Geometric control of topological dynamics in a singing saw

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The common handsaw can be converted into a bowed musical instrument capable of producing exquisitely sustained notes when its blade is appropriately bent. Acoustic modes of a thin elastic shell are inflection points known to underlie the saw’s sonorous quality, yet the origin of localization has remained mysterious. Here we uncover a topological basis for the existence of inflection line modes that relies on and is protected by spatial airvature. By combining experimental demonstrations, theory, and computation, we show bowspatial variations in blade airvature control the localization of these trapped states, allowing the saw to function as a geometrically tunable high-quality oscillator. Our work establishes an unexpected connection between the dynamics of thin elastic shells and topological insulators and offers a robust principle to design high-quality resonators across scales, from macroscopic instruments to nanoscale devices, simply through geometry.

Musical instruments, even those made from everyday objects such as sticks, saws, pans, and bowls (I), must have the ability to create sustained notes for them to be effective. While this ability is often built into the design of the instruments, the musical saw, used to make music across the world for over a century and a half (2), is unusual in that it is just a carpenter’s saw but held in an unconventional manner to allow it to sing. When a saw (Fig. IA) is either bowed or struck by a mallet, it produces a sustained sound that mimics a soprano’s lyric trill (3). Importantly, for such a note to be produced, the blade cannot be flatter bent into a J-shape (Fig. IB) but must be bent into a S-shape (Fig. IC).

This geometric transformation allows the saw to sing and is well-known to musicians who describe the presence of a “sweet spot,” i.e., the inflection curve in the S-shaped blade; bowing near it produces the clearest notes, while bowing far from it causes the saw to fall silent (3). Early works (4, 5), including notably Scott and Woodhouse (6), attempted to understand this peculiar feature by analyzing the linearized vibrational modes of a thin elastic shell (7, 8). Through a simplified asymptotic analysis, they showed that a localized vibrational eigenmode emerges at an inflection point in a shell with spatially varying curvature and is responsible for the musicality of the saw. Recent works have reproduced this result using numerical simulations (9, 10), but a deeper understanding of the origin of localization has remained elusive.

A simple demonstration of playing the saw quickly reveals the robustness of its musical quality to imperfections in the saw, irregularities in its shape, and the precise details of how the blade is flexed. Fig. 1D shows a time trace and spectrogram of the saw clamped in a saw (Fig. IB) or an S shape (Fig. 1C), where the pitch can be varied by changing the curvature of the blade while being unusually bent, we describe the presence of a sweet spot, i.e., the inflection curve in the S-shaped blade; bowing near it produces the clearest notes, while bowing far from it causes the saw to fall silent (3). Early works (4, 5), including notably Scott and Woodhouse (6), attempted to understand this peculiar feature by analyzing the linearized vibrational modes of a thin elastic shell (7, 8). Through a simplified asymptotic analysis, they showed that a localized vibrational eigenmode emerges at an inflection point in a shell with spatially varying curvature and is responsible for the musicality of the saw. Recent works have reproduced this result using numerical simulations (9, 10), but a deeper understanding of the origin of localization has remained elusive.

The lack of sensitivity to these details suggests a topological origin for the localized modes responsible for the saw’s striking sonority. That topology can have implications for band structures and the presence of edge conducting states even when the bulk is insulating, was originally explored in electronic aspects of condensed matter to explain the quantization of the Hall oonduiance (11) and led to the prediction of topological insulators (12, 13). More recently, similar ideas have been used to understand the topological properties of mechanical excitations, e.g., acoustic and floppy modes in discrete periodic lattices (14-17), in continuum elasticity (18-21), in fluid dynamics in geophysical and active mattersystems(22-25), etc. In many of the aforementioned systems, the breaking of time-reversal symmetry leads to the appearance of topologically protected modes. Alternately, in the absence of driven or active elements, spatial symmetries of a unit cell can also be used to achieve topological modes via acoustic analogs of the quantum spin or valley Hall effect (17, 26-29), although these examples rely on carefully engineered periodic lattices. Here we expand the use of topological ideas to continuum shells and show that:

Significance

The ability to sustain notes or vibrations underlies the design of most acoustic devices, ranging from musical instruments to nanomechanical resonators. Inspired by the singing saw that acquires its musical quality from its blade being unusually bent, we ask how geometry can be used to trap and insulate acoustic modes from dissipative decay in a continuum elastic medium. By using experiments and theoretical and numerical analysis, we demonstrate that spatially varying curvature in a thin shell can localize topologically protected modes at inflection lines, akin to exotic edge states in topological insulators. A key feature is the ability to geometrically control both spatial localization and the dynamics of oscillations in thin shells. Our work uncovers an unusual mechanism for designing robust, yet reconfigurable, high-quality resonators across scales.
underlying the time-reversible Newtonian dynamics of the sawing is a topological invariant that characterizes the propagation of waves in thin shells, arising from the breaking of up-down inversion symmetry by curvature.

Results

Continuum Model of ThinShell Dynamics. Thesaw is modeled as a very thin rectangular elastic shell (thickness h/4; W < L, where W, L are the width and length of the strip) made of a material with Young's modulus Y, Poisson's ratio \( \nu \), and density \( \rho \) (Fig. IE). Its geometry is characterized by a spatially varying curvature tensor (second fundamental form) \( b(x, y) \), where \( x = (x, y) \) is the spatial coordinate in the plane. As the saw is bent only along the (long) x axis, \( b_{x}(x) = b(x) \) is the sole nonvanishing curvature. To describe its dynamical response, we take advantage of its slenderness and treat the saw as a thin elastic shell that can be bent, stretched, sheared, and twisted. Before moving to a computational model that accounts for these modes of deformation as well as real boundary conditions, to gain some insight into the problem and expose the topological nature of elastic waves, it is instructive to instead consider a simplified description valid for shallow shells with slowly varying curvature.

In a thin shallow shell (h/2L ≈ 1), as bending is energetically cheaper than stretching (30), shear becomes negligible (Q(t)\_0; Fig. IE), and in-plane deformations propagate much more rapidly (at the speed of sound \( c = \sqrt{Y/\rho} \)) so that the depth-averaged stresses can be assumed to equilibrate, i.e., \( b_{\mu\nu}(t) = 0 \). In this limit, using the solution of these equations in terms of the spectrogram in Fig. 1C lies within the frequency gap. The \( \psi \)-shaped saw (Fig. IB) also minimizes low-frequency (compared to the gap) when struck, presumably through the \( \nu = 0 \) branch of delocalized flexural modes, although higher frequencies above the band gap can be excited by careful bowing (SF Appendix, Fig. SI A and BJ).

Curvature-Induced Z2 Topological Invariant. To unveil the topological structure of the vibration spectrum of the saw, we cast these second-order dynamical equations (Eqs. 1 and 2) in terms of first-order equations by taking the square root of the dynamical matrix (14, 34). Focusing on the flexural modes alone, we obtain a Schrödinger-like equation for the transverse deflections of a shallow shell (SF Appendix, section 3).

\[
\begin{align*}
\frac{i}{\hbar} & \frac{\partial}{\partial t} \Psi = H \Psi, \quad \Psi = \begin{bmatrix} 0 \\ V \end{bmatrix}, \\
\int & = \int \Psi(\mathbf{q}, \mathbf{v}) \psi(\mathbf{q}, \mathbf{v}) d^{3}q d^{3}v, \\
H & = -i \hbar \frac{\partial}{\partial \mathbf{q}} + V(\mathbf{q}, \mathbf{v}).
\end{align*}
\]

where \( \Psi = (c^{T} f, iO_{i}) \) and \( V = iJ_{i} < Y q^{2} / 2 + bt' t \) represents the conjugate transpose. The eigenvalues of the effective Hamiltonian \( H \) are given by the previously derived \( \Phi(\mathbf{q}) \), and its complex eigenvectors \( \Phi(\mathbf{q}) \) encode the topology of the band structure. The singularities in the arbitrary phase of the eigenvectors signals nontrivial band topology. To understand the phase of eigenvectors along the saw's long direction, we can consider fixing the transverse wave vector \( \mathbf{v} \), \( \mathbf{f} \), leading to an effective one-dimensional (1D) system along the x axis. Then the obstruction to continuously define the phase of the eigenvectors at every \( \mathbf{q}x \) in Fourier space while respecting all the symmetries of the problem is quantified by the ID Berry connection \( A(\mathbf{q}) = \int \mathbf{q} \cdot n_{z}(\mathbf{q}) d^{2}q, \int \mathbf{V}(\mathbf{q}) \) (the \( \mathbf{q} \) dependence is suppressed) (35, 36). However, what are the symmetries of our elastodynamic system?

One important symmetry is that imposed by classical time-reversal invariance in a passive, reciprocal material \( \mathbf{C} : \mathbf{x}, \mathbf{t} \rightarrow -\mathbf{C}, \mathbf{t} \rightarrow -\mathbf{C} \) (SF Appendix, section 3), which maps forward moving waves into backward moving ones and guarantees that eigenmodes appear in complex-conjugate pairs (34). A second symmetry special to the saw is an emergent spatial reflection symmetry in the local tangent plane \( \{ x_1, x_2, x_3 \} \rightarrow \{ x_1, -x_2, x_3 \} \) (SF Appendix, section 3), which originates from the uniaxial nature of the prescribed curvature along the x axis and the insensitivity of bending to the orientation of the local tangent plane, a symmetry that is inherited from 3D rotational invariance. The latter is easily seen by noting that the bending energy only involves an even number of gradients via \( \mathbf{b}'' \). Upon simultaneously enforcing both dynamical and spatial symmetries, a new topological obstruction posed by curvature emerges and is quantified by a \( L_{2} \) index (SF Appendix, section 3).

\[
\sin u_{\text{Air}} \rightarrow \text{topological insulators with crystalline symmetries (37–39). Ph(W) denotes the Pf.\_fian of the antisymmetric overlap matrix } W^{<}(\mathbf{q}) = W_{\mathbf{q}}(\mathbf{q}) \text{ in } tf_{\text{Air}}(\mathbf{q}), \text{ (i.e., pm)}. \text{ We note that unlike the mechanical Su-Schrieffer-Heeger chain (14) that exhibits a topological polarization in ID, the emergent tangent-plane spatial reflection symmetry in our problem forces this polarization to vanish (SF Appendix, section 3).}
\]

As we work in the continuum, only differences in the topological invariant are well defined independent of microscopic details. Across an interface at which curvature changes sign, i.e., a curvature domain wall, the jump in the topological invariant is given by

\[
(-1)^{n_{-}} = \text{sgn}(b_{x} < b_{x}),
\]
Equations (Subscripts denote symmetrization) for are interpreted as Variant. and index manipulations employ the reference metric that the two oppositely curved sections of the saw have as shape and which is that the two oppositely curved sections of the saw behave as shape, while simplifies the dynamics to \( V \cdot a = b \cdot Q - V \cdot (b - IV) \).

Numerical Mode Structure and Localization. We test these predictions by numerically computing the eigenmodes of a finite elastic strip of length \( L = 1 \) m, width \( W = 0.25 \) m, and thickness \( h = 10^{-3} \) m. For our shell model, we move away from the Kirchhoff model for shells and account for the inelastic associated with shear in addition to those associated with bending and stretching, as they effectively reduce the numerical ill-conditioning commonly seen in high-order continuum theories for slender plates and shells while allowing for numerical methods that require less smoothness and are easier to implement (SIAppendix, section 3). This expression directly demonstrates that the two oppositely curved sections of the saw have as topological nontrivial bulk systems, with a \( t.l.v = 1 \), that meet at the inflection line that functions as an internal edge. As a result, nontrivial band topology underlies the emergence of the localized midgap mode, endowing it with robustness against details of the curvature profile and weakly nonlinear deformations (SIAppendix, section 3).

Fig. 1. The musical saw in mathematical model. (A) A violin bow and mallet placed alongside the saw. We clamp the saw in two configurations: (B) J-shape and (C) S-shape, which is required to play violin. The primary distinction between the two is that a saw has an inflection point (the sweet spot) in its profile, while \( \delta \) has curvature of constant sign. (Scale bar, 5 cm.) (D) (left) The axial series of the normalized audio signals when the saw is Bisist (green) and when the saw is bisist (black). (Middle and Right) The corresponding spectrograms for both the J-shape (B) and the S-shape (C). The signal decays rapidly for the J-shape and is visible at a wider spread in frequency, while for the S-shape, a single dominant note with a 595 Hz survives the ringdown of the blade lasting several seconds. (E) A schematic of a blade of length \( L \), width \( W \), and thickness \( h \) is sketched with a uniaxial curvature profile \( h(x,y) = b(x) \), which changes along the x axis as \( \delta \). The saw can be modeled as an elastic shell whose deformations include an in-plane displacement \( u \), normal to the shell and a rotation \( \theta (\text{the local normal}^* \text{ as degrees of freedom}) \). Elastic tensors \( A' \) and \( f' \) enter the constitutive equations (Subscripts denote symmetrization) for the in-plane stress vectors, bending moment (M), and transverse shear (Q) (SIAppendix, section 2). Derivatives are interpreted as \( \partial \text{O} \text{V} \text{ariant} \), and index manipulations enable the reference metric of the shell (SIAppendix, section 2). The Kirchhoff limit for a shell simplifies the dynamics to \( a = 0, phif = M + tr(b \cdot u) \), along with \( \delta = -h(x,y) \).

where \( b < \) and \( b > \) are the curvature on either side of the interface (SIAppendix, section 3).
In Fig. 2B, the distribution of eigenmodes as a function of frequency is shown in the integrated density of states for a constant curvature shell, \( b'(z) = b_0 \) (dashed lines), and an S-shaped shell with a smooth curvature profile \( b'(z) = b_0 \tanh(z_/z_0) \) (solid lines) that varies over a width \( f \) near the inflection point at \( x = 0 \) (i.e., a curvature domain wall). In both cases, the ends of the strip are kept clamped, and the spectra are calculated using an open-source code based on the finite element method (41, 42). As the curvature of the S shape approaches a constant \( \pm b_0 \) far from the origin, the bulk spectral gap and delocalized modes match that of the constant curvature case.Figures 2C, 2D, and 2E show the localized modes (inset showing normalized deflection \( \phi \)) for increasing mode number \( m \), with a gapless spectrum for both constant curvature (dashed) and the Sigmoid profile (solid). Higher \( m \) X(10; \( m = 2 \)) exhibit a finite gap \( \sim 2 \text{kHz} \) for constant curvature (dashed), while the sigmoid profile features a localized mode \( (\omega \sim 1 \text{kHz}) \) at the inflection point within the bulk band gap. (Q Numerical eigenmodes for the sigmoid profile with the local normalized deflection plotted (dashed lines are \( \pm 10\% \) isocurvature). Low-frequency delocalized states with \( m = 0 \) (Top), \( m = 1 \) (Middle), and the first localized mode with \( m = 2 \) (Bottom), (D) Frequency of the localized modes (Inset showing normalized deflection \( \phi \)) plotted against the curvature gradient \( b' \) and the length scale of curvature variation \( l \).

\[ \text{IPR} = \frac{\left( \frac{\text{Inverse participation ratio}}{\chi^2} \right)}{\left( \frac{\text{Inverse participation ratio}}{\chi^2} \right)} \]

Within the spectral gap (solid red line, Fig. 2B). This midgap state (shown here for \( m = 2 \)) is a localized mode that is trapped in the neighborhood of the inflection line (Fig. 2C, Bnang.m). For increasing mode number \( m \), 2, similar topological modes appear within the bulk bandgap, with growing localization length (Fig. 2D, Inset) and higher frequencies (Fig. 2D), as predicted analytically (SI Appendix, section 3). Qualitatively, the presence of an inflection line in the S-shaped saw makes it geometrically soft there; the generators of cylindrical modes are now along the length of the saw, and the curved regions on either side that are geometrically stiffer serve to insulate the soft internal edge from the real damped edges.

Of particular note is that the localized modes, unlike the extended states, are visually unaffected by the boundaries and the conditions there (see SI Appendix, Fig. S2A, for eigenmodes in a strip with asymmetric boundary conditions where the left edge is clamped and the right edge is free). Spatial gradients in curvature, however, do impact the extent of localization. We demonstrate this using a piecewise continuous curvature profile that has a constant linear gradient \( b' \) over a length \( L \), the origin and adopts a constant curvature outside this region. By varying both the curvature gradient \( b' \) and the length scale \( L \), we can tune the localization of the lowest topological mode (same as Fig. 2C, Banne.m), quantified by the inverse participation ratio \( \text{IPR} = \frac{\chi^2}{\chi^2} \) (Fig. 2E). Strong localization (high
Fig. 3. Dissipative dynamics and high-quality oscillators. (A) Resonance waves for a shell with a linear curvature profile (Ins. et al.) periodically driven at the inflection point (x = 0; red) and away from it (x = 0.4L; black) for varying frequency (ω = 740 Hz corresponds to the first localized mode). (B) Numerically computed Q factor shows dramatic enhancement at localized mode frequencies (red) over delocalized modes (blue). (C and D) Experimental measurement of Q factor (see SI Appendix, section 1 for details) for the musical saw (Fig. 1B) and O'S shape (Fig. 1C). (Top) Note the normalized Fourier spectrum amplitude is on a log scale below 0.1 and linear above, with the peak frequency marked as ω. (Bottom) The average signal decay (blue curve) is fit to a single exponential (black curve). The shaded region is the SE of both C and D.

IPR) is quickly achieved for sharp gradients in curvature (IPR oc J/L/h; SI Appendix, section 3) as long as the length scale of curvature variation is not too small (E/L; ≈ 0.1, Fig. 2D), corresponding to a diffuse domain wall. In the opposite limit of b'/L fixed, a sharp domain wall with a discontinuous curvature profile b(x) = b0σgn(x), strong localization persists (SI Appendix, Fig. S3), consistent with our topological prediction and demonstrating the ease of geometric control of localization.

Geometrically Tunable High-Quality Oscillators. The boundary insensitivity of topologically localized modes has important dynamic consequences that can be harnessed to produce high-quality resonators. The primary mode of dissipation in the saw, as in nanoelectromechanical devices (43), is through substrate or anchoring losses at the boundary. Internal dissipation mechanisms (from, e.g., plasticity, thermoelastic effects, and radiation losses), although present, are considerably weaker and neglected here. To model dissipative dynamics, we retain damped boundary conditions on the left end and augment the right boundary to include a restoring spring (k) and dissipative friction (γ) for both the in-plane forces and bending moments (Fig. 3 A, Inset, and SI Appendix, section 2). Informed by F, g, 2E, we choose a linear curvature profile spanning the entire length of the shell to obtain a strongly localized mode. Upon driving the shell into steady oscillations, with a periodic point force applied at the inflection point (x = 0; Fig. 3A, red curve), we see an extremely sharp resonance peak right at the frequency of the first localized mode (Fig. 3A). In contrast, when the shell is driven closer to the boundary (x = 0.4L; Fig. 3A, black curve), the response is at least six orders of magnitude weaker than the localized mode is not excited and only the delocalized modes contribute. Localization hence protects the mode from dissipative decay, unlike extended states that dampen rapidly through the boundaries. We further quantify this using a Q factor computed from unclipped relaxation of the shell initialized in a given eigenmode (SI Appendix, section 2). Ultrahigh values of Q" = 105 to 106 are easily attained when a localized mode is excited (Fig. 3B, red), well over the Q factor of all other modes (Fig. 3B, blue). Similar results are obtained for other curvature profiles as well, such as a sigmoid curve (SI Appendix, Fig. S2B).
To compare these computational results with experiments, we perform ringdown measurements on a makeshift saw (see SI Appendix; section I, for details) damped in both the \( J \) shape (Fig. IB) and the \( S \) shape (Fig. IC). As indicated by Eq. 5, the key distinguishing feature of the \( S \)-shaped saw (compared to the \( J \) shape) is the presence of an inflection line (curvature domain wall) that engenders a well-localized domain wall mode capable of sustaining long-lived oscillations. The normaliz.ed Fourier spectra and exponential decay \( -t \) of the signal envelope are shown in Fig. 3C (\( U \) shape) and Fig. 3D (\( S \) shape) with the dominant frequency (\( \omega \)) marked. We find a factor \( \sim 15 \) enhancement in the \( Q \) factor (\( Q = \omega -\omega /2 \)) for the \( S \)-shaped saw (Fig.3D, Left) over the \( J \) shape (\( Q = 150 \); Fig.3D, Left). We emphasize I.e. that this significant factor improvement, although not as dramatic as the numerically computed \( Q \) factors (Fig. 3B), is still striking given the initial impulse (mallet strike for \( J \) shape and solely on the scale separation intrinsic to any curved thin even in the ultimate limit of atomically thin graphene (52).

in the interior hence remain vibrationally isolated and decay an alternate strategy inspired by the singing saw, which relies patterned periodic arrays of nanomembranes to control localized in nanomechanical devices, dissipation can be dominated by on intrinsic nonlinearities (49). Just as in the musical saw, providing a framework to explore not just topological mechanics but also dynamics in thin plates and shells.

The ability to control spatial geometry to trap modes at interfaces in the interior of the system offers a unique opportunity to design high-quality oscillators. As our results are material independent, they apply equally well to nanoscale electromechanical \(<euros\)onsators (47, 48) and provide a geometric approach to design high-quality resonators without relying on intrinsic nonlinearities (49). Just as in the musical saw, in nanomechanical devices, dissipation can be dominated by radiation through the clamped boundary (43). Current on-chip topological nanoelectromechanical metamaterials use carefully patterned periodic arrays of nanomembranes to control localized modes in robust acoustic waveguides (50, 51). Our work suggests an alternate strategy inspired by the singing saw, which relies solely on the scale separation intrinsic to any curved thin sheet by manipulating curvature spatially, topological modes localized in the interior hence remain vibrationally isolated and decay extremely slowly, allowing ultrahigh-quality oscillations, perhaps even in the ultimate limit of atomically thin graphene (52).

Discussion and Conclusion

Our combination of analysis, finite element simulations and experiments has demonstrated that a sawsings because its curvature generates a frequency gap in the acoustic spectrum which closes at an inflection point (line) that acts as an interior edge allowing a locali.r.ed mode to emerge within the band gap. Unlike mechanisms of weak localization (44, 45) or well-known whispering gallery modes (30, 46) that rely sensitively on details of the domain geometry, our topological argument explains the existence of locali.r.ed sound modes and their robustness against perturbations in the musical saw, providing a framework to explore not just topological mechanics but also dynamics in thin plates and shells.

Materials and Methods

Extended data on the experiments and the details of the numerical modeling and theoretical calculations are provided in SI Appendix.