

GLOBAL REGULARITY OF SKEW MEAN CURVATURE FLOW FOR SMALL DATA IN $d \geq 4$ DIMENSIONS

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ABSTRACT. The skew mean curvature flow is an evolution equation for a d dimensional manifold immersed into $\mathbb{R}^{d+2}$, and which moves along the binormal direction with a speed proportional to its mean curvature.

In this article, we prove small data global regularity in low-regularity Sobolev spaces for the skew mean curvature flow in dimensions $d \geq 4$. This extends the local well-posedness result in [7].

1. INTRODUCTION

The skew mean curvature flow (SMCF) evolves a codimension 2 submanifold along its binormal direction with a speed given by its mean curvature. Precisely speaking, assume that Σ is a d -dimensional oriented manifold and $(\mathcal{M}, g_{\mathcal{M}})$ is an $(d+2)$ -dimensional oriented Riemannian manifold, then SMCF is a family of time-dependent immersions $F : \mathbb{I} \times \Sigma \rightarrow \mathcal{M}$ satisfying

$$\begin{cases} \partial_t F = J(F)\mathbf{H}(F), & (t, x) \in \mathbb{I} \times \Sigma, \\ F(0, x) = F_0(x), \end{cases} \quad (1.1)$$

where, for each given $t \in \mathbb{I}$, $\mathbf{H}(F)$ denotes the mean curvature vector of the submanifold $\Sigma_t := F(t, \Sigma)$. Here $J(F)$, which denotes the natural induced complex structure for the normal bundle $N\Sigma_t$, can be simply defined as rotating a vector in the normal space by $\frac{\pi}{2}$ positively (notice that $N\Sigma_t$ is of rank 2). An alternative formulation of SMCF is

$$\begin{cases} (\partial_t F)^\perp = J(F)\mathbf{H}(F), \\ F(0, \cdot) = F_0. \end{cases} \quad (1.2)$$

Here, for an arbitrary vector $Z \in T\mathcal{M}$ at F , $Z^\perp$ denotes its orthogonal projection onto $N\Sigma_t$. Note that (1.1) differs from (1.2) by a time dependent diffeomorphism of Σ_t . Hence, (1.1) and (1.2) are topologically equivalent, but (1.2) has a larger gauge group consisting of all space-time changes of coordinates.

For $d = 1$, the 1-dimensional SMCF in $\mathbb{R}^3$ is the vortex filament equation $\partial_t v = \partial_s v \times \partial_s^2 v$ for $v : (s, t) \in \mathbb{R} \times \mathbb{R} \mapsto v(s, t) \in \mathbb{R}^3$, where t denotes time, s denotes the arc-length parameter of the curve $v(t, \cdot)$, and $\times$ denotes the cross product in $\mathbb{R}^3$. The vortex filament

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equation describes the free motion of a vortex filament, see Da Rios [1], Hasimoto [5]. For $d \geq 2$, the (SMCF) was deduced by both physicists and mathematicians. The physical motivations are the localized induction approximation (LIA) of high dimensional Euler equations and asymptotic dynamics of vortices in superconductivity and superfluidity, see Lin [19], Jerrard [10], Shashikanth [27], Khesin [16]. SMCF also appears in various mathematical problems, especially the Hamiltonian flow associated with Marsden-Weinstein symplectic structure [23], nonlinear Grassmannian manifolds discussed by Haller-Vizman [6], and the star mean curvature flow introduced by Terng [32]. Moreover, it is remarkable that SMCF has a deep relationship with the Schrödinger map flow (e.g. [33]), in fact, [29] proved that the Gauss map of a d -dimensional SMCF in $\mathbb{R}^{d+2}$ satisfies a Schrödinger map flow equation.

Let us briefly recall some earlier works on SMCF. The 1-d case is special, in that the problem has a semilinear, rather than quasilinear structure, and is essentially equivalent to the 1-d cubic NLS problem. For more details we refer the reader to the survey article of Vega [28].

The early work of Gomez [3] proposed a way to write SMCF as a quasilinear Schrödinger equation system by introducing a complex valued scalar mean curvature and choosing some gauge for the normal bundle. The model (1.1) was studied by Song-Sun [30], who proved the local existence of SMCF for $F : \Sigma \rightarrow \mathbb{R}^4$ with a compact oriented surface Σ . This was generalized by Song [31] to $F : \Sigma^d \rightarrow \mathbb{R}^{d+2}$ with a compact oriented manifold Σ^d for all $d \geq 2$. Moreover, [31] proved the continuous dependence of solutions on initial data in a geometric distance. However, as noted in [7, 31], (1.1) has much less freedom than (1.2). In fact, the formulation (1.2) is invariant under time dependent coordinate transformations of Σ_t , while (1.1) is only invariant under time independent coordinate transformations. This additional gauge freedom included in (1.2) will help to compensate the derivative loss in the local Cauchy theory.

The above issue was clarified in Huang-Tataru [7, 8], who proposed an alternative approach, namely to start with the formulation of SMCF in (1.2), and then to choose a favourable space-time gauge (i.e. coordinates). In this gauge there is no more loss of derivatives, and they were able to prove a full local well-posedness result in low regularity Sobolev spaces for initial data which are small perturbations of flat metrics. Precisely, the solutions obtained in [7, 8] are at regularity H^s , with $s > \frac{d}{2}$, measured at the curvature level; this is one derivative above scaling. The gauge formulation of the SMCF flow in [7, 8] closely resembles a quasilinear Schrödinger equation, coupled with several elliptic/parabolic equations. For the local well-posedness theory of general quasilinear Schrödinger equations, see the pioneer works of Kenig-Ponce-Vega [13–15] for localized initial data, as well as Marzuola-Metcalf-Tataru [24, 24, 26] for data in translation invariant H^s based spaces.

The small data global regularity problem for SMCF in the formulation (1.1) was considered in [21] which proved that Euclidean planes are stable under SMCF for small transversal perturbations in some $W^{2,q} \cap H^k$ space with some $q \in (1, 2)$ and integer $k > \max(\frac{d+7}{2}, d+1)$. In the later work [22], the $W^{2,q}$ smallness and transversal assumption of [21] were removed in $d \geq 3$, and it proved the global in time existence and scattering of small data solutions, and the existence of wave operators.

1.1. The main result. Our objective in this paper is to establish the global in time well-posedness and scattering for solutions to SMCF in the formulation (1.2) for small initial data.

Our main dynamic variable will be the *complex mean curvature* ψ for our system, which is defined in the next section, see (2.3), and stands for the representation of the scalar mean curvature relative to an orthonormal frame in $N\Sigma$ determined by our choice of gauge. The similar representation of the full second fundamental form will be denoted by λ , which is defined in (2.2), and the two are related by $\psi = \text{Tr } \lambda$.

To measure the Sobolev regularity of ψ for our global solutions we introduce the index s_d so that

$$s_d \geq 3, \quad \text{if } d = 4; \quad s_d > \frac{d+1}{2} + \frac{1}{2(d-1)}, \quad \text{if } d \geq 5. \quad (1.3)$$

To measure the (Strichartz) decay of the solutions in time we will use the exponent r_d defined by

$$r_d = \frac{2d(d-1)}{(d-2)^2}, \quad \text{for } d \geq 4. \quad (1.4)$$

Then we define the Strichartz norms $S[0, T]$ as

$$\|\psi\|_{S[0, T]} := \|\psi\|_{L^2(0, T; W^{1,4})} + \|\psi\|_{L^2(0, T; W^{s_d-2, r_d})}, \quad \text{for } d = 4, \quad (1.5)$$

and

$$\|\psi\|_{S[0, T]} := \|\psi\|_{L^2(0, T; W^{s_d-2, r_d})}, \quad \text{for } d \geq 5. \quad (1.6)$$

At this point, we content ourselves with a less precise formulation of the main result, relative to the harmonic/Coulomb gauge which was introduced in [7] and is discussed in Section 2:

Theorem 1.1 (Small data global regularity and scattering). *Let s_d and r_d be as (1.3), (1.4) respectively for $d \geq 4$. Then there exists $\epsilon_0 > 0$ sufficiently small such that, for all initial data Σ_0 with metric and mean curvature satisfying*

$$\|\partial_x(g_0 - I)\|_{H^{s_d}} + \|\mathbf{H}_0\|_{H^{s_d}} \leq \epsilon_0,$$

the skew mean curvature flow (1.2) for maps from $\mathbb{R}^d$ to the Euclidean space $(\mathbb{R}^{d+2}, g_{\mathbb{R}^{d+2}})$ is globally well-posed in the harmonic/Coulomb gauge.

Moreover, in the harmonic/Coulomb gauge, the metric and complex mean curvature satisfy the bounds

$$\|\partial_x(g - I_d)\|_{C_t H_x^{s_d+1}} + \|\psi\|_{S(\mathbb{R})} + \|\psi\|_{C_t H_x^{s_d}} \lesssim \|\psi_0\|_{H_x^{s_d}}. \quad (1.7)$$

In addition, there exists $\psi_{\pm} \in H^{s_d-2}$ such that

$$\lim_{t \rightarrow \pm\infty} \|\psi - e^{it\Delta} \psi_{\pm}\|_{H_x^{s_d-2}} = 0. \quad (1.8)$$

Remark 1.1.1. The gauge choice used for the above result is the *harmonic/Coulomb gauge*, following [7]. Here harmonic refers to the choice of coordinates on Σ at fixed time, and Coulomb applies to the choice of the orthonormal frame on $N\Sigma$. In this gauge, the surface Σ is uniquely determined up to symmetries by the complex mean curvature ψ at fixed time in an elliptic fashion. By contrast, in [8] the harmonic/Coulomb gauge is only imposed at the initial time, while a heat gauge is used forward in time.

Remark 1.1.2. One may compare the Sobolev index s_d in the theorem with the weaker restriction $s > d/2$ in [7, 8]. Here the choice of regularity s_d is more restricted due to the need to also control decay via global in time Strichartz norms. Precisely, our main control norm for the energy estimates will essentially be $\|\lambda\|_{L_t^2 L_x^\infty}$, see (1.11) below. To bound this by $\|\psi\|_{L_T^2 W^{s_d-2, r_d}}$ by elliptic estimates and Sobolev embeddings requires that $s_d > \frac{d}{r_d} + 2$. This gives the s_d threshold (1.3) for $d \geq 5$.

In dimension $d = 4$ we face an additional obstruction arising in the study of the global well-posedness for the linearized equation. For that we need Strichartz estimates in the space $L^2 W^{1,4}$, which in turn restricts the regularity to $s_d \geq 3$.

Remark 1.1.3. Compared with the result in [22], the main difference is that we work in lower regularity Sobolev spaces, which particularly also include the fractional Sobolev spaces. Indeed, with the better formulation (1.2) and new gauges, we get elliptic equations of metrics and obtain a Schrödinger type nonlinear equation of the defined scalar mean curvature ψ . There are three key factors which help us to lower the regularity assumption. First, the coefficients in the second derivative terms of ψ are two derivatives better than ψ . By contrast, the coefficients in second derivative terms in [22] are only one derivative better. Secondly, in addition to endpoint Strichartz estimates, we also apply inhomogeneous Strichartz estimates in order to estimate the nonlinear terms. Thirdly, we use the energy estimates for linearized equation and the paradifferential expansion of Sobolev norm in order to transfer energy bounds from integral to fractional Sobolev spaces. The above new ideas play crucial role in our main result. However, there is still a gap between s_d (1.3) in our theorem and the possible optimal regularity $d/2 - 1$.

The global regularity is closely related to the energy estimates and Strichartz estimates for the complex mean curvature ψ for our system. Following [7], in the harmonic/Coulomb

gauge ψ solves a quasilinear Schrödinger equation (2.27)-(2.28). We describe these estimates next, beginning with the energy estimates.

A key point in the following proposition is that we should work with the “good” energy, which is both coercive and propagates well along the flow. At integer Sobolev regularity indices there is a canonical, geometric choice, given by the L^2 norm of covariant derivatives of ψ . The intrinsic Sobolev norms are denoted as H^k with integer $k \in \mathbb{N}$, which are defined in (3.2). The challenge is then to prove coercivity, which is no longer a covariant property but depends instead on our gauge choice.

Proposition 1.2 (Energy estimates in H^k). *For each nonnegative integer k there exists an energy functional $E^k = E^k(\psi)$ defined on functions in H^k which are also small in H^s for some $s > d/2$, which has the following two properties:*

- i) [Energy growth] *If ψ is a solution of the SMCF flow (2.27) with constraints (2.3), (2.5), (2.7), (2.8), (2.19) and (2.24) then*

$$\frac{d}{dt} E^k(\psi) \leq C_E \|\lambda\|_{L^\infty}^2 \|\lambda\|_{H^k}^2. \quad (1.9)$$

- ii) [Coercivity:] *In the harmonic/Coulomb gauge we have the equivalence relation:*

$$E^k(\psi) \approx_{C_1} \|\psi\|_{H^k}^2 \approx \|\lambda\|_{H^k}^2 \approx \|\lambda\|_{H^k}^2, \quad (1.10)$$

where the constant C_1 only depends on the H^s norm of ψ .

As a consequence of (1.9), by (1.10) and Gronwall’s inequality we obtain

$$\|\psi(t)\|_{H^k}^2 \leq C_1^2 e^{\int_0^t C_E \|\lambda\|_{L^\infty}^2 ds} \|\psi_0\|_{H^k}^2. \quad (1.11)$$

This justifies the need to control the norm $\|\lambda\|_{L^2 L^\infty}$ for our global solutions.

Remark 1.2.1. The energy estimate (1.9) holds without any gauge assumptions, and was proved first in [30, Lemma 4.9]. Here we use a different method to prove this estimate, using only the Schrödinger equation for ψ and the associated constraints to gain the estimates. The gauge choice is, however, essential for the coercivity part.

Remark 1.2.2. The energies are constructed in an explicit fashion only for integer k . Nevertheless, as a consequence in our analysis in the last section of the paper, it follows that bounds of the form (1.11) hold also for all noninteger $k > 0$. However, we do this using a mechanism which is akin to a paradifferential expansion, without constructing an explicit energy functional as provided by the above theorem in the integer case.

We now turn our attention to the Strichartz estimates for ψ . Since our problem is quasilinear, here we a-priori assume that ψ remains small in H^{s_d} , and we also lose some derivatives.

Proposition 1.3 (Strichartz estimates). *Let s_d be as (1.3) and $\sigma_d = s_d - 2$ for $d \geq 4$. Assume that ψ is a solution of (2.27)-(2.28) on some interval $[0, T]$ for $T > 1$, which satisfies the smallness condition*

$$\|\psi\|_{L^\infty H^{s_d}} \leq C_0 \epsilon_0,$$

Then ψ satisfies the Strichartz bound

$$\|\psi(t)\|_{S[0,T]} \leq C_2(\|\psi_0\|_{H^{\sigma_d + \frac{d-2}{2(d-1)}}} + (C_0 \epsilon_0)^2 \|\psi(t)\|_{S[0,T]}). \quad (1.12)$$

A starting point for this result is provided by the endpoint Strichartz estimates of Keel-Tao [11]. However, in addition we also use the larger class of inhomogeneous Strichartz estimates developed by [2, 17, 18, 34]. The latter plays a key role in lowering the regularity assumptions for the initial data in our theorem.

1.2. An outline of the proof. There are several key steps in the proof of our main result:

1. *The gauge choice.* The formulation (1.2) has a key additional gauge freedom compared with the equation (1.1). Indeed, (1.2) is invariant under any time dependent diffeomorphism in Σ_t , while (1.1) is only invariant under time independent diffeomorphisms in Σ_t . This additional freedom enabled us to use the harmonic coordinate system in [7]. This is then combined with the the Coulomb gauge for the orthonormal frame on the normal bundle. This reformulation of the equation (1.2) is reviewed in Section 2, where we rewrite it as a nonlinear Schrödinger equation for a single independent variable. This independent variable, denoted by ψ , represents the trace of the second fundamental form on Σ_t , in complex notation. In addition to the independent variables, we will use several dependent variables, as follows:

- The Riemannian metric g on Σ_t .
- The (complex) second fundamental form λ for Σ_t .
- The magnetic potential A , associated to the natural connection on the normal bundle $N\Sigma_t$, and the corresponding temporal component B .
- The advection vector field V , associated to the time dependence of our choice of coordinates.

These additional variables will be viewed as uniquely determined by our independent variable ψ , provided that a suitable gauge choice was made; in our case this gauge is the combined harmonic/Coulomb gauge. Thus (1.2) reduces to

- (a) A nonlinear Schrödinger equation for ψ , see (2.27);
- (b) An elliptic fixed time system (2.28) for the dependent variables (g, λ, V, A, B) , together with suitable compatibility conditions (constraints).

At the conclusion of Section 2 we provide a gauge version of our main result, see Theorem 2.9.

2. *Elliptic estimates.* In Section 4, we then consider the space-time bounds for the elliptic system (2.28) and the associated linearized equations. Such bounds have already been proved in [7] at the level of the H^s spaces. But here we also need similar bounds at the level of the Strichartz norms, which capture the time decay of λ and $(g - I, V, A, B)$ in terms of the corresponding decay bounds for ψ . Another novelty here is that we also prove elliptic bounds for the linearized system with ψ_{lin} in H^{-1} ; this is in contrast to [7], where only nonnegative Sobolev norms were used.

3. *Energy estimates.* In Section 5, we turn our attention to the energy estimates in Proposition 1.2. Here we use the intrinsic Sobolev spaces $\mathbf{H}^k$ to define the energy functional, and give the related energy estimates. We also prove an energy estimate for the linearized Schrödinger equation, which will be needed in particular to transfer energy bounds from **integer** to fractional Sobolev spaces.

4. *Strichartz estimates.* The Strichartz estimates for ψ are proved in Section 6 using the Schrödinger system (2.27). Since this is a quasilinear problem, we cannot directly work with the linear variable coefficient system. Instead, we prove Proposition 1.3 using a bootstrap argument which is based on the Strichartz estimates for the flat Schrödinger evolution, namely Keel-Tao's endpoint Strichartz estimates and inhomogeneous Strichartz estimates, see [2, 17, 18].

5. *The final bootstrap.* In the last section of the paper, we gain the H^s solutions as a limit of solutions in higher order Sobolev spaces. Using the energy estimates in $\mathbf{H}^N$ for integer N and the energy estimates of linearized equation, we prove the improved energy bounds for ψ in fractional Sobolev spaces. This in turn allows us to close the high level bootstrap loop for both the energy estimates and the Strichartz estimates, as stated in Proposition 2.10. As a byproduct, we also obtain the scattering result Schrödinger equation for ψ in the weaker Sobolev norms H^{s_d-2} .

2. THE DIFFERENTIATED EQUATIONS AND THE GAUGE CHOICE

The goal of this section is to introduce the main independent variable ψ , which represents the trace of the second fundamental form in complex notation, as well as the following auxiliary variables: the metric g , the second fundamental form λ , the connection coefficients A, B for the normal bundle as well as the advection vector field V . For ψ we start with (1.2) and derive a nonlinear Schrödinger type system (2.27), with coefficients depending on $(\lambda, \mathcal{S})$ where $\mathcal{S} = (h, V, A, B)$ and $h = g - I_d$. Under suitable gauge conditions, the auxiliary variables $(\lambda, \mathcal{S})$ are shown to satisfy an elliptic system (2.28), as well as a natural set of constraints. We conclude the section with a gauge formulation of our main result, see Theorem 2.9. For the detailed derivation, we refer to section 2 in [7].

2.1. The Riemannian metric g and the second fundamental form. Let (Σ^d, g) be a d -dimensional oriented manifold and let $(\mathbb{R}^{d+2}, g_{\mathbb{R}^{d+2}})$ be $(d+2)$ -dimensional Euclidean space. Let $\alpha, \beta, \gamma, \dots \in \{1, 2, \dots, d\}$. Considering the immersion $F : \Sigma \rightarrow (\mathbb{R}^{d+2}, g_{\mathbb{R}^{d+2}})$, we obtain the induced metric g in Σ ,

$$g_{\alpha\beta} = \partial_{x_\alpha} F \cdot \partial_{x_\beta} F. \quad (2.1)$$

We denote the inverse of the matrix $g_{\alpha\beta}$ by $g^{\alpha\beta}$, i.e.

$$g^{\alpha\beta} := (g_{\alpha\beta})^{-1}, \quad g_{\alpha\gamma} g^{\gamma\beta} = \delta_\alpha^\beta.$$

Let ∇ be the canonical Levi-Civita connection in Σ associated with the induced metric g . A direct computation shows that on the Riemannian manifold (Σ, g) we have the Christoffel symbols

$$\Gamma_{\alpha\beta}^\gamma = \frac{1}{2} g^{\gamma\sigma} (\partial_\beta g_{\alpha\sigma} + \partial_\alpha g_{\beta\sigma} - \partial_\sigma g_{\alpha\beta}) = g^{\gamma\sigma} \partial_{\alpha\beta}^2 F \cdot \partial_\sigma F.$$

Hence, the Laplace-Beltrami operator Δ_g can be written in the form

$$\Delta_g f = \text{Tr } \nabla^2 f = g^{\alpha\beta} (\partial_{\alpha\beta}^2 f - \Gamma_{\alpha\beta}^\gamma \partial_\gamma f),$$

for any twice differentiable function $f : \Sigma \rightarrow \mathbb{R}$. The curvature tensor R on the Riemannian manifold (Σ, g) is given by

$$R_{\gamma\alpha\beta}^\sigma = \partial_\alpha \Gamma_{\beta\gamma}^\sigma - \partial_\beta \Gamma_{\alpha\gamma}^\sigma + \Gamma_{\beta\gamma}^m \Gamma_{\alpha m}^\sigma - \Gamma_{\alpha\gamma}^m \Gamma_{\beta m}^\sigma, \quad R_{\alpha\beta\gamma\sigma} = g_{\mu\alpha} R_{\beta\gamma\sigma}^\mu.$$

We will also use the Ricci curvature

$$\text{Ric}_{\alpha\beta} = R_{\alpha\sigma\beta}^\sigma = g^{\sigma\gamma} R_{\gamma\alpha\sigma\beta}.$$

Next, we compute the second fundamental form. Let $\bar{\nabla}$ be the Levi-Civita connection in $(\mathbb{R}^{d+2}, g_{\mathbb{R}^{d+2}})$ and let $\mathbf{h}$ be the second fundamental form for Σ as an embedded manifold. Then by the Gauss relation we have

$$\mathbf{h}_{\alpha\beta} = \mathbf{h}(\partial_\alpha, \partial_\beta) = \bar{\nabla}_{\partial_\alpha} \partial_\beta F - F_*(\nabla_{\partial_\alpha} \partial_\beta) = \partial_{\alpha\beta}^2 F - \Gamma_{\alpha\beta}^\gamma \partial_\gamma F.$$

This gives the mean curvature $\mathbf{H}$ at $F(x)$,

$$\mathbf{H} = \text{Tr}_g \mathbf{h} = \Delta_g F.$$

Hence, the F -equation in (1.2) is rewritten as

$$(\partial_t F)^\perp = J(F) \Delta_g F = J(F) g^{\alpha\beta} (\partial_{\alpha\beta}^2 F - \Gamma_{\alpha\beta}^\gamma \partial_\gamma F).$$

This equation is still independent of the choice of coordinates in Σ^d .

2.2. The complex structure equations. This part is inspired by Gomez [3]. We introduce a complex structure on the normal bundle $N\Sigma_t$. This is achieved by choosing $\{\nu_1, \nu_2\}$ to be an orthonormal basis of $N\Sigma_t$ such that

$$J\nu_1 = \nu_2, \quad J\nu_2 = -\nu_1.$$

Note that such a choice is not unique.

The vectors $\{F_1, \dots, F_d, \nu_1, \nu_2\}$ form a **basis** at each point on the manifold (Σ, g) , **where** $F_\alpha = \partial_\alpha F$ for $\alpha \in \{1, \dots, d\}$ **might not be orthonormal**. We define the tensors $\kappa_{\alpha\beta}$, $\tau_{\alpha\beta}$, the connection coefficients A_α and the temporal component B of the connection in the normal bundle by

$$\kappa_{\alpha\beta} := \partial_\alpha F_\beta \cdot \nu_1, \quad \tau_{\alpha\beta} := \partial_\alpha F_\beta \cdot \nu_2, \quad A_\alpha = \partial_\alpha \nu_1 \cdot \nu_2, \quad B = \partial_t \nu_1 \cdot \nu_2.$$

We then define the complex vector field m and the complex second fundamental form tensor $\lambda_{\alpha\beta}$ to be

$$m = \nu_1 + i\nu_2, \quad \lambda_{\alpha\beta} = \kappa_{\alpha\beta} + i\tau_{\alpha\beta}, \quad (2.2)$$

and define the *complex scalar mean curvature* ψ to be the trace of λ ,

$$\psi = \text{Tr } \lambda = g^{\alpha\beta} \lambda_{\alpha\beta}. \quad (2.3)$$

If we differentiate the basis, then we obtain a set of structure equations of the following type

$$\begin{cases} \partial_\alpha F_\beta = \Gamma_{\alpha\beta}^\gamma F_\gamma + \text{Re}(\lambda_{\alpha\beta} \bar{m}), \\ \partial_\alpha^A m = -\lambda_\alpha^\gamma F_\gamma. \end{cases} \quad (2.4)$$

Here $\partial_\alpha^A = \partial_\alpha + iA_\alpha$, and similarly we will denote $\nabla_\alpha^A = \nabla_\alpha + iA_\alpha$.

We then use the structure equations (2.4) to derive a set of constraints for λ and A , and hence to obtain their elliptic equations. Precisely, by (2.4) and the relations $\partial_\alpha \partial_\beta F = \partial_\beta \partial_\alpha F$, we obtain the Riemannian curvature and Ricci curvature

$$R_{\sigma\gamma\alpha\beta} = \text{Re}(\lambda_{\beta\gamma} \bar{\lambda}_{\alpha\sigma} - \lambda_{\alpha\gamma} \bar{\lambda}_{\beta\sigma}), \quad \text{Ric}_{\gamma\beta} = \text{Re}(\lambda_{\gamma\beta} \bar{\psi} - \lambda_{\gamma\alpha} \bar{\lambda}_\beta^\alpha), \quad (2.5)$$

as well as the Codazzi relations

$$\nabla_\alpha^A \lambda_\beta^\gamma = \nabla_\beta^A \lambda_\alpha^\gamma = \nabla^{A,\gamma} \lambda_{\alpha\beta}, \quad (2.6)$$

where the second fundamental form λ should also satisfy the constraint

$$\lambda_{\alpha\beta} = \lambda_{\beta\alpha}. \quad (2.7)$$

The structure equations (2.4) combined with the relations $\partial_\alpha \partial_\beta m = \partial_\beta \partial_\alpha m$ imply the compatibility condition for connection coefficients A

$$\nabla_\alpha A_\beta - \nabla_\beta A_\alpha = \text{Im}(\lambda_\alpha^\gamma \bar{\lambda}_{\beta\gamma}). \quad (2.8)$$

We state an elliptic system for the second fundamental form λ in terms of ψ , using the Codazzi relations (2.6) and $\psi = \text{Tr } \lambda$.

Lemma 2.1 (Div-curl system for λ , Lemma 2.2 [7]). *The second fundamental form λ satisfies*

$$\nabla_\alpha^A \lambda_{\beta\gamma} - \nabla_\beta^A \lambda_{\alpha\gamma} = 0, \quad \nabla^{A,\alpha} \lambda_{\alpha\beta} = \nabla_\beta^A \psi. \quad (2.9)$$

In order to both fix the gauge and obtain an elliptic system for A , we impose the Coulomb gauge condition

$$\nabla^\alpha A_\alpha = 0. \quad (2.10)$$

We state the elliptic A -equations from the Ricci equations (2.8).

Lemma 2.2 (Div-curl form for A). *Under the Coulomb gauge condition (2.10), the connection A solves*

$$\nabla_\alpha A_\beta - \nabla_\beta A_\alpha = \text{Im}(\lambda_\alpha^\gamma \bar{\lambda}_{\beta\gamma}), \quad \nabla^\alpha A_\alpha = 0. \quad (2.11)$$

As a corollary, we can derive a second order elliptic equation for A .

Corollary 2.3. *Under the Coulomb gauge condition (2.10) and harmonic coordinates (2.15), the connection A solves*

$$\nabla^\gamma \nabla_\gamma A_\alpha = \text{Re}(\lambda_\alpha^\sigma \bar{\psi} - \lambda_\alpha^\gamma \bar{\lambda}_\gamma^\sigma) A_\sigma + \nabla^\gamma \text{Im}(\lambda_\gamma^\sigma \bar{\lambda}_{\alpha\sigma}). \quad (2.12)$$

Let $\partial^\alpha = g^{\alpha\beta} \partial_\beta$, the connection A also solves

$$\partial_\gamma \partial^\gamma A_\alpha = (\partial_\gamma g^{\gamma\beta} \partial_\alpha - \partial_\alpha g^{\gamma\beta} \partial_\gamma) A_\beta - \partial_\gamma \text{Im}(\lambda_{\alpha\sigma} \bar{\lambda}^{\gamma\sigma}). \quad (2.13)$$

Proof. The first equation (2.12) has been derived in [7]. For the second equation (2.13), by harmonic coordinates (2.15), the div-curl system (2.11) for A can be rewritten as

$$g^{\gamma\beta} \partial_\alpha A_\beta - \partial^\beta A_\alpha = \text{Im}(\lambda_\alpha^\sigma \bar{\lambda}_\sigma^\gamma), \quad \partial^\alpha A_\alpha = 0.$$

Then the equation (2.13) is obtained by applying ∂_β to the above first formula. $\square$

2.3. The elliptic equation for the metric g in harmonic coordinates. Here we take the next step towards fixing the gauge, by choosing to work in harmonic coordinates. Precisely, we will require the coordinate functions $\{x^\alpha, \alpha = 1, \dots, d\}$ to be globally Lipschitz solutions of the elliptic equations

$$\Delta_g x^\alpha = 0. \quad (2.14)$$

This determines the coordinates uniquely modulo time dependent affine transformations. This remaining ambiguity will be removed later on by imposing suitable boundary conditions at infinity.

Here, we will interpret the above harmonic coordinate condition at fixed time as an elliptic equation for the metric g . The equations (2.14) can be expressed in terms of the Christoffel symbols Γ , which must satisfy the condition

$$g^{\alpha\beta} \Gamma_{\alpha\beta}^\gamma = 0, \quad \text{for } \gamma = 1, \dots, d. \quad (2.15)$$

In fact, we can obtain global harmonic coordinate by the smallness of $\partial_x h$ in H^s as follows. Here for a change of coordinate $y = x + \phi(x)$, we denote

$$\tilde{F}(y) = F(x(y)),$$

and denote its metric and Christoffel symbols as $\tilde{g}$ and $\tilde{\Gamma}$, respectively.

Lemma 2.4 (Existence of global harmonic coordinates, Proposition 8.1 [7]). *Let $d \geq 3$, $s > \frac{d}{2}$, and $F : (\mathbb{R}_x^d, g) \rightarrow (\mathbb{R}^{d+2}, g_{\mathbb{R}^{d+2}})$ be an immersion with induced metric $g = I_d + h$. Assume that $\partial_x h(x)$ is small in $H^s(dx)$, i.e. $\|\partial_x h\|_{H^s} \leq \epsilon_0$. Then there exists a unique change of coordinates $y = x + \phi(x)$ with $\lim_{x \rightarrow \infty} \phi(x) = 0$ and $\nabla \phi$ uniformly small, such that the new coordinates $\{y_1, \dots, y_d\}$ are global harmonic coordinates, namely,*

$$\tilde{g}^{\alpha\beta}(y)\tilde{\Gamma}_{\alpha\beta}^\gamma(y) = 0, \quad \text{for any } y \in \mathbb{R}^d.$$

Moreover,

$$\|\partial_x^2 \phi(x)\|_{H^s(dx)} \lesssim \|\partial_x h(x)\|_{H^s(dx)},$$

and, in the new coordinates $\{y_1, \dots, y_d\}$,

$$\|\partial_y \tilde{h}\|_{H^s(dy)} \lesssim \|\partial_x h\|_{H^s(dx)}.$$

Under the harmonic coordinates, the Ricci curvature formula (2.5) leads to an equation for the metric g :

Lemma 2.5 (Elliptic equations of g , Lemma 2.4 [7]). *In harmonic coordinates, the metric g satisfies*

$$\begin{aligned} g^{\alpha\beta} \partial_{\alpha\beta}^2 g_{\gamma\sigma} &= [-\partial_\gamma g^{\alpha\beta} \partial_\beta g_{\alpha\sigma} - \partial_\sigma g^{\alpha\beta} \partial_\beta g_{\alpha\gamma} + \partial_\gamma g_{\alpha\beta} \partial_\sigma g^{\alpha\beta}] \\ &\quad + 2g^{\alpha\beta} \Gamma_{\sigma\alpha,\nu} \Gamma_{\beta\gamma}^\nu - 2\operatorname{Re}(\lambda_{\gamma\sigma} \bar{\psi} - \lambda_{\alpha\gamma} \bar{\lambda}_\sigma^\alpha). \end{aligned} \quad (2.16)$$

2.4. The motion of the basis $\{F_1, \dots, F_d, m\}$ under (SMCF). Here we derive the equations of motion for the basis, assuming that the immersion F satisfies (1.2). Then we state the Schrödinger equation for mean curvature ψ and the elliptic equations for advection fields V and temporal connection coefficient B .

We begin by rewriting the SMCF equations in the form

$$\partial_t F = J(F) \mathbf{H}(F) + V^\gamma F_\gamma,$$

where V^γ is a vector field on the manifold Σ , whose components in general depend on the choice of coordinates. By the definition of m and $\lambda_{\alpha\beta}$, we get

$$\partial_t F = -\operatorname{Im}(\psi \bar{m}) + V^\gamma F_\gamma. \quad (2.17)$$

Applying ∂_α to (2.17), by the structure equations (2.4) and $m \perp F_\alpha = 0$ we obtain the equations of motion for the basis

$$\begin{cases} \partial_t F_\alpha = -\operatorname{Im}(\partial_\alpha^A \psi \bar{m} - i\lambda_{\alpha\gamma} V^\gamma \bar{m}) + [\operatorname{Im}(\psi \bar{\lambda}_\alpha^\gamma) + \nabla_\alpha V^\gamma] F_\gamma, \\ \partial_t^B m = -i(\partial^{A,\alpha} \psi - i\lambda_\gamma^\alpha V^\gamma) F_\alpha, \end{cases} \quad (2.18)$$

where $\partial_t^B = \partial_t + iB$.

From this we obtain the evolution equation for the metric g . Precisely, we denote

$$G_{\alpha\beta} = \text{Im}(\psi \bar{\lambda}_{\alpha\beta}) + \frac{1}{2}(\nabla_\alpha V_\beta + \nabla_\beta V_\alpha).$$

By the definition of the induced metric g (2.1) and (2.18) we have

$$\partial_t g_{\alpha\beta} = 2G_{\alpha\beta}, \quad (2.19)$$

and the evolutions of $g^{\alpha\beta}$ and $\sqrt{\det g}$ are given by

$$\partial_t g^{\alpha\beta} = -2G^{\alpha\beta}, \quad \partial_t \sqrt{\det g} = \nabla_\alpha V^\alpha \sqrt{\det g}.$$

These yield the evolution equations for Christoffel symbols

$$\partial_t \Gamma_{\alpha\beta}^\gamma = \nabla_\alpha G_\beta^\gamma + \nabla_\beta G_\alpha^\gamma - \nabla^\gamma G_{\alpha\beta}. \quad (2.20)$$

Moreover, by (2.19) and (2.20) we have

$$\partial_t (g^{\alpha\beta} \Gamma_{\alpha\beta}^\gamma) = -2G^{\alpha\beta} \Gamma_{\alpha\beta}^\gamma + 2\nabla_\alpha \text{Im}(\psi \bar{\lambda}^{\alpha\gamma}) + \Delta_g V^\gamma + \text{Re}(\lambda_\sigma^\gamma \bar{\psi} - \lambda_{\alpha\sigma} \bar{\lambda}^{\alpha\gamma}) V^\sigma. \quad (2.21)$$

So far, the choice of V has been unspecified; it depends on the choice of coordinates on our manifold as the time varies. However, once the latter is fixed via the harmonic coordinate condition (2.15), we can also derive an elliptic equation for the advection field V :

Lemma 2.6 (Elliptic equation for the vector field V , Lemma 2.5 [7]). *Under the harmonic coordinate condition (2.15), the advection field V solves*

$$\begin{aligned} \Delta_g V^\gamma = & -2\nabla_\alpha \text{Im}(\psi \bar{\lambda}^{\alpha\gamma}) - \text{Re}(\lambda_\sigma^\gamma \bar{\psi} - \lambda_{\alpha\sigma} \bar{\lambda}^{\alpha\gamma}) V^\sigma \\ & + 2(\text{Im}(\psi \bar{\lambda}^{\alpha\beta}) + \nabla^\alpha V^\beta) \Gamma_{\alpha\beta}^\gamma. \end{aligned} \quad (2.22)$$

Remark 2.6.1. We should emphasize that (2.22) is not a covariant equation but an equation for the components of V , which depend on the choices of coordinates. In view of the relation (2.21), a different choice of coordinates would yield a different type of equation of V . For example, a parabolic equation for V was obtained in [8], where the heat gauge was chosen.

Next, from the equations (2.18) of motion for the basis we derive the main Schrödinger equation and the second compatibility condition. The starting point is the commutation relation

$$[\partial_t^B, \partial_\alpha^A]m = i(\partial_t A_\alpha - \partial_\alpha B)m,$$

which can be expanded via equating the coefficients of the tangent vectors and of the normal vector m , see [7]. Using the expressions (2.4), (2.18) for the derivatives of the basis, the above formula yields the evolution equation for λ

$$\partial_t^B \lambda_\alpha^\sigma + \lambda_\alpha^\gamma (\text{Im}(\psi \bar{\lambda}_\gamma^\sigma) + \nabla_\gamma V^\sigma) = i\nabla_\alpha^A (\partial^{A,\sigma} \psi - i\lambda_\gamma^\sigma V^\gamma), \quad (2.23)$$

as well as the compatibility condition (curvature relation)

$$\partial_t A_\alpha - \partial_\alpha B = \operatorname{Re}(\lambda_\alpha^\gamma \overline{\nabla_\gamma^A \psi}) - \operatorname{Im}(\lambda_\alpha^\gamma \bar{\lambda}_{\gamma\sigma}) V^\sigma. \quad (2.24)$$

This in turn allows us to use the Coulomb gauge condition (2.10) in order to obtain an elliptic equation for B :

Lemma 2.7 (Elliptic equation for B). *The temporal connection coefficient B solves*

$$\begin{aligned} \nabla^\gamma \nabla_\gamma B = & -\nabla^\gamma \nabla_\sigma \operatorname{Re}(\lambda_\gamma^\sigma \bar{\psi}) + \frac{1}{2} \Delta_g |\psi|^2 + \nabla^\gamma [\operatorname{Im}(\lambda_\gamma^\sigma \bar{\lambda}_{\sigma\beta}) V^\beta] \\ & + (2 \operatorname{Im}(\psi \bar{\lambda}^{\beta\gamma}) + \nabla^\beta V^\gamma + \nabla^\gamma V^\beta) \partial_\beta A_\gamma. \end{aligned} \quad (2.25)$$

Proof. The equation (2.25) is obtained by Lemma 2.6 in [7] and the following relation

$$\operatorname{Re}(\lambda_\gamma^\sigma \overline{\nabla_\sigma^A \psi}) = \nabla_\sigma \operatorname{Re}(\lambda_\gamma^\sigma \bar{\psi}) - \operatorname{Re}(\nabla_\gamma^A \psi \bar{\psi}) = \nabla_\sigma \operatorname{Re}(\lambda_\gamma^\sigma \bar{\psi}) - \frac{1}{2} \nabla_\gamma |\psi|^2.$$

□

Finally, we use (2.23) to derive the main equation, i.e. the Schrödinger equation for ψ . By (2.6), contracting (2.23) yields

$$i(\partial_t^B - V^\gamma \nabla_\gamma^A) \psi + \nabla_\alpha^A \nabla^{A,\alpha} \psi = -i \lambda_\sigma^\gamma \operatorname{Im}(\psi \bar{\lambda}_\gamma^\sigma). \quad (2.26)$$

2.5. The main result for modified Schrödinger system from SMCF. To conclude, under the Coulomb gauge condition $\nabla^\alpha A_\alpha = 0$ and the harmonic coordinate condition $g^{\alpha\beta} \Gamma_{\alpha\beta}^\gamma = 0$, by (2.26), (2.9), (2.16), (2.22), (2.11) and (2.25), we obtain the Schrödinger equation for the complex mean curvature ψ

$$\begin{cases} i(\partial_t^B - V^\gamma \nabla_\gamma^A) \psi + \nabla_\alpha^A \nabla^{A,\alpha} \psi = -i \lambda_\sigma^\gamma \operatorname{Im}(\psi \bar{\lambda}_\gamma^\sigma), \\ \psi(0) = \psi_0, \end{cases} \quad (2.27)$$

where the metric g , curvature tensor λ , the advection field V , connection coefficients A and B are determined at fixed time in an elliptic fashion via the following equations

$$\begin{cases} \nabla_\alpha^A \lambda_{\beta\gamma} - \nabla_\beta^A \lambda_{\alpha\gamma} = 0, & \nabla^{A,\alpha} \lambda_{\alpha\beta} = \nabla_\beta^A \psi, \\ g^{\alpha\beta} \partial_{\alpha\beta}^2 g_{\gamma\sigma} = [-\partial_\gamma g^{\alpha\beta} \partial_\beta g_{\alpha\sigma} - \partial_\sigma g^{\alpha\beta} \partial_\beta g_{\alpha\gamma} + \partial_\gamma g_{\alpha\beta} \partial_\sigma g^{\alpha\beta}] \\ \quad + 2g^{\alpha\beta} \Gamma_{\sigma\alpha,\nu} \Gamma_{\beta\gamma}^\nu - 2 \operatorname{Re}(\lambda_{\gamma\sigma} \bar{\psi} - \lambda_{\alpha\gamma} \bar{\lambda}_\sigma^\alpha), \\ \nabla^\alpha \nabla_\alpha V^\gamma = 2 \nabla_\alpha \operatorname{Im}(\lambda^{\alpha\gamma} \bar{\psi}) - \operatorname{Re}(\lambda_\sigma^\gamma \bar{\psi} - \lambda_{\alpha\sigma} \bar{\lambda}^{\alpha\gamma}) V^\sigma \\ \quad + 2(\operatorname{Im}(\psi \bar{\lambda}^{\alpha\beta}) + \nabla^\alpha V^\beta) \Gamma_{\alpha\beta}^\gamma, \\ \nabla_\alpha A_\beta - \nabla_\beta A_\alpha = \operatorname{Im}(\lambda_\alpha^\gamma \bar{\lambda}_{\beta\gamma}), & \nabla^\alpha A_\alpha = 0, \\ \nabla^\gamma \nabla_\gamma B = -\nabla^\gamma \nabla_\sigma \operatorname{Re}(\lambda_\gamma^\sigma \bar{\psi}) + \frac{1}{2} \Delta_g |\psi|^2 + \nabla^\gamma [\operatorname{Im}(\lambda_\gamma^\sigma \bar{\lambda}_{\sigma\beta}) V^\beta] \\ \quad + (2 \operatorname{Im}(\psi \bar{\lambda}^{\beta\gamma}) + \nabla^\beta V^\gamma + \nabla^\gamma V^\beta) \partial_\beta A_\gamma. \end{cases} \quad (2.28)$$

Fixing the remaining degrees of freedom (i.e. the affine group for the choice of the coordinates as well as the time dependence of the $SU(1)$ connection) we can assume that the following conditions hold at infinity:

$$\lambda(\infty) = 0, \quad g(\infty) = I_d, \quad V(\infty) = 0, \quad A(\infty) = 0, \quad B(\infty) = 0$$

These are needed to insure the unique solvability of the above elliptic equations in a suitable class of functions. For the metric g it will be useful to use the representation

$$g = I_d + h$$

so that h vanishes at infinity.

We note that the above elliptic system (2.27) is accompanied by a large family of compatibility conditions as follows: (2.5), (2.7), (2.8), (2.10), (2.15), (2.19) (2.23) and (2.24). These conditions can all be shown to be satisfied for small solutions to the nonlinear system (2.27)-(2.28).

Now we recall the small data local well-posedness result for the (SMCF) system in [7, Theorem 2.7] in terms of the above system:

Theorem 2.8 (Small data local well-posedness in the good gauge). *Let $s > \frac{d}{2}$, $d \geq 4$. Then there exists $\epsilon_0 > 0$ sufficiently small such that, for all initial data ψ_0 satisfying*

$$\|\psi_0\|_{H^s} \leq \epsilon_0,$$

the modified Schrödinger system (2.27), with (λ, h, V, A, B) determined via the elliptic system (2.28), is locally well-posed in H^s on the time interval $I = [0, 1]$. Moreover, the mean curvature satisfies the bounds

$$\|(\psi, \lambda, \partial_x V, \partial_x A)\|_{C[0,1;H^s]} + \|\partial_x h\|_{C[0,1;H^{s+1}]} + \|\partial_x B\|_{C[0,1;H^{s-1}]} \lesssim \|\psi_0\|_{H^s}.$$

In addition, the mean curvature ψ and the auxiliary functions (λ, h, V, A, B) satisfy the constraints (2.3), (2.5), (2.7), (2.8), (2.10) and (2.15) for any fixed time $t \in [0, 1]$, and the evolutions (2.19), (2.23) and (2.24).

Actually, in [7] the system (2.27)-(2.28) was solved in a smaller function space, which includes both Sobolev regularity and local energy bounds. In the above theorem, by well-posedness we mean a full Hadamard-type well-posedness, see [9].

The main result of this paper is to extend the above local solution for small data to global for the (SMCF) system in Theorem 1.1 in terms of the above system. The next theorem represents the harmonic/Coulomb gauge form of our main result in Theorem 1.1:

Theorem 2.9 (Small data global regularity in the good gauge). *Let s_d, r_d be as (1.3) and (1.4) respectively for $d \geq 4$. Then there exists $\epsilon_0 > 0$ sufficiently small such that, for all initial data ψ_0 satisfying*

$$\|\psi_0\|_{H^{s_d}} \leq \epsilon_0, \tag{2.29}$$

the modified Schrödinger system (2.27), with (λ, h, V, A, B) determined via the elliptic system (2.28), is globally well-posed in H^{s_d} . Moreover, the mean curvature satisfies the bound (1.7) and the scattering (1.8).

This result is achieved by the following bootstrap proposition and continuity method.

Proposition 2.10 (Bootstrap proposition). *Let s_d, r_d be as (1.3) and (1.4) respectively for $d \geq 4$. Assume that $(\psi, \lambda, \mathcal{S})$ is a solution to (2.27) and (2.28) on some time interval $[0, T]$, $T \geq 1$, with initial data satisfying the smallness assumption (2.29). Assume also that the solution satisfies the bootstrap hypothesis*

$$\|\psi\|_{S[0,T]} + \|\psi\|_{L^\infty(0,T;H_x^{s_d})} \leq C_0 \|\psi_0\|_{H^{s_d}}. \quad (2.30)$$

Then the following improved bound holds:

$$\|\psi\|_{S[0,T]} + \|\psi\|_{L^\infty(0,T;H_x^{s_d})} \leq \frac{C_0}{2} \|\psi_0\|_{H^{s_d}}, \quad (2.31)$$

where $C_0 > 1$ is a large universal constant.

In the remaining sections, we will focus on the proof of this proposition, which splits in a modular fashion into an energy component and a Strichartz component, as in Propositions 1.2, 1.3 in the introduction.

3. FUNCTION SPACES AND NOTATIONS

We define the function spaces for the study of global solutions to the system (2.27)-(2.28). First we introduce the usual Sobolev spaces $W^{s,p}$, H^s and the intrinsic Sobolev spaces $\mathbf{H}^k$ for tensors on $(\mathbb{R}^d, g)$. The gauge independent intrinsic norms $\mathbf{H}^k$ are used in the energy estimates. Then we state a equivalence relation between the H^k and $\mathbf{H}^k$ norms under some assumptions on metric and magnetic potential.

For a function $u(t, x)$ or $u(x)$, let $\hat{u} = \mathcal{F}u$ denote the Fourier transform in the spatial variable x . Fix a smooth radial function $\varphi : \mathbb{R}^d \rightarrow [0, 1]$ supported in $\{x \in \mathbb{R}^d : |x| \leq 2\}$ and equal to 1 in $\{x \in \mathbb{R}^d : |x| \leq 1\}$, and for any $i \in \mathbb{Z}$, let

$$\varphi_i(x) := \varphi(x/2^i) - \varphi(x/2^{i-1}).$$

We then have the spatial Littlewood-Paley decomposition,

$$\sum_{i=-\infty}^{\infty} P_i(D) = 1, \quad \sum_{i=0}^{\infty} S_i(D) = 1,$$

where we use the differential operator $D := \frac{1}{\sqrt{-1}} \partial_x$, P_i localizes to frequency 2^i for $i \in \mathbb{Z}$, i.e.,

$$\mathcal{F}(P_i u) = \varphi_i(\xi) \hat{u}(\xi),$$

and

$$S_0(D) = \sum_{i \leq 0} P_i(D), \quad S_i(D) = P_i(D), \text{ for } i > 0.$$

For simplicity of notation, we set

$$u_j = S_j u, \quad u_{\leq j} = \sum_{i=0}^j S_i u, \quad u_{\geq j} = \sum_{i=j}^{\infty} S_i u.$$

We denote $W^{s,p}$ for $1 \leq p \leq \infty$ as the usual Sobolev spaces, and denote $H^s := W^{s,2}$. For simplicity of notation, on some time interval $[0, T]$, we define

$$\|u\|_{L_T^r W^{s,p}} = \|u\|_{L^r(0,T;W^{s,p})}, \quad 1 \leq r \leq \infty.$$

For the solution ψ of Schrödinger equation in (2.27) and the related second fundamental form λ , we will be working primarily in $L_T^\infty H^{s_d} \cap S[0, T]$ for $s_d > \frac{d}{r_d} + 2$. Here $S[0, T]$ are the Strichartz spaces defined by (1.5) and (1.6). For convenience, corresponding to the L_t^2 component of the Strichartz norms, we also define the Sobolev norm at fixed time as

$$\|\psi\|_{\mathbf{str}} := \|\psi\|_{W^{1,4}} + \|\psi\|_{W^{s_d-2,r_d}}, \quad \text{for } d = 4,$$

and

$$\|\psi\|_{\mathbf{str}} := \|\psi\|_{W^{s_d-2,r_d}}, \quad \text{for } d \geq 5.$$

For the elliptic system (2.28), at a fixed time we define the $\mathcal{H}^s$ norm as

$$\|(h, V, A, B)\|_{\mathcal{H}^s} = \|\partial_x h\|_{H^{s+1}} + \|\partial_x V\|_{H^s} + \|A\|_{H^{s+1}} + \|\partial_x B\|_{H^{s-1}}.$$

Next, we define the intrinsic Sobolev spaces $\mathbf{H}^k$ for integer $k \in \mathbb{Z}$. Since the Schrödinger equation (2.27) is a quasilinear equations with variable coefficients g , we will use the space $\mathbf{H}^k$ to derive its energy estimates later. Let g be a Riemannian metric on $\mathbb{R}^d$, and A_γ be a magnetic potential. For any complex tensor $T = T_{\beta_1 \dots \beta_s}^{\alpha_1 \dots \alpha_r} dx^{\beta_1} \otimes \dots \otimes dx^{\beta_s} \otimes \frac{\partial}{\partial x^{\alpha_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\alpha_r}}$, the covariant derivative is defined by

$$\nabla_\gamma^A T = \nabla_\gamma T + i A_\gamma T,$$

where

$$\nabla_\gamma T_{\beta_1 \dots \beta_s}^{\alpha_1 \dots \alpha_r} = \partial_\gamma T_{\beta_1 \dots \beta_s}^{\alpha_1 \dots \alpha_r} + \sum_{i=1}^r \Gamma_{\gamma \sigma}^{\alpha_i} T_{\beta_1 \dots \beta_s}^{\alpha_1 \dots \alpha_{i-1} \sigma \alpha_{i+1} \dots \alpha_r} - \sum_{j=1}^s \Gamma_{\gamma \beta_j}^\sigma T_{\beta_1 \dots \beta_{j-1} \sigma \beta_{j+1} \dots \beta_s}^{\alpha_1 \dots \alpha_r}. \quad (3.1)$$

We have

$$|\nabla^A T|_g^2 = g_{\alpha_1 \alpha'_1} \dots g_{\alpha_r \alpha'_r} g^{\beta_1 \beta'_1} \dots g^{\beta_s \beta'_s} \nabla_\gamma^A T_{\beta_1 \dots \beta_s}^{\alpha_1 \dots \alpha_r} \overline{\nabla_{A, \gamma} T_{\beta'_1 \dots \beta'_s}^{\alpha'_1 \dots \alpha'_r}}.$$

Then the intrinsic Sobolev norm $\mathbf{H}^k$ for nonnegative integer $k \in \mathbb{N}$ is defined by

$$\|T\|_{\mathbf{H}^k} = \left(\sum_{l=0}^k \int_\Sigma |\nabla^{A, l} T|_g^2 d\mu \right)^{1/2}, \quad (3.2)$$

where volume form is $d\mu = \sqrt{\det g}dx$ and $\nabla^{A,l}$ is the l -th order covariant derivative. By duality, we can also define the negative intrinsic Sobolev spaces as

$$\|T\|_{\mathbf{H}^{-k}} = \sup_{\|U\|_{\mathbf{H}^k} \leq 1} \langle T, U \rangle_{L^2}.$$

Under a suitable smallness assumption on the metric h and the magnetic field A we have the following equivalence relation between H^k and $\mathbf{H}^k$ for a range of integers k .

Lemma 3.1. *Let $d \geq 3$ and $s > \frac{d}{2}$. Assume that $\|\partial_x h\|_{H^{s-1}} + \|A\|_{H^{s-1}} \leq \epsilon$. Then for any integer $0 \leq k \leq s$ we have the equivalent relation*

$$\|T\|_{\mathbf{H}^k} \approx \|T\|_{H^k}. \quad (3.3)$$

Proof. By covariant derivative (3.1), schematically we write $\nabla^{A,k}T$ as

$$\nabla^{A,k}T = \partial^k T + \sum_{1 \leq n \leq k} \sum_{l_1 + \dots + l_{n+1} = k-n} \partial^{l_1}(\Gamma + iA) \dots \partial^{l_n}(\Gamma + iA) \cdot \partial^{l_{n+1}}T. \quad (3.4)$$

Then by the smallness of $\partial_x h$ we have

$$\begin{aligned} \|\nabla^k T\|_{L^2(d\mu)} &\lesssim \| |\nabla^k T|_g \|_{L^2(dx)} \\ &\lesssim \|\partial^k T\|_{L^2} + \sum_{1 \leq n \leq k} \sum_{l_1 + \dots + l_{n+1} = k-n} \|\partial_x^{l_1}(\Gamma + iA) \dots \partial_x^{l_n}(\Gamma + iA) \cdot \partial_x^{l_{n+1}}T\|_{L^2}. \end{aligned} \quad (3.5)$$

In order to bound the second term above, it suffices to bound $\Gamma^n T$ in H^{k-n} . For this we claim that

$$\|\Gamma T\|_{H^{k'-1}} \lesssim \|\partial_x h\|_{H^{s-1}} \|T\|_{H^{k'}}, \quad \text{for any } 1 \leq k' \leq k. \quad (3.6)$$

Then by induction we have

$$\|\Gamma^n T\|_{H^{k-n}} \lesssim \|\partial_x h\|_{H^{s-1}} \|\Gamma^{n-1} T\|_{H^{k-n+1}} \lesssim \|\partial_x h\|_{H^{s-1}}^n \|T\|_{H^k}.$$

This combined with (3.5) and the smallness of $\partial_x h$ and A in H^{s-1} implies

$$\|T\|_{\mathbf{H}^k} \lesssim \|T\|_{H^k} + \epsilon \|T\|_{H^k} \lesssim \|T\|_{H^k}.$$

We now return to prove the claim (3.6). Using a Littlewood-Paley decomposition and Bernstein's inequality we have

$$\begin{aligned} \|S_j(\Gamma T)\|_{H^{k'-1}} &\lesssim \sum_{0 \leq j_1 \leq j} 2^{(k'-1)j} \|\Gamma_{j_1}\|_{L^2} 2^{dj_1/2} \|T_{j_1}\|_{L^2} + \sum_{0 \leq j_1 \leq j} 2^{dj_1/2} \|\Gamma_{j_1}\|_{L^2} 2^{(k'-1)j} \|T_j\|_{L^2} \\ &\quad + \sum_{j_1 > j} 2^{(d/2+k'-1)(j-j_1)} \|\Gamma_{j_1}\|_{H^{d/2-1}} \|T_{j_1}\|_{H^{k'}} \\ &:= I_{1j} + I_{2j} + I_{3j}. \end{aligned}$$

For the first term we have

$$\begin{aligned}
I_{1j} &\lesssim \mathbf{1}_{<d/2}(k') \sum_{0 \leq j_1 \leq j} 2^{(d/2-1+\delta)j} \|\Gamma_j\|_{L^2} 2^{(d/2-k'+\delta)(j_1-j)} 2^{k'j_1} \|T_{j_1}\|_{L^2} \\
&\quad + \mathbf{1}_{\geq d/2}(k') \sum_{0 \leq j_1 \leq j} 2^{(k'-1+\delta)j} \|\Gamma_j\|_{L^2} 2^{(d/2-k'-\delta)j_1} 2^{k'j_1} \|T_{j_1}\|_{L^2} \\
&\lesssim \|\Gamma_j\|_{H^{s-1}} \|T\|_{H^{k'}}.
\end{aligned}$$

For the second term we have

$$I_{2j} \lesssim \sum_{0 \leq j_1 \leq j} 2^{j_1-j} 2^{(d/2-1)j_1} \|\Gamma_{j_1}\|_{L^2} 2^{k'j} \|T_j\|_{L^2} \lesssim \|\Gamma\|_{H^{s-1}} \|T_j\|_{H^{k'}}.$$

The last term I_{3j} is bounded by

$$I_{3j} \lesssim \sum_{j_1 > j} 2^{(d/2+k'-1)(j-j_1)} \|\Gamma_{j_1}\|_{H^{d/2-1}} \|T\|_{H^{k'}}.$$

Hence, these give

$$\|\Gamma T\|_{H^{k'-1}} \lesssim \left(\sum_{j \geq 0} \|S_j(\Gamma T)\|_{H^{k'-1}}^2 \right)^{1/2} \lesssim \|\Gamma\|_{H^{s-1}} \|T\|_{H^{k'}} \lesssim \|\partial_x h\|_{H^{s-1}} \|T\|_{H^{k'}}.$$

Then the claim (3.6) is obtained.

Conversely, by (3.4) we also have

$$\begin{aligned}
\|\partial^k T\|_{L^2} &\lesssim \|\nabla^{A,k} T\|_{L^2} + \sum_{1 \leq n \leq k} \sum_{l_1 + \dots + l_{n+1} = k-n} \|\partial^{l_1}(\Gamma + iA) \dots \partial^{l_n}(\Gamma + iA) \cdot \partial^{l_{n+1}} T\|_{L^2} \\
&\lesssim \|\nabla^{A,k} T|_g\|_{L^2(d\mu)} + \epsilon \|T\|_{H^k}.
\end{aligned}$$

This implies

$$\|T\|_{H^k} \lesssim \|T\|_{\mathbf{H}^k} + \epsilon \|T\|_{H^k} \lesssim \|T\|_{\mathbf{H}^k},$$

which completes the proof of the lemma. $\square$

4. ELLIPTIC ESTIMATES

In this section, we consider the elliptic system (2.28). Its solvability was already considered in [7] under the assumption that ψ is small in H^s . Here we prove some additional space-time bounds for $(\lambda, \mathcal{S})$ with $\mathcal{S} = (h, V, A, B)$, which are adapted to the Strichartz

norm we will use later on. We begin with the linearization of the solution map

$$\left\{ \begin{aligned} \nabla_\alpha^A \lambda_{lin,\beta\gamma} - \nabla_\beta^A \lambda_{lin,\alpha\gamma} &= -(\nabla_\alpha^A)_{lin} \lambda_{\beta\gamma} + (\nabla_\beta^A)_{lin} \lambda_{\alpha\gamma}, \\ \nabla^{A,\alpha} \lambda_{lin,\alpha\beta} &= -(\nabla^{A,\alpha})_{lin} \lambda_{\alpha\beta} + (\nabla_\beta^A \psi)_{lin}, \\ g^{\alpha\beta} \partial_{\alpha\beta}^2 g_{lin,\gamma\sigma} &= -g_{lin}^{\alpha\beta} \partial_{\alpha\beta}^2 g_{\gamma\sigma} + ((-\partial_\gamma g^{\alpha\beta} \partial_\beta g_{\alpha\sigma} - \partial_\sigma g^{\alpha\beta} \partial_\beta g_{\alpha\gamma} + \partial_\gamma g_{\alpha\beta} \partial_\sigma g^{\alpha\beta}) \\ &\quad + 2g^{\alpha\beta} \Gamma_{\sigma\alpha,\nu} \Gamma_{\beta\gamma}^\nu - 2\operatorname{Re}(\lambda_{\gamma\sigma} \bar{\psi} - \lambda_{\alpha\gamma} \bar{\lambda}_\sigma^\alpha))_{lin}, \\ \nabla^\alpha \nabla_\alpha V_{lin}^\gamma &= -(\nabla^\alpha \nabla_\alpha)_{lin} V^\gamma + (2\nabla_\alpha \operatorname{Im}(\lambda^{\alpha\gamma} \bar{\psi}) - \operatorname{Re}(\lambda_\sigma^\gamma \bar{\psi} - \lambda_{\alpha\sigma} \bar{\lambda}^{\alpha\gamma}) V^\sigma \\ &\quad + 2(\operatorname{Im}(\psi \bar{\lambda}^{\alpha\beta}) + \nabla^\alpha V^\beta) \Gamma_{\alpha\beta}^\gamma)_{lin}, \\ \partial_\alpha A_{lin,\beta} - \partial_\beta A_{lin,\alpha} &= \operatorname{Im}(\lambda_\alpha^\gamma \bar{\lambda}_{\beta\gamma})_{lin}, \quad g^{\alpha\beta} \partial_\alpha A_{lin,\beta} = -g_{lin}^{\alpha\beta} \partial_\alpha A_\beta, \\ \nabla^\gamma \nabla_\gamma B_{lin} &= -(\nabla^\gamma \nabla_\gamma)_{lin} B + (-\nabla^\gamma \nabla_\sigma \operatorname{Re}(\lambda_\gamma^\sigma \bar{\psi}) + \frac{1}{2} \Delta_g |\psi|^2 \\ &\quad + \nabla^\gamma [\operatorname{Im}(\lambda_\gamma^\sigma \bar{\lambda}_{\sigma\beta}) V^\beta] + (2\operatorname{Im}(\psi \bar{\lambda}^{\beta\gamma}) + \nabla^\beta V^\gamma + \nabla^\gamma V^\beta) \partial_\beta A_\gamma)_{lin}, \end{aligned} \right. \quad (4.1)$$

where $(\psi_{lin}, \lambda_{lin}, \mathcal{S}_{lin})$ is the linearization of the solution map $(\psi, \lambda, \mathcal{S})$ in (2.27)-(2.28). Then we recall the solvability result in [7]:

Theorem 4.1 ([7], Theorem 4.1). *Assume that ψ is small in H^s for $s > d/2$ and $d \geq 4$. Then the elliptic system (2.28) admits a unique small solution $(\lambda, \mathcal{S})$ in $H^s \times \mathcal{H}^s$, with*

$$\|\lambda\|_{H^s} + \|\mathcal{S}\|_{\mathcal{H}^s} \lesssim \|\psi\|_{H^s}. \quad (4.2)$$

Moreover, for the linearization of the solution map above we also have the bound:

$$\|\lambda_{lin}\|_{H^\sigma} + \|\mathcal{S}_{lin}\|_{\mathcal{H}^\sigma} \lesssim \|\psi_{lin}\|_{H^\sigma}, \quad \sigma \in \left(\frac{d}{2} - 3, s\right]. \quad (4.3)$$

Here we will supplement the above result with an additional set of estimates:

Lemma 4.2. *Let*

$$\frac{d}{r_d} < \sigma \leq s_d - 2, \quad \sigma_1 \leq s_d + 2 - \frac{d(d-2)}{2(d-1)}.$$

Let ψ be defined in the interval $[0, T]$ and satisfy the hypothesis of Proposition 2.10. Then we have

$$\|\lambda\|_{S[0,T]} + \|h\|_{L_T^2 W^{\sigma_1, 2(d-1)}} + \|(0, V, A, B)\|_{L_T^2 \mathcal{H}^{s_d}} \lesssim \|\psi\|_{S[0,T]}, \quad (4.4)$$

and

$$\|\partial_x V\|_{L_T^1 W^{\sigma, r_d}} + \|\partial_x^2 V\|_{L_T^1 L^d} \lesssim \|\psi\|_{S[0,T]}^2. \quad (4.5)$$

In addition, in dimension $d = 4$ we have

$$\|h\|_{L_T^2 W^{1,4}} \lesssim \|\psi\|_{L_T^2 W^{1,4}} \|\psi\|_{L_T^\infty H^1}. \quad (4.6)$$

We remark that in essence this is a fixed time result, where the evolution equation for ψ is never used. What we prove in effect is the corresponding bound at fixed time where all the L_T^2 norms are dropped.

Proof. Step 1: The estimate for λ in (4.4). Here we use the div-curl system (2.9), which we write schematically in the form

$$\begin{aligned}\partial_\alpha \lambda_{\alpha\beta} &= \partial_\beta \psi + A\psi + h\partial_x \lambda + \partial_x h\lambda, \\ \partial_\alpha \lambda_{\beta\gamma} - \partial_\beta \lambda_{\alpha\gamma} &= A\lambda + \partial_x h\lambda.\end{aligned}$$

By the relation

$$\widehat{\lambda}(\xi) = |\xi|^{-2}(\widehat{\lambda} \cdot \xi)\xi + |\xi|^{-2}(\widehat{\lambda}\xi^\top - \xi\widehat{\lambda}^\top) \cdot \xi,$$

we have

$$\begin{aligned}\|\lambda\|_{W^{\sigma,r_d}} &\lesssim \|\mathcal{R}(\mathcal{R} \cdot \lambda)\|_{W^{\sigma,r_d}} + \|\mathcal{R}(\mathcal{R}_\alpha \lambda_{\beta\gamma} - \mathcal{R}_\beta \lambda_{\alpha\gamma})\|_{W^{\sigma,r_d}} \\ &\lesssim \|\psi\|_{W^{\sigma,r_d}} + \| |D|^{-1}(A\psi + A\lambda + h\partial_x \lambda + \partial_x h\lambda) \|_{W^{\sigma,r_d}},\end{aligned}$$

where $\mathcal{R} = \frac{\partial_x}{|D|}$ is the Riesz transform. By Sobolev embeddings and (4.2) we can estimate

$$\begin{aligned}\| |D|^{-1}(h\partial_x \lambda + \partial_x h\lambda) \|_{W^{\sigma,r_d}} &\lesssim \|\mathcal{R}(h\lambda)\|_{W^{\sigma,r_d}} + \| |D|^{-1}(\partial_x h\lambda) \|_{W^{\sigma,r_d}} \\ &\lesssim \|h\|_{W^{\sigma,r_d}} \|\lambda\|_{L^\infty} + \|h\|_{L^\infty} \|\lambda\|_{W^{\sigma,r_d}} \\ &\quad + \| |D|^{-1}P_{\leq 0}(\partial_x h\lambda) \|_{L^{r_d}} + \|\partial_x h\lambda\|_{W^{\sigma-1,r_d}} \\ &\lesssim \|\partial_x h\|_{H^s} \|\lambda\|_{W^{\sigma,r_d}} + \|\partial_x h\lambda\|_{L^2} \\ &\lesssim \epsilon_0 \|\lambda\|_{W^{\sigma,r_d}}.\end{aligned}$$

Similarly, since $\psi = \text{Tr } \lambda$ we can bound the other terms by

$$\begin{aligned}\| |D|^{-1}(A\lambda) \|_{W^{\sigma,r_d}} &\lesssim \|A\lambda\|_{L^2} + \|A\lambda\|_{W^{\sigma-1,r_d}} \\ &\lesssim \|\partial_x A\|_{H^s} \|\lambda\|_{W^{\sigma,r_d}} + \|A\|_{W^{\sigma-1,r_d}} \|\lambda\|_{L^\infty} \\ &\lesssim \|\partial_x A\|_{H^s} \|\lambda\|_{W^{\sigma,r_d}} \\ &\lesssim \epsilon_0 \|\lambda\|_{W^{\sigma,r_d}}.\end{aligned}$$

Hence, from these estimates we obtain

$$\|\lambda\|_{L_T^2 W^{\sigma,r_d}} \lesssim \|\psi\|_{L_T^2 W^{\sigma,r_d}}. \quad (4.7)$$

Similarly, in dimensions 4 we also have

$$\|\lambda\|_{L_T^2 W^{1,4}} \lesssim \|\psi\|_{L_T^2 W^{1,4}}. \quad (4.8)$$

Step 2: The estimate for the metric g in (4.4) and (4.6). It suffices to consider the following schematic form of the equations (2.16):

$$\Delta h = h\partial_x^2 h + \partial_x h\partial_x h + h\partial_x h\partial_x h + \lambda^2.$$

For the first three terms, we use Sobolev embeddings and Hölder's inequality to estimate

$$\begin{aligned}
& \|\Delta^{-1}(h\partial_x^2 h + \partial_x h \partial_x h + h\partial_x h \partial_x h)\|_{W^{\sigma_1, 2(d-1)}} \\
& \lesssim \|h\partial_x^2 h + \partial_x h \partial_x h + h\partial_x h \partial_x h\|_{L^{\frac{2d(d-1)}{5d-4}}} + \|h\partial_x^2 h + \partial_x h \partial_x h + h\partial_x h \partial_x h\|_{W^{\sigma_1-2, 2(d-1)}} \\
& \lesssim (1 + \|\partial_x h\|_{H^{s+1}}) \|\partial_x h\|_{H^{s+1}} \|h\|_{W^{\sigma_1, 2(d-1)}} \lesssim \epsilon_0 \|h\|_{W^{\sigma_1, 2(d-1)}}.
\end{aligned}$$

For the last term, by Sobolev embeddings we have

$$\begin{aligned}
\|\Delta^{-1}(\lambda^2)\|_{W^{\sigma_1, 2(d-1)}} & \lesssim \|\lambda^2\|_{L^{\frac{2d(d-1)}{5d-4}}} + \|\lambda^2\|_{W^{\sigma_1-2, 2(d-1)}} \\
& \lesssim \|\lambda\|_{L^{d/2}} \|\lambda\|_{L^{2(d-1)}} + \|\lambda\|_{W^{\sigma_1-2, 2(d-1)}} \|\lambda\|_{L^\infty} \\
& \lesssim \|\lambda\|_{H^s} \|\lambda\|_{W^{\sigma, r_d}} \lesssim \epsilon_0 \|\lambda\|_{W^{\sigma, r_d}}.
\end{aligned}$$

Hence, by the above estimates and (4.7) we obtain

$$\|h\|_{L_T^2 W^{\sigma_1, 2(d-1)}} \lesssim \epsilon_0 \|\lambda\|_{L_T^2 W^{\sigma, r_d}} \lesssim \epsilon_0 \|\psi\|_{L_T^2 W^{\sigma, r_d}}.$$

In the same way, from (4.8) we also obtain the bound (4.6) in dimension $d = 4$.

Step 3: The estimate for the advection field V and the connection coefficients A in (4.4). Again it suffices to consider the following schematic form of the equations (2.22), (2.13)

$$\begin{aligned}
\Delta V &= h\partial_x^2 V + \partial_x h \partial_x V + \partial_x h \partial_x h V + \lambda^2(A + V + \partial_x h) + \partial_x(\lambda^2), \\
\Delta A &= \partial_x(\lambda^2) + \partial_x(h\partial_x A).
\end{aligned} \tag{4.9}$$

The estimates for V and A are similar, so we only prove the bound for V .

As in the proof of (4.2), we bound the first three terms on the right by

$$\begin{aligned}
\| |D|^{-1}(h\partial_x^2 V + \partial_x h \partial_x V + \partial_x h \partial_x h V) \|_{H^{s_d}} & \lesssim (1 + \|\partial_x h\|_{H^{s_d+1}}) \|\partial_x h\|_{H^{s_d+1}} \|\partial_x V\|_{H^{s_d}} \\
& \lesssim \epsilon_0 \|\partial_x V\|_{H^{s_d}}.
\end{aligned}$$

For the forth term in (4.9), by Sobolev embeddings we have

$$\begin{aligned}
\| |D|^{-1}(\lambda^2(A + V + \partial_x h)) \|_{H^{s_d}} & \lesssim \|\lambda^2(A + V + \partial_x h)\|_{L^{\frac{2d}{d+2}}} + \|\lambda^2(A + V + \partial_x h)\|_{H^{s_d}} \\
& \lesssim \|\lambda^2\|_{H^{s_d}} \|(\partial_x A, \partial_x V, \partial_x h)\|_{H^{s_d}} \\
& \lesssim \epsilon_0^2 \|\lambda\|_{W^{\sigma, r_d}}.
\end{aligned}$$

For the last term in (4.9), we also have

$$\|\mathcal{R}(\lambda^2)\|_{H^{s_d}} \lesssim \|\lambda\|_{H^{s_d}} \|\lambda\|_{L^\infty} \lesssim \epsilon_0 \|\lambda\|_{W^{\sigma, r_d}}.$$

Hence, we obtain

$$\|\partial_x V\|_{H^{s_d}} \lesssim \epsilon_0 \|\lambda\|_{W^{\sigma, r_d}}.$$

Step 4: The estimate for B in (4.4). Again it suffices to consider the schematic form of the equation (2.25):

$$\begin{aligned}\Delta B &= h\partial_x^2 B + \partial_x(\lambda\partial_x\lambda) + \lambda^2(\partial_x A + \partial_x V + \partial_x h(V + A)) + \lambda\partial_x\lambda(V + A + \partial_x h) \\ &\quad + \partial_x V\partial_x A + \partial_x hV\partial_x A.\end{aligned}$$

By Sobolev embeddings and (4.2), we obtain

$$\|\partial_x B\|_{L^2 H^{s_d-1}} \lesssim \epsilon_0(\|A\|_{L^2 H^{s_d+1}} + \|\partial_x V\|_{L^2 H^{s_d}}) + \epsilon_0\|\lambda\|_{L_t^2 W^{\sigma,r_d}} \lesssim \epsilon_0\|\psi\|_{L_t^2 W^{\sigma,r_d}}.$$

The proof of this bound is similar to the above *steps*, and we omit the detail.

Step 5: The estimates for V in (4.5). It suffices to consider the form

$$\Delta V = h\partial_x^2 V + \partial_x h\partial_x V + \partial_x h\partial_x hV + \lambda^2(A + V + \partial_x h) + \partial_x(\lambda^2). \quad (4.10)$$

First, we prove that

$$\|\partial_x V\|_{L_T^1 W^{\sigma_d,r_d}} \lesssim \|\lambda\|_{S[0,T]}^2 + \epsilon_0\|\partial_x V\|_{L_T^1 W^{\sigma_d,r_d}}.$$

This implies the bound (4.5) for the term $\|\partial_x V\|_{L_T^1 W^{\sigma_d,r_d}}$.

By V -equation and Sobolev embeddings we have

$$\begin{aligned}\|\partial_x V\|_{L_T^1 W^{k_0-2,r_d}} &\lesssim \| |D|^{-1} [h\partial_x^2 V + \partial_x h\partial_x V + \partial_x h\partial_x hV] \|_{L_T^1 W^{k_0-2,r_d}} \\ &\quad + \| |D|^{-1} [\lambda^2(A + V + \partial_x h)] \|_{L_T^1 W^{k_0-2,r_d}} + \|\mathcal{R}(\lambda^2)\|_{L_T^1 W^{k_0-2,r_d}} \\ &:= I + II + III,\end{aligned}$$

where $\mathcal{R}$ is Riesz transform. For the first term on the right hand side of (4.10), we easily have

$$\begin{aligned}I &\lesssim \|P_{\leq 0}(h\partial_x^2 V + \partial_x h\partial_x V + \partial_x h\partial_x hV)\|_{L_T^1 L^2} \\ &\quad + \|h\partial_x^2 V + \partial_x h\partial_x V + \partial_x h\partial_x hV\|_{L_T^1 W^{k_0-3,r_d}} \\ &\lesssim (1 + \|\partial_x h\|_{L_T^\infty H^{k_d}})(\|\partial_x h\|_{L^\infty H^{k_d}}\|\partial_x V\|_{L_T^1 W^{k_d-2,r_d}} + \|\partial_x h\|_{L^2 W^{k_d-2,r_d}}\|\partial_x V\|_{L_T^2 H^{k_d}}) \\ &\lesssim \epsilon_1\|\partial_x V\|_{L_T^1 W^{k_d-2,r_d}} + \|\lambda\|_{S[0,T]}^2.\end{aligned}$$

For the second term in (4.10), by Sobolev embeddings we have

$$\begin{aligned}II &\lesssim \|P_{\leq 0}|D|^{-1}[\lambda^2(A + V + \partial_x h)]\|_{L^{r_d}} + \|P_{> 0}[\lambda^2(A + V + \partial_x h)]\|_{L^1 W^{\sigma_d-1,r_d}} \\ &\lesssim \|\lambda\|_{L^2 L^{r_d}}\|\lambda\|_{L^2 L^\infty}\|A + V + \partial_x h\|_{L^d} + \|\lambda\|_{L^1 L^\infty}^2\|\nabla(A + V + \partial_x h)\|_{L^\infty H^{s_d-1}} \\ &\quad + \|\lambda\|_{L^2 W^{\sigma_d,r_d}}\|\lambda\|_{L^2 L^\infty}\|A + V + \partial_x h\|_{L^\infty L^\infty} \\ &\lesssim \|\lambda\|_{L^2 W^{\sigma_d,r_d}}^2\|(\partial_x A, \partial_x V, \partial_x h)\|_{L^\infty H^{s_d}} \\ &\lesssim \epsilon_0\|\lambda\|_{L^2 W^{\sigma_d,r_d}}^2.\end{aligned}$$

For the last term in (4.10), we also have

$$III \lesssim \|\lambda\|_{L_T^2 L^\infty}\|\lambda\|_{L_T^2 W^{k_0-2,r_d}} \lesssim \|\lambda\|_{L_T^2 W^{k_0-2,r_d}}^2 \lesssim \epsilon_1^2.$$

Hence, we give the bound (4.5) for $\partial_x V$.

Next, we prove that

$$\|\partial_x^2 V\|_{L^1 L^d} \lesssim \epsilon_0 \|\partial_x^2 V\|_{L^d} + \|\lambda\|_{S[0,T]}^2 \quad (4.11)$$

From the general form (4.10), we use (4.4), (4.5) for $\partial_x V$ to bound the first three terms in (4.10)

$$\begin{aligned} & \|h\partial_x^2 V + \partial_x h\partial_x V + \partial_x h\partial_x hV\|_{L^1 L^d} \\ & \lesssim \|\partial_x h\|_{L^\infty H^{s_d}} \|\partial_x^2 V\|_{L^1 L^d} + \|\partial_x h\|_{L^\infty L^d} \|\partial_x V\|_{L^1 W^{\sigma_d, r_d}} + \|\partial_x h\|_{L^2 L^\infty}^2 \|\partial_x V\|_{L^\infty H^{s_d-2}} \\ & \lesssim \epsilon_0 \|\partial_x^2 V\|_{L^d} + \epsilon_0 \|\lambda\|_{S[0,T]}^2. \end{aligned}$$

We bound the last two terms in (4.10) by

$$\begin{aligned} & \|\lambda^2(A + V + \partial_x h)\|_{L^1 L^d} + \|\partial_x(\lambda^2)\|_{L^1 L^d} \\ & \lesssim \|\lambda\|_{L^2 L^\infty}^2 \|(\partial_x A, \partial_x V, \partial_x h)\|_{L^\infty H^{s_d-1}} + \|\lambda\|_{S[0,T]}^2 \\ & \lesssim \|\lambda\|_{S[0,T]}^2 (1 + \epsilon_0). \end{aligned}$$

Then the desired bound (4.11) follows, and we obtain the bound (4.5) for $\partial_x^2 V$. $\square$

Finally, we turn our attention to the linearization of the elliptic system (2.28). This has already been studied in [7] in nonnegative Sobolev spaces. However, for our global estimates here we need instead to work with the linearized equation in H^{-1} . For this case, the elliptic estimates are as follows:

Proposition 4.3. *With the notation and hypothesis in Proposition 2.10, for the linearized equations of (2.28) we have*

$$\|\lambda_{lin}\|_{L^\infty H^{-1}} \lesssim \|\psi_{lin}\|_{L^\infty H^{-1}}, \quad (4.12)$$

$$\|\partial_x h_{lin}\|_{L^2 L^2} + \|A_{lin}\|_{L^2 L^2} + \|V_{lin}\|_{L^2 L^2} + \|B_{lin}\|_{L^2 H^{-1}} \lesssim \|\psi_{lin}\|_{L^\infty H^{-1}} \|\psi\|_{S[0,T]}. \quad (4.13)$$

Proof. Step 1: Prove the h_{lin} bound

$$\|\partial_x h_{lin}\|_{L^2 L^2} \lesssim \|\lambda_{lin}\|_{L^\infty H^{-1}} \|\lambda\|_{L^2 W^{\sigma_d, r_d}}. \quad (4.14)$$

For the h -equations in (2.28), we consider the general form

$$\Delta h = h\partial_x^2 h + \partial_x h\partial_x h + h\partial_x h\partial_x h + \lambda^2.$$

For the term $\lambda_{lin}\lambda$, by Sobolev embeddings we have

$$\begin{aligned} & \| |D|^{-1}(\lambda_{lin}\lambda) \|_{L^2} \\ & \lesssim \| |D|^{-1}(\lambda_{lin, \leq 0}\lambda) \|_{L^2} + \| |D|^{-1}(\lambda_{lin, > 0}\lambda) \|_{L^2} \\ & \lesssim \|\lambda_{lin, \leq 0}\|_{L^2} \|\lambda\|_{L^d} + \| |D|^{-1}(|D|^{-1}\lambda_{lin, > 0}|D|\lambda) + |D|^{-1}\lambda_{lin, > 0}\lambda \|_{L^2} \\ & \lesssim \|\lambda_{lin, \leq 0}\|_{L^2} \|\lambda\|_{L^d} + \| |D|^{-1}\lambda_{lin, > 0} \|_{L^2} (\| |D|\lambda \|_{L^d} + \|\lambda\|_{L^\infty}) \\ & \lesssim \|\lambda_{lin}\|_{H^{-1}} \|\lambda\|_{\mathbf{str}}. \end{aligned} \quad (4.15)$$

For the term $h_{lin}\partial^2 h$, we also have

$$\begin{aligned} \| |D|^{-1}(h_{lin}\partial^2 h) \|_{L^2} &\lesssim \| h_{lin}\partial h \|_{L^2} + \| |D|^{-1}(\partial h_{lin}\partial h) \|_{L^2} \\ &\lesssim \| h_{lin} \|_{L^{\frac{2d}{d-2}}} \| \partial h \|_{L^d} + \| |D| h_{lin} \|_{L^2} \| \partial h \|_{L^d} \\ &\lesssim \| \partial h_{lin} \|_{L^2} \| \partial h \|_{L^d}. \end{aligned}$$

The other terms are controlled at the same way. Hence, by (2.30) we obtain

$$\| \partial h_{lin} \|_{L^2} \lesssim \epsilon_0 \| \partial h_{lin} \|_{L^2} + \| \lambda_{lin} \|_{H^{-1}} \| \lambda \|_{\mathbf{str}}.$$

This implies the bound (4.14).

Step 2: Prove the bound

$$\| A_{lin} \|_{L^2 L^2} + \| V_{lin} \|_{L^2 L^2} \lesssim \| \lambda_{lin} \|_{L^\infty H^{-1}} \| \lambda \|_{S[0,T]}. \quad (4.16)$$

The estimates of V_{lin} and A_{lin} are similar, so we only prove the first one. For the V -equation, we consider the form

$$\Delta V = h\partial_x^2 V + \partial_x h \partial_x V + \partial_x h \partial_x h V + \lambda^2(A + V + \partial_x h) + \partial_x(\lambda^2). \quad (4.17)$$

For $\partial(\lambda^2)$, we have the bound (4.15). For the term $\lambda^2(A + V + \partial_x h)$, we have

$$\| \Delta^{-1}(\lambda^2 V_{lin}) \|_{L^2} \lesssim \| \lambda^2 V_{lin} \|_{L^{\frac{2d}{d+4}}} \lesssim \| \lambda \|_{L^d}^2 \| V_{lin} \|_{L^2},$$

and

$$\begin{aligned} &\| \Delta^{-1}(\lambda_{lin} \lambda V) \|_{L^2} \\ &\lesssim \| \Delta^{-1}(\langle D \rangle^{-1} \lambda_{lin} \langle D \rangle \lambda V + \langle D \rangle^{-1} \lambda_{lin} \lambda \langle D \rangle V) + \Delta^{-1} \langle D \rangle (\langle D \rangle^{-1} \lambda_{lin} \lambda V) \|_{L^2} \\ &\lesssim \| \langle D \rangle^{-1} \lambda_{lin} \langle D \rangle \lambda V + \langle D \rangle^{-1} \lambda_{lin} \lambda \langle D \rangle V \|_{L^{\frac{2d}{d+4}}} + \| \langle D \rangle^{-1} \lambda_{lin} \|_{L^2} \| \lambda \|_{L^d \cap L^\infty} \| V \|_{L^d} \\ &\lesssim \| \lambda_{lin} \|_{H^{-1}} \| \lambda \|_{\mathbf{str}} \| V \|_{L^d}. \end{aligned}$$

For the term $h\partial^2 V$, we have

$$\begin{aligned} \| \Delta^{-1}(h_{lin}\partial^2 V) \|_{L^2} &= \| \Delta^{-1}(\partial h_{lin}\partial V) + |D|^{-1}(h_{lin}\partial V) \|_{L^2} \\ &\lesssim \| \partial h_{lin} \|_{L^2} \| \partial V \|_{H^s}, \end{aligned}$$

and

$$\begin{aligned} \| \Delta^{-1}(h\partial^2 V_{lin}) \|_{L^2} &= \| \Delta^{-1}(\partial^2 h V_{lin}) + |D|^{-1}(\partial h V_{lin}) + h V_{lin} \|_{L^2} \\ &\lesssim \| \partial h \|_{H^s} \| \partial V_{lin} \|_{L^2}. \end{aligned}$$

The second and third term in (4.17) are bounded similarly. Hence, we obtain

$$\| V_{lin} \|_{L^2} \lesssim \epsilon_1 (\| V_{lin} \|_{L^2} + \| A_{lin} \|_{L^2}) + \| \lambda_{lin} \|_{H^{-1}} \| \lambda \|_{\mathbf{str}}.$$

In the same way, we also have

$$\| A_{lin} \|_{L^2} \lesssim \| \lambda_{lin} \|_{H^{-1}} \| \lambda \|_{\mathbf{str}}.$$

These two estimates imply the desired bound (4.16).

Step 3: Prove the λ_{lin} bound (4.12).

As before, it suffices to consider the simplified form of the div-curl system for λ , namely

$$\begin{aligned}\partial_\alpha \lambda_{\alpha\beta} &= \partial_\beta \psi + A\psi + h\partial_x \lambda + \partial_x h\lambda, \\ \partial_\alpha \lambda_{\beta\gamma} - \partial_\beta \lambda_{\alpha\gamma} &= A\lambda + \partial_x h\lambda.\end{aligned}$$

For the term $A\lambda$ we have

$$\| |D|^{-1}(A_{lin}\lambda) \|_{H^{-1}} \lesssim \|A_{lin}\|_{L^2} \|\lambda\|_{L^d},$$

and

$$\begin{aligned}\| |D|^{-1}(A\lambda_{lin}) \|_{H^{-1}} &\lesssim \|A\|_{L^d} \|\lambda_{lin,\leq 0}\|_{L^2} + \| |D|^{-1}(|D|A|D|^{-1}\lambda_{lin,>0}) + A|D|^{-1}\lambda_{lin,>0} \|_{L^2} \\ &\lesssim \|\partial A\|_{H^s} \|\lambda_{lin}\|_{H^{-1}}.\end{aligned}$$

The other terms are controlled by

$$\|\psi_{lin}\|_{H^{-1}} + \|\partial_x h_{lin}\|_{L^2} \|\lambda\|_{H^s} + \|\partial_x h\|_{H^s} \|\lambda_{lin}\|_{H^{-1}}.$$

Then these estimates combined with (4.14) and (4.16) yield

$$\|\lambda_{lin}\|_{H^{-1}} \lesssim \|\psi_{lin}\|_{H^{-1}} + \epsilon_0 \|\lambda_{lin}\|_{H^{-1}}.$$

This implies the bound (4.12).

Step 4: Prove the bound

$$\|B_{lin}\|_{L^2 H^{-1}} \lesssim \|\lambda_{lin}\|_{L^\infty H^{-1}} \|\lambda\|_{S[0,T]}. \quad (4.18)$$

For the B -equation we consider the general form

$$\begin{aligned}\Delta B &= h\partial_x^2 B + \partial_x^2(\lambda^2) + \partial_x(\lambda^2)(\partial_x h + V) + \lambda^2(\partial_x^2 h + \partial_x h\partial_x h + \partial_x A + \partial_x V + \partial_x hV) \\ &\quad + \partial_x V\partial_x A + \partial_x hV\partial_x A.\end{aligned}$$

For the second term $\partial^2(\lambda^2)$ we have

$$\|\lambda_{lin}\lambda\|_{H^{-1}} \lesssim \|\lambda_{lin}\|_{H^{-1}} \|\lambda\|_{\mathbf{str}}.$$

For the third term, we have

$$\|\Delta^{-1}(\partial_x(\lambda_{lin}\lambda)V)\|_{H^{-1}} \lesssim \|\lambda_{lin}\|_{H^{-1}} \|\lambda\|_{\mathbf{str}} \|\partial_x V\|_{H^s}.$$

and

$$\|\Delta^{-1}(\partial_x(\lambda^2)V_{lin})\|_{H^{-1}} \lesssim \|\lambda\|_{\mathbf{str}}^2 \|V_{lin}\|_{L^2}.$$

We control the other terms at the same way, then by (4.14), (4.16) and (2.30) we obtain

$$\|B_{lin}\|_{H^{-1}} \lesssim \epsilon_0 \|B_{lin}\|_{H^{-1}} + \|\lambda_{lin}\|_{H^{-1}} \|\lambda\|_{\mathbf{str}}.$$

This gives the bound (4.18).

In conclusion, from (4.14), (4.16), (4.18) and (4.12) we obtain the second bound (4.13). $\square$

5. ENERGY ESTIMATES

Here we consider the Schrödinger equation (2.27), and prove the energy estimates in Proposition 1.2 as well as an energy estimate of linearized Schrödinger equation. These will be needed in order to prove energy bounds (2.31) in fractional Sobolev spaces. For two tensors $\mathbb{T}$ and $\tilde{\mathbb{T}}$, we denote $\mathbb{T} * \tilde{\mathbb{T}}$ any tensor product of two tensors when we do not need the precise expression.

To start with, we define the energy functional as follows. Let the metric g and connection A be (part of) the solutions to the elliptic equations (2.28). For any nonnegative integer $k \in \mathbb{N}$, we define $E^k(\psi)$ as

$$E^k(\psi) := \|\psi\|_{\mathbb{H}^k}^2 = \left(\sum_{l=0}^k \int_{\Sigma} |\nabla^{A,l} \psi|_g^2 d\mu \right)^{1/2}. \quad (5.1)$$

We will show that this energy functional satisfies the bounds in Proposition 1.2.

a) Proof of the energy estimates (1.9).

Step 1: Prove that the time derivative of E^k has the form

$$\frac{d}{dt} E^k(\psi) = \sum_{\Sigma |\alpha_j| \leq 2k} \int \operatorname{Re} \prod_{j=1}^{J=4} \nabla^{A, \alpha_j} \lambda \, d\mu \quad (5.2)$$

with coefficients depending on the metric g so that each of the terms in the above integrand is covariant.

We recall the Schrödinger equation (2.27) first

$$i(\partial_t^B - V^\gamma \nabla_\gamma^A) \psi + \Delta_g^A \psi = -i\lambda_\sigma^\gamma \operatorname{Im}(\psi \bar{\lambda}_\gamma^\sigma). \quad (5.3)$$

On one hand, the energy (5.1) is defined by intrinsic Sobolev norm, which does not depend on the choice of gauge. On the other hand, the equation (1.2) is equivalent to (1.1) up to diffeomorphisms tangent to Σ_t . Indeed, let $F : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d+2}$ be a family of embeddings satisfying the equation (1.2), and let $x(t, y)$ be a family of diffeomorphisms of $\mathbb{R}^d$ satisfying

$$D_x F(x(t, y), t) \left(\frac{\partial x}{\partial t}(t, y) \right) = - \left(\frac{\partial F}{\partial t}(x(t, y), t) \right)^\top.$$

Then $\tilde{F}(t, y) = F(x(t, y), t)$ is a solution of (1.1) with the advection field

$$\tilde{V}^\gamma(t, y) \partial_{y^\gamma} \tilde{F} = (\partial_t \tilde{F})^\top = 0.$$

Hence, inspired by the above two properties, we can derive the energy estimates from (5.3) with the advection field $V = 0$. Then the volume form $d\mu = \sqrt{\det g}$ is preserved along time t .

Applying $\frac{d}{dt}$ to $\|\nabla^{A,k}\psi|_g(t)\|_{L^2(d\mu)}^2$, by (2.19) we obtain

$$\begin{aligned} & \frac{d}{dt} \int_{\Sigma} |\nabla^{A,k}\psi|_g^2 d\mu \\ &= \int_{\mathbb{R}^d} 2 \operatorname{Re} g(\nabla_t^B \nabla^{A,k}\psi, \overline{\nabla^{A,k}\psi}) d\mu + (\nabla_t g)(\nabla^{A,k}\psi, \overline{\nabla^{A,k}\psi}) d\mu \\ &= \int_{\mathbb{R}^d} 2 \operatorname{Re} g(\nabla_t^B \nabla^{A,k}\psi, \overline{\nabla^{A,k}\psi}) + 2G(\nabla^{A,k}\lambda, \overline{\nabla^{A,k}\psi}) d\mu. \end{aligned} \quad (5.4)$$

By the equalities (2.20) and (2.24) with $V = 0$, we have

$$[\partial_t^B, \nabla^{A,k}]\psi = \sum_{l_1+l_2+l_3=k} \nabla^{A,l_1}\lambda * \nabla^{A,l_2}\psi * \nabla^{A,l_3}\psi.$$

Moreover, note that by Gauss equation, the curvature tensor $\mathbf{R}$ on Σ can be expressed as $\mathbf{R} = \lambda * \lambda$, so the following commutator equality holds

$$[\nabla^{A,k}, \Delta^A]\psi = \sum_{i+j+m=k} \nabla^{A,i}\lambda * \nabla^{A,j}\lambda * \nabla^{A,m}\psi. \quad (5.5)$$

So by (5.3), the first term in the right-hand side of (5.4) reduces to

$$\begin{aligned} & \int_{\mathbb{R}^d} 2 \operatorname{Re} g(\nabla_t^B \nabla^{A,k}\psi, \overline{\nabla^{A,k}\psi}) d\mu \\ &= \int_{\mathbb{R}^d} 2 \operatorname{Re} g(\nabla^{A,k} \partial_t^B \psi, \overline{\nabla^{A,k}\psi}) + \sum_{l_1+l_2+l_3=k} \operatorname{Re} g(\nabla^{A,l_1}\lambda * \nabla^{A,l_2}\lambda * \nabla^{A,l_3}\psi, \overline{\nabla^{A,k}\psi}) d\mu \\ &= \int_{\mathbb{R}^d} 2 \operatorname{Re} g(\nabla^{A,k} i \Delta^A \psi, \overline{\nabla^{A,k}\psi}) + \sum_{l_1+l_2+l_3=k} \operatorname{Re} g(\nabla^{A,l_1}\lambda * \nabla^{A,l_2}\lambda * \nabla^{A,l_3}\psi, \overline{\nabla^{A,k}\psi}) d\mu \\ &= \int_{\mathbb{R}^d} -2 \operatorname{Re} i |\nabla^{A,k+1}\psi|_g^2 + \sum_{l_1+l_2+l_3=k} \operatorname{Re} g(\nabla^{A,l_1}\lambda * \nabla^{A,l_2}\lambda * \nabla^{A,l_3}\psi, \overline{\nabla^{A,k}\psi}) d\mu \\ &= \int_{\mathbb{R}^d} \sum_{l_1+l_2+l_3=k} \operatorname{Re} g(\nabla^{A,l_1}\lambda * \nabla^{A,l_2}\lambda * \nabla^{A,l_3}\psi, \overline{\nabla^{A,k}\psi}) d\mu \end{aligned}$$

Hence, we obtain the energy relation (5.2).

Step 2: Prove the energy bound (1.9).

Let us first recall the following interpolation inequality proved by Hamilton [4, Section 12].

Lemma 5.1. *If T is any tensor and if $1 \leq i \leq l-1$, then with a constant $C = C(d, l)$ depending only on dimensions d and l , which is independent of the metric g and the connection Γ , we have the estimate*

$$\int_{\mathbb{R}^d} |\nabla^i T|^{\frac{2l}{i}} d\mu \leq C |T|_{L^\infty}^{2(\frac{l}{i}-1)} \int_{\mathbb{R}^d} |\nabla^l T|^2 d\mu.$$

Then by the interpolation inequality, (5.2), (4.3) and (5.6), for each integer k we have

$$\begin{aligned} \frac{d}{dt} E^k(\psi) &\lesssim \sum_{m \leq k} \sum_{i+j+l=m} \|\nabla^{A,i} \lambda\|_{L^{2m/i}} \|\nabla^{A,j} \lambda\|_{L^{2m/j}} \|\nabla^{A,l} \psi\|_{L^{2m/l}} \|\nabla^{A,m} \psi\|_{L^2} \\ &\lesssim \|\lambda\|_{L^\infty}^2 \|\lambda\|_{\mathbf{H}^k}^2. \end{aligned}$$

Thus we obtain the energy estimates (1.9). $\square$

b) Proof of the energy equivalence relation (1.10).

The relation (1.10) for $k \leq s$ with some $s > \frac{d}{2}$ is already a consequence of (3.3). We should be more accurate here, we get a better range from (3.3).

It remains to prove (1.10) for $k > s$. Our starting point is the higher regularity bounds for the elliptic system (2.28), which were proved in [7, Section 7.6], as follows:

$$\|(\lambda, h, V, A, B)\|_{H^\sigma \times \mathcal{H}^\sigma} \lesssim \|\psi\|_{H^\sigma}, \quad \sigma \geq s. \quad (5.6)$$

This implies in particular that the Christoffel symbol Γ and connection coefficients A_α satisfy

$$\|(\Gamma, A)\|_{H^{\sigma+1}} \lesssim \|\psi\|_{H^\sigma}, \quad \sigma \geq s. \quad (5.7)$$

By the expression (3.4), Sobolev embeddings and (5.7) we have

$$\begin{aligned} \|\psi\|_{\mathbf{H}^k} &\lesssim \|\psi\|_{H^k} + \sum_{1 \leq n \leq k} \sum_{l_1 + \dots + l_{n+1} \leq k-n} \|\partial_x^{l_1}(\Gamma + iA) \cdots \partial_x^{l_n}(\Gamma + iA) \cdot \partial_x^{l_{n+1}} \psi\|_{L^2} \\ &\lesssim \|\psi\|_{H^k} + \sum_{1 \leq n \leq k} \|\Gamma + iA\|_{H^k} \|\Gamma + iA\|_{H^s}^{n-1} \|\psi\|_{H^s} + \sum_{1 \leq n \leq k} \|\Gamma + iA\|_{H^s}^n \|\psi\|_{H^k} \\ &\lesssim \|\psi\|_{H^k} + \sum_{1 \leq n \leq k} \|\psi\|_{H^k} \|\psi\|_{H^s}^n \\ &\lesssim \|\psi\|_{H^k}. \end{aligned}$$

Conversely, by (3.4) we also have

$$\begin{aligned} \|\psi\|_{H^k} &\lesssim \|\psi\|_{\mathbf{H}^k} + \sum_{1 \leq n \leq k} \sum_{l_1 + \dots + l_{n+1} \leq k-n} \|\partial^{l_1}(\Gamma + iA) \cdots \partial^{l_n}(\Gamma + iA) \cdot \partial^{l_{n+1}} \psi\|_{L^2} \\ &\lesssim \|\psi\|_{\mathbf{H}^k} + \epsilon_0 \|\psi\|_{H^k}. \end{aligned}$$

Thus we obtain the equivalence relation $\|\psi\|_{\mathbf{H}^k} \approx \|\psi\|_{H^k}$.

In the same way as the above, we also have the equivalence $\|\lambda\|_{\mathbf{H}^k} \approx \|\lambda\|_{H^k}$. By (5.6), (4.3) and the bound

$$\|\psi\|_{H^k} = \|g^{\alpha\beta} \lambda_{\alpha\beta}\|_{H^k} \lesssim \|\lambda\|_{H^k} + \epsilon_0 \|\lambda\|_{H^k} \lesssim \|\lambda\|_{H^k},$$

we also have the equivalence $\|\psi\|_{H^k} \approx \|\lambda\|_{H^k}$. Hence, the desired equivalence relations (1.10) are obtained. $\square$

Finally, we prove an energy estimate in negative Sobolev spaces H^{-1} for the linearized equation of (2.27):

$$\begin{aligned} & i(\partial_t^B - V^\gamma \nabla_\gamma^A) \psi_{lin} + \Delta_g^A \psi_{lin} \\ &= (V^\gamma \nabla_\gamma^A)_{lin} \psi - (\Delta_g^A)_{lin} \psi + B_{lin} \psi - [i\lambda_\sigma^\gamma \text{Im}(\psi \bar{\lambda}_\gamma^\sigma)]_{lin} := F_{lin}. \end{aligned} \quad (5.8)$$

Proposition 5.2. *Let $d \geq 4$. Under the assumptions (2.29) and (2.30), for the linearized equation of (2.27) we have the bound*

$$\|\psi_{lin}\|_{L_T^\infty H^{-1}} \leq C_{lin} \|\psi_{lin}(0)\|_{H^{-1}} + C_{lin} \|\psi\|_{S[0,T]}^2 \|\psi_{lin}\|_{L_T^\infty H^{-1}}. \quad (5.9)$$

For clarity, here we note that the linearized equation depends on our gauge choices. The above proposition and its proof below assume we are in the harmonic/Coulomb gauge.

Proof. Here we will treat the source term F_{lin} in (5.8) perturbatively. This allows us to split the proof of (5.9) into two parts. Precisely, it suffices to prove the linear bound

$$\|\psi_{lin}\|_{L_T^\infty H^{-1}} \lesssim \|\psi_{lin}(0)\|_{H^{-1}} + \|F_{lin}\|_{L_T^1 H^{-1}}. \quad (5.10)$$

respectively the source term estimate

$$\|F_{lin}\|_{L_T^1 H^{-1}} \lesssim \|\psi_{lin}\|_{L_T^\infty H^{-1}} \|\psi\|_{S[0,T]}^2 \quad (5.11)$$

Together, these two bounds imply the conclusion of the proposition. It remains to prove (5.10) and (5.11).

We first consider the bound (5.10), which we prove using duality. For this we need the associated adjoint equation, which has the form

$$i(\partial_t^B - V^\gamma \nabla_\gamma^A) v + \Delta_g^A v - i\nabla_\gamma V^\gamma v = \mathcal{N}, \quad (5.12)$$

The adjoint evolution is considered in the same time interval $[0, T]$, but as a backward Cauchy problem with the initial data at time T . Then we claim that this evolution is (backward) well-posed in $\mathbf{H}^1$, and satisfies the bound

$$\|v\|_{L^\infty(0,T;\mathbf{H}^1)} \lesssim \|v(T)\|_{\mathbf{H}^1} + \|\mathcal{N}\|_{L^1(0,T;\mathbf{H}^1)}. \quad (5.13)$$

Assuming this holds, then from the duality relation

$$\langle \psi_{lin}, v \rangle_0^T = \langle -i\mathcal{N}, F_{lin} \rangle$$

we have the bound

$$|\langle (\psi_{lin}(T), \psi_{lin}), (v(T), -i\mathcal{N}) \rangle| \lesssim (\|\psi_{lin}(0)\|_{\mathbf{H}^{-1}} + \|F_{lin}\|_{L^1 \mathbf{H}^{-1}}) (\|v(T)\|_{\mathbf{H}^1} + \|\mathcal{N}\|_{L^1 \mathbf{H}^1})$$

which in turn implies that

$$\|\psi_{lin}\|_{L_T^\infty \mathbf{H}^{-1}} \lesssim \|\psi_{lin}(0)\|_{\mathbf{H}^{-1}} + \|F_{lin}\|_{L_T^1 \mathbf{H}^{-1}}.$$

Since the metric $g - I_d$ and connection A are small in harmonic/Coulomb gauge, by equivalence (3.3) and duality we have

$$\|u\|_{H^{-1}} = \sup_{\|v\|_{H^1} \leq 1} \langle u, v \rangle_{L^2} \lesssim \sup_{\|v\|_{H^1} \leq 1} \langle u, v \rangle_{L^2} = \|u\|_{H^{-1}}.$$

Then the desired bound (5.10) follows.

Now we prove the bound (5.11) for the nonlinear terms F_{lin} . This is a consequence of the fixed time bound

$$\|F_{lin}\|_{H^{-1}} \lesssim \|\psi_{lin}\|_{H^{-1}} \|\psi\|_{\mathbf{str}}^2, \quad (5.14)$$

which we now prove by successively considering all the terms in F_{lin} .

Using Sobolev embeddings and (4.13) we bound the worst term $(\Delta_g)_{lin}\psi$ by

$$\begin{aligned} \|(\Delta_g)_{lin}\psi\|_{H^{-1}} &= \|h_{lin}^{\alpha\beta} \partial_{\alpha\beta}^2 \psi\|_{H^{-1}} \\ &\lesssim (\|h_{lin}\|_{L^{\frac{2d}{d-2}}} + \|\partial_x h_{lin}\|_{L^2}) \|\psi\|_{W^{1,d}} \\ &\lesssim \|\lambda_{lin}\|_{H^{-1}} \|\psi\|_{\mathbf{str}}^2 \\ &\lesssim \|\psi_{lin}\|_{H^{-1}} \|\psi\|_{\mathbf{str}}^2. \end{aligned}$$

For the term $A_{lin}^\alpha \partial_\alpha \psi$, by (4.13) we have

$$\|A_{lin}^\alpha \partial_\alpha \psi\|_{H^{-1}} \lesssim \|A_{lin}\|_{L^2} \|\partial_x \psi\|_{L^d} \lesssim \|\psi_{lin}\|_{H^{-1}} \|\psi\|_{\mathbf{str}}^2. \quad (5.15)$$

Similarly, by (4.13) we also have

$$\begin{aligned} &\|(\nabla_\alpha A^\alpha)_{lin}\psi\|_{H^{-1}} + \|(A^\alpha A_\alpha)_{lin}\psi\|_{H^{-1}} + \|(\lambda^3)_{lin}\|_{H^{-1}} \\ &\lesssim (\|A_{lin}\|_{L^2} + \|(A_\alpha A^\alpha)_{lin}\|_{L^2}) \|\psi\|_{\mathbf{str}} + \|\lambda_{lin}\|_{H^{-1}} \|\lambda\|_{\mathbf{str}}^2 \\ &\lesssim \|\psi_{lin}\|_{H^{-1}} \|\psi\|_{\mathbf{str}}^2. \end{aligned} \quad (5.16)$$

and

$$\|B_{lin}\psi\|_{H^{-1}} \lesssim \|B_{lin}\|_{H^{-1}} (\|\psi\|_{L^\infty} + \|\psi\|_{W^{1,d}}) \lesssim \|\psi_{lin}\|_{H^{-1}} \|\psi\|_{\mathbf{str}}^2.$$

For the term $(V^\gamma \nabla_\gamma^A)_{lin}\psi$, by the same argument as (5.15) and (5.16) and the estimate (4.13) we bound it by

$$\begin{aligned} \|(V^\gamma \nabla_\gamma^A)_{lin}\psi\|_{H^{-1}} &\lesssim \|V_{lin} \partial_x \psi\|_{H^{-1}} + \|(VA)_{lin}\psi\|_{H^{-1}} \\ &\lesssim \|V_{lin}\|_{L^2} \|\partial_x \psi\|_{L^d} + \|(VA)_{lin}\|_{L^2} \|\psi\|_{\mathbf{str}} \\ &\lesssim \|\psi_{lin}\|_{H^{-1}} \|\psi\|_{\mathbf{str}}^2 \end{aligned}$$

This concludes the proof of (5.14) and thus of (5.11).

Finally, we turn to the proof of the claim (5.13). Since this proof is more complicated than that of Proposition 1.2, we provide the full details. By (5.12) and integration by

parts, we have the basic energy estimate

$$\begin{aligned}
\frac{d}{dt} \|v\|_{\mathbf{L}^2}^2 &= \int 2 \operatorname{Re} \langle \partial_t^B v, v \rangle + |v|^2 \nabla_\alpha V^\alpha \, d\mu \\
&= \int 2 \operatorname{Re} \langle (V^\gamma \nabla_\gamma^A v + i \Delta_g^A v + \nabla_\gamma V^\gamma v - i \mathcal{N}), v \rangle + |v|^2 \nabla_\alpha V^\alpha \, d\mu \\
&= \int 2 \nabla_\gamma V^\gamma |v|^2 - 2 \operatorname{Re} \langle i \mathcal{N}, v \rangle \, d\mu \\
&\lesssim \|v\|_{\mathbf{L}^2}^2 \|\nabla^\gamma V_\gamma\|_{L^\infty} + \|\mathcal{N}\|_{\mathbf{L}^2} \|v\|_{\mathbf{L}^2} \\
&\lesssim \|v\|_{\mathbf{L}^2}^2 \|\lambda\|_{W^{\sigma_d, r_d}} + \|\mathcal{N}\|_{\mathbf{L}^2} \|v\|_{\mathbf{L}^2}.
\end{aligned}$$

We then derive an energy estimate for $\nabla^A v$ in $\mathbf{L}^2$. By (5.12) and integration by parts we have

$$\begin{aligned}
\frac{d}{dt} \int |\nabla^A v|^2 \, d\mu &= \int 2 \operatorname{Re} \langle [\partial_t^B, \nabla_\alpha^A] v + \nabla_\alpha^A \partial_t^B v, \nabla^{A, \alpha} v \rangle \\
&\quad + \operatorname{Re} \langle \nabla_\alpha^A v, -2G^{\alpha\beta} \nabla_\beta^A v \rangle + |\nabla^A v|^2 \nabla_\alpha V^\alpha \, d\mu \\
&= \int 2 \operatorname{Re} \langle [\partial_t^B, \nabla_\alpha^A] v, \nabla^{A, \alpha} v \rangle + \operatorname{Re} \langle \nabla_\alpha^A v, -2G^{\alpha\beta} \nabla_\beta^A v \rangle + |\nabla^A v|^2 \nabla_\alpha V^\alpha \, d\mu \\
&\quad + \int 2 \operatorname{Re} \langle \nabla_\alpha^A (V^\gamma \nabla_\gamma^A v + i \Delta_g^A v + \nabla_\gamma V^\gamma v - i \mathcal{N}), \nabla^{A, \alpha} v \rangle \, d\mu \\
&= \int 2 \operatorname{Re} \langle [\partial_t^B - i \Delta_g^A, \nabla_\alpha^A] v, \nabla^{A, \alpha} v \rangle \\
&\quad + \operatorname{Re} \langle \nabla_\alpha^A v, -2G^{\alpha\beta} \nabla_\beta^A v \rangle + 2 |\nabla^A v|^2 \nabla_\alpha V^\alpha \, d\mu \\
&\quad + \int 2 \operatorname{Re} \langle V^\gamma [\nabla_\alpha^A, \nabla_\gamma^A] v + \nabla_\alpha V^\gamma \nabla_\gamma^A v + \nabla_\alpha \nabla_\gamma V^\gamma v - i \nabla_\alpha^A \mathcal{N}, \nabla^{A, \alpha} v \rangle \, d\mu.
\end{aligned}$$

This implies

$$\begin{aligned}
\frac{d}{dt} \int |\nabla^A v|^2 \, d\mu &\lesssim \|[\partial_t^B - i \Delta_g^A, \nabla_\alpha^A] v\|_{\mathbf{L}^2} \|\nabla^A v\|_{\mathbf{L}^2} + \|V^\gamma [\nabla_\alpha^A, \nabla_\gamma^A] v\|_{\mathbf{L}^2} \|\nabla^A v\|_{\mathbf{L}^2} \\
&\quad + \|v\|_{\mathbf{H}^1}^2 (\|\lambda\|_{L^\infty}^2 + \|\nabla V\|_{L^\infty}) + \|\nabla_\alpha \nabla_\gamma V^\gamma\|_{L^d} \|v\|_{L^{2d/(d-2)}} \|v\|_{\mathbf{H}^1} \\
&\quad + \|\mathcal{N}\|_{H^1} \|v\|_{\mathbf{H}^1}.
\end{aligned} \tag{5.17}$$

We use (2.24) and (5.5) to bound the first term on the right hand side of (5.17) by

$$\begin{aligned}
\|[\partial_t^B - i \Delta_g^A, \nabla_\alpha^A] v\|_{\mathbf{L}^2} \|\nabla^A v\|_{\mathbf{L}^2} &\lesssim \|(\lambda * \nabla^A \psi + \lambda^2 V) v\|_{L^2} \|v\|_{H^1} \\
&\quad + \|\nabla^A \lambda * \lambda * v + \lambda * \lambda * \nabla v\|_{L^2} \|v\|_{H^1} \\
&\lesssim (\|\lambda * \nabla^A \lambda\|_{L^d} + \|\lambda^2 V\|_{L^d}) \|v\|_{L^{2d/(d-2)}} \|v\|_{H^1} \\
&\quad + \|\lambda\|_{L^\infty}^2 \|v\|_{H^1}^2 \\
&\lesssim \|\lambda\|_{\mathbf{str}}^2 (1 + \|\nabla V\|_{H^{s-2}}) \|v\|_{H^1}
\end{aligned}$$

For the second term in (5.17) we have

$$\|V^\gamma [\nabla_\alpha^A, \nabla_\gamma^A] v\|_{\mathbf{L}^2} \|\nabla^A v\|_{\mathbf{L}^2} = \|i V^\gamma \operatorname{Im}(\lambda_{\alpha\mu} \bar{\lambda}_\gamma^\mu) v\|_{L^2} \|v\|_{H^1} \lesssim \|\lambda\|_{\mathbf{str}}^2 \|\nabla V\|_{H^{s-2}} \|v\|_{H^1}.$$

For the third and forth terms in (5.17), by Sobolev embeddings and (4.5) we have

$$\|v\|_{\mathbf{H}^1}^2 (\|\lambda\|_{L^\infty}^2 + \|\nabla V\|_{L^\infty}) + \|\nabla_\alpha \nabla_\gamma V^\gamma\|_{L^d} \|v\|_{L^{2d/(d-2)}} \|v\|_{\mathbf{H}^1} \lesssim \|v\|_{\mathbf{H}^1} \|\lambda\|_{\mathbf{str}}^2.$$

Hence, we conclude that

$$\begin{aligned} \|v\|_{L_T^\infty \mathbf{H}^1} &\lesssim \|v_0\|_{\mathbf{H}^1} + \|\lambda\|_{L_T^2 \mathbf{str}}^2 \|v\|_{L_T^\infty \mathbf{H}^1} + \|\mathcal{N}\|_{L_T^1 \mathbf{H}^1} \\ &\lesssim \|v_0\|_{\mathbf{H}^1} + \|\lambda\|_{S[0,T]}^2 \|v\|_{L_T^\infty \mathbf{H}^1} + \|\mathcal{N}\|_{L_T^1 \mathbf{H}^1} \end{aligned}$$

By the assumption (2.30) and Hölder's inequality, this yields the claim (5.13). This completes the proof of the proposition. $\square$

6. STRICHARTZ ESTIMATES

Here we consider the Schrödinger equation (2.27), and prove the Strichartz bounds in Proposition 1.3. First, we introduce the endpoint Strichartz estimates of Keel-Tao [11] and the inhomogeneous Strichartz estimates developed by [2, 17, 18, 34]. Then we use these to bound the linear and nonlinear part, respectively.

We begin with the homogeneous Strichartz estimates obtained by Keel-Tao [11]:

$$\|e^{it\Delta} f\|_{L^q L^r} \lesssim \|f\|_{L^2}, \quad (6.1)$$

where (q, r) is Schrödinger-admissible pair, that is, $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$, $2 \leq q$, $r \leq \infty$, $(q, r, d) \neq (2, \infty, 2)$. Here we will use the endpoint pair $(q, r) = (2, \frac{2d}{d-2})$.

Next, we state the inhomogeneous Strichartz estimates, which summarize several known results, see [2, 17, 18, 34].

Definition 6.1. We say that the pair (q, r) is Schrödinger-acceptable if

$$1 \leq q < \infty, 2 \leq r \leq \infty, \quad \frac{1}{q} < \frac{d}{2} \left(1 - \frac{2}{r}\right), \quad \text{or } (q, r) = (\infty, 2).$$

Theorem 6.2 (Inhomogeneous Strichartz estimates). *Let $d \geq 3$ and p' be the duality of p with $\frac{1}{p} + \frac{1}{p'} = 1$. Assume that the pairs (q, r) and $(\tilde{q}, \tilde{r})$ are Schrödinger-acceptable pairs, and satisfy the condition*

$$\frac{1}{q} + \frac{1}{\tilde{q}} = \frac{d}{2} \left(1 - \frac{1}{r} - \frac{1}{\tilde{r}}\right).$$

In addition, assume one of the following:

i) non-sharp case:

$$\frac{1}{q} + \frac{1}{\tilde{q}} < 1, \quad \frac{d-2}{d} \frac{1}{r} \leq \frac{1}{\tilde{r}} \leq \frac{d}{d-2} \frac{1}{r}, \quad \frac{1}{r}, \frac{1}{\tilde{r}} \leq \frac{1}{2};$$

ii) sharp case:

$$\frac{1}{q} + \frac{1}{\tilde{q}} = 1, \quad \frac{d-2}{d} \frac{1}{r} < \frac{1}{\tilde{r}} < \frac{d}{d-2} \frac{1}{r}, \quad \frac{1}{r} \leq \frac{1}{q}, \quad \frac{1}{\tilde{r}} \leq \frac{1}{\tilde{q}};$$

iii) endpoint cases when $d \geq 3$:

$$\frac{1}{q} + \frac{1}{\tilde{q}} = 1, \quad \frac{1}{r} = \frac{d}{d-2} \frac{1}{\tilde{r}} \quad \text{or} \quad \frac{d-2}{d} \frac{1}{\tilde{r}}, \quad \frac{1}{r} \leq \frac{1}{q}, \quad \frac{1}{\tilde{r}} \leq \frac{1}{\tilde{q}}.$$

Then the following estimate holds

$$\left\| \int_0^t e^{i(t-s)\Delta} F(s) ds \right\|_{L^q L^r} \lesssim \|F\|_{L^{\tilde{q}'} L^{\tilde{r}'}}. \quad (6.2)$$

We now aim to prove the space-time bound $S[0, T]$ for ψ in Proposition 1.3 by combining the above Strichartz estimates with the elliptic estimates in section 4.

Proof of Proposition 1.3. By Duhamel's principle, the solution ψ of (2.27) can be expressed by

$$\psi(t) = e^{it\Delta} \psi_0 + \int_0^t e^{i(t-s)\Delta} \mathcal{N}(s) ds,$$

where

$$\mathcal{N} := h\partial_x^2 \psi + (V + A)\partial_x \psi + (B + A^2 + VA + \lambda^2)\psi.$$

Using Sobolev embeddings, the bound (6.1) with $(q, r) = (2, \frac{2d}{d-2})$ and the estimates (6.2) with pairs $(q, r) = (2, r_d)$, $(\tilde{q}, \tilde{r}) = (2, \frac{2(d-1)}{d-2})$, we have

$$\|\psi\|_{L_T^2 W^{\sigma_d, r_d}} \lesssim \|\psi_0\|_{H^{\sigma_d + \frac{d-2}{2(d-1)}}} + \|\mathcal{N}\|_{L_T^2 W^{\sigma_d, \tilde{r}'}}.$$

It remains to successively estimate the terms in $\mathcal{N}$. For the first term $h\partial_x^2 \psi$, by $\frac{1}{\tilde{r}'} = \frac{1}{2} + \frac{1}{2(d-1)}$ and Littelwood-Paley decomposition, we have

$$\begin{aligned} \|h\partial_x^2 \psi\|_{L_T^2 W^{\sigma_d, \tilde{r}'}} &\lesssim \left\| \left(\sum_k 2^{2\sigma_d k} |P_{\leq k+3} h P_k \partial_x^2 \psi|^2 \right)^{1/2} \right\|_{L_T^2 L^{\tilde{r}'}} \\ &\quad + \left\| \left(\sum_k 2^{2\sigma_d k} |P_k h P_{\leq k+3} \partial_x^2 \psi|^2 \right)^{1/2} \right\|_{L_T^2 L^{\tilde{r}'}} \\ &\quad + \left\| \sum_k 2^{\sigma_d k} \left| \sum_{l>k-3} P_l h P_{[l-3, l+3]} \partial_x^2 \psi \right| \right\|_{L_T^2 L^{\tilde{r}'}} \\ &\lesssim \|h\|_{L_T^2 L^{2(d-1)}} \|\psi\|_{L_T^\infty H^{\sigma_d+2}} + \|\partial_x^2 h\|_{L_T^\infty H^{\sigma_d}} \|\psi\|_{L_T^2 L^{2(d-1)}} \\ &\quad + \left\| \sum_k \sum_{l>k-3} 2^{\sigma_d(k-l)} 2^{\sigma_d l + 2l} \|P_l h\|_{L^2} \|P_{[l-3, l+3]} \psi\|_{L^{2(d-1)}} \right\|_{L_T^2}, \end{aligned} \quad (6.3)$$

where the last term (6.3) can be bounded by

$$\begin{aligned} (6.3) &\lesssim \left\| \sum_k \sum_{l>k-3} 2^{\sigma_d(k-l)} 2^{\sigma_d l + 2l} \|P_l h\|_{L_x^2} \right\|_{L_T^\infty} \|\psi\|_{L_T^2 L^{2(d-1)}} \\ &\lesssim \|\partial_x^2 h\|_{L_T^\infty H^{\sigma_d+\epsilon}} \|\psi\|_{L_T^2 L^{2(d-1)}} \end{aligned}$$

with $\epsilon > 0$ small. Hence, the above calculations combined with (4.4) and (4.2) yield

$$\begin{aligned} \|h\partial_x^2 \psi\|_{L_T^2 W^{\sigma_d, \tilde{r}'}} &\lesssim \|h\|_{L_T^2 L^{2(d-1)}} \|\psi\|_{L_T^\infty H^{\sigma_d+2}} + \|\partial_x^2 h\|_{L_T^\infty H^{\sigma_d+\epsilon}} \|\psi\|_{L_T^2 L^{2(d-1)}} \\ &\lesssim \epsilon_0 \|\psi\|_{L_T^2 W^{\sigma_d, r_d}}. \end{aligned}$$

For the second term $(V + A)\partial_x \psi$, by (4.4) and (4.2) we have

$$\|(V + A)\partial_x \psi\|_{L_T^2 W^{\sigma_d, \tilde{r}'}} \lesssim (\|\partial_x V\|_{L_T^2 H^{s_d}} + \|\partial_x A\|_{L_T^2 H^{s_d}}) \|\psi\|_{L_T^\infty H^{s_d}} \lesssim \epsilon_0 \|\psi\|_{L_T^2 W^{\sigma_d, r_d}}.$$

Finally, for the other terms in $\mathcal{N}$, by (4.7) we also have

$$\begin{aligned} & \|(B + A^2 + VA + \lambda^2)\psi\|_{L_T^2 W^{\sigma_d, \tilde{r}'}} \\ & \lesssim (\|\partial_x B\|_{L_T^2 H^{s_d-1}} + \|\partial_x A\|_{L_T^2 H^{s_d}} + \|\partial_x V\|_{L_T^2 H^{s_d}})(1 + \|\mathcal{S}\|_{L_t^\infty \mathcal{H}^{s_d}}) \|\psi\|_{L_T^\infty H^{s_d}} \\ & \quad + \|\lambda\|_{L_T^2 W^{\sigma_d, r_d}} \|\lambda\|_{L_T^\infty H^{s_d}} \|\psi\|_{L_T^\infty H^{s_d}} \\ & \lesssim \epsilon_0 \|\psi\|_{L_T^2 W^{\sigma_d, r_d}}. \end{aligned}$$

This concludes the proof of the bound (1.12) for $\psi \in L_T^2 W^{\sigma_d, r_d}$.

In order to obtain the bound for $\psi \in L_T^2 W^{1,4}$, by (6.1) and (6.2) we have

$$\|\psi\|_{L_T^2 W^{1,4}} \lesssim \|\psi_0\|_{H^1} + \|\mathcal{N}\|_{L_T^2 W^{1, \frac{4}{3}}}.$$

Using (4.6), (4.4), (4.8) and (2.30) we bound the nonlinear terms by

$$\begin{aligned} \|\mathcal{N}\|_{L_T^2 W^{1,4}} & \lesssim \|h\|_{L_T^2 W^{1,4}} \|\psi\|_{L_T^\infty H^3} \\ & \quad + (\|\partial_x B\|_{L_T^2 L^2} + \|\partial_x A\|_{L_T^2 L^2} + \|\partial_x V\|_{L_T^2 L^2})(1 + \|\mathcal{S}\|_{L_t^\infty \mathcal{H}^{s_d}}) \|\psi\|_{L_T^\infty H^2} \\ & \quad + \|\lambda\|_{L_T^2 W^{1,4}} \|\lambda\|_{L_T^\infty H^{s_d}} \|\psi\|_{L_T^\infty H^1} \\ & \lesssim (C\|\psi_0\|_{H^3})^2 \|\psi\|_{L_T^2 W^{1,4}}. \end{aligned}$$

These imply the bound (1.12) for $\psi \in L_T^2 W^{1,4}$ in dimensions $d = 4$. $\square$

7. ROUGH SOLUTIONS AND SCATTERING

In this section, we use elliptic estimates in section 4, energy estimates (1.11), (5.9) and Strichartz estimates in Proposition 1.3 to prove the improved energy bounds (2.31) for ψ in fractional Sobolev spaces. This closes the proof of Proposition 2.10. As a byproduct, we also obtain the scattering property (1.8).

Here we start with an equivalent definition of H^s . Since in the Hilbertian case all interpolation methods yield the same result, for the H^s norm we will use a characterization which is akin to a Littlewood-Paley decomposition, or to a discretization of the J method of interpolation. Precisely, we have

Lemma 7.1. *Let $0 \leq s \leq N$. Then H^s can be defined as the space of distributions u which admit a representation*

$$u = \sum_{j=0}^{\infty} u_j$$

with the property that the following norm is finite:

$$\| (u_j) \|_s^2 = \sum_{j=0}^{\infty} 2^{2j(s+1)} \|u_j\|_{H^{-1}}^2 + 2^{2j(s-N)} \|u_j\|_{H^N}^2$$

and with equivalent norm defined as

$$\|u\|_{H^s}^2 = \inf \| (u_j) \|_s^2, \quad (7.1)$$

where the infimum is taken with respect to all representations as above.

7.1. Regularized data. Consider an initial data $\psi_0 \in H^{s_d}$ small, and let $\{c_k\}_{k \geq 0}$ be a sharp frequency envelope for ψ_0 in H^{s_d} . In fact, the c_k can be defined as

$$c_k = 2^{-\delta k} \|\psi_0\|_{H^{s_d}} + \max_k 2^{-\delta|j-k|} \|S_j \psi_0\|_{H^{s_d}},$$

where constant δ only depends on s and the dimension d . For ψ_0 we consider a family of regularizations at frequencies $\lesssim 2^k$, i.e.

$$\psi_0^{(k)} := S_{\leq k} \psi_0 \in H^\infty := \cap_{j=0}^\infty H^j,$$

where k is a dyadic frequency parameter. This parameter can be taken either discrete or continuous, depending on whether we have access to difference bounds or only to the linearized equation. Suppose we work with differences. Then the family $\psi_0^{(k)}$ can be taken to have similar properties to Littlewood-Paley truncations:

i) Uniform bounds:

$$\|S_j \psi_0^{(k)}\|_{H^{s_d}} \lesssim c_j.$$

ii) High frequency bounds: for $\sigma > \delta$,

$$\|\psi_0^{(k)}\|_{H^{s_d+\sigma}} \lesssim 2^{\sigma k} c_k. \quad (7.2)$$

iii) Difference bounds:

$$\|\psi_0^{(k+1)} - \psi_0^{(k)}\|_{H^{-1}} \lesssim 2^{-(s_d+1)k} c_k. \quad (7.3)$$

iv) Limit as $k \rightarrow \infty$:

$$\psi_0 = \lim_{k \rightarrow \infty} \psi_0^{(k)} \quad \text{in } H^{s_d}.$$

Correspondingly, we obtain a family of smooth solutions $\psi^{(k)}$.

7.2. Uniform bounds. Corresponding to the above family of regularized data, we obtain a family of smooth solutions $\psi^{(k)}$ on $[0, T_n]$ for $T_n > 1$ by Theorem 2.8. For this we can use the energy estimates (1.11) to propagate Sobolev regularity for solutions as well as difference bounds as in Proposition 5.2. Using induction we will prove that the solutions $\psi^{(k)}$ are global as follows:

(i) We prove that the solution $\psi^{(0)}$ is global. By local well-posedness in Theorem 2.8, let T_0 be

$$T_0 = \sup_T \left\{ T : \|\psi^{(0)}\|_{S[0,T]} + \|\psi^{(0)}\|_{L_T^\infty H^{s_d}} \leq C_0 \|\psi_0\|_{H^{s_d}} \right\}.$$

Then on the interval $[0, T_0]$, by Proposition 1.3 we have

$$\|\psi^{(0)}\|_{S[0,T]} \leq 2C_2 \|\psi_0^{(0)}\|_{H^{s_d}} \leq 2C_2 \|\psi_0\|_{H^{s_d}}.$$

Using (5.9) and (1.11) we have

$$\begin{aligned} \|\psi^{(0)}\|_{L_{T_0}^\infty H^{s_d}} &\leq \|\psi^{(0)}\|_{L_{T_0}^\infty H^{-1}} + \|\psi^{(0)}\|_{L_{T_0}^\infty H^N} \\ &\leq (1 - C_{lin} C_0^2 \epsilon_0^2)^{-1} C_{lin} \|\psi_0\|_{H^{-1}} + C_1 e^{C_E C_0^2 \epsilon_0^2} \|\psi_0\|_{H^{s_d}} \\ &\leq \frac{C_0}{2} \|\psi_0\|_{H^{s_d}}. \end{aligned}$$

Here we set

$$C_0 > 4C_1 + 4C_2 + 4C_{lin}, \quad (7.4)$$

and choose ϵ_0 to be sufficiently small such that

$$(C_E + C_{lin}) C_0^2 \epsilon_0^2 < \frac{1}{4}. \quad (7.5)$$

This implies that the solution can be extended, and thus the lifespan is $T_0 = \infty$.

(ii) We prove that the solutions $\psi^{(k)}$ for any k are global. By local well-posedness in Theorem 2.8, let T_k be

$$T_k = \sup_T \left\{ T : \|\psi^{(k)}\|_{S[0,T]} + \|\psi^{(k)}\|_{L_T^\infty H^{s_d}} \leq C_0 \|\psi_0\|_{H^{s_d}} \right\}.$$

Then on the interval $[0, T_k]$, by Proposition 1.3 we have the improved Strichartz estimates in $S[0, T]$. We then prove the improved energy estimates for $\psi^{(k)}$.

By (i) we assume that $\psi^{(l)}$ for $l \leq k-1$ are global. Then we have two properties as follow:

a) High frequency bounds:

$$\|\psi^{(l)}\|_{C[0, T_k; H^{N_1}]} \leq C_1 e^{C_E C_0 \epsilon_0} 2^{(N_1 - s_d)l} c_l, \quad 0 \leq l \leq k, \quad s_d < N_1 \in \mathbb{N}. \quad (7.6)$$

b) Difference bounds:

$$\|\psi^{(l+1)} - \psi^{(l)}\|_{C[0, T_k; H^{-1}]} \leq 2C_{lin} 2^{-(s_d+1)l} c_l, \quad 0 \leq l \leq k-1. \quad (7.7)$$

The first bound is obtained from (1.11) and (7.2). The second bound (7.7) is obtained by Proposition 5.2, Proposition 1.3 and (7.3). Indeed,

$$\begin{aligned} &\|\psi^{(l+1)} - \psi^{(l)}\|_{C[0, T_k; H^{-1}]} \\ &\leq C_{lin} \|\psi_0^{(l+1)} - \psi_0^{(l)}\|_{H^{-1}} + C_{lin} \|(\psi^{(l+1)}, \psi^{(l)})\|_{S[0, T_k]}^2 \|\psi^{(l+1)} - \psi^{(l)}\|_{C[0, T_k; H^{-1}]} \\ &\leq C_{lin} 2^{-(s+1)l} c_l + 2C_{lin} (C_0 \epsilon_0)^2 \|\psi^{(l+1)} - \psi^{(l)}\|_{C[0, T_k; H^{-1}]}. \end{aligned}$$

Interpolating the two estimates (7.6) and (7.7), we obtain

$$\|\psi^{(l+1)} - \psi^{(l)}\|_{C[0, T_k; H^N]} \leq \max\{2C_{lin}, C_1 e^{C_E C_0^2 \epsilon_0^2}\} 2^{-(s_d - N)l} c_l, \quad 0 < N < N_1. \quad (7.8)$$

We use these bounds to establish uniform frequency envelope bounds for $\psi^{(k)}$,

$$\begin{aligned}
\|\psi^{(k)}\|_{C[0,T_k;H^{s_d}]}^2 &\leq \|\psi^{(0)}\|_{C[0,T_k;H^{-1}]}^2 + \sum_{l=1}^{k-1} 2^{2(s_d+1)l} \|\psi^{(l+1)} - \psi^{(l)}\|_{C[0,T_k;H^{-1}]}^2 \\
&\quad + \|\psi^{(0)}\|_{C[0,T_k;H^N]}^2 + \sum_{l=1}^{k-1} 2^{2(s_d-N)l} \|\psi^{(l+1)} - \psi^{(l)}\|_{C[0,T_k;H^N]}^2 \\
&\leq \sum_{l=0}^{k-1} (2C_{lin}c_l)^2 + \sum_{l=0}^{k-1} (\max\{2C_{lin}, C_1 e^{C_E C_0^2 \epsilon_0^2}\})^2 c_l^2 \\
&\leq \sum_{l=0}^{k-1} \left(\frac{C_0}{2} c_l\right)^2 \leq \left(\frac{C_0}{2} \|\psi_0\|_{H^{s_d}}\right)^2.
\end{aligned}$$

Here C_0 and ϵ_0 are chose as (7.4), (7.5) respectively. This implies that the solutions $\psi^{(k)}$ are also global.

Now consider the convergence of solutions $\psi^{(k)}$ in $C(\mathbb{R}; H^{s_d})$ as $k \rightarrow \infty$. From the difference bounds (7.7) we obtain convergence in H^{-1} to a limit $\psi \in C[0, \infty; H^{-1}]$, with

$$\|\psi - \psi^{(k)}\|_{C(\mathbb{R}; H^{-1})} \leq \sum_{l=k}^{\infty} \|\psi^{(l+1)} - \psi^{(l)}\|_{C(\mathbb{R}; H^{-1})} \leq \sum_{l=k}^{\infty} 2^{-(s_d+1)l} c_l \lesssim 2^{-(s+1)k}.$$

On the other hand, expanding the difference as a telescopic sum, where, in view of the above bounds (7.6) and (7.7), each summand is essentially concentrated at frequency 2^l , with H^{s_d} size c_l and exponentially decreasing tails. By the equivalent norm (7.1), (7.7) and (7.8) we have

$$\begin{aligned}
\|\psi - \psi^{(k)}\|_{C(\mathbb{R}; H^{s_d})}^2 &\leq \sum_{l=k}^{\infty} 2^{2(s_d+1)l} \|\psi^{(l+1)} - \psi^{(l)}\|_{C(\mathbb{R}; H^{-1})}^2 \\
&\quad + \sum_{l=k}^{\infty} 2^{2(s_d-N)l} \|\psi^{(l+1)} - \psi^{(l)}\|_{C(\mathbb{R}; H^N)}^2 \\
&\lesssim \sum_{l=k}^{\infty} c_l^2 = c_{\geq k},
\end{aligned}$$

so we also have convergence in $C(\mathbb{R}, H^{s_d})$.

Hence, we obtain the solution ψ as the limit of solutions $\psi^{(k)}$, and have the bound

$$\|\psi\|_{C(\mathbb{R}; H^{s_d})} \leq \lim_{k \rightarrow \infty} \|\psi^{(k)}\|_{C(\mathbb{R}; H^{s_d})} \leq \frac{C_0}{2} \|\psi_0\|_{H^{s_d}}.$$

This gives the improved energy bound in Proposition 2.10. The first improved bound in (2.31) for $\psi \in S[0, T]$ is obtained by Proposition 1.3, (7.4) and (7.5). Hence, we complete the proof of Proposition 2.10.

Finally, we prove that scattering holds.

Proposition 7.2 (Scattering). *Let s_d be as in (1.3). There exist $\psi_{\pm} \in H^{s_d-2}$ such that*

$$\lim_{t \rightarrow \pm\infty} \|\psi - e^{it\Delta} \psi_{\pm}\|_{H^{s_d-2}} = 0. \quad (7.9)$$

Proof. It is standard to deduce (7.9) from

$$\|(V_{\alpha} - 2A_{\alpha})\nabla^{\alpha}\psi\|_{L_t^2 W^{s_d-2, r_d}} + \|\mathcal{N}\|_{L_t^2 W^{s_d-2, r_d}} \lesssim 1,$$

which has been proved in Proposition 1.3. So our lemma follows. $\square$

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