Advances in Mathematics 441 (2024) 109559

Contents lists available at ScienceDirect

Advances in Mathematics

journal homepage: www.elsevier.com/locate/aim

On the equality of test ideals

Ian Aberbach [a], Craig Huneke [b], Thomas Polstra [c,*,1]

[a] Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
[b] Department of Mathematics, University of Virginia, Charlottesville, VA 22903, USA
[c] Department of Mathematics, University of Alabama, Tuscaloosa, AL 35401, USA

ARTICLE INFO

Article history:
Received 6 January 2023
Received in revised form 28 January 2024
Accepted 4 February 2024
Available online 27 February 2024
Communicated by Karen Smith

Keywords:
Tight closure
Rees algebras
Test ideals
Characteristic p
Local cohomology

ABSTRACT

We provide a natural criterion that implies equality of the test ideal and big test ideal in local rings of prime characteristic. Most notably, we show that the criterion is met by every local weakly F-regular ring whose anti-canonical algebra is Noetherian on the punctured spectrum.

1. Introduction

Suppose R is a Noetherian ring of prime characteristic $p > 0$ and let R° be the set of elements which avoid all minimal primes of R. Let $I \subseteq R$ be an ideal of R and denote by $I^{[p^e]}$ the expansion of I along the eth iterate of the Frobenius endomorphism. The

* Corresponding author.
E-mail addresses: aberbachi@missouri.edu (I. Aberbach), huneke@virginia.edu (C. Huneke), tmpolstra@ua.edu (T. Polstra).

[1] Polstra was supported in part by NSF Grant DMS #2101890 and by a grant from the Simons Foundation, Grant Number 814268, MSRI.

https://doi.org/10.1016/j.aim.2024.109559
0001-8708/

tight closure of I is the ideal I^* consisting of elements $x \in R$ such that there exists an element $c \in R^\circ$ with the property that $cx^{p^e} \in I^{[p^e]}$ for all $e \gg 0$. Unlike integral closure of ideals, the tight closure of an ideal does not commute with localization, [5]. Brenner's and Monsky's counterexample to the localization problem leaves open the intriguing problem if the property of tight closure being a trivial operation on ideals commutes with localization.

Continue to let R be a Noetherian ring of prime characteristic $p > 0$. The ring R is called *weakly F-regular* if every ideal is tight closed, that is $I = I^*$ for every ideal I.[2] A ring is called *F-regular* if every localization of R is weakly F-regular. Let F^e_*R denote the restriction of scalars of R along the eth iterate Frobenius endomorphism $F^e : R \to R$. We say that R is *strongly F-regular* if for each $c \in R^\circ$ there exists $e \in \mathbb{N}$ such that the R-linear map $R \to F^e_*R$ defined by $1 \mapsto F^e_*c$ is pure. Every strongly F-regular ring is weakly F-regular and the property of being strongly F-regular passes to localization. It is conjectured that all three notions of F-regularity agree.

Conjecture 1.1 *(The weak implies strong conjecture). If R is an excellent weakly F-regular ring of prime characteristic $p > 0$ then R is strongly F-regular.*

Williams proved Conjecture 1.1 for the class of 3-dimensional rings, [31]. Every excellent 4-dimensional F-regular ring is strongly F-regular by pairing [1, Corollary 4.4] with [4, Corollary K]. The purpose of this article is to extend the results of [1] to rings of arbitrary dimension. In particular, if the results of the prime characteristic minimal model program in dimension 3 established in [4] are valid in all dimensions, then the classes of excellent F-regular and excellent strongly F-regular rings are equivalent.

A prime characteristic ring R is weakly F-regular if and only if $R_\mathfrak{m}$ is weakly F-regular for every maximal ideal $\mathfrak{m} \in \operatorname{Spec}(R)$. Moreover, an excellent local ring is weakly F-regular if and only if its completion is weakly F-regular. Every local weakly F-regular ring is a Cohen-Macaulay normal domain. We therefore restrict our attention to the class of local Cohen-Macaulay normal domains which admit a canonical module.

Theorem A. *Let $(R, \mathfrak{m}, k)$ be an excellent Cohen-Macaulay normal domain of prime characteristic $p > 0$, of Krull dimension d, and $I \subseteq R$ an anti-canonical ideal.*[3] *Suppose that there exists an $m \in \mathbb{N}$ such that $I^{(m)}$ is principal when localized at each height 2 prime*[4] *and for each $1 \leq j \leq d-2$ there exists an ideal $\mathfrak{a}_j$ of height $d-j+1$ such that*

[2] A defining property of tight closure theory is that every regular ring is weakly F-regular.

[3] An ideal $I \subseteq R$ is an *anti-canonical ideal* if it represents the inverse of the canonical divisor in the class group of R. Equivalently, there exists a canonical ideal $J \subseteq R$, with components disjoint from that of I, so that $I \cap J$ is a principal ideal.

[4] Every excellent normal ring which is F-rational in codimension 2 admits an $m \geq 1$ with this property. Indeed, F-rational rings have pseudo-rational singularities, excellent pseudo-rational singularities are rational in codimension 2, and 2-dimensional excellent local rational singularities have torsion class group, [26,20,19].

$$\mathfrak{a}_j^{p^e} H_{\mathfrak{m}}^j\left(\frac{R}{I^{(mp^e)}}\right) = 0$$

for every $e \in \mathbb{N}$. If R is weakly F-regular then R is strongly F-regular.

Remark 1.2. The Matlis dual of the local cohomology module $H_{\mathfrak{m}}^j(R/I^{(mp^e)})$ is the completion of $\operatorname{Ext}_R^{d-j}(R/I^{(mp^e)}, J)$, a module which is not supported in codimension $d-j$ if $j \le d-2$. Hence $H_{\mathfrak{m}}^j(R/I^{(mp^e)})$ is annihilated by an ideal of height $d-j+1$. The criterion of Theorem A is therefore reasonable as it is natural to anticipate that the annihilators of $H_{\mathfrak{m}}^j(R/I^{(mp^e)})$ are of linear comparisons as $e \to \infty$.

Remark 1.3. Let $(R, \mathfrak{m}, k)$ be as in Theorem A and assume that R is Cohen-Macaulay. Suppose that $E_R(k)$ is an injective hull of the residue field. Let $0^*_{E_R(k)}$ and $0^{*fg}_{E_R(k)}$ denote the tight closure and finitistic tight closure respectively of the 0-submodule of $E_R(k)$, see Section 3 for definitions. Then $0^{*fg}_{E_R(k)} = 0^*_{E_{R(k)}}$ under the hypotheses of Theorem A, see Theorem 4.10. Therefore the test ideal and big test ideal of R agree by [14, Proposition 8.23] and [3, Theorem 3.2], cf. [22, Theorem 7.1 and Theorem 7.2]. By definition, the test ideal of R is the unit ideal if and only if R is weakly F-regular and the big test ideal of R is the unit ideal if and only if R is strongly F-regular. Therefore Theorem A is a consequence of Theorem 4.10.

Conjecture 1.1 is valid for rings R which are standard graded over a field, [21]. It would be interesting to know if such rings satisfy the hypotheses of Theorem A. Without the standard graded assumption, most established cases of Conjecture 1.1 require an assumption on R that is akin to being Gorenstein. Hochster and Huneke proved Conjecture 1.1 for the class of Gorenstein rings, [15]. Building upon Williams' proof of Conjecture 1.1 for the class of 3-dimensional rings, [31], MacCrimmon proved the weak implies strong conjecture for rings which are $\mathbb{Q}$-Gorenstein on the punctured spectrum, [9]. Singh announced that Conjecture 1.1 is valid for rings whose anti-canonical algebra[5] is Noetherian. Singh's result was never published, but has since been recaptured by others, [10]. Takagi established the validity of Conjecture 1.1 for rings that are numerically $\mathbb{Q}$-Gorenstein, as shown in [29, Main Theorem]. Furthermore, Takagi demonstrated the equality of test ideals under the numerically $\mathbb{Q}$-Gorenstein hypothesis, a hypothesis that is weaker to being $\mathbb{Q}$-Gorenstein. If the anti-canonical algebra of R is Noetherian, then the condition of numerically $\mathbb{Q}$-Gorenstein is equivalent to $\mathbb{Q}$-Gorenstein, [29, Lemma 3.5].

Singularities of prime characteristic rings are related to KLT singularities of the complex minimal model program through the process of reduction to prime characteristic, [12,28]. Theorems of the complex minimal model program establish that if R is essentially of finite type over $\mathbb{C}$ with at worst KLT singularities, then the symbolic Rees algebras associated to ideals of pure height 1 are Noetherian. It is therefore natural to conjecture

[5] Suppose that R is a normal domain and $I \subseteq R$ is an anti-canonical ideal. The *anti-canonical algebra of* R is the symbolic Rees algebra $R \oplus I \oplus I^{(2)} \oplus \cdots$, an algebra unique up to R-algebra isomorphism.

the same in strongly F-regular rings and that the hypotheses of Singh's Theorem are vacuous.

Conjecture 1.4. *If R is an excellent strongly F-regular ring of prime characteristic $p > 0$ and $I \subseteq R$ an ideal of pure height 1. Then the symbolic Rees algebra of I is Noetherian.*

Progress around Conjecture 1.4 is quite limited. An elementary and (mostly) algebraic proof of Conjecture 1.4 for the class of 2-dimensional F-regular rings can be derived from [23, Corollary 3.2]. Recent progress of the minimal model program establishes Conjecture 1.4 for the class of 3-dimensional F-regular rings, see [4, Corollary K] and [1, Proof of Corollary 4.5] for necessary details.

In light of Conjecture 1.4, it would be desirable to remove the assumption that the anti-canonical algebra of R is Noetherian in Singh's Theorem and replace it with the milder hypothesis that the anti-canonical algebra is assumed to be Noetherian at non-closed points of $\operatorname{Spec}(R)$. Such a step puts forth a much needed inductive program to establish Conjecture 1.4, or at the very least establish that the class of F-regular and strongly F-regular rings agree. This is what we accomplish and is the main contribution of this article.

Theorem B. *Let $(R, \mathfrak{m}, k)$ be an excellent weakly F-regular ring of prime characteristic $p > 0$, of Krull dimension d, and $I \subseteq R$ an anti-canonical ideal. Suppose that the anti-canonical algebra of R is Noetherian on the punctured spectrum. There exists $m \in \mathbb{N}$ so that $I^{(m)}$ is principal when localized at each height 2 prime and for each $1 \leq j \leq d-2$ there exists an ideal $\mathfrak{a}_j$ of height $d - j + 1$ such that*

$$\mathfrak{a}_j^{p^e} H^j_{\mathfrak{m}}\left(\frac{R}{I^{(mp^e)}}\right) = 0$$

for every $e \in \mathbb{N}$. In particular, the ring R is strongly F-regular by Theorem A.

Remark 1.5. The implications of the techniques employed in this article regarding the agreement between the test ideal and big test ideal of R are not explicitly clear when only considering the assumption that the anti-canonical algebra is Noetherian on the punctured spectrum. Our approach requires not only Noetherianity of the anti-canonical algebra of R on the punctured spectrum, but also the additional condition of Cohen-Macaulayness on the punctured spectrum. We observe that this condition holds true if R is weakly F-regular, see the proof of Corollary 2.7. To establish the equality of test ideals solely based on the assumption that the anti-canonical algebra is Noetherian on the punctured spectrum, one would need to appropriately modify the outcomes and methodologies presented in Section 2 to accommodate algebras that may not be Cohen-Macaulay.

2. Annihilators of local cohomology

This section is devoted to proving Theorem B. Let $(R, \mathfrak{m}, k)$ be an excellent local normal domain of Krull dimension $d \geq 3$ and $I \subseteq R$ an ideal of pure height 1. Let $W = R \setminus \bigcup_{P \in \min(I)} P$ and for each $n \in \mathbb{N}$ let $I^{(n)} = I^n R_W \cap R$ denote the nth symbolic power of the ideal I. To study the annihilators of $H^i_{\mathfrak{m}}(R/I^{(n)})$ we will approximate the ideals $I^{(n)}$ by ideals of the form $\overline{(y_1, \ldots, y_h)^n}$ where h is "small," $y_1, \ldots, y_h \in I$, and $\overline{J}$ denotes the integral closure of an ideal $J \subseteq R$.

Let $J \subseteq R$ be an ideal and $n \in \mathbb{N}$. There are short exact sequences

$$0 \to \frac{\overline{J^{n-1}}}{\overline{J^n}} \to \frac{R}{\overline{J^n}} \to \frac{R}{\overline{J^{n-1}}} \to 0,$$

and so there are exact sequences of local cohomology modules

$$H^i_{\mathfrak{m}}\left(\frac{\overline{J^{n-1}}}{\overline{J^n}}\right) \to H^i_{\mathfrak{m}}\left(\frac{R}{\overline{J^n}}\right) \to H^i_{\mathfrak{m}}\left(\frac{R}{\overline{J^{n-1}}}\right).$$

Our aim is to establish uniform annihilators of the local cohomology modules $H^i_{\mathfrak{m}}(\overline{J^{n-1}}/\overline{J^n})$ that are independent of n. For the sake of convenience, we adopt the following notation:

- $R[Jt] = \bigoplus_{n \geq 0} J^n t^n$ is the Rees algebra of J;
- $R[Jt, t^{-1}] = \bigoplus_{n \in \mathbb{N}} J^n t^n$ is the extended Rees algebra of J, i.e. $R[Jt, t^{-1}]$ agrees with the Rees algebra $R[Jt]$ in positive degree and contains copies of R in negative degree;
- $\mathcal{R}$ is the integral closure of the Rees algebra $R[Jt]$ in $R[t]$; $\mathcal{R}$ is $\mathbb{N}$-graded and the nth graded piece of $\mathcal{R}$ is $\overline{J^n}$;
- $\mathcal{R}[t^{-1}]$ is the integral closure of the extended Rees algebra $R[Jt, t^{-1}]$ in $R[t, t^{-1}]$. If $n \geq 0$ then the nth graded piece of $\mathcal{R}[t^{-1}]$ is $\overline{J^n}$. The algebra $\mathcal{R}[t^{-1}]$ contains copies of R in negative degrees.

If $x \in R$ then $xH^i_{\mathfrak{m}}(\overline{J^{n-1}}/\overline{J^n}) = 0$ for all $n \in \mathbb{N}$ if and only if

$$xH^i_{\mathfrak{m}}(\mathcal{R}[t^{-1}]/(t^{-1}\mathcal{R}[t^{-1}]) = 0.$$

The Faltings Annihilator Theorem, later generalized by Brodmann, provides a criterion to establish such annihilation properties.

Theorem 2.1 *([7, Theorem 9.5.1]). Let S be a Noetherian ring which is the homomorphic image of a regular ring, M a finitely generated S-module, and let $\mathfrak{a}, \mathfrak{b} \subseteq R$ be ideals. Then*

$$\min\{i \in \mathbb{N} \mid \nexists C : \mathfrak{a}^C H^i_{\mathfrak{b}}(M) = 0\} = \min\{\operatorname{depth}(M_P) + \operatorname{height}((\mathfrak{b} + P)/P) \mid P \notin V(\mathfrak{a})\}.$$

Our first step towards proving Theorem B is the following lemma.

Lemma 2.2. *Let* $(R, \mathfrak{m}, k)$ *be an excellent local normal domain of Krull dimension* d *and* $J \subseteq R$ *an ideal generated by at most* h *elements. Suppose that the associated graded algebra* $\bigoplus_{n\geq 0} \overline{J^n}/\overline{J^{n+1}} \otimes_R R_x$ *is Cohen-Macaulay. Then there exists a constant* C *so that*

$$x^C H^i_{\mathfrak{m}}(\overline{J^n}/\overline{J^{n+1}}) = 0$$

for every $0 \leq i \leq d - h - 1$ *and* $n \in \mathbb{N}$*.*

Proof. Without loss of generality, we may pass to the completion of R and assume that R is the homomorphic image of a regular local ring. Let $S = \mathcal{R}[t^{-1}]$ and $G = S/t^{-1}S$. The lemma is equivalent to the assertion that there exists a constant C so that $x^C H^i_{\mathfrak{m}S}(G) = 0$ for every $1 \leq i \leq d-h-1$. By Theorem 2.1, it suffices to show that if $P \in \operatorname{Spec}(S) \setminus V(xS)$ then

$$\operatorname{depth}(G_P) + \operatorname{height}\left(\frac{\mathfrak{m}S + P}{P}\right) \geq d - h.$$

If $P \notin V(xS)$ then G_P is Cohen-Macaulay. Therefore

$$\operatorname{depth}(G_P) = \dim(G_P) = \operatorname{height}_S(P) - 1.$$

Then, because S is catenary,

$$\begin{aligned}
\operatorname{depth}(G_P) + \operatorname{height}\left(\frac{\mathfrak{m}S + P}{P}\right) &= \operatorname{height}_S(P) - 1 + \dim(S/P) - \dim(S/\mathfrak{m}S + P) \\
&= \operatorname{height}_S(P) - 1 + d + 1 - \operatorname{height}_S(P) \\
&\quad - \dim(S/\mathfrak{m}S + P) \\
&= d - \dim(S/\mathfrak{m}S + P).
\end{aligned}$$

Recall that S is the integral closure of $\mathcal{R}[t^{-1}]$ in $R[t, t^{-1}]$. It follows that $S/\mathfrak{m}S$ is a finite extension of the fiber cone of J, an $R/\mathfrak{m}$-algebra of Krull dimension at most h. Therefore

$$\dim(S/\mathfrak{m}S + P) \leq \dim(S/\mathfrak{m}S) = h$$

and so $\operatorname{depth}(G_P) + \operatorname{height}\left(\frac{\mathfrak{m}S+P}{P}\right) \geq d - h$ as needed. □

Corollary 2.3. *Let* $(R, \mathfrak{m}, k)$ *be an excellent local normal domain of Krull dimension* d *and* $J \subseteq R$ *an ideal generated by at most* h *elements. Suppose that the ring* $\bigoplus \overline{J^n}/\overline{J^{n+1}} \otimes_R R_x$ *is Cohen-Macaulay. Then there exists a constant* C *so that*

$$x^{Cn}H^i_{\mathfrak{m}}(R/\overline{J^n}) = 0$$

for every $0 \le i \le d-h-1$ *and* $n \in \mathbb{N}$*.*

Proof. For every $i \ge 0$ and for every $n \in \mathbb{N}$ there are exact sequences of local cohomology modules

$$H^i_{\mathfrak{m}}(\overline{J^n}/\overline{J^{n+1}}) \to H^i_{\mathfrak{m}}(R/\overline{J^{n+1}}) \to H^i_{\mathfrak{m}}(R/\overline{J^n}).$$

By Lemma 2.2, if $i \le d-h-1$, then there exists a constant C so that x^C annihilates the left most module of the above exact sequences for all $n \ge 0$. By induction, x^{Cn} annihilates $H^i_{\mathfrak{m}}(R/\overline{J^n})$ for every $n \in \mathbb{N}$. $\square$

Remark 2.4. If we are only interested in annihilation properties of $H^1_{\mathfrak{m}}(\overline{J^n}/\overline{J^{n+1}})$, then many of the assumptions of Lemma 2.2 and Corollary 2.3 are not necessary. One only needs to assume that R is an excellent normal domain and J is generated by at most $d-2$ elements to conclude that there exists a constant C so that $\mathfrak{m}^C$ annihilates $H^1_{\mathfrak{m}}(\overline{J^n}/\overline{J^{n+1}})$ for every $n \in \mathbb{N}$. Indeed, $\text{height}(\mathfrak{m}S + P/P) \ge 1$ for all $P \in \text{Spec}(S) \setminus V(\mathfrak{m}S)$. Thus, to show

$$\text{depth}(G_P) + \text{height}(\mathfrak{m}S + P/P) \ge 2$$

for every $P \in \text{Spec}(S) \setminus V(\mathfrak{m}S)$, it suffices to show that $\text{height}(\mathfrak{m}S + P/P) \ge 2$ whenever $\text{depth}(G_P) = 0$. If $\text{depth}(G_P) = 0$ then $P \in \text{Spec}(S)$ is an associated prime of $t^{-1}S$. The ring S is normal and t^{-1} is a nonzerodivisor. Therefore every associated prime of $t^{-1}S$ is minimal and so $\dim(G_P) = 0$. One can now proceed as in the proof of Lemma 2.2 to show that $\text{height}(\mathfrak{m}S + P/P) \ge 2$.

Lemma 2.5. *Let* $(R, \mathfrak{m}, k)$ *be an excellent Noetherian local normal domain with infinite residue field,* $I \subseteq R$ *an ideal,* $P_1, \dots, P_t \in \text{Spec}(R)$ *a finite collection of non-comparable prime ideals, and* $W = R \setminus \bigcup_{i=1}^t P_i$*. Suppose that* $\ell_{R_{P_i}}(IR_{P_i}) \le h$ *for every* $1 \le i \le t$*. Then there exist elements* $y_1, \dots, y_h \in I$ *and* $x \in W$ *with the following properties:*

(1) $(y_1, \dots, y_h)R_W \subseteq IR_W$ *is a reduction of* IR_W*;*
(2) $x^n\overline{I^n} \subseteq \overline{(y_1, \dots, y_h)^n}$ *for all* $n \in \mathbb{N}$*.*

Proof. Recall the following: Suppose $(S, \mathfrak{n}, \ell)$ is a local ring and $J \subseteq I$ are ideals. Then J is reduction of I if and only if $S[Jt] \otimes_S \ell \to S[It] \otimes_S \ell$ is finite, see [27, Proposition 8.2.4]. In particular, if $J' \subseteq I$ is an ideal such that $J' \equiv J + \mathfrak{n}I$ and J is a reduction of I then J' is a reduction of I.

To prove the lemma start by choosing elements $y_{1,i}, \dots, y_{h,i} \in I$ so that $(y_{1,i}, \dots, y_{h,i})R_{P_i}$ forms a reduction of the ideal IR_{P_i}. Choose elements $r_j \in (\cap_{i \ne j} P_i) \setminus P_j$ and set $y_i = \sum r_j y_{j,i}$. Then $(y_1, \dots, y_h)R_{P_i}$ forms a reduction of IR_{P_i} for each $1 \le i \le t$

by the above discussion. Therefore $(y_1,\ldots,y_h)R_W$ forms a reduction of IR_W by [27, Propositions 8.1.1.].

Let $J=(y_1,\ldots,y_h)$. Then $\overline{J}R_W=\overline{I}R_W$ and so there exists an element $x\in W$ such that $x\overline{I}\subseteq\overline{J}$, in particular $xI\subseteq\overline{J}$. Raising the containment to the nth power we find that $x^nI^n\subseteq\overline{J^n}$ for every $n\in\mathbb{N}$. We claim that $x^n\overline{I^n}\subseteq\overline{J^n}$. Let $r\in\overline{I^n}$, then there exists a $t\in\mathbb{N}$ and an equation

$$r^t+a_1r^{t-1}+\cdots+a_{t-1}r+a_t=0$$

such that $a_j\in I^{nj}$ for each $1\leq j\leq t$. Multiplying by x^{nt} we find that there is an equation

$$(x^nr)^t+x^na_1(x^nr)^{t-1}+\cdots x^{n(t-1)}a_{t-1}+x^{nt}a_t=0.$$

The elements $x^{nj}a_j$ belong to $\overline{J^{nj}}$ and therefore $x^nr\in\overline{\overline{J^n}}=\overline{J^n}$. $\square$

Theorem 2.6. *Let $(R,\mathfrak{m},k)$ be an excellent local Cohen-Macaulay normal domain of Krull dimension $d\geq 3$ and $I\subseteq R$ an ideal of pure height 1 with the following properties:*

- *$I^nR_P=I^{(n)}R_P$ for every $P\in\operatorname{Spec}(R)\setminus\{\mathfrak{m}\}$ and every $n\in\mathbb{N}$;*
- *If $P\in\operatorname{Spec}(R)\setminus\{\mathfrak{m}\}$ and $G=\mathcal{R}[t^{-1}]/t^{-1}\mathcal{R}[t^{-1}]$ is the associated graded ring of I then G_P is Cohen-Macaulay.*

Then there exists a system of parameters $x_1,x_2,\ldots,x_d$ such that for every $3\leq t\leq d$;

$$(x_1^n,\ldots,x_t^n)H^j_{\mathfrak{m}}(R/I^{(n)})=0$$

for every $0\leq j\leq d-(t-1)$ and $n\in\mathbb{N}$.

Proof. The ideal I is locally principal at its height 1 components because R is normal. Therefore $\overline{I^n}\subseteq I^{(n)}$. Our assumptions inform us that $I^{(n)}/\overline{I^n}$ is 0-dimensional for every integer n. Therefore for every integer $i\geq 1$

$$H^i_{\mathfrak{m}}(R/\overline{I^n})\cong H^i_{\mathfrak{m}}(R/I^{(n)}).$$

Start by choosing $x_1\in I$. Then clearly $x_1^n\in\overline{I^n}$ and therefore x_1^n annihilates $H^i_{\mathfrak{m}}(R/\overline{I^n})$ for all integers i and n. If W_1 is the complement of the union of the minimal primes of x_1R then IR_{W_1} is a principal ideal. By Lemma 2.5 there exists an element $y\in I$ and $x\in W_1$ so that $x^n\overline{I^n}\subseteq y^nR$ for every $n\in\mathbb{N}$. There are short exact sequences

$$0\to\frac{\overline{I^n}}{y^nR}\to\frac{R}{y^nR}\to\frac{R}{\overline{I^n}}\to 0$$

and so $H^j_{\mathfrak{m}}(R/\overline{I^n}) \cong H^{j+1}_{\mathfrak{m}}(\overline{I^n}/y^nR)$ if $j \leq d-3$ and there is an injective map $H^{d-2}_{\mathfrak{m}}(R/\overline{I^n}) \to H^{d-1}_{\mathfrak{m}}(\overline{I^n}/y^nR)$. Therefore x^n annihilates $H^j_{\mathfrak{m}}(R/\overline{I^n})$ for every $j \leq d-2$ and we take $x_2 = x$.

If W_2 is the complement of the union of the minimal primes of (x_1, x_2) then IR_{W_2} has analytic spread at most 1, see [11, Proof of Theorem 1.5]. The ring R_{W_2} is normal, every principal ideal in a normal ring is integrally closed, and therefore IR_{W_2} is a principal ideal. We therefore proceed as before to find an element x_3 so that x_3^n annihilates $H^j_{\mathfrak{m}}(R/\overline{I^n})$ for every $j \leq d-2$ as needed.

Inductively, suppose that we have found parameter elements $x_1, \ldots, x_i$, with $i \geq 3$, so that if $3 \leq t \leq i$ then

$$(x_1^n, \ldots, x_t^n)H^j_{\mathfrak{m}}(R/\overline{I^n}) = 0$$

for every $0 \leq j \leq d-(t-1)$. It is important that $t \geq 3$ in the inductive step of the proof. If $t = 2$ then it is not the case that (x_1^n, x_2^n) annihilates $H^j_{\mathfrak{m}}(R/\overline{I^n})$ for every $0 \leq j \leq d-(2-1) = d-1$. Indeed, the annihilator of the top local cohomology module $H^{d-1}_{\mathfrak{m}}(R/\overline{I^n})$ is the height 1 ideal $I^{(n)}$ and $(x_1^n, x_2^n) \not\subseteq I^{(n)}$. If $i = d$ then we are done. Suppose that $i \leq d-1$. Our aim is to find a parameter element x_{i+1} so that

$$x_{i+1}^n H^j_{\mathfrak{m}}(R/\overline{I^n}) = 0$$

for every $j \leq d-i$.

Let W be the complement of the union of the minimal primes of the parameter ideal $(x_1, \ldots, x_i)$. Then $I^nR_W = I^{(n)}R_W$ for all integers n and so the localization of IR_W at a maximal ideal of R_W is an ideal of analytic spread at most $i-1$, see [11, Proof of Theorem 1.5]. By Lemma 2.5 there exist elements $y_1, \ldots, y_{i-1} \in I$ and $x'_{i+1} \in W$ so that

(1) $(y_1, \ldots, y_{i-1})R_W \subseteq IR_W$ is a reduction of IR_W;
(2) $(x'_{i+1})^n\overline{I^n} \subseteq \overline{(y_1, \ldots, y_{i-1})^n}$ for all $n \in \mathbb{N}$.

Let $J = (y_1, \ldots, y_{i-1})$ and consider the short exact sequences

$$0 \to \frac{\overline{I^n}}{\overline{J^n}} \to \frac{R}{\overline{J^n}} \to \frac{R}{\overline{I^n}} \to 0.$$

The element $(x'_{i+1})^n$ annihilates the left-most module in the above short exact sequence and there are exact sequences of local cohomology modules

$$H^j_{\mathfrak{m}}\left(\frac{R}{\overline{J^n}}\right) \to H^j_{\mathfrak{m}}\left(\frac{R}{\overline{I^n}}\right) \to H^{j+1}_{\mathfrak{m}}\left(\frac{\overline{I^n}}{\overline{J^n}}\right).$$

The element $(x'_{i+1})^n$ annihilates the right-most module. By our hypothesis that the associated graded ring of I is Cohen-Macaulay on the punctured spectrum of R, Corollary 2.3

implies that there exists a constant C so that $(x'_{i+1})^{Cn}$ annihilates $H^j_{\mathfrak{m}}(R/\overline{J^n})$ for every $j \leq d-(i-1)-1 = d-i$. Therefore $(x'_{i+1})^{C(n+1)}$ annihilates $H^j_{\mathfrak{m}}(R/\overline{I^n})$ for every $j \leq d-i$. Therefore $x_{i+1} = (x'_{i+1})^{2C}$ has the desired annihilation properties. □

Theorem B is a consequence of the following theorem.

Corollary 2.7. *Let $(R, \mathfrak{m}, k)$ be an excellent local Cohen-Macaulay normal domain of prime characteristic $p > 0$ and Krull dimension $d \geq 3$. Suppose that R is a splinter on the punctured spectrum of R and that the anti-canonical algebra of R is Noetherian on the punctured spectrum. Then there exists an ideal $I \subseteq R$ of pure height 1 and parameters $x_1, \ldots, x_d$ with the following properties:*

(1) *$I \cong \omega_R^{(-m)}$ for some $m \geq 1$;*
(2) *For each $1 \leq j \leq d-2$, the ideal $\mathfrak{a}_j := (x_1, \ldots, x_{d-j+1})$ is such that*

$$\mathfrak{a}_j^{[p^e]} H^j_{\mathfrak{m}}(R/I^{(p^e)}) = 0$$

for each $e \in \mathbb{N}$.

Proof. Start by choosing an ideal $I \subseteq R$ of pure height 1 so that $I \cong \omega_R^{(-1)}$ is an anti-canonical ideal. We are assuming that the anti-canonical algebra is Noetherian on the punctured spectrum. Therefore if $P \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$ then there exists an integer m such that the symbolic Rees algebra of $I^{(m)}R_P$ is standard graded. The set of prime ideals $\cup \operatorname{Ass}(R/I^{(m)n})$ is a finite set by [6], see also [17]. By prime avoidance there exists an $s \in R \setminus P$ which is contained in each non-minimal member of $\cup \operatorname{Ass}(R/I^{(m)n})$. Then $I^{(m)n}R_s = I^{(mn)}R_s$ for every $n \in \mathbb{N}$. The space $\operatorname{Spec}(R \setminus \{\mathfrak{m}\}$ is quasi-compact. Therefore there exists finitely many open sets $D(s_1), \ldots, D(S_t)$ covering $\operatorname{Spec}(R \setminus \{\mathfrak{m}\}$ and integers m_i, $1 \leq i \leq s$, so that for all $n \in \mathbb{N}$ $I^{(m_i)n}R_{s_i} = I^{(m_i n)}R_{s_i}$. If m is a common multiple of $m_1, \ldots, m_s$ and then the symbolic Rees algebra of $I^{(m)}$ is standard graded on the punctured spectrum, i.e. $I^{(m)n}R_P = I^{(mn)}R_P$ for all $n \in \mathbb{N}$ and $P \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$. We replace I by $I^{(m)}$. By Theorem 2.6, it suffices to show that R is Cohen-Macaulay and that if $G = \operatorname{Gr}_I(R)$ is the associated graded ring of I then G_P is a Cohen-Macaulay algebra for all $P \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$.

For each non-maximal prime P the localized ring R_P is strongly F-regular by [10, Corollary 5.9], see also [30, Theorem 0.1]. Therefore the localized (symbolic) Rees algebras $R[It] \otimes R_P$ are Cohen-Macaulay for all $P \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$, see [10, Lemma 6.1]. We may now conclude that G_P is a Cohen-Macaulay algebra for all $P \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$ by [16, Proposition 1.1]. □

3. Tight closure, local cohomology, and local cohomology bounds

3.1. Tight closure

Let R be a ring of prime characteristic $p > 0$ and let R° be the complement of the union of the minimal primes of R. The eth Frobenius functor, or the eth Peskine-Szpiro functor, is the functor $F^e : \mathrm{Mod}(R) \to \mathrm{Mod}(R)$ obtained by extending scalars along the eth iterate of the Frobenius endomorphism. If $N \subseteq M$ are R-modules and $m \in M$, then m is in the tight closure of N relative to M if there exists a $c \in R^\circ$ such that for all $e \gg 0$ the element m is in the kernel of the following composition of maps:

$$M \to M/N \to F^e(M/N) \xrightarrow{\cdot c} F^e(M/N).$$

In particular, if we consider an inclusion of R-modules of the form $I \subseteq R$ then $F^e(R/I) \cong R/I^{[p^e]}$ where $I^{[p^e]} = (r^{p^e} \mid r \in I)$, and an element $r \in R$ is in the tight closure of I relative to R if there exists a $c \in R^\circ$ such that $cr^{p^e} \in I^{[p^e]}$ for all $e \gg 0$. The tight closure of the module N relative to the module M is denoted N^*_M. In the case that $M = R$ and $N = I$ is an ideal then we denote the tight closure of I relative to R as I^*. We say that N is tightly closed in M if $N = N^*_M$. If an ideal is tightly closed in R then we simply say that the ideal is tightly closed. The finitistic tight closure of $N \subseteq M$ is denoted N^{*fg}_M and is the union of $(N \cap M')^*_{M'}$ where M' runs over all finitely generated submodules of M.

The notions of weak F-regularity and strong F-regularity can be compared by studying the finitistic tight closure and tight closure of the zero submodule of the injective hull of a local ring by [14, Proposition 8.23] and [25, Proposition 7.1.2]. Suppose that $(R, \mathfrak{m}, k)$ is complete local and $E_R(k)$ is the injective hull of the residue field. The finitistic test ideal of R is $\tau_{fg}(R) = \bigcap_{I \subseteq R} \mathrm{Ann}_R(I^*/I)$ and agrees with $\mathrm{Ann}_R(0^{*fg}_{E_R(k)})$. The (big) test ideal of R is $\tau(R) = \bigcap_{N \subseteq M \in \mathrm{Mod}(R)} \mathrm{Ann}_R(N^*_M/N)$ and agrees with $\mathrm{Ann}_R(0^*_{E_R(k)})$. The ring R is weakly F-regular if and only if $\tau_{fg}(R) = R$ and R is strongly F-regular if and only if $\tau(R) = R$. Thus to prove the conjectured equivalence of weak and strong F-regularity it is enough to show $0^*_{E_R(k)} = 0^{*fg}_{E_R(k)}$ under hypotheses satisfied by rings which are weakly F-regular.

To explore the tight closure of the zero submodule of $E_R(k)$ we exploit the structure of $E_R(k)$ as a direct limit of 0-dimensional Gorenstein quotients of R described in [13]. Suppose $(R, \mathfrak{m}, k)$ is a complete local Cohen-Macaulay domain of Krull dimension d and $J_1 \subsetneq R$ a canonical ideal. Let $0 \neq x_1 \in J_1$, $x_2, \ldots, x_d \in R$ a parameter sequence, and for each $t \in \mathbb{N}$ let $I_t = (x_1^{t-1}J_1, x_2^t, \ldots, x_d^t)$. The sequences of ideals $\{I_t\}$ form a decreasing sequence of irreducible $\mathfrak{m}$-primary ideals cofinal with $\{\mathfrak{m}^t\}$. Moreover, the direct limit system $\varinjlim R/I_t \xrightarrow{\cdot x_1 \cdots x_d} R/I_{t+1}$ is isomorphic to $E_R(k)$. The following lemma uses this description of the injective hull of the residue field to describe any potential difference between the modules $0^*_{E_R(k)}$ and $0^{*fg}_{E_R(k)}$. We refer the reader to the discussion at the beginning of [2, Section 2] for details.

Lemma 3.1. *Let $(R, \mathfrak{m}, k)$ be a complete Cohen-Macaulay local ring of prime characteristic $p > 0$ and of Krull dimension d. Let $J_1 \subsetneq R$ be a choice of canonical ideal and $x_1, \ldots, x_d$ a system of parameters such that $x_1 \in J$. Make the following identification of $E_R(k)$:*

$$E_R(k) \cong \varinjlim \left(\frac{R}{(x_1^{t-1}J_1, x_2^t, \cdots x_d^t)} \xrightarrow{\cdot x_1 \cdots x_d} \frac{R}{(x_1^t J_1, x_2^{t+1}, \cdots x_d^{t+1})} \right)$$

If $\eta = [r + (x_1^{t-1}J_1, x_2^t, \cdots x_d^t)] \in E_R(k)$ then

(1) *$\eta \in 0^{*fg}_{E_R(k)}$ if and only if there exists a $c \in R^\circ$ and $s \in \mathbb{N}$ such that for all $e \in \mathbb{N}$*

$$c(r(x_1x_2 \cdots x_d)^s)^{p^e} \in (x_1^{s+t-1}J_1, x_2^{s+t}, \ldots, x_d^{s+t})^{[p^e]};$$

(2) *$\eta \in 0^{*}_{E_R(k)}$ if and only if there exists a $c \in R^\circ$ such that for all $e \in \mathbb{N}$ there exists an $s = s(e)$ such that*

$$c(r(x_1x_2 \cdots x_d)^s)^{p^e} \in (x_1^{s+t-1}J_1, x_2^{s+t}, \ldots, x_d^{s+t})^{[p^e]}.$$

3.2. Local cohomology bounds

We will relate the modules $0^{*fg}_{E_R(k)}$ and $0^{*}_{E_R(k)}$ in Lemma 3.1 through the language of local cohomology bounds. To this end, suppose that M is a module over a ring R and $\underline{x} = x_1, \ldots, x_d$ a sequence of elements. For each $j \in \mathbb{N}$ let $\underline{x}^j = x_1^j, \ldots, x_d^j$ and for each pair of integers $j_1 \leq j_2$ let $\tilde{\alpha}^\bullet_{M;x_i;j_1;j_2}$ be the map of Kosul cocomplexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{\cdot x_i^{j_1}} & M & \longrightarrow & 0 \\ & & \downarrow = & & \downarrow \cdot x_i^{j_2-j_1} & & \\ 0 & \longrightarrow & M & \xrightarrow{\cdot x_i^{j_2}} & M & \longrightarrow & 0 \end{array}$$

Let $\tilde{\alpha}^\bullet_{M;\underline{x};j_1;j_2}$ be the following product:

$$\tilde{\alpha}^\bullet_{M;\underline{x};j_1;j_2} := \tilde{\alpha}^\bullet_{R;x_1;j_1;j_2} \otimes \tilde{\alpha}^\bullet_{R;x_2;j_1;j_2} \otimes \cdots \otimes \tilde{\alpha}^\bullet_{R;x_d;j_1;j_2} \otimes M.$$

Then $\tilde{\alpha}^\bullet_{M;\underline{x};j_1;j_2}$ is a map of Koszul cocomplexes

$$K^\bullet(\underline{x}^{j_1}; M) \xrightarrow{\tilde{\alpha}^\bullet_{M;\underline{x};j_1;j_2}} K^\bullet(\underline{x}^{j_2}; M).$$

Let $\alpha^i_{M;\underline{x};j_1;j_2}$ denote the induced map of Koszul cohomologies

$$H^i(\underline{x}^{j_1}; M) \xrightarrow{\alpha^i_{M;\underline{x};j_1;j_2}} H^i(\underline{x}^{j_2}; M).$$

Then

$$\varinjlim_{j_1 \le j_2} \left(H^i(\underline{x}^{j_1}; M) \xrightarrow{\alpha^i_{M;\underline{x};j_1;j_2}} H^i(\underline{x}^{j_2}; M) \right) \cong H^i_{(\underline{x})R}(M)$$

by [8, Theorem 3.5.6].

Denote by $\alpha^i_{M;\underline{x};j;\infty}$ the map

$$H^i(\underline{x}^j; M) \xrightarrow{\alpha^i_{M;\underline{x};j;\infty}} H^i_{(\underline{x})A}(M).$$

Observe that $\eta \in \operatorname{Ker}(\alpha^i_{M;\underline{x};j;\infty})$ if and only if there exists some $k \ge 0$ such that $\eta \in \operatorname{Ker}(\alpha^i_{M;\underline{x};j;j+k})$. If $\eta \in \operatorname{Ker}(\alpha^i_{M;\underline{x};j;\infty})$ we let

$$\epsilon^i_{\underline{x}^j}(\eta) = \min\{k \mid \eta \in \operatorname{Ker}(\alpha^i_{M;\underline{x};j;j+k})\}.$$

Definition 3.2. Let R be a ring, $\underline{x} = x_1, \dots, x_d$ a sequence of elements in R, and M an R-module. The ith local cohomology bound of M with respect to the sequence of elements $\underline{x}$ is

$$\operatorname{lcb}_i(\underline{x}; M) = \sup\{\epsilon^i_{\underline{x}^j}(\eta) \mid \eta \in \operatorname{Ker}(\alpha^i_{M;\underline{x};j;\infty}) \text{ for some } j\} \in \mathbb{N} \cup \{\infty\}.$$

Observe that if M is an R-module and $\underline{x}$ is a sequence of elements, then $\operatorname{lcb}_i(\underline{x}; M) = N < \infty$ simply means that if $\eta \in H^i(\underline{x}^j; M)$ represents the 0-element in the direct limit

$$\varinjlim_{j_1 \le j_2} \left(H^i(\underline{x}^{j_1}; M) \xrightarrow{\alpha^i_{M;\underline{x};j_1;j_2}} H^i(\underline{x}^{j_2}; M) \right) \cong H^i_{(\underline{x})R}(M)$$

then $\alpha^i_{M;\underline{x};j;j+N}(\eta)$ is the 0-element of the Koszul cohomology group $H^i(\underline{x}^{j+N}; M)$. Therefore finite local cohomology bounds correspond to a uniform bound of annihilation of zero elements in a choice of direct limit system defining a local cohomology module. It would be interesting to understand when local cohomology bounds are finite.

3.3. Basic properties of local cohomology bounds

Our study of local cohomology bounds begins with two useful observations.

Lemma 3.3. *Let R be a commutative Noetherian ring, M an R-module, and $\underline{x} = x_1, \dots, x_d$ a sequence of elements, then $\operatorname{lcb}_i(\underline{x}^j; M) \le \operatorname{lcb}_i(\underline{x}; M)$. Furthermore, $\operatorname{lcb}_i(\underline{x}; M) \le jm$ for some integers j, m if and only if $\operatorname{lcb}_i(\underline{x}^j; M) \le m$ where $\underline{x}^j$ is the sequence of elements $x_1^j, \dots, x_d^j$.*

Proof. One only has to observe that $\alpha^i_{M;\underline{x}^j;k,k+m} = \alpha^i_{M;\underline{x};jk,jk+jm}$. □

If $x_1, \ldots, x_d$ is a sequence of elements in a ring R and if $x_1 M = 0$ for some R-module M then the short exact sequence of Koszul cocomplexes

$$0 \to K^\bullet(x_2, \ldots, x_d; M)(-1) \to K^\bullet(x_1, x_2, \ldots, x_d; M) \to K^\bullet(x_2, \ldots, x_d; M) \to 0$$

is split and therefore $H^i(x_1, x_2, \ldots, x_d; M) \cong H^i(x_2, \ldots, x_d; M) \oplus H^{i-1}(x_2, \ldots, x_d; M)$. The content of the following lemma is a description of the behavior of the maps $\alpha^i_{M;x_1,x_2,\ldots,x_d;j,j+k}$ with respect to these isomorphisms of Koszul cohomologies.

Lemma 3.4. *Let R be a commutative Noetherian ring, M an R-module, and $x_1, x_2, \ldots, x_d$ a sequence of elements such that $x_1 M = 0$. If $i, j, k \in \mathbb{N}$ then*

$$H^i(x_1^j, x_2^j, \ldots, x_d^j; M) \cong H^i(x_2^j, \ldots, x_d^j; M) \oplus H^{i-1}(x_2^j, \ldots, x_d^j; M)$$

and the map $\alpha_{M;x_1,x_2,\ldots,x_d;j,j+k}$ is the direct sum of $\alpha^i_{M;x_2,\ldots,x_d;j,j+k}$ and the 0-map.

Proof. Let $(F^\bullet, \partial^\bullet)$ be the Koszul cocomplex $K^\bullet(x_2^j, \ldots, x_d^j; R)$ and let $(G^\bullet, \delta^\bullet)$ be the Koszul cocomplex $K^\bullet(x_1^j; R)$. Let

$$(L^\bullet, \gamma^\bullet) = K^\bullet(x_1^j, x_2^j, \ldots, x_d^j; R) \cong K^\bullet(x_1^j; R) \otimes K^\bullet(x_2^j, \ldots, x_d^j; R).$$

Then $L^i \cong (G^0 \otimes F^i) \oplus (G^1 \otimes F^{i-1}) \cong F^i \oplus F^{i-1}$. We abuse notation and let $\cdot x_1^j$ denote the multiplication map on F^i. The map γ^i can be thought of as

$$\gamma^i = \begin{pmatrix} \partial^i & 0 \\ \pm x_1^j & \partial^{i-1} \end{pmatrix} : F^i \oplus F^{i-1} \to F^{i+1} \oplus F^i.$$

In particular, if we apply $- \otimes_R M$ the map $\cdot \pm x_1^j \otimes M$ is the 0-map and therefore the ith map of the Koszul cocomplex $K^i(x_1^j, x_2^j, \ldots, x_d^j; M)$ is the direct sum of maps $(\partial^i \otimes M) \oplus (\partial^{i-1} \otimes M)$. In particular

$$H^i(x_1^j, x_2^j, \ldots, x_d^j; M) \cong H^i(x_2^j, \ldots, x_d^j; M) \oplus H^{i-1}(x_2^j, \ldots, x_d^j; M).$$

To see that $\alpha_{M;x_1,x_2,\ldots,x_d;j,j+k}$ is the direct sum of $\alpha^i_{M;x_2,\ldots,x_d;j,j+k}$ and the 0-map is similar to above argument but uses the fact that

$$\tilde{\alpha}^\bullet_{M;x_1,x_2,\ldots,x_d;j;j+k} = \tilde{\alpha}^\bullet_{R;x_2,\ldots,x_d;j;j+k} \otimes \tilde{\alpha}^\bullet_{R;x_1;j;j+k} \otimes M$$

and $\tilde{\alpha}^1_{R;x_1;j;j+k} \otimes M = 0$. $\square$

A particularly useful corollary of Lemma 3.4 is the following:

Corollary 3.5. *Let R be a commutative Noetherian ring and M an R-module. Suppose $x_1, \dots, x_d$ is a sequence of elements, $1 \le i \le d$, and $(x_1, \dots, x_{d-i})M = 0$. If $j, k \in \mathbb{N}$ then*

$$\alpha^{\ell}_{M;x_1,\dots,x_d;j,j+k} : H^{\ell}(x_1^j, \dots, x_d^j; M) \to H^{\ell}(x_1^{j+k}, \dots, x_d^{j+k}; M)$$

is the 0-map for all $\ell \ge i+1$. In particular, $\mathrm{lcb}_{\ell}(x_1, \dots, x_d; M) = 1$ for all $\ell \ge i+1$.

Proof. By multiple applications of Lemma 3.4 it is enough to observe that

$$H^{\ell}(x_{d-i+1}^j, \dots, x_d^j; M) = 0.$$

This is clearly the case since $x_{d-i+1}^j, \dots, x_d^j$ is a list of i elements and we are examining an $\ell \ge i+1$ Koszul cohomology of M with respect to this sequence. □

Suppose $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of R-modules. The next two properties of local cohomology bounds we record allow us to compare the local cohomology bounds of the modules appearing in the short exact sequence. Proposition 3.6 allows us to effectively compare the local cohomology bounds of two of the terms in the sequence provided a subset of the elements in the sequence of elements defining Koszul cohomology annihilates the third. Proposition 3.7 compares the local cohomology bounds of two of the terms in the short exact whenever the sequence of elements defining Koszul cohomology is a regular sequence on the third module.

Proposition 3.6. *Let R be a commutative Noetherian ring and*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

a short exact sequence of R-modules. Let $\underline{x} = x_1, \dots, x_d$ be a sequence of elements of R.

(1) *If $(x_1, \dots, x_{d-j})M_1 = 0$ then for all $\ell \ge j+1$*

$$\mathrm{lcb}_{\ell}(\underline{x}; M_2) \le \mathrm{lcb}_{\ell}(\underline{x}; M_3) + 1.$$

(2) *If $(x_1, \dots, x_{d-j})M_2 = 0$ then for all $\ell \ge j+1$*

$$\mathrm{lcb}_{\ell}(\underline{x}; M_3) \le \mathrm{lcb}_{\ell+1}(\underline{x}; M_1) + 1.$$

(3) *If $(x_1, \dots, x_{d-j})M_3 = 0$ then for all $\ell \ge j+2$*

$$\mathrm{lcb}_{\ell}(\underline{x}; M_1) \le \mathrm{lcb}_{\ell}(\underline{x}; M_2) + 1.$$

Proof. For each integer $j \in \mathbb{N}$ let $\underline{x}^j$ denote the sequence of elements $x_1^j, x_2^j, \ldots, x_d^j$. For (1) we consider the following commutative diagram, whose middle row is exact:

$$\begin{array}{ccccc}
 & & H^\ell(\underline{x}^j; M_2) & \longrightarrow & H^\ell(\underline{x}^j; M_3) \\
 & & \big\downarrow{\scriptstyle \alpha^\ell_{M_2;\underline{x};j;j+k}} & & \big\downarrow{\scriptstyle \alpha^\ell_{M_3;\underline{x};j;j+k}} \\
H^\ell(\underline{x}^{j+k}; M_1) & \longrightarrow & H^\ell(\underline{x}^{j+k}; M_2) & \longrightarrow & H^\ell(\underline{x}^{j+k}; M_3) \\
\big\downarrow{\scriptstyle \alpha^\ell_{M_1;\underline{x};j+k;j+k+1}} & & \big\downarrow{\scriptstyle \alpha^\ell_{M_2;\underline{x};j+k;j+k+1}} & & \\
H^\ell(\underline{x}^{j+k+1}; M_1) & \longrightarrow & H^\ell(\underline{x}^{j+k+1}; M_2) & &
\end{array}$$

By Corollary 3.5 the map $\alpha^\ell_{M_1;\underline{x};j+k;j+k+1}$ is the 0-map for all $\ell \geq j+1$. A straightforward diagram chase of the above diagram, which follows an element $\eta \in \operatorname{Ker}(\alpha^\ell_{M_2;\underline{x};j;j+k'})$ for some k', shows that $\eta \in \operatorname{Ker}(\alpha^\ell_{M_2;\underline{x};j;j+k+1})$ whenever $k \geq \operatorname{lcb}_\ell(\underline{x}; M_3)$. In particular, $\operatorname{lcb}_\ell(\underline{x}; M_2) \leq \operatorname{lcb}_\ell(\underline{x}; M_3) + 1$.

Statements (2) and (3) follow in a similar manner and the details are left to the reader. □

Proposition 3.7. *Let R be a commutative Noetherian ring, $0 \to M_1 \to M_2 \to M_3 \to 0$ a short exact sequence of R-modules, and $\underline{x} = x_1, \ldots, x_d$ a sequence of elements in R.*

(1) *If $\underline{x}$ is a regular sequence on M_1 then $\operatorname{lcb}_i(\underline{x}; M_2) = \operatorname{lcb}_i(\underline{x}; M_3)$ for all $i \leq d-1$.*
(2) *If $\underline{x}$ is a regular sequence on M_2 then $\operatorname{lcb}_i(\underline{x}; M_3) = \operatorname{lcb}_{i+1}(\underline{x}; M_1)$ for all $i \leq d-1$.*
(3) *If $\underline{x}$ is a regular sequence on M_3 then $\operatorname{lcb}_i(\underline{x}; M_1) = \operatorname{lcb}_i(\underline{x}; M_2)$ for all $i \leq d$.*

Proof. The proofs of the three statements are very similar to one another and we only provide the details of (1).

Proof of (1): For $i < d$ we have $H^i(\underline{x}^j; M_1) = 0$ and therefore if $i \leq d-2$ there are commutative diagrams

$$\begin{array}{ccc}
H^i(\underline{x}^j; M_2) & \xrightarrow{\cong} & H^i(\underline{x}^j; M_3) \\
\big\downarrow{\scriptstyle \alpha^i_{M_2;\underline{x};j;j+k}} & & \big\downarrow{\scriptstyle \alpha^i_{M_3;\underline{x};j;j+k}} \\
H^i(\underline{x}^{j+k}; M_2) & \xrightarrow{\cong} & H^i(\underline{x}^{j+k}; M_3)
\end{array}$$

whose horizontal arrows are isomorphisms. It readily follows that $\operatorname{lcb}_i(\underline{x}; M_2) = \operatorname{lcb}_i(\underline{x}; M_3)$ whenever $i \leq d-2$. Because $\underline{x}$ is a regular sequence on M_1 we have that the maps $\alpha^d_{M_1,\underline{x},j,j+k}$ are injective. Conside the following commutative diagrams whose rows are exact:

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^{d-1}(\underline{x}^j; M_2) & \xrightarrow{\pi_j} & H^{d-1}(\underline{x}^j; M_3) & \xrightarrow{\delta_j} & H^d(\underline{x}^j; M_1) \\
 & & \big\downarrow{\scriptstyle \alpha^{d-1}_{M_2;\underline{x};j;j+k}} & & \big\downarrow{\scriptstyle \alpha^{d-1}_{M_3;\underline{x};j;j+k}} & & \big\downarrow{\scriptstyle \alpha^d_{M_1;\underline{x};j;j+k}} \\
0 & \longrightarrow & H^{d-1}(\underline{x}^{j+k}; M_2) & \xrightarrow{\pi_{j+k}} & H^{d-1}(\underline{x}^{j+k}; M_3) & \xrightarrow{\delta_{j+k}} & H^d(\underline{x}^{j+k}; M_1)
\end{array}$$

If $\eta \in \operatorname{Ker}(\alpha^{d-1}_{M_2;\underline{x};j,j+k})$ then $\pi_j(\eta) \in \operatorname{Ker}(\alpha^{d-1}_{M_3;\underline{x};j,j+k})$. The maps π_{j+k} are injective. Therefore $\alpha^{d-1}_{M_2;\underline{x};j,j+k}(\eta) = 0$ whenever $k \geq \operatorname{lcb}_{d-1}(\underline{x}; M_3)$ and hence $\operatorname{lcb}_{d-1}(\underline{x}; M_2) \leq \operatorname{lcb}_{d-1}(\underline{x}; M_3)$.

To show that $\operatorname{lcb}_{d-1}(\underline{x}; M_2) \geq \operatorname{lcb}_{d-1}(\underline{x}; M_3)$ consider an element $\eta \in \operatorname{Ker}(\alpha^{d-1}_{M_3;\underline{x};j;j+k})$. Then $\delta_j(\eta) \in \operatorname{Ker}(\alpha^{d}_{M_1;\underline{x};j;j+k})$. But the maps $\alpha^{d}_{M_1;\underline{x};j;j+k}$ are injective and therefore $\delta_j(\eta) = 0$. In particular, $\eta = \pi_j(\eta')$ for some $\eta' \in H^{d-1}(\underline{x}^j; M_2)$. The maps π_{j+k} are all injective. Therefore $\eta' \in \operatorname{Ker}(\alpha^{d-1}_{M_1;\underline{x};j;j+k})$ and it follows that $\alpha^{d-1}_{M_2;\underline{x};j;j+k}(\eta) = 0$ whenever $k \geq \operatorname{lcb}_{d-1}(\underline{x}; M_2)$. Therefore $\operatorname{lcb}_{d-1}(\underline{x}; M_2) \geq \operatorname{lcb}_{d-1}(\underline{x}; M_3)$ and hence $\operatorname{lcb}_{d-1}(\underline{x}; M_2) = \operatorname{lcb}_{d-1}(\underline{x}; M_3)$. This completes the proof of (1). □

4. Equality of test ideals

Theorem A is a consequence of Theorem 4.2 and Theorem 4.9. Theorem 4.2 is an explicit relationship between local cohomology bounds and equality of test ideals. Theorem 4.9, when paired with Proposition 4.8, provides the needed local cohomology bounds described in Theorem 4.9 whenever we are able to linearly compare the annihilators of $H^i_{\mathfrak{m}}(R/I^{(n)})$ as $n \to \infty$ and $I \cong \omega_R^{(-m)}$ is a multiple of an anti-canonical ideal.

4.1. Local cohomology bounds and equality of test ideals

The content of the following lemma can be pieced together by work of the first author in [2]. We refer the reader to [24, Lemma 6.7] for a direct presentation of the lemma.[6]

Lemma 4.1. *Suppose that $(R, \mathfrak{m}, k)$ is a Cohen-Macaulay local normal domain of dimension d, and $J \subsetneq R$ an ideal of pure height 1. Let $x_1, \ldots, x_d \in R$ be a system of parameters for R, assume that $x_1 \in J$, and fix $e \in \mathbb{N}$.*

(1) *If $x_2 J \subseteq a_2 R$ for some $a_2 \in J$, then for any non-negative integers $N_2, \ldots, N_d$ with $N_2 \geq 2$, we have that*

$$\begin{aligned} &((J^{(p^e)}, x_2^{N_2 p^e}, x_3^{N_3 p^e}, \ldots, x_d^{N_d p^e}) : x_2^{(N_2-1)p^e}) \\ =\ &((J^{[p^e]}, x_2^{N_2 p^e}, x_3^{N_3 p^e}, \ldots, x_d^{N_d p^e}) : x_2^{(N_2-1)p^e}) \\ =\ &((J^{[p^e]}, x_2^{2p^e}, x_3^{N_3 p^e}, \ldots, x_d^{N_d p^e}) : x_2^{p^e}). \end{aligned}$$

(2) *Suppose $x_3^m J^{(m)} \subseteq a_3 R \subseteq J^{(m)}$ for some $m \in \mathbb{N}$, then for any non-negative integers $N_2, \ldots, N_d$ with $N_3 \geq 2$, we have that*

$$\begin{aligned} &((J^{(p^e)}, x_2^{N_2 p^e}, x_3^{N_3 p^e} \ldots, x_d^{N_d p^e}) : x_3^{(N_3-1)p^e}) \\ \subseteq\ &((J^{(p^e)}, x_2^{N_2 p^e}, x_3^{2p^e}, \ldots, x_d^{N_d p^e}) : x_1^m x_3^{p^e}). \end{aligned}$$

[6] In [24, Lemma 6.7] there is an assumption that R is complete. The lemma claims equality among certain colon ideals, and equality of ideals can be checked after completion as $R \to \widehat{R}$ is faithfully flat.

Theorem 4.2. *Let $(R, \mathfrak{m}, k)$ be a local normal Cohen-Macaulay domain of Krull dimension d and of prime characteristic $p > 0$. Assume that R has a test element. Let $J_1 \subseteq R$ be a choice of canonical ideal. Suppose $x_1, \dots, x_d$ is a system of parameters of R, $x_1 \in J_1$, and suppose that the following conditions are met:*

- *There exists element $a_2 \in J_1$ and $a_3 \in J_1^{(m)}$ such that $x_2 J_1 \subseteq a_2 R$ and $x_3^m J_1^{(m)} \subseteq a_3 R$[7];*
- *For each $e \in \mathbb{N}$ there exists an integer ℓ such that*

$$\mathrm{lcb}_{d-1}(x_2^\ell, x_3^\ell, x_4, \dots, x_d; R/J_1^{(mp^e+1)}) \leq p^e + 1.$$

Then $0^{*fg}_{E_R(k)} = 0^*_{E_R(k)}$.

Proof. Identify $E_R(k)$ as

$$E_R(k) \cong \varinjlim \left(\frac{R}{(x_1^{t-1}J_1, x_2^t, \dots, x_d^t)} \xrightarrow{\cdot x_1 x_2 \cdots x_d} \frac{R}{(x_1^t J_1, x_2^{t+1}, \dots, x_d^{t+1})} \right).$$

Suppose that $\eta = [r + (x_1^{t-1}J_1, x_2^t, \dots, x_d^t)] \in 0^*_{E_R(k)}$. Equivalently, there exists a $c \in R^\circ$ such that for all $e \in \mathbb{N}$

$$\begin{aligned} 0 = c\eta^{p^e} &= [cr^{p^e} + (x_1^{t-1}J_1, x_2^t, \dots, x_d^t)^{[p^e]}] \in F^e(E_R(k)) \\ &\cong \varinjlim \left(\frac{R}{(x_1^{t-1}J_1, x_2^t, \dots, x_d^t)^{[p^e]}} \right). \end{aligned}$$

Let $J = x_1^{t-1}J_1$ and consider the local cohomology module

$$H^{d-1}_{\mathfrak{m}}\left(\frac{R}{J^{[p^e]}}\right) = \varinjlim \left(\frac{R}{J^{[p^e]} + (x_2^t, \dots, x_d^t)} \xrightarrow{\cdot x_2 \cdots x_d} \frac{R}{J^{[p^e]} + (x_2^{t+1}, \dots, x_d^{t+1})} \right).$$

Claim 4.3. *Let*

$$\alpha^{p^e} = [r^{p^e} + (x_2^{tp^e}, \dots, x_d^{tp^e})] \in H^{d-1}_{\mathfrak{m}}\left(\frac{R}{J^{[p^e]}}\right),$$

then $c\alpha^{p^e} = [cr^{p^e} + (x_2^t, \dots, x_d^t)^{[p^e]}]$ is the 0-element of $H^{d-1}_{\mathfrak{m}}(R/J^{[p^e]})$.

[7] This property is automatic if R_P is $\mathbb{Q}$-Gorenstein for each height 2 prime ideal $P \in \mathrm{Spec}(R)$. Recall that a local normal Cohen-Macaulay domain R with canonical ideal $J \subseteq R$ is $\mathbb{Q}$-Gorenstein if there exists a $m \geq 1$ such that $J^{(m)}$ is a principal ideal. If W_1 is the complement of the union of the minimal components of J_1, then $J_1 R_{W_1}$ is principally generated by an element $a_2 \in J_1$, hence x_2 can be chosen with the property $x_2 J_1 \subseteq (a_2) \subseteq J_1$. If W_2 is then the complement of the union of the minimal components of (J_1, x_2) then R_{W_2} is a semi-local domain. Hence we can choose an m so that $J^{(m)} R_{W_2}$ is principally generated by an element $a_3 \in J_1^{(m)}$ by [18, Theorem 60]. There then exists parameter element x_3 so that $x_3 J^{(m)} \subseteq (a_3) \subseteq J^{(m)}$. We opt to use the containment $x_3^m J^{(m)} \subseteq (a_3) \subseteq J^{(m)}$ to ease computational complexity of the proof.

Proof of Claim. The element $[cr^{p^e} + (x_1^{t-1}J_1, x_2^t, \ldots, x_d^t)^{[p^e]}]$ is the 0-element of

$$\varinjlim \left(\frac{R}{(x_1^{t-1}J_1, x_2^t, \ldots, x_d^t)^{[p^e]}} \right).$$

Therefore there exists an $s \in \mathbb{N}$ such that

$$\begin{aligned} cr^{p^e}(x_1x_2\cdots x_d)^{sp^e} &\in (x_1^{t+s-1}J_1, x_2^{t+s}, \ldots, x_d^{t+s})^{[p^e]} \\ &= (x_1^{(t+s-1)p^e}J_1^{[p^e]}, x_2^{(t+s)p^e}, \ldots, x_d^{(t+s)p^e}). \end{aligned}$$

So there exists an element $j_1 \in J_1^{[p^e]}$ such that

$$cr^{p^e}(x_1x_2\cdots x_d)^{sp^e} - x_1^{(t+s-1)p^e}j_1 \in (x_2^{(t+2)p^e}, \ldots, x_d^{(t+s)p^e}).$$

The sequence $x_1, x_2, \ldots, x_d$ is a regular sequence and so

$$cr^{p^e}(x_2\cdots x_d)^{sp^e} - x_1^{(t-1)p^e}j_1 \in (x_2^{(t+s)p^e}, \ldots, x_d^{(t+s)p^e}).$$

Hence

$$cr^{p^e}(x_2\cdots x_d)^{sp^e} \in (x_1^{(t-1)p^e}J_1^{[p^e]}, x_2^{(t+s)p^e}, \ldots, x_d^{(t+s)p^e}) = (J^{[p^e]}, x_2^{(t+s)p^e}, \ldots, x_d^{(t+s)p^e}),$$

which proves the claim. $\square$

Choose $e_0 \in \mathbb{N}_{\geq 1}$ so that $p^e \geq mp^{e-e_0} + 1$ for all $e \gg 0$. If $e \gg 0$ then

$$J^{[p^e]} \subseteq J^{(p^e)} \subseteq J^{(mp^{e-e_0}+1)}.$$

Fix $e \gg 0$ and consider the local cohomology module

$$H_{\mathfrak{m}}^{d-1}\left(\frac{R}{J^{(mp^{e-e_0}+1)}}\right) \cong \varinjlim \left(\frac{R}{(J^{(mp^{e-e_0}+1)}, x_2^t, \ldots, x_d^t)} \right).$$

Let $\tilde{\alpha}^{p^e}$ denote the image of α^{p^e} in $H_{\mathfrak{m}}^{d-1}(R/J^{(mp^{e-e_0}+1)})$ induced by the projection $R/J^{[p^e]} \to R/J^{(mp^{e-e_0}+1)}$. By Claim 4.3

$$\begin{aligned} 0 = c\tilde{\alpha}^{p^e} &= [cr^{p^e} + (x_2^{tp^e}, \ldots, x_d^{tp^e})] \in H_{\mathfrak{m}}^{d-1}\left(\frac{R}{J^{(mp^{e-e_0}+1)}}\right) \\ &\cong \varinjlim \left(\frac{R}{J^{(mp^{e-e_0}+1)} + (x_2^t, \ldots, x_d^t)} \right). \end{aligned}$$

There are short exact sequences

$$0 \to \frac{R}{J_1^{(mp^{e-e_0}+1)}} \xrightarrow{\cdot x_1^{(t-1)(mp^{e-e_0}+1)}} \frac{R}{J^{(mp^{e-e_0}+1)}} \to \frac{R}{x_1^{(t-1)(mp^{e-e_0}+1)}R} \to 0.$$

Let ℓ be a choice of integer, which depends on $e-e_0$, as in the statement of the theorem. The sequence $x_2^\ell, x_3^\ell, x_4, \dots, x_d$ is a regular sequence on $R/x_1^{(t-1)(mp^{e-e_0}+1)}R$. By (3) of Proposition 3.7 we have that

$$\mathrm{lcb}_{d-1}(x_2^\ell, x_3^\ell, x_4, \dots, x_d; R/J_1^{(mp^{e-e_0}+1)}) = \mathrm{lcb}_{d-1}(x_2^\ell, x_3^\ell, x_4, \dots, x_d; R/J^{(mp^{e-e_0}+1)}),$$

and so by assumption

$$\mathrm{lcb}_{d-1}(x_2^\ell, x_3^\ell, x_4, \dots, x_d; R/J^{(mp^{e-e_0}+1)}) \le p^{e-e_0}+1 \le p^e.$$

Recall that

$$0 = [cr^{p^e} + (x_2^{tp^e}, \dots, x_d^{tp^e})] = [cr^{p^e}(x_2^t x_3^t)^{(\ell-1)p^e} + (x_2^{t\ell p^e}, x_3^{t\ell p^e}, x_4^{tp^e} \dots, x_d^{tp^e})]$$

as an element of $H_{\mathfrak{m}}^{d-1}(R/J^{(mp^{e-e_0}+1)})$. By Lemma 3.3,

$$\begin{aligned}\mathrm{lcb}_{d-1}(x_2^{t\ell}, x_3^{t\ell}, x_4^t, \dots, x_d^t; R/J^{(mp^{e-e_0}+1)}) &\le \mathrm{lcb}_{d-1}(x_2^\ell, x_3^\ell, x_4, \dots, x_d; R/J^{(mp^{e-e_0}+1)}) \\ &\le p^e,\end{aligned}$$

so

$$c(rx_4^t \cdots x_d^t)^{p^e}(x_2^t x_3^t)^{\ell p^e} \in (J^{(mp^{e-e_0}+1)}, x_2^{t(\ell+1)p^e}, x_3^{t(\ell+1)p^e}, x_4^{2tp^e}, \dots, x_d^{2tp^e}). \tag{4.1}$$

Notice that $x_1^{tp^{e_0}} \in J^{(p^{e_0})}$ and so

$$x_1^{tp^{e_0}} J^{(mp^{e-e_0}+1)} \subseteq x_1^{tp^{e_0}} J^{(p^{e-e_0})} \subseteq J^{(p^e)}.$$

We therefore multiply the containment (4.1) by $x_1^{tp^{e_0}}$ and obtain that

$$cx_1^{tp^{e_0}}(rx_4^t \cdots x_d^t)^{p^e}(x_2^t x_3^t)^{\ell p^e} \in (J^{(p^e)}, x_2^{t(\ell+1)p^e}, x_3^{t(\ell+1)p^e}, x_4^{2tp^e}, \dots, x_d^{2tp^e}).$$

Therefore

$$cx_1^{tp^{e_0}}(rx_4^t \cdots x_d^t)^{p^e}(x_2^t)^{\ell p^e} \in (J^{(p^e)}, (x_2^t)^{(\ell+1)p^e}, (x_3^t)^{(\ell+1)p^e}, x_4^{2tp^e}, \dots, x_d^{2tp^e}) : x_3^{t\ell p^e}.$$

We utilize the assumption that $x_3^m J_1^{(m)} \subseteq a_2 R \subseteq J_1^{(m)}$ to conclude that;

$$x_3^m J^{(m)} = x_3^m (x_1^{t-1} J_1)^{(m)} = x_1^{(t-1)m} x_3^m J_1^{(m)} \subseteq x_1^{(t-1)m} a_2 R \subseteq x_1^{(t-1)m} J_1^{(m)} \subseteq J^{(m)}.$$

Therefore $x_3^{tm} J^{(m)} \subseteq x_1^{(t-1)m} a_2 R \subseteq J^{(m)}$ and we apply (2) of Lemma 4.1 with respect to x_3^t and $N_3 = \ell + 1$ to conclude that

$$cx_1^{tp^{e_0}}(rx_4^t\cdots x_d^t)^{p^e}(x_2^t)^{\ell p^e} \in (J^{(p^e)}, (x_2^t)^{(\ell+1)p^e}, x_3^{2tp^e}, x_4^{2tp^e}, \dots, x_d^{2tp^e}) : x_1^{tm}x_3^{tp^e}.$$

Equivalently,

$$cx_1^{t(m+p^{e_0})}(rx_3^tx_4^t\cdots x_d^t)^{p^e}(x_2^t)^{\ell p^e} \in (J^{(p^e)}, (x_2^t)^{(\ell+1)p^e}, x_3^{2tp^e}, x_4^{2tp^e}, \dots, x_d^{2tp^e}).$$

Similarly, we are able to apply (1) of Lemma 4.1 with respect to the element x_2^t and obtain that

$$\begin{aligned} cx_1^{t(m+p^{e_0})}(rx_2^tx_3^tx_4^t\cdots x_d^t)^{p^e} &\in (J^{[p^e]}, x_2^{2tp^e}, x_3^{2tp^e}, x_4^{2tp^e}, \dots, x_d^{2tp^e}) \\ &= (J, x_2^{2t}, x_3^{2t}, \dots, x_d^{2t})^{[p^e]}. \end{aligned}$$

The element $cx_1^{t(m+p^{e_0})}$ does not depend on e and therefore

$$rx_2^tx_3^tx_4^t\cdots x_d^t \in (J, x_2^{2t}, x_3^{2t}, \dots, x_d^{2t})^*.$$

In particular,

$$\eta = [r + (x_1^{t-1}J_1, x_2^t, \dots, x_d^t)] = [rx_2^tx_3^tx_4^t\cdots x_d^t + (J, x_2^{2t}, \dots, x_d^{2t})] \in 0^{*fg}_{E_R(k)}$$

as claimed. □

4.2. S_2-ification, higher Ext-modules, and local cohomology

We begin with two lemmas that experts may already be aware of.

Lemma 4.4. *Let $(S, \mathfrak{m}, k)$ be a Cohen-Macaulay local domain and M a finitely generated S-module such that $\operatorname{ht}(\operatorname{Ann}_S(M)) = h$. Then $\operatorname{Ext}_S^h(M, S)$ is an (S_2)-module over its support.*

Proof. Let $(F_\bullet, \partial_\bullet)$ be the minimal free resolution of M, let $(-)^*$ denote $\operatorname{Hom}_S(-, S)$, and consider the dual complex $(F_\bullet^*, \partial_\bullet^*)$. Because $\operatorname{ht}(\operatorname{Ann}_S(M)) = h$ we have that the following complex is exact:

$$0 \to F_0^* \xrightarrow{\partial_1^*} F_1^* \to \dots \to F_{h-1}^* \xrightarrow{\partial_h^*} F_h^* \to \operatorname{Coker}(\partial_h^*) \to 0.$$

In particular, $\operatorname{depth}(\operatorname{Coker}(\partial_h^*)) = d - h$. Moreover, there is a short exact sequence

$$0 \to \operatorname{Ext}_S^h(M, S) \to \operatorname{Coker}(\partial_h^*) \to \operatorname{Im}(\partial_{h+1}^*) \to 0.$$

The module $\operatorname{Im}(\partial_{h+1}^*)$ is torsion-free and therefore has depth at least 1. If $d - h \geq 2$ then $\operatorname{Ext}_S^h(M, S)$ has depth at least 2. If $d - h = 1$ then the depth of $\operatorname{Ext}_S^h(M, S)$ is 1. If $d - h = 0$ then M is 0-dimensional. Therefore if $\operatorname{ht}(\operatorname{Ann}_S(M)) = h$ then $\operatorname{Ext}_S^h(M, S)$ is an (S_2)-module over its support. □

Continue to consider the ring S, the module M, and the resolution $(F_\bullet, \partial_\bullet)$ as above. Suppose further S is a regular local ring and hence every finitely generated S-module has a finite free resolution. Consider the minimal free resolution $(G_\bullet, \delta_\bullet)$ of $\mathrm{Ext}^h_S(M, S)$. If $\mathrm{depth}(M) = d - h$ is maximal, then $\mathrm{Ext}^h_S(M, S) = \mathrm{Coker}(\partial^*_h)$ and therefore $(G_\bullet, \delta_\bullet)$ is the complex

$$0 \to F^*_0 \xrightarrow{\partial^*_1} F^*_1 \to \ldots \to F^*_{h-1} \xrightarrow{\partial^*_h} F^*_h \to 0.$$

In particular, if $\mathrm{depth}(M) = d-h$ then $\mathrm{Ext}^h_S(\mathrm{Ext}^h_S(M, S), S) \cong M$. Suppose $\mathrm{depth}(M) < d - h$ and let $(F^*_\bullet, \partial^*_\bullet)_{tr}$ be the complex obtained by truncating $(F^*_\bullet, \partial^*_\bullet)$ at the hth spot. That is $(F^*_\bullet, \partial^*_\bullet)_{tr}$ is the minimal free resolution of $\mathrm{Coker}(\partial^*_h)$. Then the natural inclusion $\mathrm{Ext}^h_S(M, S) \subseteq \mathrm{Coker}(\partial^*_h)$ lifts to a map of complexes $(G_\bullet, \delta_\bullet) \to (F^*_\bullet, \partial^*_\bullet)_{tr}$ and therefore there is an induced natural map $M \to \mathrm{Ext}^h_S(\mathrm{Ext}^h_S(M, S), S)$. The map $M \to \mathrm{Ext}^h_S(\mathrm{Ext}^h_S(M, S), S)$ is an isomorphism whenever M is a (maximal) Cohen-Macaulay module over its support.

Lemma 4.5. *Let $(R, \mathfrak{m}, k)$ be a complete local normal domain of Krull dimension $d \geq 1$ and $J \subseteq R$ a pure height 1 ideal. Suppose $(S, \mathfrak{n}, k)$ is a regular local ring mapping onto R, $R \cong S/P$, and $\mathrm{ht}(P) = h$. Then for every integer i the kernel of the natural map $R/J^i \to \mathrm{Ext}^{h+1}_S(\mathrm{Ext}^{h+1}_S(R/J^i, S), S)$ is $J^{(i)}/J^i$. In particular, for every integer i there is a natural inclusion $R/J^{(i)} \subseteq \mathrm{Ext}^{h+1}_S(\mathrm{Ext}^{h+1}_S(R/J^i, S), S)$. Moreover, the natural inclusion $R/J^{(i)} \subseteq \mathrm{Ext}^{h+1}_S(\mathrm{Ext}^{h+1}_S(R/J^i, S), S)$ is an isomorphism whenever localized at prime ideal $\mathfrak{p} \in V(J)$ such that $(R/J^{(i)})_\mathfrak{p}$ is Cohen-Macaulay.*

Proof. Let $L_i \subseteq R$ be the ideal of R, containing J^i, so that L_i/J^i is the kernel of $R/J^i \to \mathrm{Ext}^{h+1}_S(\mathrm{Ext}^{h+1}_S(R/J^i, S), S)$. Then $R/L_i \subseteq \mathrm{Ext}^{h+1}_S(\mathrm{Ext}^{h+1}_S(R/J^i, S), S)$. If P is a prime component of J then R_P/J^iR_P is 0-dimensional and therefore Cohen-Macaulay. By the above the discussion, the map under analysis is an isomorphism in the Cohen-Macaulay locus, and therefore

$$R_P/J^iR_P = R_P/J^{(i)}R_P = R_P/L_{iP} = \mathrm{Ext}^{h+1}_{S_P}(\mathrm{Ext}^{h+1}_{S_P}(R_P/J^{(i)}R_P, S_P), S_P)$$

at prime P which are minimal components of J.

If P is a prime of R of height 1 which is not a component of J, then $R_P/J^iR_P = 0$ and the identifications above remain true. Therefore the height 1 components of the ideal of L_i are precisely the height 1 components of J^i, i.e. $J^{(i)}$. To conclude that $L_i = J^{(i)}$ it remains to show that the ideal L_i does not have embedded components. The module $\mathrm{Ext}^{h+1}_S(\mathrm{Ext}^{h+1}_S(R/J^i, S), S)$ is an (S_2) R/J^i-module and R/L_i is a submodule. Therefore R/L_i is an (S_1)-module and hence L_i cannot have an embedded component.

We have proven the first claim of the lemma that for each $i \in \mathbb{N}$ there is a natural inclusion $R/J^{(i)} \subseteq \mathrm{Ext}^{h+1}_S(\mathrm{Ext}^{h+1}_S(R/J^i, S), S)$. It remains to check that this inclusion is an isomorphism whenever localized at prime ideal $\mathfrak{p} \in V(J)$ such that $(R/J^{(i)})_\mathfrak{p}$ is Cohen-Macaulay. Indeed, $R/J^i \to R/J^{(i)}$ induces an isomorphism

$$\operatorname{Ext}_S^{h+1}(R/J^{(i)}, S) \xrightarrow{\cong} \operatorname{Ext}_S^{h+1}(R/J^i, S)$$

as $J^{(i)}/J^i$ is not supported at any height $h+1$ component of S. Therefore the inclusion $R/J^{(i)} \subseteq \operatorname{Ext}_S^{h+1}(\operatorname{Ext}_S^{h+1}(R/J^i, S), S)$ is the same as

$$R/J^{(i)} \to \operatorname{Ext}_S^{h+1}(\operatorname{Ext}_S^{h+1}(R/J^{(i)}, S), S)$$

and this map is an isomorphism in the Cohen-Macaulay locus by the discussion preceding the lemma. $\square$

We record a corollary of Lemma 4.5 for future reference.

Corollary 4.6. *Let $(R, \mathfrak{m}, k)$ be a complete local normal domain, $\mathbb{Q}$-Gorenstein in codimension 2, and $J_1 \subsetneq R$ a choice of canonical ideal. Let $m \in \mathbb{N}$ be an integer such that $J_1^{(m)}$ is principal in codimension 2. Suppose $(S, \mathfrak{n}, k)$ is a regular local ring mapping onto R, $R \cong S/P$, and $\operatorname{ht}(P) = h$. Then for every integer i the natural inclusion $R/J_1^{(mi+1)} \to \operatorname{Ext}_S^{h+1}(\operatorname{Ext}_S^{h+1}(R/J_1^{mi+1}, S), S)$ is an isomorphism whenever localized at a prime ideal of R of height 2 or less.*

Proof. If $\mathfrak{p}$ is prime ideal of height 2 or less then $R_\mathfrak{p}$ is Cohen-Macaulay and hence $R_\mathfrak{p}/J_1R_\mathfrak{p}$ is Gorenstein of dimension at most 1. The Corollary follows by Lemma 4.5 as $J_1^{(mi+1)}R_\mathfrak{p} \cong J_1R_\mathfrak{p}$ is a canonical ideal whenever $\mathfrak{p}$ is a prime of R of height 2 or less. $\square$

The next proposition and theorem provide the linear bound of top local cohomology bounds of the family of R-modules $\left\{R/J_1^{(mp^e+1)}\right\}$ described in Theorem 4.2 whenever there exists an ideal $I \subseteq R$ of pure height 1 and parameters $x_1, \ldots, x_d$ with the following properties:

(1) $I \cong \omega_R^{(-h)}$ for some $h \geq 1$ and I is principal in codimension 2;
(2) For each $1 \leq j \leq d-2$, the ideal $\mathfrak{a}_j := (x_1, \ldots, x_{d-j+1})$ is such that

$$\mathfrak{a}_j^{[p^e]} H_\mathfrak{m}^j(R/I^{(p^e)}) = 0$$

for each $e \in \mathbb{N}$.

We first provide a lemma. In the following lemma we let $(-)^\vee$ denote the Matlis dual functor.

Lemma 4.7. *Let $(R, \mathfrak{m}, k)$ be a local normal Cohen-Macaulay domain of Krull dimension d and $\mathbb{Q}$-Gorenstein in codimension 2. Assume that R has a test element. Let $J_1 \subseteq R$ be a choice of canonical ideal and $m \in \mathbb{N}$ such that $J_1^{(m)}$ is principal in codimension 2. Suppose S is a regular local ring of Krull dimension $d+h$ mapping onto R, $R \cong S/P$,* *and* $\mathrm{ht}(P) = h$. *Suppose that* $I_1 \subseteq R$ *is an ideal of pure height* 1 *with components disjoint from those of* J_1, $I_1 \cap J_1 = x_1 R$ *is principal. Then*

$$H^{j-1}_{\mathfrak{m}}\left(\frac{R}{I_1^{(mi)}}\right) \cong \left(\mathrm{Ext}^{d+h-j}_S\left(\mathrm{Ext}^{h+1}_S\left(\frac{R}{J_1^{(mi+1)}}, S\right), S\right)\right)^{\vee}$$

for all $j \le d-2$.

Proof. If $j \le 0$ then $H^{j-1}_{\mathfrak{m}}\left(\frac{R}{I_1^{(mi)}}\right) = \mathrm{Ext}^{d+h-j}_S\left(\mathrm{Ext}^{h+1}_S\left(\frac{R}{J_1^{(mi+1)}}, S\right), S\right) = 0$. In particular, we may assume that $d \ge 3$. There are isomorphisms

$$\mathrm{Ext}^{h+1}_S\left(\frac{R}{J_1^{(mi+1)}}, S\right) \cong \omega_{R/J_1^{(mi+1)}} \cong \mathrm{Ext}^1_R\left(\frac{R}{J_1^{(mi+1)}}, J_1\right). \tag{4.2}$$

Consider the short exact sequences

$$0 \to J_1^{(mi+1)} \to R \to \frac{R}{J_1^{(mi+1)}} \to 0.$$

The ring R is Cohen-Macaulay. Therefore $\mathrm{Ext}^1_R(R, J_1) = 0$ and there is a resulting short exact sequence

$$0 \to J_1 \to \mathrm{Hom}_R(J_1^{(mi+1)}, J_1) \to \mathrm{Ext}^1_R\left(\frac{R}{J_1^{(mi+1)}}, J_1\right) \to 0. \tag{4.3}$$

But $I_1 \cap J_1$ is principal, I_1 and J_1 have disjoint components, therefore $J_1 \cong I_1^{(mi)} \cap J_1^{(mi+1)}$ and so

$$\mathrm{Hom}_R(J_1^{(mi+1)}, J_1) \cong \mathrm{Hom}_R(J_1^{(mi+1)}, I_1^{(mi)} \cap J_1^{(mi+1)}) \cong I_1^{(mi)}. \tag{4.4}$$

The ideal J_1 is a maximal Cohen-Macaulay R-module and so $\mathrm{Ext}^{\ge h+1}_S(J_1, S) = 0$. Therefore by (4.2), (4.3), and (4.4), if $j \le d-2$ then

$$\mathrm{Ext}^{d+h-j}_S\left(\mathrm{Ext}^{h+1}_S\left(\frac{R}{J_1^{(mi+1)}}, S\right), S\right) \cong \mathrm{Ext}^{d+h-j}_S\left(I_1^{(mi)}, S\right).$$

Consider the short exact sequence

$$0 \to I_1^{(mi)} \to R \to \frac{R}{I_1^{(mi)}} \to 0.$$

Then

$$\mathrm{Ext}_S^{d+h-j}\left(I_1^{(mi)}, S\right) \cong \mathrm{Ext}_S^{d+h-(j-1)}\left(\frac{R}{I_1^{(mi)}}, S\right).$$

An application of Matlis duality now completes the proof as

$$\left(\mathrm{Ext}_S^{d+h-(j-1)}\left(\frac{R}{I_1^{(mi)}}, S\right)\right)^{\vee} \cong H_{\mathfrak{m}}^{j-1}\left(\frac{R}{I_1^{(mi)}}\right). \quad \square$$

Proposition 4.8. *Let $(R, \mathfrak{m}, k)$ be a local normal Cohen-Macaulay domain of Krull dimension d and $\mathbb{Q}$-Gorenstein in codimension 2. Let $p > 0$ be a natural number. Let $J_1 \subseteq R$ be a choice of canonical ideal and $m \in \mathbb{N}$ such that $J_1^{(m)}$ is principal in codimension 2. Suppose S is a regular local ring mapping onto R, $R \cong S/P$, and $\mathrm{ht}(P) = h$. Suppose that $I_1 \subseteq R$ is an ideal of pure height 1 with components disjoint from those of J_1, $I_1 \cap J_1 = x_1 R$ is principal, and parameters $x_1, x_2 \ldots, x_d$ with the property that for each $1 \leq j \leq d-2$, the parameter ideal $(x_2, \ldots, x_{d-j+1})$ is such that*

$$(x_2^{p^e}, \ldots, x_{d-j+1}^{p^e}) H_{\mathfrak{m}}^j(R/I_1^{(mp^e)}) = 0$$

for each $e \in \mathbb{N}$. Then for all $e \in \mathbb{N}$

$$\mathrm{lcb}_{d-1}(x_2^{d-3}, x_3^{d-3}, \ldots, x_d^{d-3}; \mathrm{Ext}_S^{h+1}(\mathrm{Ext}_S^{h+1}(R/J_1^{(mp^e+1)}, S), S)) \leq p^e.$$

Proof. Let $(F_\bullet, \partial_\bullet)$ be the minimal free S-resolution of $\mathrm{Ext}_S^{h+1}(R/J_1^{(mp^e+1)}, S)$. Denote by $(-)^*$ the functor $\mathrm{Hom}_S(-, S)$ and consider the dualized complex $(F_\bullet^*, \partial_\bullet^*)$. For every $j \geq 1$ there are short exact sequences

$$0 \to \mathrm{Ext}_S^{h+j}(\mathrm{Ext}_S^{h+1}(R/J_1^{(mp^e+1)}, S), S) \to \mathrm{Coker}(\partial_{h+j}^*) \to \mathrm{Im}(\partial_{h+j+1}^*) \to 0$$

and

$$0 \to \mathrm{Im}(\partial_{h+j+1}^*) \to F_{h+j+1}^* \to \mathrm{Coker}(\partial_{h+j+1}^*) \to 0.$$

Let $\mathcal{J}_e$ denote the preimage of $J_1^{(mp^e+1)}$ in S, an ideal of height $h+1$. The S-module $\mathrm{Coker}(\partial_{h+1}^*)$ has projective dimension $h+1$ and the ideal $\mathcal{J}_e$ annihilates the submodule $\mathrm{Ext}_S^{h+1}(\mathrm{Ext}_S^{h+1}(R/J_1^{(mp^e+1)}, S), S)$. By prime avoidance, and abuse of notation, we may lift $\underline{x} = x_2, \ldots, x_d$ to elements of S and assume that $\underline{x}$ is a regular sequence on $\mathrm{Coker}(\partial_{h+1}^*)$ and the free S-modules F_i^*.

The module $\mathrm{Ext}_S^{h+1}(R/J_1^{(mp^e+1)}, S)$ is an (S_2)-module over its support, see Lemma 4.4. In particular,

$$\mathrm{Ext}_S^{h+d}(\mathrm{Ext}_S^{h+1}(R/J_1^{(mp^e+1)}, S), S) = \mathrm{Ext}_S^{h+d-1}(\mathrm{Ext}_S^{h+1}(R/J_1^{(mp^e+1)}, S), S) = 0$$

and

$$\mathrm{Coker}(\partial^*_{h+d-2}) \cong \mathrm{Ext}^{h+d-2}_S(\mathrm{Ext}^{h+1}_S(R/J_1^{(mp^e+1)}, S), S).$$

Consider the short exact sequence

$$0 \to \mathrm{Im}(\partial^*_{h+d-2}) \to F^*_{h+d-2} \to \mathrm{Ext}^{h+d-2}_S(\mathrm{Ext}^{h+1}_S(R/J_1^{(mp^e+1)}, S), S) \to 0.$$

By our assumptions and by Lemma 4.7,

$$(x_2^{p^e}, x_3^{p^e}, \dots, x_d^{p^e})\,\mathrm{Ext}^{h+d-2}_S(\mathrm{Ext}^{h+1}_S(R/J_1^{(mp^e+1)}, S), S) = 0$$

for every $e \in \mathbb{N}$. By (2) of Proposition 3.7

$$\mathrm{lcb}_2(\mathrm{Im}(\partial^*_{h+d-2})) = \mathrm{lcb}_1(\mathrm{Ext}^{h+d-2}_S(\mathrm{Ext}^{h+1}_S(R/J_1^{(mp^e+1)}, S), S)).$$

As $\underline{x}^{p^e}$ annihilates $\mathrm{Ext}^{h+d-2}_S(\mathrm{Ext}^{h+1}_S(R/J_1^{(mp^e+1)}, S), S))$,

$$\mathrm{lcb}_1(\mathrm{Ext}^{h+d-2}_S(\mathrm{Ext}^{h+1}_S(R/J_1^{(mp^e+1)}, S), S)) \le p^e$$

by Lemma 3.3 and Corollary 3.5.

Next, we consider the short exact sequence

$$0 \to \mathrm{Ext}^{h+d-3}_S(\mathrm{Ext}^{h+1}_S(R/J_1^{(mp^e+1)}, S), S) \to \mathrm{Coker}(\partial^*_{h+d-3}) \to \mathrm{Im}(\partial^*_{h+d-2}) \to 0.$$

We established $\mathrm{lcb}_2(\underline{x}; \mathrm{Im}(\partial^*_{h+d-2})) \le p^e$. By assumption and Lemma 4.7

$$(x_2^{p^e}, \dots, x_{d-1}^{p^e})\,\mathrm{Ext}^{h+d-3}_S(\mathrm{Ext}^{h+1}_S(R/J_1^{(mp^e+1)}, S), S) = 0$$

for every $e \in \mathbb{N}$. By (1) of Proposition 3.6 and Lemma 3.3 we have

$$\mathrm{lcb}_2(\underline{x}; \mathrm{Coker}(\partial^*_{h+d-3})) \le p^e + p^e = 2p^e.$$

Next consider the short exact sequence

$$0 \to \mathrm{Im}(\partial^*_{h+d-3}) \to F^*_{h+d-3} \to \mathrm{Coker}(\partial^*_{h+d-3}) \to 0.$$

By (2) of Proposition 3.7 and knowing that $\mathrm{lcb}_2(\underline{x}; \mathrm{Coker}(\partial^*_{h+d-3})) \le 2p^e$ we see that

$$\mathrm{lcb}_3(\underline{x}; \mathrm{Im}(\partial^*_{h+d-3})) \le 2p^e.$$

Inductively, we find that

$$\mathrm{lcb}_j(\underline{x}; \mathrm{Im}(\partial^*_{h+d-j})) \le (j-1)p^e$$

and

$$\mathrm{lcb}_j(\underline{x}; \mathrm{Coker}(\partial^*_{h+d-j-1})) \leq jp^e$$

for each $2 \leq j \leq d-2$. Now consider the short exact sequence

$$0 \to \mathrm{Ext}^{h+2}_S(\mathrm{Ext}^{h+1}_S(R/J_1^{(mp^e+1)}, S), S) \to \mathrm{Coker}(\partial^*_{h+2}) \to \mathrm{Im}(\partial^*_{h+3}) \to 0.$$

By induction, $\mathrm{lcb}_{d-3}(\mathrm{Im}(\partial^*_{h+3})) \leq (d-4)p^e$, therefore by (1) of Proposition 3.6

$$\mathrm{lcb}_{d-3}(\mathrm{Coker}(\partial^*_{h+2})) \leq (d-3)p^e.$$

Now consider the short exact sequence

$$0 \to \mathrm{Im}(\partial^*_{h+2}) \to F^*_{h+2} \to \mathrm{Coker}(\partial^*_{h+2}) \to 0.$$

Apply (2) of Proposition 3.7 to conclude $\mathrm{lcb}_{d-2}(\mathrm{Im}(\partial^*_{h+2})) \leq (d-3)p^e$. Now consider one last short exact sequence:

$$0 \to \mathrm{Ext}^{h+1}_S(\mathrm{Ext}^{h+1}_S(R/J_1^{(mp^e+1)}, S), S) \to \mathrm{Coker}(\partial^*_{h+2}) \to \mathrm{Im}(\partial^*_{h+2}) \to 0.$$

We now utilize that $\underline{x}$ is a regular sequence on $\mathrm{Coker}(\partial^*_{h+2})$ and utilize (2) of Proposition 3.7 to conclude that

$$\mathrm{lcb}_{d-1}(\mathrm{Ext}^{h+1}_S(\mathrm{Ext}^{h+1}_S(R/J_1^{(mp^e+1)}, S), S)) = \mathrm{lcb}_{d-2}(\mathrm{Im}(\partial^*_{h+2})) \leq (d-3)p^e.$$

By Lemma 3.3 the parameter sequence $\underline{x}^{d-3} = x_2^{d-3}, \ldots, x_d^{d-3}$ on R/J_1 satisfies

$$\mathrm{lcb}_{d-1}(\underline{x}^{d-1}; \mathrm{Ext}^{h+1}_S(\mathrm{Ext}^{h+1}_S(R/J_1^{(mp^e+1)}, S), S))) \leq p^e$$

for each $e \in \mathbb{N}$. $\square$

Theorem 4.9. *Let $(R, \mathfrak{m}, k)$ be a local normal Cohen-Macaulay domain of Krull dimension $d \geq 4$ and $\mathbb{Q}$-Gorenstein in codimension 2. Let $p > 0$ be a natural number. Let $J_1 \subseteq R$ be a choice of canonical ideal and $m \in \mathbb{N}$ such that $J_1^{(m)}$ is principal in codimension 2. Suppose S is a regular local ring mapping onto R, $R \cong S/P$, and $\mathrm{ht}(P) = h$. Suppose that $I_1 \subseteq R$ is an ideal of pure height 1 with components disjoint from those of J_1, $I_1 \cap J_1 = x_1R$ is principal, and parameters $x_1, x_2 \ldots, x_d$ with the following properties:*

1. *$J_1R_{x_2}$ and $J_1^{(m)}R_{x_3}$ are principal in their respective localizations;*
2. *For every $1 \leq j \leq d-2$, the parameter ideal $(x_1, \ldots, x_{d-j+1})$ is such that*

$$(x_2, \ldots, x_{d-j+1})^{p^e} H^j_{\mathfrak{m}}(R/I_1^{(mp^e)}) = 0$$

   for each $e \in \mathbb{N}$.

Then the following hold:

(1) *For each* $e \in \mathbb{N}$ *there exists* $\ell \in \mathbb{N}$ *such that*

$$\operatorname{lcb}_{d-1}(x_2^{\ell(d-1)}, x_3^{\ell(d-1)}, x_4^{d-1}, \ldots, x_d^{d-1}; R/J_1^{(mp^e+1)}) \leq p^e + 1;$$

(2) *For each* $e \in \mathbb{N}$ *there exists* $\ell \in \mathbb{N}$ *such that*

$$\operatorname{lcb}_{d-1}(x_2^{\ell(d-1)}, x_3^{\ell(d-1)}, x_4^{d-1}, \ldots, x_d^{d-1}; R/J_1^{mp^e+1}) \leq p^e + 2.$$

Proof. For each $e \in \mathbb{N}$ let C_e be the cokernel of

$$R/J^{mp^e+1} \to \operatorname{Ext}_S^{h+1}(\operatorname{Ext}_S^{h+1}(R/J_1^{mp^e+1}, S), S) \cong \operatorname{Ext}_S^{h+1}(\operatorname{Ext}_S^{h+1}(R/J_1^{(mp^e+1)}, S), S)$$

and consider the short exact sequences

$$0 \to R/J_1^{(mp^e+1)} \to \operatorname{Ext}_S^{h+1}(\operatorname{Ext}_S^{h+1}(R/J_1^{mp^e+1}, S), S) \to C_e \to 0,$$

see Lemma 4.5 for details.

By Lemma 4.5 the module C_e is 0 when either x_2 or x_3 is inverted. Hence for each $e \in \mathbb{N}$ there exists an integer ℓ such that $(x_2^\ell, x_3^\ell)C_e = 0$. Because $d \geq 4$ we have that $d - 1 \geq 3$ and (3) of Proposition 3.6 implies

$$\begin{aligned}&\operatorname{lcb}_{d-1}(x_2^{\ell(d-1)}, x_3^{\ell(d-1)}, x_4^{d-1}, \ldots, x_d^{d-1}; R/J_1^{(mp^e+1)})\\ &\quad\leq \operatorname{lcb}_{d-1}(x_2^{\ell(d-1)}, x_3^{\ell(d-1)}, x_4^{d-1}, \ldots, x_d^{d-1}; \operatorname{Ext}_S^{h+1}(\operatorname{Ext}_S^{h+1}(R/J_1^{mp^e+1}, S, S)) + 1.\end{aligned}$$

Statement (1) follows by Proposition 4.8.

To prove (2) let $K_e = J_1^{(mp^e+1)}/J_1^{mp^e+1}$ and consider the short exact sequences

$$0 \to K_e \to R/J_1^{mp^e+1} \to R/J_1^{(mp^e+1)} \to 0.$$

The module K_e is 0 when either x_2 or x_3 are inverted. Hence for each $e \in \mathbb{N}$ there exists an integer ℓ such that $(x_2^\ell, x_3^\ell)K_e = 0$. By (1) of Proposition 3.6 we have that

$$\begin{aligned}\operatorname{lcb}_{d-1}(x_2^\ell, x_3^\ell, x_4, \ldots, x_d; R/J_1^{mp^e+1}) &\leq \operatorname{lcb}_{d-1}(x_2^\ell, x_3^\ell, x_4, \ldots, x_d; R/J_1^{(mp^e+1)}) + 1\\ &\leq p^e + 2. \quad \square\end{aligned}$$

Theorem A is a consequence of the following theorem.

Theorem 4.10. *Let* $(R, \mathfrak{m}, k)$ *be an excellent local Cohen-Macaulay normal domain of prime characteristic* $p > 0$*, of Krull dimension* $d \geq 4$*,* $I_1 \subseteq R$ *an anti-canonical ideal, and* $E_R(k)$ *an injective hull of the residue field. Suppose further that there exists an*

$m \in \mathbb{N}$ so that $I_1^{(m)}$ is principal in codimension 2 and for each $1 \le j \le d-2$ there exists an ideal $\mathfrak{a}_j$ of height $d-j+1$ such that

$$\mathfrak{a}_j^{p^e} H^j_{\mathfrak{m}}\left(\frac{R}{I_1^{(mp^e)}}\right) = 0$$

for every $e \in \mathbb{N}$. Then $0^{*fg}_{E_R(k)} = 0^*_{E_R(k)}$.

Proof. Our strategy is to employ Theorem 4.9 and then Theorem 4.2 to conclude $0^{*fg}_{E_R(k)} = 0^*_{E_R(k)}$. But first, we change the ideals $\mathfrak{a}_j$, if necessary, so that there are inclusions $\mathfrak{a}_{d-2} \subseteq \mathfrak{a}_{d-3} \subseteq \cdots \subseteq \mathfrak{a}_1$ and so that there exists parameter elements $x_1^m, x_2, x_3 \in \mathfrak{a}_j$ for all $1 \le j \le d-2$ with the property that $x_1R = I_1 \cap J_1$ for some canonical ideal J_1 and the ideals $I_1R_{x_2}$ and $I_1^{(m)}R_{x_3}$ are principal in their respective localizations.

The ideal $\mathfrak{a}_j \cap \mathfrak{a}_{j-1} \cap \cdots \cap \mathfrak{a}_1$ has height at least $d-j+1$. We can replace the ideal $\mathfrak{a}_j$ with $\mathfrak{a}_j \cap \mathfrak{a}_{j-1} \cap \cdots \cap \mathfrak{a}_1$ and may assume that

$$\mathfrak{a}_{d-2} \subseteq \mathfrak{a}_{d-3} \subseteq \cdots \subseteq \mathfrak{a}_1.$$

Start by choosing $x_1 \in I_1$ a generic generator so that $x_1R = I_1 \cap J_1$ and the ideals I_1 and J_1 have disjoint components. Clearly $x_1^{mp^e}$ annihilates $H^j_{\mathfrak{m}}(R/I_1^{(mp^e)})$ for every $e \in \mathbb{N}$. The ideal I_1 is principal in codimension 1, the ideal $I_1^{(m)}$ is principal in codimension 2. Therefore there exists part of a system of parameters x_2, x_3 of R/x_1R so that $I_1R_{x_2}$ and $I_1^{(m)}R_{x_3}$ are principal in their respective localizations. Moreover, we can replace x_2 and x_3 by suitable powers and can assume that there exist elements $a, b \in I_1$ so that $x_2I_1 \subseteq aR \subseteq I_1$ and $x_3I_1^{(m)} \subseteq bR \subseteq I_1^{(m)}$. Therefore $x_2^{mp^e} I_1^{(mp^e)} \subseteq a^{mp^e}R \subseteq I_1^{(mp^e)}$ and $x_3^{p^e} I_1^{(mp^e)} \subseteq b^{p^e}R \subseteq I_1^{(mp^e)}$. Consider the short exact sequences

$$0 \to \frac{I_1^{(mp^e)}}{a^{mp^e}R} \to \frac{R}{a^{mp^e}R} \to \frac{R}{I_1^{(mp^e)}} \to 0$$

and

$$0 \to \frac{I_1^{(mp^e)}}{b^{p^e}R} \to \frac{R}{b^{mp^e}R} \to \frac{R}{I_1^{(mp^e)}} \to 0.$$

The elements $x_2^{mp^e}$ and $x_3^{p^e}$ annihilate $I_1^{(mp^e)}/a^{mp^e}R$ and $I_1^{(mp^e)}/b^{p^e}R$ respectively. Examining the resulting long exact sequences of local cohomology informs us that $x_2^{mp^e}$ and $x_3^{p^e}$ annihilate $H^j_{\mathfrak{m}}(R/I_1^{(mp^e)})$ for every $1 \le j \le d-2$. Replace the element x_2 by x_2^m. Then $(x_2^{p^e}, x_3^{p^e})$ annihilates $H^j_{\mathfrak{m}}(R/I_1^{(mp^e)})$ for every $1 \le j \le d-2$. For each $e \in \mathbb{N}$ the ideal $((\mathfrak{a}_j + (x_1^m, x_2, x_3))^{4p^e}$ is generated by elements which live in either $\mathfrak{a}_j^{p^e}$ or $(x_1^m, x_2, x_3)^{[p^e]}$ and therefore annihilate $H^j_{\mathfrak{m}}(R/I_1^{(mp^e)})$. We replace $\mathfrak{a}_j$ by the ideal $(\mathfrak{a}_j + (x_1^m, x_2, x_3))^4$.

The ideal $\mathfrak{a}_j$ has height at least $d-j+1$ and $x_1^m, x_2, x_3 \in \mathfrak{a}_j$. We can extend x_1^m, x_2, x_3 to a parameter sequence $x_1^m, x_2, x_3, \ldots, x_{d-j+1}$ in $\mathfrak{a}_j \subseteq \mathfrak{a}_{j-1} \subseteq \cdots \subseteq \mathfrak{a}_1$. By Theorem 4.9, for each $e \in \mathbb{N}$ there exists an ℓ so that

$$\operatorname{lcb}_{d-1}(x_2^{\ell(d-1)}, x_3^{\ell(d-1)}, x_d^{d-1}, \ldots, x_d^{d-1}; R/J_1^{(mi+1)}) \leq p^e + 1.$$

Therefore $0^{*fg}_{E_R(k)} = 0^*_{E_R(k)}$ by Theorem 4.2. $\square$

Acknowledgment

We are thankful to an anonymous referee for carefully reading our article and providing feedback which improved the article's exposition and content.

References

[1] I. Aberbach, T. Polstra, Local cohomology bounds and the weak implies strong conjecture in dimension 4, J. Algebra 605 (2022) 37–57.
[2] I.M. Aberbach, Some conditions for the equivalence of weak and strong F-regularity, Commun. Algebra 30 (4) (2002) 1635–1651.
[3] I.M. Aberbach, F. Enescu, Test ideals and base change problems in tight closure theory, Trans. Am. Math. Soc. 355 (2) (2003) 619–636.
[4] B. Bhatt, L. Ma, Z. Patakfalvi, K. Schwede, K. Tucker, J. Waldron, J. Witaszek, Globally +-regular varieties and the minimal model program for threefolds in mixed characteristic, 2020.
[5] H. Brenner, P. Monsky, Tight closure does not commute with localization, Ann. Math. (2) 171 (1) (2010) 571–588.
[6] M. Brodmann, Asymptotic stability of $\operatorname{Ass}(M/I^nM)$, Proc. Am. Math. Soc. 74 (1) (1979) 16–18.
[7] M.P. Brodmann, R.Y. Sharp, Local Cohomology, second edition, Cambridge Studies in Advanced Mathematics, vol. 136, Cambridge University Press, Cambridge, 2013, An algebraic introduction with geometric applications.
[8] W. Bruns, J. Herzog, Cohen-Macaulay Rings, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993.
[9] B. Cameron Maccrimmon, Strong F-Regularity and Boundedness Questions in Tight Closure, Thesis (Ph.D.)–University of Michigan, ProQuest LLC, Ann Arbor, MI, 1996.
[10] A. Chiecchio, F. Enescu, L.E. Miller, K. Schwede, Test ideals in rings with finitely generated anti-canonical algebras, J. Inst. Math. Jussieu 17 (1) (2018) 171–206.
[11] S. Dale Cutkosky, J. Herzog, H. Srinivasan, Asymptotic growth of algebras associated to powers of ideals, Math. Proc. Camb. Philos. Soc. 148 (1) (2010) 55–72.
[12] N. Hara, K.-I. Watanabe, F-regular and F-pure rings vs. log terminal and log canonical singularities, J. Algebraic Geom. 11 (2) (2002) 363–392.
[13] M. Hochster, Cyclic purity versus purity in excellent Noetherian rings, Trans. Am. Math. Soc. 231 (2) (1977) 463–488.
[14] M. Hochster, C. Huneke, Tight closure, invariant theory, and the Briançon-Skoda theorem, J. Am. Math. Soc. 3 (1) (1990) 31–116.
[15] M. Hochster, C. Huneke, Tight closure of parameter ideals and splitting in module-finite extensions, J. Algebraic Geom. 3 (4) (1994) 599–670.
[16] C. Huneke, On the associated graded ring of an ideal, Ill. J. Math. 26 (1) (1982) 121–137.
[17] C. Huneke, I. Smirnov, Prime filtrations of the powers of an ideal, Bull. Lond. Math. Soc. 47 (4) (2015) 585–592.
[18] I. Kaplansky, Commutative Rings, revised edition, University of Chicago Press, Chicago, Ill.-London, 1974.
[19] J. Lipman, Rational singularities, with applications to algebraic surfaces and unique factorization, Publ. Math. Inst. Hautes Études Sci. 36 (1969) 195–279.
[20] J. Lipman, Desingularization of two-dimensional schemes, Ann. Math. (2) 107 (1) (1978) 151–207.

[21] G. Lyubeznik, K.E. Smith, Strong and weak F-regularity are equivalent for graded rings, Am. J. Math. 121 (6) (1999) 1279–1290.

[22] G. Lyubeznik, K.E. Smith, On the commutation of the test ideal with localization and completion, Trans. Am. Math. Soc. 353 (8) (2001) 3149–3180.

[23] T. Polstra, A theorem about maximal Cohen-Macaulay modules, Int. Math. Res. Not. 3 (2022) 2086–2094.

[24] T. Polstra, K. Tucker, F-signature and Hilbert-Kunz multiplicity: a combined approach and comparison, Algebra Number Theory 12 (1) (2018) 61–97.

[25] K.E. Smith, Tight closure of parameter ideals and F-rationality, Thesis (Ph.D.)–University of Michigan, ProQuest LLC, Ann Arbor, MI, 1993.

[26] K.E. Smith, F-rational rings have rational singularities, Am. J. Math. 119 (1) (1997) 159–180.

[27] I. Swanson, C. Huneke, Integral Closure of Ideals, Rings, and Modules, London Mathematical Society Lecture Note Series, vol. 336, Cambridge University Press, Cambridge, 2006.

[28] S. Takagi, An interpretation of multiplier ideals via tight closure, J. Algebraic Geom. 13 (2) (2004) 393–415.

[29] S. Takagi, Finitistic test ideals on numerically $\mathbb{Q}$-Gorenstein varieties, J. Algebra 571 (2021) 266–279.

[30] K. Watanabe, Infinite cyclic covers of strongly F-regular rings, in: Commutative Algebra: Syzygies, Multiplicities, and Birational Algebra, South Hadley, MA, 1992, in: Contemp. Math., vol. 159, Amer. Math. Soc., Providence, RI, 1994, pp. 423–432.

[31] L.J. Williams, Uniform stability of kernels of Koszul cohomology indexed by the Frobenius endomorphism, J. Algebra 172 (3) (1995) 721–743.