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Cohomological jump loci and duality in local algebra

Benjamin Briggs[1] · Daniel McCormick[1] · Josh Pollitz[1]

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Abstract

In this article a higher order support theory, called the *cohomological jump loci*, is introduced and studied for dg modules over a Koszul extension of a local dg algebra. The generality of this setting applies to dg modules over local complete intersection rings, exterior algebras and certain group algebras in prime characteristic. This family of varieties generalizes the well-studied support varieties in each of these contexts. We show that cohomological jump loci satisfy several interesting properties, including being closed under (Grothendieck) duality. The main application of this support theory is that over a local ring the homological invariants of Betti degree and complexity are preserved under duality for finitely generated modules having finite complete intersection dimension.

Keywords Betti degree · Complete intersection dimension · Cohomology operators · Complete intersection · Duality · Koszul complex · Jump loci · Support

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Introduction

Over a local complete intersection ring the minimal free resolution of a finitely generated module has polynomial growth. More precisely the Betti numbers are eventually modeled

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✉ Josh Pollitz
pollitz@math.utah.edu

Benjamin Briggs
briggs@math.utah.edu

Daniel McCormick
mccormic@math.utah.edu

1 Department of Mathematics, University of Utah, Salt Lake City, UT 84112, USA

by a quasi-polynomial of period two. A striking result of Avramov and Buchweitz in [4], implicitly contained in [1], is that the degrees of the quasi-polynomials corresponding to the Betti numbers of a finitely generated module and its (derived) dual coincide; see also [31, 38]. In this article we strengthen this result by showing their leading terms also agree.

Throughout we fix a surjective map $\varphi\colon A \to B$ of local rings with common residue field k. We assume φ is complete intersection of codimension c in the sense that its kernel is generated by an A-regular sequence of length c. Let M be a finitely generated B-module that has finite projective dimension over A.

Classical results of Eisenbud [22] and Gulliksen [26] associate to φ a ring of cohomology operators $\mathcal{S} = k[\chi_1, \dots, \chi_c]$, with each χ_i residing in cohomological degree 2, in a way that the graded k-space $\mathsf{Ext}_B(M, k)$ is naturally a *finitely generated* graded $\mathcal{S}$-module. The Hilbert–Serre theorem implies that the Krull dimension of $\mathsf{Ext}_B(M, k)$ over $\mathcal{S}$ is the degree of the quasi-polynomial eventually governing the sequence of Betti numbers $\beta_i^B(M)$ for M. This value is called the complexity of M over B, denoted $\operatorname{cx}^B M$; see Definition 3.1 for a precise definition.

This article concerns the behaviour of this quasi-polynomial with respect to the derived duality $M^* = \mathsf{RHom}_B(M, B)$. When M is maximal Cohen-Macaulay, this coincides with the ordinary B-dual module.

In this notation, it was shown in [4] that the supports of $\mathsf{Ext}_B(M, k)$ and $\mathsf{Ext}_B(M^*, k)$ over $\mathcal{S}$ are the same and hence $\operatorname{cx}^B M = \operatorname{cx}^B M^*$; see also [31, 38] for different proofs. However the methods in *loc. cit.* are not fine enough to show that the leading coefficients of the quasi-polynomials corresponding to the Betti numbers of two B-modules agree.

Theorem A *Let $\varphi\colon A \to B$ be a surjective complete intersection map with common residue field k. For a finitely generated B-module M whose projective dimension over A is finite, the multiplicities of $\mathsf{Ext}_B(M, k)$ and $\mathsf{Ext}_B(M^*, k)$ over $\mathcal{S}$ coincide. In particular, the leading terms of the quasi-polynomials eventually modeling $\beta_i^B(M)$ and $\beta_i^B(M^*)$ agree.*

The theorem above is contained in Theorem 3.6 where it is stated in terms of Betti degrees; see Definition 3.1. These values, studied in [1–3, 27], are normalized leading coefficients for the quasi-polynomials eventually corresponding to sequences of Betti numbers. Theorem A can also be proven using work of Eisenbud, Peeva and Schreyer [23]; see Remark 3.7 for a discussion of this connection. From Theorem A we deduce that the Betti degree of a module of finite complete intersection dimension and its dual coincide; see Corollary 3.10. Another consequence is the following.

Corollary B *If A is Gorenstein, the leading terms of quasi-polynomials eventually modeling the Betti numbers and the Bass numbers of M are the same.*

The proof of Theorem A is geometric in nature, and does not rely on special properties of resolutions with respect to the duality functor. We show that the Betti degree is encoded in a sequence of varieties, refining the support theory of Avramov and Buchweitz, studied and extended by many others in local algebra [1, 4, 11, 19, 28, 35]. Cohomological supports have yielded applications in revealing asymptotic properties for local complete intersection maps in *loc. cit.*, and more recently, their utility has been detecting the complete intersection property among surjective maps and maps of essentially finite type [15, 16, 34]. Below we discuss properties of the support theory presented in this article, and direct the curious reader to their construction in Definition 1.6.

We associate to M a nested sequence of Zariski closed subsets of $\mathbb{P}_k^{c-1}$, called the *cohomological jump loci* of M,

$$\mathbb{P}_k^{c-1} = \mathrm{V}_\varphi^0(M) \supseteq \mathrm{V}_\varphi^1(M) \supseteq \mathrm{V}_\varphi^2(M) \supseteq \dots .$$

The first jump locus $V^1_\varphi(M)$ is the support of $\mathsf{Ext}_B(M,k)$ over $\mathcal{S}$, and hence it coincides with the cohomological support of M studied by Avramov *et. al.* From a geometric perspective, the sequence of cohomological jump loci can be arbitrarily complicated: any nested sequence of closed subsets of $\mathbb{P}^{c-1}_k$ can be realized as the sequence jump loci of some B-module, up to re-indexing; see Theorem 1.14.

This theory is analogous to the jump loci in [18] for differential graded Lie algebras which have found numerous applications in geometry and topology. The cohomological jump loci in the present article have several interesting properties. For example, they respect the triangulated structure of derived categories in an "additive" sense; see Proposition 2.10 for a precise formulation. We highlight two properties here. First, upon a reduction to the case of maximal complexity, the sequence of cohomological jump loci for M encodes its Betti degree; this is the content of Lemma 3.5. The second property, found in Theorem 2.8, is the following.

Theorem C *Let $\varphi\colon A \to B$ be a surjective complete intersection map. If M is a finitely generated B-module whose projective dimension over A is finite, then there are the equalities $\mathrm{V}^i_\varphi(M) = \mathrm{V}^i_\varphi(M^*)$ for all $i \geq 0$.*

Outline

In Sect. 1 we introduce the theory of cohomological jump loci. This is done in greater generality than discussed above. Namely we let A be a local differential graded (=dg) algebra and consider a Koszul complex B on a finite list of elements in A_0; in this context M is a dg B-module that is perfect over A. A number of examples are provided and we conclude the section with our realizability result, discussed above, in Theorem 1.14.

In Sect. 2 we establish basic and important properties of cohomological jump loci. The main result of the section is that the cohomogical jump loci of M and M^* are the same; this is the subject of Theorem 2.8. Finally, Sect. 3 specializes to the context of the introduction, and to modules of finite complete intersection dimension. This contains applications to local algebra like Theorem A and Corollary B, discussed above.

1 Definitions and examples

Throughout this article $(A, \mathfrak{m}, k)$ is a fixed commutative noetherian local dg algebra. That is, $A = \{A_i\}_{i\geq 0}$ is a nonnegatively graded, strictly graded-commutative dg algebra with $(A_0, \mathfrak{m}_0, k)$ a commutative noetherian local ring, and the homology modules $\mathsf{H}_i(A)$ are finitely generated over $\mathsf{H}_0(A)$.

We fix a list of elements $\boldsymbol{f} = f_1, \ldots, f_c$ in $\mathfrak{m}_0$ and set

$$B := A\langle e_1, \ldots, e_c \mid \partial e_i = f_i\rangle$$

to be the Koszul complex on $\boldsymbol{f}$ over A—that is, B is the exterior algebra over A on exterior variables $e_1, \ldots, e_n$ of degree 1 with differential uniquely determined, via the Leibniz rule, by $\partial e_i = f_i$. This will be regarded as a dg A-algebra in the standard fashion, and we let $\varphi\colon A \to B$ be the structure map.

We will also denote throughout

$$\mathcal{S} := k[\chi_1, \ldots, \chi_c],$$

the graded polynomial algebra over k generated by polynomial variables χ_i of cohomological degree 2. We refer to $\mathcal{S}$ as the ring of cohomology operators (over k) corresponding to φ; this name is justified in 1.4.

Remark 1.1 If A is a local ring (that is, concentrated in degree 0), as in the introduction, then B is quasi-isomorphic to $A/(\boldsymbol{f})$ under the additional assumption that $\boldsymbol{f}$ is an A-regular sequence. In this case, everything that follows directly translates to the setting where we instead define $B = A/(\boldsymbol{f})$ from the beginning, cf. [24, Theorem 6.10].

We let $\mathsf{D}(B)$ denote the derived category of dg B-modules. This is a triangulated category in the usual way; see [5, Section 3]. Restricting along the structure map $A \to B$ defines a functor $\mathsf{D}(B) \to \mathsf{D}(A)$. Through this map objects of $\mathsf{D}(B)$ are regarded as objects of $\mathsf{D}(A)$. It will be convenient for us to work in the following subcategory of $\mathsf{D}(B)$.

Definition 1.2 Let $\mathsf{D}^{\mathrm{b}}(B/A)$ denote the full subcategory of $\mathsf{D}(B)$ consisting of dg B-modules which are perfect when restricted to $\mathsf{D}(A)$. That is, M belongs to $\mathsf{D}^{\mathrm{b}}(B/A)$ provided that, while viewed as an object of $\mathsf{D}(A)$, it belongs to the smallest thick subcategory containing A. This category is denoted $\mathsf{D}^{\varphi-\mathrm{b}}(B)$ in [25].

Remark 1.3 When A is a regular local ring, $\mathsf{D}^{\mathrm{b}}(B/A)$ is simply the bounded derived category of dg B-modules; namely, $\mathsf{D}^{\mathrm{b}}(B/A)$ is exactly the full subcategory of $\mathsf{D}(B)$ consisting of dg B-modules with finitely generated homology over the ring A, which is often denoted $\mathsf{D}^{\mathrm{b}}(B)$.

The utility of this category is due to a theorem of Gulliksen [26, Theorem 3.1] which is recast in the following construction.

1.4. If M is an object of $\mathsf{D}^{\mathrm{b}}(B/A)$ then $\mathsf{RHom}_B(M, k)$ can naturally be given the structure of a perfect dg $\mathcal{S}$-module.

Indeed, $\mathsf{RHom}_B(M, k)$ is quasi-isomorphic to $\mathsf{Hom}_A(F, k) \otimes_k \mathcal{S}$, with the twisted differential

$$\partial^{\mathsf{Hom}(F,k)} \otimes 1 + \sum_{i=1}^{n} \mathsf{Hom}(e_i -, k) \otimes \chi_i$$

where $F \xrightarrow{\simeq} M$ is a semifree resolution of M over B. This defines a dg $\mathcal{S}$-module structure that is independent of choice of F up to quasi-isomorphism; cf. [3, Section 2]. When we need to refer to this dg $\mathcal{S}$-module explicitly, it will be denoted $\mathsf{RHom}_A(M, k) \otimes_k^{\tau} \mathcal{S}$; this notation is used, for example, in Theorem 2.8.

We point out that $F \xrightarrow{\simeq} M$ can be taken to be any dg B-module map where the underlying graded A-module of F is a finite coproduct of shifts of A, provided such an F exists. When A is a ring, the existence of such a resolution is contained in [3, 2.1]. If such a resolution exists, then one can show that $\mathsf{Hom}_A(F, k) \otimes_k^{\tau} \mathcal{S}$ is a perfect dg $\mathcal{S}$-module arguing as in [7, 9, 35]. However, at this level of generality, the existence of such resolutions has not been established, and so we argue in a different fashion.

Under the identification of $\mathsf{RHom}_B(M, k)$ with $\mathsf{RHom}_A(M, k) \otimes_k^{\tau} \mathcal{S}$ we have the following quasi-isomorphism

$$\frac{\mathsf{RHom}_B(M, k)}{(\boldsymbol{\chi})\, \mathsf{RHom}_B(M, k)} \simeq \mathsf{RHom}_A(M, k)$$

and because M is perfect over A, the homology module $\mathsf{H}(\mathsf{RHom}_A(M, k))$ is a finite k-space. It follows by a homological version of Nakayama's lemma, see for example [35, Theorem 3.2.4], that $\mathsf{Ext}_B(M,k) = \mathsf{H}(\mathsf{RHom}_B(M,k))$ is finitely generated over $\mathcal{S}$. Finally, since $\mathcal{S}$ has finite global dimension, we conclude that $\mathsf{RHom}_B(M,k)$ is perfect when regarded as a dg $\mathcal{S}$-module as claimed.

When A is an (orindary) ring and $\boldsymbol{f}$ is an A-regular sequence, $\mathcal{S}^2$ is the usual k-space of operators associated to $\boldsymbol{f}$ in the works of Avramov [1], Eisenbud [22], Gulliksen [26], Mehta [33], and others; this is clarified in [12].

Remark 1.5 While our focus is on the $\mathcal{S}$-action on $\mathsf{Ext}_B(M,k)$, the cohomology operators $\boldsymbol{\chi}$ do lift to elements of $\mathsf{Ext}_B(M,M)$, and we will use this in 1 below.

Indeed, mimicking the proof of [3, Proposition 2.6], it follows that the operators $\boldsymbol{\chi}$ defining $\mathcal{S}$ can be realized as elements in the Hochschild cohomology of B over A. More precisely, with B^e_A denoting the enveloping dg algebra of B over A, there is an isomorphism of dg algebras

$$\mathsf{RHom}_{B^e_A}(B,B) \simeq B[\chi_1,\ldots,\chi_c]$$

where each χ_i is in cohomological degree 2. This quasi-isomorphism yields a homomorphism $B[\chi_1,\ldots,\chi_c] \to \mathsf{RHom}_B(M,M)$, through which $\mathsf{Ext}_B(M,M)$ obtains an action of the cohomology operators. Furthermore, the natural projection $\pi : B[\chi_1,\ldots,\chi_c] \to \mathcal{S}$ determines the same $\mathcal{S}$-action as the one discussed in 1.4 on $\mathsf{RHom}_B(M,k)$ for any dg B-module M.

Let $\operatorname{Spec}\mathcal{S}$ denote the set of homogeneous prime ideals of $\mathcal{S}$ with the Zariski topology, having closed sets of the form

$$\mathcal{V}(\eta_1,\ldots,\eta_t) = \{\mathfrak{p} \in \operatorname{Spec}\mathcal{S} : \eta_i \in \mathfrak{p} \text{ for all } i\}$$

for some list of homogeneous elements $\eta_1,\ldots,\eta_t$ in $\mathcal{S}$. For a graded $\mathcal{S}$-module X and $\mathfrak{p} \in \operatorname{Spec}\mathcal{S}$ we write $X_\mathfrak{p}$ for the (homogeneous) localization of X at $\mathfrak{p}$. Furthermore, $\kappa(\mathfrak{p})$ will be the graded field $\kappa(\mathfrak{p}) := \mathcal{S}_\mathfrak{p}/\mathfrak{p}\mathcal{S}_\mathfrak{p}$.

Given a graded field κ, any finitely generated κ-module X has the form κ^r for some r, and below we use the notation $\operatorname{rank}_\kappa X = r$.

Definition 1.6 Let $\mathfrak{p}$ be in $\operatorname{Spec}\mathcal{S}$ and M be in $\mathsf{D}(B)$. Define the *cohomological rank of M at $\mathfrak{p}$* to be

$$\operatorname{crk}_\mathfrak{p}(M) := \operatorname{rank}_{\kappa(\mathfrak{p})} \mathsf{H}(\mathsf{RHom}_B(M,k) \otimes^{\mathsf{L}}_{\mathcal{S}} \kappa(\mathfrak{p})).$$

The i^{th} *cohomological jump locus* of M is defined to be

$$\mathrm{V}^i_\varphi(M) := \{\mathfrak{p} \in \operatorname{Spec}\mathcal{S} : \operatorname{crk}_\mathfrak{p}(M) \geq i\}.$$

Remark 1.7 For a dg B-module M, trivially $\mathrm{V}^0_\varphi(M) = \operatorname{Spec}\mathcal{S}$ and there is a descending chain of subsets of $\operatorname{Spec}\mathcal{S}$:

$$\mathrm{V}^0_\varphi(M) \supseteq \mathrm{V}^1_\varphi(M) \supseteq \mathrm{V}^2_\varphi(M) \supseteq \ldots . \tag{1.7.1}$$

Hence when M is in $\mathsf{D}^{\mathrm{b}}(B/A)$, this chain must stabilize at $\varnothing$ since $\mathsf{RHom}_B(M,k)$ is perfect over $\mathcal{S}$ by 1.4.

If M is in $\mathsf{D}^{\mathrm{b}}(B/A)$ we have that $\mathrm{V}^1_\varphi(M)$ is simply the support of $\mathsf{Ext}_B(M,k)$ regarded as a graded $\mathcal{S}$-module; this is contained in [20, Theorem 2.4]. That is,

$$\begin{aligned}\mathrm{V}^1_\varphi(M) &= \{\mathfrak{p} \in \operatorname{Spec}\mathcal{S} : \mathsf{Ext}_B(M,k)_\mathfrak{p} \neq 0\} \\ &= \mathcal{V}(\eta_1,\ldots,\eta_t)\end{aligned}$$

where $\eta_1, \dots, \eta_t$ generate $\operatorname{ann}_{\mathcal{S}} \mathsf{Ext}_B(M, k)$. In particular, $\mathrm{V}^1_\varphi(M)$ is a closed subset of $\operatorname{Spec} \mathcal{S}$, provided M is in $\mathsf{D}^{\mathrm{b}}(B/A)$. Looking ahead, in Proposition 2.1, we show that $\mathrm{V}^i_\varphi(M)$ is closed for all i, whenever M is in $\mathsf{D}^{\mathrm{b}}(B/A)$.

Remark 1.8 When A is a ring, $\mathrm{V}^1_\varphi(M)$ is the cohomological support of M over B as defined in [34, 35]; these are derived versions of the support varieties in local algebra studied in [1, 4, 11, 28].

1.9. Let X be a dg $\mathcal{S}$-module with finitely generated homology. The *total Betti number* of X is

$$\beta^{\mathcal{S}}_{\mathrm{total}}(X) = \sum_{i \in \mathbb{Z}} \operatorname{rank}_k \mathsf{Tor}^{\mathcal{S}}_i(X, k);$$

the sum is only over finitely many integers as $\mathcal{S}$ has finite global dimension.

Example 1.10 Assume M is a perfect dg B-module and $r = \beta^{\mathcal{S}}_{\mathrm{total}}(\mathsf{RHom}_B(M, k))$. It follows directly that

$$\mathrm{V}^i_\varphi(M) = \begin{cases} \operatorname{Spec} \mathcal{S} & i = 0 \\ \{(\boldsymbol{\chi})\} & 1 \leq i \leq r \\ \varnothing & i > r. \end{cases}$$

Example 1.11 Let ν denote the embedding dimension of A_0 and let K^A denote the Koszul complex on a minimal generating set for the maximal ideal of A_0 over A. As $\boldsymbol{f}$ is contained in $\mathfrak{m}_0$, there is a dg B-module structure on K^A which is explained below: Fixing a minimal generating set $\boldsymbol{x} = x_1, \dots, x_\nu$ for $\mathfrak{m}_0$ with $\partial e'_i = x_i$ in K^A and writing each

$$f_i = \sum_{j=1}^{\nu} a_{ij} x_j,$$

determines a B-action on K^A by

$$e_i \cdot \omega = \left(\sum_{j=1}^{\nu} a_{ij} e'_j \right) \omega.$$

In particular, if $\boldsymbol{f} \subseteq \mathfrak{m}_0^2$ it follows from 1.4 that there is the following isomorphism of graded $\mathcal{S}$-modules

$$\mathsf{RHom}_B(K^A, k) \cong \mathsf{Hom}_A(K^A, k) \otimes_k \mathcal{S} \cong \bigwedge \left(\Sigma^{-1} k^\nu \right) \otimes_k \mathcal{S}$$

and hence, $\operatorname{crk}_{\mathfrak{p}}(K^A) = 2^\nu$. Therefore, there are the following equalities

$$\mathrm{V}^i_\varphi\left(K^A\right) = \begin{cases} \operatorname{Spec} \mathcal{S} & i \leq 2^\nu \\ \varnothing & i > 2^\nu. \end{cases}$$

When A is a regular local ring, we have calculated the sequence of jump loci $\mathrm{V}^i_\varphi(k)$ since $K^A \xrightarrow{\simeq} k$ as dg B-modules.

Example 1.12 Assume A is a regular local ring (or more generally, a UFD) and consider $R := A/(\boldsymbol{f})$ where $\boldsymbol{f} = f_1, f_2$. When $\boldsymbol{f}$ is a regular sequence, $B \xrightarrow{\simeq} R$ and so from Example 1.10 we have the equalities

$$\mathrm{V}^i_{\varphi}(R) = \begin{cases} \operatorname{Spec} \mathcal{S} & i = 0 \\ \{(\boldsymbol{\chi})\} & i = 1 \\ \varnothing & i > 1. \end{cases}$$

Now assume $\boldsymbol{f}$ does not form an A-regular sequence; in this case there exists an A-regular sequence f_1', f_2' with $f_i = f_i' g$ for some $g \in \mathfrak{m}_0$. It follows that

$$0 \to A \xrightarrow{\begin{pmatrix} -f_2' \\ f_1' \end{pmatrix}} A^2 \xrightarrow{\begin{pmatrix} f_1 & f_2 \end{pmatrix}} A \to 0$$

is an A-free resolution of R, and this has a dg B-module structure with the e_1 and e_2 action indicated by

$$e_1 : \quad 0 \leftarrow A \xleftarrow{\begin{pmatrix} 0 & g \end{pmatrix}} A^2 \xleftarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} A \leftarrow 0$$

$$e_2 : \quad 0 \leftarrow A \xleftarrow{\begin{pmatrix} -g & 0 \end{pmatrix}} A^2 \xleftarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} A \leftarrow 0.$$

It follows easily, using 1.4, that $\mathsf{RHom}_B(R, k)$ is isomorphic to the complex of free $\mathcal{S}$-modules:

$$0 \to \Sigma^{-4}\mathcal{S} \xrightarrow{0} \Sigma^{-2}\mathcal{S}^{\oplus 2} \xrightarrow{\begin{pmatrix} \chi_1 & \chi_2 \end{pmatrix}} \mathcal{S} \to 0.$$

Therefore, assuming k is algebraically closed,

$$\mathrm{V}^i_{\varphi}(R) = \begin{cases} \operatorname{Spec} \mathcal{S} & i \le 2 \\ \{(\chi_1, \chi_2)\} & i = 3, 4 \\ \varnothing & i > 4. \end{cases}$$

Example 1.13 Let $A = k[\![x, y, z]\!]$ and set $\boldsymbol{f} = x^3, y^3, z^3$. For the $A/(\boldsymbol{f})$-module $M = A/(\boldsymbol{f}, xz, yz^2)$. Using similar calculations as the ones in Example 1.12 it follows that

$$\mathrm{V}^i_{\varphi}(M) = \begin{cases} \operatorname{Spec} \mathcal{S} & i \le 8 \\ \mathcal{V}(\chi_1) & 9 \le i \le 12 \\ \mathcal{V}(\chi_1, \chi_2) & 13 \le i \le 14 \\ \{(\boldsymbol{\chi})\} & 15 \le i \le 16 \\ \varnothing & i > 16. \end{cases}$$

In particular, this example produces a complete flag in $\mathbb{A}^3$ from an indecomposable $A/(\boldsymbol{f})$-module.

We end this section with the following realizability theorem that, roughly speaking, says there is essentially no restriction on the sequence of closed subsets that appear as the sequence of jump loci for a fixed dg B-module. This is a higher order version of the realizability results for supports corresponding to a deformation (or Koszul complex); see [8, 19, 35].

Theorem 1.14 *If $f \subseteq \mathfrak{m}_0^2$, then for every descending chain of closed subsets*

$$\operatorname{Spec} \mathcal{S} = W_0 \supsetneq W_1 \supsetneq W_2 \supsetneq \ldots \supsetneq W_t = \varnothing$$

there exists M in $\mathsf{D}^b(B/A)$ and an increasing sequence of integers $0 = j_0 < j_1 < \ldots < j_t$ such that

$$\mathrm{V}_\varphi^j(M) = W_i$$

for $j_i \leq j < j_{i+1}$.

For a fixed dg B-module M, we call the numbers $j_0, \ldots, j_t$ in Theorem 1.14, at which the jump loci change, *the jump numbers of M*. It follows from Lemma 3.5 below that the first jump number is always even. The last jump number j_t is always $\beta^{\mathcal{S}}_{\text{total}}(\mathsf{RHom}_B(M, k))$; see 1.9.

An essential ingredient in the proof of Theorem 1.14 is the theory of Koszul objects introduced by Avramov and Iyengar in [8].

1.15. Fix a dg B-module M and η as in $\mathcal{S}$. Lifting η to $B[\chi_1, ..., \chi_c]$ along π in Remark 1.5 determines a morphism $\tilde{\eta}$ in $\mathsf{D}(B)$

$$M \xrightarrow{\tilde{\eta}} \Sigma^{|\eta|} M.$$

A Koszul object on M with respect to η is the mapping cone of $\tilde{\eta}$, denoted $M /\!\!/ \eta$; we point out that $M /\!\!/ \eta$ is *not* unique, even up to isomorphism, in $\mathsf{D}(B)$. Given a sequence $\boldsymbol{\eta} = \eta_1, \ldots, \eta_n$ in $\mathcal{S}$ we define $M /\!\!/ \boldsymbol{\eta}$ inductively as M_n where

$$M_{i+1} := M_i /\!\!/ \eta_{i+1} \quad \text{with} \quad M_0 = M.$$

It is a direct calculation that $\mathsf{RHom}_B(M /\!\!/ \boldsymbol{\eta}, k)$ is isomorphic to

$$\mathsf{Kos}^{\mathcal{S}}(\boldsymbol{\eta}) \otimes_{\mathcal{S}} \mathsf{RHom}_B(M, k)$$

as dg $\mathcal{S}$-modules, up to a shift; in particular, $\mathsf{RHom}_B(M /\!\!/ \boldsymbol{\eta}, k)$ is independent of the chosen lifts $\tilde{\eta}_i$ of each η_i along π.

Proof of Theorem 1.14 Write each W_i as $\mathcal{V}(\boldsymbol{\eta}^i)$ for some list of elements $\boldsymbol{\eta}^i$ from $\mathcal{S}$ of length n_i. Define M^i to be $K^A /\!\!/ \boldsymbol{\eta}^i$; see 1.15. It follows from Example 1.11 that $\mathsf{RHom}_B(M^i, k)$ is isomorphic to

$$\mathsf{Kos}^{\mathcal{S}}(\boldsymbol{\eta}^i) \otimes_k \bigwedge \Sigma^{-1} k^{\nu}$$

as dg $\mathcal{S}$-modules, up to shift, where ν denotes the minimal number of generators for $\mathfrak{m}_0$. From here it is clear that

$$\mathrm{V}_\varphi^j(M^i) = \mathcal{V}(\boldsymbol{\eta}^i)$$

for all $j = 1, \ldots, n_i$ and $\mathrm{V}_\varphi(M^i) = \varnothing$ for all $j > n_i$. The dg B-module

$$M := M^1 \oplus \ldots \oplus M^{t-1}$$

has the desired properties. □

2 Basic properties

We adopt the notation set in Sect. 1. In this section we show the support theory introduced in the previous section satisfies several important properties.

Proposition 2.1 *Let M be in $\mathsf{D}^b(B/A)$. For each $i \geq 0$, the jump locus $\mathrm{V}^i_\varphi(M)$ is a Zariski closed subset of* $\operatorname{Spec} \mathcal{S}$.

This follows from the following standard lemmas.

Lemma 2.2 *Fix a graded field κ, and let X be a finitely generated dg κ-module. Then*

$$\operatorname{rank}_\kappa \mathsf{H}(X) = 2 \operatorname{rank}_\kappa \left(\operatorname{coker} \partial^X\right) - \operatorname{rank}_\kappa X.$$

Proof. Let B and Z denote the boundaries and cycles of X. Since rank is additive on exact sequences, the desired statements follow immediately from the following diagram with exact rows and columns.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & B \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \mathsf{H}(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \operatorname{coker} \partial^X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \Sigma B \arrow[r] \arrow[d] & \Sigma B \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

□

Lemma 2.3 *Let X be a dg $\mathcal{S}$-module which, upon forgetting its differential, is free of rank of r over $\mathcal{S}$, and set $C = \operatorname{coker} \partial^X$. For each $i \geq 0$, there is an equality*

$$\operatorname{Supp}_{\mathcal{S}} \left(\bigwedge^{r+i} (C \oplus C) \right) = \{\mathfrak{p} \in \operatorname{Spec} \mathcal{S} : \operatorname{rank}_{\kappa(\mathfrak{p})} \mathsf{H}(X \otimes_{\mathcal{S}} \kappa(\mathfrak{p})) \geq i\},$$

and so, in particular, the right-hand set above is a Zariski closed subset of $\operatorname{Spec} \mathcal{S}$.

Proof. Fix $\mathfrak{p} \in \operatorname{Spec} \mathcal{S}$. Applying Lemma 2.2 to $X \otimes_{\mathcal{S}} \kappa(\mathfrak{p})$ gives

$$\operatorname{rank}_{\kappa(\mathfrak{p})} \mathsf{H}(X \otimes_{\mathcal{S}} \kappa(\mathfrak{p})) = 2 \operatorname{rank}_{\kappa(\mathfrak{p})} (C \otimes_{\mathcal{S}} \kappa(\mathfrak{p})) - r,$$

from which we obtain the equivalence

$$\operatorname{rank}_{\kappa(\mathfrak{p})} \mathsf{H}(X \otimes_{\mathcal{S}} \kappa(\mathfrak{p})) \geq i \iff \operatorname{rank}_{\kappa(\mathfrak{p})} ((C \oplus C) \otimes_{\mathcal{S}} \kappa(\mathfrak{p})) \geq r + i.$$

We are done once noting the latter statement is true precisely when

$$\left(\bigwedge^{r+i} (C \oplus C) \right) \otimes_{\mathcal{S}} \kappa(\mathfrak{p}) = \bigwedge^{r+i} ((C \oplus C) \otimes_{\mathcal{S}} \kappa(\mathfrak{p})) \neq 0. \quad \square$$

Proof of Proposition 2.1 First, since M is perfect as a dg A-module, $\mathsf{RHom}_B(M,k)$ is perfect as dg $\mathcal{S}$-module by 1.4. This means there is a quasi-isomorphism of dg $\mathcal{S}$-modules $\mathsf{RHom}_B(M,k) \simeq X$, where X is a dg $\mathcal{S}$-module with underlying $\mathcal{S}$-module being free of finite rank; see [5, Theorem 4.8]. Hence we may apply Lemma 2.3 to X to obtain

$$\mathrm{V}_\varphi^i(M) = \mathrm{Supp}_{\mathcal{S}} \bigwedge^{r+i}(C \oplus C)$$

where $C = \operatorname{coker} \partial^X$ and r is the rank of X regarded as a free $\mathcal{S}$-module. □

2.4. Let $\psi\colon A_0 \to A_0'$ be a flat local extension, and write k' for the residue field of A_0'. Denote the corresponding dg algebras by $A' = A \otimes_{A_0} A_0'$ and $B' = B \otimes_{A_0} A_0'$, the induced homomorphism by $\varphi'\colon A' \to B'$, and the corresponding ring of cohomology operators by $\mathcal{S}' = \mathcal{S} \otimes_k k'$. Then there is an induced map on spectra

$$\psi^*\colon \operatorname{Spec} \mathcal{S}' \to \operatorname{Spec} \mathcal{S}.$$

The next result explains how the cohomological jump loci behave with respect to these maps.

Lemma 2.5 *With notation as in 2.4 above, if M is an object of $\mathsf{D}^b(B/A)$ then $M' = M \otimes_A A'$ is an object of $\mathsf{D}^b(B'/A')$ and for all i*

$$\mathrm{V}_\varphi^i(M) = \psi^*\big(\mathrm{V}_{\varphi'}^i(M')\big).$$

Proof Let $\mathfrak{p}'$ be a prime of $\operatorname{Spec} \mathcal{S}'$ and set $\mathfrak{p} = \psi^*\mathfrak{p}'$. There are isomorphisms

$$\begin{aligned} \mathsf{RHom}_{B'}(M',k') \otimes^{\mathsf{L}}_{\mathcal{S}'} \kappa(\mathfrak{p}') &\cong \mathsf{RHom}_B(M,k) \otimes^{\mathsf{L}}_{\mathcal{S}} \mathcal{S}' \otimes^{\mathsf{L}}_{\mathcal{S}'} \kappa(\mathfrak{p}') \\ &\cong \mathsf{RHom}_B(M,k) \otimes^{\mathsf{L}}_{\mathcal{S}} \kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}'). \end{aligned}$$

Knowing this, the lemma follows directly from the definition of cohomological jump loci; see Definition 1.6. □

Lemma 2.6 *Let M, N be in $\mathsf{D}^b(B/A)$. Suppose*

$$q\colon \mathsf{RHom}_A(M,k) \otimes_k \mathcal{S} \to \mathsf{RHom}_A(N,k) \otimes_k \mathcal{S}$$

is a dg $\mathcal{S}$-module map such that the underlying map of $\mathcal{S}$-modules remains a chain map between the twisted complexes

$$q^\tau\colon \mathsf{RHom}_A(M,k) \otimes_k^\tau \mathcal{S} \to \mathsf{RHom}_A(N,k) \otimes_k^\tau \mathcal{S}.$$

Then q is a quasi-isomorphism if and only if q^τ is a quasi-isomorphism.

Proof This follows directly from the Eilenberg–Moore comparison theorem [39, Theorem 5.5.11] following the observation that the ordinary and twisted complexes coincide upon passing to their associated graded complexes with respect to the $(\boldsymbol{\chi})$-adic filtration. □

Lemma 2.7 *Consider, for some $1 \le c' \le c$, the factorization $A \to B' \to B$ where $B' = A\langle e_1, \ldots, e_{c'} \mid \partial e_i = f_i\rangle$. Then for any M in $\mathsf{D}^b(B/A)$ we have*

$$\mathsf{RHom}_{B'}(M,k) \simeq \mathsf{RHom}_B(M,k) \otimes^{\mathsf{L}}_{\mathcal{S}} \mathcal{S}/\mathfrak{p}$$

as dg $\mathcal{S}$-modules where $\mathfrak{p} = (\chi_{c'+1}, \ldots, \chi_c) \subseteq \mathcal{S}$.

Proof Let $\mathcal{S}' = k[\chi_1, \dots, \chi_{c'}]$ denote the ring of cohomology operators corresponding to $A \to B'$. By direct inspection of the construction in 1.4, we see

$$\begin{aligned}\mathsf{RHom}_{B'}(M,k) &= \mathsf{RHom}_A(M,k) \otimes_k^\tau \mathcal{S}' \\ &\simeq \mathsf{RHom}_A(M,k) \otimes_k^\tau \mathcal{S}/\mathfrak{p} \\ &\simeq \left(\mathsf{RHom}_A(M,k) \otimes_k^\tau \mathcal{S}\right) \otimes_{\mathcal{S}}^{\mathsf{L}} \mathcal{S}/\mathfrak{p} \\ &\simeq \mathsf{RHom}_B(M,k) \otimes_{\mathcal{S}}^{\mathsf{L}} \mathcal{S}/\mathfrak{p}.\end{aligned}$$ □

The next result is the main one from this section. For what follows, we reserve the notation

$$(-)^* := \mathsf{RHom}_B(-, B)$$

for the duality functor on $\mathsf{D}(B)$. However, as $A \to B$ is a Koszul extension, B-duality coincides with $\Sigma^c\, \mathsf{RHom}_A(-, A)$. Thus $(-)^*$ restricts to an endofunctor on $\mathsf{D}^{\mathsf{b}}(B/A)$.

Theorem 2.8 *For any M in $D^b(B/A)$, there are equalities $\mathrm{V}^i_\varphi(M) = \mathrm{V}^i_\varphi(M^*)$ for each positive integer i. Hence $\operatorname{crk}_{\mathfrak{p}} M = \operatorname{crk}_{\mathfrak{p}} M^*$ for all primes $\mathfrak{p}$ of $\mathcal{S}$.*

Proof First, we may assume that the residue field k is algebraically closed by Lemma 2.5 and by [14, Appendice, §2] (see also [30, Theorem 10.14]). Since the jump loci are closed, conical subsets of $\operatorname{Spec} \mathcal{S}$ by Proposition 2.1, it follows that $\mathrm{V}^i_\varphi(M)$ is either empty, $\{(\boldsymbol{\chi})\}$, or the closure of the coheight one primes it contains. Therefore it suffices to show that $\operatorname{crk}_{\mathfrak{p}} M = \operatorname{crk}_{\mathfrak{p}} M^*$ for all coheight one primes $\mathfrak{p}$ of $\operatorname{Spec} \mathcal{S}$ and for $\mathfrak{p} = (\boldsymbol{\chi})$. The proof of the latter is essentially contained in the former, so we will proceed assuming $\mathfrak{p}$ is coheight one. Using the Nullstellensatz and a linear change of variables, we may further assume $\mathfrak{p} = (\chi_2, \dots, \chi_c)$.

Next, let B' denote the dg subalgebra $A\langle e_1\rangle \subseteq B$ and $\mathcal{S}' = k[\chi_1]$ denote the corresponding ring of cohomology operators for $A \to B'$. Since $\mathcal{S}' = \mathcal{S}/\mathfrak{p}$, if we let κ' denote the residue field of $\mathcal{S}'$ at (0), then $\kappa' = \kappa(\mathfrak{p})$ and hence by Lemma 2.7,

$$\begin{aligned}\mathsf{RHom}_B(M,k) \otimes_{\mathcal{S}}^{\mathsf{L}} \kappa(\mathfrak{p}) &\simeq \mathsf{RHom}_B(M,k) \otimes_{\mathcal{S}}^{\mathsf{L}} \mathcal{S}' \otimes_{\mathcal{S}'}^{\mathsf{L}} \kappa' \\ &\simeq \mathsf{RHom}_{B'}(M,k) \otimes_{\mathcal{S}'}^{\mathsf{L}} \kappa'.\end{aligned}$$

Once we recall the fact that for a perfect dg $\mathcal{S}'$-module N one has the equality

$$\operatorname{rank}_{\kappa'} \mathsf{H}\left(N \otimes_{\mathcal{S}'}^{\mathsf{L}} \kappa'\right) = \operatorname{rank}_{\kappa'} \mathsf{H}\left(\mathsf{RHom}_{\mathcal{S}'}(N, \mathcal{S}') \otimes_{\mathcal{S}'}^{\mathsf{L}} \kappa'\right),$$

we see that it is sufficient to show

$$\mathsf{RHom}_{B'}(M^*, k) \simeq \mathsf{RHom}_{\mathcal{S}'}(\mathsf{RHom}_{B'}(M,k), \mathcal{S}').$$

To this end, observe that we have the following isomorphisms of dg $\mathcal{S}'$-modules:

$$\begin{aligned}\mathsf{RHom}_{B'}(M^*, k) &\simeq \mathsf{RHom}_A(M^*, k) \otimes_k^\tau \mathcal{S}' \\ &\simeq \mathsf{Hom}_k(k \otimes_A^{\mathsf{L}} M^*, k) \otimes_k^\tau \mathcal{S}' \\ &\simeq \mathsf{Hom}_k(\mathsf{RHom}_A(M,k), k) \otimes_k^\tau \mathcal{S}';\end{aligned}$$

the second one being nothing more than adjunction, while the third uses the dg B-module isomorphism $\mathsf{RHom}_A(M,k) \simeq k \otimes_A^{\mathsf{L}} M^*$ which is one place the assumption that M is perfect over A is being invoked. Furthermore, the natural maps

$$\begin{array}{c}\mathsf{Hom}_k(\mathsf{RHom}_A(M,k),k)\otimes_k \mathcal{S}' \\ \downarrow\wr \\ \mathsf{Hom}_k(\mathsf{RHom}_A(M,k),\mathcal{S}') \\ \downarrow\wr \\ \mathsf{RHom}_{\mathcal{S}'}(\mathsf{RHom}_A(M,k)\otimes_k \mathcal{S}',\mathcal{S}')\end{array}$$

are each quasi-isomorphisms of dg $\mathcal{S}'$-modules. A direct computation shows that the composite map is compatible with the twisted differential, inducing a map

$$\mathsf{Hom}_k(\mathsf{RHom}_A(M,k),k)\otimes_k^\tau \mathcal{S}' \to \mathsf{RHom}_{\mathcal{S}'}(\mathsf{RHom}_A(M,k)\otimes_k^\tau \mathcal{S}',\mathcal{S}'),$$

which, by Lemma 2.6, is also a quasi-isomorphism. Combining this quasi-isomorphism with the already established ones above, we obtain the desired result. □

Remark 2.9 In the case that A is a local ring and $B = A/(\boldsymbol{f})$ is the quotient by a regular sequence $\boldsymbol{f} = f_1,\ldots,f_c$, we indicate here how to interpret the above theory more classically in terms of matrix factorizations.

Fix a nonzero point $(a_1,\ldots,a_c)$ in k^c and choose lifts $\tilde{a}_i$ of each a_i to A. Any complex M in $\mathsf{D}(B)$ be regarded as a $A/(\sum \tilde{a}_i f_i)$-module through the factorization

$$A \to A/\big(\textstyle\sum \tilde{a}_i f_i\big) \to B.$$

For ease of notation, let $A_{\tilde{a}}$ denote $A/(\sum \tilde{a}_i f_i)$. By [11, Theorem 2.1], for lifts $\tilde{a}$ and $\tilde{a}'$ of a point a in k^c there is an equality of Betti numbers $\beta_i^{A_{\tilde{a}}}(M) = \beta_i^{A_{\tilde{a}'}}(M)$ for each integer i. Hence we simply write $\beta_i^a(M)$ for $\beta_i^{A_{\tilde{a}}}(M)$. Furthermore, when M is in $\mathsf{D}^{\mathrm{b}}(B/A)$ the sequence of values $\beta_i^a(M)$ eventually stabilizes; this stable value is denoted $\underline{\beta}^a(M)$, called the stable Betti number of M at a. Moreover $\underline{\beta}^a(M)$ is exactly the rank of the free modules appearing in a matrix factorization describing the tail of a free $A_{\tilde{a}}$-module resolution of M; cf. [22, 37]. When A is Gorenstein, this is also the k-rank of each *stable* (or *Tate*) cohomology module $\underline{\mathsf{Ext}}_B^i(M,k)$.

When k is algebraically closed, by invoking the Nullstellensatz, the (inhomogeneous) maximal ideals of $\operatorname{Spec}\mathcal{S}$ correspond to k^c, affine c-space over k. In light of the discussion above, for each nonnegative integer i, it is sensible to consider the following subset of k^c:

$$\{a \in k^c : 2\underline{\beta}^a(M) \geq i\} \cup \{0\}. \tag{2.9.1}$$

The proof of Theorem 2.8 shows that the closed points of the cone over $\mathrm{V}_\varphi^i(M)$ correspond exactly with the subset in 2.9.1. When $i = 1$, the subset 2.9.1 is the classical support variety from [1, 3.11].

We end this section with an *accoutrement* demonstrating an a priori surprising property of the cohomological jump loci when taken in total. There are general axioms for a support theory on a triangulated category; see, for example, the conditions specified in [13, Theorem 1]. Two such axioms are: first, that the support takes direct sums to unions, and second, the so-called *two-out-of-three property* on the supports of objects in an exact triangle. The following proposition says that the jump loci all together satisfy a higher-order generalization of these usual containment properties.

Proposition 2.10 *Given an exact triangle $L \to M \to N \to$ in $\mathsf{D}^b(B/A)$ there is the following containment of jump loci*

$$\mathrm{V}^l_\varphi(M) \subseteq \bigcup_{i+j=l} \mathrm{V}^i_\varphi(L) \cap \mathrm{V}^j_\varphi(N);$$

equality holds when $M \to N$ admits a section.

Proof This follows directly from the exact triangle obtained by applying $- \otimes^{\mathsf{L}}_{\mathcal{S}} \kappa(\mathfrak{p})$ to the exact triangle $L \to M \to N \to$, and noting that when $M \to N$ admits a section, so does the corresponding induced map. □

Remark 2.11 In light of Proposition 2.10, the higher jump loci V^i_φ for $i > 1$ do not respect containment among thick subcategories of $\mathsf{D}^{\mathrm{b}}(B/A)$. This should be contrasted with usual support varieties V^1_φ which can even be used to classify the thick subcategories of $\mathsf{D}^{\mathrm{b}}(B/A)$ when A is a regular ring and $\boldsymbol{f}$ is an A-regular sequence; see [31, 38].

3 Applications to betti degree

In this section $(A, \mathfrak{m}, k)$ is a local ring, $\boldsymbol{f} = f_1, \ldots, f_c$ is an A-regular sequence. Set $B = A/(\boldsymbol{f})$, and let $\varphi : A \to B$ be the canonical projection. As noted in Remark 1.1, we can freely apply the results from the preceding sections while studying Ext-modules over B in the present section.

Definition 3.1 ([1, (3.1),(4.1)]) Let M be an object of $\mathsf{D}(B)$. The *complexity* of M, denoted $\mathrm{cx}^B(M)$, is the smallest natural number b such that the sequence $\{\beta^B_i(M)\}^\infty_{n=0}$ of Betti numbers over B, given by $\beta^B_i(M) = \mathrm{rank}_k\, \mathsf{Ext}^i_B(M, k)$, is eventually bounded by a polynomial of degree $b - 1$. If no such integer exists one sets $\mathrm{cx}^B(M)$ to be infinity.

If M has finite complexity $\mathrm{cx}^B(M) = n + 1$, the *Betti degree* of M (over B) is defined to be

$$\beta\mathrm{deg}^B(M) = 2^n n! \limsup_{i\to\infty} \frac{\beta^B_i(M)}{i^n}. \tag{3.1.1}$$

3.2. According to 1.4, if M is in $\mathsf{D}^{\mathrm{b}}(B/A)$ then $\mathsf{Ext}_B(M, k)$ is a finitely generated graded $\mathcal{S}$-module. In particular, by the Hilbert-Serre Theorem, $\mathrm{cx}^B(M)$ is exactly the Krull dimension of $\mathsf{Ext}_B(M, k)$ over $\mathcal{S}$. Hence, $\mathrm{cx}^B(M) \le c$, and by the Nullsetellsatz, $\mathrm{cx}^B(M)$ is the dimension of the Zariski closed subset $\mathrm{V}^1_\varphi(M)$; cf. [1, 4]. It is worth remarking that the above assertions hold at the level of generality in Sect. 1; however, the next discussion is one place we are forced to specialize to the setting of the present section.

3.3. Let M be in $\mathsf{D}^{\mathrm{b}}(B/A)$ with $\mathrm{cx}^B(M) = n + 1$. Then there exist polynomials q_{ev} and q_{odd} of degree n whose leading coefficients agree such that for all $i \gg 0$

$$\beta^B_i(M) = \begin{cases} q_{\mathrm{ev}}(i) & i \text{ is even} \\ q_{\mathrm{odd}}(i) & i \text{ is odd;} \end{cases}$$

see [1, Remark 4.2]. In particular, the sequence defining $\beta\mathrm{deg}^B(M)$ in Definition 3.1 converges and the leading coefficient of both q_{ev} and q_{odd} is $\beta\mathrm{deg}^B(M)/2^n n!$.

Finally up to further scaling $\beta\mathrm{deg}^B(M)$ can also be realized as the multiplicity of $\mathsf{Ext}_B(M, k)$ over $\mathcal{S}$.

3.4. Fix M in $\mathsf{D}^{\mathrm{b}}(B/A)$ with complexity $\operatorname{cx}^B(M) = n+1$. Let S be the polynomial ring $\mathcal{S}$ regraded so that the variables χ_i are in cohomological degree 1. We may define E to be the graded S-module consisting of the even degrees of $\mathsf{Ext}_B(M,k)$, i.e.

$$E^i = \mathsf{Ext}_B^{2i}(M,k).$$

When endowed with the degree filtration, $E^{\geq n} = \bigoplus_{i\geq n} E^i$, the associated Hilbert polynomial is $q_{\mathrm{ev}}(2t)$ as defined in 3.3. In particular, the leading term is given by

$$\frac{\beta\deg^B(M)/2^n}{n!}(2t)^n = \frac{\beta\deg^B(M)}{n!}t^n.$$

When endowed with the $(\boldsymbol{\chi})$-adic filtration, the leading term of the associated Hilbert polynomial is of the form

$$\frac{e(E)}{n!}t^n$$

where $e(E)$ is the multiplicity of E as an S-module. Since E is finitely generated over S, for all n sufficiently large, $E_{n+1} = (\boldsymbol{\chi})E_n$, and hence the leading terms of the two Hilbert polynomials agree, so

$$e(E) = \beta\deg^B(M).$$

This is the reason for the normalization factor of $2^n n!$ in the definition 3.1.1; in particular the number $\beta\deg^B(M)$ is always a positive integer.

Finally, since S is a regular integral domain, we obtain the equality [32, Theorem 14.8]

$$e(E) = e(S)\cdot \operatorname{rank}_{S_{(0)}} E_{(0)} = \operatorname{rank}_{S_{(0)}} E_{(0)}.$$

Repeating this process for the module consisting of the odd degrees of $\mathsf{Ext}_B(M,k)$ yields

$$\operatorname{rank}_{\mathcal{S}_{(0)}} \mathsf{Ext}_B(M,k)_{(0)} = 2\beta\deg^B(M).$$

Lemma 3.5 *An object M of $\mathsf{D}^b(B/A)$ has maximal complexity c if and only if $\mathrm{V}^1_\varphi(M) = \operatorname{Spec}\mathcal{S}$, and in this case*

$$2\beta deg^B(M) = \max\left\{i : \mathrm{V}^i_\varphi(M) = \operatorname{Spec}\mathcal{S}\right\}.$$

Proof Recall from 3.3, that $\operatorname{cx}^B(M) = \dim V^1_\varphi(M)$. From this, we see that maximal complexity of M is equivalent to $\mathrm{V}^1_\varphi(M) = \operatorname{Spec}\mathcal{S}$.

Since the jump loci are closed, $\mathrm{V}^i_\varphi(M) = \operatorname{Spec}\mathcal{S}$ if and only if $(0) \in \mathrm{V}^i_\varphi(M)$. However,

$$\operatorname{rank}_{\kappa(0)} \mathsf{H}(\mathsf{RHom}_B(M,k)\otimes^{\mathsf{L}}_{\mathcal{S}}\kappa(0)) = \operatorname{rank}_{\mathcal{S}_{(0)}} \mathsf{Ext}_B(M,k)_{(0)}$$

hence

$$\max\left\{i : \mathrm{V}^i_\varphi(M) = \operatorname{Spec}\mathcal{S}\right\} = \operatorname{rank}_{\mathcal{S}_{(0)}} \mathsf{Ext}_B(M,k)_{(0)}.$$

The lemma now follows from 3.4. □

We remind the reader that we use the notation $(-)^* = \mathsf{RHom}_B(-,B)$ for B-duality throughout, and that up to a shift, this coincides with the A-duality $\mathsf{RHom}_A(-,A)$.

Theorem 3.6 *Let $A \to B$ be a surjective map of local rings whose kernel is generated by an A-regular sequence. If M is in $\mathsf{D}^b(B/A)$ then $\beta deg^B(M) = \beta deg^B(M^*)$.*

Proof We first reduce to the case of full complexity. Since Betti numbers, and hence the Betti degree, are unchanged by flat base change, we may assume that the residue field k is infinite. Recall from 3.2 that the Krull dimension of $\mathsf{Ext}_B(M,k)$ over $\mathcal{S}$ is equal to $\mathrm{cx}(M) = c'$. By Noether normalisation we can make a linear change of coordinates and assume that $\mathsf{Ext}_B(M,k)$ is finite over $k[\chi_1,\ldots,\chi_{c'}]$. Writing $B' = A/(f_{c'+1},\ldots,f_c)$ and $\mathfrak{p} = (\chi_1,\ldots,\chi_{c'}) \subseteq \mathcal{S}$ it follows from Lemma 2.7 that

$$\mathsf{RHom}_{B'}(M,k) \simeq \mathsf{RHom}_B(M,k) \otimes^{\mathsf{L}}_{\mathcal{S}} \mathcal{S}/\mathfrak{p}.$$

The right-hand-side has cohomology which is finite over $k[\chi_1,\ldots,\chi_{c'}]$ (since it is built by $\mathsf{RHom}_B(M,k)$), and simultaneously annihilated by $\mathfrak{p}$; therefore it must be finite dimensional. This means that $\mathsf{Ext}_{B'}(M,k)$ is bounded, and we conclude that M is in $\mathsf{D}^{\mathsf{b}}(B/B')$, and it has the maximal complexity c' among objects of this category.

We may now assume that M has maximal complexity within $\mathsf{D}^{\mathsf{b}}(B/A)$, so we can use Lemma 3.5 and Theorem 2.8 to deduce

$$\begin{aligned} 2\beta\mathrm{deg}^B(M) &= \max\left\{i : \mathrm{V}^i_\varphi(M) = \operatorname{Spec}\mathcal{S}\right\} \\ &= \max\left\{i : \mathrm{V}^i_\varphi(M^*) = \operatorname{Spec}\mathcal{S}\right\} = 2\beta\mathrm{deg}^B(M^*). \end{aligned}$$

From this we obtain the desired equality $\beta\mathrm{deg}^B(M) = \beta\mathrm{deg}^B(M^*)$. □

Remark 3.7 Let M be a module over a deformation $A \to B$, as in the setup of Theorem 3.6. Eisenbud, Peeva and Schreyer prove in [23] that the Betti degree of M is equal to the rank of a minimal matrix factorization for M, of a generically chosen relation in an intermediate deformation A' (chosen as in the proof of Theorem 3.6); see [23, Theorem 4.3] for a precise statement. Our Theorem 3.6 can also be deduced from this result. Conversely, [23, Theorem 4.3] can alternatively be proven using the cohomological jump loci along the lines of Theorem 3.6.

Eisenbud, Peeva and Schreyer make essential use of the theory of higher matrix factorizations in their work. This raises the question of the connection between the data visible in a higher matrix factorization of a module M and its cohomological jump loci.

The conclusion in Theorem 3.6 for the quasi-polynomials governing the Betti numbers of M and M^* cannot be improved. That is to say, the lower order terms of the respective quasi-polynomials need not agree.

Example 3.8 Consider $A = k[\![x,y]\!]$ and $B = A/(x^3,y^3)$. For $M = B/(x^2,xy,y^2)$ and $i \geq 0$ there are equalities

$$\beta_i^B(M) = \begin{cases} \frac{3}{2}i + 1 & i \text{ even} \\ \frac{3}{2}i + \frac{3}{2} & i \text{ odd} \end{cases} \quad \text{and} \quad \beta_i^B(M^*) = \begin{cases} \frac{3}{2}i + 2 & i \text{ even} \\ \frac{3}{2}i + \frac{3}{2} & i \text{ odd.} \end{cases}$$

3.9. We now fix a local ring B (and we forget A for a brief moment). Following the work of Avramov, Gasharov and Peeva [6] and Sather-Wagstaff [36], a complex M of B-modules is said to have *finite ci-dimension* if there exists a diagram of local rings

$$A \longrightarrow B' \longleftarrow B$$

in which $B \to B'$ is flat and $A \to B'$ is a surjective deformation, and such that $M \otimes_B B'$ is isomorphic in $\mathsf{D}(A)$ to a bounded complex of projective modules.

Corollary 3.10 *If B is a local ring and M is a complex of B-modules with finitely generated homology and finite ci-dimension, then $\beta deg^B(M) = \beta deg^B(M^*)$.*

Proof Both the duality and the Betti degree are preserved by flat base change, so we may assume that B admits a deformation $\varphi\colon A \to B$ such that M is in $\mathsf{D}^{\mathrm{b}}(B/A)$, and the statement follows from Theorem 3.6. □

Remark 3.11 Let M be a maximal Cohen-Macaulay module over B which has finite ci-dimension. It is well known that M admits a complete resolution over B, in the sense of [17] that M is the cokernel of a differential in a acyclic complex of projective B-modules. The two ends of this complete resolution (the projective resolution and coresolution of M) grow quasi-polynomially with the same degree; see, for example, [4, 9, 31]. Corollary 3.10 asserts that moreover the leading terms of these two quasi-polynomials are the same. This is in stark contrast with the results of [29], where modules are exhibited with complete resolutions that have wildly asymmetric growth. All such modules must have infinite ci-dimension.

We now move on to our final result. If we specialize to the case where A is a Gorenstein ring, then Gorenstein duality allows us to form a connection between the Betti numbers of a module and its Bass numbers as a direct corollary to Theorem 3.6.

Definition 3.12 ([1, (5.1)]) Let M be an object of $\mathsf{D}(B)$. Recall that the i-th Bass number of M is defined to be

$$\mu^i_B(M) := \operatorname{rank}_k \operatorname{Ext}^i_B(k, M).$$

The *cocomplexity* (or *plexity* as used in [4, 10]) of M, denoted $\operatorname{px}_B(M)$, is defined to be the smallest nonnegative integer b such that the sequence $\{\operatorname{rank}_k \operatorname{Ext}^n_B(k, M)\}_{n=0}^{\infty}$ is eventually bounded by a polynomial of degree $b-1$.

Suppose $\operatorname{px}_B(M) = n+1$. Define the *Bass degree* of M over B to be

$$\mu\mathrm{deg}_B := 2^n n! \limsup_{i\to\infty} \frac{\mu^i_B(M)}{i^n}.$$

Corollary 3.13 *If A is Gorenstein then for any M in $\mathsf{D}^b(B/A)$*

$$\mu deg_B(M) = \beta deg^B(M).$$

Proof This is an easy consequence of Gorenstein-duality and Theorem 3.6. Namely, A being Gorenstien forces B to be Gorenstein and so there is an isomorphism of graded k-spaces

$$\operatorname{Ext}_B(M^*, k) \cong \Sigma^s \operatorname{Ext}_B(k, M)$$

for some integer s. Hence $\mu\mathrm{deg}_B(M) = \beta\mathrm{deg}^B(M^*)$ and so now applying Theorem 3.6, we obtain the desired equality. □

Question 3.14 Let M and N be two dg B-modules, each perfect over A, and assume that $\operatorname{Ext}_B(M, N)$ is degree-wise of finite length (in large degrees). In this context the numbers $\operatorname{length}_B \operatorname{Ext}^i_B(M, N)$ are also eventually modelled by quasi-polynomial $q(M, N)$ of period two; cf. [17, 10.3] and [21]. In the case that A is regular, Avramov and Buchweitz prove, using the theory of support varieties for pairs of modules, that $q(M, N)$ and $q(N, M)$ have equal degrees [4]. Corollary 3.13 suggests the following question: Assuming A is Gorenstein, what is the relationship between the leading terms of $q(M, N)$ and $q(N, M)$?

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