PNAS RESEAROIMncu | APPLIED PHYSICAL SCIENCES

Geometric control of topological dynamics in a singing saw

Suraj Shankar[a], G. Petur Bryde[a,1], and L. Mahadevan[a,b,c]

Edited by Vlnceruo Vitelll, University of Chicago. Chicago, IL; received S<,ptember 19, 2021; aa:epted March 3. 2022, by Edltorlal Board Member Paul Chalkln

The oommon handsaw can be oonverted into a bowed musical instrument capable of producing exquisitely susruned notes wben its blade is appropriately bent. Aooustic modes locali7.edat an inflection pointare lmown to underlie tbe saw's sonorous quality, yet tbe origin of localuation bas remained mysterious. Here we uncover a topological basis for tbe existence of locali7.ed modes tbat relies on and is protected by spatial air-vature. By combining experimental demonstrations, theory, and oomputation, we sbow bowspatial variations in blade airvature oontrol tbe localii.ation of tbese trapped states, allowing tbesaw to function as a geometrically tunablehigh-quality oscillator. Our work establishesan unexpected oonnection between tbe dynamics of tbinsbe!Js and topological insulatorsand offersa robust principle to design high-quality resonators acrossscales, &om macroscopic instruments to nanoscale devices, simply tbrougb geometry.

mUSicalacoU:Stics | topological insulators | thin elastic shells

Significance

The ability to sustain notes or vibrations underlies the design of **most acoustic devices, ranging** from musical instruments to nanomechanlcal resonators. Inspired by the singing saw that acquires ks musical quality from Its blade being unusually bent, we ask howgeometry canbe used to trap and insulate acoustic modes from dissipative decay In a continuum elastic medium.By using experiments and theoretical and numerical analysis,we demonstrate that spatially varying curvature in a thinshell can localize topologically protected modes atInflection lines, akin to exotic edge states in topological insulators. A key feature is the ability to geometrically control bothspatial localization and the dynamics of oscillations In thin shells. Our work uncovers an unusual mechanism for designing robust, *yet* reconfigurable, high-quality resonators across scales.

Musical instruments, even those made from everyday objects such as sticks, saws, pans, and bowls (l}, must have the ability to cteate sustained notes for them to be effective. While this ability is often built into the design of the instruments, the musical saw, used to make music across the world for over a century and a half (2), is unusual in that it is **just a carpenter's saw but held in an unconventional manner to allow it to sing. When** a saw (Fig. IA) is either bowed or struck by a mallet, it produces a sustained sound that mimics a•soprano's lyric trUI" (3). Importantly, for such a note to be produced, the blade cannot be flator bent intoa J shape (Fig. I*B)* but must be bent intoan *S* shape (Fig. IC). **This geometric transformation allows the saw to sing and is we.U known to musicians** who describe the presence of a•sweet spot," i.e., the inflection curve in the S-shaped blade; bowing near it produces the clearest notes, while bowing far ftom it causes the saw to fall silent (3). Early works (4, 5}, including notably Scott and Woodhouse (6), attempted to understand this peculiar feature byanalyzing the leariud vibrational modes of a thin elastic shell (7, 8}. Through a simplified asymptotic analysis, they showed that a localized vibrational eigenmode emerges at an inflection point in a shell with spatially varying curvature and is responsible for the musicality of the saw. Recent works have reproduced this result using numerical simulations (9, 10), but a deeper understanding of **the origin of localii.acion has remained elusive.**

Asimple demonstration of playing the saw quickly reveals the robustness of its musical quality to imperfections in the saw, irregularities in its shape, and the precise details of how the blade is flexed. Fig. ID shows a time trace and spectrogram of the sawclamped in either a J shape or an *S* shape (Fig. I Band C) when struck or bowed near the sweetspot. The dull and short-lived sound (Audio I) associated with the J shape might be contrasted with the nearly pure tone (,s,595 Hz) lasting several seoonds {Audio 2} when the saw is bowed whileshaped like an S. While the pitch can be varied by changing the curvature of the saw, the sustained quality of the note is largely indifferent to the manner of excitation and the specific nature of the clamps, as long as the inflection point is present.

The lack of sensitivity to thesedetailssuggestsa topological origin for the localiud mode responsible for the saw's strikingsonority. That topology can have implications for band structuresand the presence of edge oonductingstates even when the bulk is insulating was originally explored in electronic aspects of condensed matter to explain the quantization of the Hall oonduaance (11) and led to the prediction of topological insulators (12, 13}. More recently, similar ideas have been used to understand the topological properties **of mechanical excitations, e.g.. acoustic and floppy modes in discrete periodic lattices** (14-17}, in continuum elasticity (18-21}, in fluid dynamics in geophysical and active mattersystems(22-25). etc.In manyof the aforementionedsystems, the breaking of time-reversal symmetry leads to the appearance of topologically protected modes. Alternately, **in the absence of driven or active elements, spatial symmetries of a unit cell can also be** used to achieve topological modes via acoustic analogs of the quantum spin or valley Hall effect (I7. 26-29), although these examples rely on carefully engineered periodic lartices. Here we expand the use of topological ideas to continuum shells and show that

AUthot c:ontrb.Jtiors: LM. ttSe<ll'ch; S.S.., P.B., and LM. designed resea S.S. and P.B. perlotmed resean:tc S.S.. P.B.. and LM. contrtbuted nf!# analytJc S.S., P.B..and LM.analyzed data;andS.S., P.B.,aod LM.'M'Ote the paper.

'Theauthotsdedate nocom lntetest

lhiS ¥tidetsa PNAS Direct Stb'l'llS:sion. V.V. ts a g.Jest ecltor nftedby the Editodal Boatd.

[1]s.s.and P.B.cootlbrtede(JJ,illtyto thiswont.

11'0 whOm COffespondence may be adci'essed.. Emal: hlahade.@g.hatvKd.edu.

lhiS altide oontainS "« lnfotmatien onlne at httpS'//ww#-4)nas.otgllooicup/ Udot 10.10731pnas. 21172411191-/DCSupplemental

Aptl21,20ll.

underlying the time-reversible Newtonian dynamicsof thesinging **saw is a topological invariant that characteri?.eS the propagation** of waves in thin shells, arising from the breaking of up-down **inversion symmetry by curvature.**

Results

Continuum Model of ThinShell Dynamics. Thesawis modeled as a very thin rectangular elastic shell (thickness*h* /4: *W* < *L,* where *W, Lare* the width and length of the strip) made of a material with Young's modulus *Y,* Poisson ratio *v,* and density *p* (Fig. *IE).* Its geometry is characterized by a spatially varying curvature tensor (second fundamental form) **b(x),** where **x** = *(x, y) is* the spatial coordinate in the plane. *As* the saw is bent only along the {long) *x* axis, *b..,(x)* = *b(x)* is the sole nonvanishing curvature. To describe its dynamical response, we take advantage of its slenderness and treat the sawas a thin elasticshell thatcan be bent, stretched, sheared, and twisted. Before movingto a computational model that accounts for these modes ofdeformation as well as real boundary conditions, to gain some insight into the problem and **expose the topological nature of elastic waves, it is instructive to** instead consider a simplified description valid for shallow shells with slowly varying curvature.

In a thin shallow shell *(hlb,11*/4: 1), as bending *is* energetically cheaper than stretching (30), shear becomes negligible **(Q""0;** Fig. *IE),*and in-plane deformations propagate much more rapidly (at the speed of sound c = ,J'Vlµ) so that the depth-averaged stresses can be assumed to equilibrated, i.e., $B_1 u,_1 = 0$ (6,31). In **th.is limit, usingthesolution of these equations in terms of the *Ai*** stress function *x (u,₁* = *'P,/J²x,* where *'Pt:1* = *6'1 - B,0₁f'V* is a projection operator; *SI Append;x,* section 3), the in-plane geometric compatibility relation and the linearized dynamical equations for transverse motions can be written as (7. 32)

$$\frac{1}{Yh}\text{"}i1^4 x = -'P,1\text{"}il^2(b,11), \qquad [I]$$

$$phB'fj = -1\text{'"}i1^4 f + b,1'Pt:1\text{"}i1^2 x. \qquad (2)$$

Here *f* is the out-of-plane deflection of the shell (Fig. IE) and the bending rigidicy"-= $Yh^3/(12(1-v^2))$. Crucially, in-plane and flexural (out-of-plane} modes remain geometrically coupled in the presence of curvature even in the linearized setting (Eqs. I and 2). For a shell bent with constant curvature along the *x* axis, i.e., a section of a uniform cylinder, *b(x)* = bo is a constant. In the bulk of the system, disregarding boundaries, we can Fourier transform Eqs. I and 2 using the solution ansatz/ = *f,e-<w<+<••* to obl:tin the dispersion relation for flexural waves to bew±(q) = ±*J(1'/ph)q*⁴ + *c²b6(qy/q)*⁴ (Fig. 2A}, where *q* = lql. When **<Jv = 0, i.e.. the sheet is undeformed in the transverse direction,** it remains developable (with generators that run parallel to the *y* direction), and ,he bending waves are gapless, i.e., *w* 0 as *q* 0. Howeve.r, when *qy ,f* 0, a finite frequency gap ~ clbol (in addition to finite *qy* corrections) controlled by the speed of sound and the curvature of the shell emerges as *q* 0 (Fig. 2A). **Intuitively, this arises due to the geometric coupling berween** bending and stretching deformations in acurvedshell which leads to an effective stiffening that forbids wave propagation below a frequency threshold. Similar spectral gaps appear in curved filaments and doubly curved shells as well (31, 33).

For the $-shaped saw, curvature scales of *b*~0.4 to 0.8 m-¹ are easily achievable (as in Fig. I *B* and C), while the typical sound speed in steel is c ~5 to 6 x 103*mis* so that the frequency gap is of order 2 to 5 kHz.. Comparing these estimates to the spectrogram in Fig. *ID* (further quantified in Fig. 3) suggests that the localized mode excited upon bowing the S-shaped saw (Fig. IC) lieswithin the frequencygap. The}-shaped saw(Fig. IBJ alsovibratesat lowfi-equencies(compared to thegap)whenstruck, presumably through the *'iv* = 0 branch of delocalized flexural modes, although higher frequencies above the band gap can be excited by careful bowing *(Sf Appendix,* Fig. SI *A* and BJ.

Curvature-Induced Z_2 Topological Invariant. To unveil the topo**logical structure of the vibration spectrum of the saw, we cast** thesecond-order dynamical equations (Eqs. I and 2) in terms of first-order equations by raking the square root of the dynamical matrix (14,34). Focusingon the flexural modes alone, we obtain a Schrodinger-like equation for the transverse deflections of a shallow shell *(Sf Append;x,* section 3),

$$\frac{i}{c}\partial_t \Psi = \mathcal{H}\Psi, \quad 'Ii = \begin{pmatrix} 0 & \nu t \\ V & 0 \end{pmatrix}, \qquad (3)$$

where1¥ = *(cV¹f, iO,/)*and*V* = *iJi</ Yh"i/²* + *bt:1'Pt:1* and t represents the conjugate transpose. Theeigenvalues of the effective Hamiltonian *1i* are given by the previously derived **(w±(q)},** and its complex eigenvectors**W±(q)** encode the topology of the band structure. The singularities in the arbitrary phase of the eigenvectors signals nontrivial band topology. To understand the phase of eigenvectors along the saw's long direction, we can consider fixing the transverse wave vector *'iv ,f* 0, leading to an effective one-dimensional (ID) system along the *x* axis.Then the **obstruction to continuously define the phase of the eigenvectors at every *qx* in Fourier space while respecting all the symmetries of** the problem is quantified by the ID Berry connection *A(q.)* = i En=± *'¥n(q,)¹0q,*'Vn(*q.)* (the *qy* dependence is suppressed) (35, 36). However, what are the symmetriesof our elastodynamic **system?**

One important symmetry is that imposed by classical time**reversal invariance in a passive. reciprocal material** {C : *x x, t -t, '¥ w•* ; *Sf Appmdix,* section 3), which maps forward **moving waves into backward moving ones and guarantees that** eigenmodes appear in complex-conjugate pairs (34). A second symmetryspecial to the saw is an emergent spatial reflection symmetry in the local tangent plane (11: *x -x, t t, '¥ '¥* ; *Sf Appendix,* section 3), which originates from the uniaxial nature of the prescribed curvature along the *x* axisand the insensitivity of bending to the orientation of the local tangent plane, a symmetry **that is inherited from 3D rotational invariance. The latter *is* easily** seen by noting that the bending energy only involves an even number of gradients via "i/² *f.* Upon simulraneously enforcing both dynamical and spatial symmetries, a new topological obstruction posed by curvature emerges and is quantified by a *'Li.* index *(Sf Appendix,* section 3).

$$(-1)^{\text{"}} = \exp\left[i\int^{\text{""}} dq,A(q,)l\_ : \right)' \qquad (4)$$

simUar to topological insulators with crystalline symmetries (37-39). Pf(W) denotes the Pf.,ffian of the antisymmetric overlap matrix *W<:1(qz)* = *w,(q,)ICITw₁(q.) (i,j* =±).We note that unlike the mechanical Su-Schrieffer-Heeger chain (14) that exhibits a topological polariution in ID. the emergent tangent-plane spatial reflection symmetry in our problem forces this polariiation to vanish *(Sf Appendix,* section 3).

***As* we work in the continuum, only di.ffe.rences in the topo**logical invariant are well defined indepe.ndent of microscopic **details. Across an interface at which curvature changes sign. i.e .. a curvature domain wall, the jump in the topological invariant is** given by

$$(-1f\text{"} = \text{sgn}(b<b>), \qquad (5)$$

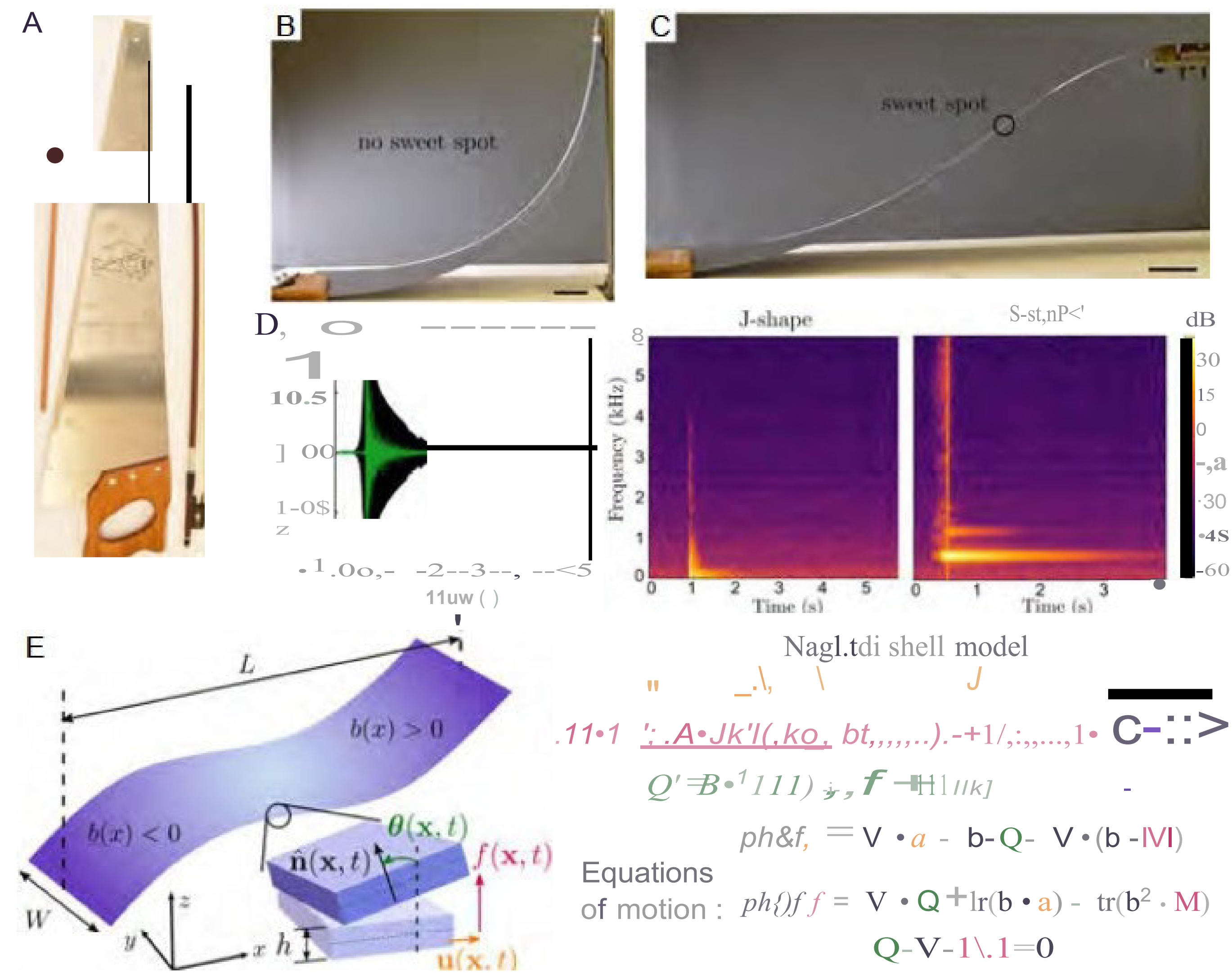

Fig. 1. Themusical sawand its mathematic.al model.(A) Aviolin bow andmallet placedalongside the saw. Weclamp the saw in two configurations:(B}aJ shapeand(Qan Sshape. whichis required to pl.l)tmusic. The primarydistinct:ion between thehw is d\atC hasan inflection point(thesweet spot) in its profile. while 8 hascurvature ofconstant sign. (Scale bar, S cm.) (D)(left) nme series of the rormalizedaudio signalwhen the saw in Bisst.rude{green)andwhen the s.1w in C is bcwed (black).(Middleand Rigtt) The corresponding spectrograms for both th! J shape(8) and the S shape (Q. The signal decays rapidly for the J Shapewith a wider spread in frequency, while for the S shape, a singledominant note with w 595 Hz survives the ringdown of the blade lasting several seconds. (E)A sc.hematic of a blade o(length L. widthW, and thickness h is sketchedwith a uniaxial auvature profile bxit(x) = b(x) th.atchanges signalorg the x axis as in 8. The saw can be modeled as an elastic shell whose def'ormations include an in-plane displacement u.a midsu.rface deflection/ ncrmal to the shell and a rotation 80(the local normal" asdegreesof freedom [x =(x.y} is the sp,1tial coordinate). Elastic tensors A ' and f:i/ enter theconstitutive equations (Subscriptsdenotes symmetrization} for the in-plane stress(o}. bending moment (M). and transverse sh-ear(Q) (SIAppendix. section2). Der'ivcltives are interpreted as <OVariant. and index manipulations emplOJthe reference metrico(the shell (SIAppendix. section2). The Kirchhoff limit for a sn.tllcw shell simplifies thedynamicsto a =0. phtif'f= . M +tr(b •u).alongwith 8 = . SIAppendix. sections2 and 3}.

where $b_<$ and $b_>$ are the curvature on either side of the interface (*SJAppmdix,*section 3). This expression directly demonstrates that the two oppositely rurved sections of the saw be.have as topologically nontrivial bulk systems, with a $t:i.v = 1$, that meet at the inflection line that functions as an inte,mal edge. *As* a result, nontrivial band topology underlies the emergence of the localized midgap mode, endowing it with robustness against details of the rurvature profile and weakly nonlinear deformations (*SJAppendix,*section 3).

Numeric.al Mode Structure and Localization. We test these predictions by numerically computing the eigenmodes of a finite elastic strip of length $L = 1$ m, width $W = 0.25$ m, and thickness $h = 10^{-3}$ m. For our shell model, we move away from the K;rchhoff model for shells and account for the ldnematia associated with shear in addition to those associated with bending and stretching, as they effectively reduce the numerical ill-conditioning commonly seen in high-order continuum theories for slender plates and shells while allowing for numerical methods that require less smoothness and are easier to implement (*SJAppend;x);* together, these allow for better computational acruracy.This framework fonns the basis for the Naghdishell model (40) (see *SJ Append;x,*section 2, for details) and accounts for an in-plane displacement vector alongthe midsurface $\mathbf{u}(\mathbf{x}, t)$, an out-of-plane deffection $f(\mathbf{x}, t)$ nornul to the shell, and an additional roration $\theta(\mathbf{x}, t)$ of the local normal itself (Fig. IE). These modes of deformations lead to depth-averaged stress resultants associated with stretching *(u),* bending **(M)**, and shear (Q) as shown in Fig. IE. The resulting covariant nonlinear shell theory along with inertial Newtonian dynamics provides an accurate and computationally tractable description of the elastodynamics of thin shells (Fig. IE and *SJ Appen,i;x,*section 2). To highlight the topological robustness of our results, in our calrularions we vary both the boundary conditions and curvature profiles imposed on the saw.

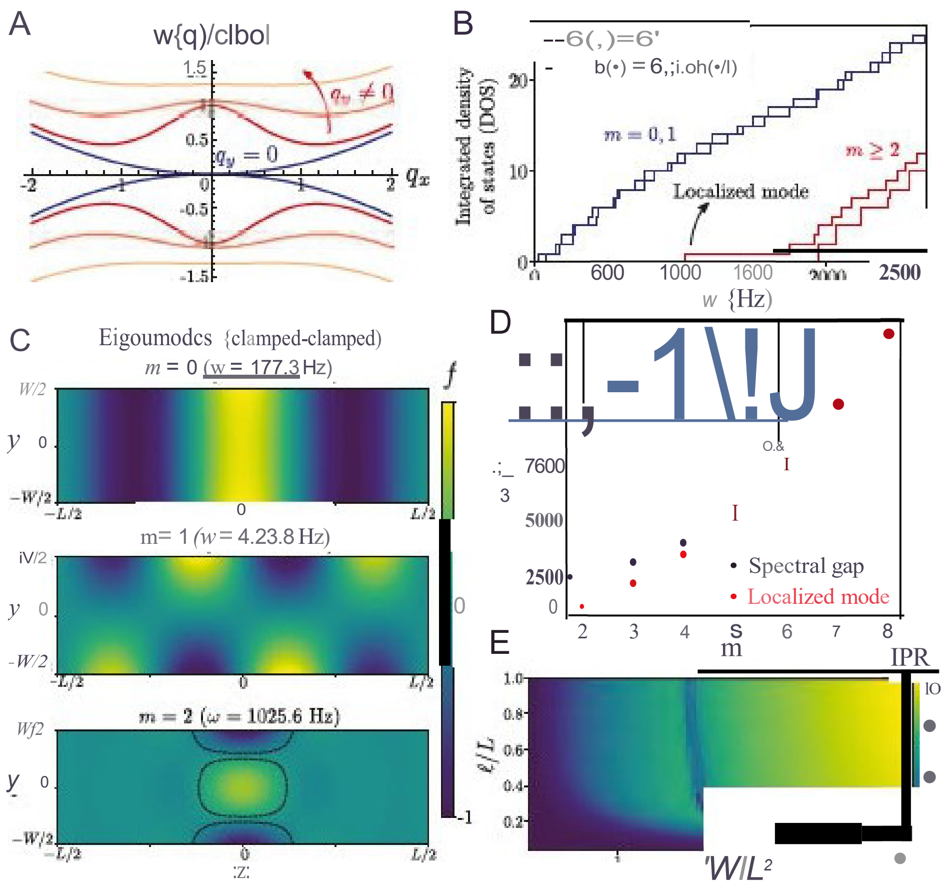

Fig. 2. Eigenmodes, bandstru<t.u.re, andtopological loc.alization.(A)Analytical dispersioo relation mmputed for an infinitely long strip with constant QJNature alorg the x axis(hlbol =5 x 10-s. v =1/3). The blue ru.rvescorrespond to theqy =0 gapless modes, and the red ru.rves withqy ¢ 0 havea finite frequency gap.(8) Numerically computed integrated density ofstates for a finiterurvedstrip (boL =05, t/L =0.1, L =1 m)with clamped-cfamped bourdarycondtions. Developable eigenmodes(blue; labeledby discrete mode numbers m =O.1, akin to =0 in thecontinuum)aregapless for bothconstant curvatu.re(dashed) and the Sigmoid profile (solid). Higher m<Xles (red; m 2) exhibit a finite gap ~2 kHz for constant cu.rvature {dashed). while the sigmoid profile features a localized mode (w ~1 kHz) at the inflection point witt.n the bulk band gap. (Q Numer'ic.al eigenmodes for the sigmoid profile with the local normalized deflec.tion/ plotted (dashed linesare 10% isoc.ortou.rs}. Low-frequency delocalized states with m =O(Top). m =1 *(Middle,)* and the first localized mode with m =2 (£bttom). (D) Frequen<yof the localized modes *(Inset* sh<Ml"s normalized deflectionat y =W/2 along x) and <«responding spectral gap for inc:reasirg transverse mode nt.m.ber m 2.(E) Inverse participaticr,;ratio *of* the first localized mode for a piecewise linear curvature profile.plcttedagainst the a.uvature gradient b' and the length scale of curvatll'e variation t.

In Fig. 2B, the distribution of eigenmodes as a function of frequency is shown in the integrated demiryofstatesfor a constant curvature shell, b(:z:) = bo (dashed lines), and an S-shaped shell with a smooth curvarure profile b(:z:) = bo tmh(:z:/t) (solid lines) that varies over a width t near the inflection point at x =0 (i.e., a curvature domain wall). In both cases, the ends of the strip are kept clamped, and the spectra are calculated using an open-source code based on the finite element method (41, 42). As the curvature of the S shape approaches a constant ±bofar from the origin, the bulk spectral gap and delocaliud modes match that of the constant curvature case.. FJexuraJ modes that vary at most linearly in *they* direction· (labeled by discrete mode numbers m = 0, 1 due to the lack of translational invarianc,,) correspond to linearized isometries; they delocalize over the entire ribbon {Fig. 2 C, *Top* and *Middle)* and populate states all the way to zero frequency, i.e., with a gaplessspectrum. This is true for both constant curvature (dashed blue line, Fig. 2B) and the S-shaped shell (solid blue line, Fig. 2B) as these bulk modes are unaffected by curvature. In contrast, all other modes that bend in both directions (m ;:: 2) are generically gapped fora constant curvature profile (dashed red line, Fig. 2/J) as expected. However., for the S shape, in addition to the gapped bulk modes, a new modeappears within the spectral gap (solid red line, Fig. 2B). This midgap state (shown here for m = 2) is a localized mode that is trapped in the neighborhood of the inflection line {Fig. 2 C, *Bom,m).* For increasing mode number m ;:: 2, simUar topological modes appear within the bulk bandgap, withgrowing localii.ation lengths (Fig. 2 *D, lnret)* and higher frequencies (Fig. 2D), as predicted analytically *(SI Append;x,* section 3). Qualitative.ly, the presence of an inflection line in the S-shaped saw makes it geometrically soft there; the generators of cylindrical modes are now along the length of the saw, and the curved regions on either side that are geometrically stiffserve to insulate the soft internal edge from the real damped edges.

Of panirular note is that the localized modes, unlike the extended states, are vinually unaffected by the boundaries and the conditions there (see *SI Appendix,* Fig. S2A, for eigenmodes in a strip with asymmetric boundary conditions where the left edge is clamped and the right edge is free). Spatialgradients in curvature, however, do impact the extent of localii.ation. We demonstrate this using a piecewise continuous curvamre profile that has a constant linear gradient b' over a Jength t, aero the origin and adopts a constant curvature outside this region. By varying both the curvarure gradient b' and the length scale t, we can tune the localization of the lowest topological mode {same as Fig. 2 C, *Borum,),* quantified b the inverse participation ratio IPR= Jclxl/(x)l⁴ /(Jclxl/(x)l)2 (Fig. 2£). Strong localization (high

"EigetYnodesina *trite* sawnudtovaiy a least<JJ,adtatk.aly , -e., m 1) in«dl'J"to bendil'lg.w'ldue: defOffll.dons t-.the 5t'l8l

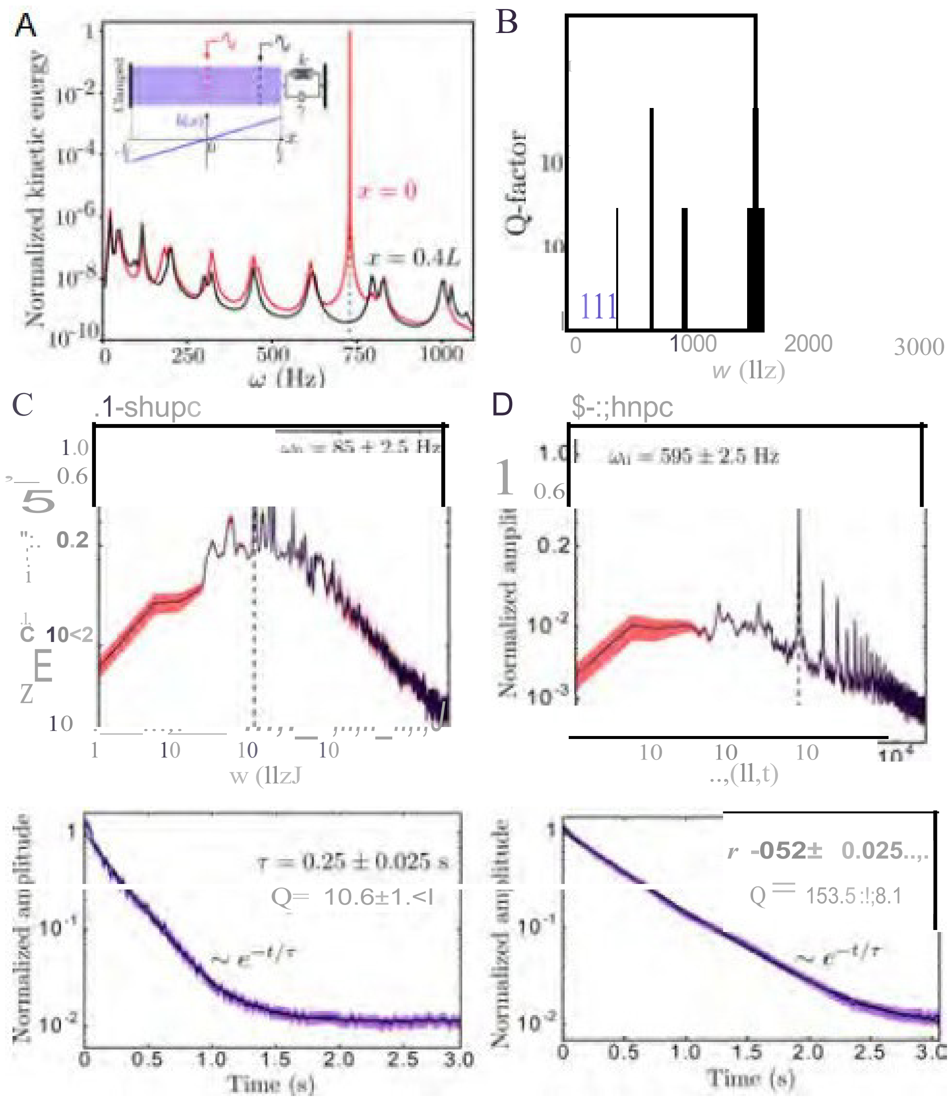

Fig. 3. **Dissipative dynamicsand high-quality oscillators. (A) Rescnance wrves for a shell with a linear curvature profile (Ins.et) periodically driven at the inflection point (x =O; red) and awa, from it (x =0.4L; blac:k) for varying frequen<y *(w* 740 Hz mrresponds to the first localized mode). (B) Numerically computed Q factor shewsdramatic enhancement at localized mode frequencies (red) O\erdelocalized modes (blue). (CandD} Experimental measuremert *d* Q factor {see *SI Appendix.* section1 for detailS) for the musical sawina (QJ shape (Fig. 18) and (O)S shape (Fig. 1C} (Top) Note the normalized Fourier spectrum amplitude ison a log scale below 0.1 and linear above., with the peak fre(Jlency marked as .(Boltom) The average signal decay (blue a.uve) is fit to a single dec.ayil'€ exponential (black OJ.rve). The st\3ded region is the SE in both *C* and0.**

IPR) is quickly achieved for sharp gradients in curvature (IPR oc JI/II/h; *SI Appendix,* section 3) as long as the length scale of curvature variation is not too small (£/ *L;?:* 0.1, Fig. 2£}, corresponding to a diffuse domain wall. In the opposite limit *oft->* 0 for *b'l* = *ho* fixed, i.e., a sharp domain wall with a discontinuous curvature profile *b(x)* = bosgn(x), strong locali1.ation persists *(SI Appendix,* Fig. S3), consistent with our topological prediaion **and demonstrating the ease of geometric control of localization.**

Geometrically Tunable High-Quality Osc.illatol'\$. The boundary insensitivity of topologically localized modes has important dynamic consequences that can be harnessed to produce high-**quality resonators. The primary mode of di ipation in the saw,** as in nanoelectromechanical devices (43), is through substrate or anchoring losses at the boundary. Internal dissipation mechanisms (from, e.g., plasticity, thermoelastic effects, and radiation losses), although present, are considerably weaker and neglected here. To model dissipative dynamics, we retain damped boundary conditions on the lefr end and augment the right boundary to include a restoring spring {k} and dissipative friction (-y) for both the in-plane forces and bending moments (Fig. 3 *A, Inset,* and *SI Appendix,* section 2). Informed by F',g. 2£, we choose a linear rurvature profile spanning the entire length of the shell to obtain a strongly localized mode. Upon driving the shell into steady oscillations, with a periodic point force applied at the inflection point (x = 0 ; Fig. 3A, red curve}, we see an extremely sharp resonance peak right at the frequency of the first localized mode (Fig. 3A). In contrast, when the shell is driven closer to the boundary *(x = 0.4L;* Fig.3A, black curve}, the response is ac least sixordersof magnitude weaker as the localized mode is not excited and only the delocalized modes contribute. Locali1.acion hence protects the mode from dissipative decay, unlike extended states that dampen rapidly through the boundaries. We further quantify this using a Q factor computed from unclriven relaxation of the shell initialized in a given eigenmode *(SI Appendix,* section 2). Ultrahigh values of Q''' 10^5 to 106 are easily attained when a localized mode is excited (Fig. 3B, red}, well over the Q factor of all other modes (Fig. 3B, blue}. Similar results are obouned **for other curvature profiles as well., such as a sigmoid curve** *(SI Appendix,* Fig. S2B).

To compare these computational results with experiments. we perform ringdown measurements on a musicaf saw {see *SI Appendix,* section I, for details) damped in both the J shape {Fig. *IB)* and the S shape {Fig. IC). *As* indicated by Eq. 5. the keydistinguishing feature of theS-shaped saw {compared to the J shape) is the presence of an inflection line (curvarure domain wall) that engenders a well-localired domain waJI mode capableof sustaining long-lived oscUlations. The normali:r.edFourier spectra and exponential decay {-r) of the signal envelope are shown in Fig. 3C U shape) and Fig. 3D (S shape) with the dominant frequency *(wo)* marked. We find a factor ~15 enhancement in the Q factor(Q = *wo-r* /2) forthe S-shaped saw (Q""150; Fig.3 *D, Left)*over theJshape (Q""10; Fig. 3 C, *Left)*. We emphasi1.e that thissignificant quality factor improvement, although not as dramatic as the numericallycomputed Q factors (Fig. 3B). is still striking given the initial impulse {mallet strike for J shape and bow for S shape; see *SI Appendix,* Fig. SI. for other cases) excites an uncontrolled range of frequencies and othersources of energy loss including internal damping are presumably also present

Discussion and Conclusion

Our combination of analysis, finite element simulations and experiments has demonstrated that a sawsings because its curvature generates a frequency gap in the acoustic spectrum which closes at an inflection point (line) that acts asan interior edge allowing a locali:r.ed mode to emerge within the band gap. Unlike mechanisms of weak localization (44, 45) or well-known whispering gallerymodes {30,46) that relysensitivelyon details of thedomain geometry, our topological argument explains the existence of locali:r.edsound modes and their robustness against petrurbations in the musical saw, providing a framework to explore not just topologicalmechanics but also dynamics in thin platesand shells.

The ability to control spatial geometry to trap modes at inte.rfaces in the interior of the system offers a unique opporwnity to design high-quality oscillators. *As* our results are material independent, they apply equally well to nanoscale electromechanical <esonators (47, 48) and provide a geometric approach to design high-quality resonators without relying on intrinsic nonlinearities (49). Just as in the musical saw, in nanomechanical devices, dissipation can be dominated by radiation through the clamped boundary (43). Current on-chip topological nanoelectromechanical metamaterials use carefully patterned periodic arrays of nanomembranes to control localized modes in robust acoustic waveguides (50, 51).Our worksuggests an alternate strategy inspired by the singing saw, which relies solely on the scale separation intrinsic to any curved thin *sheet* by manipulating curvawre spatially, topological modes localized in the interior hence remain vibrationally isolated and decay extremely slowly, allowing ultrahigh-quality oscillations, perhaps even in the ultimare limit ofatomically thin graphene (52).

Materials and Methods

Extended datl on theexperimentsandthedetlilsofthe numerirarnodelingand theoretiralralculationsare providedin *SI Appendix*.

Saw E>cperhnents. Thewooden handleof themusiral saw (Wentworth) was clampedontoanoptiral tlble, whilethetlpeced end of thebladewasattached toa sliding metal brackethrn ountedontoa vertiralguide rail.Corl:discs(around 2anindiameterand0.5to1 aninthickness)wereusedtocushionandsoftly supporttheclampedendofthe blade.This helpeddampoutoscillations at the saw end and reducedany high-frequencyringingarising from direcrn etlk>n· metalcontlct.Thesaw bladewasbentintotwoconfigurations,aJshape(Fig.18) andan5shape(Fig.1C),andh anuallyeitherstruckw ithamalletor bowedw ith aviolinbowat*the*straight*edge,* bothnearthecenteroftheblade.Thebladewas allowed to*freely* ringdown postextitltion.The audiowas recordedusinga U5B microphone(Fifinetechnology,K669·K669B,samplingfrequency*f,* = 44.1kHz) placednearthesawandanalyzedusingthesoftwareAudacity.

Multiplemeasurementsofthe ringdownsignal,eachlasting 5 to6 s,were madewitha gapofa fewseconds between runs.Aseparate 10to15s audio samplew itha stationarysaw was usedasa templatetofilter any background noise usingthein-built noisereductionfunctionalityinAudacity.The denoised audiosampleswere thenanalyzed usinga customPythoncode.Bothleft and right (stereo) channels arestronglycorrelatedwith each other, sowe simply awragedthetwotogetthesignalforeachrun.Uponusinga Hannwindowand Fourier transformingeachsignal,we binnedthefrequencyaxiswitha binsize of *t>w* = 5 Hzandawraged the normalized(by thm aximum)magnitudeof the Fouriertransformowrdifferent runs*(N* = 26: J shape,mallet *N*= 28: S shape, bow).

The average spectrum(normalized) is plotted in Fig. 3 *C* and *D, ropw* ith the shaded region correspondingto!he SE *over* the independent runs. The spectrograms in Fig. 1 *D, Middle* and *Right,* were computed for individual audiosignals usingmatplotlib's specgramfunctionwithoptions Nffl= 512 Hz (numberof fast Fourier transformdatl points per block), pad.lo = 8,192 Hz, and n rfap = 256 Hz.In order tocomputethe deraytimeof thesound, we normalizedeach timeseries by itsm aximum(inmagn ude)andlined them up sot= 0 is at them aximumof thesignal.Weaveraged theabsolutevalueofthe temporallyalignedsignalsoverindependent runsand performed anadditional moving *average over* a time step *tJ.t*= 0.025 s to smooth out all the high-frequencyoscillations,leavingbehindonlylherequirede lope.lbissmoothed *awrage*cuM!(onceagainnormalizedbyitsmaximum)isshown indarkbluein Fig.3 *C*and *D,*Bottom.Asimilarralculationandsmoothingisalsodoneforlhe SE computed*over*independent runsandisplottedastheshadedregionaboutthe *awrage*.Thesmoothed ave<age timeseriesisthenfittoan exponentialfunction witha constant offset usingSciPy's in-builtnonlinearcurve fittingfunction.The errors on ourestimatefor thedominantfrequency(Wo)and the decay time(r) arise primarilyfrom thechosen resolutionofoursmoothingwindows(tJ.w, tJ.l) as other sources ofm easurement errorare much smaller.We have nonetheless checked that our choice of thew indowsize(t>w,tJ.t} isoptimumaschangingit bysmallamountsdoes notaffect our results,but decreasing*tJ.t* byan orderof magnitudesignifirantlydegradestheexponentialfit

Data ""•UabUity. Codeand datl reproducing the resultsin this paper haw been deposited on Figshare, https://doi.org/10.6084/m9.figshare.19441385, andaredescribedintheartideandsupportinginformation.

ACKNOWLEDGMENTS. S.S. is supported by the Harvard Societyof Fellows. S.S.thanks VincenzoVitelli for helpfuldiscussionsandgratefullyacknowledges usefulinteractionsduring thevirtual2021 KavliInstituteof TheoretiralPhysics program'The Physicsof Elastic Films: From Biologiral Membranesto Extreme Mechanics; supportedin partbythe NSF undecGrant NSFPHY-1748958. L.M. acknowledgesGrants NSFOMR2011754and NSFOMR1922321, theSimons Foundation,andthe HenriSeydouxFundforpartialfinancialsupport.

Author a.ffllatlons: •oepaJ'IJ'Ylent of Physk:s, HaMld l.Wl,,tt \'- caMbtdge, MA 02138; "Pa Sd'lod of EOdneeftng and Applied Sdeoces, liatv¥d - CM'lbtdgt!, W.02138;.r'd<-oe ofOtgantsmkMd E-.dlJtionaty BiOfO!)I, liatvatdlA\t.H!rSlt)', Cimbtktge, MA02138

1. H.H.Fleb:bK,lROSWlg,lhefflysjcsofMU'Siallnstn.mel'Jls(SpringKNew'fcn,2008i
2. G.Johnson,Si-it:M.usicaf(GrcwM usicOnline,(b.iordLkiifflsityPrtSS,.2001}.
3. J. J.leonard,J.GliebM,*SctJthMr*Bact:*Al'icrrJriJ HislotyofrheMusial Sawlf1d Howto Play*.lt. (KaleidoscopePress).ed.1 (1989).
4. R.Cook.Viirationofasegmentofa noo-Orcularcylindricalshell:The•roosicalsnl'problem. *J. Sound* Wbrat1'6,339-341(1991).
5. A.llilis.R.l themusicalrabOllilfeaturesandsir111ledynamic.al tfleory.1*Acoust* Soc. *Am.*71,S82-S83(1982}.
6. J.F.M.Scott,JW oodhol.M,VibAlionUanelasticstripwMvar,;ng CllMtuft..*Phil.*kans. *R.*Soc*A* 339. S87-62S(1992f
7. W.l J.oirer,"AconsisteatfifSI prOMlalioo• thegtt1erallhluyof thinebsticshelis-i-n *PIOct'EdvlgsofW.1.AM.*SJ'l¥)0SllNDM *theTbeotyoflhinUastic Shells*W.l Koitel,Ed. (NortMiolland,.AmstHdam,1960i pp.12-33.

8. . 8..Audoly,Y.Pomeau,*Uisticiryafldlieomef1y: FtomCudsrorheNon-lineatRes.por,s,eo/Shelfs* (Oxford UniveM)'Pres.s.201oi
9. R.Worland,Themusicalsawand theflei:atorie: Ane:cperimNJtalstudyofconfilleDd rational modes inmd.1 1esofvariablecuMttn.J.AcoosStoc.Am:.139,2011•2011(2016}.
10. R.Worland,Tbtmusicalsaw:Musialacousticsd !rappedWilrationalmodesinacurwdbladt.J. *Acoust.Soc.Am.*1'5,1750-1750(2019).
11. 0. J.lhooless.M.S:ohmoto,l.l P.Nigbtiogak-, M.dKI Nijs.Ouantiled hallcoodudanceina m>-dimensionalperiocic pottltial.*Phys..Rev.Lelt*49,405(198n
12. M. Z.Hasan,C..L lane,Colloquillffl:Topologicalinsulabn *Ref.Mod.*fflJs.12, XM.5(2010).
13. X.L 0., 5.C.Zhaog, Topologicailn:sulatocsandS141eroonductors.Rw.*Mod.Pfr,s.*83,1057(2011i
14. o:ane,l lti>ensky,Topologicalbooodacymodesinis clatticts.NatfflJs.10, 39-45(2014i
15. S..D.Huber, Topologicalmechania..Natffiys.12,621 (2016).
16. X.Mao,l C.lubensky, MallWelllatticesandtopologiaml e<haniG--AnooR. « *CtJndMs.Mi«N* PJiys. **9**, **413-all**(201Bi
17. 6. M.Xiao,C.l Chan,topdogicalpbas,esinacwsticaodmechanicalsr.,aems.*Nat Rev.*ff1Js1, **281-294(1019)**
18. 0.Bartolo,0..Carpealier, Toj:dogicalelasticityofDOOOriMtat'eribbons.Pflys.*RNX*9,0C1058 (2019}.
19. A.Sarerni,Z.Roclml,Topologicalelasticifyof flniblI'struellMes..fflys.MNX10,011052(2020).
20. S:.Sun, X.Mao,Continwmtheo,yfortopologialedgesoftmodes.*ff',ys.RN. Lei.*12',207601 (2020).
21. S:.Sun, X.Mao, Fractionaleltitationsin11on.fudideanelasticplates.Fflys.*Rev.*l.elt127,098001 (2021}.
22. P.Oeli:kt, J.8.Maotori,A.Vffiaile, Topologicaloriginofequatofi.lwaves.Sdero> 358. 107S..10n(2017}.
23. S..Shanbr, M.*l* bid:, M.C.Man:he Topologicalsooodandffotiigona.vedsurfaces..ff1Js. Jl><X7.0l1039(2017}.
24. A.SwslO¥, U>asbiswas,l,U1uchartS. .VaikuwnatbaQV,.'flteli,Topdogicalwmsinftuidswithodd """"·*l'l,ys.*Jl><*l"1* **t22,**128001(201Qi
2S. S..Shriar, A..*Souslcrt*,M.J.SOMct.l.l C..Mardletti,*V.*Vitelli,TopologicaaldN'tmatt«.arXiv (Pt.. mtJ(2020ihl1p5:/l abs/2010.0036<IAae<sed 7 Apnl2022)
26. R.Susstrunl:,S. 0..Huber,Obsttvation af ononichelicaledgesta.tesina methnc.11topological Wldator.SciencPM9, 47.50(201S).
27. M.l.liniaci,R.Pal,8..Morvan,l,l Ruuene, &perimentalobservation oftopologiallyi:wot«1ed helical edgemodesinpattemed elasicplates.*l'trjsk.* XI,031074(2018).
28. R.l. Pal, M. RwMe,Edgewawsinplateswithresonat«s:Anelasticanalogued the valley hallelecLNewJ. *Pt,ys.*19,025001(201n.
29. H. Massa,«.*al,* Nonrecipn,cityinacousticand elasticmaterials.Nakt Malet 5, 667-68S(20'10).
30. J. W.S. 8. Rayteigh,*lhelneoryolSoond*(Maonillan,1896},wt *l*
31. A.A.E'Val'JS,A.J. lecle,ReH«tiM and relractiooafflexuralwavesatgeomeuicbouodaries.*Pfrys.*Re,: *Lei.*111,038101(2013).
32. P.G.Ciarlet,*Theotyol Shelfs.Mathematical8astitity.*(« Science, 2cm).
31. J.J'.emes,A.J. Le...,e, EffectsofcurvatureM thepn,pagationofoodulalorywawsin lcr#er dimensionalelasticmaterials.*Pbys.Rtt E*103,01300'1(2021).
34. R.Sus:strunk,S.0. Hubel,Classificationaftopologicapl honoosinliDNmredwnic.lrnetamaierials. Fkx.Nal.lod.Sd.US.A.113,E476J.E477S(2016).
35. lfrankt-1, lTle*GHJm«ryof fflysics:NI* inaodtx:tiM(Cunbridge lklivecsityPress,2011).
36. . M. Nabhara,GeometJy.1opok,*gafnd*Physics(CRCPress,201Bi
37. L Fu,C.l Kane,Topologicalinsulab:wwsithW!Vfflionsymmett)' *Pfr/sk.* *8* Conde?s.Maller*Mit.el. f!>ys.*7',04Sl01(2007i
38. . Lfu,Topologicalaysulline insulators.fflys.Revl..elt11M,106802(201n
39. l l Hughes.l Prodan,8..A.Sem g,hwe,sion,symmefflctopologicailnsulators.*l'trjs.Ref.8 Condens..*Matte,*Mat«.Phys.*13, 245132(2011i
40. P.l.l H4di,"1'1:uidationsofelasticshell*tbeor(* inJ;ogress*NJ SolidM«hinia,*t N.Sneddon, ll Hill,Eds.(North.fiolland,Amsterdam,1963),vol..t,pp.1-90.
41. M.S..Alnas« *al,* Thef&iCSi:woied version15.Atdt.*NIITJ'lt'f.Sob-*.3,9.23(201si
42. J.S..Hale, M.Btu .S..P.A.Bordas,C.Maurini,Simpleandextensibleteaodshellfinile d«nel'II modets throuau1omaticcodegMerationtools.*Comput.*StJuc.209,163-181(2018).
U t Wilson-Rae*et a.,*Higb-0nanomemanic.s,;a destruaiweinteriNenctoCelislicwaws.*PfTfs.Rtt* W.1°',04720S(2011J.
44. S.M.HEilman,R.S..Strichw.localizedeigenfooctions:Here yousee ffl\theseyoudon't.*Not.Am.* —·Soc.57,62• · (2010)
45. M. Filoche,S. Ma)'boroda,LffliversalmechanismtorAnde,son andwe;tlocalintion.Aw.*NatlAud. Sci.USA.*109,14761-14766(2012).
.&6.. L Rayleigh,Ull. lheproblemofthew.fiisperigall«y *lood.E6nb.DoblinFf7ilos.Mag.1Sci.*20, 1001-100<(1910}.
47. H.G.Craighead, Nanoelectromechanic.lsymms,*SciMce*290,1532•1536(2000i
,48.. l.LEkinO, Eleelromechanicaltransducersatthe nanosale:AcbJationandSfflSingIXmotion in aanoeledromechani<.a9lJ:SttmS(NEMSiSmJN1, 7U.797(20C 1
49. Rlffilz,M.C..Cross."Nonlinea.dt ynamics*d* nanomemanic.laodmioomed'lanicaliesonatocs• in *RefitwolHoolirNNJrDynamicsC/JmpJetitf,*H.G.Schustef, Ed.(Wiley, 2008},pp.1-48.
SO. J. Cha, l. W.Kirn,C..Daraio,&perimentalieealiation*d* olKhiptopologica11WOOelee1tomechanical metamaterials.Naru,eSU,229•2ll(201Bi
51. J. Ma,X.Xi,Y.li, X.Sun,Haoomechanical IOpOlogicalinsulatowithanau:w:ilia,yorbital degreeof teedorn.Hat.*N4'10ledinal.*1. 4,57 583(2021).
52. J.S..Bunch*et a.,*EledromedwnicaresonatorsfromgraphMesheets.*Soena*31S,490-493(2007}.