NONLINEAR STABILITY FOR THE 2D INCOMPRESSIBLE MHD SYSTEM WITH FRACTIONAL DISSIPATION IN THE HORIZONTAL DIRECTION

WEN FENG[1], WEINAN WANG[2] AND JIAHONG WU[3]

ABSTRACT. This paper focuses on a 2D magnetohydrodynamic (MHD) system with fractional horizontal dissipation in the domain $\Omega = \mathbb{T} \times \mathbb{R}$ with $\mathbb{T} = [0, 1]$ being a periodic box. The goal here is to understand the stability problem on perturbations near any fixed magnetic field $A = (A_1, A_2)$, where $A_1, A_2 \in \mathbb{R}$. Due to the lack of vertical dissipation, this stability problem is difficult. This paper solves the desired stability problem by simultaneously exploiting two smoothing and stabilizing mechanisms: the enhanced dissipation due to the coupling between the velocity and the magnetic fields, and the strong Poincaré type inequalities for the oscillation part of the solution, namely the difference between the solution and its horizontal average. In addition, the oscillation part of the solution is shown to converge exponentially to zero in H^2 as $t \to \infty$. As a consequence, the solution converges to its horizontal average asymptotically.

1. INTRODUCTION

Let $\Omega = \mathbb{T} \times \mathbb{R}$ with $\mathbb{T} = [0, 1]$ being a one-dimensional (1D) periodic domain and $\mathbb{R}$ being the real line. Consider the 2D incompressible magnetohydrodynamic (MHD) equations with horizontal fractional dissipation

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P + \nu \Lambda_1^{2\alpha} u = B \cdot \nabla B, \\ \partial_t B + u \cdot \nabla B + \eta \Lambda_1^{2\beta} B = B \cdot \nabla u, \\ \nabla \cdot u = \nabla \cdot B = 0, \end{cases} \tag{1.1}$$ E:MHD1

where $\alpha \geq 0$ and $\beta \geq 0$. Here u represents the velocity field, P the total pressure and B the magnetic field, and ν and η denote the viscosity and the magnetic damping coefficients, respectively. The fractional partial operator $\Lambda_1^{2\alpha}$ is defined by the Fourier transform

$$\widehat{\Lambda_1^{2\alpha} f}(\xi) = \xi_1^{2\alpha} \hat{f}(\xi).$$

In particular, $\Lambda_1^{2\alpha}$ with $\alpha = 0$ becomes the identity operator.

(1.1) admits a special class of steady-state solutions represented by the background magnetic field. Attention is focused on the steady-state solution

$$u^{(0)} = (0, 0), \quad B^{(0)}(x) = A = (A_1, A_2),$$

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where A_1 and A_2 are arbitrarily fixed real numbers. The perturbation (u, b) around this steady solution with $b = B - A$ obeys

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P + \nu \Lambda_1^{2\alpha} u = b \cdot \nabla b + A \cdot \nabla b, \\ \partial_t b + u \cdot \nabla b + \eta \Lambda_1^{2\beta} b = b \cdot \nabla u + A \cdot \nabla u, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x,0) = u_0(x), \ b(x,0) = b_0(x). \end{cases} \tag{1.2}$$ E:MHD2

The corresponding vorticity $\omega = \nabla \times u$ and the current density $j = \nabla \times b$ satisfy

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega + \nu \Lambda_1^{2\alpha} \omega = b \cdot \nabla j + A \cdot \nabla j, \\ \partial_t j + u \cdot \nabla j + \eta \Lambda_1^{2\beta} j = b \cdot \nabla \omega + Q + A \cdot \nabla \omega \end{cases}$$

with

$$Q = 2\partial_1 b_1 (\partial_2 u_1 + \partial_1 u_2) - 2\partial_1 u_1 (\partial_2 b_1 + \partial_1 b_2).$$

We first remark that the coupling and interaction in the MHD system (1.2) leads to the smoothing and stabilizing in the direction of the background magnetic field A. This is reflected in the wave equations derived from (1.2) via the following simple process. Applying the Helmholtz-Leray projection operator

$$\mathbb{P} := I - \nabla \Delta^{-1} \nabla \cdot$$

to the velocity equation in (1.2), we eliminate the pressure to obtain

$$\partial_t u + \nu \Lambda_1^{2\alpha} u - A \cdot \nabla b = N_1, \qquad N_1 = \mathbb{P}(-u \cdot \nabla u + b \cdot \nabla b). \tag{1.3}$$ ueq

By separating the linear terms from the nonlinear ones in (1.2), the equation of b can be written as

$$\partial_t b + \eta \Lambda_1^{2\beta} b - A \cdot \nabla u = N_2, \qquad N_2 = -u \cdot \nabla b + b \cdot \nabla u. \tag{1.4}$$ beq

Thus, (1.2) can be written as

$$\begin{cases} \partial_t u + \nu \Lambda_1^{2\alpha} u - A \cdot \nabla b = N_1, \\ \partial_t b + \eta \Lambda_1^{2\beta} b - A \cdot \nabla u = N_2, \\ \nabla \cdot u = \nabla \cdot b = 0. \end{cases}$$

Differentiating (1.3) and (1.4) in time and making several substitutions, we find

$$\begin{cases} \partial_{tt} u + (\nu \Lambda_1^{2\alpha} + \eta \Lambda_1^{2\beta}) \partial_t u - ((A \cdot \nabla)^2 u - \eta \nu \Lambda_1^{2\alpha} \Lambda_1^{2\beta} u) = N_3, \\ \partial_{tt} b + (\nu \Lambda_1^{2\alpha} + \eta \Lambda_1^{2\beta}) \partial_t b - ((A \cdot \nabla)^2 b - \eta \nu \Lambda_1^{2\alpha} \Lambda_1^{2\beta} b) = N_4, \end{cases} \tag{1.5}$$ wave

where N_3 and N_4 are given by

$$N_3 = (\partial_t + \eta \Lambda_1^{2\beta}) N_1 + (A \cdot \nabla) N_2, \qquad N_4 = (\partial_t + \nu \Lambda_1^{2\alpha}) N_2 + (A \cdot \nabla) N_1.$$

Both u and b are found to satisfy nonhomogeneous wave type equations with exactly the same linear parts. Moreover, (1.5) exhibits much more regularization than its original counterpart in (1.2). In particular, the terms $(A \cdot \nabla)^2 u$ and $(A \cdot \nabla)^2 b$ generate the smoothing and stability in the direction of A. Together with the fractional horizontal dissipation in (1.2), this allows us to control the nonlinearity. This explains the mechanism of the stability for this anisotropically dissipated MHD system. We remark that the stabilizing effect of the magnetic field on electrically conducting fluids have been observed in physical experiments and numerical simulations (see, e.g., [1–3, 12–14, 27, 28]).

In order to understand the desired stability, we need to distinguish the horizontal zeroth Fourier mode and the rest of the horizontal modes. The spatial domain here is $\Omega = \mathbb{T} \times \mathbb{R}$ and we take full advantage of the geometry of this domain. The horizontal direction is periodic and we can separate the zeroth Fourier mode from the non-zero ones. The zeroth Fourier mode corresponds to the horizontal average. This hints the decomposition of the physical quantities into the horizontal averages and the corresponding oscillation parts. More precisely, for a function f that is integrable in $x \in \mathbb{T}$, we define

$$\bar{f}(x_2) = \int_{\mathbb{T}} f(x_1, x_2)\, dx_1, \qquad f = \bar{f} + \widetilde{f}.$$

This decomposition is orthogonal in the Sobolev space $H^k(\Omega)$ for any integer $k \geq 0$ (see Lemma 2.3 in Section 2). More crucially, we prove in this paper that the oscillation part $\widetilde{f}$ obeys very general strong Poincaré type inequalities, for any $\sigma \geq 0$,

$$\|\widetilde{f}\|_{L^2(\Omega)} \leq C\, \|\Lambda_1^\sigma \widetilde{f}\|_{L^2(\Omega)}$$

and, for any $\gamma > 0$,

$$\|\widetilde{f}\|_{L^\infty(\Omega)} \leq C\, \|\Lambda_1^\gamma \widetilde{f}\|_{H^1(\Omega)}.$$

Detailed statements and proofs can be found in Lemma 2.1 in the subsequent section. These inequalities allow us to control some of the nonlinear parts in terms of the horizontal dissipation. By invoking the decompositions

$$u = \bar{u} + \widetilde{u}, \quad b = \bar{b} + \widetilde{b}$$

and applying the aforementioned Poincaré inequalities together with various anisotropic inequalities, we are able to successfully bound the nonlinearity and establish the following stability result.

onlinearstability⟩ **Theorem 1.1.** *Let η, $\nu > 0$, $\alpha > 0$ and $\beta > 0$. Consider (1.2) with the initial data $(u_0, b_0) \in H^3(\Omega)$, and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Then, there exists a constant $\varepsilon_0 := \varepsilon_0(\nu, \eta) > 0$ such that if $\varepsilon \leq \varepsilon_0$ and*

$$\|u_0\|_{H^3} + \|b_0\|_{H^3} \leq \varepsilon,$$

then the global classical solution $(u, b) \in C(0, \infty; H^3)$ satisfying, for any $t > 0$,

$$\|u(t)\|_{H^3}^2 + \|b(t)\|_{H^3}^2 + \int_0^t \left(\|\Lambda_1^\alpha u\|_{H^3}^2 + \|\Lambda_1^\beta b\|_{H^3}^2 \right)\, d\tau \leq C\, \varepsilon^2$$

for some universal constant.

We remark that the existence of solution (u, b) in Theorem 1.1 can be established following a standard procedure. The first step is to establish the local-in-time existence via an approximation procedure and local energy estimates. This step doesn't require any dissipation and works even for inviscid equations. One can mimic the details on the local existence proof on solutions to the Navier-Stokes and the Euler equations (see Chatper 3 of [44]). The second step is to establish the global existence of solutions by combining the local existence result with the global *a priori* bound obtained in the proof of Theorem 1.1.

Besides the stability, we can also show that the oscillation part actually decays exponentially to zero.

⟨exp_decay⟩ **Theorem 1.2.** *Let $u_0, b_0 \in H^3(\Omega)$ with $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Assume that $\|u_0\|_{H^3} + \|b_0\|_{H^3} \le \varepsilon$ for sufficiently small $\varepsilon > 0$. Let (u, b) be the corresponding solution of (1.2). Then the H^2 norm of the oscillation part $(\widetilde{u}, \widetilde{b})$ decays exponentially in time,*

$$\|\widetilde{u}(t)\|_{H^2} + \|\widetilde{b}(t)\|_{H^2} \le C(\|u_0\|_{H^2} + \|b_0\|_{H^2})e^{-Ct}, \tag{1.6}$$ dec

for some constant $C > 0$ and for all $t > 0$.

Due to physical applications and mathematical importance, global regularity and stability problems on the MHD equations with partial dissipation has attracted wide attention and there have been substantial recent developments. The pioneering work of Duvaut and Lions [21] established the local existence of classical solutions to the MHD equations with full dissipation while Sermange and Temam [49] obtained the global existence of weak solutions. The situation when the MHD equations involve only partial or fractional dissipation is more subtle. The global existence and regularity has been obtained for the 2D MHD equations with various partial dissipation in many different functional settings (see, e.g., [8–10, 20, 22, 23, 34, 37, 38, 58, 64]). The global regularity problem on the MHD equations with fractional dissipation was investigated in [16, 17, 65–67]. Studies on the stability problem concerning the MHD equations near a background magnetic field or other steady state solutions have flourished, and significant progress has been made for many partially dissipated MHD systems (see, e.g., [4–7, 11, 15, 24–26, 29–31, 33, 35, 36, 39, 41, 42, 46–48, 50, 53, 55–57, 60, 63, 68–70]). Considerable efforts have also been devoted to the MHD boundary layer problem (see, e.g., [43]) and the compressible MHD systems (see, e.g., [32,59,61]). The stability and large-time behavior problem on the 2D anisotropic Navier-Stokes equations with dissipation in only one direction in the domain $\mathbb{T} \times \mathbb{R}$ was first successfully solved in [18]. A systematic method with all necessary techniques were developed in [18] to tackle such problems. This approach was then used to establish the stability and exponential results on the 2D Boussinesq as well as the 3D Boussinesq equations [19, 62]. This approach was also efficient in dealing with the stability problems on the 2D MHD equations when the dissipation and the magnetic diffusion are in the same one direction [45, 52]. In addition, [40] considered the stability of the 2D MHD system with partial mixed velocity dissipation and horizontal magnetic diffusion. In comparison with [45], [52] and [40], the main contributions of this paper are that it allows any fractional horizontal dissipation and the background can be any fixed 2D vector. New tools are developed in this paper to handle this very general type of fractional horizontal dissipation. In particular, sharp strong Poincaré type inequalities for the oscillation part are derived (Lemma 2.1).

We use the bootstrapping argument (see, e.g., [54]) to prove the nonlinear stability in Theorem 1.1. We first define a suitable energy functional

$$E(t) := \sup_{0\le\tau\le t}(\|u(t)\|_{H^3}^2 + \|b(t)\|_{H^3}^2) + 2\nu\int_0^t \|\Lambda_1^\alpha u(\tau)\|_{H^3}^2 d\tau + 2\eta\int_0^t \|\Lambda_1^\beta b(\tau)\|_{H^3}^2 d\tau.$$

Here $E(t)$ represents the standard energy consisting of the H^3-norm of (u, b) and the associated time integrals parts in u and b. Our main efforts are devoted to proving that, for any $t > 0$,

$$E(t) \le E(0) + CE(t)^{3/2}. \tag{1.7}$$ bootstrap

Once we have (1.7), the bootstrapping argument implies that if

$$\|u_0\|_{H^3} + \|b_0\|_{H^3} \le \varepsilon \quad \text{or} \quad E(0) \le \varepsilon^2,$$

then there exists a constant $C > 0$ such that

$$E(t) \leq C\varepsilon^2, \quad \forall\, t \geq 0.$$

The main efforts are devoted to proving (1.7). The proof makes use of the aforementioned orthogonal decomposition and Poincaré type inequalities derived in this paper. The exponential decay result is shown by making use of the evolution equations of the oscillation part of the stable solution.

The rest of this paper is divided into three sections. The second section serves as a preparation for the proof of Theorem 1.1. It provides properties related to the decomposition and triple product estimates for the domain Ω. We also derive the sharp strong Poincaré type inequalities for the oscillation part. The third section is devoted to the proof of Theorem 1.1 while the Section 4 proves Theorem 1.2.

2. Preliminaries

⟨pre⟩ This section states several properties on the decomposition defined in the introduction and provides several anisotropic inequalities to be used in the proofs of Theorems 1.1. Some of the materials presented here can be found in [9, 18, 19]. But the result and the proof of Lemma 2.1 are new.

We recall the definition of the horizontal average and the oscillation part. Let $\Omega = \mathbb{T}\times\mathbb{R}$ and let $f = f(x_1, x_2)$ with $(x_1, x_2) \in \Omega$ be sufficiently smooth, say integrable in $x_1 \in \mathbb{T}$. The horizontal average $\overline{f}$ is given by

$$\overline{f}(x_2) = \int_{\mathbb{T}} f(x_1, x_2)\, dx_1. \tag{2.1}$$ E:fbar

We decompose f into $\overline{f}$ and the oscillation portion $\widetilde{f}$,

$$f = \overline{f} + \widetilde{f}. \tag{2.2}$$ decomp

The oscillation part obeys the following Poincaré type inequalities. This lemma significantly sharps the corresponding ones in [19]. We no longer require $\sigma = \gamma = 1$. The proof presented here is completely different and new.

⟨E:p1estimate⟩⟨pp⟩ **Lemma 2.1.** *Let $\Omega = \mathbb{T}\times\mathbb{R}$ and let $\widetilde{f}$ be defined as above. Then, for any $\sigma \geq 0$,*

$$\|\widetilde{f}\|_{L^2(\Omega)} \leq \|\Lambda_1^\sigma \widetilde{f}\|_{L^2(\Omega)}.$$

For any $\gamma > 0$,

$$\|\widetilde{f}\|_{L^\infty(\Omega)} \leq C\|\Lambda_1^\gamma \widetilde{f}\|_{H^1(\Omega)},$$

where

$$C := \sqrt{\pi}\left(\sum_{k\in\mathbb{Z}, k\neq 0} |k|^{-2\gamma}(1+k^2)^{-\frac{1}{2}}\right)^{\frac{1}{2}} < \infty.$$

Proof of Lemma 2.1. These inequalities can be shown by the Fourier transform.

$$\|\widetilde{f}\|_{L^2(\Omega)}^2 = \sum_{k\in\mathbb{Z}, k\neq 0} \int_{\mathbb{R}} |\widehat{f}(k,\eta)|^2 d\eta \leq \sum_{k\in\mathbb{Z}, k\neq 0} \int_{\mathbb{R}} |k|^\sigma |\widehat{f}(k,\eta)|^2 d\eta = \|\Lambda_1^\sigma \widetilde{f}\|_{L^2(\Omega)}^2,$$

where $\widehat{f}(k,\eta)$ is the Fourier transform of f,

$$\widehat{f}(k,\eta) = \int_{\mathbb{R}}\int_{\mathbb{T}} f(x,y)\, e^{ikx+i\eta y}\, dxdy. \tag{2.3}$$ ff

By the definition of the Fourier transform,

$$\begin{aligned}
\|\widetilde{f}\|_{L^\infty(\Omega)} &\le \sum_{k\in\mathbb{Z},k\neq 0} \int_{\mathbb{R}} |\widehat{f}(k,\eta)|d\eta \\
&= \sum_{k\in\mathbb{Z},k\neq 0} |k|^{-\gamma} \int_{\mathbb{R}} (1+k^2+\eta^2)^{-\frac12} |k|^{\gamma}(1+k^2+\eta^2)^{\frac12}\, |\widehat{f}(k,\eta)|d\eta \\
&\le \sum_{k\in\mathbb{Z},k\neq 0} |k|^{-\gamma} \left(\int_{\mathbb{R}} \frac{1}{1+k^2+\eta^2}\, d\eta\right)^{\frac12} \\
&\qquad \times \left(\int_{\mathbb{R}} |k|^{2\gamma}(1+k^2+\eta^2)\, |\widehat{f}(k,\eta)|^2 d\eta\right)^{\frac12}.
\end{aligned}$$

The first integral can be computed as follows. By setting $\eta = (1+k^2)^{\frac12}\xi$,

$$\int_{\mathbb{R}} \frac{1}{1+k^2+\eta^2}\, d\eta = (1+k^2)^{-\frac12} \int_{\mathbb{R}} \frac{1}{1+\xi^2} d\xi = \pi(1+k^2)^{-\frac12}.$$

Therefore,

$$\begin{aligned}
\|\widetilde{f}\|_{L^\infty(\Omega)} &\le \sqrt{\pi} \sum_{k\in\mathbb{Z},k\neq 0} |k|^{-\gamma}(1+k^2)^{-\frac14} \left(\int_{\mathbb{R}} |k|^{2\gamma}(1+k^2+\eta^2)\, |\widehat{f}(k,\eta)|^2 d\eta\right)^{\frac12} \\
&\le \sqrt{\pi} \left(\sum_{k\in\mathbb{Z},k\neq 0} |k|^{-2\gamma}(1+k^2)^{-\frac12}\right)^{\frac12} \\
&\qquad \times \left(\sum_{k\in\mathbb{Z},k\neq 0} \int_{\mathbb{R}} |k|^{2\gamma}(1+k^2+\eta^2)\, |\widehat{f}(k,\eta)|^2 d\eta\right)^{\frac12} \\
&= C\, \|\Lambda_1^{\gamma}\widetilde{f}\|_{H^1(\Omega)},
\end{aligned}$$

where, for $\gamma > 0$,

$$C := \sqrt{\pi} \left(\sum_{k\in\mathbb{Z},k\neq 0} |k|^{-2\gamma}(1+k^2)^{-\frac12}\right)^{\frac12} < \infty.$$

This completes the proof of Lemma 2.1. □

The following lemma is a direct consequence of (2.1) and (2.2).

⟨E:decomp⟩ **Lemma 2.2.** *The average operator and the oscillation operator commute with the partial derivatives, for $i = 1, 2$,*

$$\partial_i \overline{f} = \overline{\partial_i f}, \quad \partial_i \widetilde{f} = \widetilde{\partial_i f}, \quad \partial_1 \overline{f} = 0, \quad \overline{\widetilde{f}} = 0,$$

As a special consequence, if $\nabla\cdot f = 0$, then

$$\nabla\cdot\overline{f} = 0, \quad \nabla\cdot\widetilde{f} = 0.$$

The second lemma states that the decomposition in (2.2) is orthogonal in any homogeneous Sobolev space $\dot{H}^k(\Omega)$. Here $\dot{H}^k(\Omega)$ is defined as follows. For any $s \in \mathbb{R}$, the homogeneous Sobolev space $\dot{H}^k(\Omega)$ consists of square integrable functions on Ω such that

$$\|f\|_{\dot{H}^s(\Omega)} = \left(\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} (k^2+\eta^2)^s \, |\widehat{f}(k,\eta)|^2 d\eta \right)^{\frac{1}{2}} < \infty,$$

where $\widehat{f}(k,\eta)$ denotes the Fourier transform of f defined in (2.3). More details on the proof of the following lemma can be found in [19].

⟨or⟩ **Lemma 2.3.** *Let $\Omega = \mathbb{T}\times\mathbb{R}$. Let $k \geq 0$ be an integer. Let $f \in \dot{H}^k(\Omega)$. Then $\overline{f}$ and $\widetilde{f}$ are orthogonal in $\dot{H}^k(\Omega)$, namely*

$$(\overline{f}, \widetilde{f})_{\dot{H}^k} := \int_\Omega D^k\overline{f}\cdot D^k\widetilde{f}\; dx = 0. \qquad \|f\|^2_{\dot{H}^k(\Omega)} = \|\overline{f}\|^2_{\dot{H}^k(\Omega)} + \|\widetilde{f}\|^2_{\dot{H}^k(\Omega)}$$

In particular, $\|\overline{f}\|_{\dot{H}^k} \leq \|f\|_{\dot{H}^k}$ *and* $\|\widetilde{f}\|_{\dot{H}^k} \leq \|f\|_{\dot{H}^k}$.

Next we present several anisotropic inequalities. Anisotropic upper bounds for triple products are frequently used to bound the nonlinear terms when only partial dissipation is present. In the case when the spatial domain is the whole space $\mathbb{R}^2$, Cao and Wu [9] showed and applied the following inequality

$$\left|\int_{\mathbb{R}^2} f\,g\,h\right| \leq C\,\|f\|^{\frac12}_{L^2(\mathbb{R}^2)}\,\|\partial_1 f\|^{\frac12}_{L^2(\mathbb{R}^2)}\,\|g\|^{\frac12}_{L^2(\mathbb{R}^2)}\,\|\partial_2 g\|^{\frac12}_{L^2(\mathbb{R}^2)}\,\|h\|_{L^2(\mathbb{R}^2)}. \tag{2.4}$$ oo

In fact, (2.4) is a consequence of the elementary 1D inequality

$$\|f\|_{L^\infty(\mathbb{R})} \leq \sqrt{2}\|f\|^{\frac12}_{L^2(\mathbb{R})}\,\|f'\|^{\frac12}_{L^2(\mathbb{R})}. \tag{2.5}$$ jj

Another consequence of (2.5) is the following inequality

$$\|f\|_{L^\infty(\mathbb{R}^2)} \leq C\|f\|^{\frac14}_{L^2(\mathbb{R}^2)}\,\|\partial_1 f\|^{\frac14}_{L^2(\mathbb{R}^2)}\|\partial_2 f\|^{\frac14}_{L^2(\mathbb{R}^2)}\,\|\partial_1\partial_2 f\|^{\frac14}_{L^2(\mathbb{R}^2)}.$$

When the 1D spatial domain is a bounded domain, say $\mathbb{T}$,

$$\|f\|_{L^\infty(\mathbb{T})} \leq C\,\|f\|^{\frac12}_{L^2(\mathbb{T})}\left(\|f\|_{L^2(\mathbb{T})} + \|f'\|_{L^2(\mathbb{T})}\right)^{\frac12}.$$

Since the oscillation part $\widetilde{f}$ has mean zero, for $\widetilde{f} \in H^1(\mathbb{T})$,

$$\|\widetilde{f}\|_{L^\infty(\mathbb{T})} \leq C\|\widetilde{f}\|^{\frac12}_{L^2(\mathbb{T})}\|(\widetilde{f})'\|^{\frac12}_{L^2(\mathbb{T})}.$$

As a consequence of these elementary inequalities, the following two lemmas hold. Complete proofs of the following two lemmas can be found in [18, 19].

⟨E:triple1⟩ **Lemma 2.4.** *Let $\Omega = \mathbb{T}\times\mathbb{R}$. For any $f, g, h \in L^2(\Omega)$ with $\partial_1 f \in L^2(\Omega)$ and $\partial_2 g \in L^2(\Omega)$, then*

$$\int_\Omega |fgh|\; dx \leq C\|f\|^{\frac12}_{L^2}(\|f\|_{L^2} + \|\partial_1 f\|_{L^2})^{\frac12}\|g\|^{\frac12}_{L^2}\|\partial_2 g\|^{\frac12}_{L^2}\|h\|_{L^2}.$$

For any $f \in H^2(\Omega)$, we have

$$\begin{aligned}\|f\|_{L^\infty}(\Omega) \leq\ & C\|f\|^{\frac14}_{L^2(\Omega)}(\|f\|_{L^2(\Omega)} + \|\partial_1 f\|_{L^2(\Omega)})^{\frac14}\|\partial_2 f\|^{\frac14}_{L^2(\Omega)}\\ & \times(\|\partial_2 f\|_{L^2(\Omega)} + \|\partial_1\partial_2 f\|_{L^2(\Omega)})^{\frac14}.\end{aligned}$$

After replacing f by the oscillation part, we have the following inequalities.

⟨E:triple2⟩ **Lemma 2.5.** *Let $\Omega = \mathbb{T}\times\mathbb{R}$. For any $f, g, h \in L^2(\Omega)$ with $\partial_1 f \in L^2(\Omega)$ and $\partial_2 g \in L^2(\Omega)$, then*

$$\int_\Omega |\widetilde{f} g h| \, dx \le C \|\widetilde{f}\|_{L^2}^{\frac12} \|\partial_1 \widetilde{f}\|_{L^2}^{\frac12} \|g\|_{L^2}^{\frac12} \|\partial_2 g\|_{L^2}^{\frac12} \|h\|_{L^2}.$$

For any $f \in H^2(\Omega)$, we have

$$\|\widetilde{f}\|_{L^\infty(\Omega)} \le C \|\widetilde{f}\|_{L^2(\Omega)}^{\frac14} \|\partial_1 \widetilde{f}\|_{L^2(\Omega)}^{\frac14} \|\partial_2 \widetilde{f}\|_{L^2(\Omega)}^{\frac14} \|\partial_1\partial_2 \widetilde{f}\|_{L^2(\Omega)}^{\frac14}.$$

Finally, we present a bound for triple products that repeatedly appear in the proof of Theorem 1.1.

⟨refs⟩ **Lemma 2.6.** *Let f, g, h be of sufficient regularity. Then, for any σ_1, σ_2, $\sigma_3 > 0$, there exists a universal (in terms of f, g, h) constant $C > 0$, such that*

$$\begin{aligned}
&\sum_{k=1}^{3} C_3^k \int_\Omega \partial_2^k f \cdot \partial_2^{3-k} \nabla g \cdot \partial_2^3 h \, dx \\
&\le C\Big(\|f\|_{H^3} \|\Lambda_1^{\sigma_2} g\|_{H^3} \|\Lambda_1^{\sigma_3} h\|_{H^3} + \|g\|_{H^3} \|\Lambda_1^{\sigma_1} h\|_{H^3} \|\Lambda_1^{\sigma_3} h\|_{H^3} \\
&\qquad + \|h\|_{H^3} \|\Lambda_1^{\sigma_1} f\|_{H^3} \|\Lambda_1^{\sigma_2} g\|_{H^3}\Big).
\end{aligned}$$

Proof of Lemma 2.6. We write

$$f = \overline{f} + \widetilde{f}, \quad g = \overline{g} + \widetilde{g}, \quad h = \overline{h} + \widetilde{h}$$

and use the simple fact that the integral of any triple product with two averages is zero, namely

$$\int_\Omega \partial_2^k \overline{f} \cdot \partial_2^{3-k} \nabla \overline{g} \cdot \partial_2^3 \widetilde{h} \, dx = 0,$$

we obtain

$$\begin{aligned}
\sum_{k=1}^{3} C_3^k \int_\Omega \partial_2^k f \cdot \partial_2^{3-k} \nabla g \cdot \partial_2^3 h \, dx =& \sum_{k=1}^{3} C_3^k \int_\Omega \partial_2^k \overline{f} \cdot \partial_2^{3-k} \nabla \widetilde{g} \cdot \partial_2^3 \widetilde{h} \, dx \\
&+ \sum_{k=1}^{3} C_3^k \int_\Omega \partial_2^k \widetilde{f} \cdot \partial_2^{3-k} \nabla \overline{g} \cdot \partial_2^3 \widetilde{h} \, dx \\
&+ \sum_{k=1}^{3} C_3^k \int_\Omega \partial_2^k \widetilde{f} \cdot \partial_2^{3-k} \nabla \widetilde{g} \cdot \partial_2^3 \overline{h} \, dx \\
&+ \sum_{k=1}^{3} C_3^k \int_\Omega \partial_2^k \widetilde{f} \cdot \partial_2^{3-k} \nabla \widetilde{g} \cdot \partial_2^3 \widetilde{h} \, dx. \qquad (2.6)
\end{aligned}$$
ju1

The first term can be explicitly written as

$$\begin{aligned}
\sum_{k=1}^{3} C_3^k \int_\Omega \partial_2^k \overline{f} \cdot \partial_2^{3-k} \nabla \widetilde{g} \cdot \partial_2^3 \widetilde{h} \, dx =& 3\int_\Omega \partial_2 \overline{f} \cdot \partial_2^2 \nabla \widetilde{g} \cdot \partial_2^3 \widetilde{h} \, dx + 3\int_\Omega \partial_2^2 \overline{f} \cdot \partial_2 \nabla \widetilde{g} \cdot \partial_2^3 \widetilde{h} \, dx \\
&+ \int_\Omega \partial_2^3 \overline{f} \cdot \nabla \widetilde{g} \cdot \partial_2^3 \widetilde{h} \, dx.
\end{aligned}$$

By Lemma 2.1 and Sobolev's inequality,

$$\begin{aligned}\int_\Omega \partial_2\overline{f}\cdot\partial_2^2\nabla\widetilde{g}\cdot\partial_2^3\widetilde{h}\,dx &\le \|\partial_2\overline{f}\|_{L^\infty}\,\|\partial_2^2\nabla\widetilde{g}\|_{L^2}\,\|\partial_2^3\widetilde{h}\|_{L^2}\\ &\le C\,\|f\|_{H^3}\,\|\Lambda_1^{\sigma_2}\partial_2^2\nabla\widetilde{g}\|_{L^2}\,\|\Lambda_1^{\sigma_3}\partial_2^3\widetilde{h}\|_{L^2}\\ &\le C\,\|f\|_{H^3}\,\|\Lambda_1^{\sigma_2}g\|_{H^3}\|\Lambda_1^{\sigma_3}h\|_{H^3}.\end{aligned}$$

Similarly,

$$\begin{aligned}\int_\Omega \partial_2^2\overline{f}\cdot\partial_2\nabla\widetilde{g}\cdot\partial_2^3\widetilde{h}\,dx &\le \|\partial_2^2\overline{f}\|_{L^2}\,\|\partial_2\nabla\widetilde{g}\|_{L^\infty}\,\|\partial_2^3\widetilde{h}\|_{L^2}\\ &\le C\,\|f\|_{H^3}\,\|\Lambda_1^{\sigma_2}\partial_2\nabla\widetilde{g}\|_{H^1}\,\|\Lambda_1^{\sigma_3}\partial_2^3\widetilde{h}\|_{L^2}\\ &\le C\,\|f\|_{H^3}\,\|\Lambda_1^{\sigma_2}g\|_{H^3}\|\Lambda_1^{\sigma_3}h\|_{H^3}\end{aligned}$$

and

$$\begin{aligned}\int_\Omega \partial_2^3\overline{f}\cdot\nabla\widetilde{g}\cdot\partial_2^3\widetilde{h}\,dx &\le \|\partial_2^3\overline{f}\|_{L^2}\,\|\nabla\widetilde{g}\|_{L^\infty}\,\|\partial_2^3\widetilde{h}\|_{L^2}\\ &\le C\,\|f\|_{H^3}\,\|\Lambda_1^{\sigma_2}\nabla\widetilde{g}\|_{H^1}\,\|\Lambda_1^{\sigma_3}\partial_2^3\widetilde{h}\|_{L^2}\\ &\le C\,\|f\|_{H^3}\,\|\Lambda_1^{\sigma_2}g\|_{H^3}\|\Lambda_1^{\sigma_3}h\|_{H^3}.\end{aligned}$$

We thus have shown that the first term on the right-hand side of (2.6) satisfies the desired bound. The other three terms can be bounded very similarly and we omit the details. This completes the proof of Lemma 2.6. □

3. Proof of Theorem 1.1

This section proves Theorem 1.1. Our main efforts are devoted to establishing (1.7).

Due to the equivalence of the norm $\|(u,b)\|_{H^3}$ with the norm $\|(u,b)\|_{L^2}+\|(u,b)\|_{\dot H^3}$, it suffices to estimate the L^2 and homogeneous $\dot H^3$-bound of (u,b). By multiplying (1.2) by (u,b) and integrating over Ω, we have, after integrating by parts and using the divergence free condition,

$$\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2+2\nu\int_0^t\|\Lambda_1^\alpha u\|_{L^2}^2\,d\tau+2\eta\int_0^t\|\Lambda_1^\beta b\|_{L^2}^2\,d\tau=\|u_0\|_{L^2}^2+\|b_0\|_{L^2}^2.$$

To estimate the homogeneous norm $\|(u,b)\|_{\dot H^3}$, we apply $\partial_i^3(i=1,2)$ to (1.2) and then multiply by $(\partial_i^3u,\partial_i^3b)$ and integrate the resulting equation to obtain

$$\begin{aligned}&\frac12\frac{d}{dt}\sum_{i=1}^2(\|\partial_i^3u\|_{L^2}^2+\|\partial_i^3b\|_{L^2}^2)+\sum_{i=1}^2\nu\|\partial_i^3\Lambda_1^\alpha u\|_{L^2}^2+\sum_{i=1}^2\eta\|\partial_i^3\Lambda_1^\beta b\|_{L^2}^2\\ &:=J+K+L+M+N,\end{aligned}\tag{3.1}$$

EQ:JKLMN

where

$$J=\sum_{i=1}^2\int_\Omega \partial_i^3\partial_1 b\cdot\partial_i^3u+\partial_i^3\partial_1u\cdot\partial_i^3b\,dx,$$

$$K=-\sum_{i=1}^2\int_\Omega\partial_i^3(u\cdot\nabla u)\cdot\partial_i^3u\,dx,$$

$$L = \sum_{i=1}^{2} \int_\Omega (\partial_i^3(b\cdot\nabla b) - b\cdot\nabla\partial_i^3 b)\cdot\partial_i^3 u\, dx,$$

$$M = -\sum_{i=1}^{2} \int_\Omega \partial_i^3(u\cdot\nabla b)\cdot\partial_i^3 b\, dx,$$

$$N = \sum_{i=1}^{2} \int_\Omega (\partial_i^3(b\cdot\nabla u) - b\cdot\nabla\partial_i^3 u)\cdot\partial_i^3 b\, dx.$$

By integration by parts, $J = 0$. For the term K, we split K into two terms,

$$\begin{aligned} K &= -\int_\Omega \partial_1^3(u\cdot\nabla u)\cdot\partial_1^3 u\, dx - \int_\Omega \partial_2^3(u\cdot\nabla u)\cdot\partial_2^3 u\, dx, \\ &= K_1 + K_2. \end{aligned}$$

We first estimate K_1.

$$\begin{aligned} K_1 &= -\int_\Omega \partial_1^3(u\cdot\nabla u)\cdot\partial_1^3 u\, dx \\ &= -\sum_{k=1}^{3} C_3^k \int_\Omega \partial_1^k u\cdot\partial_1^{3-k}\nabla u\cdot\partial_1^3 u\, dx - \int_\Omega u\cdot\partial_1^3\nabla u\cdot\partial_1^3 u\, dx \\ &= K_{1,1} + K_{1,2}, \end{aligned}$$

where $C_3^k = \frac{3!}{k!(3-k)!}$ is the binomial coefficient. By Hölder's inequality, Lemma 2.1 and Lemma 2.5,

$$\begin{aligned} K_{1,1} &= 3\int_\Omega \partial_1\widetilde{u}\cdot\partial_1^2\nabla\widetilde{u}\cdot\partial_1^3\widetilde{u}\, dx + 3\int_\Omega \partial_1^2\widetilde{u}\cdot\partial_1\nabla\widetilde{u}\cdot\partial_1^3\widetilde{u}\, dx + \int_\Omega \partial_1^3\widetilde{u}\cdot\nabla u\cdot\partial_1^3\widetilde{u}\, dx \\ &\le C\|\partial_1\widetilde{u}\|_{L^\infty}\|\partial_1^2\nabla\widetilde{u}\|_{L^2}\|\partial_1^3\widetilde{u}\|_{L^2} \\ &\quad + C\|\partial_1^3\widetilde{u}\|_{L^2}\|\partial_1\nabla\widetilde{u}\|_{L^2}^{1/2}\|\partial_1\partial_1\nabla\widetilde{u}\|_{L^2}^{1/2}\|\partial_1^2\widetilde{u}\|_{L^2}^{1/2}\|\partial_1^2\partial_2\widetilde{u}\|_{L^2}^{1/2} \\ &\quad + C\|\nabla u\|_{L^\infty}\|\partial_1^3\widetilde{u}\|_{L^2}^2 \\ &\le C\|u\|_{H^3}\|\Lambda_1^\alpha u\|_{H^3}^2. \end{aligned}$$

By integration by parts and the divergence-free condition,

$$K_{1,2} = -\int_\Omega u\cdot\partial_1^3\nabla u\cdot\partial_1^3 u\, dx = -\frac{1}{2}\int_\Omega u\cdot\nabla(\partial_1^3 u)^2\, dx = 0.$$

It follows that

$$K_1 \le C\|u\|_{H^3}\|\Lambda_1^\alpha u\|_{H^3}^2. \tag{3.2}$$ K1bound

To bound K_2, we further decompose it into four terms,

$$\begin{aligned} K_2 &= -\int_\Omega \partial_2^3(u\cdot\nabla u)\cdot\partial_2^3 u\, dx \\ &= -\sum_{k=0}^{3} C_3^k \int_\Omega \partial_2^k u\cdot\partial_2^{3-k}\nabla u\cdot\partial_2^3 u\, dx \\ &= K_{2,1} + K_{2,2} + K_{2,3} + K_{2,4}. \end{aligned}$$

Due to integration by parts and divergence condition,

$$K_{2,1} = \int_\Omega u\cdot\partial_2^3\nabla u\cdot\partial_2^3 u\, dx = 0.$$

$K_{2,2}$ can be split into four terms by divergence-free condition of u,

$$\begin{aligned}
K_{2,2} &= 3\int_\Omega \partial_2 u\cdot\partial_2^2\nabla u\cdot\partial_2^3 u\ dx\\
&= 3\int_\Omega \partial_2 u_1\partial_2^2\partial_1 u_1\partial_2^3 u_1\ dx + 3\int_\Omega \partial_2 u_1\partial_2^2\partial_1 u_2\partial_2^3 u_2\ dx\\
&\quad + 3\int_\Omega \partial_2 u_2\partial_2^3 u_1\partial_2^3 u_1\ dx + 3\int_\Omega \partial_2 u_2\partial_2^3 u_2\partial_2^3 u_2\ dx\\
&= 3\int_\Omega \partial_2 u_1\partial_2^2\partial_1 u_1\partial_2^3 u_1\ dx - 3\int_\Omega \partial_2 u_1\partial_2^2\partial_1 u_2\partial_2^2\partial_1 u_1\ dx\\
&\quad - 3\int_\Omega \partial_1 u_1\partial_2^3 u_1\partial_2^3 u_1\ dx - 3\int_\Omega \partial_1 u_1\partial_2^2\partial_1 u_1\partial_2^2\partial_1 u_1\ dx\\
&= K_{2,2,1}+K_{2,2,2}+K_{2,2,3}+K_{2,2,4}.
\end{aligned}$$

Using $\partial_1 u_1=\partial_1\widetilde{u}_1$ and writing $u_1=\bar{u}_1+\widetilde{u}_1$, we have

$$\begin{aligned}
K_{2,2,1} &= 3\int_\Omega \partial_2 u_1\partial_2^2\partial_1 u_1\partial_2^3 u_1\ dx\\
&= 3\int_\Omega \partial_2\widetilde{u}_1\partial_2^2\partial_1\widetilde{u}_1\partial_2^3\widetilde{u}_1\ dx\\
&\quad + 3\int_\Omega \partial_2\widetilde{u}_1\,\partial_2^2\partial_1\widetilde{u}_1\partial_2^3\bar{u}_1\ dx + 3\int_\Omega \partial_2\bar{u}_1\,\partial_2^2\partial_1\widetilde{u}_1\partial_2^3\widetilde{u}_1\ dx,
\end{aligned}$$

where we have used the fact that

$$\int_\Omega \partial_2\bar{u}_1\,\partial_2^2\partial_1\widetilde{u}_1\partial_2^3\bar{u}_1\ dx = 0.$$

By Hölder's inequality and Lemma 2.1, for any $\alpha>0$,

$$\begin{aligned}
K_{2,2,1} &\le C\,\|\partial_2\widetilde{u}_1\|_{L^\infty}\,\|\partial_2^2\partial_1\widetilde{u}_1\|_{L^2}\,\|\partial_2^3\widetilde{u}_1\|_{L^2} + C\,\|\partial_2\widetilde{u}_1\|_{L^\infty}\,\|\partial_2^2\partial_1\widetilde{u}_1\|_{L^2}\,\|\partial_2^3\bar{u}_1\|_{L^2}\\
&\quad + C\,\|\partial_2\bar{u}_1\|_{L^\infty}\,\|\partial_2^2\partial_1\widetilde{u}_1\|_{L^2}\,\|\partial_2^3\widetilde{u}_1\|_{L^2}\\
&\le C\,\|\partial_1\partial_2\widetilde{u}_1\|_{H^1}^2\|u\|_{H^3} + C\,\|u\|_{H^3}\|\Lambda_1^\alpha\partial_2^2\partial_1\widetilde{u}_1\|_{L^2}\,\|\Lambda_1^\alpha\partial_2^3\widetilde{u}_1\|_{L^2}\\
&\le C\,\|u\|_{H^3}\|\Lambda_1^\alpha u\|_{H^3}^2.
\end{aligned} \tag{3.3}$$

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By Hölder's inequality and Lemma 2.4,

$$\begin{aligned}
K_{2,2,2}+K_{2,2,4} &= -3\int_\Omega \partial_2 u_1\partial_2^2\partial_1 u_2\partial_2^2\partial_1 u_1\ dx - 3\int_\Omega \partial_1 u_1\partial_2^2\partial_1 u_1\partial_2^2\partial_1 u_1\ dx\\
&\le C\|\partial_2 u_1\|_{L^\infty}\|\partial_2^2\partial_1 u_2\|_{L^2}\|\partial_2^2\partial_1 u_1\|_{L^2} + C\|\partial_1 u_1\|_{L^\infty}\|\partial_2^2\partial_1 u_1\|_{L^2}\|\partial_2^2\partial_1 u_1\|_{L^2}\\
&\le C\|\Lambda_1^\alpha u\|_{H^3}^2\|u\|_{H^3}.
\end{aligned}$$

Similarly, by $u_1=\bar{u}_1+\widetilde{u}_1$ and Lemma 2.1, for any $\alpha>0$,

$$\begin{aligned}
K_{2,2,3} &= -3\int_\Omega \partial_1 u_1\partial_2^3 u_1\partial_2^3 u_1\ dx\\
&= -3\int_\Omega \partial_1\widetilde{u}_1(\partial_2^3\widetilde{u}_1)^2\ dx - 6\int_\Omega \partial_1\widetilde{u}_1\partial_2^3\widetilde{u}_1\,\partial_2^3\bar{u}_1\ dx\\
&\le C\,\|\partial_1\widetilde{u}_1\|_{L^\infty}\,\|\partial_2^3\widetilde{u}_1\|_{L^2}\|u\|_{H^3} \le C\|\Lambda_1^\alpha u\|_{H^3}^2\|u\|_{H^3},
\end{aligned} \tag{3.4}$$

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Thus,

$$K_{2,2}\le C\|u\|_{H^3}\|\Lambda_1^\alpha u\|_{H^3}^2.$$

By the divergence-free condition $\partial_2 u_2 = -\partial_1 u_1$, by Lemma 2.1 and Lemma 2.5,

$$\begin{aligned}
K_{2,3} &= -3\int_\Omega \partial_2^2 u\cdot\partial_2\nabla u\cdot\partial_2^3 u\,dx\\
&= -3\int_\Omega \partial_2^2 u_1\partial_2\partial_1 u_1\partial_2^3 u_1\,dx - 3\int_\Omega \partial_2^2 u_1\partial_2\partial_1 u_2\partial_2^3 u_2\,dx\\
&\quad - 3\int_\Omega \partial_2^2 u_2\partial_2\partial_2 u_1\partial_2^3 u_1\,dx - 3\int_\Omega \partial_2^2 u_2\partial_2^2 u_2\partial_2^3 u_2\,dx\\
&= -3\int_\Omega \partial_2^2 u_1\partial_2\partial_1 u_1\partial_2^3 u_1\,dx + 3\int_\Omega \partial_2^2 u_1\partial_2\partial_1 u_2\partial_2^2\partial_1 u_1\,dx\\
&\quad + 3\int_\Omega \partial_2\partial_1 u_1\partial_2^2 u_1\partial_2^3 u_1\,dx + 3\int_\Omega \partial_2\partial_1 u_1\partial_2\partial_1 u_1\partial_2^2\partial_1 u_1\,dx\\
&= 3\int_\Omega \partial_2^2 u_1\partial_2\partial_1 u_2\partial_2^2\partial_1 u_1\,dx + 3\int_\Omega \partial_2\partial_1 u_1\partial_2\partial_1 u_1\partial_2^2\partial_1 u_1\,dx\\
&= 3\int_\Omega \partial_2^2 u_1\partial_2\partial_1 \widetilde{u}_2\partial_2^2\partial_1 u_1\,dx + 3\int_\Omega \partial_2\partial_1 \widetilde{u}_1\partial_2\partial_1 \widetilde{u}_1\partial_2^2\partial_1 u_1\,dx\\
&\le C\|\partial_2^2\partial_1 u_1\|_{L^2}\|\partial_2\partial_1\widetilde{u}_2\|_{L^2}^{1/2}\|\partial_1^2\partial_2\widetilde{u}_2\|_{L^2}^{1/2}\|\partial_2^2 u_1\|_{L^2}^{1/2}\|\partial_2^3 u_1\|_{L^2}^{1/2}\\
&\quad + C\|\partial_2^2\partial_1 u_1\|_{L^2}\|\partial_2\partial_1\widetilde{u}_1\|_{L^2}^{1/2}\|\partial_1^2\partial_2\widetilde{u}_1\|_{L^2}^{1/2}\|\partial_2\partial_1\widetilde{u}_1\|_{L^2}^{1/2}\|\partial_2^2\partial_1\widetilde{u}_1\|_{L^2}^{1/2}\\
&\le C\|u\|_{H^3}\|\Lambda_1^\alpha u\|_{H^3}^2.
\end{aligned}$$

By divergence free condition of u, $K_{2,4}$ can be decomposed into four terms,

$$\begin{aligned}
K_{2,4} &= -\int_\Omega \partial_2^3 u\cdot\nabla u\cdot\partial_2^3 u\,dx\\
&= -\int_\Omega \partial_2^3 u_1\partial_1 u_1\partial_2^3 u_1\,dx - \int_\Omega \partial_2^3 u_1\partial_1 u_2\partial_2^3 u_2\,dx\\
&\quad - \int_\Omega \partial_2^3 u_2\partial_2 u_1\partial_2^3 u_1\,dx - \int_\Omega \partial_2^3 u_2\partial_2 u_2\partial_2^3 u_2\,dx\\
&= -\int_\Omega \partial_2^3 u_1\partial_1 u_1\partial_2^3 u_1\,dx + \int_\Omega \partial_2^3 u_1\partial_1 u_2\partial_2^2\partial_1 u_1\,dx\\
&\quad + \int_\Omega \partial_2^2\partial_1 u_1\partial_2 u_1\partial_2^3 u_1\,dx + \int_\Omega \partial_2^2\partial_1 u_1\partial_1 u_1\partial_2^2\partial_1 u_1\,dx\\
&= K_{2,4,1} + K_{2,4,2} + K_{2,4,3} + K_{2,4,4}.
\end{aligned}$$

Since $K_{2,4,1} = \frac{1}{3}K_{2,2,3}$ and $K_{2,4,3} = \frac{1}{3}K_{2,2,1}$, by (3.3) and (3.4), we obtain

$$K_{2,4,1} + K_{2,4,3} \le C\|\Lambda_1^\alpha u\|_{H^3}^2\|u\|_{H^3}.$$

By Lemma 2.1, Lemma 2.4 and Hölder's inequality,

$$\begin{aligned}
K_{2,4,2} + K_{2,4,4} &= \int_\Omega \partial_2^3 u_1\partial_1\widetilde{u}_2\partial_2^2\partial_1 u_1\,dx + \int_\Omega \partial_2^2\partial_1 u_1\partial_1 u_1\partial_2^2\partial_1 u_1\,dx\\
&\le C\|\partial_2^2\partial_1 u_1\|_{L^2}\|\partial_1\widetilde{u}_2\|_{L^\infty}\|\partial_2^3 u_1\|_{L^2} + C\|\partial_1 u_1\|_{L^\infty}\|\partial_2^2\partial_1 u_1\|_{L^2}^2\\
&\le C\|\Lambda_1^\alpha u\|_{H^3}\|\partial_1^2\widetilde{u}_2\|_{H^1}\|u\|_{H^3} + C\|u\|_{H^3}\|\partial_2^2\partial_1 u_1\|_{L^2}^2\\
&\le C\|u\|_{H^3}\|\Lambda_1^\alpha u\|_{H^3}^2.
\end{aligned}$$

Then we have

$$K_{2,4} \le C\|u\|_{H^3}\|\Lambda_1^\alpha u\|_{H^3}^2.$$

Collecting the bounds for $K_{2,1}$, $K_{2,2}$, $K_{2,3}$, $K_{2,4}$ and K_1 estimates in (3.2), we obtain

$$K \le C\|u\|_{H^3}\|\Lambda_1^\alpha u\|_{H^3}^2. \tag{3.5}$$ Kbound

To bound L, we decompose it as

$$\begin{aligned}
L &= \sum_{i=1}^{2}\left(\int_\Omega \partial_i^3(b\cdot\nabla b)\cdot\partial_i^3 u\,dx - \int_\Omega b\cdot\nabla\partial_i^3 b\cdot\partial_i^3 u\,dx\right)\\
&= \sum_{i=1}^{2}\sum_{k=1}^{3} C_3^k\int_\Omega \partial_i^k b\cdot\partial_i^{3-k}\nabla b\cdot\partial_i^3 u\,dx\\
&= \sum_{k=1}^{3} C_3^k\int_\Omega \partial_1^k b\cdot\partial_1^{3-k}\nabla b\cdot\partial_1^3 u\,dx + \sum_{k=1}^{3} C_3^k\int_\Omega \partial_2^k b\cdot\partial_2^{3-k}\nabla b\cdot\partial_2^3 u\,dx\\
&= L_1 + L_2.
\end{aligned}$$

By Hölder's inequality, Lemma 2.1, Lemma 2.2, Lemma 2.5 and Young's inequality,

$$\begin{aligned}
L_1 &= 3\int_\Omega \partial_1\widetilde{b}\cdot\partial_1^2\nabla\widetilde{b}\cdot\partial_1^3\widetilde{u}\,dx + 3\int_\Omega \partial_1^2\widetilde{b}\cdot\partial_1\nabla\widetilde{b}\cdot\partial_1^3\widetilde{u}\,dx + \int_\Omega \partial_1^3\widetilde{b}\cdot\nabla b\cdot\partial_1^3\widetilde{u}\,dx\\
&\le C\|\partial_1\widetilde{b}\|_{L^\infty}\|\partial_1^2\nabla\widetilde{b}\|_{L^2}\|\partial_1^3\widetilde{u}\|_{L^2}\\
&\quad + C\|\partial_1^3\widetilde{u}\|_{L^2}\|\partial_1^2\widetilde{b}\|_{L^2}^{1/2}\|\partial_1^3\widetilde{b}\|_{L^2}^{1/2}\|\partial_1\nabla\widetilde{b}\|_{L^2}^{1/2}\|\partial_1\partial_2\nabla\widetilde{b}\|_{L^2}^{1/2}\\
&\quad + C\|\nabla b\|_{L^\infty}\|\partial_1^3\widetilde{b}\|_{L^2}\|\partial_1^3\widetilde{u}\|_{L^2}\\
&\le C\|b\|_{H^3}\|\Lambda_1^\beta\widetilde{b}\|_{H^3}\|\Lambda_1^\alpha\widetilde{u}\|_{H^3} \le C\|b\|_{H^3}(\|\Lambda_1^\alpha u\|_{H^2}^2 + \|\Lambda_1^\beta b\|_{H^3}^2).
\end{aligned} \tag{3.6}$$ L1bound

L_2 can be split into three terms.

$$\begin{aligned}
L_2 &= 3\int_\Omega \partial_2 b\cdot\partial_2^2\nabla b\cdot\partial_2^3 u\,dx + 3\int_\Omega \partial_2^2 b\cdot\partial_2\nabla b\cdot\partial_2^3 u\,dx + \int_\Omega \partial_2^3 b\cdot\nabla b\cdot\partial_2^3 u\,dx\\
&= L_{2,1} + L_{2,2} + L_{2,3}.
\end{aligned}$$

By Lemma 2.2 and the divergence-free conditions of u and b,

$$\begin{aligned}
L_{2,1} &= 3\int_\Omega \partial_2 b\cdot\partial_2^2\nabla b\cdot\partial_2^3 u\,dx\\
&= 3\int_\Omega \partial_2 b_1\partial_2^2\partial_1 b\cdot\partial_2^3 u\,dx + 3\int_\Omega \partial_2 b_2\partial_2^3 b\cdot\partial_2^3 u\,dx\\
&= 3\int_\Omega \partial_2 b_1\partial_2^2\partial_1 b\cdot\partial_2^3 u\,dx - 3\int_\Omega \partial_1\widetilde{b}_1\partial_2^3 b\cdot\partial_2^3 u\,dx\\
&= 3\int_\Omega \partial_2 b_1\partial_2^2\partial_1 b_1\partial_2^3 u_1\,dx - 3\int_\Omega \partial_2 b_1\partial_2^2\partial_1 b_2\partial_2^2\partial_1 u_1\,dx\\
&\quad - 3\int_\Omega \partial_1\widetilde{b}_1\partial_2^3 b_1\partial_2^3 u_1\,dx + 3\int_\Omega \partial_1\widetilde{b}_1\partial_2^2\partial_1 b_1\partial_2^2\partial_1 u_1\,dx\\
&= L_{2,1,1} + L_{2,1,2} + L_{2,1,3} + L_{2,1,4}.
\end{aligned}$$

By Hölder's inequality, $u = \bar{u} + \widetilde{u}$ and $b = \bar{b} + \widetilde{b}$, Lemma 2.1 and Lemma 2.4,

$$\begin{aligned} L_{2,1,1} + L_{2,1,3} &= 3\int_\Omega \partial_2 b_1 \partial_2^2 \partial_1 \widetilde{b}_1 \partial_2^3 u_1 \ dx - 3\int_\Omega \partial_1 \widetilde{b}_1 \partial_2^3 b_1 \partial_2^3 u_1 \ dx \\ &= 3\int_\Omega \partial_2 \widetilde{b}_1 \partial_2^2 \partial_1 \widetilde{b}_1 \partial_2^3 \widetilde{u}_1 \ dx + 3\int_\Omega \partial_2 \bar{b}_1 \partial_2^2 \partial_1 \widetilde{b}_1 \partial_2^3 \widetilde{u}_1 \ dx \\ &\quad + 3\int_\Omega \partial_2 \widetilde{b}_1 \partial_2^2 \partial_1 \widetilde{b}_1 \partial_2^3 \bar{u}_1 \ dx - 3\int_\Omega \partial_1 \widetilde{b}_1 \partial_2^3 \widetilde{b}_1 \partial_2^3 \widetilde{u}_1 \ dx \\ &\quad - 3\int_\Omega \partial_1 \widetilde{b}_1 \partial_2^3 \bar{b}_1 \partial_2^3 \widetilde{u}_1 \ dx - 3\int_\Omega \partial_1 \widetilde{b}_1 \partial_2^3 \widetilde{b}_1 \partial_2^3 \bar{u}_1 \ dx \\ &\leq C\|\partial_2^2 \partial_1 b_1\|_{L^2} \|\Lambda_1^\alpha \partial_2^3 u_1\|_{L^2} \|b\|_{H^3} + C\|u\|_{H^3} \|\Lambda_1^\beta b_1\|_{H^3}^2 \\ &\quad + C\|b\|_{H^3} \|\Lambda_1^\beta b_1\|_{H^3} \|\Lambda_1^\alpha u_1\|_{H^3} \\ &\leq C(\|u\|_{H^3} + \|b\|_{H^3})(\|\Lambda_1^\alpha u\|_{H^3}^2 + \|\Lambda_1^\beta b\|_{H^3}^2). \end{aligned} \tag{3.7}$$

By Hölder's inequality, Lemma 2.1, Lemma 2.4 and Young's inequality,

$$\begin{aligned} L_{2,1,2} + L_{2,1,4} &= -3\int_\Omega \partial_2 b_1 \partial_2^2 \partial_1 b_2 \partial_2^2 \partial_1 u_1 \ dx + 3\int_\Omega \partial_1 \widetilde{b}_1 \partial_2^2 \partial_1 b_1 \partial_2^2 \partial_1 u_1 \ dx \\ &\leq C\|\partial_2 b_1\|_{L^\infty} \|\partial_2^2 \partial_1 b_2\|_{L^2} \|\partial_2^2 \partial_1 u_1\|_{L^2} + C\|\partial_1 \widetilde{b}_1\|_{L^\infty} \|\partial_2^2 \partial_1 b_1\|_{L^2} \|\partial_2^2 \partial_1 u_1\|_{L^2} \\ &\leq C\|b\|_{H^3} \|\partial_1 b\|_{H^2} \|\partial_1 u\|_{H^2} \\ &\leq C\|b\|_{H^3} (\|\Lambda_1^\alpha u\|_{H^3}^2 + \|\Lambda_1^\beta b\|_{H^3}^2). \end{aligned} \tag{3.8}$$

Then we have

$$L_{2,1} \leq C(\|u\|_{H^3} + \|b\|_{H^3})(\|\Lambda_1^\alpha u\|_{H^3}^2 + \|\Lambda_1^\beta b\|_{H^3}^2). \tag{3.9}$$

By Lemma 2.5, Young's inequality and the divergence-free conditions of u and b,

$$\begin{aligned} L_{2,2} &= 3\int_\Omega \partial_2^2 b \cdot \partial_2 \nabla b \cdot \partial_2^3 u \ dx \\ &= 3\int_\Omega \partial_2^2 b_1 \partial_2 \partial_1 b_1 \partial_2^3 u_1 \ dx + 3\int_\Omega \partial_2^2 b_1 \partial_2 \partial_1 b_2 \partial_2^3 u_2 \ dx \\ &\quad + 3\int_\Omega \partial_2^2 b_2 \partial_2^2 b_1 \partial_2^3 u_1 \ dx + 3\int_\Omega \partial_2^2 b_2 \partial_2^2 b_2 \partial_2^3 u_2 \ dx \\ &= 3\int_\Omega \partial_2^2 b_1 \partial_2 \partial_1 b_1 \partial_2^3 u_1 \ dx - 3\int_\Omega \partial_2^2 b_1 \partial_2 \partial_1 b_2 \partial_2^2 \partial_1 u_1 \ dx \\ &\quad - 3\int_\Omega \partial_2 \partial_1 b_1 \partial_2^2 b_1 \partial_2^3 u_1 \ dx - 3\int_\Omega \partial_2 \partial_1 b_1 \partial_2 \partial_1 b_1 \partial_2^2 \partial_1 u_1 \ dx \\ &= -3\int_\Omega \partial_2^2 b_1 \partial_2 \partial_1 \widetilde{b}_2 \partial_2^2 \partial_1 u_1 \ dx - 3\int_\Omega \partial_2 \partial_1 b_1 \partial_2 \partial_1 \widetilde{b}_1 \partial_2^2 \partial_1 u_1 \ dx \\ &\leq C\|\partial_2^2 \partial_1 u_1\|_{L^2} \|\partial_2 \partial_1 \widetilde{b}_2\|_{L^2}^{1/2} \|\partial_1^2 \partial_2 \widetilde{b}_2\|_{L^2}^{1/2} \|\partial_2^2 b_1\|_{L^2}^{1/2} \|\partial_2^3 b_1\|_{L^2}^{1/2} \\ &\quad + C\|\partial_2^2 \partial_1 u_1\|_{L^2} \|\partial_2 \partial_1 \widetilde{b}_1\|_{L^2}^{1/2} \|\partial_1^2 \partial_2 \widetilde{b}_1\|_{L^2}^{1/2} \|\partial_2 \partial_1 b_1\|_{L^2}^{1/2} \|\partial_2^2 \partial_1 b_1\|_{L^2}^{1/2} \\ &\leq C\|\partial_1 u\|_{H^2} \|\partial_1 b\|_{H^2} \|b\|_{H^3} \\ &\leq C\|b\|_{H^3} (\|\Lambda_1^\alpha u\|_{H^3}^2 + \|\Lambda_1^\beta b\|_{H^3}^2). \end{aligned} \tag{3.10}$$

The last equality is due to the cancellation of the first and the third term. Similarly, by divergence-free condition of u and b, we can split $L_{2,3}$ into four terms.

$$\begin{aligned}
L_{2,3} &= \int_\Omega \partial_2^3 b \cdot \nabla b \cdot \partial_2^3 u \; dx \\
&= \int_\Omega \partial_2^3 b_1 \partial_1 b_1 \partial_2^3 u_1 \; dx + \int_\Omega \partial_2^3 b_1 \partial_1 b_2 \partial_2^3 u_2 \; dx \\
&\quad + \int_\Omega \partial_2^3 b_2 \partial_2 b_1 \partial_2^3 u_1 \; dx + \int_\Omega \partial_2^3 b_2 \partial_2 b_2 \partial_2^3 u_2 \; dx \\
&= \int_\Omega \partial_2^3 b_1 \partial_1 \widetilde{b}_1 \partial_2^3 u_1 \; dx - \int_\Omega \partial_2^3 b_1 \partial_1 \widetilde{b}_2 \partial_2^2 \partial_1 u_1 \; dx \\
&\quad - \int_\Omega \partial_2^2 \partial_1 b_1 \partial_2 b_1 \partial_2^3 u_1 \; dx - \int_\Omega \partial_2^2 \partial_1 b_1 \partial_1 b_1 \partial_2^2 \partial_1 u_1 \; dx \\
&= L_{2,3,1} + L_{2,3,2} + L_{2,3,3} + L_{2,3,4}
\end{aligned}$$

Since $L_{2,1,3} = -3L_{2,3,1}$, $L_{2,1,1} = 3L_{2,3,3}$, $L_{2,1,4} = -3L_{2,3,4}$, by (3.7) and (3.8), we obtain,

$$L_{2,3,1} + L_{2,3,3} + L_{2,3,4} \le C(\|u\|_{H^3} + \|b\|_{H^3})(\|\Lambda_1^\alpha u\|_{H^3}^2 + \|\Lambda_1^\beta b\|_{H^3}^2).$$

For $L_{2,3,2}$, by Lemma 2.1, Hölder's inequality and Young's inequality,

$$\begin{aligned}
L_{2,3,2} &= -\int_\Omega \partial_2^3 b_1 \partial_1 \widetilde{b}_2 \partial_2^2 \partial_1 u_1 \; dx \le C\|\partial_2^3 b_1\|_{L^2} \|\partial_2^2 \partial_1 u_1\|_{L^2} \|\partial_1 \widetilde{b}_2\|_{L^\infty} \\
&\le C\|b\|_{H^3} \|\partial_1 u\|_{H^2} \|\partial_1^2 \widetilde{b}_2\|_{H^1} \le C\|b\|_{H^3} \|\Lambda_1^\alpha u\|_{H^3} \|\Lambda_1^\beta b\|_{H^3} \\
&\le C\|b\|_{H^3}(\|\Lambda_1^\alpha u\|_{H^3}^2 + \|\Lambda_1^\beta b\|_{H^3}^2).
\end{aligned}$$

Hence

$$L_{2,3} \le C(\|u\|_{H^3} + \|b\|_{H^3})(\|\Lambda_1^\alpha u\|_{H^3}^2 + \|\Lambda_1^\beta b\|_{H^3}^2). \tag{3.11}$$ L23bound

By (3.6), (3.9), (3.10) and (3.11),

$$L \le C(\|u\|_{H^3} + \|b\|_{H^3})(\|\Lambda_1^\alpha u\|_{H^3}^2 + \|\Lambda_1^\beta b\|_{H^3}^2). \tag{3.12}$$ Lbound

Now we estimate M,

$$\begin{aligned}
M &= -\sum_{i=1}^2 \int_\Omega \partial_i^3 (u \cdot \nabla b) \cdot \partial_i^3 b \; dx, \\
&= -\int_\Omega \partial_1^3 (u \cdot \nabla b) \cdot \partial_1^3 b \; dx - \int_\Omega \partial_2^3 (u \cdot \nabla b) \cdot \partial_2^3 b \; dx, \\
&= M_1 + M_2.
\end{aligned}$$

We can rewrite

$$\begin{aligned}
M_1 &= -\sum_{k=1}^3 C_3^k \int_\Omega \partial_1^k u \cdot \partial_1^{3-k} \nabla b \cdot \partial_1^3 b \; dx - \int_\Omega u \cdot \partial_1^3 \nabla b \cdot \partial_1^3 b \; dx \\
&= M_{1,1} + M_{1,2}.
\end{aligned}$$

By Lemma 2.2, Lemma 2.1, Lemma 2.5, Hölder's inequality and Young's inequality,

$$\begin{aligned}
M_{1,1} &= -3\int_\Omega \partial_1 u \cdot \partial_1^2 \nabla b \cdot \partial_1^3 b \; dx - 3\int_\Omega \partial_1^2 \widetilde{u} \cdot \partial_1 \nabla \widetilde{b} \cdot \partial_1^3 \widetilde{b} \; dx - \int_\Omega \partial_1^3 u \cdot \nabla b \cdot \partial_1^3 b \; dx \\
&\le C\|\partial_1 u\|_{L^\infty} \|\partial_1^2 \nabla b\|_{L^2} \|\partial_1^3 b\|_{L^2} + C\|\partial_1^3 \widetilde{b}\|_{L^2} \|\partial_1^2 \widetilde{u}\|_{L^2}^{1/2} \|\partial_1^3 \widetilde{u}\|_{L^2}^{1/2} \|\partial_1 \nabla \widetilde{b}\|_{L^2}^{1/2} \|\partial_2 \partial_1 \nabla \widetilde{b}\|_{L^2}^{1/2} \\
&\quad + C\|\partial_1^3 u\|_{L^2} \|\partial_1^3 b\|_{L^2} \|\nabla b\|_{L^\infty}
\end{aligned}$$

$$\leq C\|\partial_1 b\|_{H^2}\|\partial_1 u\|_{H^2}\|b\|_{H^3} \leq C\|b\|_{H^3}(\|\Lambda_1^\alpha u\|_{H^3}^2 + \|\Lambda_1^\beta b\|_{H^3}^2).$$

By integration by parts and the divergence-free condition $M_{1,2} = 0$,

$$M_{1,2} = -\int_\Omega u\cdot\partial_1^3\nabla b\cdot\partial_1^3 b\,dx = -\frac{1}{2}\int_\Omega u\cdot\nabla(\partial_1^3 b)^2\,dx = 0.$$

To estimate M_2, we split it into four terms,

$$\begin{aligned} M_2 &= -\int_\Omega \partial_2^3(u\cdot\nabla b)\cdot\partial_2^3 b\,dx,\\ &= -\sum_{k=1}^3 C_3^k\int_\Omega \partial_2^k u\cdot\partial_2^{3-k}\nabla b\cdot\partial_2^3 b\,dx - \int_\Omega u\cdot\partial_2^3\nabla b\cdot\partial_2^3 b\,dx\\ &= M_{2,1}+M_{2,2}+M_{2,3}+M_{2,4}. \end{aligned}$$

$M_{2,4} = 0$ due to $\nabla\cdot u = 0$. We decompose $M_{2,1}$ into four parts,

$$\begin{aligned} M_{2,1} &= -3\int_\Omega \partial_2 u\cdot\partial_2^2\nabla b\cdot\partial_2^3 b\,dx\\ &= -3\int_\Omega \partial_2 u_1\partial_2^2\partial_1 b_1\partial_2^3 b_1\,dx - 3\int_\Omega \partial_2 u_1\partial_2^2\partial_1 b_2\partial_2^3 b_2\,dx\\ &\quad - 3\int_\Omega \partial_2 u_2\partial_2^2\partial_2 b_1\partial_2^3 b_1\,dx - 3\int_\Omega \partial_2 u_2\partial_2^3 b_2\partial_2^3 b_2\,dx\\ &= -3\int_\Omega \partial_2 u_1\partial_2^2\partial_1 b_1\partial_2^3 b_1\,dx + 3\int_\Omega \partial_2 u_1\partial_2^2\partial_1 b_2\partial_2^2\partial_1 b_1\,dx\\ &\quad + 3\int_\Omega \partial_1 u_1\partial_2^3 b_1\partial_2^3 b_1\,dx + 3\int_\Omega \partial_1 u_1\partial_2^2\partial_1 b_1\partial_2^2\partial_1 b_1\,dx\\ &= M_{2,1,1}+M_{2,1,2}+M_{2,1,3}+M_{2,1,4}. \end{aligned}$$

By $u = \bar{u}+\widetilde{u}$ and $b = \bar{b}+\widetilde{b}$ and Lemma 2.1,

$$\begin{aligned} M_{2,1,1} &= -3\int_\Omega \partial_2\widetilde{u}_1\partial_2^2\partial_1\widetilde{b}_1\partial_2^3\widetilde{b}_1\,dx - 3\int_\Omega \partial_2\widetilde{u}_1\partial_2^2\partial_1\widetilde{b}_1\partial_2^3\bar{b}_1\,dx\\ &\quad - 3\int_\Omega \partial_2\bar{u}_1\partial_2^2\partial_1\widetilde{b}_1\partial_2^3\widetilde{b}_1\,dx\\ &\leq C\|\partial_2^2\partial_1\widetilde{b}_1\|_{L^2}\|\partial_2^3\widetilde{b}_1\|_{L^2}\|\partial_2 u_1\|_{L^\infty} + C\|\partial_2^2\partial_1\widetilde{b}_1\|_{L^2}\|\partial_2^3 b_1\|_{L^2}\|\partial_2\widetilde{u}_1\|_{L^\infty}\\ &\leq C(\|u\|_{H^3}+\|b\|_{H^3})(\|\Lambda_1^\alpha u\|_{H^3}^2+\|\Lambda_1^\beta b\|_{H^3}^2). \end{aligned} \tag{3.13}$$

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By Hölder's inequality and Lemma 2.4,

$$\begin{aligned} M_{2,1,2}+M_{2,1,4} &= 3\int_\Omega \partial_2 u_1\partial_2^2\partial_1 b_2\partial_2^2\partial_1 b_1\,dx + 3\int_\Omega \partial_1 u_1\partial_2^2\partial_1 b_1\partial_2^2\partial_1 b_1\,dx\\ &\leq C\|\partial_2 u_1\|_{L^\infty}\|\partial_2^2\partial_1 b_2\|_{L^2}\|\partial_2^2\partial_1 b_1\|_{L^2} + C\|\partial_1 u_1\|_{L^\infty}\|\partial_2^2\partial_1 b_1\|_{L^2}^2\\ &\leq C\|u\|_{H^3}\|\partial_1 b\|_{H^2}^2 \leq C\|u\|_{H^3}\|\Lambda_1^\beta b\|_{H^3}^2. \end{aligned} \tag{3.14}$$

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As in (3.13),

$$\begin{aligned} M_{2,1,3} &= 3\int_\Omega \partial_1 u_1\partial_2^3 b_1\partial_2^3 b_1\,dx\\ &\leq C(\|u\|_{H^3}+\|b\|_{H^3})(\|\Lambda_1^\alpha u\|_{H^3}^2+\|\Lambda_1^\beta b\|_{H^3}^2). \end{aligned}$$

Thus, by the bounds of $M_{2,1,1}$, $M_{2,1,2}$, $M_{2,1,3}$ and $M_{2,1,4}$,

$$M_{2,1} \le C\|(u,b)\|_{H^3}(\|\Lambda_1^\beta b\|_{H^3}^2 + \|\Lambda_1^\alpha u\|_{H^3}^2).$$

We rewrite $M_{2,2}$ by using divergence free condition of u and b,

$$\begin{aligned}
M_{2,2} &= -3\int_\Omega \partial_2^2 u\cdot\partial_2\nabla b\cdot\partial_2^3 b\ dx\\
&= -3\int_\Omega \partial_2^2 u_1\partial_2\partial_1 b_1\partial_2^3 b_1\ dx - 3\int_\Omega \partial_2^2 u_1\partial_2\partial_1 b_2\partial_2^3 b_2\ dx\\
&\quad - 3\int_\Omega \partial_2^2 u_2\partial_2^2 b_1\partial_2^3 b_1\ dx - 3\int_\Omega \partial_2^2 u_2\partial_2^2 b_2\partial_2^3 b_2\ dx\\
&= -3\int_\Omega \partial_2^2 u_1\partial_2\partial_1 b_1\partial_2^3 b_1\ dx + 3\int_\Omega \partial_2^2 u_1\partial_2\partial_1 b_2\partial_2^2\partial_1 b_1\ dx\\
&\quad + 3\int_\Omega \partial_2\partial_1 u_1\partial_2^2 b_1\partial_2^3 b_1\ dx + 3\int_\Omega \partial_2\partial_1 u_1\partial_2\partial_1 b_1\partial_2^2\partial_1 b_1\ dx\\
&= M_{2,2,1} + M_{2,2,2} + M_{2,2,3} + M_{2,2,4}.
\end{aligned}$$

By Lemma 2.5,

$$\begin{aligned}
M_{2,2,2} + M_{2,2,4} &= 3\int_\Omega \partial_2^2 u_1\partial_2\partial_1\widetilde{b}_2\partial_2^2\partial_1 b_1\ dx + 3\int_\Omega \partial_2\partial_1 u_1\partial_2\partial_1\widetilde{b}_1\partial_2^2\partial_1 b_1\ dx\\
&\le C\|\partial_2^2\partial_1 b_1\|_{L^2}\|\partial_2\partial_1\widetilde{b}_2\|_{L^2}^{1/2}\|\partial_2\partial_1^2\widetilde{b}_2\|_{L^2}^{1/2}\|\partial_2^2 u_1\|_{L^2}^{1/2}\|\partial_2^3 u_1\|_{L^2}^{1/2}\\
&\quad + C\|\partial_2^2\partial_1 b_1\|_{L^2}\|\partial_2\partial_1\widetilde{b}_1\|_{L^2}^{1/2}\|\partial_2\partial_1^2\widetilde{b}_1\|_{L^2}^{1/2}\|\partial_2\partial_1 u_1\|_{L^2}^{1/2}\|\partial_2^2\partial_1 u_1\|_{L^2}^{1/2}\\
&\le C\|\partial_1 b\|_{H^2}^2\|u\|_{H^3} \le C\|u\|_{H^3}\|\Lambda_1^\beta b\|_{H^3}^2.
\end{aligned} \tag{3.15}$$

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By integration by parts twice, Lemma 2.5, Hölder's inequality, Lemma 2.1 and Young's inequality,

$$\begin{aligned}
M_{2,2,1} &= -3\int_\Omega \partial_2^2 u_1\partial_2\partial_1\widetilde{b}_1\partial_2^3 b_1\ dx = -3\int_\Omega \partial_2^2\widetilde{u}_1\partial_2\partial_1\widetilde{b}_1\partial_2^3 b_1\ dx - 3\int_\Omega \partial_2^2\overline{u_1}\partial_2\partial_1\widetilde{b}_1\partial_2^3 b_1\ dx\\
&= -3\int_\Omega \partial_2^2\widetilde{u}_1\partial_2\partial_1\widetilde{b}_1\partial_2^3 b_1\ dx + 3\int_\Omega \partial_2^2\overline{u_1}\partial_2\widetilde{b}_1\partial_2^3\partial_1 b_1\ dx\\
&= -3\int_\Omega \partial_2^2\widetilde{u}_1\partial_2\partial_1\widetilde{b}_1\partial_2^3 b_1\ dx - 3\int_\Omega \partial_2^3\overline{u_1}\partial_2\widetilde{b}_1\partial_2^2\partial_1 b_1\ dx - 3\int_\Omega \partial_2^2\overline{u_1}\partial_2^2\widetilde{b}_1\partial_2^2\partial_1 b_1\ dx\\
&\le \|\partial_2^3 b_1\|_{L^2}\|\partial_2^2\widetilde{u}_1\|_{L^2}^{1/2}\|\partial_1\partial_2^2\widetilde{u}_1\|_{L^2}^{1/2}\|\partial_1\partial_2\widetilde{b}_1\|_{L^2}^{1/2}\|\partial_1\partial_2^2\widetilde{b}_1\|_{L^2}^{1/2}\\
&\quad + C\|\partial_2\widetilde{b}_1\|_{L^\infty}\|\partial_2^3\overline{u_1}\|_{L^2}\|\partial_2^2\partial_1 b_1\|_{L^2}\\
&\quad + C\|\partial_2^2\partial_1 b_1\|_{L^2}\|\partial_2^2\widetilde{b}_1\|_{L^2}^{1/2}\|\partial_1\partial_2^2\widetilde{b}_1\|_{L^2}^{1/2}\|\partial_2^2\overline{u_1}\|_{L^2}^{1/2}\|\partial_2^3\overline{u_1}\|_{L^2}^{1/2}\\
&\le \|\partial_2^3 b_1\|_{L^2}\|\partial_1\partial_2^2\widetilde{u}_1\|_{L^2}\|\partial_1\partial_2\widetilde{b}_1\|_{L^2}^{1/2}\|\partial_1\partial_2^2\widetilde{b}_1\|_{L^2}^{1/2}\\
&\quad + C\|\partial_1\partial_2\widetilde{b}_1\|_{H^1}\|\partial_2^3\overline{u_1}\|_{L^2}\|\partial_2^2\partial_1 b_1\|_{L^2} + C\|\partial_2^2\partial_1 b_1\|_{L^2}\|\partial_1\partial_2^2\widetilde{b}_1\|_{L^2}\|\partial_2^2\overline{u_1}\|_{L^2}^{1/2}\|\partial_2^3\overline{u_1}\|_{L^2}^{1/2}\\
&\le C\|\Lambda_1^\alpha u\|_{H^3}\|\Lambda_1^\beta b\|_{H^3}\|b\|_{H^3} + C\|u\|_{H^3}\|\Lambda_1^\beta b\|_{H^3}^2\\
&\le C\|(u,b)\|_{H^3}(\|\Lambda_1^\alpha u\|_{H^3}^2 + \|\Lambda_1^\beta b\|_{H^3}^2)
\end{aligned} \tag{3.16}$$

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and

$$\begin{aligned}
M_{2,2,3} &= 3\int_\Omega \partial_2\partial_1 u_1 \partial_2^2 b_1 \partial_2^3 b_1 \ dx \\
&= 3\int_\Omega \partial_2\partial_1 \widetilde{u}_1 \partial_2^2 \widetilde{b}_1 \partial_2^3 b_1 \ dx + 3\int_\Omega \partial_2\partial_1 \widetilde{u}_1 \partial_2^2 \overline{b_1} \partial_2^3 b_1 \ dx \\
&= 3\int_\Omega \partial_2\partial_1 \widetilde{u}_1 \partial_2^2 \widetilde{b}_1 \partial_2^3 b_1 \ dx - 3\int_\Omega \partial_2 \widetilde{u}_1 \partial_2^2 \overline{b_1} \partial_2^3 \partial_1 b_1 \ dx \\
&= 3\int_\Omega \partial_2\partial_1 \widetilde{u}_1 \partial_2^2 \widetilde{b}_1 \partial_2^3 b_1 \ dx + 3\int_\Omega \partial_2^2 \widetilde{u}_1 \partial_2^2 \overline{b_1} \partial_2^2 \partial_1 b_1 \ dx + 3\int_\Omega \partial_2 \widetilde{u}_1 \partial_2^3 \overline{b_1} \partial_2^2 \partial_1 b_1 \ dx \\
&\le C\|\partial_2^3 b_1\|_{L^2} \|\partial_2^2 \widetilde{b}_1\|_{L^2}^{1/2} \|\partial_1 \partial_2^2 \widetilde{b}_1\|_{L^2}^{1/2} \|\partial_2 \partial_1 \widetilde{u}_1\|_{L^2}^{1/2} \|\partial_2^2 \partial_1 \widetilde{u}_1\|_{L^2}^{1/2} \\
&\quad + C\|\partial_2^2 \partial_1 b_1\|_{L^2} \|\partial_2^2 \widetilde{u}_1\|_{L^2}^{1/2} \|\partial_1 \partial_2^2 \widetilde{u}_1\|_{L^2}^{1/2} \|\partial_2^2 \overline{b_1}\|_{L^2}^{1/2} \|\partial_2^3 \overline{b_1}\|_{L^2}^{1/2} \\
&\quad + C\|\partial_2 \widetilde{u}_1\|_{L^\infty} \|\partial_2^3 \overline{b_1}\|_{L^2} \|\partial_2^2 \partial_1 b_1\|_{L^2} \\
&\le C\|b\|_{H^3} \|\partial_1 \partial_2^2 \widetilde{b}_1\|_{L^2} \|\partial_1 u\|_{H^2} + C\|\partial_1 b\|_{H^2} \|\partial_1 \partial_2^2 \widetilde{u}_1\|_{L^2} \|b\|_{H^3} \\
&\quad + C\|\partial_1 \partial_2 \widetilde{u}_1\|_{H^1} \|b\|_{H^3} \|\partial_1 b\|_{H^2} \\
&\le C\|b\|_{H^3} \|\Lambda_1^\beta b\|_{H^3} \|\Lambda_1^\alpha u\|_{H^3} \le C\|b\|_{H^3} (\|\Lambda_1^\alpha u\|_{H^3}^2 + \|\Lambda_1^\beta b\|_{H^3}^2).
\end{aligned}$$

It follows that

$$M_{2,2} \le C\|(u,b)\|_{H^3} (\|\Lambda_1^\beta b\|_{H^3}^2 + \|\Lambda_1^\alpha u\|_{H^3}^2).$$

$M_{2,3}$ admits the same bound as the one for $L_{2,3}$, by (3.11),

$$M_{2,3} \le C\|b\|_{H^3} (\|\Lambda_1^\alpha u\|_{H^3}^2 + \|\Lambda_1^\beta b\|_{H^3}^2).$$

Combining the estimates for M_1 and M_2, we obtain

$$M \le C\|(u,b)\|_{H^3} (\|\Lambda_1^\alpha u\|_{H^3}^2 + \|\Lambda_1^\beta b\|_{H^3}^2). \tag{3.17}$$ Mbound

Now we estimate the term N,

$$\begin{aligned}
N &= \sum_{i=1}^2 \left(\int_\Omega \partial_i^3 (b\cdot\nabla u)\cdot \partial_i^3 b \ dx - \int_\Omega b\cdot\nabla \partial_i^3 u \cdot \partial_i^3 b \ dx \right) \\
&= \sum_{i=1}^2 \sum_{k=1}^3 C_3^k \int_\Omega \partial_i^k b \cdot \partial_i^{3-k} \nabla u \cdot \partial_i^3 b \ dx \\
&= \sum_{k=1}^3 C_3^k \int_\Omega \partial_1^k b \cdot \partial_1^{3-k} \nabla u \cdot \partial_1^3 b \ dx + \sum_{k=1}^3 C_3^k \int_\Omega \partial_2^k b \cdot \partial_2^{3-k} \nabla u \cdot \partial_2^3 b \ dx \\
&= N_1 + N_2.
\end{aligned}$$

By Hölder's inequality, Lemma 2.2, Lemma 2.4, Lemma 2.5 and Young's inequality,

$$\begin{aligned}
N_1 &= 3\int_\Omega \partial_1 \widetilde{b} \cdot \partial_1^2 \nabla u \cdot \partial_1^3 b \ dx + 3\int_\Omega \partial_1^2 \widetilde{b} \cdot \partial_1 \nabla u \cdot \partial_1^3 \widetilde{b} \ dx + \int_\Omega \partial_1^3 b \cdot \nabla u \cdot \partial_1^3 b \ dx \\
&\le C\|\partial_1 \widetilde{b}\|_{L^\infty} \|\partial_1^2 \nabla u\|_{L^2} \|\partial_1^3 b\|_{L^2} \\
&\quad + C\|\partial_1^3 \widetilde{b}\|_{L^2} \|\partial_1^2 \widetilde{b}\|_{L^2}^{1/2} \|\partial_1^3 \widetilde{b}\|_{L^2}^{1/2} \|\partial_1 \nabla u\|_{L^2}^{1/2} \|\partial_2 \partial_1 \nabla u\|_{L^2}^{1/2} + C\|\nabla u\|_{L^\infty} \|\partial_1^3 b\|_{L^2}^2 \\
&\le C\|\partial_1 b\|_{H^2} \|\partial_1 u\|_{H^2} \|b\|_{H^3} + C\|\partial_1 b\|_{H^2}^2 \|u\|_{H^3} \\
&\le C\|(u,b)\|_{H^3} (\|\Lambda_1^\beta b\|_{H^3}^2 + \|\Lambda_1^\alpha u\|_{H^3}^2).
\end{aligned}$$

To bound N_2, we split it into three terms,

$$\begin{aligned}N_2 &= 3\int_\Omega \partial_2 b\cdot\partial_2^2\nabla u\cdot\partial_2^3 b\ dx + 3\int_\Omega \partial_2^2 b\cdot\partial_2\nabla u\cdot\partial_2^3 b\ dx + \int_\Omega \partial_2^3 b\cdot\nabla u\cdot\partial_2^3 b\ dx\\ &= N_{2,1}+N_{2,2}+N_{2,3}.\end{aligned}$$

By Hölder's inequality and Lemma 2.4,

$$\begin{aligned}N_{2,1} &= 3\int_\Omega \partial_2 b\cdot\partial_2^2\nabla u\cdot\partial_2^3 b\ dx\\ &= 3\int_\Omega \partial_2 b_1\partial_2^2\partial_1 u\cdot\partial_2^3 b\ dx + 3\int_\Omega \partial_2 b_2\partial_2^2\partial_2 u\cdot\partial_2^3 b\ dx\\ &= N_{2,1,1}+N_{2,1,2}.\end{aligned}$$

By the divergence-free condition $\nabla\cdot b=0$,

$$\begin{aligned}N_{2,1,1} &= 3\int_\Omega \partial_2 b_1\partial_2^2\partial_1 u\cdot\partial_2^3 b\ dx\\ &= 3\int_\Omega \partial_2 b_1\partial_2^2\partial_1 u_1\partial_2^3 b_1\ dx + 3\int_\Omega \partial_2 b_1\partial_2^2\partial_1 u_2\partial_2^3 b_2\ dx\\ &= 3\int_\Omega \partial_2\overline{b_1}\partial_2^2\partial_1 u_1\partial_2^3 b_1\ dx + 3\int_\Omega \partial_2\widetilde{b}_1\partial_2^2\partial_1 u_1\partial_2^3 b_1\ dx - 3\int_\Omega \partial_2 b_1\partial_2^2\partial_1 u_2\partial_2^2\partial_1 b_1\ dx\\ &= N_{2,1,1,1}+N_{2,1,1,2}+N_{2,1,1,3}.\end{aligned}$$

By Lemma 2.1, Lemma 2.4, Hölder's inequality and Young's inequality,

$$\begin{aligned}N_{2,1,1,2}+N_{2,1,1,3} &= 3\int_\Omega \partial_2\widetilde{b}_1\partial_2^2\partial_1 u_1\partial_2^3 b_1\ dx - 3\int_\Omega \partial_2 b_1\partial_2^2\partial_1 u_2\partial_2^2\partial_1 b_1\ dx\\ &\le C\|\partial_2\widetilde{b}_1\|_{L^\infty}\|\partial_2^2\partial_1 u_1\|_{L^2}\|\partial_2^3 b_1\|_{L^2} + C\|\partial_2 b_1\|_{L^\infty}\|\partial_2^2\partial_1 u_2\|_{L^2}\|\partial_2^2\partial_1 b_1\|_{L^2}\\ &\le C\|\partial_1\partial_2\widetilde{b}_1\|_{H^1}\|\Lambda_1^\alpha u\|_{H^3}\|b\|_{H^3} + C\|\partial_1 b\|_{H^2}\|\Lambda_1^\alpha u\|_{H^3}\|b\|_{H^3}\\ &\le C\|b\|_{H^3}(\|\Lambda_1^\alpha u\|_{H^3}^2+\|\Lambda_1^\beta b\|_{H^3}^2).\end{aligned}$$

By integration by parts twice, Lemma 2.2, Lemma 2.1, Lemma 2.5, anisotropic Hölder's inequality and Young's inequality,

$$\begin{aligned}N_{2,1,1,1} &= 3\int_\Omega \partial_2\overline{b_1}\partial_2^2\partial_1\widetilde{u}_1\partial_2^3 b_1\ dx\\ &= 3\int_\Omega \partial_2\overline{b_1}\partial_2^2\partial_1\widetilde{u}_1\partial_2^3\widetilde{b}_1\ dx\\ &\le \|\partial_2\overline{b_1}\|_{L^\infty}\ \|\partial_2^2\partial_1\widetilde{u}_1\|_{L^2}\|\partial_2^3\widetilde{b}_1\|_{L^2}\\ &\le C\|b\|_{H^3}\ \|\Lambda_1^\alpha\partial_1\partial_2^2\widetilde{u}_1\|_{L^2}\|\Lambda_1^\beta\partial_2^3\widetilde{b}_1\|_{L^2}\\ &\le C\|b\|_{H^3}(\|\Lambda_1^\alpha u\|_{H^3}^2+\|\Lambda_1^\beta b\|_{H^3}^2).\end{aligned}$$

Hence

$$N_{2,1,1}\le C\|b\|_{H^3}(\|\Lambda_1^\alpha u\|_{H^3}^2+\|\Lambda_1^\beta b\|_{H^3}^2).$$

Similarly,

$$\begin{aligned}N_{2,1,2} &= -3\int_\Omega \partial_1\widetilde{b}_1\partial_2^2\partial_2 u\cdot\partial_2^3 b\ dx\\ &\le C\ \|(u,b)\|_{H^3}(\|\Lambda_1^\alpha u\|_{H^3}^2+\|\Lambda_1^\beta b\|_{H^3}^2).\end{aligned}$$

Combining the estimates for $N_{2,1,1}$ and $N_{2,1,2}$ leads to

$$N_{2,1} \le C\|(u,b)\|_{H^3}(\|\Lambda_1^\alpha u\|_{H^3}^2 + \|\Lambda_1^\beta b\|_{H^3}^2).$$

We rewrite $N_{2,2}$,

$$\begin{aligned}
N_{2,2} &= 3\int_\Omega \partial_2^2 b \cdot \partial_2 \nabla u \cdot \partial_2^3 b \, dx \\
&= 3\int_\Omega \partial_2^2 b_1 \partial_2 \partial_1 u \cdot \partial_2^3 b \, dx + 3\int_\Omega \partial_2^2 b_2 \partial_2^2 u \cdot \partial_2^3 b \, dx \\
&= N_{2,2,1} + N_{2,2,2}.
\end{aligned}$$

We decompose $N_{2,2,1}$ into three terms by Lemma 2.2,

$$\begin{aligned}
N_{2,2,1} &= 3\int_\Omega \partial_2^2 b_1 \partial_2 \partial_1 u \cdot \partial_2^3 b \, dx \\
&= 3\int_\Omega \partial_2^2 b_1 \partial_2 \partial_1 u_1 \partial_2^3 b_1 \, dx + 3\int_\Omega \partial_2^2 b_1 \partial_2 \partial_1 u_2 \partial_2^3 b_2 \, dx \\
&= 3\int_\Omega \partial_2^2 \overline{b_1} \partial_2 \partial_1 \widetilde{u}_1 \partial_2^3 b_1 \, dx + 3\int_\Omega \partial_2^2 \widetilde{b}_1 \partial_2 \partial_1 \widetilde{u}_1 \partial_2^3 b_1 \, dx - 3\int_\Omega \partial_2^2 b_1 \partial_2 \partial_1 \widetilde{u}_2 \partial_2^2 \partial_1 b_1 \, dx \\
&= N_{2,2,1,1} + N_{2,2,1,2} + N_{2,2,1,3}.
\end{aligned}$$

By applying integration by parts twice, Hölder's inequality, Lemma 2.1, Lemma 2.4, Lemma 2.5

$$\begin{aligned}
N_{2,2,1,1} &= 3\int_\Omega \partial_2^2 \overline{b_1} \partial_2 \partial_1 \widetilde{u}_1 \partial_2^3 b_1 \, dx = -3\int_\Omega \partial_2^2 \overline{b_1} \partial_2 \widetilde{u}_1 \partial_2^3 \partial_1 b_1 \, dx \\
&= 3\int_\Omega \partial_2^3 \overline{b_1} \partial_2 \widetilde{u}_1 \partial_2^2 \partial_1 b_1 \, dx + 3\int_\Omega \partial_2^2 \overline{b_1} \partial_2 \partial_2 \widetilde{u}_1 \partial_2^2 \partial_1 b_1 \, dx \\
&\le C\|\partial_2 \widetilde{u}_1\|_{L^\infty} \|\partial_2^3 \overline{b_1}\|_{L^2} \|\partial_2^2 \partial_1 b_1\|_{L^2} \\
&\quad + C\|\partial_1 \partial_2^2 b_1\|_{L^2} \|\partial_2^2 \widetilde{u}\|_{L^2}^{1/2} \|\partial_1 \partial_2^2 \widetilde{u}\|_{L^2}^{1/2} \|\partial_2^2 \overline{b_1}\|_{L^2}^{1/2} \|\partial_2^3 \overline{b_1}\|_{L^2}^{1/2} \\
&\le C\|\partial_1 \partial_2 \widetilde{u}_1\|_{H^1} \|b\|_{H^3} \|\Lambda_1^\beta b\|_{H^3} + C\|b\|_{H^3} \|\partial_1 \partial_2^2 \widetilde{u}\|_{L^2} \|\Lambda_1^\beta b\|_{H^3} \\
&\le C\|b\|_{H^3}(\|\Lambda_1^\alpha u\|_{H^3}^2 + \|\Lambda_1^\beta b\|_{H^3}^2).
\end{aligned}$$

By Lemma 2.1, Lemma 2.5 and Young's inequality,

$$\begin{aligned}
N_{2,2,1,2} &= 3\int_\Omega \partial_2^2 \widetilde{b}_1 \partial_2 \partial_1 \widetilde{u}_1 \partial_2^3 b_1 \, dx \\
&\le C\|\partial_2^3 b_1\|_{L^2} \|\partial_2 \partial_1 \widetilde{u}_1\|_{L^2}^{1/2} \|\partial_2^2 \partial_1 \widetilde{u}_1\|_{L^2}^{1/2} \|\partial_2^2 \widetilde{b}_1\|_{L^2}^{1/2} \|\partial_1 \partial_2^2 \widetilde{b}_1\|_{L^2}^{1/2} \\
&\le C\|b\|_{H^3} \|\partial_1 u\|_{H^2} \|\partial_1 \partial_2^2 \widetilde{b}_1\|_{L^2} \\
&\le C\|b\|_{H^3} \|\Lambda_1^\beta b\|_{H^3} \|\partial_1 u\|_{H^2} \le C\|b\|_{H^3}(\|\Lambda_1^\alpha u\|_{H^3}^2 + \|\Lambda_1^\beta b\|_{H^3}^2).
\end{aligned}$$

Similarly,

$$\begin{aligned}
N_{2,2,1,3} &= -3\int_\Omega \partial_2^2 b_1 \partial_2 \partial_1 \widetilde{u}_2 \partial_2^2 \partial_1 b_1 \, dx \\
&\le C\|\partial_2^2 \partial_1 b_1\|_{L^2} \|\partial_2 \partial_1 \widetilde{u}_2\|_{L^2}^{1/2} \|\partial_2 \partial_1^2 \widetilde{u}_2\|_{L^2}^{1/2} \|\partial_2^2 b_1\|_{L^2}^{1/2} \|\partial_2^3 b_1\|_{L^2}^{1/2} \\
&\le C\|b\|_{H^3} \|\Lambda_1^\beta b\|_{H^3} \|\partial_1 u\|_{H^2} \le C\|b\|_{H^3}(\|\Lambda_1^\alpha u\|_{H^3}^2 + \|\Lambda_1^\beta b\|_{H^3}^2).
\end{aligned}$$

Thus we obtain

$$N_{2,2,1} \le C\|b\|_{H^3}(\|\Lambda_1^\alpha u\|_{H^3}^2 + \|\Lambda_1^\beta b\|_{H^3}^2).$$

By divergence free condition,

$$\begin{aligned}N_{2,2,2} &= -3\int_\Omega \partial_2\partial_1\widetilde{b}_1\partial_2^2 u\cdot\partial_2^3 b\ dx = -3\int_\Omega \partial_2\partial_1\widetilde{b}_1\partial_2^2 u_1\partial_2^3 b_1\ dx - 3\int_\Omega \partial_2\partial_1\widetilde{b}_1\partial_2^2 u_2\partial_2^3 b_2\ dx\\ &= -3\int_\Omega \partial_2\partial_1\widetilde{b}_1\partial_2^2 u_1\partial_2^3 b_1\ dx - 3\int_\Omega \partial_2\partial_1\widetilde{b}_1\partial_2\partial_1 u_1\partial_2^2\partial_1 b_1\ dx\\ &= N_{2,2,2,1}+N_{2,2,2,2}\end{aligned}$$

We can see that $N_{2,2,2,1}=M_{2,2,1}$, $N_{2,2,2,2}=-M_{2,2,4}$, by the estimates of $M_{2,2,1}$ in (3.16) and $M_{2,2,4}$ in (3.15), we have

$$N_{2,2,2}\le C\|(u,b)\|_{H^3}(\|\Lambda_1^\alpha u\|_{H^3}^2+\|\Lambda_1^\beta b\|_{H^3}^2).$$

Further, combining the estimates of $N_{2,2,1}$ and $N_{2,2,2}$,

$$N_{2,2}\le C\|(u,b)\|_{H^3}(\|\Lambda_1^\alpha u\|_{H^3}^2+\|\Lambda_1^\beta b\|_{H^3}^2).$$

We can rewrite $N_{2,3}$,

$$\begin{aligned}N_{2,3} &= \int_\Omega \partial_2^3 b\cdot\nabla u\cdot\partial_2^3 b\ dx\\ &= \int_\Omega \partial_2^3 b_1\partial_1 u_1\partial_2^3 b_1+\partial_2^3 b_1\partial_1 u_2\partial_2^3 b_2+\partial_2^3 b_2\partial_2 u_1\partial_2^3 b_1+\partial_2^3 b_2\partial_2 u_2\partial_2^3 b_2\ dx\\ &= \int_\Omega \partial_2^3 b_1\partial_1 u_1\partial_2^3 b_1-\partial_2^3 b_1\partial_1\widetilde{u}_2\partial_2^2\partial_1 b_1-\partial_2^2\partial_1 b_1\partial_2 u_1\partial_2^3 b_1-\partial_2^2\partial_1 b_1\partial_1 u_1\partial_2^2\partial_1 b_1\ dx\\ &= N_{2,3,1}+N_{2,3,2}+N_{2,3,3}+N_{2,3,4}.\end{aligned}$$

By $u=\bar{u}+\widetilde{u}$ and $b=\bar{b}+\widetilde{b}$ and Lemma 2.1,

$$\begin{aligned}N_{2,3,1} &= \int_\Omega \partial_2^3 b_1\partial_1\widetilde{u}_1\partial_2^3 b_1\ dx = 2\int_\Omega \partial_2^3\widetilde{b}_1\partial_1\widetilde{u}_1\partial_2^3\bar{b}_1\ dx+\int_\Omega \partial_2^3\widetilde{b}_1\partial_1\widetilde{u}_1\partial_2^3\widetilde{b}_1\ dx\\ &\le C\|(u,b)\|_{H^3}(\|\Lambda_1^\beta b\|_{H^3}^2+\|\Lambda_1^\alpha u\|_{H^3}^2).\end{aligned}$$

By Hölder's inequality, Lemma 2.1 and Young's inequality,

$$\begin{aligned}N_{2,3,2} &= -\int_\Omega \partial_2^3 b_1\partial_1\widetilde{u}_2\partial_2^2\partial_1 b_1\ dx \le C\|\partial_1\widetilde{u}_2\|_{L^\infty}\|\partial_2^3 b_1\|_{L^2}\|\partial_2^2\partial_1 b_1\|_{L^2}\\ &\le C\|\partial_1^2\widetilde{u}_2\|_{H^1}\|\partial_1 b\|_{H^2}\|b\|_{H^3}\le C\|\Lambda_1^\alpha u\|_{H^3}\|\Lambda_1^\beta b\|_{H^3}\|b\|_{H^3}\\ &\le C\|b\|_{H^3}(\|\Lambda_1^\beta b\|_{H^3}^2+\|\Lambda_1^\alpha u\|_{H^3}^2).\end{aligned}$$

We can see that

$$N_{2,3,3}=\frac{1}{3}M_{2,1,1},\quad N_{2,3,4}=-\frac{1}{3}M_{2,1,4}.$$

By using the estimates in (3.13) and (3.14),

$$N_{2,3,3}+N_{2,3,4}\le C\|(u,b)\|_{H^3}(\|\Lambda_1^\beta b\|_{H^3}^2+\|\Lambda_1^\alpha u\|_{H^3}^2).$$

It follows that

$$N_{2,3}\le C\|(u,b)\|_{H^3}(\|\Lambda_1^\beta b\|_{H^3}^2+\|\Lambda_1^\alpha u\|_{H^3}^2).$$

Collecting the bounds from $N_{2,1}$, $N_{2,2}$, $N_{2,3}$ and N_1, we obtain

$$N\le C\|(u,b)\|_{H^3}(\|\Lambda_1^\beta b\|_{H^3}^2+\|\Lambda_1^\alpha u\|_{H^3}^2). \tag{3.18}$$

Nbound

Combining the bounds in (3.5),(3.12), (3.17) and (3.18) above leads to

$$J+K+L+M+N\le C\|(u,b)\|_{H^3}(\|\Lambda_1^\beta b\|_{H^3}^2+\|\Lambda_1^\alpha u\|_{H^3}^2).$$

Inserting the upper bound for $J+K+L+M+N$ in (3.1), Integrating in time, we get

$$\begin{aligned}
&\|(u,b)\|_{H^3}^2 + 2\nu\int_0^t \|\Lambda_1^\alpha u(\tau)\|_{H^3}^2 d\tau + 2\eta\int_0^t \|\Lambda_1^\beta b(\tau)\|_{H^3}^2 d\tau \\
&\qquad \le \|(u_0,b_0)\|_{H^3}^2 + C\int_0^t \|(u,b)\|_{H^3}(\|\Lambda_1^\beta b\|_{H^3}^2 + \|\partial_1^\alpha u\|_{H^3}^2)\, d\tau \\
&\qquad \le \|(u_0,b_0)\|_{H^3}^2 + C\sup_{0\le\tau\le t}\|(u,b)\|_{H^3}\int_0^t (\|\Lambda_1^\beta b\|_{H^3}^2 + \|\partial_1^\alpha u\|_{H^3}^2)\, d\tau \\
&\qquad \le E(0) + CE(t)^{\frac{3}{2}}.
\end{aligned}$$

Thus this completes the proof of (1.7). As a preparation for the application of the bootstrapping argument, we briefly explain why $\|u(t)\|_{H^3}$ and $\|b(t)\|_{H^3}$ are continuous functions of t. The desired continuity can be shown by following a procedure outlined in Chapter 3 of [44]. One starts with the weak continuity, $u, b \in C_W(0,\infty; H^3)$ and then show the continuity at $t = t_0$ by obtaining bounds for $\|u(t)\|_{H^3}$ and $\|b(t)\|_{H^3}$ in terms of $\|u(t_0)\|_{H^3}$, $\|b(t_0)\|_{H^3}$ and $|t-t_0|$.

Then an application of the bootstrapping argument to (1.7) leads to the desired upper bound in Theorem 1.1. We set

$$\varepsilon_0^2 := \frac{1}{16C^2}.$$

Assume $0 < \varepsilon \le \varepsilon_0$ and the initial data (u_0, b_0) satisfies

$$E(0) = \|(u_0,b_0)\|_{H^3}^2 < \varepsilon^2. \tag{3.19}$$ e0bound

To apply the bootstrapping argument, we make the ansatz that, for $t > 0$,

$$E(t) \le 4\varepsilon^2. \tag{3.20}$$ ansatz

It then follows from (1.7) that

$$E(t) \le E(0) + 2C\varepsilon\, E(t) \le E(0) + 2C\varepsilon_0\, E(t) = E(0) + \frac{1}{2}E(t) \quad \text{or} \quad E(t) \le 2E(0).$$

By (3.19), for all $t > 0$,

$$E(t) \le 2\varepsilon^2,$$

which is just half of the bound in the ansatz (3.20). The bootstrapping argument then asserts that this bound actually holds for all $t > 0$. It is worth commenting that even though $E(t)$ only involves the L^2 norms of ∂_i^3 $(i = 1,2)$ derivatives of $u, b, \Lambda_1^\alpha u, \Lambda_1^\beta b$, the boundedness of the L^2 norms of any other derivatives of order three follows via interpolation inequalities. For example,

$$\|\partial_1\partial_2^2 u\|_{L^2}^2 \le \frac{1}{3}\|\partial_1^3 u\|_{L^2}^2 + \frac{2}{3}\|\partial_2^3 u\|_{L^2}^2.$$

Thus, we obtain the desired global uniform bound on $\|(u(t),b(t))\|_{H^3}$. This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

⟨deca⟩ This section proves Theorem 1.2, which assesses that the oscillation part $(\widetilde{u},\widetilde{b})$ decays exponentially to zero in the H^2-norm as $t \to \infty$. We consider the equations of $(\widetilde{u},\widetilde{b})$ and apply the properties of the orthogonal decomposition and several anisotropic inequalities.

Proof of Theorem 1.2. We first write the equation of $(\bar u, \bar b)$. By taking the average of (1.2), we have

$$\begin{cases} \partial_t \bar u + \overline{u\cdot\nabla\widetilde u} + \begin{pmatrix}0\\ \partial_2\bar p\end{pmatrix} = \nu\Lambda_1^{2\alpha}\bar u + \overline{b\cdot\nabla\widetilde b} + \overline{A\cdot\nabla b},\\ \partial_t\bar b + \overline{u\cdot\nabla\widetilde b} = \eta\Lambda_1^{2\beta}\bar b + \overline{b\cdot\nabla\widetilde u}+\overline{A\cdot\nabla u},\end{cases} \tag{4.1}$$ E:MHDbar

where we have invoked the fact $\overline{u_2\partial_2\overline{u}} = 0$. The divergence-free condition $\partial_1\overline{u_1} + \partial_2\overline{u_2} = 0$ and the fact that $\overline{u_1}$ is only a function of x_2 yield that $\overline{u_2}$ is a constant. It must be zero due to $\overline{u_2} \in L^2(\Omega)$. Since $\partial_2\overline{u}$ is a function of x_2 alone,

$$\overline{u_2\partial_2\overline{u}} = \int_{\mathbb{T}} u_2\partial_2\overline{u}\,dx_1 = \partial_2\overline{u}\int_{\mathbb{T}} u_2 dx_1 = \partial_2\overline{u}\,\overline{u_2} = 0.$$

Taking the difference of (1.2) and (4.1), we obtain

$$\begin{cases}\partial_t\widetilde u + \widetilde{u\cdot\nabla\widetilde u} + u_2\partial_2\bar u + \nabla\widetilde p - \nu\Lambda_1^{2\alpha}\widetilde u - \widetilde{b\cdot\nabla\widetilde b} - b_2\partial_2\bar b - \widetilde{A\cdot\nabla b} = 0,\\ \partial_t\widetilde b + \widetilde{u\cdot\nabla\widetilde b} + u_2\partial_2\bar b - \eta\Lambda_1^{2\beta}\widetilde b - \widetilde{b\cdot\nabla\widetilde u} - b_2\partial_2\bar u - \widetilde{A\cdot\nabla u}=0.\end{cases} \tag{4.2}$$ E:MHDtilde

Taking the inner product of (4.2) with $(\widetilde u, \widetilde b)$, after integration by parts and divergence-free conditions, we find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\widetilde u\|_{L^2}^2 + \|\widetilde b\|_{L^2}^2) + \nu\|\Lambda_1^\alpha\widetilde u\|_{L^2}^2 + \eta\|\Lambda_1^\beta\widetilde b\|_{L^2}^2\\
&= -\int_\Omega \widetilde{u\cdot\nabla\widetilde u}\cdot\widetilde u\,dx - \int_\Omega u_2\partial_2\bar u\cdot\widetilde u\,dx - \int_\Omega \widetilde{u\cdot\nabla\widetilde b}\cdot\widetilde b\,dx\\
&\quad + \int_\Omega \widetilde{b\cdot\nabla\widetilde b}\cdot\widetilde u\,dx + \int_\Omega b_2\partial_2\bar b\cdot\widetilde u\,dx - \int_\Omega u_2\partial_2\bar b\cdot\widetilde b\,dx\\
&\quad + \int_\Omega \widetilde{b\cdot\nabla\widetilde u}\cdot\widetilde b\,dx + \int_\Omega b_2\partial_2\bar u\cdot\widetilde b\,dx\\
&:= R_1 + R_2 + R_3 + R_4 + R_5 + R_6 + R_7 + R_8.
\end{aligned} \tag{4.3}$$ E:R1R8

By Lemma 2.2 and divergence free condition of u,

$$R_1 = -\int_\Omega u\cdot\nabla\widetilde u\cdot\widetilde u\,dx + \int_\Omega \overline{u\cdot\nabla\widetilde u}\cdot\widetilde u\,dx = 0.$$

By Lemma 2.1 and Lemma 2.4,

$$\begin{aligned}
R_2 &= -\int_\Omega \widetilde u_2\partial_2\bar u\cdot\widetilde u\,dx \le \|\partial_2\bar u\|_{L^\infty}\|\Lambda_1^\alpha\widetilde u_2\|_{L^2}\|\Lambda_1^\alpha\widetilde u\|_{L^2}\\
&\le C\|u\|_{H^3}\|\Lambda_1^\alpha\widetilde u\|_{L^2}^2.
\end{aligned}$$

Similar to R_1, we also have $R_3 = 0$. By Lemma 2.2 and the divergence-free conditions,

$$\begin{aligned}
R_4 + R_7 &= \int_\Omega \widetilde{b\cdot\nabla\widetilde b}\cdot\widetilde u\,dx + \int_\Omega \widetilde{b\cdot\nabla\widetilde u}\cdot\widetilde b\,dx\\
&= \int_\Omega b\cdot\nabla\widetilde b\cdot\widetilde u\,dx + \int_\Omega b\cdot\nabla\widetilde u\cdot\widetilde b\,dx - \int_\Omega \overline{b\cdot\nabla\widetilde b}\cdot\widetilde u\,dx - \int_\Omega \overline{b\cdot\nabla\widetilde u}\cdot\widetilde b\,dx = 0.
\end{aligned}$$

By Hölder's inequality, Lemma 2.1, Lemma 2.4 and Young's inequality,

$$\begin{aligned}
R_5 &= \int_\Omega \widetilde b_2\partial_2\bar b\cdot\widetilde u\,dx\\
&\le C\|\partial_2\bar b\|_{L^\infty}\|\widetilde b_2\|_{L^2}\|\widetilde u\|_{L^2} \le C\|\partial_2\bar b\|_{L^\infty}\|\Lambda_1^\beta\widetilde b\|_{L^2}\|\Lambda_1^\alpha\widetilde u\|_{L^2}
\end{aligned}$$

$$\leq C\|b\|_{H^3}(\|\Lambda_1^\beta\widetilde{b}\|_{L^2}^2+\|\Lambda_1^\alpha\widetilde{u}\|_{L^2}^2).$$

Similarly,

$$R_6=-\int_\Omega \widetilde{u}_2\partial_2\bar{b}\cdot\widetilde{b}\,dx\leq C\|b\|_{H^3}(\|\Lambda_1^\beta\widetilde{b}\|_{L^2}^2+\|\Lambda_1^\alpha\widetilde{u}\|_{L^2}^2),$$
$$R_8\leq C\|u\|_{H^3}\|\Lambda_1^\beta\widetilde{b}\|_{L^2}^2.$$

Inserting the estimates for R_1 through R_8 in (4.3), we obtain

$$\frac{d}{dt}(\|\widetilde{u}\|_{L^2}^2+\|\widetilde{b}\|_{L^2}^2)+(2\nu-C\|(u,b)\|_{H^3})\|\Lambda_1^\alpha\widetilde{u}\|_{L^2}^2+(2\eta-C\|(u,b)\|_{H^3})\|\Lambda_1^\beta\widetilde{b}\|_{L^2}\leq 0.$$

According to Theorem 1.1, if $\varepsilon>0$ is sufficiently small and $\|u_0\|_{H^3}+\|b_0\|_{H^3}\leq\varepsilon$, then $\|u\|_{H^3}+\|b\|_{H^3}\leq C\varepsilon$ and

$$2\nu-C\|(u,b)\|_{H^3}\geq\nu,\qquad 2\eta-C\|(u,b)\|_{H^3}\geq\eta.$$

Thus we have

$$\frac{d}{dt}(\|\widetilde{u}\|_{L^2}^2+\|\widetilde{b}\|_{L^2}^2)+\nu\|\Lambda_1^\alpha\widetilde{u}\|_{L^2}^2+\eta\|\Lambda_1^\beta\widetilde{b}\|_{L^2}^2\leq 0.$$

By Lemma 2.1,

$$\|\widetilde{u}\|_{L^2}\leq\|\Lambda_1^\alpha\widetilde{u}\|_{L^2},\quad \|\widetilde{b}\|_{L^2}\leq\|\Lambda_1^\beta\widetilde{b}\|_{L^2},$$

we obtain by Gronwall's Lemma,

$$\|\widetilde{u}(t)\|_{L^2}+\|\widetilde{b}(t)\|_{L^2}\leq C(\|u_0\|_{L^2}+\|b_0\|_{L^2})e^{-C_1t},\tag{4.4}$$ L2estimatestilde

where $C_1=C_1(\nu,\eta)>0$.

Next we consider the exponential decay for $\|(\nabla\widetilde{u}(t),\nabla\widetilde{b}(t))\|_{L^2}$. Taking the gradient of (4.2) yields

$$\begin{cases}\partial_t\nabla\widetilde{u}+\nabla(\widetilde{u\cdot\nabla\widetilde{u}})+\nabla(u_2\partial_2\bar{u})+\nabla\nabla\widetilde{p}-\nu\partial_1^{2\alpha}\nabla\widetilde{u}\\ \qquad\qquad -\nabla(\widetilde{b\cdot\nabla\widetilde{b}})-\nabla(b_2\partial_2\bar{b})-\nabla(\widetilde{A\cdot\nabla b})=0,\\ \partial_t\nabla\widetilde{b}+\nabla(\widetilde{u\cdot\nabla\widetilde{b}})+\nabla(u_2\partial_2\bar{b})-\eta\partial_1^{2\beta}\nabla\widetilde{b}-\nabla(\widetilde{b\cdot\nabla\widetilde{u}})\\ \qquad\qquad -\nabla(b_2\partial_2\bar{u})-\nabla(\widetilde{A\cdot\nabla u})=0.\end{cases}\tag{4.5}$$ E:MHDnatilde

Multiplying (4.5) with $(\nabla\widetilde{u},\nabla\widetilde{b})$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla\widetilde{u}\|_{L^2}^2+\|\nabla\widetilde{b}\|_{L^2}^2)+\nu\|\Lambda_1^\alpha\nabla\widetilde{u}\|_{L^2}^2+\eta\|\Lambda_1^\beta\nabla\widetilde{b}\|_{L^2}^2\\ &=-\int_\Omega\nabla(\widetilde{u\cdot\nabla\widetilde{u}})\cdot\nabla\widetilde{u}\,dx-\int_\Omega\nabla(u_2\partial_2\bar{u})\cdot\nabla\widetilde{u}\,dx-\int_\Omega\nabla(\widetilde{u\cdot\nabla\widetilde{b}})\cdot\nabla\widetilde{b}\,dx\\ &\quad+\int_\Omega\nabla(\widetilde{b\cdot\nabla\widetilde{b}})\cdot\nabla\widetilde{u}\,d+\int_\Omega\nabla(b_2\partial_2\bar{b})\cdot\nabla\widetilde{u}\,dx-\int_\Omega\nabla(u_2\partial_2\bar{b})\cdot\nabla\widetilde{b}\,dx\\ &\quad+\int_\Omega\nabla(\widetilde{b\cdot\nabla\widetilde{u}})\cdot\nabla\widetilde{b}\,dx+\int_\Omega\nabla(b_2\partial_2\bar{u})\cdot\nabla\widetilde{b}\,dx\\ &:=S_1+S_2+S_3+S_4+S_5+S_6+S_7+S_8.\end{aligned}\tag{4.6}$$ E:S1S8

By Lemma 2.1, Lemma 2.2, Lemma 2.4 and divergence free condition of u,

$$S_1=-\int_\Omega\nabla(\widetilde{u\cdot\nabla\widetilde{u}})\cdot\nabla\widetilde{u}\,dx$$

$$
\begin{aligned}
&= -\int_\Omega \nabla(u\cdot\nabla\widetilde{u})\cdot\nabla\widetilde{u}\,dx + \int_\Omega \nabla(\overline{u\cdot\nabla\widetilde{u}})\cdot\nabla\widetilde{u}\,dx\\
&= -\int_\Omega \nabla u\cdot\nabla\widetilde{u}\cdot\nabla\widetilde{u}\,dx \le C\|u\|_{H^3}\|\Lambda_1^\alpha\nabla\widetilde{u}\|_{L^2}^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
S_3 &= -\int_\Omega \nabla(\widetilde{u\cdot\nabla\widetilde{b}})\cdot\nabla\widetilde{b}\,dx\\
&= -\int_\Omega \nabla(u\cdot\nabla\widetilde{b})\cdot\nabla\widetilde{b}\,dx + \int_\Omega \nabla(\overline{u\cdot\nabla\widetilde{b}})\cdot\nabla\widetilde{b}\,dx\\
&= -\int_\Omega \nabla u\cdot\nabla\widetilde{b}\cdot\nabla\widetilde{b}\,dx \le C\|u\|_{H^3}\|\Lambda_1^\beta\nabla\widetilde{b}\|_{L^2}^2.
\end{aligned}
$$

In order to bound S_2, we rewrite it as

$$
\begin{aligned}
S_2 &= -\int_\Omega \nabla(u_2\partial_2\bar{u})\cdot\nabla\widetilde{u}\,dx\\
&= -\int_\Omega \nabla u_2\partial_2\bar{u}\cdot\nabla\widetilde{u}\,dx - \int_\Omega u_2\nabla\partial_2\bar{u}\cdot\nabla\widetilde{u}\,dx\\
&= S_{2,1} + S_{2,2}.
\end{aligned}
$$

Similarly to the method estimating S_1,

$$
\begin{aligned}
S_{2,1} &= -\int_\Omega \nabla u_2\partial_2\bar{u}\cdot\nabla\widetilde{u}\,dx = -\int_\Omega \nabla\widetilde{u}_2\partial_2\bar{u}\cdot\nabla\widetilde{u}\,dx\\
&\le C\|u\|_{H^3}\|\Lambda_1^\alpha\nabla\widetilde{u}\|_{L^2}^2.
\end{aligned}
$$

By Lemma 2.1,

$$
\begin{aligned}
S_{2,2} &= -\int_\Omega \widetilde{u}_2\nabla\partial_2\bar{u}\cdot\nabla\widetilde{u}\,dx \le C\|\widetilde{u}_2\|_{L^\infty}\|\nabla\partial_2\bar{u}\|_{L^2}\|\nabla\widetilde{u}\|_{L^2}\\
&\le C\|\Lambda_1^\alpha\widetilde{u}_2\|_{H^1}\|\nabla\partial_2\bar{u}\|_{L^2}\|\nabla\widetilde{u}\|_{L^2} \le C(\|\Lambda_1^\alpha\widetilde{u}_2\|_{L^2} + \|\Lambda_1^\alpha\nabla\widetilde{u}_2\|_{L^2})\|\nabla\partial_2\bar{u}\|_{L^2}\|\nabla\widetilde{u}\|_{L^2}\\
&\le C(\|\Lambda_1\Lambda_1^\alpha\widetilde{u}_2\|_{L^2} + \|\Lambda_1^\alpha\nabla\widetilde{u}_2\|_{L^2})\|\nabla\partial_2\bar{u}\|_{L^2}\|\Lambda_1^\alpha\nabla\widetilde{u}\|_{L^2} \le C\|u\|_{H^3}\|\Lambda_1^\alpha\nabla\widetilde{u}\|_{L^2}^2.
\end{aligned}
$$

Thus, we have

$$
S_2 \le C\|u\|_{H^3}\|\Lambda_1^\alpha\nabla\widetilde{u}\|_{L^2}^2.
$$

By Lemma 2.2, Lemma 2.1, Lemma 2.4, the divergence-free condition $\nabla\cdot b = 0$, and Young's inequality,

$$
\begin{aligned}
S_4 + S_7 &= \int_\Omega \nabla(\widetilde{b\cdot\nabla\widetilde{b}})\cdot\nabla\widetilde{u}\,dx + \int_\Omega \nabla(\widetilde{b\cdot\nabla\widetilde{u}})\cdot\nabla\widetilde{b}\,dx\\
&= \int_\Omega \nabla(b\cdot\nabla\widetilde{b})\cdot\nabla\widetilde{u}\,dx + \int_\Omega \nabla(b\cdot\nabla\widetilde{u})\cdot\nabla\widetilde{b}\,dx - \int_\Omega \nabla(\overline{b\cdot\nabla\widetilde{b}})\cdot\nabla\widetilde{u}\,dx\\
&\quad - \int_\Omega \nabla(\overline{b\cdot\nabla\widetilde{u}})\cdot\nabla\widetilde{b}\,dx\\
&= \int_\Omega \nabla b\cdot\nabla\widetilde{b}\cdot\nabla\widetilde{u}\,dx + \int_\Omega \nabla b\cdot\nabla\widetilde{u}\cdot\nabla\widetilde{b}\,dx + \int_\Omega b\cdot\nabla^2\widetilde{b}\cdot\nabla\widetilde{u}\,dx\\
&\quad + \int_\Omega b\cdot\nabla^2\widetilde{u}\cdot\nabla\widetilde{b}\,dx\\
&= \int_\Omega \nabla b\cdot\nabla\widetilde{b}\cdot\nabla\widetilde{u}\,dx + \int_\Omega \nabla b\cdot\nabla\widetilde{u}\cdot\nabla\widetilde{b}\,dx
\end{aligned}
$$

$$\leq C\|b\|_{H^3}\|\Lambda_1^\alpha\nabla\widetilde{u}\|_{L^2}\|\Lambda_1^\beta\nabla\widetilde{b}\|_{L^2} \leq C\|b\|_{H^3}(\|\Lambda_1^\alpha\nabla\widetilde{u}\|_{L^2}^2 + \|\Lambda_1^\beta\nabla\widetilde{b}\|_{L^2}^2),$$

S_5 is bounded by

$$\begin{aligned} S_5 &= \int_\Omega \nabla(b_2\partial_2\bar{b})\cdot\nabla\widetilde{u}\,dx = \int_\Omega \nabla b_2\partial_2\bar{b}\cdot\nabla\widetilde{u}\,dx + b_2\partial_2\nabla\bar{b}\cdot\nabla\widetilde{u}\,dx \\ &\leq C\|b\|_{H^3}(\|\Lambda_1^\alpha\nabla\widetilde{u}\|_{L^2}^2 + \|\Lambda_1^\beta\nabla\widetilde{b}\|_{L^2}^2). \end{aligned}$$

Similarly, S_6 and S_8 are bounded by

$$\begin{aligned} &S_6 \leq C\|b\|_{H^3}(\|\Lambda_1^\alpha\nabla\widetilde{u}\|_{L^2}^2 + \|\Lambda_1^\beta\nabla\widetilde{b}\|_{L^2}^2), \\ &S_8 \leq C\|u\|_{H^3}\|\Lambda_1^\beta\nabla\widetilde{b}\|_{L^2}^2. \end{aligned}$$

Inserting the estimates for S_1 through S_8 in (4.6) gives

$$\begin{aligned} \frac{d}{dt}(\|\nabla\widetilde{u}\|_{L^2}^2 + \|\nabla\widetilde{b}\|_{L^2}^2) &+ (2\nu - C\|(u,b)\|_{H^3})\|\Lambda_1^\alpha\nabla\widetilde{u}\|_{L^2}^2 \\ &+ (2\eta - C\|(u,b)\|_{H^3})\|\Lambda_1^\beta\nabla\widetilde{b}\|_{L^2} \leq 0. \end{aligned}$$

Choosing $\varepsilon > 0$ sufficiently small and by Theorem 1.1, if $\|u_0\|_{H^3} + \|b_0\|_{H^3} \leq \varepsilon$, then $\|u\|_{H^3} + \|b\|_{H^3} \leq C\varepsilon$ and

$$2\nu - C\|(u,b)\|_{H^3} \geq \nu, \qquad 2\eta - C\|(u,b)\|_{H^3} \geq \eta.$$

Thus we could have

$$\frac{d}{dt}(\|\nabla\widetilde{u}\|_{L^2}^2 + \|\nabla\widetilde{b}\|_{L^2}^2) + \nu\|\Lambda_1^\alpha\nabla\widetilde{u}\|_{L^2}^2 + \eta\|\Lambda_1^\beta\nabla\widetilde{b}\|_{L^2} \leq 0.$$

By Lemma 2.1,

$$\|\nabla\widetilde{u}\|_{L^2} \leq \|\Lambda_1^\alpha\nabla\widetilde{u}\|_{L^2}, \quad \|\nabla\widetilde{b}\|_{L^2} \leq \|\Lambda_1^\beta\nabla\widetilde{b}\|_{L^2},$$

we obtain by Gronwall's Lemma,

$$\|\nabla\widetilde{u}(t)\|_{L^2} + \|\nabla\widetilde{b}(t)\|_{L^2} \leq C(\|\nabla u_0\|_{L^2} + \|\nabla b_0\|_{L^2})e^{-C_2 t}, \tag{4.7}$$ H1tilde

where $C_2 = C_2(\nu,\eta) > 0$.

Then we consider the exponential decay for $\|(\Delta\widetilde{u}(t), \Delta\widetilde{b}(t))\|_{L^2}$. Due to the standard Hessian-Laplace inequality (see, e.g., [51])

$$\|\nabla^2 f\|_{L^2} \leq C\,\|\Delta f\|_{L^2},$$

it suffices to estimate $\|(\Delta u, \Delta b)\|_{L^2}$. Taking the divergence of (4.5) yields

$$\begin{cases} \partial_t\Delta\widetilde{u} + \Delta(\widetilde{u\cdot\nabla\widetilde{u}}) + \Delta(\widetilde{u_2\partial_2\bar{u}}) + \Delta\nabla\widetilde{p} - \nu\partial_1^{2\alpha}\Delta\widetilde{u} \\ \qquad\qquad -\Delta(\widetilde{b\cdot\nabla\widetilde{b}}) - \Delta(b_2\partial_2\bar{b}) - \Delta(\widetilde{A\cdot\nabla b}) = 0, \\ \partial_t\Delta\widetilde{b} + \Delta(\widetilde{u\cdot\nabla\widetilde{b}}) + \Delta(u_2\partial_2\bar{b}) - \eta\partial_1^{2\beta}\Delta\widetilde{b} - \Delta(\widetilde{b\cdot\nabla\widetilde{u}}) \\ \qquad\qquad -\Delta(b_2\partial_2\bar{u}) - \Delta(\widetilde{A\cdot\nabla u}) = 0. \end{cases} \tag{4.8}$$ E:MHDdetilde

Multiplying (4.8) with $(\Delta\widetilde{u}, \Delta\widetilde{b})$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\Delta\widetilde{u}\|_{L^2}^2 + \|\Delta\widetilde{b}\|_{L^2}^2) + \nu\|\Lambda_1^\alpha\Delta\widetilde{u}\|_{L^2}^2 + \eta\|\Lambda_1^\beta\Delta\widetilde{b}\|_{L^2}^2 \\
&= -\int_\Omega \Delta(\widetilde{u\cdot\nabla\widetilde{u}})\cdot\Delta\widetilde{u}\,dx - \int_\Omega \Delta(u_2\partial_2\bar{u})\cdot\Delta\widetilde{u}\,dx - \int_\Omega \Delta(\widetilde{u\cdot\nabla\widetilde{b}})\cdot\Delta\widetilde{b}\,dx \\
&\quad + \int_\Omega \Delta(\widetilde{b\cdot\nabla\widetilde{b}})\cdot\Delta\widetilde{u}\,d + \int_\Omega \Delta(b_2\partial_2\bar{b})\cdot\Delta\widetilde{u}\,dx - \int_\Omega \Delta(u_2\partial_2\bar{b})\cdot\Delta\widetilde{b}\,dx \\
&\quad + \int_\Omega \Delta(\widetilde{b\cdot\nabla\widetilde{u}})\cdot\Delta\widetilde{b}\,dx + \int_\Omega \Delta(b_2\partial_2\bar{u})\cdot\Delta\widetilde{b}\,dx \\
&:= T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 + T_8.
\end{aligned}
\tag{4.9}
$$

E:T1T8

We can rewrite T_1 as

$$
\begin{aligned}
T_1 &= -\int_\Omega \Delta(\widetilde{u\cdot\nabla\widetilde{u}})\cdot\Delta\widetilde{u}\,dx \\
&= -\int_\Omega \Delta(u\cdot\nabla\widetilde{u})\cdot\Delta\widetilde{u}\,dx + \int_\Omega \Delta(\overline{u\cdot\nabla\widetilde{u}})\cdot\Delta\widetilde{u}\,dx \\
&= -\int_\Omega \Delta u\cdot\nabla\widetilde{u}\cdot\Delta\widetilde{u}\,dx - 2\int_\Omega \nabla u\cdot\nabla^2\widetilde{u}\cdot\Delta\widetilde{u}\,dx - \int_\Omega u\cdot\nabla\Delta\widetilde{u}\cdot\Delta\widetilde{u}\,dx \\
&= -\int_\Omega \Delta u\cdot\nabla\widetilde{u}\cdot\Delta\widetilde{u}\,dx - 2\int_\Omega \nabla u\cdot\nabla^2\widetilde{u}\cdot\Delta\widetilde{u}\,dx \\
&= T_{1,1} + T_{1,2}.
\end{aligned}
$$

By Lemma 2.1,

$$
\begin{aligned}
T_{1,1} &= -\int_\Omega \Delta u\cdot\nabla\widetilde{u}\cdot\Delta\widetilde{u}\,dx \le C\|u\|_{H^3}\|\Lambda_1^\alpha\Delta\widetilde{u}\|_{L^2}^2, \\
T_{1,2} &= -2\int_\Omega \nabla u\cdot\nabla^2\widetilde{u}\cdot\Delta\widetilde{u}\,dx \le C\|u\|_{H^3}\|\Lambda_1^\alpha\Delta\widetilde{u}\|_{L^2}^2.
\end{aligned}
$$

Similarly, T_3 can be split into two terms,

$$
\begin{aligned}
T_3 &= -\int_\Omega \Delta(\widetilde{u\cdot\nabla\widetilde{b}})\cdot\Delta\widetilde{b}\,dx \\
&= -\int_\Omega \Delta(u\cdot\nabla\widetilde{b})\cdot\Delta\widetilde{b}\,dx + \int_\Omega \nabla(\overline{u\cdot\nabla\widetilde{b}})\cdot\nabla\widetilde{b}\,dx \\
&= -\int_\Omega \Delta u\cdot\nabla\widetilde{b}\cdot\Delta\widetilde{b}\,dx - 2\int_\Omega \nabla u\cdot\nabla^2\widetilde{b}\cdot\Delta\widetilde{b}\,dx - \int_\Omega u\cdot\nabla\Delta\widetilde{b}\cdot\Delta\widetilde{b}\,dx \\
&= -\int_\Omega \Delta u\cdot\nabla\widetilde{b}\cdot\Delta\widetilde{b}\,dx - 2\int_\Omega \nabla u\cdot\nabla^2\widetilde{b}\cdot\Delta\widetilde{b}\,dx \\
&= T_{3,1} + T_{3,2}.
\end{aligned}
$$

By Lemma 2.1,

$$
T_3 \le C\|u\|_{H^3}\|\Lambda_1^\beta\Delta\widetilde{b}\|_{L^2}^2.
$$

To bound T_2, we rewrite it as

$$
\begin{aligned}
T_2 &= -\int_\Omega \Delta(u_2\partial_2\bar{u})\cdot\Delta\widetilde{u}\,dx \\
&= -\int_\Omega \Delta u_2\partial_2\bar{u}\cdot\Delta\widetilde{u}\,dx - 2\int_\Omega \nabla u_2\partial_2\nabla\bar{u}\cdot\Delta\widetilde{u}\,dx - \int_\Omega u_2\partial_2\Delta\bar{u}\cdot\Delta\widetilde{u}\,dx
\end{aligned}
$$

$$= T_{2,1} + T_{2,2} + T_{2,3}.$$

By Lemma 2.1,

$$T_{2,1} = -\int_\Omega \Delta u_2 \partial_2 \bar{u} \cdot \Delta \widetilde{u}\, dx \le C\|u\|_{H^3}\|\Lambda_1^\alpha \Delta \widetilde{u}\|_{L^2}^2,$$

$$T_{2,2} = -2\int_\Omega \nabla \widetilde{u}_2 \partial_2 \nabla \bar{u} \cdot \Delta \widetilde{u}\, dx \le C\|\nabla \widetilde{u}_2\|_{L^\infty}\|\partial_2 \nabla \bar{u}\|_{L^2}\|\Delta \widetilde{u}\|_{L^2} \le C\|u\|_{H^3}\|\Lambda_1^\alpha \Delta \widetilde{u}\|_{L^2}^2,$$

$$T_{2,3} = -\int_\Omega \widetilde{u}_2 \partial_2 \Delta \bar{u} \cdot \Delta \widetilde{u}\, dx \le C\|\widetilde{u}_2\|_{L^\infty}\|\partial_2 \Delta \bar{u}\|_{L^2}\|\Delta \widetilde{u}\|_{L^2} \le C\|u\|_{H^3}\|\Lambda_1^\alpha \Delta \widetilde{u}\|_{L^2}^2.$$

So it follows that

$$T_2 \le C\|u\|_{H^3}\|\Lambda_1^\alpha \Delta \widetilde{u}\|_{L^2}^2.$$

We combine T_4 and T_7,

$$\begin{aligned}
T_4 + T_7 &= \int_\Omega \Delta(\widetilde{b \cdot \nabla \widetilde{b}}) \cdot \Delta \widetilde{u}\, dx + \int_\Omega \Delta(\widetilde{b \cdot \Delta \widetilde{u}}) \cdot \Delta \widetilde{b}\, dx \\
&= \int_\Omega \Delta(b \cdot \nabla \widetilde{b}) \cdot \Delta \widetilde{u}\, dx + \int_\Omega \Delta(b \cdot \nabla \widetilde{u}) \cdot \Delta \widetilde{b}\, dx - \int_\Omega \Delta(\overline{b \cdot \nabla \widetilde{b}}) \cdot \Delta \widetilde{u}\, dx \\
&\quad - \int_\Omega \Delta(\overline{b \cdot \nabla \widetilde{u}}) \cdot \Delta \widetilde{b}\, dx \\
&= \int_\Omega \Delta b \cdot \nabla \widetilde{b} \cdot \Delta \widetilde{u}\, dx + 2\int_\Omega \nabla b \cdot \nabla^2 \widetilde{b} \cdot \Delta \widetilde{u}\, dx + \int_\Omega b \cdot \nabla \Delta \widetilde{b} \cdot \Delta \widetilde{u}\, dx \\
&\quad + \int_\Omega \Delta b \cdot \nabla \widetilde{u} \cdot \Delta \widetilde{b}\, dx + 2\int_\Omega \nabla b \cdot \nabla^2 \widetilde{u} \cdot \Delta \widetilde{b}\, dx + \int_\Omega b \cdot \nabla \Delta \widetilde{u} \cdot \Delta \widetilde{b}\, dx \\
&= \int_\Omega \Delta b \cdot \nabla \widetilde{b} \cdot \Delta \widetilde{u}\, dx + 2\int_\Omega \nabla b \cdot \nabla^2 \widetilde{b} \cdot \Delta \widetilde{u}\, dx + \int_\Omega \Delta b \cdot \nabla \widetilde{u} \cdot \Delta \widetilde{b}\, dx \\
&\quad + 2\int_\Omega \nabla b \cdot \nabla^2 \widetilde{u} \cdot \Delta \widetilde{b}\, dx \\
&= T_{4,1} + T_{4,2} + T_{4,3} + T_{4,4}.
\end{aligned}$$

By Lemma 2.1,

$$T_{4,1} \le C\|b\|_{H^3}(\|\Lambda_1^\alpha \Delta \widetilde{u}\|_{L^2}^2 + \|\Lambda_1^\beta \Delta \widetilde{b}\|_{L^2}^2).$$

Similarly,

$$T_{4,3} \le C\|b\|_{H^3}(\|\Lambda_1^\alpha \Delta \widetilde{u}\|_{L^2}^2 + \|\Lambda_1^\beta \Delta \widetilde{b}\|_{L^2}^2),$$

$$\begin{aligned}
T_{4,2} + T_{4,4} &= 2\int_\Omega \nabla b \cdot \nabla^2 \widetilde{b} \cdot \Delta \widetilde{u}\, dx + 2\int_\Omega \nabla b \cdot \nabla^2 \widetilde{u} \cdot \Delta \widetilde{b}\, dx \\
&\le C\|b\|_{H^3}(\|\Lambda_1^\alpha \Delta \widetilde{u}\|_{L^2}^2 + \|\Lambda_1^\beta \Delta \widetilde{b}\|_{L^2}^2).
\end{aligned}$$

Combining the estimates of $T_{4,1}$, $T_{4,2}$, $T_{4,3}$ and $T_{4,4}$, we have

$$T_4 + T_7 \le C\|b\|_{H^3}(\|\Lambda_1^\alpha \Delta \widetilde{u}\|_{L^2}^2 + \|\Lambda_1^\beta \Delta \widetilde{b}\|_{L^2}^2).$$

To bound T_5, we write

$$\begin{aligned}
T_5 &= \int_\Omega \Delta(b_2 \partial_2 \bar{b}) \cdot \Delta \widetilde{u}\, dx \\
&= \int_\Omega \Delta b_2 \partial_2 \bar{b} \cdot \Delta \widetilde{u}\, dx + 2\int_\Omega \nabla b_2 \partial_2 \nabla \bar{b} \cdot \Delta \widetilde{u}\, dx + \int_\Omega b_2 \partial_2 \Delta \bar{b} \cdot \Delta \widetilde{u}\, dx \\
&= T_{5,1} + T_{5,2} + T_{5,3}.
\end{aligned}$$

By Lemma 2.1,

$$T_{5,1} = \int_\Omega \Delta b_2 \partial_2 \bar{b} \cdot \Delta\widetilde{u}\, dx = \int_\Omega \Delta\widetilde{b}_2 \partial_2 \bar{b} \cdot \Delta\widetilde{u}\, dx \\ \leq C\|b\|_{H^3}(\|\Lambda_1^\alpha \Delta\widetilde{u}\|_{L^2}^2 + \|\Lambda_1^\beta \Delta\widetilde{b}\|_{L^2}^2),$$

$$T_{5,2} = 2\int_\Omega \nabla\widetilde{b}_2 \partial_2 \nabla\bar{b} \cdot \Delta\widetilde{u}\, dx \\ \leq C\|b\|_{H^3}(\|\Lambda_1^\alpha \Delta\widetilde{u}\|_{L^2}^2 + \|\Lambda_1^\beta \Delta\widetilde{b}\|_{L^2}^2),$$

$$T_{5,3} = -\int_\Omega \widetilde{b}_2 \partial_2 \Delta\bar{b} \cdot \Delta\widetilde{u}\, dx \\ \leq C\|b\|_{H^3}(\|\Lambda_1^\alpha \Delta\widetilde{u}\|_{L^2}^2 + \|\Lambda_1^\beta \Delta\widetilde{b}\|_{L^2}^2).$$

Thus, we obtain

$$T_5 = -\int_\Omega \Delta(b_2\partial_2\bar{b}) \cdot \Delta\widetilde{u}\, dx \leq C\|b\|_{H^3}(\|\Lambda_1^\alpha \Delta\widetilde{u}\|_{L^2}^2 + \|\Lambda_1^\beta \Delta\widetilde{b}\|_{L^2}^2).$$

Similarly, T_6 and T_8 are bounded by

$$T_6 = -\int_\Omega \Delta(u_2\partial_2\bar{b}) \cdot \Delta\widetilde{b}\, dx \leq C\|b\|_{H^3}(\|\Lambda_1^\alpha \Delta\widetilde{u}\|_{L^2}^2 + \|\Lambda_1^\beta \Delta\widetilde{b}\|_{L^2}^2)$$
$$T_8 = \int_\Omega \Delta(b_2\partial_2\bar{u}) \cdot \Delta\widetilde{b}\, dx \leq C\|u\|_{H^3}\|\Lambda_1^\beta \Delta\widetilde{b}\|_{L^2}^2.$$

Plugging the estimates for T_1 through T_8 in (4.9),

$$\frac{d}{dt}(\|\Delta\widetilde{u}\|_{L^2}^2 + \|\Delta\widetilde{b}\|_{L^2}^2) + (2\nu - C\|(u,b)\|_{H^3})\|\Lambda_1^\alpha \Delta\widetilde{u}\|_{L^2}^2 \\ + (2\eta - C\|(u,b)\|_{H^3})\|\Lambda_1^\beta \Delta\widetilde{b}\|_{L^2} \leq 0.$$

Choosing $\varepsilon > 0$ sufficiently small and by Theorem 1.1, if $\|u_0\|_{H^3} + \|b_0\|_{H^3} \leq \varepsilon$, then $\|u\|_{H^3} + \|b\|_{H^3} \leq C\varepsilon$ and

$$2\nu - C\|(u,b)\|_{H^3} \geq \nu, \qquad 2\eta - C\|(u,b)\|_{H^3} \geq \eta.$$

Then we obtain

$$\frac{d}{dt}(\|\Delta\widetilde{u}\|_{L^2}^2 + \|\Delta\widetilde{b}\|_{L^2}^2) + \nu\|\Lambda_1^\alpha \Delta\widetilde{u}\|_{L^2}^2 + \eta\|\Lambda_1^\beta \Delta\widetilde{b}\|_{L^2} \leq 0.$$

By Lemma 2.1,

$$\|\Delta\widetilde{u}\|_{L^2} \leq \|\Lambda_1^\alpha \Delta\widetilde{u}\|_{L^2}, \quad \|\Delta\widetilde{b}\|_{L^2} \leq \|\Lambda_1^\beta \Delta\widetilde{b}\|_{L^2},$$

we obtain by Gronwall's Lemma,

$$\|\Delta\widetilde{u}(t)\|_{L^2} + \|\Delta\widetilde{b}(t)\|_{L^2} \leq C(\|\Delta u_0\|_{L^2} + \|\Delta b_0\|_{L^2})e^{-C_3 t}, \tag{4.10}$$ H2tilde

where $C_3 = C_3(\nu,\eta) > 0$. Combining the estimates in (4.4), (4.7) and (4.10), we obtain the desired decay result (1.6) in Theorem 1.2. □

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[1] Department of Mathematics, 5795 Lewiston Rd, Niagara University, Lewiston, NY 14109, United States

Email address: wfeng@niagara.edu

[2] Department of Mathematics, University of Arizona, Tucson, AZ 85721, United States

Email address: weinanwang@math.arizona.edu

[3] Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, United States

Email address: jiahong.wu@okstate.edu