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Coarse geometry of operator spaces and complete isomorphic embeddings into ℓ_1 and c_0-sums of operator spaces

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Abstract
The nonlinear geometry of operator spaces has recently started to be investigated. Many notions of nonlinear embeddability have been introduced so far, but, as noticed before by other authors, it was not clear whether they could be considered "correct notions". The main goal of these notes is to provide the missing evidence to support that *almost complete coarse embeddability* is "a correct notion". This is done by proving results about the complete isomorphic theory of ℓ_1-sums of certain operators spaces. Several results on the complete isomorphic theory of c_0-sums of operator spaces are also obtained.

Keywords Operator spaces · Coarse geometry · Embeddings

Mathematics Subject Classification 47L25 · 46L07 · 46B80

1 Introduction

This article concerns the nonlinear theory of operator spaces, which recently started to be investigated [3, 4, 7], as well as their isomorphic theory. We refer the reader to Sect. 2 for the background in operator space theory needed for these notes. For now, we only recall some basics: an *operator space* is a Banach space X together with an isometric embedding into $B(H)$—the space of bounded operators on some (complex) Hilbert space H. This isometric

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embedding allows us to see X as a subspace of $B(H)$ and each $\mathrm{M}_n(X)$ as a subspace of $\mathrm{M}_n(B(H)) \cong B(H^{\oplus n})$; where $\mathrm{M}_n(X)$ denotes the space of n-by-n matrices with entries in X and $H^{\oplus n}$ denotes the ℓ_2-sum of n copies of H. The operator norm of $B(H^{\oplus n})$ induces a Banach norm on each $\mathrm{M}_n(X)$, which we denote by $\|\cdot\|_{\mathrm{M}_n(X)}$. Given another operator space Y, a subset $A \subset X$, and a map $f \colon A \to Y$, the *n-th amplification of f* is the map $f_n \colon \mathrm{M}_n(A) \to \mathrm{M}_n(Y)$ given by

$$f_n([x_{ij}]) = [f(x_{ij})] \ \text{ for all } \ [x_{ij}] \in \mathrm{M}_n(X).$$

If f is a linear operator, so is each f_n and we denote its operator norm by $\|f_n\|_n$. We then say that f is *completely bounded* if its *cb-norm* is finite, i.e., if

$$\|f\|_{cb} = \sup_{n\in\mathbb{N}} \|f_n\|_n < \infty.$$

Completely bounded maps then induce the notions of complete isomorphisms and complete isomorphic embeddings between operator spaces.

1.1 Nonlinear theory of operator spaces

Complete boundedness naturally inspires the following version of coarseness for nonlinear maps between operator spaces: a map $f \colon X \to Y$ between operator spaces is *completely coarse* if for all $r > 0$ there is $s > 0$ such that

$$\|[x_{ij}] - [y_{ij}]\|_{\mathrm{M}_n(X)} \le r \ \text{ implies } \ \|[f(x_{ij})] - [f(y_{ij})]\|_{\mathrm{M}_n(Y)} \le s$$

for all $n \in \mathbb{N}$ and all $[x_{ij}], [y_{ij}] \in \mathrm{M}_n(X)$.[1] However, as shown in [3, Theorem 1.1], completely coarse maps are automatically $\mathbb{R}$-linear. Therefore, this notion does not lead to an interesting nonlinear theory of operator spaces, which has led to a search for "correct" notions of nonlinear morphisms, embeddings, and equivalences between operator spaces. We point out that one should not expect to recover $\mathbb{C}$-linearity from complete coarseness. In fact, Bourgain showed that there are nonisomorphic $\mathbb{C}$-Banach spaces which are isomorphic as $\mathbb{R}$-Banach spaces [6] and Ferenczi strengthened this result showing that there are $\mathbb{C}$-Banach spaces which are $\mathbb{R}$-linearly isomorphic to each other but totally incomparable as complex spaces [10].

In order to remedy this issue, [3] proposed to look at the coarse version of *almost* complete isomorphic embeddings, instead of complete isomorphic embeddings. Precisely:

Definition 1.1 ([3, Definition 4.1]) Let X and Y be operator spaces.

(1) A sequence $(f^n \colon X \to Y)_n$ of linear operators is called an *almost complete isomorphic embedding* if the amplifications

$$\Big(f^n_n \colon \mathrm{M}_n(X) \to \mathrm{M}_n(Y)\Big)_n$$

are equi-isomorphic embeddings.[2]

(2) A sequence $(f^n \colon X \to Y)_n$ of maps is called an *almost complete coarse embedding* if the amplifications

$$\Big(f^n_n \colon \mathrm{M}_n(X) \to \mathrm{M}_n(Y)\Big)_n$$

[1] The definition of a *coarse* map between Banach spaces is precisely this one for $n = 1$.

[2] We call a sequence $(g_n : X_n \to Y_n)_n$ of isomorphic embeddings *equi-isomorphic* if $\sup_n \|g_n\|, \sup_{n\in\mathbb{N}} \|g_n^{-1}\| < \infty$, where g_n^{-1} is defined only on the image of g_n.

are equi-coarse embeddings.[3]

The main results of [3, 4] show that almost complete coarse embeddability is strictly weaker than almost complete isomorphic embeddability but still strong enough to capture linear aspects of operator space structures (see, for instance, [3, Theorem 1.2] and [4, Theorems 1.4 and 1.6]).

There is however a weakness in the methods of [4]. Precisely, although [4, Theorem 1.4] gives examples of operator spaces X and Y such that X almost completely coarsely embeds into Y but does not almost completely isomorphically embed into Y, the restriction for the latter happens already in the Banach level, i.e., X does not even isomorphically embed into Y. The next definition was introduced by the first named author in [7] precisely to fix this problem and it is the notion of nonlinear embeddability which we deal with in these notes. Given a Banach space X, B_X denotes its closed unit ball.

Definition 1.2 ([7, Definitions 1.2 and 1.6]) Let X and Y be operator spaces.

(1) We say that the *bounded subsets of X almost completely coarsely embed into Y* if there is a sequence of maps $(f^n : n \cdot B_X \to Y)_n$ such that the amplifications

$$\left(f_n^n \restriction_{n \cdot B_{\mathrm{M}_n(X)}} : n \cdot B_{\mathrm{M}_n(X)} \to \mathrm{M}_n(Y)\right)_n$$

are equi-coarse embeddings.[4]

(2) We say that the *bounded subsets of X and Y are almost completely coarsely equivalent* if there is a sequence of bijections $(f^n : X \to Y)_n$ such that, letting $g^n = (f^n)^{-1}$, the amplifications

$$\left(f_n^n \restriction_{n \cdot B_{\mathrm{M}_n(X)}} : n \cdot B_{\mathrm{M}_n(X)} \to \mathrm{M}_n(Y)\right)_n \text{ and } \left(g_n^n \restriction_{n \cdot B_{\mathrm{M}_n(Y)}} : n \cdot B_{\mathrm{M}_n(Y)} \to \mathrm{M}_n(X)\right)_n$$

are equi-coarse embeddings.

Although these notions of embeddability and equivalence are very weak, they are still strong enough to capture some of the linear aspects of operator spaces. For instance, if the bounded subsets of an operator space X almost completely coarsely embed into G. Pisier's operator Hilbert space OH, then X completely isomorphically embeds into OH ([7, Corollary 1.5]). Also, if R and C are the row and the column operator spaces, respectively,[5] and $(R, C)_\theta$ is the complex interpolation space with parameter θ (see [16, Sect. 2.7] for interpolation of operator spaces), where $\theta \in [0, 1]$, then [7, Theorem 1.3] shows that the family $((R, C)_\theta)_{\theta \in [0,1]}$ is incomparable with respect to almost complete coarse embeddability of bounded subsets.

On the other hand, this notion of embeddability is indeed weaker than complete isomorphic embeddability and the reason for that does not occur in the Banach level. Precisely:

Theorem 1.3 ([7, Theorem 1.8]) *There are operator spaces X and Y such that*

(1) *X linearly isomorphically embeds into Y,*
(2) *X does not completely isomorphically embed into Y, and*

[3] Recall, a family of maps $(f_n : X_n \to Y_n)$ between metric spaces (X_n, d_n) and (Y_n, ∂_n) are *equi-coarse embeddings* if for all $r > 0$ there is $s > 0$ such that (1) $d_n(x, z) \leq r$ implies $\partial_n(f_n(x), f_n(z)) \leq s$ and (2) $d_n(x, z) \geq s$ implies $\partial_n(f_n(x), f_n(z)) \geq r$.

[4] Notice that $B_{\mathrm{M}_n(X)}$ is contained in $\mathrm{M}_n(B_X)$, so $f_n^n \restriction_{n \cdot B_{\mathrm{M}_n(X)}}$ is well defined.

[5] See Sect. 2 for the precise definition of those operator spaces.

(3) *the bounded subsets of X and Y are almost completely coarsely equivalent.*

We point out however that Theorem 1.3 only partially fixes the issue brought up above with the current state of the nonlinear theory of operator spaces. Indeed, if X and Y are given by Theorem 1.3, then, while the bounded subsets of X almost completely coarsely embed into Y, X does not completely isomorphically embed into Y, and the reason for that does not happen in the Banach level, the methods in [7] were not strong enough to guarantee that X does not *almost* completely isomorphically embed into Y—which would be a more desirable result since the bounded subsets of X only *almost* completely coarsely embed into Y.

The main goal of these notes is to resolve this issue. Philosophically speaking, this shows that the notion of almost completely coarse embeddability of subsets is "a correct one"—we emphasise the indefinite article here. With this interpretation in mind, this paper finishes the work initiated in [3], and continued in [4, 7], of showing that there is a genuinely interesting and highly nontrivial nonlinear theory for operator spaces.

1.2 Embeddings into ℓ_1-sums

We now describe our main result on the nonlinear theory of operator spaces. The following strengthens Theorem 1.3 and answers the question asked in the paragraph following [7, Theorem 1.8].

Theorem 1.4 *There are separable operator spaces X and Y such that*

(1) *X linearly isomorphically embeds into Y,*
(2) *X does not almost completely isomorphically embed into Y, and*
(3) *the bounded subsets of X and Y are almost completely coarsely equivalent.*

We point out that Theorem 1.4 strengthens Theorem 1.3 not only since X does not *almost* completely isomorphically embed into Y, but also because the spaces given by Theorem 1.4 are separable, which is not the case in Theorem 1.3.

We briefly describe our approach to Theorem 1.4. We prove this theorem by studying the complete isomorphic theory of ℓ_1-sums of operator spaces. Precisely, let $Q : \mathrm{MAX}(L_1) \to \mathrm{MIN}(\ell_2)$ be a completely bounded surjection given by the composition of surjections $L_1 \to \ell_1$ and $\ell_1 \to \ell_2$. This allow us to define operator spaces $(Y_i)_{i\in\mathbb{N}}$ such that the norm of each Y_i is given by

$$\|y\|_{Y_i} = \max\{\|y\|, 2^i\|Q(y)\|\}$$

(see Sect. 3 for details). Our main technical result of Sect. 3 then shows that $\mathrm{MIN}(\ell_2)$ cannot almost completely isomorphically embed into $(\bigoplus_i Y_i)_{\ell_1}$. This proof is quite technical and it is done in Lemma 3.3 below. Together with [7, Theorem 4.3] (restated below as Theorem 3.1), this will allow us to obtain Theorem 1.4.

1.3 Embeddings into c_0-sums

In the second part of this paper, we leave the nonlinear theory aside and move to study the complete isomorphic embeddability of certain operator spaces into certain c_0-sums $(\bigoplus_n Y_n)_{c_0}$. Precisely, Sect. 4 deals with operator space versions (and counterexamples) of the following classic result from Banach theory, whose proof follows from a standard gliding hump argument (cf. [11, Proposition 2.c.4] or Proposition 4.2 below).

Proposition 1.5 (Folklore) *Suppose X and $(Y_i)_{i\in\mathbb{N}}$ are Banach spaces and assume that X is infinite dimensional and does not contain an isomorphic copy of c_0. If X isomorphically embeds into $(\bigoplus_i Y_i)_{c_0}$, then some infinite dimensional subspace of X isomorphically embeds into some Y_i.*

We show that Proposition 1.5 has an operator space version for homogeneous Hilbertian operator spaces as long as we restrict ourselves to c_0-sums of a single operator space.[6] Precisely:

Theorem 1.6 *Let X and Y be homogeneous Hilbertian spaces and assume that X has infinite dimension. If X completely isomorphically embeds into $c_0(Y)$, then X completely isomorphically embeds into Y. In particular, if X and Y have the same density character, then X and Y are completely isomorphic.*

The restriction in Theorem 1.6 of only considering c_0-sums of a single operator space is not superfluous. Precisely, we show the following:

Theorem 1.7 *There are operator spaces $(Y_i)_{i\in\mathbb{N}}$ all of which are completely isomorphic to* $\mathrm{MIN}(\ell_2)$ *and a separable homogeneous Hilbertian operator space X such that X completely isomorphically embeds into $(\bigoplus_n Y_n)_{c_0}$, but X is not completely isomorphic to* $\mathrm{MIN}(\ell_2)$.

Notice that, since the operator space X in Theorem 1.7 is homogeneous and Hilbertian, the conclusion of this theorem implies that no infinite subspace of X completely isomorphically embeds into any of the Y_i's. Hence, Theorem 1.7 does indeed show that the operator space version of Proposition 1.5 does not hold in general.

Although Theorem 1.7 says that we cannot generalize Theorem 1.6 to arbitrary c_0-sums, we show that this can be done at least for some specific operator spaces X. In the next theorem, R and C denote the *row* and *column operator spaces*, respectively, and $R\cap C$ their *intersection operator space* (see Sect. 2 for precise definitions).

Theorem 1.8 *Let $(Y_i)_{i\in\mathbb{N}}$ be operator spaces all of which are completely isomorphic to* $\mathrm{MIN}(\ell_2)$. *If $X\in\{R, C, R\cap C, \mathrm{MAX}(\ell_2)\}$, then X does not embed completely isomorphically into $(\bigoplus_i Y_i)_{c_0}$.*

Finally, we observe that, if, in the setting of Theorem 1.7, the homogeneity of X is not assumed, then we can embed an "unexpected" space X not just into a c_0-sum of Y_i's, but also into the simpler space $c_0(Y)$:

Theorem 1.9 *There are separable Hilbertian operator spaces X and Y such that X completely isometrically embeds into $c_0(Y)$, but X does not completely isomorphically embed into Y.*

We point that, although X does not completely isomorphically embeds into Y in the theorem above, our example produces an X with infinite dimensional subspaces which do completely isomorphically embed into Y.

Remark 1.10 The questions addressed by this paper can also be investigated for real operator spaces (we refer to [17] for an introduction into this topic). However, we focus on operator spaces over the complex field, because in this setting, the general theory is much better developed, and we can rely on a vast array of known results.

[6] Recall, an operator space X is *Hilbertian* if it is isomorphic (as a Banach space) to ℓ_2. Also, given $\lambda \geq 1$, X is λ-*homogeneous* if $\|u\|_{cb} \leq \lambda\|u\|$ for all operators $u\colon X\to X$. We then say X is *homogeneous* if it is λ-homogeneous for some $\lambda\geq 1$.

2 Preliminaries

In this section, we recall the basics of operator space theory which will be used throughout the paper. We refer the reader to [16] for a monograph on this theme. We start by saying that, for each $k \in \mathbb{N}$, M_k denotes the space of k-by-k matrix with complex entries, i.e., $\mathrm{M}_k = \mathrm{M}_k(\mathbb{C})$.

Let $(X_\lambda)_{\lambda\in\Lambda}$ be a family of operator spaces. Then $(\bigoplus_\lambda X_\lambda)_{\ell_\infty}$ denotes the ℓ_∞-*sum of* $(X_\lambda)_{\lambda\in\Lambda}$, i.e.,

$$\left(\bigoplus_\lambda X_\lambda\right)_{\ell_\infty} = \left\{(x_\lambda)_\lambda \in X^\Lambda \mid \sup_{\lambda\in\Lambda} \|x_\lambda\| < \infty\right\}$$

together with the operator space structure given by

$$\|[x_{ij}]\|_{\mathrm{M}_k((\bigoplus_\lambda X_\lambda)_{\ell_\infty})} = \sup_{\lambda\in\Lambda} \|[x_{ij}(\lambda)]\|_{\mathrm{M}_k(X_\lambda)}$$

for all $k \in \mathbb{N}$ and all $[x_{ij}] = ([x_{ij}(\lambda)])_\lambda \in \mathrm{M}_k((\bigoplus_\lambda X_\lambda)_{\ell_\infty})$. If $\Lambda = \mathbb{N}$, the c_0-*sum of* $(X_n)_n$, denoted by $(\bigoplus_n X_n)_{c_0}$, is the operator subspace of $(\bigoplus_n X_n)_{\ell_\infty}$ consisting of all $(x_n)_n$ such that $\lim_n \|x_n\| = 0$. If all X_n's are the same, say $X = X_n$ for all $n \in \mathbb{N}$, we simply write $c_0(X)$ for $(\bigoplus_n X_n)_{c_0}$. Also, if the sequence $(X_n)_n$ is finite, say $X_1, \ldots, X_k$, we simply write $X_1 \oplus_\infty \ldots \oplus_\infty X_k$ for their c_0-sum.

Similarly, $(\bigoplus_n X_n)_{\ell_1}$ denotes the ℓ_1-*sum of* $(X_n)_n$, i.e.,

$$\left(\bigoplus_n X_n\right)_{\ell_1} = \left\{(x_n)_n \in X^\mathbb{N} \mid \sum_{n\in\mathbb{N}} \|x_n\| < \infty\right\}$$

together with the Banach norm $\|(x_n)_n\|_{(\bigoplus_n X_n)_{\ell_1}} = \sum_{n\in\mathbb{N}} \|x_n\|$ and the operator space structure given by the isometric embedding

$$J\colon \left(\bigoplus_n X_n\right)_{\ell_1} \to \left(\bigoplus_{u\in P} B(H_u)\right)_{\ell_\infty},$$

where P denotes the family of all sequences $u = (u_n)_n$ of completely contractive[7] maps $u_n : X_n \to B(H_u)$ and $J((x_n)_n) = (u_n(x_n))_n$ for all $(x_n)_n \in (\bigoplus_n X_n)_{\ell_1}$ (we can restrict ourselves to the Hilbert spaces H_u whose density character does not exceed that of X). If all X_n's are the same, say $X = X_n$ for all $n \in \mathbb{N}$, we simply write $\ell_1(X)$ for $(\bigoplus_n X_n)_{\ell_1}$. If the sequence $(X_n)_n$ is finite, say $X_1, \ldots, X_k$, we simply write $X_1 \oplus_1 \ldots \oplus_1 X_k$ for their ℓ_1-sum (see [16, Sect. 2.6] for details on direct sums of operator spaces).

We also need to define the minimal and maximal operator space structures on a given Banach space. Below we recall the basics; for more detail, we refer the reader to [9, Sect. 3.3] and [16, Chap. 3].

Given $\lambda \geq 1$, an operator space Z is λ-*minimal* if for any operator space Y and any operator $u : Y \to Z$, we have that $\|u\|_{cb} \leq \lambda\|u\|$. We say that Z is *minimal* if it is λ-minimal for some $\lambda \geq 1$.[8] It is easy to see that any λ-minimal operator space is also λ-homogeneous.

It turns out that, for any Banach space X, there is a unique 1-minimal operator space which has X as an underlying Banach space; we denote this space by $\mathrm{MIN}(X)$. The matricial structure of $\mathrm{MIN}(X)$ is determined by the identity $\mathrm{M}_n(\mathrm{MIN}(X)) = B(X^*, \mathrm{M}_n)$. A concrete

[7] Recall, an operator $u : X \to Y$ between operator spaces is *completely contractive* if $\|u\|_{cb} \leq 1$.

[8] We point out that authors interested in the isometric theory of operator spaces often use the term *minimal* to refer to what we are calling a 1-*minimal* operator space.

representation of $\mathrm{MIN}(X)$ can be obtained, for instance, from the canonical embedding of X into $C(B_{X^*})$—the Banach space of weak*-continuous functions on the closed unit ball of X^*, hereby denoted by B_{X^*}. Since $C(B_{X^*})$ is a C*-algebra, we can see it as a C*-subalgebra of $B(H)$, for some Hilbert space H. Moreover, as C*-algebras have unique C*-norms, the operator space structure on $C(B_{X^*})$ is independent of H and of the embedding of $C(B_{X^*})$ in $B(H)$. The operator space structure on X thus inherited from $C(B_{X^*})$ is precisely that of $\mathrm{MIN}(X)$.

On the other hand, a Banach space X can be endowed with the operator space structure given by the isometric embedding

$$X \to \left(\bigoplus_{u\in P} B(H_u)\right)_{\ell_\infty},$$

where P denotes the family of all contractions $u : X \to B(H_u)$ and $J(x) = (u(x))_{u\in P}$. This is the maximal operator space structure on X, denoted by $\mathrm{MAX}(X)$. To justify this terminology, observe that, for any operator space Y, and for any operator $u : \mathrm{MAX}(X) \to Y$, we have $\|u\| = \|u\|_{cb}$. Consequently, $\mathrm{MAX}(X)$ is 1-homogeneous.

Abusing the notation slightly, we sometimes write $\mathrm{MAX}(X)$ or $\mathrm{MIN}(X)$ for an operator space X; this refers to $\mathrm{MAX}(X')$ (respectively, $\mathrm{MIN}(X')$), where X' denotes the underlying Banach space of X.

Throughout this note, R and C denote the *row* and the *column operator spaces*, respectively. That is, let $(e_i)_i$ denote the canonical orthonormal basis of ℓ_2 and let $(e_{ij})_{i,j\in\mathbb{N}}$ denote the *matrix units*—that is, the operators in $B(\ell_2)$ such that $e_{ij}e_j = e_i$ and $e_{ij}e_k = 0$ for all $i, j, k \in \mathbb{N}$ with $k \neq j$. Then

$$R = \overline{\mathrm{span}}\{e_{1i} \mid i \in \mathbb{N}\} \ \text{ and } \ C = \overline{\mathrm{span}}\{e_{i1} \mid i \in \mathbb{N}\}.$$

Clearly, both R and C are isometric to ℓ_2; this can be seen since the maps $r\colon \ell_2 \to R$ and $c\colon \ell_2 \to C$ determined by $r(e_i) = e_{1i}$ and $c(e_i) = e_{i1}$, for all $i \in \mathbb{N}$, are isometries. The operator space $R\cap C$ is the Banach space ℓ_2 together with the operator space structure given by the isometric embedding

$$x \in \ell_2 \mapsto (r(x), c(x)) \in R \oplus_\infty C.$$

At last, $R + C$ denotes the operator space given by the quotient $R \oplus_1 C/\Delta$, where $\Delta = \{(r(x), -c(x)) \mid x \in \ell_2\}$ (see [16, p. 194]).

For more information about the row and column spaces, we refer the reader to [9, Sect. 3.4] or [16, Chap. 1].

3 Embeddings into certain ℓ_1-sums

The main goal of this section is to prove Theorem 1.4. For that, we must recall some results obtained in [7]. Let Y and X be operator spaces and consider a completely bounded map $Q\colon Y \to X$ which is also a Banach quotient map. For each $m \in \mathbb{N}$, let Y_m be the Banach space such that $Y_m = Y$ as a vector space and with norm given by

$$\|y\|_{Y_m} = \max\{\|y\|, 2^m\|Q(y)\|\}$$

for all $y \in Y_m$. Moreover, endow Y_m with the operator space structure given by

$$\|[y_{ij}]\|_{\mathrm{M}_k(Y_m)} = \max\{\|[y_{ij}]\|_{\mathrm{M}_k(Y)}, 2^m\|[Q(y_{ij})]\|_{\mathrm{M}_k(X)}\}$$

for all $k \in \mathbb{N}$ and all $[y_{ij}] \in \mathrm{M}_k(Y_m)$. It follows from Ruan's Theorem that this indeed induces an operator space structure on each Y_m ([16, Sect. 2.2]). Notice also that each Y_m is completely isomorphic to Y.

Let $\mathcal{Z}(Q) = (\bigoplus_m Y_m)_{\ell_1}$. Then, by the universal properties of ℓ_1-directed sums of operator spaces, there is a completely bounded map

$$\tilde{Q} \colon \mathcal{Z}(Q) \to X$$

such that $\tilde{Q} \circ i_m = Q$ for all $m \in \mathbb{N}$; where each $i_m \colon Y_m \hookrightarrow \mathcal{Z}(Q)$ denotes the canonical inclusion ([5, Subsect. 1.4.13]). Clearly, $\tilde{Q}$ is also a Banach quotient map.

We can now state one of the main technical results from [7].

Theorem 3.1 ([7, Theorem 4.3]) Let X and Y be operator spaces, $Q \colon Y \to X$ be a completely bounded map which is also a Banach quotient, and let $\tilde{Q} \colon \mathcal{Z}(Q) \to X$ be as above. Then, the bounded subsets of $\mathcal{Z}(Q)$ and $X \oplus \ker(\tilde{Q})$ are almost completely coarsely equivalent.

Remark 3.2 We point out to the reader that, strictly speaking, the operator spaces Y_m were defined in [7] to have norm

$$|||y|||_{Y_m} = \max\{2^{-m}\|y\|, \|Q(y)\|\}.$$

This is however just a formal difference since $(Y_m, \|\cdot\|_{Y_m})$ and $(Y_m, |||\cdot|||_{Y_m})$ are clearly completely isometric to each other. So, Theorem 3.1 remains valid under this slight change in the definition of the $(Y_m)_m$'s.

We will prove Theorem 1.4 by looking at an appropriate quotient map $Q \colon Y \to X$. Suppose $Y = \mathrm{MAX}(L_1)$ and $X = \mathrm{MIN}(\ell_2)$. Since every separable Banach space is a quotient of ℓ_1 ([2, Theorem 2.3.1]) and ℓ_1 embeds into L_1 complementably ([2, Proposition 5.7.2]), there is a bounded map $Q \colon Y \to X$ which is also a Banach quotient. As $X = \mathrm{MIN}(\ell_2)$, it is automatic that Q is actually completely bounded.

Lemma 3.3 *Let $Y = \mathrm{MAX}(L_1)$ and $X = \mathrm{MIN}(\ell_2)$, and let $Q \colon Y \to X$ be the map described above. Let $(Y_m)_m$ be as defined above for Y. Then X does not almost completely isomorphically embed into $\mathcal{Z}(Q) = (\bigoplus_m Y_m)_{\ell_1}$.*

Proof We start by noticing that $\mathcal{Z}(Q)$ can be viewed as being embedded into $\ell_1(Y) \oplus_\infty \ell_1(X)$. Indeed, for each $m \in \mathbb{N}$, let $J(m) \colon Y_m \to Y \oplus_\infty X$ be the map given by

$$J(m)(y) = (y, 2^m Q(y))$$

for all $y \in Y_m$. So, each $J(m)$ is a complete isometric embedding and this allows us to view each Y_m as a subspace of $Y \oplus_\infty X$. Consequently, we can view $\mathcal{Z}(Q)$ as a subspace of $\ell_1(Y \oplus_\infty X)$. Since $\ell_1(Y \oplus_\infty X)$ is completely isomorphic to $\ell_1(Y) \oplus_\infty \ell_1(X)$, we can view $\mathcal{Z}(Q)$ as being embedded in it. Let

$$P_1 \colon \ell_1(Y) \oplus_\infty \ell_1(X) \to \ell_1(Y) \text{ and } P_2 \colon \ell_1(Y) \oplus_\infty \ell_1(X) \to \ell_1(X)$$

denote the standard projections. □

Claim 3.4 *Given any linear map $u = \oplus_m u(m) \colon X \to \mathcal{Z}(Q)$, there exists an orthonormal sequence $(\xi_i)_i$ in X such that $\lim_i \sum_m 2^m \|Qu(m)\xi_i\| = 0$. Consequently, $\lim_i \|P_2 u \xi_i\| = 0$.*

Proof We start by setting some notation. For each $m \in \mathbb{N}$, let $v(m) = 2^m Qu(m)$. As $X = \ell_2$ as a Banach space, we have that $v(m) \in B(\ell_2)$. As u takes values in $\mathcal{Z}(Q)$, the definition of the norm in this space implies that $v = \oplus_m v(m) \in B(\ell_2, \ell_1(\ell_2))$. Finally, for each $I \subset \mathbb{N}$, let $v(I) = \oplus_{m \in I} v(m)$ and $v(I)^\perp = \oplus_{m \notin I} v(m)$. We now point out the following basic facts:

- For any finite dimensional $E \subset \ell_2$ and any $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $\|v(\{1,\ldots,k\})^{\perp}\xi\| \leq \varepsilon\|\xi\|$ for any $\xi \in E$.
- As Q is the composition of an operator $L_1 \to \ell_1$ with an operator $\ell_1 \to \ell_2$ and any operator $\ell_1 \to \ell_2$ is strictly singular[9] ([2, Theorem 2.1.9]), we have that each $v(n)$ is strictly singular. Consequently, each $v(I)$ is strictly singular for any finite $I \subset \mathbb{N}$.

Fix $\varepsilon > 0$. Notice that for any finite dimensional $E \subset \ell_2$, there exists an infinite dimensional $F \subset E^{\perp}$ such that $\|v\xi\| \leq \varepsilon\|\xi\|$ for any $\xi \in F$. Indeed, suppose this statement is false. Then, since each $v(I)$, for $I \subset \mathbb{N}$ finite, is strictly singular, we can find sequences $(I_i)_i$ and $(\xi_i)_i$ such that

(1) each I_i is a finite subset of $\mathbb{N}$ and $\max(I_i) < \min(I_{i+1})$ for all $i \in \mathbb{N}$,
(2) $(\xi_i)_i$ is a normalized sequence in ℓ_2 equivalent to its standard unit basis and such that $\|v\xi_i\| \geq \varepsilon$ for all $i \in \mathbb{N}$, and
(3) $\|v\xi_i - v(I_i)\xi_i\| \leq 2^{-i}$ for all $i \in \mathbb{N}$.

Therefore, up to a constant $C > 0$ independent on k, we have that

$$\left\| v\left(\sum_{i=1}^{k} \xi_i\right)\right\| \geq Ck$$

for all $k \in \mathbb{N}$. On the other hand, since $(\xi_i)_i$ is equivalent to the standard unit basis of ℓ_2, we have that,

$$\left\|\sum_{i=1}^{k} \xi_i\right\| \leq Dk^{1/2}$$

for all $k \in \mathbb{N}$, where $D > 0$ is another constant independent on k. This gives us a contradiction since v is bounded.

We now construct the required orthonormal sequence $(\xi_i)_i$ recursively as follows. Pick a norm one $\xi_1 \in \ell_2 = X$ with finite support (with respect to the canonical basis) and such that $\|v\xi_1\| \leq 2^{-1}$. Suppose finitely supported normalized vectors $\xi_1, \ldots, \xi_i \in \ell_2$ have been chosen so that

(1) $\mathrm{supp}(\xi_j) < \mathrm{supp}(\xi_{j+1})$ for all $j \in \{1,\ldots,i-1\}$, and
(2) $\|v\xi_j\| \leq 2^{-j}$ for all $j \in \{1,\ldots,i\}$.

By the previous paragraph, we can choose a norm one finitely supported $\xi_{n+1} \in \ell_2$ such that $\mathrm{supp}(\xi_{i+1}) > \mathrm{supp}(\xi_i)$ and $\|v\xi_{i+1}\| < 2^{-i-1}$. It follows straightforwardly from the definition of v and the norm in $\ell_1(\ell_2)$ that $\lim_i \sum_m 2^m \|Qu(m)\xi_i\| = 0$. □

Claim 3.5 *For every $\gamma > 0$ there exists $m \in \mathbb{N}$ such that any operator $u\colon X \to \mathcal{Z}(Q)$ with $\|u^{-1}\| \leq 1$ satisfies $\|u_m\| \geq \gamma$.*

Proof Fix $\gamma > 0$. Fix $u\colon X \to \mathcal{Z}(Q)$ with $\|u^{-1}\| \leq 1$—we will find $m \in \mathbb{N}$ below which does not depend on u. By Claim 3.4, there exists an orthonormal sequence $(\xi_i)_i$ in ℓ_2 such that $\|P_2u\xi_i\| < 2^{-i}$ for any $i \in \mathbb{N}$. Therefore,

$$\|P_1u\xi_i\| \geq \|u^{-1}\|\|P_1u\xi_i\| = \|u^{-1}\|\|u\xi_i\| \geq \|\xi_i\| \geq 1$$

9 Recall, an operator $u : X \to Y$ between Banach spaces is *strictly singular* if none of its restrictions to an infinite dimensional subspaces of X is an isomorphic embedding. For more on strictly singular operators, see [11, Sect. 2.c].

for all $i \in \mathbb{N}$.

Notice that, by the definition of the ℓ_1-sum operator space structure, the range of P_1 can be identified with

$$\ell_1(\mathrm{MAX}(L_1)) = \mathrm{MAX}(L_1(\mathbb{N} \times (0,1))),$$

where $\mathbb{N} \times (0,1)$ is considered together with its canonical measure, which we denote by μ. As usual, we write $L_1(\mu) = L_1(\mathbb{N} \times (0,1))$.

Let $c > 0$ be a constant to be determined later. Since $R+C$ is not a minimal operator space, the identity $\mathrm{MIN}(\ell_2) \to R + C$ is not completely bounded. Hence, there are $m, N \in \mathbb{N}$, and $a_1, \ldots, a_N \in \mathrm{M}_m$, such that

$$\left\| \sum_i a_i \otimes \xi_i \right\|_{\mathrm{M}_m \otimes \mathrm{MIN}(\ell_2)} = 1 \text{ and } \left\| \sum_i a_i \otimes \xi_i \right\|_{\mathrm{M}_m \otimes (R+C)} > c$$

(by [16], one can take $N \sim c^2$). Let $\mathbb{T}$ denote the unit torus, i.e., $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. Then, by the 1-homogeneity of $\mathrm{MIN}(\ell_2)$, we have that

$$\begin{aligned} \left\| \sum_i \omega_i a_i \otimes \xi_i \right\|_{\mathrm{M}_m \otimes \mathrm{MIN}(\ell_2)} &= \left\| \sum_i a_i \otimes \omega_i \xi_i \right\|_{\mathrm{M}_m \otimes \mathrm{MIN}(\ell_2)} \\ &= \left\| \sum_i a_i \otimes \xi_i \right\|_{\mathrm{M}_m \otimes \mathrm{MIN}(\ell_2)} = 1 \end{aligned}$$

for all$\omega = (\omega_i)_{i=1}^N \in \mathbb{T}^N$. We will now show that there exists $\omega = (\omega_i)_{i=1}^N \in \mathbb{T}^N$ such that

$$\left\| \sum_{i=1}^N \omega_i a_i \otimes P_1 u \xi_i \right\|_{\mathrm{M}_m \otimes L_1(\mu)} \geq \kappa c,$$

where κ is a universal constant. From this inequality, it follows that $\|u_m\| \geq \kappa c$. Therefore, taking $c = \gamma/\kappa$, the proof of the proposition will be completed. Indeed, notice that $m \in \mathbb{N}$ depends only on c, therefore, since κ is a universal constant, m depends only on γ and not on u.

Let ν be the rotation invariant probability measure on $\mathbb{T}^N$. For each $i \in \{1, \ldots, N\}$, let $\phi_i : \mathbb{T}^N \to \mathbb{T}$ be the canonical projection onto the i-th coordinate of $\mathbb{T}^N$. We claim that $(\phi_i \otimes P_1 u \xi_i)_{i=1}^N$ is a 1-*completely unconditional basic sequence* in $L_1(\nu \otimes \mu)$—that is, for any sequence $(\alpha_i)_{i=1}^N$ of complex numbers, with $\max_i |\alpha_i| \leq 1$, the "diagonal" map Φ_α on $F = \mathrm{span}[\phi_i \otimes P_1 u \xi_i : 1 \leq i \leq N]$, taking $\phi_i \otimes P_1 u \xi_i$ to $\alpha_i \phi_i \otimes P_1 u \xi_i$, is completely contractive. By convexity, it suffices to show this holds if $|\alpha_i| = 1$ for every $i \leq N$.

To evaluate $\|\Phi_\alpha\|_{cb}$ in this setting, it is more convenient to work not with the injective tensor product with matrix spaces M_n, but rather, to consider the "dual" setting. For each $n \in \mathbb{N}$, S_1^n denotes the operator space of $n \times n$ trace class operators and, if E is another operator space, then $S_1^n[E]$ denotes the projective operator space tensor product $S_1^n \otimes^\wedge E$. The reader is referred to [9, Chap. 7], [15], or [16, Chap. 4] for a detailed treatment of this tensor product. Here we mention two properties important for us.

(1) For any map $T : E \to F$ between operator spaces, $\|T\|_{cb} = \sup_n \|I \otimes T : S_1^n[E] \to S_1^n[F]\|$ (this follows from [15, Lemma 1.7], or by duality from [9, Proposition 7.1.6]).
(2) For any σ-finite measure μ, $S_1^n[L_1(\mu)]$ is isometrically identified with $L_1(\mu, S_1^n)$ (see [15, Proposition 2.1]).

Notice that, if $(y_i)_{i=1}^N$ is in S_1^n, then

$$\left\|\sum_{i=1}^N y_i \otimes \phi_i \otimes P_1 u\xi_i\right\|_{S_1^n[L_1(\nu\otimes\mu)]} = \int \left\|\sum_{i=1}^N y_i \otimes \phi_i(\omega) P_1 u\xi_i\right\|_{L_1(\mu,S_1^n)} d\nu(\omega)$$
$$= \int \left\|\sum_{i=1}^N y_i \otimes \omega_i P_1 u\xi_i\right\|_{L_1(\mu,S_1^n)} d\nu(\omega).$$

Now suppose $|\alpha_i| = 1$ for $1 \leq i \leq N$. By the rotation invariance of ν, the right hand side equals

$$\int \left\|\sum_{i=1}^N y_i \otimes \alpha_i\omega_i P_1 u\xi_i\right\|_{L_1(\mu,S_1^n)} d\nu(\omega) = \left\|\sum_{i=1}^N y_i \otimes \alpha_i\phi_i \otimes P_1 u\xi_i\right\|_{S_1^n[L_1(\nu\otimes\mu)]}$$
$$= \left\|(I \otimes \Phi_\alpha)\sum_{i=1}^N y_i \otimes \phi_i \otimes P_1 u\xi_i\right\|_{S_1^n[L_1(\nu\otimes\mu)]}.$$

Thus, Φ_α is a complete isometry on $\operatorname{span}[\phi_i \otimes P_1 u\xi_i : 1 \leq i \leq N]$. This establishes the desired unconditionality.

Notice that

$$\|\phi_i \otimes P_1 u\xi_i\|_{L_1(\nu\otimes\mu)} = \|P_1 u\xi_i\|_{L_1(\mu)} \geq 1$$

for all $i \in \{1,\ldots,N\}$. Hence, [12, Proposition 4.3] gives a constant $\kappa > 0$ (independent on u and N) such that the operator $T : F \to R + C$ determined by $\phi_i \otimes P_1 u\xi_i \mapsto e_i$, for each $i \in \mathbb{N}$, has c.b. norm at most $1/\kappa$; here $(e_i)_i$ denotes the canonical basis of $R + C$. As noted above, for every n we have

$$\|\mathrm{Id} \otimes T : S_1^n[F] \to S_1^n[R + C]\| \leq \|T\|_{cb} \leq \frac{1}{\kappa}.$$

As $\|\sum_i a_i \otimes \xi_i\|_{\mathrm{M}_m\otimes(R+C)} > c$, it follows from the 1-homogeneity of $R + C$ that $\|\sum_i a_i \otimes e_i\|_{\mathrm{M}_m\otimes(R+C)} > c$. Therefore, [15, Lemma 1.7] gives us $m \times m$ matrices b_1 and b_2 of Hilbert–Schmidt norm one, such that

$$\left\|\sum_i b_1 a_i b_2 \otimes e_i\right\|_{S_1^m[R+C]} > c.$$

By 1-homogeneity of $R + C$ again, the same inequality holds if each a_i above is replaced by $\omega_i a_i$, where $(\omega_i)_{i=1}^N$ is an arbitrary element of $\mathbb{T}^N$. Consequently,

$$\int \left\|\sum_i \omega_i b_1 a_i b_2 \otimes P_1 u\xi_i\right\|_{S_1^m[L_1(\mu)]} d\nu(\omega)$$
$$= \left\|\sum_i b_1 a_i b_2 \otimes \phi_i \otimes P_1 u\xi_i\right\|_{S_1^m[L_1(\nu\otimes\mu)]}$$
$$\geq \|\mathrm{Id} \otimes T\|^{-1} \left\|\sum_i b_1 a_i b_2 \otimes e_i\right\|_{S_1^m[R+C]}$$
$$> \kappa c.$$

Thus, there exists $\omega = (\omega_i)_{i=1}^N \in \mathbb{T}^N$ such that

$$\left\| \sum_i \omega_i b_1 a_i b_2 \otimes P_1 u \xi_i \right\|_{S_1^m[L_1(\mu)]} \geq \kappa c.$$

Applying [15, Lemma 1.7] again, we conclude that

$$\left\| \sum_i \omega_i a_i \otimes P_1 u \xi_i \right\|_{\mathrm{M}_m \otimes L_1(\mu)} \geq \kappa c$$

and, by the discussion above, this proves the claim.

Claim 3.5 immediately implies that there is no almost completely isomorphic embedding of X into $\mathcal{Z}(Q)$; so we are done. □

Proof of Theorem 1.4 Let $Y = \mathrm{MAX}(L_1)$ and $X = \mathrm{MIN}(\ell_2)$. Let $Q\colon Y \to X$ be the quotient map described before Lemma 3.3. Let $(Y_m)_m$, $\mathcal{Z}(Q)$ and $\tilde{Q}$ be as above. By Theorem 3.1, the bounded subsets of $\mathcal{Z}(Q)$ and $X \oplus \ker(\tilde{Q})$ are almost completely coarsely equivalent.

Note that the formal identity between $(\oplus_{m\geq 1} Y_m)_{\ell_1}$ and $(\oplus_{m\geq 2} Y_m)_{\ell_1}$ is a (complete) isomorphism. Consequently, $\mathcal{Z}(Q)$ is (completely) isomorphic to $Y \oplus \mathcal{Z}(Q)$.

It is well known that ℓ_2 isometrically embeds into L_1 ([2, Theorem 6.4.17]). So, in our case, $X \oplus \ker(\tilde{Q})$ isomorphically embeds into $\mathcal{Z}(Q)$. We are left to notice that $X \oplus \ker(\tilde{Q})$ does not almost completely isomorphically embed into $\mathcal{Z}(Q)$. For that, it is enough to show that X does not almost completely isomorphically embed into $\mathcal{Z}(Q)$. This is precisely Lemma 3.3, so we are done. □

4 Embeddings into certain c_0-sums

In this section, we leave the nonlinear theory aside and concentrate fully on the completely isomorphic theory of c_0-sums of operator spaces. Precisely, we study the extent to which Proposition 1.5 remains valid for operator spaces. In fact, a slightly stronger, and more technical, result will be the main focus of this section. Its proof follows from a simple gliding hump argument (cf. [11, Proposition 2.c.4]).

Proposition 4.1 (Folklore) *Suppose X and $(Y_i)_{i\in\mathbb{N}}$ be Banach spaces and assume that X is infinite dimensional and does not contain an isomorphic copy of c_0. If all operators $X \to Y_i$, for $i \in \mathbb{N}$, are strictly singular, then X does not isomorphically embed into $(\bigoplus_i Y_i)_{c_0}$.*

As a warm up for what is to come, we start with an elementary proposition:

Proposition 4.2 *Suppose X and $(Y_i)_{i\in\mathbb{N}}$ are operator spaces such that X has no subspace isomorphic to c_0 and every completely bounded map from X to Y_i, $i \in \mathbb{N}$, is strictly singular. Then X does not completely isomorphically embed into $(\bigoplus_i Y_i)_{c_0}$.*

As the expert will notice, the proof of Proposition 4.2 is essentially the one of Proposition 4.1. We point out however that Proposition 4.2 is *not* the operator space version of Proposition 4.1. Indeed, a truely operator space version would only assume that all operators $X \to Y_i$ are *completely strictly singular*, meaning that their restrictions to infinite subspaces of X are not complete isomorphic embeddings (but they can be isomorphic embeddings).

Proof of Proposition 4.2 Suppose, for the sake of contradiction, that there exists a complete isomorphic embedding $u : X \to (\bigoplus_i Y_i)_{c_0}$. Write $u = \oplus_i u_i$; so, the hypothesis imply that each $u_i : X \to Y_i$ is strictly singular. A standard gliding hump argument produces a normalized sequence $(x_j)_j \subset X$, for which there exists a sequence $(I_j)_j$ of intervals of $\mathbb{N}$ such that $\max(I_j) < \min(I_{j+1})$ and $\|ux_j - Q_j ux_j\| < 4^{-j}$ for all $j \in \mathbb{N}$, where each Q_j denotes the canonical projection $(\bigoplus_i Y_i)_{c_0} \to (\bigoplus_{i\in I_j} Y_i)_{c_0}$. In particular, it follows that $(ux_j)_j$ is equivalent to the c_0-basis, hence the same must be true for $(x_j)_j$, which is impossible since X does not contain an isomorphic copy of c_0. □

From this we deduce:

Corollary 4.3 *Let $(Y_i)_i$ be a sequence of operator spaces all of which are completely isomorphic to* $\mathrm{MAX}(H)$*, for a certain Hilbert space H. Then any homogeneous Hilbertian subspace $X \subset (\bigoplus_i Y_i)_{c_0}$ is completely isomorphic to* $\mathrm{MAX}(X)$.

Proof In the proof, we shall use the folklore observation that a homogeneous Hilbertian space Y is completely isomorphic to $\mathrm{MAX}(Y)$ if and only if W is completely isomorphic to $\mathrm{MAX}(W)$ for a certain (equivalently, any) separable infinite dimensional subspace $W \subset Y$.

Consider now a homogeneous Hilbertian X from the statement of this Corollary. As X cannot contain a copy of c_0, by Proposition 4.2 there exists a c.b. map $u : X \to Y_i$ which is not strictly singular. Then there is an infinite dimensional $Z \subset X$ such that $u \restriction Z$ is an isomorphic embedding. The space Y_i is completely isomorphic to $\mathrm{MAX}(H)$, and therefore, $Z' := u(Z) \subset Y_i$ is completely isomorphic to $\mathrm{MAX}(Z')$, hence Z is completely isomorphic to $\mathrm{MAX}(Z)$. By the first paragraph, we conclude that X is completely isomorphic to $\mathrm{MAX}(X)$. □

4.1 Embeddings into c_0-sums of a single operator space

In this subsection, we prove Theorems 1.6 and 1.9. The next subsection will focus on embeddings into the c_0-sum of a sequence of operator spaces.

Proof of Theorem 1.6 Let $u : X \to c_0(Y)$ be a complete isomorphic embedding and, for each $i \in \mathbb{N}$, let $u_i : X \to Y$ be the composition of u with the canonical projection $c_0(Y) \to Y$ onto the i-th coordinate of $c_0(Y)$. As X has infinite dimension, so does Y. Hence, $\mathrm{dens}(Y) = \mathrm{dens}(c_0(Y))$, which in turn implies that $\mathrm{dens}(X) \le \mathrm{dens}(Y)$. Therefore, as both X and Y are Hilbertian, we can assume that $X \subset Y$ as vector spaces and that there is $L \ge 1$ such that

$$L^{-1}\|x\|_X \le \|x\|_Y \le L\|x\|_X$$

for all $x \in X$. Moreover, to simplify notation, we fix infinite sets Γ and Λ with $\Gamma \subset \Lambda$ and assume that $X = \ell_2(\Gamma)$ and $Y = \ell_2(\Lambda)$ as vector spaces. Let $(e_\lambda)_{\lambda\in\Lambda}$ be the standard unit basis of $\ell_2(\Lambda)$. □

Case 1 The formal inclusion $X \hookrightarrow Y$ is completely bounded.

Fix $\theta \ge 1$ such that Y is θ-homogeneous. If the inverse of the inclusion $X \hookrightarrow Y$ is also completely bounded, then $X \hookrightarrow Y$ is a complete isomorphic embedding and the conclusion follows. If not, there are $k \in \mathbb{N}$ and matrices $(a_\gamma)_{\gamma\in\Gamma}$ in M_k such that

- $\|\sum_{\gamma\in\Gamma} a_\gamma \otimes e_\gamma\|_{\mathrm{M}_k(Y)} = 1$ and
- $\|\sum_{\gamma\in\Gamma} a_\gamma \otimes e_\gamma\|_{\mathrm{M}_k(X)} > \theta L\|u\|\|u^{-1}\|_{cb}$.

For each $i \in \mathbb{N}$, let $v_i : Y \to Y$ be the linear map defined by

$$v_i(e_\lambda) = \begin{cases} u_i(e_\lambda), & \text{if } \lambda \in \Gamma \\ 0, & \text{if } \lambda \in \Lambda \setminus \Gamma. \end{cases}$$

Clearly, $\|v_i\| \leq L\|u_i\| \leq L\|u\|$ for all $i \in \mathbb{N}$. By our choice of θ, we have that

$$\sup_{i\in\mathbb{N}} \|v_i\|_{cb} < \theta L\|u\|.$$

Therefore, it follows that

$$\begin{aligned} \left\| \sum_{\gamma\in\Gamma} a_\gamma \otimes u(e_\gamma) \right\|_{\mathrm{M}_k(c_0(Y))} &= \sup_{i\in\mathbb{N}} \left\| \sum_{\gamma\in\Gamma} a_\gamma \otimes u_i(e_\gamma) \right\|_{\mathrm{M}_k(Y)} \\ &= \sup_{i\in\mathbb{N}} \left\| \sum_{\gamma\in\Gamma} a_\gamma \otimes v_i(e_\gamma) \right\|_{\mathrm{M}_k(Y)} \\ &\leq \sup_{i\in\mathbb{N}} \|v_i\|_{cb} \left\| \sum_{\gamma\in\Gamma} a_\gamma \otimes e_\gamma \right\|_{\mathrm{M}_k(Y)} \\ &\leq \theta L\|u\|. \end{aligned}$$

However, as

$$\left\| \sum_{\gamma\in\Gamma} a_\gamma \otimes e_\gamma \right\|_{\mathrm{M}_k(X)} \leq \|u^{-1}\|_{cb} \left\| \sum_{\gamma\in\Gamma} a_\gamma \otimes u(e_\gamma) \right\|_{\mathrm{M}_k(c_0(Y))},$$

this contradicts our choice of $(a_\gamma)_{\gamma\in\Gamma}$.

Case 2 Case 1 does not hold.

We start by noticing that for each $i \in \mathbb{N}$ and each finite codimensional subspace $Z \subset X$, the restriction $u_i \restriction_Z : Z \to u_i(Z)$ is not an isomorphism. Indeed, suppose this is not the case and fix offenders, say $i \in \mathbb{N}$ and $Z \subset X$. As X is a homogeneous Hilbertian space and Z has finite codimension in X, there is a complete isomorphism $v : X \to Z$. Clearly, the basic sequences $(u_i(v(e_\gamma)))_{\gamma\in\Gamma}$ and $(e_\gamma)_{\gamma\in\Gamma}$ are in Y, hence, as Y is a homogeneous Hilbertian space there is a complete isomorphic embedding $w : u_i(Z) \to Y$ such that $w(u_i(v(e_\gamma))) = e_\gamma$, for all $\gamma \in \Gamma$. Since

$$w \circ u_i \circ v : X \to Y$$

is the inclusion, this inclusion must be completely bounded. This is a contradiction since we assume Case 1 does not hold.

Since $u_i \restriction_Z : Z \to u_i(Z)$ is not an isomorphism for all $i \in \mathbb{N}$ and all finite codimensional $Z \subset X$, a standard gliding hump argument from Banach space theory gives that the basis of ℓ_2 and c_0 are equivalent; contradiction.

The next result shows that homogeneity is necessary for Theorem 1.6 to hold.

Proof of Theorem 1.9 Let $Y = \mathrm{MIN}(\ell_2) \oplus \mathrm{MAX}(\ell_2)$ and fix a partition $(S_k)_k$ of $\mathbb{N}$ into finite subsets with $\lim_k |S_k| = \infty$. Let $(e_i)_i$ be the canonical unit basis of ℓ_2 and, given

$x = \sum_i a_i e_i \in \ell_2$ and $k \in \mathbb{N}$, we let

$$x \upharpoonright_{S_k} = \sum_{i \in S_k} a_i e_i \in \ell_2(S_k).$$

We then let X be the operator space consisting of ℓ_2 with the operator space structure given by the isometric embedding

$$x \in \ell_2 \mapsto (x, (x \upharpoonright_{S_k})_k) \in \mathrm{MIN}(\ell_2) \oplus \left(\bigoplus_k \mathrm{MAX}(\ell_2(S_k)) \right)_{c_0}.$$

Since $\mathrm{MIN}(\ell_2) \oplus (\bigoplus_k \mathrm{MAX}(\ell_2(S_k)))_{c_0}$ completely isometrically embeds into $c_0(Y)$, it is clear that X completely isometrically embeds into $c_0(Y)$.

We are left to notice that X does not completely isomorphically embed into Y. Suppose for a contradiction that such embedding $u \colon X \to Y$ exists. Let $p \colon Y \to \mathrm{MIN}(\ell_2)$ and $q \colon Y \to \mathrm{MAX}(\ell_2)$ denote the canonical projections. For each $k \in \mathbb{N}$, let

$$X_k = \mathrm{span}\{e_i \in X \mid i \in S_k\}.$$

So, $X_k = \mathrm{MAX}(\ell_2(S_k))$ completely isometrically. For each $k \in \mathbb{N}$, let $v_k = p \circ u \upharpoonright_{X_k}$ and let

$$v_{k,k} \colon \mathrm{M}_{1,k}(X_k) \to \mathrm{M}_{1,k}(\mathrm{MIN}(\ell_2))$$

be its 1-by-k amplification.[10] □

Claim 4.4 *The maps $(v_{k,k})_k$ are not equi-isomorphisms.*

Proof If $(v_{k,k})_k$ are equi-isomorphisms, then so are $(v_k)_k$. For each $k \in \mathbb{N}$, let $w_k \colon v_k(X_k) \to \mathrm{MIN}(\ell_2(S_k))$ be the linear map defined by $w_k(v_k(e_i)) = e_i$ for all $i \in S_k$. As $\mathrm{MIN}(\ell_2)$ is homogeneous and $(v_k)_k$ are equi-isomorphisms, $(w_k)_k$ are equi-complete isomorphisms. Therefore, letting

$$w_{k,k} \colon \mathrm{M}_{1,k}(v_k(X_k)) \to \mathrm{M}_{1,k}(\mathrm{MIN}(\ell_2(S_k)))$$

be the 1-by-k amplification of w_k, we obtain that the maps $(w_{k,k} \circ v_{k,k})_k$ are equi-isomorphisms. However, $w_{k,k} \circ v_{k,k}$ is precisely the identity

$$\mathrm{M}_{1,k}(\mathrm{MAX}(\ell_2(S_k))) \to \mathrm{M}_{1,k}(\mathrm{MIN}(\ell_2(S_k))).$$

This is a contradiction since those identities are not equi-isomorphisms. □

As $(v_{k,k})_k$ are equi-completely bounded, the previous claim implies that there are a sequence $(n_k)_k$ in $\mathbb{N}$ and a sequence $(x_k)_k$ such that

- $x_k \in \mathrm{M}_{1,n_k}(X_{n_k})$ and $\|x_k\|_{\mathrm{M}_{1,n_k}(X_{n_k})} = 1$, and
- $\delta = \inf_k \|q(u(x_k))\|_{\mathrm{M}_{1,n_k}(\mathrm{MAX}(\ell_2))} > 0$.

Therefore, since the 1-by-k amplifications of the identity $\mathrm{MAX}(\ell_2) \to R$ are isometries, by letting $N_k = n_1 + \cdots + n_k$, we have that

$$\| \left[q(u(x_1)) \; \ldots \; q(u(x_k)) \right] \|_{\mathrm{M}_{N_k}(\mathrm{MAX}(\ell_2))} \geq \delta k^{1/2}$$

[10] If E is an operator space, then $\mathrm{M}_{1,k}(E)$ denotes the subspace of $\mathrm{M}_k(E)$ consisting of all operators whose only nonzero rows are their first one. The space $\mathrm{M}_{k,1}(E)$ is defined similarly, but with the word "columns" substituting "rows".

for all $k \in \mathbb{N}$. On the other hand, as each x_k is in X_k, it follows that

$$\| \big[x_1 \ \ldots \ x_k\big] \|_{\mathrm{M}_{N_k}(X)} = \| \big[x_1 \ \ldots \ x_k\big] \|_{\mathrm{M}_{N_k}(\mathrm{MIN}(\ell_2))} = 1$$

for all $k \in \mathbb{N}$. It then follows that $\delta k^{1/2} \leq \|u\|_{cb}$ for all $k \in \mathbb{N}$; contradiction.

4.2 Embeddings into the c_0-sum of a sequence of operator spaces

We now move to study what happens with Theorems 1.6 and 1.9 if one replaces $c_0(Y)$ by $(\bigoplus_{n\in\mathbb{N}} Y_n)_{c_0}$.

Proof of Theorem 1.7 For each $m \in \mathbb{N}$, let $R^{[m]}$ be ℓ_2 as a Banach space, equipped with the operator space structure given by

$$\|[x_{ij}]\|_{\mathrm{M}_k(R^{[m]})} = \sup_p \|[p(x_{ij})]\|_{\mathrm{M}_k(R)}, \ \ \text{forall} \ \ k \in \mathbb{N},$$

where the supremum runs over all projections $p : \ell_2 \to \ell_2$ of rank m. For each $n \in \mathbb{N}$, let $Y_n = R^{[2^n]}$. So, $(Y_n)_n$ are all completely isomorphic to each other, and to $\mathrm{MIN}(\ell_2)$ (although the isomorphism constants are not uniformly bounded). Moreover, since R is 1-homogeneous, it is clear that each Y_n is also 1-homogeneous.

Define X as the image of ℓ_2 under the isometry $u\colon \ell_2 \to (\bigoplus_n Y_n)_{c_0}$ given by $u(\xi) = (\xi/n)_{n=1}^{\infty}$ for all $\xi \in \ell_2$. So, X is clearly Hilbertian and, as each Y_n is 1-homogeneous, X is also 1-homogeneous. We are left to notice that X does not completely isomorphically embeds into $\mathrm{MIN}(\ell_2)$. Since $\mathrm{MIN}(\ell_2)$ is homogeneous, it is enough to show that the identity $X \to \mathrm{MIN}(\ell_2)$ is not a complete isomorphism. For that, let $(e_n)_n$ be the canonical basis of ℓ_2 and for each $n \in \mathbb{N}$ let

$$x_n = \big[e_1 \ \ldots \ e_{2^n}\big] \in \mathrm{M}_{1,2^n}(\mathrm{MIN}(\ell_2)).$$

So, $\|x_n\|_{\mathrm{M}_{1,2^n}(\mathrm{MIN}(\ell_2))} = 1$ but $\|x_n\|_{\mathrm{M}_{1,2^n}(X)} \geq 2^{n/2}/n$ for all $n \in \mathbb{N}$. □

We are left to prove Theorem 1.8. For that, we will now turn our attention from embeddings to quotients. But first, we recall a definition: an operator $T : X \to Y$ between Banach spaces is called *strictly cosingular* if for all infinite dimensional Banach spaces Y_0 and all operators $q : Y \to Y_0$, we have that qT not surjective. Strictly cosingular operators form an ideal; see e.g. [1, Sect. 3.4] for more information.

Proposition 4.5 *Suppose X and $(Y_i)_{i\in\mathbb{N}}$ are operator spaces so* $\dim(X) = \infty$ *and any c.b. map from Y_i to X, $i \in \mathbb{N}$, is strictly cosingular. If there exists a c.b. surjection from $(\bigoplus Y_i)_{c_0}$ onto X, then X contains a copy of* $\mathrm{MIN}(c_0)$.

Before proving Proposition 4.5, we need a lemma. The following easy lemma is known to experts, but we provide a proof of it for the readers convenience.

Lemma 4.6 *For every surjection $T : F \to E$ between Banach spaces, there exists $\delta > 0$ so that $T + S$ is surjective for any operator $S : F \to E$ with $\|S\| < \delta$.*

Proof It is well known that an operator U is surjective if and only if its adjoint is bounded below. Thus, we can find $\delta > 0$ so that $\|T^*e^*\| \geq \delta\|e^*\|$ for any $e^* \in E^*$. If $\|S\| < \delta$, then

$$\|(T + S)^*e^*\| \geq \|T^*e^*\| - \|S\|\|e^*\| \geq (\delta - \|S\|)\|e^*\|$$

holds for any $e^* \in E^*$, showing the surjectivity of $T + S$. □

Proof of Proposition 4.5 For the sake of convenience, write $Z = (\bigoplus_i Y_i)_{c_0}$ and, for each $n \in \mathbb{N}$, let $Z_n = \bigoplus_{i \leq n} Y_i$ and $Z^n = \bigoplus_{i>n} Y_i$; as usual, we view Z_n and Z^n as subspaces of Z in the canonical way. Suppose $u : Z \to X$ is a completely bounded surjection. Then $\inf_n \|u|_{Z^n}\| > 0$. Indeed, otherwise, by Lemma 4.6, $u - u \restriction_{Z^n} = u \restriction_{Z_n}$ is surjective for n large enough. However, being a finite sum of strictly consingular operators, $u \restriction_{Z_n}$ is also strictly cosingular and, in particular, it cannot be surjective since $\dim(X) = \infty$.

A gliding hump argument then produces an increasing sequence $(n_j)_j$ of naturals and a sequence $(z_j)_j$ of normalized vectors such that $z_j \in \bigoplus_{n_{j-1}<i\leq n_j} Y_i$, for all $j \in \mathbb{N}$, and $\inf_j \|uz_j\| > 0$. Clearly, $(z_j)_j$ is weakly null, hence so is $(uz_j)_j$. By [2, Proposition 1.5.4], going to a subsequence if necessary, we can assume that $(uz_j)_j$ is a basic sequence. For simplicity of notation, let $x_j = uz_j$ for all $j \in \mathbb{N}$.

Fix $k \in \mathbb{N}$. Then, using the canonical identification $\mathrm{M}_k(Z) = \mathrm{M}_k \otimes Z$, for any $(a_k)_k \subset \mathrm{M}_k$, we have

$$\Big\| \sum_k a_k \otimes z_{j_k} \Big\|_{\mathrm{M}_k(Z)} \leq \sup_k \|a_k\|.$$

Therefore, we have that

$$\Big\| \sum_k a_k \otimes x_k \Big\|_{\mathrm{M}_k(Z)} \leq \|u\|_{cb} \sup_k \|a_k\|.$$

On the other hand, since $(x_k)_k$ is a basic sequence, one can find functionals $x_k^* \in X^*$, biorthogonal to x_k's[11] and such that $\sup_k \|x_k^*\| < \infty$. As for rank one operators the operator and c.b. norms coincide, we have that

$$\|a_m\| \leq \|x_m^*\| \Big\| \sum_k a_k \otimes x_k \Big\|_{\mathrm{M}_k(Z)}$$

for all $m \in \mathbb{N}$. As $k \in \mathbb{N}$ was arbitrary, $(x_k)_{k\in\mathbb{N}}$ spans a copy of $\mathrm{MIN}(c_0)$ in X. □

Note that, under the hypothesis of Proposition 4.5, X cannot embed completely complementably in $(\bigoplus_i Y_i)_{c_0}$. Moreover, we have:

Corollary 4.7 *Suppose X_0 is either R or C, and X is a Hilbertian operator space for which there exists a completely bounded surjection from X onto X_0. Further, let $(Y_i)_{i\in\mathbb{N}}$ be operator spaces such that any completely bounded operator from Y_i to X_0, $i \in \mathbb{N}$, is strictly cosingular. Then X does not completely isomorphically embed into $(\bigoplus_i Y_i)_{c_0}$.*

Proof Suppose, for the sake of contradiction, that $(\bigoplus_i Y_i)_{c_0}$ contains a subspace X' completely isomorphic to X. Find a completely bounded surjection $v : X' \to X_0$. By the injectivity of X_0, this extends to a c.b. surjection $\widetilde{v} : (\bigoplus_i Y_i)_{c_0} \to X_0$. However, such maps cannot exist, by Proposition 4.5. □

We highlight a corollary of the previous corollary:

Proof of Theorem 1.8 This follows from Corollary 4.7 since if $X_0 \in \{R, C\}$ we have that any completely bounded operator $\mathrm{MIN}(\ell_2) \to X_0$ is compact and, in particular, strictly cosingular. Indeed, suppose $u : E \to F$ is a compact operator and $q : F \to Y_0$ is a surjective

11 I.e., $x_k^*(x_k) = 1$ and $x_k^*(x_j) = 0$ for all $k \neq j$.

operator with $\dim(Y_0) = \infty$. Then, the open mapping theorem implies that the induced map $\widetilde{qu} : F/\ker(qu) \to Y_0$ is an isomorphism. However this is impossible since $\widetilde{qu}$ is compact and $\dim(Y_0) = \infty$. □

Remark 4.8 A version of Theorem 1.8 was proven in the proof of [3, Theorem 4.2]. However, this proof contains a gap in its last paragraph. Theorem 1.8 fixes this gap and [3, Theorem 4.2] remains valid with no changes in its statement. We also point out that, fixing the gap of [3, Theorem 4.2] is much easier and requires only some small modifications in its proof. But we chose to present this different proof here since it leads to a more general result (Proposition 4.5) and applications (Corollaries 4.7 and 4.9).

We finish this paper with another application of Proposition 4.5:

Corollary 4.9 *Suppose E and $(F_i)_{i\in\mathbb{N}}$ are Banach spaces, with* $\dim E = \infty$. *Let $(Y_i)_{i\in\mathbb{N}}$ be a sequence of operator spaces such that Y_i is completely isomorphic to* $\mathrm{MIN}(F_i)$*, R, or C. Then there is no c.b. surjection from $(\bigoplus_i Y_i)_{c_0}$ onto* $\mathrm{MAX}(E)$.

Proof Note that $\mathrm{MIN}(c_0)$ (just as any minimal space) is exact, hence, by [12, Corollary 2.9], $\mathrm{MAX}(E)$ does not contain a completely isomorphic copy of $\mathrm{MIN}(c_0)$. To apply Proposition 4.5, one therefore has to verify that, for each i, every completely bounded operator $Y_i \to \mathrm{MAX}(E)$ is strictly cosingular.

First assume Y_i is completely isomorphic to $\mathrm{MIN}(F_i)$. By [13, Sect. 4], $\mathrm{CB}(Y_i, \mathrm{MAX}(E))$ coincides with $\Gamma_2^*(F_i, E)$; this ideal consists of operators $T : F_i \to E$ for which there exists a factorization $T = vu$, with $u \in \Pi_2(F_i, H)$ (H is a Hilbert space) and $v \in \Pi_2^*(H, E)$ (that is, v^* is 2-summing); see [8, Chap. 7] for more information. Therefore, we have to show that, for any E and F, any element of $\Gamma_2^*(F, E)$ is strictly cosingular. For that, fix $T \in \Gamma_2^*(F, E)$. By the preceding discussion, T^* is 2-summing. If $q : E \to E_0$ is a quotient map and $\dim E_0 = \infty$, then T^*q^* is 2-summing as well, hence not an isomorphic injection. Thus, qT is not a surjection.

Now suppose Y_i is completely isomorphic to either C or R. By [14, Proposition 5.11], any c.b. map from $\mathrm{MIN}(G)$, for any operator space G, to R or C is 2-summing. If $T : Y_i \to \mathrm{MAX}(F)$ is c.b., then T^* is 2-summing. Then T^* is not an isomorphism on any infinite dimensional subspace; this establishes the strict cosingularity of T. □

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