



Surface Houghton groups

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Abstract

For every $n \geq 2$, the *surface Houghton group* $\mathcal{B}_n$ is defined as the asymptotically rigid mapping class group of a surface with exactly n ends, all of them non-planar. The groups $\mathcal{B}_n$ are analogous to, and in fact contain, the braided Houghton groups. These groups also arise naturally in topology: every monodromy homeomorphism of a fibered component of a depth-1 foliation of closed 3-manifold is conjugate into some $\mathcal{B}_n$. As countable mapping class groups of infinite type surfaces, the groups $\mathcal{B}_n$ lie somewhere between classical mapping class groups and big mapping class groups. We initiate the study of surface Houghton groups proving, among other things, that $\mathcal{B}_n$ is of type F_{n-1} , but not of type FP_n , analogous to the braided Houghton groups.

1 Introduction and results

For $n \geq 2$, denote by Σ_n the connected orientable surface with exactly n ends, all of them non-planar. We view Σ_n as constructed by first gluing a torus with two boundary components (called a *piece*) to every connected component of a sphere with n boundary components, and then inductively gluing a piece to every boundary component of the

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surface from the previous step. The *surface Houghton group*, $\mathcal{B}_n$, is the subgroup of the mapping class group $\text{Map}(\Sigma_n)$ whose elements *eventually* send pieces to pieces, in a *trivial* manner; see Sect. 2 for a precise definition.

Viewed in this light, the groups $\mathcal{B}_n$ are natural analogs of the asymptotic mapping class groups of Cantor surfaces considered in [1, 2, 14–16]. In addition, these groups are closely related to Houghton groups and their *braided* relatives [8]. In fact, using Funar’s description of the braided Houghton group $\text{br}H_n$ as an asymptotically rigid mapping class group [13], in Remark 2.2 below we briefly explain how to see that $\text{br}H_n$ may be realized as a subgroup of $\mathcal{B}_n$.

Beyond this analogy, the groups $\mathcal{B}_n$ arise naturally in the study of depth-1 foliations of closed 3-manifolds. More precisely, the non-product components are mapping tori of *end-periodic homeomorphisms* (see [12]), and every such homeomorphism is conjugate into some $\mathcal{B}_n$ (see [11, Corollary 2.9]).

1.1 Finiteness properties

Recall that a group is of type F_d if it has a classifying space with finite d -skeleton, and that it is of type FP_d if the integers, regarded as a trivial module over the group, have a projective resolution that is of finite type in dimensions up to d . Such a resolution can be obtained for a group of type F_d by using the cellular chain complex of the universal cover of a classifying space. Thus, F_d implies FP_d .

In [16], Genevois–Lonjou–Urech proved that the braided Houghton group $\text{br}H_n$ is of type F_{n-1} but not of type FP_n . Our first result is the analog of this result in our setting.

Theorem 1.1 *$\mathcal{B}_n$ is of type F_{n-1} but not of type FP_n .*

In order to prove Theorem 1.1, as is often the case with this type of result, we will make use of a classical criterion of Brown [6], expressed through the language of discrete Morse theory; see Sect. 3 for details. More precisely, for each $n \geq 2$ we will construct a finite-dimensional contractible cube complex on which $\mathcal{B}_n$ acts, and an invariant discrete Morse function on the complex, such that the descending links are highly connected. In the construction of the complex as well as in the analysis of descending links, we make heavy use of methods developed in [1].

1.2 Abelianization

A well-known theorem of Powell [26] asserts that the *pure* mapping class group $\text{PMap}(S)$ of a finite-type surface S of genus at least three has trivial abelianization. For the surfaces Σ_n , a result of Patel, Vlamis, and the first author [3, Corollary 6] implies that $H^1(\text{PMap}(\Sigma_n), \mathbb{Z}) \cong \mathbb{Z}^{n-1}$. More generally, the abelianization of (pure) mapping class groups of infinite type surfaces is more complicated; see [9, 10, 21, 23, 27].

Using [3], we compute the abelianization $\mathcal{B}_n^{ab}$ of $\mathcal{B}_n$, as well as that of their pure counterparts $P\mathcal{B}_n$.

Theorem 1.2 *For all $n \geq 2$, $\mathcal{B}_n^{ab} = \{0\}$ and $P\mathcal{B}_n^{ab} \cong \mathbb{Z}^{n-1}$.*

From the description of the abelianizations, we also describe all finite quotients of $\mathcal{B}_n$, proving that they are highly constrained; see Proposition 6.2. This has the following consequence.

Corollary 1.3 *For all $m, n \in \mathbb{N}$, the groups $\mathcal{B}_n$ and $\mathrm{br}H_m$ are not commensurable.*

1.3 Marking graphs

By Theorem 1.1, $\mathcal{B}_n$ is finitely generated for all $n \geq 2$, and so it has a well-defined coarse geometry. In the finite type setting, a quasi-isometric model for the mapping class group which has proven quite useful is Masur and Minsky's marking graph [22]. For the surfaces Σ_n , the marking graph is no longer a good model for several reasons. A trivial issue is that it is disconnected, and the orbit of a marking may lie in different components. There are $\mathcal{B}_n$ -invariant components, but even these are problematic since they are no longer locally finite. Worse, the orbit map to any such component fails to be a quasi-isometric embedding (see Sect. 7). On the other hand, there are (many) locally finite subgraphs which do serve as quasi-isometric models, by the following and the Milnor–Švarc Lemma (see e.g. [5]).

Theorem 1.4 *For all $n \geq 2$, there exist locally finite subgraphs of the marking graph on which $\mathcal{B}_n$ acts cocompactly. For $n \geq 3$, any marking μ is contained in such a subgraph.*

The final statement is not quite true for $n = 2$ since there are markings on Σ_2 with infinite stabilizer in $\mathcal{B}_n$.

Plan of the paper. Sect. 2 contains the relevant background on surfaces, mapping class groups, and the definition of $\mathcal{B}_n$, and in Sect. 3 we recall a classical result of Brown [6] about finiteness properties of groups, and describe the relevant tools in our setting. Section 4 constructs a contractible cube complex on which $\mathcal{B}_n$ acts nicely, and which is similar to that of [1, 16]. In Sect. 5 we will prove Theorem 1.1. We then prove Theorem 1.2 and Corollary 1.3 in Sect. 6 and Theorem 1.4 in Sect. 7.

2 Surfaces

Throughout this paper, all surfaces will be assumed to be connected, orientable and second-countable. A surface is said to have *finite type* if its fundamental group is finitely generated; otherwise, it has *infinite type*. In this paper, the primary surfaces of infinite type we will consider are those that have a finite number of ends, all non-planar; see Sect. 2.3 for an explicit construction.

2.1 Curves and arcs

By a *curve* on a surface S we mean the isotopy class of a simple closed curve on S . All curves will be *essential*, meaning they do not bound a disk or once-punctured disk,

nor do they cobound an annulus with a component of ∂S . An *arc* is the isotopy class (relative to ∂S) of an embedded path that connects two boundary components of S . We say that two curves/arcs are *disjoint* if they can be isotoped off each other. As usual, we will not distinguish between curves (resp. arcs) and their representatives.

2.2 Mapping class groups

The mapping class group $\text{Map}(S)$ of S is the group of isotopy classes of orientation-preserving homeomorphisms of S ; here all homeomorphisms and isotopies are assumed to fix the boundary of S pointwise. The *pure* mapping class group $\text{PMap}(S)$ is the subgroup of $\text{Map}(S)$ whose elements fix every end of S . When S has finite type, then $\text{Map}(S)$ is well-known to be of type F_∞ [17]. On the other hand, when S has infinite type, then $\text{Map}(S)$ is uncountable.

When S has infinite type, an important subgroup of $\text{Map}(S)$ (and of $\text{PMap}(S)$, in fact) is the *compactly-supported* mapping class group $\text{Map}_c(S)$, which consists of those mapping classes that are the identity outside some compact subset of S . By a result of Patel–Vlamis [25], $\text{Map}_c(S)$ is dense (in the compact-open topology) inside $\text{PMap}(S)$ if and only if S has at most one non-planar end; otherwise, $\text{PMap}(S)$ is topologically generated by $\text{Map}_c(S)$ plus the set of *handle-shifts*.

2.3 Rigid structures and surface Houghton groups

The goal of this subsection is to describe the surfaces Σ_n and the additional structure alluded to in the introduction necessary to define their asymptotic mapping class groups. These definitions, along with the toolkit needed to present them, closely follow [1, Sect. 3].

Fix an integer $n \geq 2$. Let O_n be a sphere with n boundary components, labelled $b_1, \dots, b_n$, and T a torus with two boundary components, denoted ∂^-T and ∂^+T ; we refer to ∂^+T as the *top* boundary component of T . We fix, once and for all, an orientation-reversing homeomorphism $\lambda : \partial^-T \rightarrow \partial^+T$, and orientation-reversing homeomorphisms $\mu_i : \partial^-T \rightarrow b_i$, for $i = 1, \dots, n$. We construct a sequence of compact, connected, orientable surfaces $(M^i)_i$ as follows:

- $M^1 = O_n$;
- M^2 is the result of gluing a copy of T to each boundary component of O_n , using the homeomorphisms μ_i ;
- For each $i \geq 3$, M^i is the result of gluing a copy of T along each of the boundary components of M^{i-1} , using the homeomorphism λ .

The surface Σ_n is the union of the surfaces M^i above. The closure of each of the connected components of $M^i \setminus M^{i-1}$, for $i \geq 2$, is called a *piece*. By construction, each piece $B \subset M^i \setminus M^{i-1}$ is one of the glued on copies of T , and so is equipped with a canonical homeomorphism $i_B : B \rightarrow T$. We call the unique boundary component of B that belongs to M^{i-1} the *top* boundary component of B as it maps by i_B to ∂_+T . The set

$$\{\iota_B : B \text{ is a piece}\}$$

is called the *rigid structure* on Σ_n .

A subsurface of Σ_n is *suited* if it is connected and is the union of O_n and finitely many pieces. A boundary component of a suited subsurface is called a *suited curve*.

Let $f : \Sigma_n \rightarrow \Sigma_n$ be a homeomorphism. We say that f is *asymptotically rigid* if there exists a suited subsurface $Z \subset \Sigma_n$, called a *defining surface* for f , such that:

- $f(Z)$ is a suited subsurface, and
- f is *rigid away from* Z , that is, for every piece $B \subset \overline{\Sigma_n \setminus Z}$, we have that $f(B)$ is a piece, and $f|_B \equiv \iota_{f(B)}^{-1} \circ \iota_B$.

Definition 2.1 (*Surface Houghton group*) The surface Houghton group $\mathcal{B}_n$ is the subgroup of the mapping class group $\text{Map}(\Sigma_n)$ whose elements have an asymptotically rigid representative.

We note that $\mathcal{B}_n$ is a subgroup. Indeed, if f and f' are asymptotically rigid, with defining surfaces Z and Z' , respectively, then $f(Z \cup Z')$ is a suited subsurface, and serves as a defining surface for $f' \circ f^{-1}$, exhibiting it as an asymptotically rigid homeomorphism.

We will denote by $P\mathcal{B}_n$ the intersection of $\mathcal{B}_n$ with the pure mapping class group $\text{PMap}(\Sigma_n)$.

Remark 2.2 In [13], Funar described the braided Houghton groups $\text{br}H_n$ as asymptotically rigid mapping class groups of certain planar surfaces. Replacing punctures of this planar surface with boundary components, and then doubling the resulting surface, determines a homomorphism $\text{br}H_n \rightarrow \mathcal{B}_n$ analogous to the construction of Ivanov–McCarthy [20, Sect. 2].

3 Finiteness properties

3.1 Brown's criterion for finiteness properties

In this section we recall a classical criterion, due to Brown [6], for a group to (not) have certain finiteness properties. We remark that our formulation of the criterion differs from the original [6], as we use the language of *discrete Morse theory* as developed in [4]. We recall here the basic notions and definitions, referring the interested reader to [4] or [1, Appendix A] for a thorough discussion.

Our setting is a group G acting on a piecewise euclidean CW-complex X by cell-permuting homeomorphisms that restrict to isometries on cells. A *Morse function* on X is a cell-wise affine map $h : X \rightarrow \mathbb{R}$ satisfying the condition that on each closed cell it attains a unique maximum (at a vertex, the *top vertex* of the cell). The *descending link* $\text{lk}^\downarrow(v)$ of a vertex v is the part of its link spanned by the links of cells that contain v as their top vertex. The value $h(v)$ is often referred to as the *height* of v .

We consider the vertices the *critical points* in X and the images of vertices under h are the *critical values*. A Morse function $h : X \rightarrow \mathbb{R}$ is *discrete* if the set of its critical values is a closed discrete subset of $\mathbb{R}$. Now the first theorem can be stated as follows.

Theorem 3.1 (Brown) *Let G be a group acting by cell-wise isometries on a contractible piecewise euclidean CW-complex X . Assume X is equipped with a discrete G -invariant Morse function $h: X \rightarrow \mathbb{R}$, and let $X^{\leq s}$ denote the largest subcomplex of X fully contained in the preimage $h^{-1}(-\infty, s]$. Suppose that*

- *The quotient of $X^{\leq s}$ by G is finite for all critical values s .*
- *Every cell stabilizer is of type F_∞ .*
- *There exists $d \geq 1$ such that, for sufficiently large critical values s and for every vertex $v \in X$ with $h(v) \geq s$, the descending link of v in X is d -spherical, i.e., $(d-1)$ -connected and of dimension d .*
- *For each critical value s , there exists a vertex v with non-contractible descending link at height $h(v) \geq s$.*

Then G is of type F_d but not of type FP_{d+1} .

Proof This proof mimics Brown's proof of [6, Corollary 3.3].

As the set of critical values is discrete and closed, we can index them in order. Now, we consider the filtration of X by sublevel complexes $X_i := X^{\leq s_i}$ where s_i runs through the critical values in order. The filtration step X_{i+1} is obtained from X_i up to homotopy by coning off descending links. Thus, the hypothesis that descending links are $(d-1)$ -connected implies that the inclusion of X_i into X_{i+1} induces isomorphisms in homotopy (and homology) groups up to dimension $d-1$ and an epimorphism in homotopy (and homology) in dimension d . As we assume the complex X to be contractible, the isomorphisms in dimensions below d must eventually all be trivial maps. It follows that for sufficiently large i , the sublevel complex X_i is $(d-1)$ -connected. By [6, Propositions 1.1 and 3.1] implies that G is of type F_d .

For the negative direction, we focus on the system $\{H_d(X_i)\}_i$ of homology groups in dimension d . We already observed that for i large enough, the morphisms in the system are onto. Hence, the system can only be essentially trivial if the homology groups $H_d(X_i)$ vanish for all large enough i . If, however, at the transition from X_i to X_{i+1} , we encounter a non-contractible $(d-1)$ -connected descending link of dimension d , this descending link will have non-trivial homology in dimension d . Hence the descending link contains a d -cycle. If this cycle was a boundary in X_i , coning off the descending link would provide another way of bounding it in X_{i+1} , thus creating a non-trivial $(d+1)$ -cycle, which is impossible as that element of H_{d+1} could never be killed in the future as we only ever cone off d -dimensional links. Hence, $H_d(X_i) \neq 0$ for infinitely many i . $\square$

It is clear from the criterion that tools for the analysis of connectivity properties of spaces can be useful. We collect the tools that we will need in this case.

3.2 Propagating connectivity properties

We use the convention that every space is (-2) -connected and that any non-empty space is (-1) -connected.

3.2.1 Complete joins

We will deduce connectivity properties of complexes from those of other complexes, and maps between them. One way to do this is formalized through the notion of a *complete join complex*, introduced by Hatcher and Wahl in [19].

Definition 3.2 (*Complete join complex*) Let Y and X be simplicial complexes, and $\pi : Y \rightarrow X$ a simplicial map. We say that Y is a *complete join complex* over X (with respect to the map π) if the following properties are satisfied

- π is surjective, and is injective on individual simplices.
- For each simplex $\sigma = \langle v_0, \dots, v_d \rangle$ in X , its preimage can be written as a join of fibers over vertices:

$$\pi^{-1}(\sigma) = \pi^{-1}(v_0) * \dots * \pi^{-1}(v_d).$$

Using a slight abuse of notation, we will often just say that the map “ $\pi : Y \rightarrow X$ is a complete join” to mean that Y is a complete join complex over X with respect to the simplicial map π .

Since π is injective on simplices, the π -preimages of vertices are discrete sets of vertices in Y . They are non-empty since π is surjective. It follows that for each d -simplex σ in X , the π -preimage $\pi^{-1}(\sigma)$ is a join of $d + 1$ non-empty discrete sets.

We make use of a complete join in two places, once to transfer known connectivity properties from X to Y , and once to go the other way. The first direction is the difficult one, and has been established by Hatcher–Wahl [19].

Before stating the result, recall that a simplicial complex is *weakly Cohen–Macaulay* of dimension $k + 1$ if it is k -connected and if the link of every simplex σ is $(k - \dim(\sigma) - 1)$ -connected.

Proposition 3.3 [19, Proposition 3.5] *Suppose $\pi : Y \rightarrow X$ is a complete join. If X is weakly Cohen–Macaulay of dimension $k + 1$, then so is Y .*

Going forward is the easy direction (and is implicit in the argument given by Hatcher and Wahl [19]). See [1, Remark A.15] for an explicit account.

Proposition 3.4 *Suppose $\pi : Y \rightarrow X$ is a complete join. Then X is a retract of Y and inherits all properties that can be expressed by the vanishing of group-valued functors or cofunctors. In particular, if Y is k -connected, then so is X .*

Similarly, complete joins are easily prevented from being contractible.

Observation 3.5 *Let $\pi : Y \rightarrow X$ be a complete join, and suppose that there is a top-dimensional simplex $\sigma = \langle v_0, \dots, v_d \rangle$ in X such that each vertex fiber $\pi^{-1}(v_i)$ contains at least two points. Then, the fiber over σ contains a d -sphere which defines a cycle in the homology of Y that cannot be a boundary as Y does not contain simplices of dimension $d + 1$. In particular, Y is not contractible.*

3.2.2 The bad simplex argument

A map from a complete join is a particularly nice projection. The *bad simplex argument*, introduced by Hatcher and Vogtmann [18], uses the inclusion map of a subcomplex together with additional local information to transfer connectivity properties from the ambient complex to the subcomplex. We do not follow the original exposition of the argument, since we find the language introduced in [1] more convenient.

Let X be a simplicial complex and assume that we are given a map $\sigma \mapsto \bar{\sigma}$ that assigns to each simplex σ in X a (possibly empty) face, $\bar{\sigma}$, of σ . We assume that the following two conditions are satisfied:

- Monotonicity: $\sigma \leq \tau \implies \bar{\sigma} \leq \bar{\tau}$.
- Idempotence: $\bar{\bar{\sigma}} = \bar{\sigma}$.

We call the simplex σ *good* if $\bar{\sigma}$ is empty and *bad* if $\bar{\sigma} = \sigma$. Note that by monotonicity, the good simplices in X form a subcomplex X^{good} , which we call the *good subcomplex*.

The *good link* of a bad simplex σ is the geometric realization of the poset of those proper cofaces $\tau > \sigma$ for which $\bar{\tau} = \bar{\sigma} = \sigma$ holds, i.e.,

$$\text{lk}^{\text{good}}(\sigma) := \{ \tau > \sigma \mid \bar{\tau} = \bar{\sigma} \}$$

The following proposition is due to Hatcher–Vogtmann [18, Sect. 2.1], but this wording is taken from [1, Proposition A.7].

Proposition 3.6 *Assume that for some number $k \geq -1$ and every bad simplex σ , the good link $\text{lk}^{\text{good}}(\sigma)$ is $(k - \dim(\sigma))$ -connected. Then the inclusion $X^{\text{good}} \hookrightarrow X$ induces isomorphisms in π_d for all $d \leq k$ and an epimorphism in π_{k+1} .*

4 A contractible cube complex

The strategy for proving Theorem 1.1 is well-known, and is similar to that used in [1, 7, 16], for instance. Namely, we will construct a contractible complex $\mathfrak{X}$ on which $\mathcal{B}_n$ acts in such way that we can apply Brown’s [6] criterion described in Sect. 3 above.

We now proceed to do this. The definitions of the main objects are the word-for-word adaptations of the ones in [1], and the only differences with our situation will occur when we analyze the connectivity properties of descending links. For this reason, we will briefly recall the constructions from [1, Sect. 5], referring to that article for a more thorough presentation.

Consider ordered pairs (Z, f) , where $Z \subset \Sigma_n$ is a suited subsurface and $f \in \mathcal{B}_n$. Two such pairs (Z_1, f_1) and (Z_2, f_2) are said to be equivalent if $f_2^{-1} \circ f_1(Z_1) = Z_2$ and $f_2^{-1} \circ f_1$ is rigid on the complement of Z_1 . Intuitively, the equivalence class of (Z, f) records the “non-rigid” behavior of f outside Z . For example, if $f \in \mathcal{B}_n$ is the identity outside a suited subsurface Z , then (Z, f) is equivalent to (Z, id) . As another useful example to keep in mind, observe that (Z, f_1) and (Z, f_2) are equivalent if $f_2^{-1} \circ f_1$ leaves Z invariant and is rigid outside Z .

We denote the equivalence class of (Z, f) by $[Z, f]$, and the set of all equivalence classes by $\mathcal{S}$. The group $\mathcal{B}_n$ acts on $\mathcal{S}$, by setting

$$g \cdot [Z, f] = [Z, g \circ f].$$

We define the *complexity* of a pair (Z, f) as above to be the genus of Z . Alternatively, observe that since Z is a suited subsurface, it is the union of O_n and some number of pieces. As each piece contributes 1 to the genus, the complexity is simply the number of pieces in Z . Clearly, equivalent pairs have equal complexity, and the action preserves complexity, so we have a $\mathcal{B}_n$ -invariant *complexity function*

$$h: \mathcal{S} \rightarrow \mathbb{Z} \subset \mathbb{R}.$$

Given vertices $x_1, x_2 \in \mathcal{S}$, we say that $x_1 \leq x_2$ if there are representatives (Z_i, f_i) of x_i , for $i = 1, 2$, so that $f_1 = f_2$, $Z_1 \subset Z_2$, and $Z_2 \setminus Z_1$ is a (possibly empty) disjoint union of pieces. The relation $\leq$ is not transitive, so is not a partial order.

The relation $\leq$ can be used to construct a cube complex $\mathfrak{X}$ for which $\mathcal{S}$ is the 0-skeleton. Given $x_1 \leq x_2$, the set $\{x \mid x_1 \leq x \leq x_2\}$ are the vertices of a d -cube, with $d = h(x_2) - h(x_1)$. We call x_1 the *bottom* of the cube, as it uniquely minimizes complexity over all its vertices. Since Σ_n has exactly n ends, the complex $\mathfrak{X}$ is n -dimensional.

Observe that the action $\mathcal{B}_n$ on $\mathcal{S}$ preserves the cubical structure, and that the complexity function h extends affinely over cubes to a $\mathcal{B}_n$ -invariant complexity function (of the same name)

$$h: \mathfrak{X} \rightarrow \mathbb{R}_+.$$

For $k \geq 1$, write $\mathfrak{X}^{\leq k}$ for the subcomplex of $\mathfrak{X}$ spanned by those vertices with complexity $\leq k$. A direct translation of the arguments of Proposition 5.7 and Lemmas 6.2 and 6.3 of [1] yields the following:

Theorem 4.1 *The cube complex $\mathfrak{X}$ is contractible, and the action of $\mathcal{B}_n$ on $\mathfrak{X}$ satisfies:*

- *Let C be a cube with bottom vertex $x = [Z, f]$. Then the $\mathcal{B}_n$ -stabilizer of C is isomorphic to a finite extension of $\text{Map}(Z)$. In particular, every cube stabilizer is of type F_∞ .*
- *For every $k \geq 1$, the quotient of $\mathfrak{X}^{\leq k}$ by $\mathcal{B}_n$ is compact.*

In light of the theorem above, in order to apply Brown's Theorem 3.1, we need to prove that descending links have the correct connectivity properties. As was the case in [1], the connectivity properties of descending links are determined by those of *piece complexes*, whose definition we now recall:

Definition 4.2 (*Piece complex*) Let Z be a compact surface with boundary, and let Q be a collection of boundary circles. We define the *piece complex* $\mathcal{P}(Z, Q)$ to be the simplicial subcomplex of the curve complex of Z whose vertices are separating curves which, together with a boundary circle from Q , bound a genus 1 subsurface. If $Q = \partial Z$, we will write $\mathcal{P}(Z, Q) = \mathcal{P}(Z)$.

The relation between the two complexes is encapsulated by the following result, whose proof is exactly the same as that of [1, Proposition 6.6].

Proposition 4.3 *Let $x = [Z, \text{id}]$ be a vertex of $\mathfrak{X}$. Then, the descending link $\text{lk}^\downarrow(x)$ is a complete join over the piece complex $\mathcal{P}(Z)$.*

An immediate consequence of Propositions 4.3 and 3.3 is the following.

Corollary 4.4 *If $\mathcal{P}(Z)$ is weakly Cohen–Macaulay of dimension k , then so is the descending link $\text{lk}^\downarrow(Z)$.*

Before we end this section, we will need a little more information about this complete join for the proof of the negative part of Theorem 1.1. To explain, we first recall that for $x = [Z, \text{id}]$, the complete join map $\eta: \text{lk}^\downarrow(x) \rightarrow \mathcal{P}(Z)$ is defined as follows. Given $[W, g] \in \text{lk}^\downarrow(x)$, there is a piece $Y \subset \Sigma_n$ so that $W \cup Y$ is a suited subsurface and $[W \cup Y, g] = x = [Z, \text{id}]$. It follows that $g(W \cup Y) = Z$ and $g(Y)$ is thus a vertex of $\mathcal{P}(Z)$. We then define $\eta([W, g]) = g(Y)$. With this, we now state the lemma we will need.

Lemma 4.5 *For every vertex $X \in \mathcal{P}(Z)$, the fiber in $\text{lk}^\downarrow(x)$ is infinite.*

Proof Given $X = \eta([W, g]) = g(Y)$ as above, we need to show that $\eta^{-1}(X)$ is infinite. For this, we let $h: Y \rightarrow Y$ be any homeomorphism representing an element of $\text{Map}(Y)$. We extend h by the identity outside Y to a homeomorphism of the same name, which thus represents an element of $\mathcal{B}_n$. Observe that $g \circ h(W \cup Y) = g(W \cup Y) = Z$ and $g \circ h(Y) = g(Y) \subset Z$, while $g \circ h$ is rigid outside $Z = W \cup Y$, so $[W, g \circ h] = [Z, \text{id}]$, and thus $[W, g \circ h] \in \text{lk}^\downarrow(x)$ with $\eta([W, g \circ h]) = g \circ h(Y) = g(Y) = X$. That is, $[W, g \circ h] \in \eta^{-1}(X)$ is another vertex. Moreover, $[W, g \circ h] = [W, g]$ if and only if $g^{-1} \circ g \circ h = h$ is rigid outside W (up to isotopy). Since Y is a piece outside W , this can only happen if h is isotopic (in Σ_n) to a homeomorphism which restricts to the identity in Y . This is only possible if the original homeomorphism h of Y represents the identity in $\text{Map}(Y)$, modulo Dehn twisting in the essential component of ∂Y in Z (which can be “absorbed” into W). Since $\text{Map}(Y)$ modulo the (central) subgroup generated by Dehn twisting in this component of ∂Y is infinite, it follows that $\eta^{-1}(X)$ is infinite, as required. $\square$

5 Connectivity properties of piece complexes

In this section, we shall establish connectivity properties of piece complexes and finally deduce the finiteness properties of $\mathcal{B}_n$. Given a compact surface Z , we let $g(Z)$ denote the genus of Z .

Theorem 5.1 *The piece complex $\mathcal{P}(Z, Q)$ of a compact surface is k -connected, provided that $g(Z) \geq 2k + 3$ and $|Q| \geq k + 2$.*

Let us first observe that this implies that piece complexes are *weakly Cohen–Macaulay*, as defined above.

Corollary 5.2 *The piece complex $\mathcal{P}(Z, Q)$ is weakly Cohen–Macaulay of dimension $k + 1$, provided that $g(Z) \geq 2k + 3$ and $|Q| \geq k + 2$.*

Proof Observe that the link of a d -simplex σ in $\mathcal{P}(Z, Q)$ is isomorphic to the piece complex $\mathcal{P}(Z', Q')$ where Z' is obtained from Z by cutting off the pieces in σ and Q' is the set of those boundary circles in Q that still exist in Z' . Then,

$$g(Z') = g(Z) - (\dim(\sigma) + 1) \geq 2k + 3 - \dim(\sigma) - 1 \geq 2(k - \dim(\sigma) - 1) + 3$$

and $|Q'| = |Q| - (\dim(\sigma) + 1) \geq (k - \dim(\sigma) - 1) + 2$. This implies the link of σ is $(k - \dim(\sigma) - 1)$ -connected, in view of Theorem 5.1, as required. $\square$

To analyze the connectivity properties of piece complexes, we shall introduce two more complexes: the *injective tethered handle complex* $\mathcal{TH}_1(Z, Q)$, which we will see is a complete join over $\mathcal{P}(Z, Q)$; and the *tethered handle complex* $\mathcal{TH}(Z, Q)$, which contains the injective tethered handle complex as a subcomplex. As before, if $Q = \partial Z$ we will simply write $\mathcal{TH}_1(Z, Q) = \mathcal{TH}_1(Z)$ and $\mathcal{TH}(Z, Q) = \mathcal{TH}(Z)$. A diagram of the maps we use reads as follows:

$$\begin{array}{ccc} \text{lk}^\downarrow([Z, f]) & & \mathcal{TH}_1(Z) \longrightarrow \mathcal{TH}(Z) \\ & \searrow \quad \swarrow & \\ & \mathcal{P}(Z) & \end{array}$$

We have used the left arrow to pull back connectivity from the piece complex to the descending link. We shall use the middle arrow to push forward connectivity from the injective handle complex to the piece complex, and we will use the inclusion of $\mathcal{TH}_1(Z)$ into $\mathcal{TH}(Z)$ to apply a *bad simplex argument*, pulling back connectivity from the tethered handle complex to the injective handle complex. The connectivity of the tethered handle complex itself has been analyzed in [1].

5.1 Tethered handle complexes

Let Z denote a compact connected orientable surface with boundary. By a *handle* in Z we mean a subsurface of Z that avoids the boundary ∂Z and is homeomorphic to a one-holed torus.

Given a handle T , consider (the isotopy class of) a simple arc α that joins ∂T to a component $b \subset Z$. We remark that the isotopy class of α is not taken relative to its endpoints, as is sometimes the case. Observe that the regular neighborhood of $T \cup \alpha \cup b$ is a piece. We will refer to the union of T and α as a *tethered handle* tethered to b with handle T and *tether* α .

Definition 5.3 (*Tethered handle complex*) Let Z be a compact orientable connected surface, and let Q be a collection of boundary circles of Z . The *tethered handle complex* $\mathcal{TH}(Z, Q)$ is the simplicial complex whose d -simplices are sets of $d + 1$ pairwise disjoint tethered handles, each tethered to an element of Q .

The following is Lemma 10.12 of [1], which builds upon work of Hatcher–Vogtmann [18]:

Lemma 5.4 *The tethered handle complex $\mathcal{TH}(Z, Q)$ is k -connected, provided that Q is not empty and $g(Z) \geq 2k + 3$.*

We consider the following subcomplex of the tethered handle complex.

Definition 5.5 (*Injective tethered handle complex*) The injective tethered handle complex $\mathcal{TH}_1(Z, Q)$ is the subcomplex of $\mathcal{TH}(Z, Q)$ consisting of those simplices where the involved handles are tethered to pairwise distinct boundary components in Q .

The reason why we are interested in this subcomplex is that it is a complete join over the piece complex, which allows us to push forward its connectivity. Indeed, we already observed that a small regular neighborhood of a tethered handle together with its boundary component is homeomorphic to a piece. In this way we obtain a simplicial map

$$\pi : \mathcal{TH}_1(Z, Q) \rightarrow \mathcal{P}(Z, Q).$$

The following is [1, Lemma 8.8], and the proof applies verbatim to this setting:

Proposition 5.6 *The map $\pi : \mathcal{TH}_1(Z, Q) \rightarrow \mathcal{P}(Z, Q)$ is a complete join.*

In particular, if $\mathcal{TH}_1(Z, Q)$ is k -connected, then so is $\mathcal{P}(Z, Q)$ by Proposition 3.4. Thus, we have now reduced the problem to analyzing the connectivity properties of the injective tethered handle complex $\mathcal{TH}_1(Z, Q)$.

Theorem 5.7 *The injective tethered handle complex $\mathcal{TH}_1(Z, Q)$ is k -connected, provided that $g(Z) \geq 2k + 3$ and $|Q| \geq k + 2$.*

Proof We induct on k starting at $k = -1$, which is trivial. For $k \geq 0$, we use the bad simplex argument for the inclusion of $\mathcal{TH}_1(Z, Q)$ into $\mathcal{TH}(Z, Q)$.

Consider a simplex σ in $\mathcal{TH}(Z, Q)$ and define $\bar{\sigma}$ to consist of those tethered handles in σ that are tethered to the same boundary component as another handle in σ . Note that $\mathcal{TH}_1(Z, Q)$ consists precisely of those simplices σ for which $\bar{\sigma}$ is empty, i.e., the good simplices of $\mathcal{TH}(Z, Q)$. See Fig. 1.

Now consider a bad simplex $\sigma = \bar{\sigma}$. It takes at least two handles tethered to the same boundary component to make a bad simplex. Hence, $\dim(\sigma) \geq 1$. Consequently, $2 \dim(\sigma) \geq 1 + \dim(\sigma)$ or, equivalently, $\dim(\sigma) \geq \frac{1 + \dim(\sigma)}{2}$.

Note that the good link $\text{lk}^{\text{good}}(\sigma)$ is isomorphic to $\mathcal{TH}_1(Z', Q')$, where Z' is obtained from Z by cutting off, for each boundary circle used in σ , a neighborhood of the boundary circle and all tethers and handles of σ that it meets. The set Q' consists of those boundary circles in Q that still exist in Z' . See Fig. 1. Then, we note the following inequalities which allow us to apply the induction hypothesis.

$$\begin{aligned} g(Z') &= g(Z) - (\dim(\sigma) + 1) \\ &\geq g(Z) - 2 \dim(\sigma) \end{aligned}$$

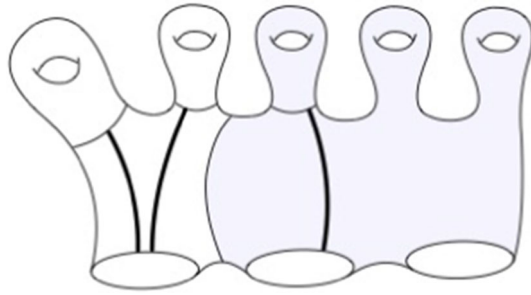


Fig. 1 The figure illustrates an example of Z where $g = 5$ and $Q = \partial Z$ has three components. There are three tethered handles, which define a 2-simplex σ , while $\tilde{\sigma}$ is a 1-simplex spanned by the two tethered handles on the left. The surface Z' is the shaded subsurface, which has genus 3 and three boundary components, but Q' consists of just the two components on the right

$$\begin{aligned} &\geq 2k + 3 - 2 \dim(\sigma) \\ &= 2(k - \dim(\sigma)) + 3 \end{aligned}$$

In a bad simplex each used boundary component tethers to at least two handles, whence we also obtain the following estimate:

$$\begin{aligned} |Q'| &\geq |Q| - \frac{\dim(\sigma) + 1}{2} \\ &\geq |Q| - \dim(\sigma) \\ &\geq k + 2 - \dim(\sigma) \\ &= (k - \dim(\sigma)) + 2 \end{aligned}$$

By the induction hypothesis, $\text{lk}^{\text{good}}(\sigma)$ is therefore $(k - \dim(\sigma))$ -connected. By Proposition 3.6, the inclusion $\mathcal{TH}_1(Z, Q) \hookrightarrow \mathcal{TH}(Z, Q)$ induces isomorphisms in π_d for all $d \leq k$. As $\mathcal{TH}(Z, Q)$ is k -connected by Lemma 5.4, $\mathcal{TH}_1(Z, Q)$ is k -connected.

□

Proof of Theorem 5.1 The result follows from the combination of Proposition 5.6 and Theorem 5.7, appealing to Proposition 3.4.

□

5.2 Proof of Theorem 1.1

The group $\mathcal{B}_n$ acts on $\mathfrak{X}$ leaving the discrete Morse function $h: \mathfrak{X} \rightarrow \mathbb{R}$ invariant. Sublevel complexes are $\mathcal{B}_n$ -cocompact, and cell stabilizers are of type F_∞ by Theorem 4.1.

By the above analysis, descending links of vertices are of dimension $n - 1$, and for vertices of height $h \geq 2n$, they are $(n - 2)$ -connected. By Proposition 4.3 and Lemma 4.5, the descending links are complete joins with infinite vertex fibers over a complex of finite dimension, hence they are not contractible by Observation 3.5.

Now the hypotheses of Brown's criterion 3.1 have been verified and the group $\mathcal{B}_n$ is of type F_{n-1} but not of type FP_n .

6 Abelianization

In this section we prove Theorem 1.2. The arguments rely on construction of non-trivial, integer-valued homomorphisms from pure mapping class groups of [3], which we briefly recall here for the particular case of the n -pronged surfaces Σ_n .

Let γ be an oriented curve that separates an end E of Σ_n from the rest, where the orientation is chosen so that the component of $\Sigma_n - \gamma$ to the right of γ is a neighborhood of E . Then, γ defines a nonzero element $[\gamma] \in H_1^{sep}(S, \mathbb{Z})$, the subgroup of $H_1(S, \mathbb{Z})$ spanned by homology classes of separating curves.

To every $f \in \text{PMap}(S)$ and $\gamma \in H_1^{sep}(S, \mathbb{Z})$, the authors of [3] associate an integer $\varphi_{[\gamma]}(f)$, which can be considered as the “signed genus” between γ and $f(\gamma)$ (see Section 3 in [3] for the details). They then show that the map

$$\varphi_{[\gamma]}: \text{PMap}(S) \rightarrow \mathbb{Z}$$

so obtained is a well-defined, nontrivial homomorphism which depends only on the homology class $[\gamma]$, see [3, Proposition 3.3]. Furthermore, by [3, Proposition 4.4], this induces a surjective homomorphism

$$\Phi: \text{PMap}(S) \rightarrow H_{sep}^1(S, \mathbb{Z}),$$

by the rule $\Phi(f)([\gamma]) = \varphi_{[\gamma]}(f)$, where $H_{sep}^1(S, \mathbb{Z})$ has been identified with $\text{Hom}(H_1^{sep}(S, \mathbb{Z}), \mathbb{Z})$ by the Universal Coefficient Theorem. Informally speaking, $\Phi(f)$ tells us “how much genus has been shifted” on each end by f . It follows from (the proof of) [3, Theorem 5] that Φ is a surjective homomorphism whose kernel is precisely $\text{PMap}_c(S)$, namely the closure of the compactly-supported mapping class group. At this point, in our particular setting, one has:

Theorem 6.1 (Corollary 6 of [3]) $\text{PMap}(\Sigma_n) = \overline{\text{PMap}_c(\Sigma_n)} \rtimes \mathbb{Z}^{n-1}$.

When we restrict Φ to the pure subgroup $P\mathcal{B}_n$, the image of Φ is generated by $n - 1$ handle shifts $\rho_1, \dots, \rho_{n-1}$, which belong to $P\mathcal{B}_n$. Even though we do not have the semi-direct product structure in Theorem 6.1 when restricting to $P\mathcal{B}_n$, the projection onto $\mathbb{Z}^{n-1}$ from that theorem still defines a surjective homomorphism fitting into a short exact sequence:

$$1 \rightarrow (\mathcal{B}_n)_c \rightarrow P\mathcal{B}_n \rightarrow \mathbb{Z}^{n-1} \rightarrow 1,$$

where $(\mathcal{B}_n)_c$ is the intersection of $\mathcal{B}_n$ with $\overline{\text{PMap}_c(\Sigma_n)}$, which is precisely the compactly supported elements of $\mathcal{B}_n$. Since $(\mathcal{B}_n)_c$ is a direct limit of mapping class groups of compact surfaces, Powell’s result [26] implies that $(\mathcal{B}_n)_c^{ab} = \{0\}$, and therefore $P\mathcal{B}_n^{ab} \cong \mathbb{Z}^{n-1}$. At this point, the fact that $\mathcal{B}_n^{ab} = \{0\}$ follows from the above and the natural short exact sequence

$$1 \rightarrow P\mathcal{B}_n \rightarrow \mathcal{B}_n \rightarrow \text{Sym}(n) \rightarrow 1,$$

where $\text{Sym}(n)$ is the symmetric group on n elements, and $\mathcal{B}_n \rightarrow \text{Sym}(n)$ comes from the action on the n ends of Σ_n . This is because the action can be used to conjugate generators of $P\mathcal{B}_n^{ab}$ to their inverses. This proves Theorem 1.2.

We observe that $(\mathcal{B}_n)_c$ is a normal subgroup of $\mathcal{B}_n$, since the conjugate of a compactly supported homeomorphism is compactly supported. From the homomorphisms above, the quotient $G = \mathcal{B}_n/(\mathcal{B}_n)_c$ admits a homomorphism to $\text{Sym}(n)$ with kernel $\mathbb{Z}^{n-1}$. It is not hard to explicitly construct a splitting of the associated short exact sequence proving that $G \cong \mathbb{Z}^{n-1} \rtimes \text{Sym}(n)$. We thus have a short exact sequence

$$1 \rightarrow (\mathcal{B}_n)_c \rightarrow \mathcal{B}_n \rightarrow (\mathbb{Z}^{n-1} \rtimes \text{Sym}(n)) \rightarrow 1.$$

On the other hand, the proof of [25, Theorem 4.6] of Patel and Vlamis (which itself relies on a result of Paris [24]) can be applied verbatim to show that $(\mathcal{B}_n)_c$ has no nontrivial finite quotients, proving the following.

Proposition 6.2 *Every finite quotient of $\mathcal{B}_n$ factors through the homomorphism to $\mathbb{Z}^{n-1} \rtimes \text{Sym}(n)$.* $\square$

An application of this and Theorem 1.1 proves Corollary 1.3.

Proof of Corollary 1.3 First observe that if $n \neq m$, then Theorem 1.1 and [16, Theorem 5.24] imply that the finiteness properties of the groups are different, and so they cannot be commensurable.

If $n = m$, and $\mathcal{B}_n$ and $\text{br}H_n$ are commensurable, then after passing to the intersection with the finite index pure subgroups, we find finite index subgroups $K < P\mathcal{B}_n$ and $J < P\text{br}H_n$ so that $K \cong J$. We note here that $P\text{br}H_n$ is the kernel of the corresponding action on the n (non-isolated) ends of the underlying surface, and is not the subgroup consisting of pure braids.

Applying Proposition 6.2, we see that $K^{ab} \cong \mathbb{Z}^{n-1}$, and hence $J^{ab} \cong \mathbb{Z}^{n-1}$ as well. That is, both abelianizations must simply be the restrictions of the abelianization of $P\mathcal{B}_n$, and thus also $P\text{br}H_n$, respectively. The kernels of the abelianizations must therefore be finite index subgroups of $(\mathcal{B}_n)_c$ and $(\text{br}H_n)_c$, respectively. The former has no finite index subgroups, whereas $(\text{br}H_n)_c$ admits a homomorphism to $\mathbb{Z}$ (being the direct limit of compactly supported braid groups), which therefore has infinitely many nontrivial finite index subgroups. This contradiction proves that $\mathcal{B}_n$ and $\text{br}H_n$ are not commensurable. $\square$

7 Marking graphs

A marking μ on a surface is a pants decomposition called the *base* of the marking, $\text{base}(\mu) = \bigcup_i \alpha_i$, together with a choice of transverse curves β_i for each α_i ; that is, a curve β_i so that $i(\alpha_i, \beta_j) = 0$ if $i \neq j$ and $i(\alpha_i, \beta_i) = 1$ or 2 , depending on whether α_i and β_i fill a one-holed torus or four-holed sphere, respectively. Masur and Minsky [22] define a graph whose vertices are markings and so that two markings are connected by an edge if they differ by an *elementary move*, which essentially swaps the roles of α_i and β_i (together with a certain “clean up” operation to ensure the result is again a

marking). Let $\mathcal{M}_n$ denote the marking graph of Σ_n . The image of a marking under a mapping class is again a marking, and the definition of elementary move implies that the mapping class group acts on the marking graph.

For a surface S of finite type, its marking graph $\mathcal{M}(S)$ is locally finite and the orbit map is a quasi-isometry, since the action is cocompact. However, the action of $\mathcal{B}_n$ on an invariant component of $\mathcal{M}_n$ is not cocompact since one can find arbitrarily many distinct homeomorphism types of markings. Moreover, the orbit map is not even a quasi-isometric embedding. Indeed, if μ is a marking, and t_i is the Dehn twist in $\alpha_i \in \text{base}(\mu)$, then the distance from μ to $t_i(\mu)$ is uniformly bounded, while the distance from the identity to t_i tends to infinity as $i \rightarrow \infty$ (since these are all distinct elements in a finitely generated group).

To prove Theorem 1.4, we need the following.

Lemma 7.1 *For all $n \geq 2$ and any marking μ , the stabilizer of μ in $\mathcal{B}_n$ is either finite, or contains a finite index subgroup that acts on Σ_n by covering transformations. In particular, the stabilizer is finite if $n \geq 3$.*

Proof There is a hyperbolic metric on Σ_n so that all $\alpha_i \in \text{base}(\mu)$ have length 1 and all β_i meet α_i orthogonally. Then any element of the stabilizer of μ acts on Σ_n by isometries. The lemma now follows since the isometry group of Σ_n is necessarily discrete, and hence finite for $n \geq 3$, and either finite or virtually cyclic acting by covering transformations for $n = 2$. $\square$

It is straightforward to construct $\mathcal{B}_n$ -invariant components of $\mathcal{M}_n$, for all $n \geq 2$ to which the following theorem applies, and which immediately implies Theorem 1.4.

Theorem 7.2 *For any μ in a $\mathcal{B}_n$ -invariant component $\mathcal{M}_n^0 \subset \mathcal{M}_n$ with finite stabilizer, there is a locally finite subgraph $\mathcal{X} \subset \mathcal{M}_n^0$ containing μ so that $\mathcal{B}_n$ acts properly and cocompactly on $\mathcal{G}$.*

Proof We let $\mathcal{G}$ be a finite subgraph which is the union of paths connecting μ to its image under each generator from some fixed finite generating set for $\mathcal{B}_n$. Further, we assume that each vertex in $\mathcal{G}$ has finite stabilizer as well. This is automatic for $n \geq 3$, and is easy to arrange for $n = 2$. Now set $\mathcal{X} = \mathcal{B}_n \cdot \mathcal{G}$.

The fact that $\mathcal{B}_n$ acts cocompactly on $\mathcal{X}$ is immediate, since the $\mathcal{G}$ -translates cover $\mathcal{X}$ by construction. The only thing we must prove is that $\mathcal{X}$ is locally finite. For this, it suffices to show that

$$K = \{g \in \mathcal{B}_n \mid g \cdot \mathcal{G} \cap \mathcal{G} \neq \emptyset\}$$

is finite. Suppose there exists an infinite sequence of distinct elements $\{g_n\} \subset K$. Let $x_n \in \mathcal{G}$ be a vertex so that $y_n = g_n \cdot x_n \in \mathcal{G}$. There are only finitely many vertices of $\mathcal{G}$, and so after passing to a subsequence (and reindexing), $x_n = x$ and $y_n = y$ for some $x, y \in \mathcal{G}$. Thus, $g_1^{-1} g_n \cdot x = x$ for all n , and hence the stabilizer of x is infinite, a contradiction. $\square$

Remark 7.3 The utility in proving that $\text{Map}(S)$ is quasi-isometric to the marking graph for a finite type surface S is that it allows one to provide a coarse estimate for distances

in terms of local information via Masur and Minsky's subsurface projections and their distance formula [22]. The Dehn twisting examples above imply that one cannot expect a similar distance formula for $\mathcal{B}_n$. However, one may wonder if there is some restricted set of subsurfaces for which one can prove a distance formula. Or perhaps there is still a distance formula for all of $\mathcal{M}_n$ (which simply does not transfer to $\mathcal{B}_n$ because it is not quasi-isometric to $\mathcal{M}_n$).

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