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Removing Splitting/Modeling Error in Projection/Penalty Methods for Navier-Stokes Simulations with Continuous Data Assimilation

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Abstract. We study continuous data assimilation (CDA) applied to projection and penalty methods for the Navier-Stokes (NS) equations. Penalty and projection methods are more efficient than consistent NS discretizations, however are less accurate due to modeling error (penalty) and splitting error (projection). We show analytically and numerically that with measurement data and properly chosen parameters, CDA can effectively remove these splitting and modeling errors and provide long time optimally accurate solutions.

AMS subject classifications: 65M12, 65M15, 65M60

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1 Introduction

Data assimilation has become a critical tool to improve simulations of many physical phenomena, from climate science to weather prediction to environmental

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forecasting and beyond [17, 38, 40]. While there are many types of data assimilation, one with perhaps the strongest mathematical foundation for use with PDEs that predict physical behavior is called continuous data assimilation (CDA). CDA was developed by Azouani, Olson, and Titi in 2014 [1], and has since been successfully used on a wide variety of problems including Navier-Stokes equations [1], Benard convection [20], planetary geostrophic models [22], turbulence [9, 13, 23, 42], Cahn-Hilliard [18] and many others. Many improvements to CDA itself have also been made, through techniques for parameter recovery [10], parameter estimation [11, 21, 44, 45, 48] sensitivity analysis with CDA [14], numerical analysis [18, 28, 30, 36, 37, 39, 51], and efficient nudging methods [51], to name just a few. You can also find various extensions of CDA of Azouani, Olson, Titi in [3–7, 12, 26].

CDA is typically applied in the following manner. Suppose the following PDE is the correct model for a particular physical phenomenon with solution $u(x,t)$:

$$\begin{aligned} &u_t+F(u)=f,\\ &u(x,t)|_{\partial\Omega}=0,\\ &u(x,0)=u_0(x). \end{aligned}$$

Suppose further that part of the true solution is known from measurements or observables, so that $I_H(u)$ is known at all times, with I_H representing an appropriate interpolant with max point spacing H. Then the CDA model takes the form

$$\begin{aligned} &v_t+F(v)+\mu I_H(v-u)=f,\\ &v(x,t)|_{\partial\Omega}=0,\\ &v(x,0)=v_0(x), \end{aligned}$$

where $\mu>0$ is a user selected nudging parameter. For many such systems, given enough measurement values it can be proven that the solution v is long time accurate regardless of the accuracy of the initial condition v_0 (often CDA analyses assume $v_0=0\neq u_0$). In numerical analyses, accuracy results of CDA enhanced discretizations can often avoid error growth in time since application of the Gronwall inequality can be avoided, leading to long time optimal accuracy results [27, 28, 51].

The purpose of this paper is to study CDA together with two commonly used discretizations of the Navier-Stokes equations (NSE), the projection method and the penalty method. The projection method is a classical splitting method for the NSE developed independently by Chorin and Temam [15, 60], and is based on a Hodge decomposition. The penalty method removes the divergence constraint but replaces it with a divergence penalty in the momentum equation. Both of