M, and closed set $X \subset M$, the group $\mathrm{Homeo}(M, X)$ is a closed subgroup of the Polish group $\mathrm{Homeo}(M)$ (endowed with the compact-open topology), hence is Polish, and since $\mathrm{Homeo}_0(M, X)$ is a closed, normal subgroup of $\mathrm{Homeo}(M, X)$, the quotient $\mathrm{Homeo}(M, X)/\mathrm{Homeo}_0(M, X)$ is also Polish. The context that originally prompted this note is the case where M is a closed surface and $X \subset M$ homeomorphic to a closed subset of a Cantor set. In this case, $\mathrm{Homeo}(M, X) \cong \mathrm{Homeo}(M - X)$ and $M - X$ is a finite genus topological surface (of infinite type, provided that X, its space of ends, is infinite). [1]

An easy argument (see [10, Cor. 3]) shows that, provided that G is a Polish, Steinhaus group, any Polish quotient group H of G is also Steinhaus. Thus, we have the following consequence of Theorem 1.2.

Corollary 2.1. Let M be either compact or homeomorphic to the interior of a compact manifold with boundary, and $X \subset M$ the union of a Cantor set and finite set. Then the *mapping class group* $\mathrm{Homeo}(M, X)/\mathrm{Homeo}_0(M, X)$ is Steinhaus, hence has the automatic continuity property.

However, there are many examples of pairs where $\mathrm{Homeo}(M, X)$ fails to have automatic continuity, even in the case where $M = S^2$ and X is a compact set. To see this, we recall first an example from [9].

Example 2.2 (Example 1.4 of [9])**.** Suppose F is a finite group. Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$. Then $H := \{(f_n) \in F^{\mathbb{N}} : \mathcal{U}_n(f_n) = 1\}$ is a finite index (of index equal to $|F|$) subgroup of the infinite product $F^{\mathbb{N}}$ that is not open. Thus, the permutation action of $F^{\mathbb{N}}$ on cosets of H gives a discontinuous homomorphism from $F^{\mathbb{N}}$ to the symmetric group $\mathrm{Sym}(F^{\mathbb{N}}/H)$.

[1] For this case where M is a surface, that the mapping class group $\mathrm{Homeo}(M, X)/Homeo_0(M, X)$ is Polish was observed earlier in [1] using the property that these groups are the automorphism groups of a countable structure, namely the curve complex of the surface.

Example 2.3. Let X be a compact subset of S^2, homeomorphic to the disjoint union of a Cantor set C and a countable set Q, with the properties that

- The space of accumulation points of Q in C is homeomorphic to the one point compactification of $\mathbb{N}$.
- For each $n \in \mathbb{N}$, $\overline{Q} \cap C$ has exactly k points of rank n (in the sense of Cantor-Bendixon rank).
- For each $n \in \mathbb{N}$, the homeomorphism group of X acts transitively on the k points of rank n in $\overline{Q} \cap C$.

Such a set is easily constructed, for instance one may take for each $n \in \mathbb{N}$ the union of a Cantor set and the set $\omega^n k + 1$ with the order topology, glued along the k points of $\omega^n k + 1$ of maximal rank; declare the nth such set to have diameter 2^{-n}, and have them converge (in the Hausdorff sense) to a single point.

Then $\mathrm{Homeo}(S^2, X)$ acts on X by homeomorphisms, and the map $\mathrm{Homeo}(S^2, X) \to \mathrm{Homeo}(X)$ is surjective (see e.g. [7] for a general proof that the homeomorphism group of a surface surjects to the space of homeomorphisms of its ends). Considering only the invariant set $\overline{Q} \cap C$ gives a natural surjective homeomorphism $\mathrm{Homeo}(X) \to (\mathrm{Sym}_k)^{\mathbb{N}}$. The map $\mathrm{Homeo}(S^2, X) \to (\mathrm{Sym}_k)^{\mathbb{N}}$ is open, so the composition with the homomorphism furnished by Example 2.2 gives a discontinuous homomorphism from this group to a finite symmetric group.

Question 2.4. For which infinite type surfaces Σ does $\mathrm{Map}(\Sigma)$ have automatic continuity?

One might wish to treat separately the case where Σ is finite genus and infinite genus, or the special case of finitely many ends but infinite genus.

The argument for the non-examples above passes through the action of the homeomorphism group on the space of ends of the surface, suggesting two additional parallel questions.

Question 2.5. Let X be a closed subset of a Cantor set. What topological conditions on X ensure that $\mathrm{Homeo}(X)$ has automatic continuity? Is some degree of local homogeneity required?

Question 2.6. Let Σ be an infinite type surface. Under what conditions on the topology of Σ does the subgroup of homeomorphisms fixing each end of Σ, and/or its identity component the *pure mapping class group* have automatic continuity?

After a preprint of this work was circulated, Domat and Dickmann constructed examples of discontinuous homomorphisms from the mapping class group of the one-ended surface of infinite genus [3]. More generally, they show that many *pure* mapping class groups admit discontinuous homomorphisms to the rationals, coming from their first homology. This suggests that Question 2.4 may depend quite subtley on the topology of Σ and the algebraic structure of its mapping class group.

3. General tools

This section contains some standard and some preliminary results to be used in the proof of Theorem 1.2. We assume that manifolds are metrizable, but otherwise arbitrary. For a topological manifold M we take the standard compact-open topology on Homeo(M), which is separable and completely metrizable, i.e. Polish. As recalled above in the introduction, if $X \subset M$ is a closed set, then Homeo(M, X) is a closed subgroup of Homeo(M), so also Polish, and in particular a Baire space. The following lemma is widely used in automatic continuity arguments.

Lemma 3.1. Let M be a manifold, $X \subset M$ a closed set, and $W \subset$ Homeo(M, X) a symmetric set such that Homeo(M, X) is a countable union of left translates of W. Then there exists a neighborhood U of the identity in Homeo(M, X) such that W^2 is dense in U.

Proof. Since Homeo(M, X) is a Baire space, and each left translate of W is homeomorphic to W, it follows that the set W is not meagre, so dense in some open set of Homeo(M, X). Since W is symmetric, it follows that W^2 is dense in a neighborhood of the identity.

□

The following technical lemma generalizes [6, Lemma 3.8], which is modeled after arguments of Rosendal from [8]. Recall that the *support* of a homeomorphism f is the closure of the set $\{x \mid f(x) \neq x\}$, and a group G is said to have *commutator length* p if each element of G can be written as a product of p commutators in G.

Lemma 3.2. Let M be a manifold, and $W \subset$ Homeo(M) a symmetric set such that Homeo(M) is a countable union of left translates of W. Let $\mathcal{A}$ be a family of open subsets of M satisfying:

(1) There exists an infinite family of pairwise disjoint, closed sets $U_i \subset M$ such that each set U_i contains an infinite family of pairwise disjoint sets belonging to $\mathcal{A}$.
(2) There exists $p \in \mathbb{N}$ such that, for each $A \in \mathcal{A}$, the group of homeomorphisms with support on A has commutator length bounded by p.

Then there exists $A \in \mathcal{A}$ such that each homeomorphism supported on A is contained in W^{8p}.

Proof. Let $G =$ Homeo(M), and for any set $U \subset M$, let $G(U)$ denote the homeomorphisms with support contained in U. Suppose that $G = \bigcup_{i \in \mathbb{N}} g_i W$, for some symmetric set W, and let $U_1, U_2, ...$ be the closed, disjoint sets from the statement of the lemma.

Step 1. We first claim that there is some U_i such that, for each $f \in G(U_i)$, there exists $w_f \in g_i W$, with support in the closure of $\bigcup_{i \in \mathbb{N}} U_i$, and such that the restriction of w_f to U_i agrees with f. For if this statement does not hold, then there is a sequence of counterexamples $f_i \in G(U_i)$ such that each f_i does not agree with the restriction to U_i of any element of $g_i W$ supported on $\bigcup_{i \in \mathbb{N}} U_i$. Define a homeomorphism $F(x)$ by

$$F(x) = \begin{cases} f_i(x) & \text{if } x \in U_i \text{ for some } i \\ x & \text{otherwise} \end{cases}$$

By assumption, for some i we have $F \in g_i W$, but F restricts to f_i on U_i. This gives the desired contradiction.

Now, given $f \in G(U_i)$, consider the homeomorphisms w_{id} and w_f obtained above. Then the restriction of $(w_{id})^{-1} w_f$ to U_i agrees with f on U_i, and $(w_{id})^{-1} w_f \in W g_i^{-1} g_i W = W^2$. Thus, we conclude that for any $f \in G(U_i)$, there exists some element in W^2 agreeing with f on U_i.

Step 2. Let $U = U_i$ be the open set obtained from step 1, and let $A_1, A_2, ...$ be a countable disjoint family of open sets in U, with each $A_i \in \mathcal{A}$. We may apply the argument from Step 1 above to the family consisting of the closures of the A_i in place of $U_1, U_2, ...$ to conclude that there exists i such that for every $f \in G(A_i)$, there exists an element in W^2, with support contained in the closure of the union of the A_i, and agreeing with f on U. Forgetting subscripts, let $A = A_i$ denote this set.

Let $f \in G(A)$. Using the bounded commutator length assumption, we can write $f = [a_1, b_1]...[a_p, b_p]$, where each a_j and b_j also have support in A. Since $A \subset U$, each a_j has support in U so by Step 1 there exists $w_{a_j} \in W^2$ such that the restriction of w_{a_j} to U agrees with a_j. In particular, w_{a_j} agrees with the identity map in $U - A$. By step 2, there also exists an element $w_{b_j} \in W^2$ with support contained in the closure of the union of the A_i in U, such that the restriction of w_{b_j} to A agrees with b_j. Thus, $\mathrm{supp}(w_{a_j}) \cap \mathrm{supp}(w_{b_j}) \subset A$, and hence the commutator $[w_{a_j}, w_{b_j}]$ agrees with the identity map outside of A. Since w_{a_j} and w_{b_j} agree with a_j and b_j respectively on A, we have $[a_j, b_j] = [w_{a_j}, w_{b_j}]$, and hence $f = [w_{a_1}, w_{b_1}]...[w_{a_p}, w_{b_p}] \in W^{8p}$.

□

The other general tool that we will use in the proof is the following.

Proposition 3.3. Let M be a manifold, and suppose that $\mathrm{Homeo}(M) \subset \bigcup_{i \in \mathbb{N}} g_i W$ for some symmetric set W. Then there exists n, depending only on the dimension of M, and a neighborhood U of the identity in $\mathrm{Homeo}(M)$ so that the following holds: If $K \subset M$ is any compact set, then any homeomorphism in U with support contained in K is an element of W^n.

This follows directly from the proof of the main theorem of automatic continuity for compact manifolds in [6].

4. First case: X finite or empty.

We begin with the case $X = \emptyset$; given the results of [6], this reduces to the case where M is homeomorphic to the interior of a compact manifold with boundary. A major technical ingredient for the proof is adapted from Rosendal–Solecki's proof of automatic continuity for the group of order-preserving automorphisms of $\mathbb{Q}$ in [10]. The argument here also covers the case where X is finite, by the following remark.

Remark 4.1 ($X = \emptyset$ case implies X finite case)**.** Suppose M is the interior of a compact manifold $\bar{M}$, and $X \subset M$ is a finite set. Since $\mathbb{R}^n - \{0\}$ is homeomorphic to $(0,1) \times S^{n-1}$, the manifold $M - X$ is homeomorphic to the interior of a compact manifold $\bar{N}$ with $\partial \bar{N}$ the union of $\partial \bar{M}$ and a disjoint union of spheres, one for each point of X. Though homeomorphisms of N do not necessarily extend to homeomorphisms of $\bar{N}$, they do extend to the space obtained by one-point-compactifying each spherical end, and this gives a topological isomorphism $\mathrm{Homeo}(N) \to \mathrm{Homeo}(M, X)$.

Proof of automatic continuity for M noncompact, $X = \emptyset$. Let M be homeomorphic to the interior of a compact manifold $\bar{M}$. In this case, the identity component $\mathrm{Homeo}_0(M)$ is open in $\mathrm{Homeo}(M)$, so we may work with the identity component $\mathrm{Homeo}_0(M)$, and show that this is Steinhaus.

Let $W \subset \mathrm{Homeo}_0(M)$ be a symmetric set such that $\mathrm{Homeo}_0(M) = \bigcup_{i \in \mathbb{N}} g_i W$.

Identify a neighborhood of the ends of M with $([0,\infty)\times\partial\bar{M})\subset M$. Let U be a neighborhood of the identity in $\mathrm{Homeo}_0(M)$ small enough so that, if $f\in U$, then f can be written as $f=k\circ h$ where k has support on the compact set $K=M-((3,\infty)\times\partial M)$, and h has support on $[2,\infty)\times\partial\bar{M}$. Proposition 3.3 shows that, provided U is chosen small enough, there exists n depending only on the dimension of M such that $k\in W^n$. We wish to prove the same is true for h.

Any homeomorphism h of $[0,\infty)\times\partial M$ can be factored as h_1h_2 where each h_i has support on a set of the form $X_i\times\partial\bar{M}$, and $X_i\subset[0,\infty)$ is an infinite disjoint union of open intervals homeomorphic in $[0,\infty)$ to $\bigcup_{n=1}^{\infty}(2n,2n+1)$. (See e.g. [5, Prop. 5.1].) Thus, it suffices to find n such that W^n contains any homeomorphism with support on a set of the form $X\times\partial\bar{M}\subset[0,\infty)\times\partial\bar{M}$ where X is homeomorphic to $\bigcup_{n\in\mathbb{N}}(2n,2n+1)$.

Let X be such a set. Reparameterizing $\mathbb{R}$, we may assume that in fact

$$X=\bigcup_{n\in\mathbb{N}}(2n,2n+1)\times\partial\bar{M}.$$

We begin by applying Lemma 3.2 with the class $\mathcal{A}$ consisting of sets of the form $A_\Lambda=Y_\Lambda\times\partial\bar{M}$, where $Y_\Lambda=\bigcup_{n\in\Lambda}(2n,2n+1)$ for some infinite set $\Lambda\subset\mathbb{N}$. Note that $\mathcal{A}$ satisfies the hypotheses of the lemma, since

(1) We may write $\mathbb{N}$ as a countable disjoint union of infinite sets Λ_i, and define U_i to be the closure of $Y_{\Lambda_i}\times\partial\bar{M}$. Each such set contains a countable union of disjoint elements of $\mathcal{A}$.
(2) Any element supported on such a set A_Λ may be written as a single commutator.

The proof of item ii) is a standard argument, we briefly recall this for completeness: given f supported on such a set $Y_\Lambda\times\partial\bar{M}$, let $I_n\subset(2n,2n+1)$ be closed intervals such that $\mathrm{supp}(f)\subset\bigcup_{n\in\Lambda}I_n\times\partial M$. Let $T:\mathbb{R}\to\mathbb{R}$ be a homeomorphism with support in Y_Λ such that $T(I_n)\cap I_n=\emptyset$; abusing notation, identify T with the homeomorphism $T\times id$ of $[0,\infty)\times\partial\bar{M}$. Let $a=\prod_{j\geq 0}T^jfT^{-j}$, which has support in $\bigcup_{n\in\Lambda}I_n$. Then $a\circ Ta^{-1}T^{-1}=f$.

Thus we conclude (using the notation from Lemma 3.2) that for some such set $A_\Lambda\in\mathcal{A}$, we have $G(A_\Lambda)\subset W^8$. Now we apply a trick used in [10]. For $\alpha\in\mathbb{R}$, let Λ_α be infinite subsets of Λ, such that $\Lambda_\alpha\cap\Lambda_\beta$ is finite for all $\alpha\neq\beta$. Such a collection may be obtained, for example, by putting Λ in bijective correspondence with $\mathbb{Q}$, and choosing Λ_α to be a sequence of distinct rational numbers converging to α.

For each α, let $f_\alpha\in\mathrm{Homeo}[0,\infty)$ satisfy $f_\alpha(2n)=2n+1$, and $f_\alpha(2n+1)\in 2\Lambda_\alpha$ for all $n\in\Lambda_\alpha$, so that f_α maps each interval of Y_{Λ_α} to the interior of a connected component of $\mathbb{R}-Y_{\Lambda_\alpha}$. Again abusing notation, identify these with the homeomorphisms $f_\alpha\times id$ of $[0,\infty)\times\partial\bar{M}$. Since $\mathbb{R}$ is uncountable, there is some α and some β such that f_α and f_β are in the same left-translate g_iW. Thus, $f_\alpha^{-1}f_\beta$ and $f_\beta^{-1}f_\alpha$ are both in W^2. We will now use these two elements to conjugate (a suitable decomposition of) homeomorphisms supported on $X\times\partial\bar{M}$ into A_Λ, this will complete the proof.

If $n\in\mathbb{N}-\Lambda_\alpha$, then $f(2n,2n+1)\subset(2m,2m+1)$ for some $m\in\Lambda_\alpha$. If $m\notin\Lambda_\beta$, then $f_\beta^{-1}f_\alpha(2n,2n+1)$ is contained in an interval of the form $(2k,2k+1)$ where $k\in\Lambda_\beta$. Since $\Lambda_\alpha\cap\Lambda_\beta$ is finite, we conclude that, with the exception of finitely many values of n, the map $f_\beta^{-1}f_\alpha$ takes intervals of the form $(2n,2n+1)$ where $n\notin\Lambda_\alpha$ into some interval in $Y_{\Lambda_\alpha} \subset Y_\Lambda$. Reversing the role of α and β, the same argument shows that, with only finitely many exceptions, $f_\alpha^{-1} f_\beta$ takes every interval of the form $(2n, 2n+1), n \notin \Lambda_\beta$, into some interval of Y_Λ. Let F denote the union of these two exceptional sets of integers. Thus we can decompose X as the union of $X_1 := \bigcup_{n \notin (\Lambda_\alpha \cup F)} (2n, 2n+1) \times \bar{M}$, with $X_2 := \bigcup_{n \notin (\Lambda_\beta \cup F)} (2n, 2n+1) \times \bar{M}$ and $X_3 := \bigcup_{n \in F} (2n, 2n+1) \times \bar{M}$, hence $G(X) = G(X_1)G(X_2)G(X_3)$.

As we have shown above,

$$f_\beta^{-1} f_\alpha G(X_1)(f_\beta^{-1} f_\alpha)^{-1} \subset G(Y_\Lambda) \subset W^8, \text{and similarly}$$

$$f_\alpha^{-1} f_\beta G(X_2)(f_\alpha^{-1} f_\beta^{-1}) \subset G(Y_\Lambda) \subset W^8.$$

Thus $G(X_1) \subset W^{12}$ for $i = 1, 2$. Since X_3 is compact, we also have that $G(X_3) \subset W^n$ by Proposition 3.3. It follows that $G(X) \subset W^{24+n}$, which proves the theorem.

□

5. General case

Let M be a topological manifold, either compact or homeomorphic to the interior of a compact manifold with boundary, and let $X \subset M$ be the union of a Cantor set and a (possibly empty) finite set. Remark 4.1 reduces the proof to the case where X is a Cantor set, since if X' is the set of isolated points of X, then $M - X'$ is homeomorphic to interior of a compact manifold with boundary, and $\text{Homeo}(M, X) \cong \text{Homeo}(M - X', X - X')$. Thus, we assume going forward that X is a Cantor set. Unlike in the previous case, here the identity component of $\text{Homeo}(M, X)$ is not open so we need to work with the full group.

Proof for $X \subset M$ a Cantor set. Let $W \subset \text{Homeo}(M, X)$ be a symmetric set such that $\text{Homeo}(M, X) = \bigcup_{i \in \mathbb{N}} g_i W$. For convenience, fix a metric on M. Then a neighborhood basis of the identity on M is given by sets of the form

$$U_{K,\delta} := \{f \mid d(f(x), x) < \delta \text{ for all } x \in K\}$$

as K ranges over compact sets of M, and $\delta > 0$.

Call a closed ball in M a *separating ball* if its boundary is disjoint from X and separates X into two components. By Lemma 3.1, W^2 is dense in a neighborhood of the identity of $\text{Homeo}(M, X)$. Thus, we may choose $\epsilon > 0$ such that the following holds:

Lemma 5.1. Let $x_1, x_2, ... x_n$ be any collection of points in X, with $d(x_i, x_j) > 2\epsilon$ for all $i \neq j$. If D_i and E_i are both separating balls contained in the ϵ-ball about x_i, then for any $\delta > 0$, there exists $f \in W^2$ such that, for all $i = 1, 2, ..., n$ the ball $f(D_i)$ is Hausdorff distance at most δ from E_i.

The proof is immediate, and the statement also holds if we replace the condition that D_i and E_i are separating balls with the condition that $\overline{D_i} \cap X = \emptyset$ and $\overline{E_i} \cap X = \emptyset$ holds for all i.

Now take two 2ϵ-separated sets, say $\{x_1, x_2, ..., x_m\}$ and $\{x'_1, x'_2, ..., x'_{m'}\}$, as in Lemma 5.1, and let $D_i \subset B_\epsilon(x_i)$ and $D'_i \subset B_\epsilon(x'_i)$ be separating balls such that

1. The union of the D_i and the D'_i cover X, and
2. Each point of X is contained in at most one element of $\{D_1, ..., D_n, D'_1, ..., D'_{m'}\}$.

We now apply Lemma 3.2 to sets consisting of separating balls. Let $\mathcal{A}$ be the collection of sets satisfying the following condition: $A \in \mathcal{A}$ iff A consists of a union of disjoint separating balls, with exactly one ball in each metric ball $B_\epsilon(x_i)$. Let $\mathcal{A}'$ be the analogous set for the metric balls $B_\epsilon(x_i')$. It is easily checked that $\mathcal{A}$ and $\mathcal{A}'$ satisfy the conditions of Lemma 3.2, so we conclude that there exists $A_0 \in \mathcal{A}$ and $A_0' \in \mathcal{A}'$ so that $G(A_0) \subset W^8$ and $G(A_0') \subset W^8$.

Let Z be a connected, simply connected neighborhood of X such that the interior of $M - Z$ together with the union of the discs D_i and D_i' covers M. By the fragmentation lemma (following Edwards–Kirby [4]), there exists a neighborhood U of the identity in M such that any $f \in U$ can be written as $g_0 g_1 g_2$ with g_0 supported on Z, g_1 supported on $\bigcup_i D_i$ and g_2 supported on $\bigcup_i D_i'$.

Since each point of X is contained in at most one such supporting set, we additionally have that if $f(X) = X$, then each g_i must preserve X as well. Repeating the proof (verbatim) from the first case where $X = \emptyset$ shows that $g_0 \in W^n$, where n is a constant that depends only on the dimension of M. The set A_0 consists of a union of separating balls, one in each ϵ-ball about x_i (for $i = 1, \ldots, m$). Construct a new set A_1 by replacing each ball B_0 of A_0 with a smaller separating ball B_1 whose closure is contained in B_0. Fix δ small enough so that the closed δ-neighborhood of A_1 is contained in A_0. Now Lemma 5.1 states that we can find $h_1 \in W^2$ such that $h_1(D_i)$ lies in the δ-neighborhood of A_1, hence inside of A_0. Similarly, we may find h_2 such that $\bigcup_i D_i' \subset h_2(A_0')$. Thus, for $i = 1, 2$ we have that $h_i g_i h_i^{-1}$ is supported in A_0 or A_0', hence an element of W^8. We conclude that $g_i \in W^{12}$. This shows that $f \in W^{24+n}$, which is what we needed to show. □

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