Investigating Quantum Entanglement Using Canonical Quantization and Scattering Theory

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***Abstract*—In this work, we propose two approaches of investigating quantum entanglement. In the first approach, canonical quantization with numerical mode decomposition is applied to inhomogeneous dispersive media. Nonlocal dispersion cancellation effect of energy-time-entangled photon pair is demonstrated. In the second approach, we study the effect of scattering on spatial-entangled photon pair from spontaneous parametric down-conversion. Schmidt number is calculated numerically. Migration of entanglement between amplitude and phase is studied.**

I. Introduction

Quantum entanglement based on photons has important applications in quantum communication, quantum radar and quantum sensing. To describe entangled photons, formulations of quantum electromagnetics applicable to complex media are needed. In this work, we propose two approaches of investigating quantum entanglement.

In the first approach, canonical quantization with numerical mode decomposition is performed to rigorously quantize the Hamiltonian for finite-sized dispersive media [1]. A generalized Hermitian eigenvalue problem for electromagnetic fields coupled to nonuniformly distributed Lorentz oscillators is developed. Eigenmodes are obtained with arbitrary geometric complexity using computational electromagnetics methods. Second-order correlation function is calculated. Nonlocal dispersion cancellation effect through energy-time entanglement of photon pair is studied.

In the second approach, we leverage electromagnetic scattering theory to study spatial entanglement of photon pair. A computational method to track the evolution of two-photon amplitude from spontaneous parametric down-conversion (SPDC) after the photon pair hit the scatterer is proposed. Schmidt decomposition is performed to evaluate the degree of entanglement. Migration of entanglement between amplitude and phase is studied.

II. Canonical Quantization Approach

A. Formulation

When dispersive media are present in the system, auxilliary fields should be included. They act as Lorentz oscillators placed in dispersive media interacting with electromagnetic fields. Hence the conjugate pairs are defined accordingly [1]

$$\mathbf{q} = [\mathbf{A}, \Pi_\Phi, \Pi_P]^T \tag{1}$$

$$\mathbf{p} = [\mathbf{\Pi}_{AP}, \Phi, -\mathbf{P}]^T \tag{2}$$

where $\mathbf{A}$, Φ and $\mathbf{P}$ are the vector potential, scalar potential and polarization current, $\mathbf{\Pi}_{AP}$, Π_Φ and Π_P are their conjugate momenta, respectively. Compared to [2], the reordering of conjugate pair eliminates the cross-coupling term, so that the Hamiltonian can be recasted as

$$H = \frac{1}{2}\int_V \begin{bmatrix}\mathbf{q}\\ \mathbf{p}\end{bmatrix}^T \cdot \begin{bmatrix}\overline{\mathbf{K}} & 0\\ 0 & \overline{\mathbf{M}}\end{bmatrix} \cdot \begin{bmatrix}\mathbf{q}\\ \mathbf{p}\end{bmatrix} \tag{3}$$

where

$$\overline{\mathbf{K}} = \begin{bmatrix}\nabla\times\frac{1}{\mu_0}\nabla\times -\epsilon_0\nabla\frac{1}{\chi_0}\nabla\cdot\epsilon_0 & 0 & 0\\ 0 & -\frac{1}{\chi_0} & 0\\ 0 & 0 & \frac{\beta(\mathbf{r})}{\epsilon_0}\end{bmatrix} \tag{4}$$

$$\overline{\mathbf{M}} = \begin{bmatrix}\frac{1}{\epsilon_0} & 0 & \frac{1}{\epsilon_0}\\ 0 & \nabla\cdot\epsilon_0\nabla & -\nabla\cdot\\ \frac{1}{\epsilon_0} & \nabla & \frac{f(\mathbf{r})+1}{\epsilon_0}\end{bmatrix} \tag{5}$$

and $f(\mathbf{r}) = \omega_0^2(\mathbf{r})/\omega_p^2(\mathbf{r})$ and $\beta(\mathbf{r}) = 1/\omega_p^2(\mathbf{r})$, where $\omega_p^2(\mathbf{r})$ and $\omega_0^2(\mathbf{r})$ are the plasma and resonant frequencies of the local Lorentz oscillator. The elimination of cross coupling between $\mathbf{q}$ and $\mathbf{p}$ allows one to derive the decoupled equations of motion (EoMs). In frequency domain, the EoM for $\mathbf{q}$ is

$$\omega^2\,\overline{\mathbf{M}}^{-1}\cdot\mathbf{q} = \overline{\mathbf{K}}\cdot\mathbf{q}. \tag{6}$$

This reduction saves computational cost. Hence, the resulting quantum observable can be written in terms eigenmodes as

$$\hat{\mathbf{q}}(\mathbf{r}, t) = \sum_\omega \tilde{\mathbf{q}}_\omega(\mathbf{r})\sqrt{\frac{\hbar}{\omega}}\hat{d}_\omega(t) + h.c. \tag{7}$$

where $\tilde{\mathbf{q}}_\omega(\mathbf{r})$ is the eigenmode with eigenfrequency ω, and $\hat{d}_\omega(t)$ is the time-harmonic creation operator for this mode.

B. Nonlocal Dispersion Cancellation

We study the nonlocal dispersion cancellation effect [3]. We consider a 1-D problem geometry, as illustrated in Fig. 1(a). An energy-time entangled photon pair is initialized in the center and propagate to left and right. The second order correlation function is computed to characterize the degree of coincidence from two photodetections at x_1, t_1 and x_2, t_2. Finite difference method with Bloch-Floquet boundary condition is used to solve (6) numerically.

As shown in Fig. 1(c), for non-entangled photon pair, the coincidence curve in the presence of both dispersive media is wider than in the one-sided dispersive medium cases. Thus it does not exhibit dispersion cancellation effect. However with entangled photon pair, the coincidence curve becomes narrower in the presence of both dispersive media, as shown in Fig. 1(b).

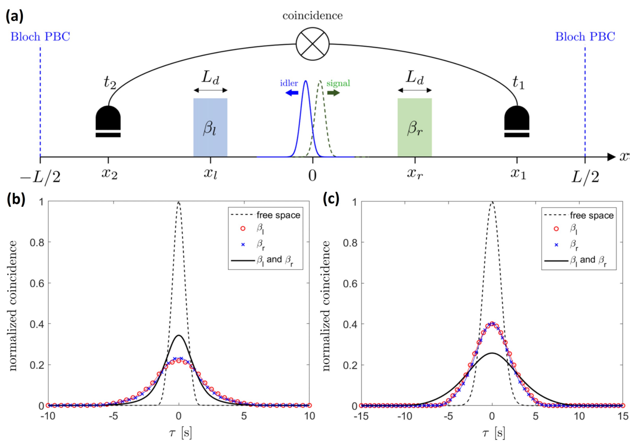

Fig. 1. (a) Problem geometry of 1-D simulations to observe nonlocal dispersion cancellation. Coincidence versus time difference for (b) entangled and (c) nonentangled photon pair. In courtesy of [1].

III. Scattering of Spatial-Entangled Photon Pair

A. Formulation

We then consider the spatial entanglement as the photon pair propagate through arbitrary scatterer. Assuming a strong, collimated pump beam propagating in z direction and degenerate down-conversion, the biphoton wave function in momentum space is [4]

$$\Phi(\mathbf{q}_1, \mathbf{q}_2) = \mathcal{N} \operatorname{sinc}\left(\frac{L}{4k_p}|\mathbf{q}_1 - \mathbf{q}_2|^2\right) e^{-\sigma_p^2|\mathbf{q}_1+\mathbf{q}_2|^2} \tag{8}$$

where $\mathcal{N}$ is the normalization constant, k_p is the momentum of pump field, $\mathbf{q}_i = k_{x,i}\hat{\mathbf{k}}_{x,i} + k_{y,i}\hat{\mathbf{k}}_{y,i}$ is the transverse momentum of photon i, and σ_p is the beam width in coordinate space. After performing the 4D Fourier transform, the wave function in coordinate space is

$$\Psi(\rho_1, \rho_2) = \mathcal{N}' \operatorname{Ssi}\left(\frac{k_p}{4L}|\rho_1 - \rho_2|^2\right) e^{-\frac{|\rho_1+\rho_2|^2}{8\sigma_p^2}} \tag{9}$$

where $\mathcal{N}'$ is another normalization constant, ρ_i is the transverse coordinate of photon i, and Ssi is the shifted sine integral. Given that $\Phi(\mathbf{q}_1, \mathbf{q}_2)$ is nonzero only around a narrow region of transverse momentum, $\Phi(\mathbf{q}_1, \mathbf{q}_2)$ is approximately the plane wave expansion of $\Psi(\rho_1, \rho_2)$ [5]

$$\Psi(\rho_1, \rho_2) = \int d\hat{\mathbf{k}}_1 d\hat{\mathbf{k}}_2 \Phi(\mathbf{q}_1, \mathbf{q}_2) e^{i(\mathbf{k}_1\cdot\mathbf{r}_1 + \mathbf{k}_2\cdot\mathbf{r}_2)}. \tag{10}$$

As the photon pair propagate near the paraxial axis, the wave function at another optical plane $\Phi(\mathbf{q}_1, \mathbf{q}_2; z_1, z_2)$ can be found by multiplying the paraxial free-space transfer function.

Now we let the photon pair hit a scatterer, and use the above wave function as the incident wave, as illustrated in Fig. 2(a). Considering far-field limit, the scattered biphoton wave function can be written as [6]

$$\Phi_s(\mathbf{q}_1, \mathbf{q}_2) = \int d\hat{\mathbf{k}}_1 d\hat{\mathbf{k}}_2 \langle \mathbf{q}_1 | T | \mathbf{q}_1' \rangle \langle \mathbf{q}_2 | T | \mathbf{q}_2' \rangle \Phi_i(\mathbf{q}_1', \mathbf{q}_2') \tag{11}$$

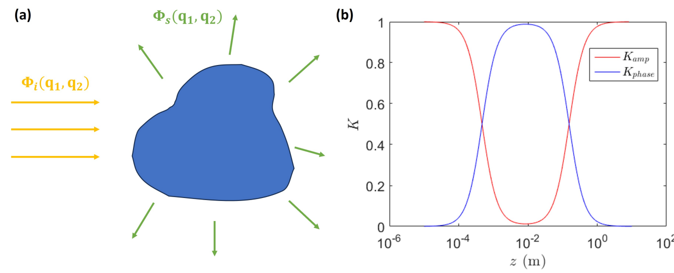

Fig. 2. (a) Problem setup of an entangled photon pair hit a scatterer. (b) Migration of entanglement between amplitude and phase.

where $\langle \mathbf{q} | T | \mathbf{q}' \rangle$ is the momentum space T-matrix. It can be found by solving a scattering problem with an incident plane-wave in the $\mathbf{q}'$ direction, using integral equation. Observing the scattered field in the far-field in the $\mathbf{q}$ direction, after factoring out e^{ikr}/r, the remaining part is $\langle \mathbf{q} | T | \mathbf{q}' \rangle$.

B. Degree of Entanglement

To characterize the degree of entanglement of the entangled photon pair, we perform the Schmidt decomposition

$$\Phi(\mathbf{q}_1, \mathbf{q}_2) = \sum_n \sqrt{\lambda_n} \phi_1(\mathbf{q}_1) \phi_2(\mathbf{q}_2) \tag{12}$$

where $\sum_n \lambda_n = 1$. When there are morn than one nonzero λ_n, Schmidt number $K = (\sum_n \lambda_n^2)^{-1}$ is a general criterion to evaluate the degree of entanglement. Larger K indicates higher nonseparability, i.e., higher degree of entanglement.

The migration of entanglement between amplitude and phase has been calculated in [7], as shown in Fig. 2(b). In this work, we will show the total Schmidt number and its distribution between amplitude and phase, as the photon pair undergo scattering.

Acknowledgement

This work was supported in part by the National Science Foundation under Grant 2202389, and in part by the Consortium for Electromagnetic Science and Technology at Purdue University. Jie Zhu thanks the teaching assistantship support from Elmore Family School of Electrical and Computer Engineering at Purdue University.

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