

RECOVERY OF BLACK HOLE MASS FROM A SINGLE QUASINORMAL MODE

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ABSTRACT. We study the determination of the mass of a de Sitter-Schwarzschild black hole from one quasinormal mode. We prove a local uniqueness result with a Hölder type stability estimate.

1. INTRODUCTION

The possibility of inferring black hole parameters from quasinormal modes (QNMs) has been explored in the physics literature, see Section 9 of the review paper [2]. For example, for slowly rotating black holes, Detweiler showed by numerical calculation in [7] that the QNM nearest to the real axis (called the fundamental QNM) is an injective function of the black hole parameters. Later, Echeverria in [10] investigated the stability issue. Since the success of gravitational wave interferometers, the topic has gained increasing attention, see e.g. [3]. One particular motivation for the study is to verify the *black hole no hair theorem* for which two QNMs are needed: one QNM is used to recover the black hole parameter and another QNM is used to test the theorem. We refer to [2, Section 9.7] for a review and [16] for the state of the art. Despite some convincing evidence, it seems that the theoretical justification is not complete. For example, most of the analysis in the literature is done for the fundamental modes corresponding to small spherical harmonic indices. However, it is generally not known which modes are excited and are extractable from the actual black hole ring down signals, see [3, 2]. In this short note, we aim to provide a mathematical justification of the recovery of black hole parameters from a single QNM.

We consider the model of a non-rotating de Sitter-Schwarzschild black hole (M, g_{dS}) :

$$(1) \quad \begin{aligned} M &= \mathbb{R}_t \times X^\circ, \quad X = (r_{bH}, r_{sI}) \times \mathbb{S}^2 \\ g_{dS} &= \alpha^2 dt^2 - \alpha^{-2} dr^2 - r^2 dw^2 \end{aligned}$$

where dw^2 denotes the standard metric on $\mathbb{S}^2$ and

$$(2) \quad \alpha = \left(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\right)^{\frac{1}{2}}.$$

Here, $m > 0$ is the mass of the black hole and $\Lambda > 0$ is the cosmological constant. They satisfy $0 < 9m^2\Lambda < 1$. r_{bH}, r_{sI} are the two positive roots of

$\alpha(r) = 0$ which corresponds to horizons. Throughout the note, we assume that Λ is known. Consider the d'Alembertian on (M, g_{dS}) :

$$(3) \quad \square_M = \alpha^{-2}(D_t^2 - \alpha^2 r^{-2} D_r(r^2 \alpha^2) D_r - \alpha^2 r^{-2} \Delta_{\mathbb{S}^2})$$

where $D_r = -i\partial_r$ and $\Delta_{\mathbb{S}^2}$ the positive Laplacian on $\mathbb{S}^2$. The stationary scattering is governed by the operator

$$(4) \quad \Delta_X = \alpha^2 r^{-2} D_r(r^2 \alpha^2) D_r + \alpha^2 r^{-2} \Delta_{\mathbb{S}^2},$$

see [20]. On $L^2(X; \Omega)$ with measure $\Omega = \alpha^{-2} r^2 dr dw$, Δ_X is an essentially self-adjoint, non-negative operator, see [18]. Consider the resolvent

$$(5) \quad R_X(\lambda) = (\Delta_X - \lambda^2)^{-1}.$$

Here, we use λ^2 as the spectral parameter and take $\text{Im } \lambda \geq 0$ to be the physical plane such that $R_X(\lambda)$ is bounded on $L^2(X; \Omega)$ for $\text{Im } \lambda \gg 0$, according to the spectral theorem. Sá Barreto and Zworski demonstrated in [20, Proposition 2.1] that $R_X(\lambda)$ has a meromorphic continuation as operators from $C_0^\infty(X)$ to $C^\infty(X)$ from $\text{Im } \lambda \geq 0$ to $\mathbb{C}$ with poles of finite rank. The poles of $R_X(\lambda)$ are called resonances. The fact that they are equivalent to the quasinormal modes defined by using Zerilli's equation (see e.g. [6, 5]) is discussed in [20], see also [4, 18].

We set $\mathcal{O} = (0, 1/(3\sqrt{\Lambda}))$. For $m \in \mathcal{O}$, we denote the set of resonances by $\mathcal{R}(m)$. Below, $\mathcal{Q}(m)$ is a discrete set of $i(-\infty, 0] \cap \mathcal{R}(m)$ defined in (22). Our main result is:

Theorem 1.1. *Let $\Lambda > 0$ and $m \in \mathcal{O}$. For any $\lambda \in \mathcal{R}(m) \setminus (\mathcal{Q}(m) \cup -i\sqrt{\Lambda/3}\mathbb{N})$, there exists $\delta > 0$ (depending on λ) such that for any $\tilde{m} \in \mathcal{O}$ with $|\tilde{m} - m| < \delta$, if $\lambda \in \mathcal{R}(\tilde{m})$ then $m = \tilde{m}$. Moreover, if $\tilde{\lambda} \in \mathcal{R}(\tilde{m}) \setminus (\mathcal{Q}(\tilde{m}) \cup -i\sqrt{\Lambda/3}\mathbb{N})$ is sufficiently close to λ , then*

$$|\tilde{m} - m| \leq C |\tilde{\lambda} - \lambda|^{1/N}$$

for some $C > 0$ and $N \in \mathbb{N}$ depending on λ .

We note that a discrete set of $i(-\infty, 0]$ is excluded from the theorem due to the following reasons. First, there is the possibility that some resonances cannot be used to determine the mass. We call λ a trivial resonance if $\lambda \in \mathcal{R}(m)$ for all $m \in \mathcal{O}$. For example, 0 is a trivial resonance, see [18]. Thanks to a recent work [14] (see also [13]), we know that for $m \searrow 0$, the set $\mathcal{R}(m)$ converges to $-i\sqrt{\Lambda/3}\mathbb{N} \cup \{0\}$. So these are the only possible trivial resonances. The numerical study [14, Fig. 6(b)] seems to indicate that such points except 0 are not trivial, but this remains to be proved rigorously. Second, our method does not apply to the resonances in $\mathcal{Q}(m)$, see Section 5. However, we point out that all these points are purely imaginary and they seem to be less relevant in practical cases, see for instance [16].

We also remark that for recovering black hole parameters, it is common to use only one or a few QNMs. This is very different from the usual inverse spectral/resonance problem for which the whole set $\mathcal{R}(m)$ is used

to determine the parameters. In fact, there is a large literature on the distribution of resonances in the high energy regime, by which we mean resonances λ with $|\operatorname{Re} \lambda| \gg 1$. For example, Theorem in [20] states that there exists $K > 0, \theta > 0$ such that for any $C > 0$ there is an injective map $\tilde{b}$ from the set of pseudo-poles

$$(\pm l \pm \frac{1}{2} - \frac{i}{2}(k + \frac{1}{2})) \frac{(1 - 9\Lambda m^2)^{\frac{1}{2}}}{3^{3/2}m}$$

to $\mathcal{R}(m, \Lambda)$ such that all the poles in

$$\Omega_C = \{\lambda : \operatorname{Im} \lambda > -C, |\lambda| > K, \operatorname{Im} \lambda > -\theta |\operatorname{Re} \lambda|\}$$

are in the image of $\tilde{b}$ and for $\tilde{b}(\mu) \in \Omega_C$, we have $\tilde{b}(\mu) - \mu \rightarrow 0$ as $|\mu| \rightarrow \infty$. See Figure 1. By looking at the sequence of resonances for large l, k , one

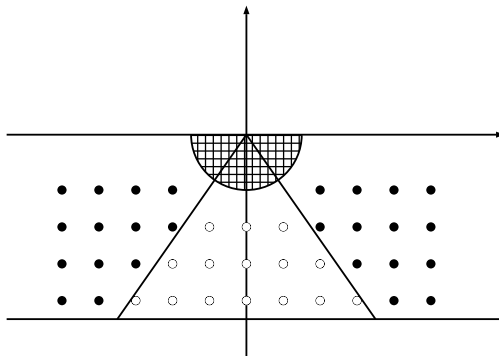


FIGURE 1. Resonances for de Sitter-Schwarzschild black holes. The black dots are resonances captured by Theorem in [20]. The hollow dots and resonances in the shaded region are not.

recovers m . Similar results exist for rotating black holes, see for example [8].

Our proof of the theorem is based on an analytic perturbation argument, by observing that the coefficients of the operator Δ_X are analytic functions in m . There are some resonance perturbation theories, see for instance Agmon [1], Howland [15], which are developed upon perturbation theory for eigenvalues, see for example [19]. Here, we use that Δ_X has asymptotically hyperbolic structure near the two horizons to construct a parametrix modulo a trace-class error term, following Mazzeo and Melrose [17]. We then use the Fredholm determinant and its analyticity in m to finish the proof. This approach has the benefit of not relying on the spherical symmetry of the black hole metric. For example, one can add general metric or potential perturbations with suitable decay at the horizons and obtain similar results to Theorem 1.1.

The note is organized as follows. We begin in Section 2 with a scattering problem to demonstrate the possibility of recovering parameters from

a single resonance. In Section 3, we discuss the asymptotically hyperbolic structure and the analyticity. We construct the resolvent in Section 4 and finish the proof in Section 5.

2. AN EXAMPLE: THE POTENTIAL BARRIER

In this section, we give an example of a scattering system depending on one parameter, for which a single resonance recovers the parameter. The example was actually used by Chandrasekhar and Detweiler in [6] to illustrate the concept of quasinormal modes.

Consider

$$u''(x) - V(x)u(x) + \sigma^2 u(x) = 0, \quad x \in \mathbb{R}$$

where σ is constant and V is the rectangular barrier

$$V(x) = \begin{cases} 1, & x \in [-L, L] \\ 0, & \text{otherwise.} \end{cases}$$

See Figure 2. Note that the potential is characterized by L . In this case, the scattering resonances can be defined as the poles of the scattering matrix. It is a standard exercise in scattering theory to find the scattering matrix. Let us look at a wave traveling to the right, hitting the potential and getting reflected and transmitted. In this case, the solution looks like

$$u_R(x) = \begin{cases} e^{i\sigma x} + r e^{-i\sigma x}, & x < -L \\ t e^{i\sigma x}, & x > L \end{cases}$$

Here, r is the reflection coefficient and t is the transmission coefficient. Similarly, we can consider a wave traveling to the left of the form

$$u_L(x) = \begin{cases} t' e^{-i\sigma x}, & x < -L \\ e^{-i\sigma x} + r' e^{i\sigma x}, & x > L \end{cases}$$

with r', t' the reflection, transmission coefficient respectively. The scattering matrix is

$$S = \begin{pmatrix} t & r \\ r' & t' \end{pmatrix}.$$

By matching the solution and its derivatives at $x = L, -L$, we can find the coefficients as

$$\begin{aligned} r = r' &= \frac{e^{-2i\sigma L + 2iqL} - e^{-2i\sigma L - 2iqL}}{K}, \\ t = t' &= e^{-2i\sigma L - 2iqL} + r \frac{q + \sigma}{q - \sigma} e^{-2iqL} \end{aligned}$$

where

$$K = \frac{q + \sigma}{q - \sigma} e^{-2iqL} - \frac{q - \sigma}{q + \sigma} e^{2iqL}.$$

Thus, the resonances are solutions of $K = 0$ or equivalently

$$(6) \quad \left(\frac{q + \sigma}{q - \sigma} \right)^2 = e^{4iqL}$$

where $q^2 = \sigma^2 - 1$. Suppose we have σ such that $\text{Im } q \neq 0$. Then we can take modulus of (6) to find L as

$$(7) \quad L = -\frac{1}{2 \text{Im } q} \ln \left| \frac{q + \sigma}{q - \sigma} \right|.$$

This shows that one can recover L from one resonance.

Now we provide a numerical verification. We compute resonances using a Matlab code from [21] and identify L using (7). It is important to note that the code from [21] does not calculate resonances by solving (7). Take $L = 1.3$. The potential and resonances are plotted in Figure 2. The numerical values of the four resonances nearest to the origin with positive real parts are

$$\begin{aligned} \lambda_1 &= 1.2127 - 0.4432i, & \lambda_2 &= 2.2120 - 1.1135i, \\ \lambda_3 &= 3.4242 - 1.4810i, & \lambda_4 &= 4.6501 - 1.7230i. \end{aligned}$$

Using any of these resonances in (7), we find $L = 1.3$ with a 10^{-4} error.

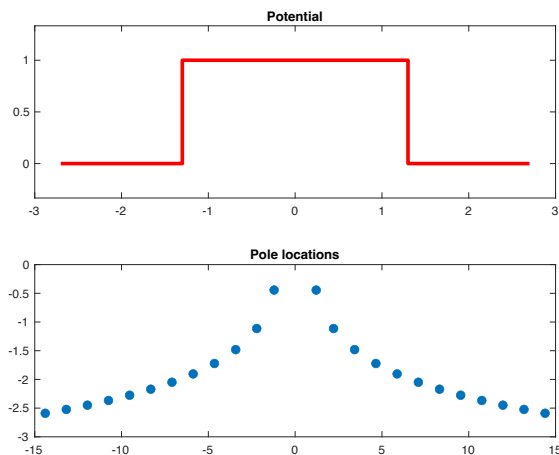


FIGURE 2. A rectangular barrier and its resonances.

3. THE ASYMPTOTICALLY HYPERBOLIC STRUCTURE AND ANALYTICITY

It is known that Δ_X in (4) can be essentially viewed as perturbed Laplacians associated with some asymptotically hyperbolic metrics near ∂X . We follow the presentation in [18]. Let X be a compact manifold of dimension $n + 1$ with boundary ∂X . Let ρ be a boundary defining function such that $\rho > 0$ in X , $\rho = 0$ at ∂X , $d\rho \neq 0$ at ∂X . A metric g on X is called conformally compact if $G = \rho^2 g$ is a non-degenerate Riemannian metric on the closure $\overline{X}$. If in addition $|d\rho|_G^2|_{\partial X} = K$ is constant, the metric g is called asymptotically hyperbolic. In this case, the sectional curvature approaches $-K$ along any curve towards ∂X , see [17, Lemma (2.5)]. There is a normal form of the metric near ∂X , see e.g. Graham [11]. In particular,

there is a choice of boundary defining function x such that in a neighborhood $U = [0, \epsilon)_x \times Y$, $Y \subset \partial X$ of $p \in \partial X$, we can use local coordinates (x, y) , $y \in Y$ and get

$$(8) \quad g = \frac{dx^2 + h(x, y, dy)}{x^2}.$$

Now we consider Δ_X in (12) on X . We define

$$\beta = \frac{1}{2} \frac{d\alpha^2}{dr} = \frac{m}{r^2} - \frac{\Lambda}{3} r.$$

We see that β is a smooth function of r on $[r_{bH}, r_{sI}]$ and analytic in $m \in \mathcal{O}$. We set $\beta_{bH} = \beta(r_{bH}) > 0$, $\beta_{sI} = \beta(r_{sI}) < 0$. Here, we recall that

$$(9) \quad \begin{aligned} r_{bH} &= \text{Im}(\sqrt{1 - (3m\sqrt{\Lambda})^2} + i3m\sqrt{\Lambda})^{1/3} / \sqrt{\Lambda}, \\ r_{sI} &= \text{Im}(-\sqrt{1 - (3m\sqrt{\Lambda})^2} + i3m\sqrt{\Lambda})^{1/3} / \sqrt{\Lambda}, \end{aligned}$$

see page 6 of [20]. Thus β_{sI}, β_{bH} are both analytic functions of $m \in \mathcal{O}$. Now we write (4) as

$$(10) \quad \Delta_X = \beta r^{-2} \alpha D_\alpha (\beta r^2 \alpha D_\alpha) + \alpha^2 r^{-2} \Delta_{\mathbb{S}^2}.$$

For convenience, we denote $\partial X = \partial X_{sI} \cup \partial X_{bH}$ with

$$\partial X_{sI} = \{r_{sI}\} \times \mathbb{S}^2, \quad \partial X_{bH} = \{r_{bH}\} \times \mathbb{S}^2.$$

Note that α only vanishes at ∂X . We let ρ be a boundary defining function defined through

$$(11) \quad \begin{aligned} \alpha &= 2r_{bH} \beta_{bH} \rho \text{ near } r = r_{bH} \\ \text{and } \alpha &= 2r_{sI} \beta_{sI} \rho \text{ near } r = r_{sI}. \end{aligned}$$

Here, the smooth structure on X is changed. Before, $r - r_{bH}$ is a smooth boundary defining function near ∂X_{bH} but now we think of $(r - r_{bH})^{\frac{1}{2}}$ as a smooth boundary defining function, see [20, Section 2]. By using ρ , (10) becomes

$$(12) \quad \begin{aligned} \Delta_X &= \beta r^{-2} \rho D_\rho (\beta r^2 \rho D_\rho) + 4\rho^2 \beta_{bH}^2 r_{bH}^2 r^{-2} \Delta_{\mathbb{S}^2} \text{ near } \partial X_{bH}, \\ \Delta_X &= \beta r^{-2} \rho D_\rho (\beta r^2 \rho D_\rho) + 4\rho^2 \beta_{sI}^2 r_{sI}^2 r^{-2} \Delta_{\mathbb{S}^2} \text{ near } \partial X_{sI}. \end{aligned}$$

Let g_{bH} be the metric defined in a neighborhood of ∂X_{bH} given by

$$(13) \quad g_{bH} = \frac{d\rho^2}{\beta^2 \rho^2} + \frac{r^2}{(2\beta_{bH} r_{bH})^2} \frac{dw^2}{\rho^2},$$

and let g_{sI} be the metric defined in a neighborhood of ∂X_{sI} given by

$$(14) \quad g_{sI} = \frac{d\rho^2}{\beta^2 \rho^2} + \frac{r^2}{(2\beta_{sI} r_{sI})^2} \frac{dw^2}{\rho^2}.$$

These can be viewed as metric perturbations of the hyperbolic metrics

$$g_{bH,0} = \frac{4dz^2}{\beta_{bH}^2(1 - |z|^2)} \quad g_{sI,0} = \frac{4dz^2}{\beta_{sI}^2(1 - |z|^2)}$$

on $\mathbb{B}^3 = \{z \in \mathbb{R}^3 : |z| \leq 1\}$ with constant negative sectional curvature $-\beta_{bH}^2$ and $-\beta_{sI}^2$ respectively. Here, $(1 - |z|^2)^{\frac{1}{2}}$ is the boundary defining function. Also, g_{bH}, g_{sI} are even asymptotically hyperbolic metrics as defined in Guilmarmou [12].

After some calculation, see [18, Proposition 8.1], we conclude that there are two smooth functions W_{bH}, W_{sI} such that

$$(15) \quad \begin{aligned} \alpha \Delta_X \alpha^{-1} &= \rho \Delta_X \rho^{-1} = \Delta_{g_{bH}} + \rho^2 W_{bH} - \beta_{bH}^2, \text{ near } \partial X_{bH}, \\ \alpha \Delta_X \alpha^{-1} &= \rho \Delta_X \rho^{-1} = \Delta_{g_{sI}} + \rho^2 W_{sI} - \beta_{sI}^2, \text{ near } \partial X_{sI}. \end{aligned}$$

This shows the asymptotically hyperbolic structure of $\alpha \Delta_X \alpha^{-1}$ near ∂X .

Consider the dependency of the operator $\alpha \Delta_X \alpha^{-1}$ on m . Note that the manifold X varies when varying m . We change the notation from X to $X(m)$. The dependency can be fixed by transforming $X(m)$ to a fixed reference manifold. Let $\mathcal{X} = (1, 2) \times \mathbb{S}^2$ and let

$$\Psi : \mathcal{X} \rightarrow X(m)$$

be a diffeomorphism defined by $(s, w) = \Psi(s, w) = (\psi(s), w)$ with

$$\psi(s) = (s - 1)r_{sI} + (2 - s)r_{bH}.$$

Note that Ψ extends smoothly to $\bar{\mathcal{X}} \rightarrow \overline{X(m)}$. Since r_{bH}, r_{sI} are analytic functions of $m \in \mathcal{O}$, ψ and Ψ are also analytic in $m \in \mathcal{O}$. Now, the pull back $\rho^* = \psi^*(\rho)$ is a family of smooth boundary defining functions for $\partial \mathcal{X}$. To see their dependency on m , we write (2) as

$$\alpha = \sqrt{\Lambda/3}(r - r_{bH})^{\frac{1}{2}}(r - r_{sI})^{\frac{1}{2}}(r - r_0)^{\frac{1}{2}}$$

where r_{bH}, r_{sI} are two positive roots of $\alpha = 0$ and r_0 is the third negative root. Using (11) and on $\mathcal{X}$ near $\partial \mathcal{X}_{bH}$, we have

$$(16) \quad \begin{aligned} \rho^* &= \frac{\sqrt{\Lambda/3}(r - r_{bH})^{\frac{1}{2}}(r - r_{sI})^{\frac{1}{2}}(r - r_0)^{\frac{1}{2}}}{2\beta_{bH}r_{bH}} \\ &= \frac{\sqrt{\Lambda/3}(r_{sI} - r_{bH})(s - 1)^{\frac{1}{2}}(2 - s)^{\frac{1}{2}}((s - 1)r_{sI} + (2 - s)r_{bH} - r_0)^{\frac{1}{2}}}{2\beta_{bH}r_{bH}} \\ &= (s - 1)^{\frac{1}{2}}A_{bH}(s, m) \end{aligned}$$

where $A_{bH}(s, m)$ is defined through the last two lines. It is clear that A_{bH} is smooth in s . Since $(s - 1)r_{sI} + (2 - s)r_{bH} - r_0 > 0$ for $s \in [1, 2]$, we see that A_{bH} , hence ρ^* , is analytic in $m \in \mathcal{O}$. Near $\partial \mathcal{X}_{sI}$, we have a similar form

$$(17) \quad \rho^* = (2 - s)^{\frac{1}{2}}A_{sI}(s, m).$$

From (13), (14), we see that the pull-back of the metrics are

$$(18) \quad \begin{aligned} \Psi^* g_{bH} &= \frac{(d\rho^*)^2}{(r_{sI} - r_{bH})^2(\beta^*)^2(\rho^*)^2} + \frac{[(s - 1)r_{sI} + (2 - s)r_{bH}]^2}{(2\beta_{bH}r_{bH})^2} \frac{dw^2}{(\rho^*)^2} \\ \Psi^* g_{sI} &= \frac{(d\rho^*)^2}{(r_{sI} - r_{bH})^2(\beta^*)^2(\rho^*)^2} + \frac{[(s - 1)r_{sI} + (2 - s)r_{bH}]^2}{(2\beta_{sI}r_{sI})^2} \frac{dw^2}{(\rho^*)^2} \end{aligned}$$

near $\partial X_{bH}, \partial X_{sI}$ respectively. Here, $\beta^* = \psi^*(\beta)$.

Let $\mathcal{V}_0(\mathcal{X})$ be the Lie algebra of smooth vector fields on $\mathcal{X}$ vanishing at $\partial\mathcal{X}$. In local coordinates (x, y) near $\partial\mathcal{X}$ with x being the boundary defining function, $\mathcal{V}_0(\mathcal{X})$ is generated by $x\partial_x, x\partial_y$. For $\kappa \in \mathbb{N}$, the space of 0-differential operators of order κ on $\mathcal{X}$, denoted by $\text{Diff}_0^\kappa(\mathcal{X})$, is generated by up to κ -fold products of vector fields in $\mathcal{V}_0(\mathcal{X})$. From the analyticity of the diffeomorphism Ψ in m , we see that the pull-back of $\alpha\Delta_X\alpha^{-1}$ to $\mathcal{X}$ is a differential operator on $\mathcal{X}$ with coefficients analytic in m . To see that it belongs to $\text{Diff}_0^2(\mathcal{X})$ with coefficients analytic in $m \in \mathcal{O}$, we consider $\alpha\Delta_X\alpha^{-1}$ near $\partial\mathcal{X}$ for example near $s = 1$. We change the boundary defining function from ρ^* to $\gamma = (s - 1)^{\frac{1}{2}}$ so $\rho^* = \gamma A_{bH}(\gamma^2, m)$. Then the metric in (18) becomes

$$\Psi^*g_{bH} = \frac{(1 + \partial_\gamma A_{bH})(d\gamma)^2}{(r_{sI} - r_{bH})^2(\beta^*)^2 A_{bH}^2 \gamma^2} + \frac{[(s - 1)r_{sI} + (2 - s)r_{bH}]^2 dw^2}{(2\beta_{bH}r_{bH})^2 A_{bH}^2 \gamma^2}$$

which is a family of Riemannian metrics on $\mathcal{X}$ analytic in m .

4. THE RESOLVENT CONSTRUCTION

We obtain an approximation of $R_X(\lambda)$ in (5) following Mazzeo-Melrose [17]. In fact, we will find the resolvent of $\alpha\Delta_X\alpha^{-1}$ on $\mathcal{X}$. We will be using operators acting on half densities on $\mathcal{X}$. For convenience, we introduce an auxiliary Riemannian metric $g_\mathcal{X}$ on $\mathcal{X}$ which equals g_{bH}, g_{sI} near $\partial\mathcal{X}_{bH}, \partial\mathcal{X}_{sI}$ respectively. Such a metric can be obtained by gluing g_{bH}, g_{sI} near $\partial\mathcal{X}$ and some Riemannian metric in the interior of $\mathcal{X}$. The choice is clearly not unique and its dependency on m is not important. We use $g_\mathcal{X}$ to trivialize the (zero) one-density bundle Ω_0 , that is we take Ω_0 to be the volume form $|dg_\mathcal{X}|$. The half-density bundle is $\Omega_0^{\frac{1}{2}}$. Let x be a boundary defining function such that in local coordinates $(x, y), x \geq 0, y \in \mathbb{S}^2$ near $\partial\mathcal{X}$, $g_\mathcal{X}$ is expressed in form of (8). In these coordinates,

$$\Omega_0^{\frac{1}{2}} = H(x, y) \left| \frac{dx}{x} \frac{dy}{x} \right|^{\frac{1}{2}}$$

for some smooth function H . Now we consider $\alpha\Delta_X\alpha^{-1}$ acting on smooth sections $C^\infty(\mathcal{X}; \Omega_0^{\frac{1}{2}})$ in the following way:

$$\alpha\Delta_X\alpha^{-1}(u\Omega_0^{\frac{1}{2}}) = (\alpha\Delta_X\alpha^{-1}u)\Omega_0^{\frac{1}{2}}.$$

The resolvent $R_\alpha(\lambda) = (\alpha\Delta_X\alpha^{-1} - \lambda^2)^{-1}$ acts on $\Omega_0^{\frac{1}{2}}$ in the same way.

The parametrix is constructed on the 0-double space of $\mathcal{X} \times \mathcal{X}$ as in [17]. Let $\text{Diag} = \{(z, z) \in \mathcal{X} \times \mathcal{X}\}$ be the diagonal of $\mathcal{X} \times \mathcal{X}$. Let $\partial\text{Diag} = \text{Diag} \cap (\partial\mathcal{X} \times \partial\mathcal{X})$ which has two (disjoint) connected components. As a set, the 0-double space is

$$\mathcal{X} \times_0 \mathcal{X} = (\mathcal{X} \times \mathcal{X}) \setminus \partial\text{Diag} \sqcup S_{++}(\partial\text{Diag})$$

where $S_{++}(\partial\text{Diag})$ denotes the inward pointing spherical bundle of $T_{\partial\text{Diag}}^*(\mathcal{X} \times \mathcal{X})$. Let

$$(19) \quad \beta_0 : \mathcal{X} \times_0 \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$$

be the blow-down map. Then $\mathcal{X} \times_0 \mathcal{X}$ is equipped with a topology and smooth structure of a manifold with corners for which β_0 is smooth. The manifold $\mathcal{X} \times_0 \mathcal{X}$ has the following boundary hyper-surfaces: the left and right faces $L = \overline{\beta_0^{-1}(\partial\mathcal{X} \times \mathcal{X})}$, $R = \overline{\beta_0^{-1}(\mathcal{X} \times \partial\mathcal{X})}$, and the front face $\text{ff} = \overline{\beta_0^{-1}(\partial\text{Diag})}$. Since $\partial\mathcal{X} = \partial\mathcal{X}_{bH} \cup \partial\mathcal{X}_{sI}$ where the asymptotic behavior of the resolvent is different at each connected component, it is convenient to introduce

$$\begin{aligned} L_{bH} &= \overline{\beta_0^{-1}(\partial\mathcal{X}_{bH} \times \mathcal{X})}, & L_{sI} &= \overline{\beta_0^{-1}(\partial\mathcal{X}_{sI} \times \mathcal{X})}, \\ R_{bH} &= \overline{\beta_0^{-1}(\mathcal{X} \times \partial\mathcal{X}_{bH})}, & R_{sI} &= \overline{\beta_0^{-1}(\mathcal{X} \times \partial\mathcal{X}_{sI})}, \end{aligned}$$

so $L = L_{bH} \cup L_{sI}$, $R = R_{bH} \cup R_{sI}$, see Figure 3. The lifted diagonal is denoted by $\text{Diag}_0 = \overline{\beta_0^{-1}(\text{Diag} \setminus \partial\text{Diag})}$. $\mathcal{X} \times_0 \mathcal{X}$ has co-dimension two corners at the intersection of two of the boundary faces L, R, ff and co-dimension three corners given by the intersection of all the three faces. See Figure 3.

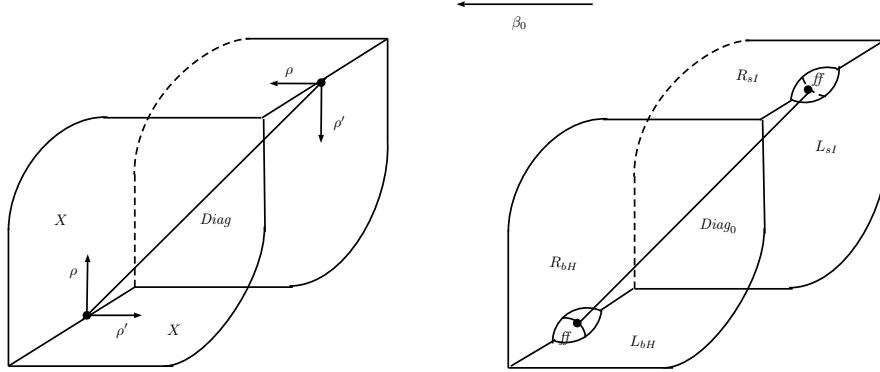


FIGURE 3. The 0-double space. The blown-up at the two components of ∂Diag_0 are shown.

Now we introduce spaces of operators on $\mathcal{X} \times_0 \mathcal{X}$. First, let $\mathcal{K}_0^\kappa(\mathcal{X}) \subset \mathcal{D}'(\mathcal{X} \times_0 \mathcal{X}; \Omega_0^{\frac{1}{2}})$ be the space of distributional sections of the bundle $\Omega_0^{\frac{1}{2}}$ which are conormal to Diag_0 and vanish to infinite order at L, R . Here, it is understood that $\Omega_0^{\frac{1}{2}}$ denotes the half-density bundle lifted from the one on $\mathcal{X} \times \mathcal{X}$ by β_0 . The corresponding class of pseudo-differential operators is denoted by $\Psi_0^\kappa(\mathcal{X}, \Omega_0^{\frac{1}{2}})$. Next, let $\mathcal{V}_b$ be the space of smooth vector fields on $\mathcal{X} \times_0 \mathcal{X}$ which are tangent to each of the boundary faces L, R, ff . Let $\rho_\bullet, \bullet = L_{bH}, L_{sI}, R_{bH}, R_{sI}, \text{ff}$ be boundary defining functions. We set

$$(20) \quad \begin{aligned} \mathcal{A}^{a,b,c,d}(\mathcal{X} \times_0 \mathcal{X}) &= \{u \in \mathcal{D}'(\mathcal{X} \times_0 \mathcal{X}) : V_1 \cdots V_k u \in \\ &\rho_{L_{bH}}^a \rho_{R_{bH}}^b \rho_{L_{sI}}^c \rho_{R_{sI}}^d L^\infty(\mathcal{X} \times_0 \mathcal{X}), V_i \in \mathcal{V}_b, i = 1, 2, \dots, k, \forall k \geq 0\}. \end{aligned}$$

Then define

$$\mathcal{K}_0^{-\infty, a, b, c, d}(\mathcal{X} \times_0 \mathcal{X}) = \mathcal{A}^{a, b, c, d}(\mathcal{X} \times_0 \mathcal{X}) \otimes C^\infty(\mathcal{X} \times_0 \mathcal{X}; \Omega_0^{\frac{1}{2}}).$$

Finally, define

$$\mathcal{K}_0^{\kappa, a, b, c, d}(\mathcal{X}) = \mathcal{K}_0^{-\infty, a, b, c, d}(\mathcal{X} \times_0 \mathcal{X}) + \mathcal{K}_0^\kappa(\mathcal{X}).$$

Then we let $\Psi_0^{\kappa, a, b, c, d}(\mathcal{X})$ be the space of operators on $\mathcal{X}$ whose Schwartz kernel when lifted to $\mathcal{X} \times_0 \mathcal{X}$ belongs to $\mathcal{K}_0^{\kappa, a, b, c, d}(\mathcal{X})$.

Below, we let $\rho \in C^\infty(\mathcal{X})$ be such that $\rho = \rho_{sI}$ near $\partial\mathcal{X}_{sI}$ and $\rho = \rho_{bH}$ near $\partial\mathcal{X}_{bH}$. We have the following result.

Proposition 4.1. *There is a family of operators $M(\lambda, m) \in \Psi_0^{-2, a, a, b, b}(\mathcal{X})$ with*

$$(21) \quad a = 1 + \frac{\lambda}{\beta_{bH}}i, \quad b = 1 + \frac{\lambda}{|\beta_{sI}|}i,$$

analytic in $m \in \mathcal{O}$ and holomorphic in $\lambda \in \mathbb{C} \setminus \mathcal{Q}(m)$ with

$$(22) \quad \mathcal{Q}(m) \doteq \frac{-i}{\beta_{bH}}\mathbb{N} \cup \frac{-i}{|\beta_{sI}|}\mathbb{N} \cup \{0\}$$

such that

$$(23) \quad (\alpha\Delta_X\alpha^{-1} - \lambda^2)M(\lambda, m) = \text{Id} + E(\lambda, m).$$

Here, $E(\lambda, m) \in \rho_{\text{ff}}^\infty \Psi_0^{-\infty, \infty, \infty, a, \infty, b}(\mathcal{X})$ is trace class on $\rho^l L^2(\mathcal{X})$ for $l > 1 - \min(\text{Re } a, \text{Re } b)$. Moreover, its Schwarz kernel is holomorphic in $\lambda \in \mathbb{C} \setminus \mathcal{Q}(m)$, and analytic in $m \in \mathcal{O}$.

Proof. For fixed m , the construction of $M(\lambda, m)$ and $E(\lambda, m)$ and their holomorphy in λ is essentially contained in Proposition (7.4) of [17], which applies to the Laplacian of asymptotically hyperbolic metrics. As argued in Proposition 2.2 of [20], the result applies to $\alpha\Delta_X\alpha^{-1} - \lambda^2$ as the normal operator is elliptic. Because we argued in Section 3 that $\alpha\Delta_X\alpha^{-1} \in \text{Diff}_0^2(\mathcal{X})$ with coefficients analytic in $m \in \mathcal{O}$, the construction in [17] produces $M(\lambda, m)$, $E(\lambda, m)$ analytic in m . For the meromorphic properties in λ , it suffices to consider the operators near $\partial\mathcal{X}_{bH}$, $\partial\mathcal{X}_{sI}$ respectively. Write $g_{bH} = \rho^{-2}h_{bH}$. From [17, Theorem (7.1)], see also [12, Theorem 1.1], we know that the resolvent of

$$(24) \quad |d\rho|_{h_{bH}}^{-2} \Delta_{g_{bH}} + \zeta(\zeta - 2)$$

belongs to $\Psi_0^{-2, \zeta, \zeta}(\mathcal{X})$ and is meromorphic in ζ with poles at $\frac{-i}{\beta_{bH}}\mathbb{N} \cup \{0\}$. Here, we followed [17] and used a different spectral parameter ζ . Near $\partial\mathcal{X}_{bH}$, $-|d\rho|_{h_{bH}}^2$ approaches $-\beta_{bH}^2$. Comparing (24) with (12), we get

$$\beta_{bH}^2 \zeta(\zeta - 2) = -\lambda^2 - \beta_{bH}^2$$

which gives $\zeta = 1 + i\lambda/\beta_{bH}$. This gives a , and $b = 1 + i\lambda/|\beta_{sI}|$ can be found in the same way near $\partial\mathcal{X}_{sI}$. The set $\mathcal{Q}(m)$ comes from the poles of the resolvent of

$$\beta_{bH}^{-2}\Delta_0 - \lambda^2 \text{ and } \beta_{sI}^{-2}\Delta_0 - \lambda^2$$

acting between weighted C^∞ spaces on $\mathbb{B}^3$, see Lemma (6.15) of [17] for the precise statement. Here, Δ_0 denotes the positive Laplacian of the standard hyperbolic metric. By rescaling the operator, we find the set $\mathcal{Q}(m)$.

Finally, we consider the mapping properties of $E(\lambda, m)$. We recall a result [17, Lemma (5.24)] that the push forward of the space $\rho_{\text{ff}}^\infty \mathcal{K}_0^{-\infty, a, b, c, d}(\mathcal{X} \times_0 \mathcal{X})$ is

$$\mathcal{A}_0^{a, b, c, d}(\mathcal{X} \times \mathcal{X}; \Omega_0^{\frac{1}{2}} \otimes \Omega_0^{\frac{1}{2}}) = \bigcap_p \tau^p \mathcal{A}^{a, b, c, d}(\mathcal{X} \times \mathcal{X}), \tau^2 = |y - y'|^2 + \rho^2 + (\rho')^2$$

with $\mathcal{A}^{a, b, c, d}(\mathcal{X} \times \mathcal{X})$ defined similarly to (20). Let $K_E(z, z')$ be the Schwarz kernel of E . As $E \in \rho_{\text{ff}}^\infty \Psi_0^{-\infty, \infty, a, \infty, b}(\mathcal{X})$, we have the estimate

$$|\rho^{-l}(z)K_E(z, z', \lambda, m)\rho^l(z')| \leq C\rho^{l+\min(\text{Re } a, \text{Re } b)}(z')$$

for some $C > 0$. By applying Schur's lemma more precisely Lemma 6.2 of [18], we conclude that $E(\lambda, m)$ is bounded on $\rho^l L^2$ for $l > 1 - \min(\text{Re } a, \text{Re } b)$. To see $E(\lambda, m)$ is trace class on $\rho^l L^2$, we write

$$K_E(z, z, \lambda, m) = \rho^N \rho^a \rho^b F_N(z, \lambda, m)$$

where $F_N \in C^\infty(\overline{\mathcal{X}})$ is analytic in $m \in \mathcal{O}$. The integral $\int_{\mathcal{X}} |K_E(z, z)| dg_{\mathcal{X}}(z)$ is finite for N large so E is of trace class. $\square$

5. PROOF OF THEOREM 1.1

We apply the resolvent to (23) to get

$$(25) \quad M(\lambda, m) = R_\alpha(\lambda)(\text{Id} + E(\lambda, m)).$$

Since $E(\lambda, m)$ is compact on $x^l L^2(\mathcal{X})$, using the analytic Fredholm theorem, we see that for any $m \in \mathcal{O}$, $(\text{Id} + E(\lambda, m))^{-1}$ is a family of bounded operators on $x^l L^2(\mathcal{X})$, meromorphic in $\lambda \in \mathbb{C} \setminus \mathcal{Q}(m)$. The poles are the resonances.

We use the determinant of $\text{Id} + E$ to analyze the poles. We recall that if A is a trace class operator on a Hilbert space $\mathcal{H}$ with eigenvalues $\lambda_k, k = 1, 2, \dots$ with $|\lambda_1| \geq |\lambda_2| \geq \dots \geq 0$, then the Fredholm determinant $\det(\text{Id} + A) = \prod_{k=1}^\infty (1 + \lambda_k)$. See [9, Appendix B]. Also, $\text{Id} + A$ is invertible if and only if $\det(\text{Id} + A)$ is non-zero, see [9, Proposition B.28]. Therefore, the set of resonances of $R_\alpha(\lambda)$ is contained in the zero set of

$$K(\lambda, m) = \det(\text{Id} + E(\lambda, m)).$$

Using the argument at the end of [9, Section B.5], we conclude that $K(\lambda, m)$ is a function holomorphic in $\lambda \in \mathbb{C} \setminus \mathcal{Q}(m)$, and analytic in $m \in \mathcal{O}$.

Now we suppose λ_0 is a resonance so $K(\lambda_0, m) = 0$. By the analyticity in m , either $K(\lambda_0, m)$ is identically zero for all m which means λ_0 is a resonance for all $m \in \mathcal{O}$, or m is the only (discrete) zero locally. For the first

alternative, we can apply Theorem 1.1 of [13] (with zero angular momentum) to conclude that the trivial resonances must be contained in $-i\sqrt{\Lambda/3}\mathbb{N}\cup\{0\}$. This proves the first assertion of Theorem 1.1.

For the stability, we write $K(\lambda, m) = (m - m_0)^N f(\lambda, m)$ for some $N \geq 0$ and f analytic in m with $f(\lambda_0, m_0) \neq 0$. Now we set $t = (m - m_0)^N$ and get

$$K(\lambda, t) = K(\lambda, m) = tf(\lambda, m_0 + t^{1/N}).$$

We see that

$$\partial_t K(\lambda, t)|_{t=0} = f(\lambda, m_0) \neq 0.$$

Using the implicit function theorem, we get that $t = g(\lambda)$ is differentiable in a neighborhood of λ_0 . Thus, $|t| \leq C|\lambda - \tilde{\lambda}|$ which implies

$$|m - \tilde{m}| \leq C|\lambda - \tilde{\lambda}|^{1/N}.$$

This completes the proof of the Theorem 1.1.

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