GLOBAL WELL-POSEDNESS OF THE 2D MHD EQUATIONS OF DAMPED WAVE TYPE IN SOBOLEV SPACE

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ABSTRACT. The magnetohydrodynamic system of damped wave type (abbreviated as MHD-wave system) is formally derived from Maxwell's equations of electromagnetism by keeping the usually ignored small term involving the product of permittivity and magnetic permeability. When this term is ignored in the context of non-relativistic charged fluid, one obtains the standard MHD system. This extra term in the MHD-wave system assumes the form $\gamma\partial_{tt}b$ with $\gamma > 0$ being a small constant and b the magnetic field. Mathematically this term makes the global well-posedness problem much more challenging than the corresponding MHD system. Even the global existence and regularity problem for the 2D MHD-wave system appears to be open. This paper solves the global well-posedness problem in a critical Sobolev setting when γ and the size of the initial data satisfy a suitable constraint. In addition, the solution of the MHD-wave system is shown to converge to that of the corresponding MHD system with an explicit rate. The energy method does not work here and this paper presents a new approach.

1. Introduction

This paper focuses on the magnetohydrodynamic system of the damped wave type (or simply MHD-wave system),

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nabla(p + \frac{|b|^2}{2}) = \nu\Delta u + b\cdot\nabla b, \quad x\in\mathbb{R}^2,\, t>0, \\ \gamma\partial_{tt}b + \partial_t b + u\cdot\nabla b = \eta\Delta b + b\cdot\nabla u, \\ \nabla\cdot u = \nabla\cdot b = 0, \\ u(x,0) = u_0(x), \quad b(x,0) = b_0(x), \quad (\partial_t b)(x,0) = a_0(x). \end{cases} \tag{1.1}$$

where u denotes the velocity field, b the magnetic field and p the pressure, and $\gamma > 0$, $\nu > 0$ and $\eta > 0$ are real parameters. The goal of this paper is to solve the global existence and regularity problem of the 2D MHD-wave system when γ and the size of the initial data satisfy a suitable constraint. The spatial domain is taken to be the whole space $\mathbb{R}^2$. In contrast to the standard 2D MHD equations, the equation of b in (1.1) is hyperbolic and the extra term $\gamma\partial_{tt}b$ is mathematically a bad term in the sense of energy estimates. As a consequence, the approach of establishing global bounds on solutions of (1.1) via energy estimates does not work. This paper presents a new strategy to understand the global regularity problem on (1.1).

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The MHD-wave system can be formally derived from the coupled Maxwell–Ohm equations

$$\begin{cases} \nabla\times b-\epsilon\mu\partial_t E=\mu j,\\ \partial_t b=-\nabla\times E,\\ \nabla\cdot b=0,\\ \frac{1}{\sigma}j=E+u\times b, \end{cases} \tag{1.2}$$

where E denotes the electric field, j the current density, ϵ the permittivity, μ the magnetic permeability and σ the electrical conductivity. When the process is not relativistic, $\epsilon\mu$ is small. The formal derivation (1.1) from (1.2) is actually quite simple.

$$\begin{aligned} \partial_t b &= -\nabla\times E=-\nabla\times(\frac{1}{\sigma}j-u\times b)\\ &= -\frac{1}{\sigma\,\mu}\nabla\times\nabla\times b+\frac{\epsilon}{\sigma}\nabla\times\partial_t E+\nabla\times(u\times b)\\ &= -\frac{1}{\sigma\,\mu}\nabla\times\nabla\times b-\frac{\epsilon}{\sigma}\partial_{tt}b+\nabla\times(u\times b) \end{aligned}$$

By the identities

$$\begin{aligned} &\nabla\times\nabla\times b=\nabla(\nabla\cdot b)-\Delta b,\\ &\nabla\times(u\times b)=b\cdot\nabla u+u(\nabla\cdot b)-u\cdot\nabla b-b(\nabla\cdot u), \end{aligned}$$

we obtain

$$\frac{\epsilon}{\sigma}\partial_{tt}b+\partial_t b-\frac{1}{\sigma\,\mu}\Delta b+u\cdot\nabla b=b\cdot\nabla u.$$

Letting

$$\frac{\epsilon}{\sigma}=\gamma,\quad \frac{1}{\sigma\,\mu}=\eta$$

yields (1.1). In particular, when we set $\epsilon=0$, we obtain the standard incompressible MHD equations.

The incompressible MHD system is given by

$$\begin{cases} \partial_t u+u\cdot\nabla u+\nabla(p+\frac{|b|^2}{2})=\nu\Delta u+b\cdot\nabla b, \quad x\in\mathbb{R}^2,\, t>0,\\ \partial_t b+u\cdot\nabla b=\eta\Delta b+b\cdot\nabla u,\\ \nabla\cdot u=\nabla\cdot b=0, \end{cases} \tag{1.3}$$

which differs from (1.1) only in the term $\gamma\partial_{tt}b$. The MHD system is the primary model for electrically conducting fluids and has been studied rather extensively (see, e.g., [2,9,28]). In particular, the 2D MHD equation have been shown to be globally well-posedness [30]. There have been substantial recent developments on various fundamental issues concerning the MHD systems (see, e.g., [1,3–8,10–25,27,29,31–42]).

Rigorous mathematical studies on the MHD-wave equations are more recent. Important results such as the small data global well-posedness of the MHD-wave equations in Fourier-Sobolev spaces have been obtained ([26]). But many fundamental issues remain open. One natural question is whether or not general large solutions to the 2D MHD-wave equations are always global in time. This is not a trivial problem. Due to the presence of the extra term $\gamma\partial_{tt}b$ in the MHD-wave equations, the approach of the energy method no longer works for the MHD-wave equations. The 2D MHD equations in (1.3) is globally well-posed in any space $H^k(\mathbb{R}^2)$ for any $k \geq 0$. This is a direct consequence of the global *a priori* bound

$$\|(u,b)(t)\|_{H^k}^2 + \nu \int_0^t \|\nabla u\|_{H^k}^2 \, d\tau + \eta \int_0^t \|\nabla b(\tau)\|_{H^k}^2 \, d\tau \leq C \, \|(u_0,b_0)\|_{H^k}^2$$

obeyed by any solution (u,b) of (1.3).

However, it does not appear to be possible to obtain any global H^k-bound with $k \geq 0$ for the solutions of (1.1). In fact, we do not even know whether the L^2-norm of (u,b) is bounded for all time. If we perform the standard energy estimate, we would obtain

$$\gamma \int_{\mathbb{R}^2} b \cdot \partial_{tt} b \, dx + \frac{1}{2}\frac{d}{dt}\|(u,b)\|_{L^2}^2 + \nu\|\nabla u\|_{L^2}^2 + \eta\|\nabla b\|_{L^2}^2 = 0. \tag{1.4}$$

Here we have used $\nabla \cdot u = \nabla \cdot b = 0$ to eliminate the nonlinear terms. Naturally one attempts to deal with the bad term

$$\gamma \int_{\mathbb{R}^2} b \cdot \partial_{tt} b \, dx$$

by combining with the estimate of $\|\partial_t b\|_{L^2}^2$. Taking the L^2-inner product of the b-equation with $\partial_t b$ yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\gamma\|\partial_t b\|_{L^2}^2 + \eta\|\nabla b\|_{L^2}^2) + \|\partial_t b\|_{L^2}^2 \\ &= -\langle \partial_t b, u \cdot \nabla b\rangle + \langle \partial_t b, b \cdot \nabla u\rangle, \end{aligned} \tag{1.5}$$

where $\langle f, g\rangle$ denotes the L^2-inner product. Multiplying (1.5) by 2γ and adding to (1.4) yield

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2 + \|b\|_{L^2}^2 + 2\gamma\langle b, \partial_t b\rangle + 2\gamma^2\|\partial_t b\|_{L^2}^2 + 2\gamma\eta\|\nabla b\|_{L^2}^2) \\ &+ \nu\|\nabla u\|_{L^2}^2 + \eta\|\nabla b\|_{L^2}^2 + \gamma\,\|\partial_t b\|_{L^2}^2 \\ &= -2\gamma\langle \partial_t b, u \cdot \nabla b\rangle + 2\gamma\langle \partial_t b, b \cdot \nabla u\rangle. \end{aligned}$$

In order to obtain suitable upper bounds for the two terms on the right-hand side, we need higher derivatives. Therefore, direct energy estimates do not lead to the desired global bounds even for the L^2-norm of the solution to (1.1).

The new idea of this paper is to examine the difference between the solution of the MHD-wave equations in (1.1) and that of the MHD equations in (1.3). The goal is to show this difference is globally bounded for sufficiently small $\gamma > 0$. Since the solution of the 2D MHD equations is known to be global in time, the global bound on the difference leads to the global bound on the solution of the MHD-wave equation. This global bound allows us to establish the following global existence and regularity result.

Theorem 1.1. *Consider the 2D MHD-wave system (1.1) with $\gamma > 0$, $\nu > 0$ and $\eta > 0$. Assume the initial data $(u_0, b_0, a_0) \in L^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$, and $\nabla \cdot u_0 = \nabla \cdot b_0 =$*

$\nabla \cdot a_0 = 0$. *Assume that, for a suitable constant $C_0 > 0$ and $\gamma_0 > 0$,*

$$\gamma_0^{\frac{s}{2}-\frac{1}{4}} H(\nu,\eta)\, (\|u_0\|_{L^2} + \|b_0\|_{H^1} + \|a_0\|_{L^2}) \le C_0, \tag{1.6}$$

where $\frac{1}{2} < s < 1$ and $H(\nu,\eta)$ is an explicit function of ν and η (their representations can be found in Section 6). Then (1.1) with any $\gamma \le \gamma_0$ has a unique global solution (u_γ, b_γ) satisfying, for any $2 \le q \le \infty$ and $\frac{1}{2} < s < 1$,

$$u_\gamma \in C(0,\infty; L^2) \cap L^q(0,\infty; H^{\frac{2}{q}});$$
$$b_\gamma \in C(0,\infty; H^s) \cap L^2(0,\infty; H^{s+1}) \cap L^q(0,\infty; H^{\frac{2}{q}}).$$

In addition, as $\gamma \to 0$, (u_γ, b_γ) converges to the corresponding solution (u,b) of the 2D MHD system (1.3) with an explicit rate, for any $0 < T < \infty$,

$$\|(u_\gamma - u, b_\gamma - b)\|_{L^4(0,T;\dot H^{\frac{1}{2}})} \le C(\nu,\eta,\|u_0\|_{L^2},\|b_0\|_{H^1},\|a_0\|_{L^2})\, \gamma^{\frac{s}{2}-\frac{1}{4}}.$$

To prove Theorem 1.1, the approach of energy estimates would not work, as explained before. Instead, we make use of integral representations of the MHD system and MHD-wave system. To avoid notational confusion, we use (u_γ, b_γ) for the solution of the MHD-wave system (1.1) and (u,b) for that of the MHD system (1.3). To develop an integral representation for the MHD-wave system, we first solve a general damped wave equation and then represent the solution (u_γ, b_γ) as

$$\begin{aligned} u_\gamma &= e^{\nu t\Delta} u_0 - \int_0^t e^{\nu(t-s)\Delta} \mathbb{P}\nabla\cdot(u_\gamma \otimes u_\gamma)(s)\,ds \\ &\quad + \int_0^t e^{\nu(t-s)\Delta} \mathbb{P}\nabla\cdot(b_\gamma \otimes b_\gamma)(s)\,ds, \\ b_\gamma &= (K_0^\gamma + \frac{1}{2}K_1^\gamma) b_0 + K_1^\gamma(\gamma\, a_0) \\ &\quad - \int_0^t K_1^\gamma(t-s)(\nabla\cdot(u_\gamma\otimes b_\gamma) - \nabla\cdot(b_\gamma \otimes u_\gamma))(s)\,ds, \end{aligned}$$

where $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ denotes the projection operator onto divergence-free vector fields, and K_0^γ and K_1^γ are the solution operators of the linear damped wave equation

$$\gamma\partial_{tt} b + \partial_t b = \eta\Delta b, \qquad b(x,0) = b_0(x), \quad (\partial_t b)(x,0) = a_0(x).$$

More precisely, the kernel functions K_0^γ and K_1^γ are given by

$$\widehat{K_0}^\gamma = \frac{1}{2}(e^{\lambda_+ t} + e^{\lambda_- t}), \qquad \widehat{K_1}^\gamma = \frac{1}{\gamma}\,\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}.$$

with $\lambda_\pm$ being the roots of

$$\gamma\lambda^2 + \lambda + \eta|\xi|^2 = 0$$

or

$$\lambda_\pm = \frac{-1 \pm \sqrt{1 - 4\gamma\eta|\xi|^2}}{2\gamma}.$$

The integral representation of the MHD equations (1.3) is clearly given by

$$u(t) = e^{\nu t\Delta} u_0 - \int_0^t e^{\nu(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes u)(s)\,ds$$

$$
\begin{aligned}
& + \int_0^t e^{\nu(t-s)\Delta}\mathbb{P}\nabla\cdot(b\otimes b)(s)\,ds,\\
b(t) &= e^{\eta t\Delta}b_0 - \int_0^t e^{\eta(t-s)\Delta}\nabla\cdot(u\otimes b)(s)\,ds\\
& + \int_0^t e^{\eta(t-s)\Delta}\nabla\cdot(b\otimes u)(s)\,ds.
\end{aligned}
$$

Then the difference $(u_\gamma - u, b_\gamma - b)$ satisfies

$$
\begin{aligned}
u_\gamma - u = & - \int_0^t e^{\nu(t-s)\Delta}\mathbb{P}\nabla\cdot((u_\gamma - u)\otimes u_\gamma)(s)\,ds\\
& - \int_0^t e^{\nu(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes(u_\gamma - u))(s)\,ds\\
& + \int_0^t e^{\nu(t-s)\Delta}\mathbb{P}\nabla\cdot((b_\gamma - b)\otimes b_\gamma)(s)\,ds\\
& + \int_0^t e^{\nu(t-s)\Delta}\mathbb{P}\nabla\cdot(b\otimes(b_\gamma - b))(s)\,ds,
\end{aligned}
\tag{1.7}
$$

$$
\begin{aligned}
& b_\gamma - b = (K_0^\gamma + \frac{1}{2}K_1^\gamma - e^{\eta\Delta t})b_0 + K_1^\gamma(\gamma\, a_0)\\
& + \int_0^t (K_1^\gamma(t-s) - e^{\eta(t-s)\Delta})(\nabla\cdot(b\otimes u) - \nabla\cdot(u\otimes b))\,ds\\
& + \int_0^t K_1^\gamma(t-s)(\nabla\cdot(b_\gamma\otimes(u_\gamma - u)) + (\nabla\cdot((b_\gamma - b)\otimes u))\,ds\\
& + \int_0^t K_1^\gamma(t-s)(\nabla\cdot((u - u_\gamma)\otimes b)) + \nabla\cdot(u_\gamma\otimes(b - b_\gamma)))\,ds.
\end{aligned}
\tag{1.8}
$$

For any initial data $u_0 \in L^2(\mathbb{R}^2)$, $b_0 \in H^1(\mathbb{R}^2)$ and $a_0 \in L^2(\mathbb{R}^2)$, we evaluate the difference $(u_\gamma - u, b_\gamma - b)$ in the functional setting

$$X := L^4(0, T; \dot{H}^{\frac{1}{2}}(\mathbb{R}^2)),$$

where $T > 0$ and $\dot{H}^s$ denotes the standard homogeneous Sobolev space. To facilitate the estimates, we first derive suitable upper bounds for K_0^γ and K_1^γ in the frequency space. K_0^γ and K_1^γ have different behaviors at different frequencies. K_0^γ and K_1^γ at high frequencies exhibit damping effects while they share similar behavior with the heat operator at lower frequencies. Explicit upper bounds on K_0^γ and K_1^γ are presented in Lemma 3.1.

Another crucial ingredient in the estimate of $(u_\gamma - u, b_\gamma - b)$ is the fact that $e^{\nu t}$, K_0^γ and K_1^γ are bounded operators on $L^q(0, T; \dot{H}^{s+\frac{2}{q}})$ for any $2 \le q \le \infty$ and $s \in \mathbb{R}$. More precisely, we have

$$\|e^{\nu t\Delta}v_0\|_{L^p(0,T;\dot{H}^{s+\frac{2}{p}})} \le \frac{1}{2\nu^{\frac{1}{p}}}\|v_0\|_{\dot{H}^s},$$

$$\|K_0^\gamma v_0\|_{L^p(0,T;\dot H^{s+\frac{2}{p}})} \le C\,\gamma^{\frac{1}{p}}\,\|v_0\|_{\dot H^s\cap \dot H^{s+1}} + C\,\eta^{-\frac{1}{p}}\|v_0\|_{\dot H^s},$$
$$\|K_1^\gamma v_0\|_{L^p(0,T;\dot H^{s+\frac{2}{p}})} \le C\,\eta^{-\frac{1}{p}}\,\|v_0\|_{\dot H^s}.$$

Detailed proof of these inequalities are provided in Section 4.

The representation of $b_\gamma - b$ involves the difference between the solution of the heat equation and that of the linear damped wave equation. We are able to show that this difference in $L^4(0,T;\dot H^{\frac12}(\mathbb{R}^2))$ admits an upper bound depending linearly on $\gamma^{\frac14}$ and γ. More precisely, we have

$$\left\|(K_0^\gamma + \frac{1}{2}K_1^\gamma - e^{\eta\Delta t})b_0 + K_1^\gamma(\gamma\, a_0)\right\|_{L^4(0,T;\dot H^{\frac12})} \le C\,\gamma^{\frac14}\|b_0\|_{\dot H^1\cap L^2} + C\,\eta^{-\frac12}\gamma\|a_0\|_{L^2}$$

This estimate is a consequence of a crucial lemma obtained in Section 5.

With these preparations at our disposal, we then estimate the difference $(u_\gamma - u, b_\gamma - b)$ in $L^4(0,T;\dot H^{\frac12}(\mathbb{R}^2))$. After lengthy and tedious estimates, we obtain, for $\frac12 < s < 1$ (close to 1),

$$\begin{aligned}
&\|(u_\gamma - u, b_\gamma - b)\|_{L^4_T\dot H^{\frac12}} \le C\,\gamma^{\frac14}\|b_0\|_{\dot H^1\cap L^2} + C\,\eta^{-\frac12}\gamma\|a_0\|_{L^2}\\
&+ C\,\gamma^{\frac14}\eta^{-\frac12}(\|u\|_{L^2_T\dot H^1}\,\|b\|_{L^\infty_T\dot H^{\frac12}} + \|u\|_{L^4_T\dot H^{\frac12}}\,\|b\|_{L^\infty_T\dot H^{\frac12}\cap L^2\dot H^{\frac32}})\\
&+ C\,\gamma^{\frac{s}{2}-\frac14}\,\eta^{-\frac34+\frac{s}{2}}\,(\|u\|_{L^4_T\dot H^{\frac12}}\|b\|_{L^2_T\dot H^1\cap L^2_T\dot H^{1+s}}\\
&\qquad\qquad + \|u\|_{L^2_T\dot H^1}\,\|b\|_{L^2_T\dot H^1\cap L^\infty_T\dot H^{2s-1}})\\
&+ C\,(\nu^{-\frac34} + \eta^{-\frac34})(\|u\|_{L^4_T\dot H^{\frac12}} + \|b\|_{L^4_T\dot H^{\frac12}})\|(u_\gamma - u, b_\gamma - b)\|_{L^4_T\dot H^{\frac12}}\\
&+ C\,(\nu^{-\frac34} + \eta^{-\frac34})\|(u_\gamma - u, b_\gamma - b)\|^2_{L^4_T\dot H^{\frac12}}.
\end{aligned}\tag{1.9}$$

The upper bound in (1.9) consists of four main parts. The first part $C\,\gamma^{\frac14}\|b_0\|_{\dot H^1\cap L^2} + C\,\eta^{-\frac12}\gamma\|a_0\|_{L^2}$ involves the initial data (b_0, a_0) and has a factor $\gamma^{\frac14}$. The second part, consisting of the next two terms in (1.9), depends on the solution (u, b) of the MHD equation and has factors $\gamma^{\frac14}$ and $\gamma^{\frac{s}{2}-\frac14}$ with $0 < s < 1$ (close to 1). The solution (u, b) can be bounded uniformly in terms of the initial norm $\|u_0\|_{L^2} + \|b_0\|_{H^1}$. The third part is linear in terms of $\|(u_\gamma - u, b_\gamma - b)\|_{L^4_T\dot H^{\frac12}}$ and will be absorbed by the term on the left-hand side of (1.9). The fourth part in (1.9) is quadratic in $\|(u_\gamma - u, b_\gamma - b)\|_{L^4_T\dot H^{\frac12}}$.

To eliminate the upper bound in (1.9) that is linear in $\|(u_\gamma - u, b_\gamma - b)\|_{L^4_T\dot H^{\frac12}}$, we apply a basic fact from real analysis to obtain that there are $T_1 > 0$ and $T_2 > 0$ such that $\|(u,b)\|_{L^4(\rho,\rho+T_1;\dot H^{\frac12})}$ for any $\rho \ge 0$ and $\|u\|_{L^4(T_2,\infty;\dot H^{\frac12})} + \|b\|_{L^4(T_2,\infty;\dot H^{\frac12})}$ are small. In particular,

$$C\,(\nu^{-\frac34} + \eta^{-\frac34})(\|u\|_{L^4(\rho,\rho+T_1;\dot H^{\frac12})} + \|b\|_{L^4(\rho,\rho+T_1;\dot H^{\frac12})}) \le \frac12,$$
$$C\,(\nu^{-\frac34} + \eta^{-\frac34})(\|u\|_{L^4(T_2,\infty;\dot H^{\frac12})} + \|b\|_{L^4(T_2,\infty;\dot H^{\frac12})}) \le \frac12.$$

(1.9) is then reduced to

$$\begin{aligned}
&\|(u_\gamma - u, b_\gamma - b)\|_{L^4(0,T_1;\dot H^{\frac12})} \\
&\le C_1\, \gamma^\beta G(\nu,\eta)(\|u_0\|_{L^2} + \|b_0\|_{\dot H^1\cap L^2} + \|a_0\|_{L^2}) \\
&+ C_1\,(\nu^{-\frac34} + \eta^{-\frac34})\|(u_\gamma - u, b_\gamma - b)\|^2_{L^4(0,T_1;\dot H^{\frac12})}.
\end{aligned}$$

A bootstrapping argument then yields the desired bound for $\|(u_\gamma - u, b_\gamma - b)\|_{L^4(0,T_1;\dot H^{\frac12})}$. Repeating this process on a finite number of time intervals $[T_1, 2T_1]$, $[2T_1, 3T_1]$, $\cdots$, and $[T_2,\infty)$ leads to the global bound on $(0,\infty)$. As a special consequence, we obtain the global bound for $\|(u_\gamma, b_\gamma)\|_{L^4(0,\infty;\dot H^{\frac12})}$ as well as the desired convergence rate in Theorem 1.1.

As explained in Section 7, the proof for the uniqueness part in Theorem 1.1 does not follow from the energy method. Instead, the proof makes use of the integral representation for (u_γ, b_γ). We find that the difference between any two solutions in $L^4(0,T;\dot H^{\frac12})$ can be bounded in terms of the initial difference, which, especially, implies the desired uniqueness.

The rest of this paper is divided into seven sections. Section 2 solves a general linear damped wave equation and derives the integral representation of (1.1). Section 3 bounds the kernel functions representing the solution of the linear damped wave equation in the frequency space. Section 4 shows that the heat operator as well as the solution operators of the damped wave equation are bounded on the space-time space $L^p(0,T;\dot H^{s+\frac2p})$ for any $2\le p\le\infty$ and $s\in\mathbb{R}$, and explicit upper bounds are presented. Section 5 estimates the difference between solutions to the heat equation and the solution of the linear damped wave equation. Section 6 establishes the existence part of Theorem 1.1. Section 7 proves the uniqueness part of Theorem 1.1 via the integral representation in the functional setting $L^4(0,T;\dot H^{\frac12})$. Section 8 asserts the higher regularity of b_γ of the solution (u_γ, b_γ) to (1.1).

2. Integral representation

This section first derives the solution formula for a general damped linear nonhomogeneous wave equation and then applies this formula to obtain an integral representation of (1.1).

Proposition 2.1. *Assume P and Q are Fourier multiplier operators. Consider the initial-value problem for the linear non-homogeneous wave equation*

$$\begin{cases}(\partial_{tt} + P(D)\partial_t + Q(D))u = f, \quad x\in\mathbb{R}^2,\ t>0,\\ u(0)=u_0, \quad (\partial_t u)(0) = u_1, \quad x\in\mathbb{R}^2.\end{cases} \tag{2.1}$$

Then the solution of (2.1) is given by

$$u(t) = K_1(D,t)u_1 + K_2(D,t)u_0 + \int_0^t K_1(D,t-\tau)f(\tau)\,d\tau,$$

where K_1 and K_2 are Fourier multiplier operators given by

$$K_1(D) = \frac{e^{\lambda_+(D)t} - e^{\lambda_-(D)t}}{\lambda_+(D) - \lambda_-(D)}, \qquad K_2(D) = \frac{\lambda_+(D)e^{\lambda_-(D)t} - \lambda_-(D)e^{\lambda_+(D)t}}{\lambda_+(D) - \lambda_-(D)}.$$

Here the symbols $\lambda_\pm(i\xi)$ are the roots of

$$\lambda^2 + P(i\xi)\lambda + Q(i\xi) = 0$$

or

$$\lambda_\pm(i\xi) = \frac{-P(i\xi) \pm \sqrt{P(i\xi)^2 - 4Q(i\xi)}}{2}.$$

Proof. The associated characteristic polynomial is

$$\lambda^2 + P(i\xi)\lambda + Q(i\xi) = 0,$$

whose roots are given by

$$\lambda_\pm(i\xi) = \frac{-P(i\xi) \pm \sqrt{P(i\xi)^2 - 4Q(i\xi)}}{2}.$$

Then the wave equation in (2.1) with $f = 0$ can be decomposed as

$$(\partial_t - \lambda_+(D))(\partial_t - \lambda_-(D))u = 0. \tag{2.2}$$

To solve (2.2), we can rewrite it as a system of equations in two different ways

$$\begin{cases} (\partial_t - \lambda_-(D))u = F, \\ (\partial_t - \lambda_+(D))\, F = 0, \end{cases} \qquad \begin{cases} (\partial_t - \lambda_+(D))u = G, \\ (\partial_t - \lambda_-(D))\, G = 0. \end{cases} \tag{2.3}$$

It then follows from the first equations of these systems in (2.3) that

$$F(0) = u_1 - \lambda_-(D)u_0, \quad G(0) = u_1 - \lambda_+(D)u_0. \tag{2.4}$$

The second equations in (2.3) yield

$$F(t) = e^{\lambda_+(D)t}F(0), \qquad G(t) = e^{\lambda_-(D)t}G(0) \tag{2.5}$$

and

$$(\lambda_+(D) - \lambda_-(D))\, u = F - G. \tag{2.6}$$

As a consequence of (2.4), (2.5) and (2.6),

$$\begin{aligned} u &= (\lambda_+(D) - \lambda_-(D))^{-1} \left(e^{\lambda_+(D)t}F(0) - e^{\lambda_-(D)t}G(0)\right) \\ &= \frac{e^{\lambda_+(D)t} - e^{\lambda_-(D)t}}{\lambda_+(D) - \lambda_-(D)}\, u_1 + \frac{\lambda_+(D)e^{\lambda_-(D)t} - \lambda_-(D)e^{\lambda_+(D)t}}{\lambda_+(D) - \lambda_-(D)}\, u_0 \\ &:= K_1(D,t)u_1 + K_2(D,t)u_0. \end{aligned}$$

By the Duhamel principle,

$$(\partial_{tt} + P(D)\partial_t + Q(D))u = f$$

is solved by

$$u(t) = K_1(D,t)u_1 + K_2(D,t)u_0 + \int_0^t K_1(D, t-\tau) f(\tau)\, d\tau.$$

This completes the proof of Proposition 2.1. □

Next we apply the solution formula in Proposition 2.1 to solve the equation of b to provide an integral representation of (1.1).

Proposition 2.2. *(1.1) can be converted into the following integral representation*

$$\begin{cases} u(t) = e^{\nu t\Delta}u_0 - \displaystyle\int_0^t e^{\nu(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes u)(s)\,ds \\ \qquad\qquad + \displaystyle\int_0^t e^{\nu(t-s)\Delta}\mathbb{P}\nabla\cdot(b\otimes b)(s)\,ds, \\ b(t) = (K_0^\gamma + \dfrac{1}{2}K_1^\gamma)b_0 + K_1^\gamma(\gamma\, a_0) \\ \qquad\qquad - \displaystyle\int_0^t K_1^\gamma(t-s)(\nabla\cdot(u\otimes b) - \nabla\cdot(b\otimes u))(s)\,ds. \end{cases} \tag{2.7}$$

where $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ *denotes the projection operator onto divergence-free vector fields, and the kernel functions* K_0^γ *and* K_1^γ *are given by*

$$\widehat{K}_0^\gamma = \frac{1}{2}\left(e^{\lambda_+ t} + e^{\lambda_- t}\right), \qquad \widehat{K}_1^\gamma = \frac{1}{\gamma}\,\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} = \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\sqrt{1-4\gamma\eta|\xi|^2}}.$$

with

$$\lambda_\pm = \frac{-1 \pm \sqrt{1-4\gamma\eta|\xi|^2}}{2\gamma}.$$

Proof. We rewrite the equation of b as

$$\begin{cases} \partial_{tt}b + \frac{1}{\gamma}\partial_t b - \frac{\eta}{\gamma}\Delta b = \frac{1}{\gamma}(b\cdot\nabla u - u\cdot\nabla b), \\ b(x,0) = b_0(x), \quad \partial_t b(x,0) = a_0(x). \end{cases} \tag{2.8}$$

The characteristic equation is

$$\lambda^2 + \frac{1}{\gamma}\lambda + \frac{\eta}{\gamma}|\xi|^2 = 0$$

and its two roots are

$$\lambda_\pm = \frac{-1 \pm \sqrt{1-4\gamma\eta|\xi|^2}}{2\gamma}.$$

Applying Proposition 2.1 to (2.8) yields

$$\widehat{b}(t) = \widehat{K}_1(\xi,t)\widehat{a}_0 + \widehat{K}_2(\xi,t)\widehat{b}_0 + \int_0^t K_1(\xi,t-\tau)\frac{1}{\gamma}(b\cdot\widehat{\nabla u - u}\cdot\nabla b)\,d\tau, \tag{2.9}$$

where

$$\widehat{K}_1(\xi,t) = \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}, \qquad \widehat{K}_2(\xi,t) = \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-}$$

We rewrite the kernel functions to get more concise forms. Clearly

$$\lambda_+ - \lambda_- = \frac{1}{\gamma}\sqrt{1-4\gamma\eta|\xi|^2}.$$

Then

$$\widehat{K}_1(\xi,t)\widehat{a}_0 = \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\sqrt{1-4\gamma\eta|\xi|^2}}(\gamma\widehat{a}_0) = \widehat{K}_1^\gamma(\gamma\widehat{a}_0)$$

where we have written

$$\widehat{K}_1^\gamma = \frac{1}{\gamma}\,\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} = \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\sqrt{1-4\gamma\eta|\xi|^2}}. \tag{2.10}$$

Similarly,

$$\widehat{K}_1(\xi, t-\tau)\frac{1}{\gamma}(b\cdot\nabla \widehat{u - u}\cdot\nabla b) = \widehat{K}_1^\gamma(t-\tau)(b\cdot\nabla \widehat{u - u}\cdot\nabla b).$$

It is easy to check that

$$\widehat{K}_2(\xi,t) = \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} = \widehat{K}_0^\gamma + \frac{1}{2}\widehat{K}_1^\gamma$$

where

$$\widehat{K}_0^\gamma = \frac{1}{2}\left(e^{\lambda_+ t} + e^{\lambda_- t}\right) \tag{2.11}$$

Thus (2.9) becomes

$$\widehat{b}(t) = \widehat{K}_0^\gamma + \frac{1}{2}\widehat{K}_1^\gamma)\widehat{b}_0 + \widehat{K}_1^\gamma(\gamma\widehat{a}_0) + \int_0^t \widehat{K}_1^\gamma(\xi, t-\tau)(b\cdot\nabla\widehat{u - u}\cdot\nabla b)\,d\tau \tag{2.12}$$

with K_0^γ and K_1^γ given in (2.11) and (2.10), respectively.

Applying the projection operator $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ to the velocity equation in (1.1) to eliminate the pressure and then representing the resulting equation in the heat operator yields

$$\begin{aligned} u(t) =& e^{\nu t\Delta}u_0 - \int_0^t e^{\nu(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes u)(s)\,ds \\ &+ \int_0^t e^{\nu(t-s)\Delta}\mathbb{P}\nabla\cdot(b\otimes b)(s)\,ds. \end{aligned} \tag{2.13}$$

Combining (2.13) and (2.12) yields (2.7). This completes the proof of Proposition 2.2. □

3. Upper bounds on the kernels

This section presents upper bounds on the symbols of the operators K_0^γ and K_1^γ. Low and high frequencies of $\widehat{K}_0^\gamma(\xi,t)$ and $\widehat{K}_1^\gamma(\xi,t)$ behave differently and obey different upper bounds. The main upper bounds are presented in the following lemma.

Lemma 3.1. *Let*

$$\lambda_\pm = \frac{-1\pm\sqrt{1-4\gamma\eta|\xi|^2}}{2\gamma}$$

and

$$\widehat{K}_0^\gamma = \frac{1}{2}(e^{\lambda_+ t} + e^{\lambda_- t}), \qquad \widehat{K}_1^\gamma = \frac{1}{\gamma}\,\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} = \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\sqrt{1-4\gamma\eta|\xi|^2}}.$$

Define

$$S_1 = \left\{\xi\in\mathbb{R}^d:\ 4\gamma\eta|\xi|^2 \ge \frac{3}{4}\right\}, \qquad S_2 = \mathbb{R}^d\setminus S_1.$$

(1) *For $\xi \in S_1$,*

$$Re\lambda_- \le -\frac{1}{2\gamma}, \quad Re\lambda_+ \le -\frac{1}{4\gamma},$$

$$|\widehat{K}_0^\gamma(\xi,t)|,\, |\widehat{K}_1^\gamma(\xi,t)| \le C\, e^{-\frac{1}{8\gamma}t}, \tag{3.1}$$

Alternatively, for $\xi \in S_1$,

$$|\widehat{K}_1^\gamma(\xi,t)| \le C\,\gamma^{-\frac{1}{2}}\eta^{-\frac{1}{2}}\,|\xi|^{-1}\,e^{-\frac{1}{8\gamma}t}. \tag{3.2}$$

or more generally, for any $0 \le \theta \le 1$,

$$|\widehat{K}_1^\gamma(\xi,t)| \le C\,\gamma^{-\frac{\theta}{2}}\eta^{-\frac{\theta}{2}}|\xi|^{-\theta}\,e^{-\frac{1}{8\gamma}t}.$$

(2) *For $\xi \in S_2$,*

$$\lambda_- \le -\frac{3}{4\gamma}, \quad \lambda_+ \le -\frac{2\eta|\xi|^2}{1+\sqrt{1-4\gamma\eta|\xi|^2}} \le -\eta|\xi|^2$$

and

$$|\widehat{K}_0^\gamma(\xi,t)| \le C\,(e^{-\frac{3}{4\gamma}t} + e^{-\eta|\xi|^2 t}),$$

$$|\widehat{K}_1^\gamma(\xi,t)| \le C\,(e^{-\frac{3}{4\gamma}t} + e^{-\eta|\xi|^2 t}).$$

Trivially, for $\xi \in S_2$ and any $0 \le \theta \le 1$,

$$|\widehat{K}_1^\gamma(\xi,t)| \le C\,\gamma^{-\frac{\theta}{2}}\eta^{-\frac{\theta}{2}}\,|\xi|^{-\theta}\,e^{-\frac{3}{4\gamma}t} + C\,e^{-\eta|\xi|^2 t}.$$

We remark that the alternative upper bound for $\widehat{K}_1^\gamma$ is a sharper estimate and allows us to gain one derivative. This fact is very useful in the proof of our main results.

Proof. For $\xi \in S_1$, $Re\,\lambda_- \le -\frac{1}{2\gamma}$ follows directly from the definition of λ_-. Using the fact that, for $\xi \in S_1$,

$$Re\,\sqrt{1-4\gamma\eta|\xi|^2} \le \frac{1}{2},$$

we have

$$Re\,\lambda_+ = \frac{-1+\sqrt{1-4\gamma\eta|\xi|^2}}{2\gamma} \le -\frac{1}{4\gamma}.$$

Then

$$|\widehat{K}_0^\gamma(\xi,t)| \le C\,e^{-\frac{1}{4\gamma}t}.$$

To prove the bound for $|K_1^\gamma(\xi,t)|$, we consider two cases

$$(a)\ 4\gamma\eta|\xi|^2 > 1, \qquad (b)\ \frac{3}{4} \le 4\gamma\eta|\xi|^2 \le 1.$$

In the first case (a), $\sqrt{1-4\gamma\eta|\xi|^2}$ is imaginary and

$$\begin{aligned}\widehat{K}_1^\gamma(\xi,t) &= \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\sqrt{1-4\gamma\eta|\xi|^2}} \\ &= e^{-\frac{1}{2\gamma}t}\,\frac{2\sin t\sqrt{4\gamma\eta|\xi|^2-1}}{\sqrt{1-4\gamma\eta|\xi|^2}} \\ &\le 2te^{-\frac{1}{2\gamma}t}\end{aligned}$$

$$\le C\,\gamma\, e^{-\frac{1}{4\gamma}t},$$

where we have used $|\sin\beta| \le |\beta|$ for any $\beta \in \mathbb{R}$. In the second case (b), we use the mean-value theorem to obtain, for $\lambda_- \le c \le \lambda_+$,

$$\widehat{K}_1^\gamma(\xi,t) = 2t\, e^{ct} \le C\,\gamma\, e^{-\frac{1}{8\gamma}t}.$$

To obtain the alternative estimate (3.2), we consider two cases:

$$4\gamma\eta|\xi|^2 \ge 2, \qquad \frac{3}{4} \le 4\gamma\eta|\xi|^2 < 2.$$

When $4\gamma\eta|\xi|^2 \ge 2$, the quantity $\sqrt{1-4\gamma\eta|\xi|^2}$ is imaginary and

$$\begin{aligned}\left|\widehat{K}_1^\gamma(\xi,t)\right| &= e^{-\frac{1}{2\gamma}t}\,\frac{2\left|\sin t\sqrt{4\gamma\eta|\xi|^2-1}\right|}{\left|\sqrt{4\gamma\eta|\xi|^2-1}\right|}\\ &\le C\,\gamma^{-\frac12}\eta^{-\frac12}\,|\xi|^{-1}\,e^{-\frac{1}{2\gamma}t},\end{aligned}$$

where we have used the fact that

$$\frac{1}{\sqrt{4\gamma\eta|\xi|^2-1}} \le \frac{1}{\sqrt{2}}\,\gamma^{-\frac12}\eta^{-\frac12}|\xi|^{-1}.$$

For $\frac34 \le 4\gamma\eta|\xi|^2 < 2$, we have

$$\frac{\sqrt{3}}{4}\gamma^{-\frac12}\eta^{-\frac12} \le |\xi| < \frac{1}{\sqrt{2}}\gamma^{-\frac12}\eta^{-\frac12}$$

and

$$\left|\widehat{K}_1^\gamma(\xi,t)\right| \le C\,e^{-\frac{1}{8\gamma}t} \le C\,|\xi|^{-1}\,\gamma^{-\frac12}\eta^{-\frac12}\,e^{-\frac{1}{8\gamma}t}.$$

Interpolating the bounds in (3.1) and (3.2), we have, for any $\theta \in [0,1]$,

$$\left|\widehat{K}_1^\gamma(\xi,t)\right| \le C\,\gamma^{-\frac{\theta}{2}}\eta^{-\frac{\theta}{2}}|\xi|^{-\theta}\,e^{-\frac{1}{8\gamma}t}.$$

We now prove (2). For $\xi \in S_2$ or $4\gamma\,\eta|\xi|^2 < \frac34$, we have

$$\sqrt{1-4\gamma\,\eta|\xi|^2} \ge \frac12.$$

Clearly, $\lambda_- \le -\frac{3}{4\gamma}$.

$$\begin{aligned}\lambda_+ &= -\frac{1}{2\gamma}\left(1-\sqrt{1-4\gamma\eta|\xi|^2}\right) = -\frac{1}{2\gamma}\frac{4\gamma\eta|\xi|^2}{1+\sqrt{1-4\gamma\eta|\xi|^2}}\\ &\le -\eta\,|\xi|^2.\end{aligned}$$

Then,

$$|\widehat{K}_0^\gamma(\xi,t)|,\ |\widehat{K}_1^\gamma(\xi,t)| \le C\,(e^{-\frac{3}{4\gamma}t} + e^{-\eta|\xi|^2 t}).$$

This completes the proof of Lemma 3.1. □

4. Heat and wave operators on $L^q(0,T;\dot H^{s+\frac{2}{q}})$

This section intends to understand the behavior of the operators $e^{\nu\Delta t}$, K_0^γ and K_1^γ when they act on functions in the space $\dot H^s$ for any $s\in\mathbb{R}$. Maximal regularity results are established. These bounds constitute some of the essential ingredients when we estimate the solution expressed in the integral representation.

The first proposition provides bounds for the heat operator on Sobolev spaces.

Proposition 4.1. *Let $\nu>0$, $s\in\mathbb{R}$ and $v_0\in\dot H^s$. Let $T>0$ and $f\in L^2(0,T;\dot H^{s-1})$. Then, for any $2\le p\le\infty$,*

$$\|e^{\nu t\Delta}v_0\|_{L^p(0,T;\dot H^{s+\frac{2}{p}})}\le\frac{1}{2\nu^{\frac{1}{p}}}\|v_0\|_{\dot H^s},$$
$$\left\|\int_0^t e^{\nu(t-\tau)\Delta}f(\tau)\,d\tau\right\|_{L^p(0,T;\dot H^{s+\frac{2}{p}})}\le\frac{1}{\nu^{\frac{1}{2}+\frac{1}{p}}}\|f\|_{L^2(0,T;\dot H^{s-1})}.$$

In addition,

$$e^{\nu t\Delta}v_0,\qquad\int_0^t e^{\nu(t-\tau)\Delta}f(\tau)\,d\tau\in C([0,T];\dot H^s).$$

Proof. We start with the two special cases $p=2$ and $p=\infty$. The general case $2\le p\le\infty$ can be shown via interpolation. When $p=2$,

$$\begin{aligned}\|e^{\nu t\Delta}v_0\|^2_{L^2(0,T;\dot H^{s+1})}&=\int_0^T\int_{\mathbb{R}^d}|\xi|^{2(s+1)}\,e^{-2\nu t|\xi|^2}\,|\widehat{v}_0|^2\,d\xi dt\\&=\int_{\mathbb{R}^d}|\xi|^{2(s+1)}\,|\widehat{v}_0|^2\int_0^T e^{-2\nu t|\xi|^2}dt\,d\xi\\&\le\frac{1}{2\nu}\int_{\mathbb{R}^d}|\xi|^{2(s+1)-2}\,|\widehat{v}_0|^2\,d\xi=\frac{1}{2\nu}\|v_0\|^2_{\dot H^s}\end{aligned}$$

or

$$\|e^{\nu t\Delta}v_0\|_{L^2(0,T;\dot H^{s+1})}\le\frac{1}{(2\nu)^{1/2}}\|v_0\|_{\dot H^s}.$$

When $p=\infty$,

$$\|e^{\nu t\Delta}v_0\|_{L^\infty(0,T;\dot H^s)}=\sup_{0\le t\le T}\|e^{-\nu t|\xi|^2}\widehat{v}_0\|_{\dot H^s}\le\|v_0\|_{\dot H^s}.$$

By the interpolation inequality,

$$\begin{aligned}\|e^{\nu t\Delta}v_0\|_{L^p(0,T;\dot H^{s+\frac{2}{p}})}&\le\|e^{\nu t\Delta}v_0\|^{\frac{2}{p}}_{L^2(0,T;\dot H^{s+1})}\,\|e^{\nu t\Delta}v_0\|^{1-\frac{2}{p}}_{L^\infty(0,T;\dot H^s)}\\&\le\left(\frac{1}{(2\nu)^{1/2}}\|v_0\|_{\dot H^s}\right)^{\frac{2}{p}}\,(\|v_0\|_{\dot H^s})^{1-\frac{2}{p}}\\&=\frac{1}{(2\nu)^{1/p}}\,\|v_0\|_{\dot H^s}.\end{aligned}$$

This establishes the first inequality. To prove the second one, we start with the case when $p=2$. By Young's inequality for the time convolution,

$$\left\|\int_0^t e^{\nu(t-\tau)\Delta}f(\tau)\,d\tau\right\|_{L^2(0,T;\dot H^{s+1})}$$

$$
\begin{aligned}
&\le \left\| \left\| \int_0^t e^{-\nu(t-\tau)|\xi|^2} |\xi|^{s+1} \widehat{f}(\xi,\tau)\, d\tau \right\|_{L^2(\mathbb{R}^2)} \right\|_{L^2(0,T)} \\
&= \left\| \left\| \int_0^t e^{-\nu(t-\tau)|\xi|^2} |\xi|^{s+1} \widehat{f}(\xi,\tau)\, d\tau \right\|_{L^2(0,T)} \right\|_{L^2(\mathbb{R}^2)} \\
&\le \left\| \|e^{-\nu t|\xi|^2}\|_{L^1(0,T)}\, |\xi|^{s+1} \|\widehat{f}(\xi,\tau)\|_{L^2(0,T)} \right\|_{L^2(\mathbb{R}^2)} \\
&\le \left\| \frac{1}{\nu|\xi|^2}\, |\xi|^{s+1} \|\widehat{f}(\xi,\tau)\|_{L^2(0,T)} \right\|_{L^2(\mathbb{R}^2)} \\
&= \frac{1}{\nu} \left\| |\xi|^{s-1} \|\widehat{f}(\xi,\tau)\|_{L^2(0,T)} \right\|_{L^2(\mathbb{R}^2)} = \frac{1}{\nu} \|f\|_{L^2(0,T;\dot{H}^{s-1})}.
\end{aligned}
$$

For $p=\infty$, by Minkowski's inequality and Young's inequality,

$$
\begin{aligned}
&\left\| \int_0^t e^{\nu(t-\tau)\Delta} f(\tau)\, d\tau \right\|_{L^\infty(0,T;\dot{H}^s)} \\
&= \left\| \left\| \int_0^t e^{-\nu(t-\tau)|\xi|^2} |\xi|^{s} \widehat{f}(\xi,\tau)\, d\tau \right\|_{L^2(\mathbb{R}^2)} \right\|_{L^\infty(0,T)} \\
&\le \left\| \left\| \int_0^t e^{-\nu(t-\tau)|\xi|^2} |\xi|^{s} \widehat{f}(\xi,\tau)\, d\tau \right\|_{L^\infty(0,T)} \right\|_{L^2(\mathbb{R}^2)} \\
&\le \left\| \|e^{-\nu t|\xi|^2}\|_{L^2(0,T)}\, |\xi|^{s} \|\widehat{f}(\xi,t)\|_{L^2(0,T)} \right\|_{L^2(\mathbb{R}^2)} \\
&\le \left\| \frac{1}{(2\nu)^{1/2}|\xi|}\, |\xi|^{s} \|\widehat{f}(\xi,t)\|_{L^2(0,T)} \right\|_{L^2(\mathbb{R}^2)} \\
&= \frac{1}{(2\nu)^{1/2}} \left\| \| |\xi|^{s-1} \|\widehat{f}(\xi,t)\|_{L^2(0,T)} \right\|_{L^2(\mathbb{R}^2)} = \frac{1}{(2\nu)^{1/2}} \|f\|_{L^2(0,T;\dot{H}^{s-1})}.
\end{aligned}
$$

It then follows from an interpolation inequality that

$$
\begin{aligned}
&\left\| \int_0^t e^{\nu(t-\tau)\Delta} f(\tau)\, d\tau \right\|_{L^p(0,T;\dot{H}^{s+\frac{2}{p}})} \\
&\le \left(\frac{1}{\nu} \|f\|_{L^2(0,T;\dot{H}^{s-1})} \right)^{\frac{2}{p}} \left(\frac{1}{(2\nu)^{1/2}} \|f\|_{L^2(0,T;\dot{H}^{s-1})} \right)^{1-\frac{2}{p}} \\
&= \frac{1}{2^{\frac{1}{2}-\frac{1}{p}} \nu^{\frac{1}{2}+\frac{1}{p}}} \|f\|_{L^2(0,T;\dot{H}^{s-1})}.
\end{aligned}
$$

The continuity in time

$$
e^{\nu t\Delta} v_0, \quad \int_0^t e^{\nu(t-\tau)\Delta} f(\tau)\, d\tau \in C([0,T];\dot{H}^s).
$$

follows from the dominated convergence theorem. This completes the proof of Proposition 4.1. □

The next two propositions assess the bounds when the operators K_0^γ and K_1^γ act on Sobolev functions. In addition, the maximal regularity estimates are also obtained.

Proposition 4.2. *Let $\gamma > 0$, $\eta > 0$ and $s \in \mathbb{R}$*

(1) *There is a constant $C > 0$ such that, for any $v_0 \in \dot H^s \cap \dot H^{s+1}$,*

$$\begin{aligned}\|K_0^\gamma v_0\|_{L^p(0,T;\dot H^{s+\frac{2}{p}})} &\le C\,\|v_0\|_{\dot H^s}^{1-\frac{2}{p}}\left(\gamma^{\frac{1}{p}}\|v_0\|_{\dot H^{s+1}}^{\frac{2}{p}} + \eta^{-\frac{1}{p}}\|v_0\||_{\dot H^s}^{\frac{2}{p}}\right)\\ &\le C\,\gamma^{\frac{1}{p}}\,\|v_0\|_{\dot H^s\cap \dot H^{s+1}} + C\,\eta^{-\frac{1}{p}}\|v_0\|_{\dot H^s}.\end{aligned} \tag{4.1}$$

(2) *There is a constant $C > 0$ such that, for any $v_0 \in \dot H^s$,*

$$\|K_1^\gamma v_0\|_{L^p(0,T;\dot H^{s+\frac{2}{p}})} \le C\,\eta^{-\frac{1}{p}}\,\|v_0\|_{\dot H^s}. \tag{4.2}$$

(4.2) implies that K_1^γ shares the same upper bound as a heat operator, and the bound is even independent of γ!

Proof. We start with the two special cases. For $p = \infty$, by Lemma 3.1,

$$\begin{aligned}\|K_0^\gamma v_0\|_{L^\infty(0,T;\dot H^s)} &= \Big\| \||\xi|^s \widehat{K_0^\gamma}\widehat{v_0}||_{L^2(\mathbb{R}^2)}\Big\|_{L^\infty(0,T)}\\ &\le \Big\| \||\xi|^s \widehat{K_0^\gamma}\widehat{v_0}\|_{L^2(S_1)} + \||\xi|^s \widehat{K_0^\gamma}\widehat{v_0}||_{L^2(S_2)}\Big\|_{L^\infty(0,T)}\\ &\le C\,\||\xi|^s\widehat{v_0}\|_{L^2(\mathbb{R}^2)} = C\,\|v_0\|_{\dot H^s}.\end{aligned}$$

For $p = 2$,

$$\begin{aligned}\|K_0^\gamma v_0\|_{L^2(0,T;\dot H^{s+1})}^2 &= \int_0^T\int_{\mathbb{R}^2} ||\xi|^{s+1}\widehat{K_0^\gamma}\widehat{v_0}|^2\,d\xi\,dt\\ &= \int_0^T\int_{S_1} ||\xi|^{s+1}\widehat{K_0^\gamma}\widehat{v_0}|^2\,d\xi\,dt + \int_0^T\int_{S_2} ||\xi|^{s+1}\widehat{K_0^\gamma}\widehat{v_0}|^2\,d\xi\,dt\\ &\le \int_0^T\int_{S_1} |\xi|^{2(s+1)}e^{-\frac{1}{4\gamma}t}\,|\widehat{v_0}|^2\,d\xi dt\\ &\quad + \int_0^T\int_{S_2} |\xi|^{2(s+1)}(e^{-\frac{3}{4\gamma}t} + e^{-\eta|\xi|^2 t})^2|\widehat{v_0}|^2\,d\xi dt\\ &\le C\,\gamma\|v_0\|_{\dot H^{s+1}}^2 + C\,\eta^{-1}\|v_0\|_{\dot H^s}^2.\end{aligned}$$

Therefore,

$$\|K_0^\gamma v_0\|_{L^2(0,T;\dot H^{s+1})} \le C\gamma^{\frac{1}{2}}\|v_0\|_{\dot H^{s+1}} + C\,\eta^{-\frac{1}{2}}\|v_0\|_{\dot H^s}.$$

(4.1) then follows from interpolation.

The proof of (4.2) is similar. For $p = \infty$,

$$\|K_1^\gamma v_0\|_{L^\infty(0,T;\dot H^s)} \le C\,\|v_0\|_{\dot H^s}.$$

For $p = 2$,

$$\|K_1^\gamma v_0\|_{L^2(0,T;\dot H^{s+1})}^2 \le C\,\gamma^{-1}\eta^{-1}\int_0^T\int_{S_1}|\xi|^{2(s+1)}|\xi|^{-2}e^{-\frac{1}{4\gamma}t}\,|\widehat{v_0}|^2\,d\xi dt$$

$$+\int_0^T \int_{S_2} |\xi|^{2(s+1)}(\gamma^{-1/2}\eta^{-1/2}|\xi|^{-1}e^{-\frac{3}{4\gamma}t} + e^{-\eta|\xi|^2 t})^2 |\widehat{v}_0|^2 \, d\xi dt$$
$$\le C\,\eta^{-1}\|v_0\|_{\dot H^s}^2$$

Therefore,

$$\|K_1^\gamma v_0\|_{L^2(0,T;\dot H^{s+1})} \le C\,\eta^{-\frac12}\|v_0\|_{\dot H^s}.$$

(4.2) then follows from interpolation. This completes the proof of Proposition 4.2. □

The following proposition assesses the maximal regularity on K_1^γ.

Proposition 4.3. *Let $\gamma > 0$, $\eta > 0$, $s \in \mathbb{R}$ and $2 \le p \le \infty$. Then there is a constant $C > 0$ such that, for any $T > 0$ and $f \in L^2(0,T;\dot H^{s-1})$,*

$$\left\|\int_0^t K_1^\gamma(t-\tau)\, f(\tau)\, d\tau\right\|_{L^p(0,T;\dot H^{s+\frac2p})} \le \frac{C}{\eta^{\frac12+\frac1p}}\|f\|_{L^2(0,T;\dot H^{s-1})}$$

Proof. For $p = \infty$, by Lemma 3.1,

$$\begin{aligned}
&\left\|\int_0^t K_1^\gamma(t-\tau)\, f(\tau)\, d\tau\right\|_{L^\infty(0,T;\dot H^s)}\\
&= \left\|\left\|\int_0^t |\xi|^s |\widehat{K_1^\gamma}(t-\tau)\widehat{f}(\xi,\tau)\, d\tau\right\|_{L^2(\mathbb{R}^d)}\right\|_{L^\infty(0,T)}\\
&\le \left\|\left\|\int_0^t |\xi|^s |\widehat{K_1^\gamma}(t-\tau)\widehat{f}(\xi,\tau)\, d\tau\right\|_{L^\infty(0,T)}\right\|_{L^2(S_1)}\\
&\quad + \left\|\left\|\int_0^t |\xi|^s |\widehat{K_1^\gamma}(t-\tau)\widehat{f}(\xi,\tau)\, d\tau\right\|_{L^\infty(0,T)}\right\|_{L^2(S_2)}\\
&\le \left\||\xi|^s \|\widehat{K_1^\gamma}(t)\|_{L^2(0,T)}\|\widehat{f}(\xi,t)\|_{L^2(0,T)}\right\|_{L^2(S_1)}\\
&+ \left\||\xi|^s \|\widehat{K_1^\gamma}(t)\|_{L^2(0,T)}\|\widehat{f}(\xi,t)\|_{L^2(0,T)}\right\|_{L^2(S_2)}\\
&\le C\,\eta^{-\frac12}\|f\|_{L^2(0,T;\dot H^{s-1})}.
\end{aligned}$$

Similarly, for $p = 2$,

$$\left\|\int_0^t K_1^\gamma(t-\tau)\, f(\tau)\, d\tau\right\|_{L^2(0,T;\dot H^{s+1})} \le \frac{1}{\eta}\|f\|_{L^2(0,T;\dot H^{s-1})}$$

The general case is obtained via interpolation. This proves Proposition 4.3. □

5. Linearized MHD and linearized MHD-wave equations

This section estimates the difference between solutions to the heat equation and the solution of the linear damped wave equation. More precisely, we establish the following proposition.

Proposition 5.1. *Let $\eta > 0$ and $\gamma > 0$. Let $s \in \mathbb{R}$. Assume that $B_0 \in \dot{H}^s \cap \dot{H}^{s+1}$ and $a_0 \in \dot{H}^s$. Let B be the solution of the heat equation*

$$\begin{cases} \partial_t B - \eta \Delta B = 0, & x \in \mathbb{R}^2,\ t > 0, \\ B(x,0) = B_0(x), & x \in \mathbb{R}^2. \end{cases} \tag{5.1}$$

Let B_γ be the solution of the damped wave equation

$$\begin{cases} \partial_{tt} B_\gamma + \partial_t B_\gamma - \eta \Delta B_\gamma = 0, & x \in \mathbb{R}^2,\ t > 0, \\ B_\gamma(x,0) = B_0(x), \quad (\partial_t B_\gamma)(x,0) = a_0(x), & x \in \mathbb{R}^2. \end{cases} \tag{5.2}$$

Let $2 \le q \le \infty$. Then there exists a constant $C > 0$ independent of γ and η such that, for any $T > 0$,

$$\|B_\gamma - B\|_{L^q(0,T;\dot{H}^{s+\frac{2}{q}})} \le C\,\gamma^{\frac{1}{q}} \|B_0\|_{\dot{H}^s \cap \dot{H}^{s+1}} + C\,\gamma\,\eta^{-\frac{1}{q}} \|a_0\|_{\dot{H}^s}.$$

The proof of Proposition 5.1 relies crucially on the following lemma.

Lemma 5.2. *Let $\gamma > 0$ and $\eta > 0$. Let*

$$\lambda_\pm = \frac{-1 \pm \sqrt{1 - 4\gamma\eta|\xi|^2}}{2\gamma}$$

and

$$\widehat{K}_0^\gamma = \frac{1}{2}(e^{\lambda_+ t} + e^{\lambda_- t}), \qquad \widehat{K}_1^\gamma = \frac{1}{\gamma}\,\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} = \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\sqrt{1 - 4\gamma\eta|\xi|^2}}.$$

Let $2 \le q \le \infty$. There is a constant $C > 0$ independent of γ and η such that, for any $\xi \in \mathbb{R}^2$ and any $0 < T \le \infty$,

$$\left\| \widehat{K}_0^\gamma + \frac{1}{2}\widehat{K}_1^\gamma - e^{-\eta|\xi|^2 t} \right\|_{L^q(0,T)} \le C\,\gamma^{\frac{1}{q}}, \tag{5.3}$$

$$\left\| \widehat{K}_1^\gamma - e^{-\eta|\xi|^2 t} \right\|_{L^q(0,T)} \le C\,\gamma^{\frac{1}{q}}. \tag{5.4}$$

In the case when $q = 2$, the sharp coefficient is $C = \frac{1}{\sqrt{2}}$.

Proof. We first prove (5.3). We divide our consideration into two cases: $\xi \in S_1$ and $\xi \in S_2$. Here S_1 and S_2 are defined in Lemma 3.1. For $\xi \in S_1$ (the high frequency case),

$$4\gamma\eta|\xi|^2 \ge \frac{3}{4} \quad \text{or} \quad \eta^{-1}|\xi|^{-2} \le \frac{16}{3}\gamma.$$

We do not need to reply on the difference in this case. By Lemma 3.1,

$$\begin{aligned} &\left\| \widehat{K}_0^\gamma + \frac{1}{2}\widehat{K}_1^\gamma - e^{-\eta|\xi|^2 t} \right\|_{L^q(0,T)} \\ &\le \left\| \widehat{K}_0^\gamma + \frac{1}{2}\widehat{K}_1^\gamma \right\|_{L^q(0,T)} + \left\| e^{-\eta|\xi|^2 t} \right\|_{L^q(0,T)} \\ &\le C\,\|e^{-\frac{1}{8\gamma}t}\|_{L^q(0,T)} + \left\| e^{-\eta|\xi|^2 t} \right\|_{L^q(0,T)} \\ &\le C\gamma^{\frac{1}{q}} + C\,\eta^{-\frac{1}{q}}|\xi|^{-\frac{2}{q}} \le C\,\gamma^{\frac{1}{q}}. \end{aligned}$$

For $\xi \in S_2$ (the low frequency case) or

$$4\gamma\eta|\xi|^2 < \frac{3}{4},$$

we need to make use of the difference and it does not appear possible to perform a direct estimate. The idea here is to use the equations they satisfy instead of the solution formula. We recall that, according to Proposition 2.2,

$$b = (K_0^\gamma + \frac{1}{2}K_1^\gamma)b_0 + K_1^\gamma(\gamma a_0)$$

solves the linear wave equation

$$\begin{cases} \gamma\partial_{tt}b + \partial_t b - \eta\Delta b = 0, \\ b(x,0) = b_0(x), \quad (\partial_t b)(x,0) = a_0(x). \end{cases} \tag{5.5}$$

In particular, if we set $b_0 = 1$ and $a_0 = 0$, we find that $F = K_0^\gamma + \frac{1}{2}K_1^\gamma$ solves

$$\begin{cases} \gamma\partial_{tt}F + \partial_t F - \eta\Delta F = 0, \\ F(x,0) = 1, \quad (\partial_t F)(x,0) = 0. \end{cases} \tag{5.6}$$

Similarly, $G = K_1^\gamma$ solves

$$\begin{cases} \gamma\partial_{tt}G + \partial_t G - \eta\Delta G = 0, \\ G(x,0) = 0, \quad (\partial_t G)(x,0) = \frac{1}{\gamma}. \end{cases} \tag{5.7}$$

Therefore $A := F - e^{\eta\Delta t}$ satisfies

$$\begin{cases} \gamma\partial_{tt}\widehat{A} + \partial_t\widehat{A} + \eta|\xi|^2\widehat{A} = -\gamma\partial_{tt}(e^{-\eta|\xi|^2 t}), \\ \widehat{A}(\xi,0) = 0, \quad \widehat{\partial_t A}(\xi,0) = \eta|\xi|^2. \end{cases} \tag{5.8}$$

We can solve (5.8) to get

$$\widehat{A}(\xi,t) = \widehat{K}_1^\gamma(\gamma\eta|\xi|^2) - \gamma\eta^2|\xi|^4 \int_0^t \widehat{K}_1^\gamma(\xi, t-\tau)e^{-\eta|\xi|^2\tau}\, d\tau.$$

Taking the $L^q(0,\infty)$ in time and applying Young's inequality yield

$$\begin{aligned} \|\widehat{A}(\xi,t)\|_{L^q(0,T)} &\le \gamma\eta|\xi|^2\|\widehat{K}_1^\gamma\|_{L^q(0,T)} + \gamma\eta^2|\xi|^4\|\widehat{K}_1^\gamma\|_{L^q(0,\infty)}\,\eta^{-1}|\xi|^{-2} \\ &\le 2\gamma\eta|\xi|^2\|\widehat{K}_1^\gamma\|_{L^q(0,T)} \\ &\le C\,\gamma\eta|\xi|^2\left\|e^{-\frac{3}{4\gamma}t} + e^{-\eta|\xi|^2 t}\right\|_{L^q(0,T)} \\ &\le C\gamma^{\frac{1}{q}} + C\,\gamma\eta|\xi|^2\,(\eta|\xi|^2)^{-\frac{1}{q}} \\ &\le C\gamma^{\frac{1}{q}} + C\,\gamma\gamma^{-(1-\frac{1}{q})} \\ &\le C\,\gamma^{\frac{1}{q}}. \end{aligned}$$

This completes the proof of (5.3).

We now turn to the proof of (5.4). For high frequencies, say $\xi \in S_1$, we do not need to make use of the difference since each part can be bounded suitably. For $\xi \in S_1$,

$$\|\widehat{K}_1^\gamma - e^{-\eta|\xi|^2 t}\|_{L^q(0,T)} \le \|\widehat{K}_1^\gamma\|_{L^q(0,T)} + \|e^{-\eta|\xi|^2 t}\|_{L^q(0,T)}$$

$$\le C \,\|e^{-\frac{1}{8\gamma}t}\|_{L^q(0,T)} + \left\|e^{-\eta|\xi|^2 t}\right\|_{L^q(0,T)}$$
$$\le C\gamma^{\frac{1}{q}} + C\,(\eta\,|\xi|^2)^{-\frac{1}{q}} \le C\,\gamma^{\frac{1}{q}}.$$

Now we consider the low frequency case $\xi \in S_2$,

$$4\gamma\eta|\xi|^2 < \frac{3}{4}.$$

We make use of the equation of

$$H = K_1^\gamma - e^{\eta\Delta t} \quad \text{or} \quad \widehat{H} = \widehat{K}_1^\gamma - e^{-\eta|\xi|^2 t}.$$

$\widehat{H}$ satisfies

$$\gamma\partial_{tt}\widehat{H} + \partial_t\widehat{H} + \eta|\xi|^2\widehat{H} = -\gamma\eta^2|\xi|^4 e^{-\eta|\xi|^2 t},$$
$$\widehat{H}(\xi,0) = -1, \quad \partial_t\widehat{H}(\xi,0) = \frac{1}{\gamma} + \eta|\xi|^2.$$

Solving this equation yields

$$\begin{aligned}\widehat{H} &= (\widehat{K}_0^\gamma + \frac{1}{2}\widehat{K}_1^\gamma)(-1) + \widehat{K}_1^\gamma(1+\gamma\eta|\xi|^2)\\ &\quad - \gamma\eta^2|\xi|^4\int_0^t \widehat{K}_1^\gamma(t-\tau)e^{-\eta|\xi|^2\tau}\,d\tau\\ &= (\frac{1}{2}\widehat{K}_1^\gamma - \widehat{K}_0^\gamma) + \gamma\eta|\xi|^2\widehat{K}_1^\gamma\\ &\quad - \gamma\eta^2|\xi|^4\int_0^t \widehat{K}_1^\gamma(t-\tau)e^{-\eta|\xi|^2\tau}\,d\tau.\end{aligned}$$

Therefore,

$$\|\widehat{H}\|_{L^q(0,T)} \le \|\frac{1}{2}\widehat{K}_1^\gamma - \widehat{K}_0^\gamma\|_{L^q(0,T)} + C\,\gamma\eta|\xi|^2\|\widehat{K}_1^\gamma\|_{L^q(0,T)},$$

where we used Young's inequality in the estimate of the last part

$$\gamma\eta^2|\xi|^4\left\|\int_0^t \widehat{K}_1^\gamma(t-\tau)e^{-\eta|\xi|^2\tau}\,d\tau\right\|_{L^q(0,T)}$$
$$\le \gamma\eta^2|\xi|^4\|\widehat{K}_1^\gamma\|_{L^q(0,\infty)}\|e^{-\eta|\xi|^2 t}\|_{L^1(0,T)} \le \gamma\eta|\xi|^2\|\widehat{K}_1^\gamma\|_{L^q(0,T)}.$$

For $\xi \in S_2$, according to Lemma 3.1,

$$\widehat{K}_1^\gamma \le C\,e^{-\frac{3}{4\gamma}t} + C\,e^{-\eta|\xi|^2 t}.$$

Thus

$$\|\widehat{K}_1^\gamma\|_{L^q(0,T)} \le C\,\gamma^{\frac{1}{q}} + C\,(\eta|\xi|^2)^{-\frac{1}{q}}$$

and, for $\xi \in S_2$ or $4\gamma\eta|\xi|^2 < \frac{3}{4}$,

$$\gamma\eta|\xi|^2\|\widehat{K}_1^\gamma\|_{L^q(0,T)} \le C\,\gamma^{\frac{1}{q}} + \gamma(\eta|\xi|^2)^{1-\frac{1}{q}} \le C\,\gamma^{\frac{1}{q}}.$$

Recall that

$$\frac{1}{2}\widehat{K}_1^\gamma - \widehat{K}_0^\gamma = \frac{1}{2\sqrt{1-4\gamma\eta|\xi|^2}}(e^{\lambda_+ t} - e^{\lambda_- t}) - \frac{1}{2}(e^{\lambda_+ t} + e^{\lambda_- t})$$

$$= \frac{1}{2}(\frac{1}{\sqrt{1-4\gamma\eta|\xi|^2}} - 1)e^{\lambda_+ t} - \frac{1}{2}(\frac{1}{\sqrt{1-4\gamma\eta|\xi|^2}} + 1)e^{\lambda_- t}$$

For $\xi \in S_2$,

$$4\gamma\eta|\xi|^2 < \frac{3}{4}, \qquad \frac{1}{\sqrt{1-4\gamma\eta|\xi|^2}} \le 2$$

and thus

$$\frac{1}{2}\left(\frac{1}{\sqrt{1-4\gamma\eta|\xi|^2}} - 1\right) = \frac{2\gamma\eta|\xi|^2}{\sqrt{1-4\gamma\eta|\xi|^2}(1+\sqrt{1-4\gamma\eta|\xi|^2})} \le 4\gamma\eta|\xi|^2.$$

Therefore, if we use the upper bounds for λ_+ and λ_- in Lemma 3.1, we have

$$\lambda_+ = -\frac{1}{2\gamma} + \frac{1}{2\gamma}\sqrt{1-4\gamma\eta|\xi|^2} = -\frac{2\eta|\xi|^2}{1+\sqrt{1-4\gamma\eta|\xi|^2}} \le -\eta|\xi|^2$$
$$\lambda_- \le -\frac{1}{2\gamma}.$$

Thus

$$\left|\frac{1}{2}\widehat{K}_1^\gamma - \widehat{K}_0^\gamma\right| \le 4\gamma\eta|\xi|^2 e^{-\eta|\xi|^2 t} + \frac{3}{2} e^{-\frac{1}{2\gamma}t}$$

and

$$\left\|\frac{1}{2}\widehat{K}_1^\gamma - \widehat{K}_0^\gamma\right\|_{L^q(0,T)} \le C\,\gamma\eta|\xi|^2(\eta|\xi|^2)^{-\frac{1}{q}} + C\,\gamma^{\frac{1}{q}} \le C\,\gamma\gamma^{-1+\frac{1}{q}} + C\,\gamma^{\frac{1}{q}}$$
$$\le C\,\gamma^{\frac{1}{q}}.$$

This completes the proof of Lemma 5.2. □

We remark that, in the case of $q = 2$, (5.3) and (5.4) in Lemma 5.2 can be alternatively shown via direct calculations for $\xi \in \mathbb{R}^2$. The calculations are tedious and can not be extended to $q \neq 2$. For notational convenience, set

$$\alpha = \frac{1}{2\gamma}\sqrt{1-4\gamma\eta|\xi|^2}$$

and rewrite $\widehat{K}_0^\gamma$ and $\widehat{K}_1^\gamma$ as

$$\widehat{K}_0^\gamma = \frac{1}{2}\left(e^{-\frac{1}{2\gamma}t+\alpha t} + e^{-\frac{1}{2\gamma}t-\alpha t}\right), \quad \widehat{K}_1^\gamma = \frac{1}{2\gamma\alpha}\left(e^{-\frac{1}{2\gamma}t+\alpha t} - e^{-\frac{1}{2\gamma}t-\alpha t}\right).$$

Then

$$\widehat{K}_0^\gamma + \frac{1}{2}\widehat{K}_1^\gamma = \left(\frac{1}{2} + \frac{1}{4\gamma\alpha}\right)e^{-\frac{1}{2\gamma}t+\alpha t} + \left(\frac{1}{2} - \frac{1}{4\gamma\alpha}\right)e^{-\frac{1}{2\gamma}t-\alpha t}.$$

Thus

$$\left\|\widehat{K}_0^\gamma + \frac{1}{2}\widehat{K}_1^\gamma - e^{-\eta|\xi|^2 t}\right\|_{L^2(0,\infty)}^2$$
$$= \int_0^\infty \left(\left(\frac{1}{2} + \frac{1}{4\gamma\alpha}\right)e^{-\frac{1}{2\gamma}t+\alpha t} + \left(\frac{1}{2} - \frac{1}{4\gamma\alpha}\right)e^{-\frac{1}{2\gamma}t-\alpha t} - e^{-\eta|\xi|^2 t}\right)^2 dt$$

$$
\begin{aligned}
&=\frac{1}{4}(1+\frac{1}{2\gamma\alpha})^2\frac{1}{\frac{1}{\gamma}-2\alpha}+\frac{1}{4}(1-\frac{1}{2\gamma\alpha})^2\frac{1}{\frac{1}{\gamma}+2\alpha}\\
&\quad+\frac{1}{2}(1-\frac{1}{4\gamma^2\alpha^2})\gamma+\frac{1}{2\eta|\xi|^2}\\
&\quad-(1+\frac{1}{2\gamma\alpha})\frac{1}{\frac{1}{2\gamma}-\alpha+\eta|\xi|^2}-(1-\frac{1}{2\gamma\alpha})\frac{1}{\frac{1}{2\gamma}+\alpha+\eta|\xi|^2}.
\end{aligned}
$$

After regrouping the terms, we have

$$
\begin{aligned}
&\left\|\widehat{K}_0^\gamma+\frac{1}{2}\widehat{K}_1^\gamma-e^{-\eta|\xi|^2t}\right\|^2_{L^2(0,\infty)}\\
&=\frac{1}{4}\left(\frac{1}{\frac{1}{\gamma}-2\alpha}+\frac{1}{\frac{1}{\gamma}+2\alpha}+\frac{2}{\frac{1}{\gamma}}\right)+\frac{1}{16\gamma^2\alpha^2}\left(\frac{1}{\frac{1}{\gamma}-2\alpha}+\frac{1}{\frac{1}{\gamma}+2\alpha}-\frac{2}{\frac{1}{\gamma}}\right)\\
&\quad+\frac{1}{2\eta|\xi|^2}+\frac{1}{4\gamma\alpha}\left(\frac{1}{\frac{1}{\gamma}-2\alpha}-\frac{1}{\frac{1}{\gamma}+2\alpha}\right)\\
&\quad-\left(\frac{1}{\frac{1}{2\gamma}-\alpha+\eta|\xi|^2}+\frac{1}{\frac{1}{2\gamma}+\alpha+\eta|\xi|^2}\right)\\
&\quad-\frac{1}{2\gamma\alpha}\left(\frac{1}{\frac{1}{2\gamma}-\alpha+\eta|\xi|^2}-\frac{1}{\frac{1}{2\gamma}+\alpha+\eta|\xi|^2}\right)\\
&=\frac{\gamma(1-2\gamma^2\alpha^2)}{1-4\gamma^2\alpha^2}+\frac{\gamma}{2(1-4\gamma^2\alpha^2)}\\
&\quad+\frac{1}{2\eta|\xi|^2}+\frac{\gamma}{1-4\gamma^2\alpha^2}\\
&\quad-\frac{4\gamma+8\gamma^2\eta|\xi|^2}{(1+2\gamma\eta|\xi|^2)^2-4\gamma^2\alpha^2}-\frac{4\gamma}{(1+2\gamma\eta|\xi|^2)^2-4\gamma^2\alpha^2}.
\end{aligned}
$$

Recall that

$$2\gamma\alpha=\sqrt{1-4\gamma\eta|\xi|^2}\quad\text{or}\quad 4\gamma^2\alpha^2=1-4\gamma\eta|\xi|^2.$$

Then

$$
\begin{aligned}
&\frac{\gamma(1-2\gamma^2\alpha^2)}{1-4\gamma^2\alpha^2}+\frac{\gamma}{2(1-4\gamma^2\alpha^2)}+\frac{1}{2\eta|\xi|^2}+\frac{\gamma}{1-4\gamma^2\alpha^2}\\
&=\frac{\gamma}{2}+\frac{1}{\eta|\xi|^2},\\
&-\frac{4\gamma+8\gamma^2\eta|\xi|^2}{(1+2\gamma\eta|\xi|^2)^2-4\gamma^2\alpha^2}-\frac{4\gamma}{(1+2\gamma\eta|\xi|^2)^2-4\gamma^2\alpha^2}\\
&=-\frac{1}{\eta|\xi|^2}\frac{2+2\gamma\eta|\xi|^2}{2+\gamma\eta|\xi|^2}.
\end{aligned}
$$

Therefore,

$$\begin{aligned}&\left\|\widehat{K}_0^\gamma + \frac{1}{2}\widehat{K}_1^\gamma - e^{-\eta|\xi|^2 t}\right\|^2_{L^2(0,\infty)}\\ &= \frac{\gamma}{2} + \frac{1}{\eta|\xi|^2} - \frac{1}{\eta|\xi|^2}\frac{2+2\gamma\eta|\xi|^2}{2+\gamma\eta|\xi|^2} = \frac{\gamma}{2} - \frac{\gamma}{2+\gamma\eta|\xi|^2} \le \frac{\gamma}{2}.\end{aligned}$$

This completes the proof of (5.3) in the case of $q = 2$. The second one is similar.

We are now ready to prove Proposition 5.1.

Proof of Proposition 5.1. The solutions B and B_γ can be represented as

$$B = e^{\eta\Delta t}B_0, \quad B_\gamma = (K_0^\gamma + \frac{1}{2}K_1^\gamma)B_0 + K_1^\gamma(\gamma a_0).$$

Therefore,

$$\begin{aligned}\|B_\gamma - B\|_{L^q(0,T;\dot{H}^{s+\frac{2}{q}})} &\le \left\|(K_0^\gamma + \frac{1}{2}K_1^\gamma - e^{\eta\Delta t})B_0\right\|_{L^q(0,T;\dot{H}^{s+\frac{2}{q}})}\\ &\quad + \|K_1^\gamma(\gamma a_0)\|_{L^q(0,T;\dot{H}^{s+\frac{2}{q}})}\\ &:= I_1 + I_2.\end{aligned}$$

For $2 \le q \le \infty$, by Minkowski's inequality,

$$\begin{aligned}I_1 &= \left\|\left\||\xi|^{s+\frac{2}{q}}(\widehat{K}_0^\gamma + \frac{1}{2}\widehat{K}_1^\gamma - e^{-\eta|\xi|^2 t})\widehat{B}_0\right\|_{L^2}\right\|_{L^q(0,T)}\\ &\le \left\|\left\|(\widehat{K}_0^\gamma + \frac{1}{2}\widehat{K}_1^\gamma - e^{-\eta|\xi|^2 t})\right\|_{L^q(0,T)} |\xi|^{s+\frac{2}{q}}|\widehat{B}_0|\right\|_{L^2}\\ &\le C\,\gamma^{\frac{1}{q}}\left\||\xi|^{s+\frac{2}{q}}|\widehat{B}_0|\right\|_{L^2}\\ &= C\,\gamma^{\frac{1}{q}}\|B_0\|_{\dot{H}^s\cap\dot{H}^{s+1}}.\end{aligned}$$

By Proposition 4.2,

$$I_2 \le C\,\gamma\,\eta^{-\frac{1}{q}}\|a_0\|_{\dot{H}^s}.$$

This completes the proof of Proposition 5.1. □

6. Global existence

This section establishes the global existence part of Theorem 1.1 for the MHD-wave system. The idea here is to examine the difference

$$(u_\gamma - u, b_\gamma - b)$$

between the solution (u_γ, b_γ) of the MHD-wave equation (1.1) and the solution (u, b) of the 2D MHD system. We make use of the integral representation. We prove via the bootstrapping argument that this difference is bounded globally in time. Since the solution (u, b) of the 2D MHD system is known to be bounded for all time, we obtain a global bound for (u_γ, b_γ).

For the purpose of comparing with the solution of the MHD-wave system, we first provide a global existence and uniqueness result in the functional setting suitable for our purpose. An interesting problem associated with this result is the maximal regularity that the solution can achieve when the initial velocity u_0 and the magnetic field b_0 have different regularities.

We consider the 2D MHD equation with the initial data (u_0, b_0),

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nabla(p + \frac{|b|^2}{2}) = \nu\Delta u + b\cdot\nabla b, \quad x\in\mathbb{R}^2,\, t>0, \\ \partial_t b + u\cdot\nabla b = \eta\Delta b + b\cdot\nabla u, \\ \nabla\cdot u = \nabla\cdot b = 0, \\ u(x,0) = u_0(x), \quad b(x,0) = b_0(x). \end{cases} \tag{6.1}$$

Clearly, for $u_0 \in L^2(\mathbb{R}^2)$ and $b_0 \in H^1(\mathbb{R}^2)$, the MHD equations (6.1) has a unique and global solution.

Proposition 6.1. *Assume that $(u_0, b_0) \in L^2(\mathbb{R}^2)$ and $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$. Then the 2D MHD equations (6.1) has a unique global solution (u, b) satisfying*

$$u \in C([0,\infty); L^2) \cap \bigcap_{p=2}^{\infty} L^p(0,\infty; \dot{H}^{\frac{2}{p}}),$$

$$b \in C([0,\infty); L^2) \cap \bigcap_{p=2}^{\infty} L^p(0,\infty; \dot{H}^{\frac{2}{p}}).$$

In particular,

$$u, b \in L^4(0,\infty; \dot{H}^{\frac{1}{2}}).$$

In addition, if $b_0 \in L^2 \cap \dot{H}^1$, then, for any $0 < s < 1$,

$$b \in C([0,\infty); \dot{H}^s) \cap L^2(0,\infty; \dot{H}^{1+s}). \tag{6.2}$$

Since the global existence and uniqueness can be obtained by a rather standard approach, we omit the details. We shall only provide the proof of (6.2), which can be shown by direct energy estimate.

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\Lambda^s b\|_{L^2}^2 + \eta\|\Lambda^{1+s} b\|_{L^2}^2 \\ &= -\int \Lambda^s(u\cdot\nabla b)\cdot\Lambda^s b\,dx + \int \Lambda^s(b\cdot\nabla u)\cdot\Lambda^s b\,dx \\ &= \int \Lambda^s(u\otimes b)\cdot\Lambda^s\nabla b\,dx - \int \Lambda^s(b\otimes u)\cdot\Lambda^s\nabla b\,dx \\ &\le \|\Lambda^s(u\otimes b)\|_{L^2}\,\|\Lambda^{1+s} b\|_{L^2} + \|\Lambda^s(b\otimes u)\|_{L^2}\,\|\Lambda^{1+s} b\|_{L^2} \\ &\le \Big(\|\Lambda^s u\|_{L^{\frac{2}{s}}}\|b\|_{L^{\frac{2}{1-s}}} + \|\Lambda^s b\|_{L^4}\|u\|_{L^4}\Big)\|\Lambda^{1+s} b\|_{L^2} \\ &\le \|\nabla u\|_{L^2}\|\Lambda^s b\|_{L^2}\|\Lambda^{1+s} b\|_{L^2} \\ &\quad + \|\Lambda^s b\|_{L^2}^{\frac{1}{2}}\|\Lambda^{1+s} b\|_{L^2}^{\frac{3}{2}}\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}} \end{aligned}$$

$$\leq \frac{\eta}{2}\|\Lambda^{1+s}b\|_{L^2}^2 + C\,\|\nabla u\|_{L^2}^2\|\Lambda^s b\|_{L^2}^2 + C\,\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2\|\Lambda^s b\|_{L^2}^2.$$

Therefore,

$$\frac{d}{dt}\|\Lambda^s b\|_{L^2}^2 + \eta\|\Lambda^{1+s}b\|_{L^2}^2 \leq C\,\|\nabla u\|_{L^2}^2\|\Lambda^s b\|_{L^2}^2 + C\,\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2\|\Lambda^s b\|_{L^2}^2.$$

Integrating in time and using the fact that $u \in L^\infty(0,\infty;L^2)\cap L^2(0,\infty;\dot{H}^1)$, we obtain the global bound for b in (6.2).

We are now ready to prove the global existence part.

Proof of Theorem 1.1 (Global Existence part). First we represent $(u_\gamma - u, b_\gamma - b)$ in integral form. The solution (u,b) of the MHD equation (6.1) is given by

$$\begin{aligned} u(t) =& e^{\nu t\Delta}u_0 - \int_0^t e^{\nu(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes u)(s)\,ds \\ &+ \int_0^t e^{\nu(t-s)\Delta}\mathbb{P}\nabla\cdot(b\otimes b)(s)\,ds, \\ b(t) =& e^{\eta t\Delta}b_0 - \int_0^t e^{\eta(t-s)\Delta}\nabla\cdot(u\otimes b)(s)\,ds \\ &+ \int_0^t e^{\eta(t-s)\Delta}\nabla\cdot(b\otimes u)(s)\,ds. \end{aligned}$$

By Proposition 2.2,

$$\begin{aligned} u_\gamma =& e^{\nu t\Delta}u_0 - \int_0^t e^{\nu(t-s)\Delta}\mathbb{P}\nabla\cdot(u_\gamma\otimes u_\gamma)(s)\,ds \\ &+ \int_0^t e^{\nu(t-s)\Delta}\mathbb{P}\nabla\cdot(b_\gamma\otimes b_\gamma)(s)\,ds, \\ b_\gamma =& (K_0^\gamma + \frac{1}{2}K_1^\gamma)b_0 + K_1^\gamma(\gamma\,a_0) \\ &- \int_0^t K_1^\gamma(t-s)(\nabla\cdot(u_\gamma\otimes b_\gamma) - \nabla\cdot(b_\gamma\otimes u_\gamma))(s)\,ds. \end{aligned}$$

Taking the difference yields the equation of $(u_\gamma - u, b_\gamma - b)$,

$$\begin{aligned} u_\gamma - u =& -\int_0^t e^{\nu(t-s)\Delta}\mathbb{P}\nabla\cdot((u_\gamma - u)\otimes u_\gamma)(s)\,ds \\ &+ \int_0^t e^{\nu(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes(u-u_\gamma))(s)\,ds \\ &+ \int_0^t e^{\nu(t-s)\Delta}\mathbb{P}\nabla\cdot((b_\gamma - b)\otimes b_\gamma)(s)\,ds \\ &+ \int_0^t e^{\nu(t-s)\Delta}\mathbb{P}\nabla\cdot(b\otimes(b_\gamma - b))(s)\,ds, \end{aligned} \tag{6.3}$$

$$b_\gamma - b = (K_0^\gamma + \frac{1}{2}K_1^\gamma - e^{\eta\Delta t})b_0 + K_1^\gamma(\gamma\,a_0)$$

$$+\int_0^t (K_1^\gamma(t-s) - e^{\eta(t-s)\Delta})(\nabla\cdot(b\otimes u) - \nabla\cdot(u\otimes b))\,ds$$
$$+\int_0^t K_1^\gamma(t-s)(\nabla\cdot(b_\gamma\otimes(u_\gamma-u)) + (\nabla\cdot((b_\gamma-b)\otimes u))\,ds$$
$$+\int_0^t K_1^\gamma(t-s)(\nabla\cdot((u-u_\gamma)\otimes b)) + \nabla\cdot(u_\gamma\otimes(b-b_\gamma)))\,ds. \quad (6.4)$$

We now estimate $\|u_\gamma - u\|_{L^4_T\dot H^{\frac12}}$. We shall write $L^4_T\dot H^{\frac12}$ for $L^4(0,T;\dot H^{\frac12})$. Taking the norm $L^4_T\dot H^{\frac12}$ of (6.3) and invoking Proposition 4.1 yields

$$\begin{aligned}
\|u_\gamma - u\|_{L^4_T\dot H^{\frac12}} \le & C\,\nu^{-\frac34}\|\nabla\cdot((u_\gamma-u)\otimes u_\gamma)\|_{L^2_T\dot H^{-1}}\\
&+ C\,\nu^{-\frac34}\|\nabla\cdot(u\otimes(u-u_\gamma))\|_{L^2_T\dot H^{-1}}\\
&+ C\,\nu^{-\frac34}\|\nabla\cdot((b_\gamma-b)\otimes b_\gamma)\|_{L^2_T\dot H^{-1}}\\
&+ C\,\nu^{-\frac34}\|\nabla\cdot(b\otimes(b_\gamma-b))\|_{L^2_T\dot H^{-1}}\\
\le & C\,\nu^{-\frac34}\|u_\gamma-u\|_{L^4_T\dot H^{\frac12}}\|u_\gamma\|_{L^4_T\dot H^{\frac12}}\\
&+ C\,\nu^{-\frac34}\|u\|_{L^4_T\dot H^{\frac12}}\,\|u_\gamma-u\|_{L^4_T\dot H^{\frac12}}\\
&+ C\,\nu^{-\frac34}\|b_\gamma-b\|_{L^4_T\dot H^{\frac12}}\|b_\gamma\|_{L^4_T\dot H^{\frac12}}\\
&+ C\,\nu^{-\frac34}\|b\|_{L^4_T\dot H^{\frac12}}\|b_\gamma-b\|_{L^4_T\dot H^{\frac12}},
\end{aligned}$$

where we have also used the simple inequality

$$\|f\,g\|_{L^2(\mathbb{R}^2)} \le \|f\|_{L^4(\mathbb{R}^2)}\,\|g\|_{L^4(\mathbb{R}^2)} \le C\,\|f\|_{\dot H^{\frac12}(\mathbb{R}^2)}\,\|g\|_{\dot H^{\frac12}(\mathbb{R}^2)}.$$

Writing $u_\gamma = u_\gamma - u + u$ and $b_\gamma = b_\gamma - b + b$ yields

$$\begin{aligned}
\|u_\gamma - u\|_{L^4_T\dot H^{\frac12}} \le & C\,\nu^{-\frac34}(\|u\|_{L^4_T\dot H^{\frac12}} + \|b\|_{L^4_T\dot H^{\frac12}})\|(u_\gamma-u, b_\gamma-b)\|_{L^4_T\dot H^{\frac12}}\\
&+ C\,\nu^{-\frac34}\|(u_\gamma-u, b_\gamma-b)\|^2_{L^4_T\dot H^{\frac12}},
\end{aligned}$$

where we have written

$$\|(f,g)\|^2_{L^4_T\dot H^{\frac12}} := \|f\|^2_{L^4_T\dot H^{\frac12}} + \|g\|^2_{L^4_T\dot H^{\frac12}}.$$

We now take $L^4_T\dot H^{\frac12}$-norm of (6.4) and estimate the right-hand side term by term. By Lemma 5.2,

$$\begin{aligned}
&\left\|(K_0^\gamma + \frac12 K_1^\gamma - e^{\eta\Delta t})b_0\right\|_{L^4_T\dot H^{\frac12}} = \left\|\left\||\xi|^{\frac12}(\widehat{K}_0^\gamma + \frac12\widehat{K}_1^\gamma - e^{-\eta|\xi|^2 t})\widehat{b}_0\right\|_{L^2}\right\|_{L^4_T}\\
&\le \left\|\left\|(\widehat{K}_0^\gamma + \frac12\widehat{K}_1^\gamma - e^{-\eta|\xi|^2 t})\right\|_{L^4_T}\;|\xi|^{\frac12}|\widehat{b}_0|\right\|_{L^2}\\
&\le C\,\gamma^{\frac14}\left\||\xi|^{\frac12}|\widehat{b}_0|\right\|_{L^2} = C\,\gamma^{\frac14}\|b_0\|_{\dot H^{\frac12}} \le C\,\gamma^{\frac14}\|b_0\|_{\dot H^1\cap L^2}.
\end{aligned}$$

By Lemma 4.2,

$$\|K_1^\gamma(\gamma\, a_0)\|_{L^4_T \dot H^{\frac12}} \le C\,\eta^{-\frac12}\gamma\|a_0\|_{L^2}.$$

We now estimate the nonlinear terms. We start with

$$\left\|\int_0^t (K_1^\gamma(t-s) - e^{\eta(t-s)\Delta})(\nabla\cdot(b\otimes u) - \nabla\cdot(u\otimes b))\,ds\right\|_{L^4_T \dot H^{\frac12}}.$$

The estimate of this term is very involved. The goal here is to obtain a bound with γ to a positive power so that this term can be made small for small $\gamma > 0$. We divide our consideration into the high frequency case and the low frequency case. They are handled differently. We split the spatial integral into two parts,

$$\begin{aligned}
&\left\|\int_0^t (K_1^\gamma(t-s) - e^{\eta(t-s)\Delta})(\nabla\cdot(b\otimes u) - \nabla\cdot(u\otimes b))\,ds\right\|_{L^4_T \dot H^{\frac12}}\\
&= \left\|\left\||\xi|^{\frac12}\int_0^t \left(\widehat{K}_1^\gamma(t-s) - e^{-\eta(t-s)|\xi|^2}\right)|\xi|(\widehat{b\otimes u} + \widehat{u\otimes b})\right\|_{L^2}\right\|_{L^4_T}\\
&\le \left\|\left\|\int_0^t \left(\widehat{K}_1^\gamma(t-s) - e^{-\eta(t-s)|\xi|^2}\right)|\xi|^{\frac32}(\widehat{b\otimes u} + \widehat{u\otimes b})\right\|_{L^4_T}\right\|_{L^2}\\
&= M_1 + M_2,
\end{aligned}$$

where M_1 and M_2 are given by

$$\begin{aligned}
M_1 &= \left\|\left\|\int_0^t \left(\widehat{K}_1^\gamma(t-s) - e^{-\eta(t-s)|\xi|^2}\right)|\xi|^{\frac32}(\widehat{b\otimes u} + \widehat{u\otimes b})ds\right\|_{L^4_T}\right\|_{L^2(S_1)},\\
M_2 &= \left\|\left\|\int_0^t \left(\widehat{K}_1^\gamma(t-s) - e^{-\eta(t-s)|\xi|^2}\right)|\xi|^{\frac32}(\widehat{b\otimes u} + \widehat{u\otimes b})ds\right\|_{L^4_T}\right\|_{L^2(S_2)}.
\end{aligned}$$

Recall that

$$\begin{aligned}
|\xi| &\ge \frac{\sqrt3}{4}\gamma^{-\frac12}\eta^{-\frac12} \quad \text{for any } \xi\in S_1,\\
|\xi| &< \frac{\sqrt3}{4}\gamma^{-\frac12}\eta^{-\frac12} \quad \text{for any } \xi\in S_2.
\end{aligned}$$

According to Lemma 3.1, for $\xi\in S_1$,

$$|\widehat{K}_1^\gamma(\xi,t)| \le C\,\gamma^{-\frac12}\eta^{-\frac12}\,|\xi|^{-1}\,e^{-\frac{1}{8\gamma}t}.$$

By Young's inequality for convolution,

$$\begin{aligned}
M_1 &\le \left\|(\|\widehat{K}_1^\gamma(t)\|_{L^{\frac43}_T} + \|e^{-\eta t|\xi|^2}\|_{L^{\frac43}_T})|\xi|^{\frac32}\|\widehat{b\otimes u} + \widehat{u\otimes b}\|_{L^2_T}\right\|_{L^2(S_1)}\\
&\le C\left\|(\gamma^{-\frac12}\eta^{-\frac12}\,|\xi|^{-1}\gamma^{\frac34} + \eta^{-\frac34}|\xi|^{-\frac32})|\xi|^{\frac32}\|\widehat{b\otimes u} + \widehat{u\otimes b}\|_{L^2_T}\right\|_{L^2(S_1)}\\
&\le C\left\|(\gamma^{\frac14}\eta^{-\frac12}\,|\xi|^{-1} + \eta^{-\frac34}(\eta^{\frac14}\gamma^{\frac14})|\xi|^{-1})|\xi|^{\frac32}\|\widehat{b\otimes u} + \widehat{u\otimes b}\|_{L^2_T}\right\|_{L^2(S_1)}
\end{aligned}$$

$$
\begin{aligned}
&\le C\,\gamma^{\frac14}\eta^{-\frac12}\left\| |\xi|^{\frac12}\|\widehat{b\otimes u}+\widehat{u\otimes b}\|_{L^2_T}\right\|_{L^2(S_1)}\\
&\le C\,\gamma^{\frac14}\eta^{-\frac12}\left\| |\xi|^{\frac12}\|\widehat{b\otimes u}+\widehat{u\otimes b}\|_{L^2_T}\right\|_{L^2(\mathbb{R}^2)}\\
&\le C\,\gamma^{\frac14}\eta^{-\frac12}(\|\Lambda^{\frac12}(b\otimes u)\|_{L^2_TL^2}+\|\Lambda^{\frac12}(u\otimes b)\|_{L^2_TL^2})\\
&\le C\,\gamma^{\frac14}\eta^{-\frac12}(\|\Lambda^{\frac12}u\|_{L^2_TL^4}\|b\|_{L^\infty_TL^4}+\|u\|_{L^4_TL^4}\|\Lambda^{\frac12}b\|_{L^4_TL^4})\\
&\le C\,\gamma^{\frac14}\eta^{-\frac12}(\|u\|_{L^2_T\dot H^1}\|b\|_{L^\infty_T\dot H^{\frac12}}+\|u\|_{L^4_T\dot H^{\frac12}}\|b\|_{L^\infty_T\dot H^{\frac12}\cap L^2\dot H^{\frac32}}).
\end{aligned}
$$

By Young's inequality, Lemma 5.2 and the fact that $|\xi|<\frac{\sqrt3}{4}\gamma^{-\frac12}\eta^{-\frac12}$ for any $\xi\in S_2$, we have, for any $\frac12<s<1$,

$$
\begin{aligned}
M_2 &\le \left\| \|\widehat{K}_1^\gamma(t)-e^{-\eta t|\xi|^2}\|_{L^2_T}|\xi|^{\frac32-s}|\xi|^s\|\widehat{b\otimes u}+\widehat{u\otimes b}\|_{L^{\frac43}_T}\right\|_{L^2(S_2)}\\
&\le C\,\gamma^{\frac12}\left(\gamma^{-\frac12}\eta^{-\frac12}\right)^{\frac32-s}\left\| |\xi|^s\|\widehat{b\otimes u}+\widehat{u\otimes b}\|_{L^{\frac43}_T}\right\|_{L^2(S_2)}\\
&\le C\,\gamma^{\frac{s}{2}-\frac14}\,\eta^{-\frac34+\frac{s}{2}}\,\left\| \||\xi|^s(\widehat{b\otimes u}+\widehat{u\otimes b})\|_{L^2(S_2)}\right\|_{L^{\frac43}_T}\\
&\le C\,\gamma^{\frac{s}{2}-\frac14}\,\eta^{-\frac34+\frac{s}{2}}\,\|\|\Lambda^s(b\otimes u)\|_{L^2}+\|\Lambda^s(u\otimes b)\|_{L^2}\|_{L^{\frac43}_T}\\
&\le C\,\gamma^{\frac{s}{2}-\frac14}\,\eta^{-\frac34+\frac{s}{2}}\,\left(\|u\|_{L^4_TL^4}\|\Lambda^sb\|_{L^2_TL^4}+\|\Lambda^su\|_{L^2_TL^{\frac2s}}\|b\|_{L^4_TL^{\frac{2}{1-s}}}\right)\\
&\le C\,\gamma^{\frac{s}{2}-\frac14}\,\eta^{-\frac34+\frac{s}{2}}\,(\|u\|_{L^4_T\dot H^{\frac12}}\|b\|_{L^2_T\dot H^{\frac12+s}}+\|u\|_{L^2_T\dot H^1}\|b\|_{L^4_T\dot H^s})\\
&\le C\,\gamma^{\frac{s}{2}-\frac14}\,\eta^{-\frac34+\frac{s}{2}}\,(\|u\|_{L^4_T\dot H^{\frac12}}\|b\|_{L^2_T\dot H^1\cap L^2_T\dot H^{1+s}}\\
&\qquad+\|u\|_{L^2_T\dot H^1}\|b\|_{L^2_T\dot H^1\cap L^\infty_T\dot H^{2s-1}}).
\end{aligned}
$$

In summary, we have obtained, for $\frac12<s<1$,

$$
\begin{aligned}
&\left\|\int_0^t(K_1^\gamma(t-s)-e^{\eta(t-s)\Delta})(\nabla\cdot(b\otimes u)-\nabla\cdot(u\otimes b))\,ds\right\|_{L^4_T\dot H^{\frac12}}\\
&\le C\,\gamma^{\frac14}\eta^{-\frac12}(\|u\|_{L^2_T\dot H^1}\|b\|_{L^\infty_T\dot H^{\frac12}}+\|u\|_{L^4_T\dot H^{\frac12}}\|b\|_{L^\infty_T\dot H^{\frac12}\cap L^2\dot H^{\frac32}})\\
&\quad+C\,\gamma^{\frac{s}{2}-\frac14}\,\eta^{-\frac34+\frac{s}{2}}\,(\|u\|_{L^4_T\dot H^{\frac12}}\|b\|_{L^2_T\dot H^1\cap L^2_T\dot H^{1+s}}\\
&\qquad\qquad+\|u\|_{L^2_T\dot H^1}\|b\|_{L^2_T\dot H^1\cap L^\infty_T\dot H^{2s-1}}).
\end{aligned}
$$

We deal with the second time integral term of (6.4) in $L^4_T\dot H^{\frac12}$. By Proposition 4.3,

$$
\begin{aligned}
&\left\|\int_0^tK_1^\gamma(t-s)(\nabla\cdot(b_\gamma\otimes(u_\gamma-u))+(\nabla\cdot((b_\gamma-b)\otimes u))\,ds\right\|_{L^4_T\dot H^{\frac12}}\\
&\le C\,\eta^{-\frac34}\|\nabla\cdot(b_\gamma\otimes(u_\gamma-u))+(\nabla\cdot((b_\gamma-b)\otimes u)\|_{L^2_T\dot H^{-1}}\\
&\le C\,\eta^{-\frac34}(\|b_\gamma\otimes(u_\gamma-u)\|_{L^2_TL^2}+\|(b_\gamma-b)\otimes u\|_{L^2_TL^2})
\end{aligned}
$$

$$\leq C\,\eta^{-\frac{3}{4}}(\|b_\gamma\|_{L^4_T\dot H^{\frac12}}\|u_\gamma-u\|_{L^4_T\dot H^{\frac12}}+\|u\|_{L^4_T\dot H^{\frac12}}\|b_\gamma-b\|_{L^4_T\dot H^{\frac12}})$$
$$\leq C\,\eta^{-\frac{3}{4}}\,(\|u\|_{L^4_T\dot H^{\frac12}}+\|b\|_{L^4_T\dot H^{\frac12}})\|(u_\gamma-u,b_\gamma-b)\|_{L^4_T\dot H^{\frac12}}$$
$$+\,C\,\eta^{-\frac{3}{4}}\,\|(u_\gamma-u,b_\gamma-b)\|^2_{L^4_T\dot H^{\frac12}}.$$

The last term in (6.4) is bounded similarly,

$$\Big\|\int_0^t K_1^\gamma(t-s)(\nabla\cdot((u-u_\gamma)\otimes b))+\nabla\cdot(u_\gamma\otimes(b-b_\gamma)))\,ds\Big\|_{L^4_T\dot H^{\frac12}}$$
$$\leq C\,\eta^{-\frac{3}{4}}\,(\|u\|_{L^4_T\dot H^{\frac12}}+\|b\|_{L^4_T\dot H^{\frac12}})\|(u_\gamma-u,b_\gamma-b)\|_{L^4_T\dot H^{\frac12}}$$
$$+\,C\,\eta^{-\frac{3}{4}}\,\|(u_\gamma-u,b_\gamma-b)\|^2_{L^4_T\dot H^{\frac12}}.$$

Combining the estimates above, we obtain, for $\frac12<s<1$ (close to 1),

$$\begin{aligned}
&\|(u_\gamma-u,b_\gamma-b)\|_{L^4_T\dot H^{\frac12}}\leq C\,\gamma^{\frac14}\|b_0\|_{\dot H^1\cap L^2}+C\,\eta^{-\frac12}\gamma\|a_0\|_{L^2}\\
&+C\,\gamma^{\frac14}\eta^{-\frac12}(\|u\|_{L^2_T\dot H^1}\,\|b\|_{L^\infty_T\dot H^{\frac12}}+\|u\|_{L^4_T\dot H^{\frac12}}\,\|b\|_{L^\infty_T\dot H^{\frac12}\cap L^2\dot H^{\frac32}})\\
&+C\,\gamma^{\frac{s}{2}-\frac14}\,\eta^{-\frac34+\frac{s}{2}}\,(\|u\|_{L^4_T\dot H^{\frac12}}\|b\|_{L^2_T\dot H^1\cap L^2_T\dot H^{1+s}}\\
&\qquad\qquad+\|u\|_{L^2_T\dot H^1}\,\|b\|_{L^2_T\dot H^1\cap L^\infty_T\dot H^{2s-1}})\\
&+C\,(\nu^{-\frac34}+\eta^{-\frac34})(\|u\|_{L^4_T\dot H^{\frac12}}+\|b\|_{L^4_T\dot H^{\frac12}})\|(u_\gamma-u,b_\gamma-b)\|_{L^4_T\dot H^{\frac12}}\\
&+C\,(\nu^{-\frac34}+\eta^{-\frac34})\|(u_\gamma-u,b_\gamma-b)\|^2_{L^4_T\dot H^{\frac12}}.
\end{aligned}\tag{6.5}$$

Invoking the fact that the solution (u,b) of the MHD equations is bounded uniformly,

$$\|u\|_{L^2_T\dot H^1}\leq\nu^{-\frac12}\|u_0\|_{L^2},\quad \|b\|_{L^\infty_T\dot H^{\frac12}\cap L^2\dot H^{\frac32}}\leq\eta^{-\frac12}\|b_0\|_{H^1},$$
$$\|u\|_{L^4_T\dot H^{\frac12}}\leq\nu^{-\frac14}\|u_0\|_{L^2},\quad \|b\|_{L^2_T\dot H^1\cap L^\infty_T\dot H^{2s-1}}\leq\eta^{-\frac12}\|b_0\|_{H^1},$$

we reduce (6.5) to

$$\begin{aligned}
&\|(u_\gamma-u,b_\gamma-b)\|_{L^4_T\dot H^{\frac12}}\\
&\leq C_1\,\gamma^{\frac{s}{2}-\frac14}G(\nu,\eta)(\|u_0\|_{L^2}+\|b_0\|_{\dot H^1\cap L^2}+\|a_0\|_{L^2})\\
&+C_1\,(\nu^{-\frac34}+\eta^{-\frac34})(\|u\|_{L^4_T\dot H^{\frac12}}+\|b\|_{L^4_T\dot H^{\frac12}})\|(u_\gamma-u,b_\gamma-b)\|_{L^4_T\dot H^{\frac12}}\\
&+C_1\,(\nu^{-\frac34}+\eta^{-\frac34})\|(u_\gamma-u,b_\gamma-b)\|^2_{L^4_T\dot H^{\frac12}}.
\end{aligned}\tag{6.6}$$

Here $\frac{s}{2}-\frac14<\frac14$ for $\frac12<s<1$, and C_1 is an absolute constant (independent of γ,ν and η), $G(\nu,\eta)$ is a function of ν,η only. We apply the bootstrapping argument to establish a uniform global bound for $\|(u_\gamma-u,b_\gamma-b)\|_{L^4_T\dot H^{\frac12}}$. Due to the presence of the term in (6.6)

$$C_1\,(\nu^{-\frac34}+\eta^{-\frac34})(\|u\|_{L^4_T\dot H^{\frac12}}+\|b\|_{L^4_T\dot H^{\frac12}})\|(u_\gamma-u,b_\gamma-b)\|_{L^4_T\dot H^{\frac12}},$$

we need to implement this process on a finite number of sub-intervals of $(0,\infty)$. We recall a basic fact from real analysis.

Lemma 6.2. *Let $(X,\mathcal{B},\mu)$ be a complete measure space. Let f be integrable with respect to the measure μ. Then, for any $\varepsilon>0$, there is a $\delta>0$ such that, if $A\in\mathcal{B}$ and $\mu(A)\le\delta$, then*

$$\int_A |f(x)|\,d\mu(x)<\varepsilon.$$

Since the solution (u,b) of the MHD equation satisfies

$$\|u\|_{L^4(0,\infty;\dot H^{\frac12})}<\infty,\quad \|b\|_{L^4(0,\infty;\dot H^{\frac12})}<\infty,$$

there are $T_1>0$ such that, for any $\rho\ge 0$, $\|(u,b)\|_{L^4(\rho,\rho+T_1;\dot H^{\frac12})}$ is small. In particular, we choose $T_1>0$ such that

$$C\,(\nu^{-\frac34}+\eta^{-\frac34})(\|u\|_{L^4(\rho,\rho+T_1;\dot H^{\frac12})}+\|b\|_{L^4(\rho,\rho+T_1;\dot H^{\frac12})})\le\frac12. \tag{6.7}$$

In addition, there is $T_2>0$ such that

$$C\,(\nu^{-\frac34}+\eta^{-\frac34})(\|u\|_{L^4(T_2,\infty;\dot H^{\frac12})}+\|b\|_{L^4(T_2,\infty;\dot H^{\frac12})})\le\frac12.$$

Obviously, there is a positive integer $k_0>0$ such that

$$k_0T_1\ge T_2.$$

We first apply the bootstrapping argument on $[0,T_1]$ and then repeat this process on the time intervals $[T_1,2T_1]$, $[2T_1,3T_1]$, $\cdots$, $[(k_0-1)T_1,k_0T_1]$ and $[T_2,\infty)$ to obtain a global bound. Inserting (6.7) in (6.6) yields

$$\begin{aligned}
&\|(u_\gamma-u,b_\gamma-b)\|_{L^4(0,T_1;\dot H^{\frac12})}\\
&\le C_1\,\gamma^{\frac{s}{2}-\frac14}G(\nu,\eta)(\|u_0\|_{L^2}+\|b_0\|_{\dot H^1\cap L^2}+\|a_0\|_{L^2})\\
&+C_1\,(\nu^{-\frac34}+\eta^{-\frac34})\|(u_\gamma-u,b_\gamma-b)\|^2_{L^4(0,T_1;\dot H^{\frac12})}.
\end{aligned}\tag{6.8}$$

If we make the ansatz that

$$\|(u_\gamma-u,b_\gamma-b)\|_{L^4(0,T_1;\dot H^{\frac12})}\le C_2, \tag{6.9}$$

where C_2 satisfies

$$C_1\,C_2\,(\nu^{-\frac34}+\eta^{-\frac34})\le\frac12.$$

Inserting (6.9) in the right-hand side of (6.8) yields

$$\begin{aligned}
&\|(u_\gamma-u,b_\gamma-b)\|_{L^4(0,T_1;\dot H^{\frac12})}\\
&\le 2C_1\,\gamma^{\frac{s}{2}-\frac14}G(\nu,\eta)(\|u_0\|_{L^2}+\|b_0\|_{\dot H^1\cap L^2}+\|a_0\|_{L^2})
\end{aligned}$$

For γ_0 satisfying (1.6) in Theorem 1.1, namely

$$\gamma_0^{\frac{s}{2}-\frac14}H(\nu,\eta)(\|u_0\|_{L^2}+\|b_0\|_{\dot H^1\cap L^2}+\|a_0\|_{L^2})\le C_0$$

for a suitable function H of ν and η, and sufficiently small C_0, we have, for $\gamma\le\gamma_0$,

$$\|(u_\gamma-u,b_\gamma-b)\|_{L^4(0,T_1;\dot H^{\frac12})}$$

$$\le 2C_1\,\gamma^{\frac{s}{2}-\frac{1}{4}}G(\nu,\eta)(\|u_0\|_{L^2}+\|b_0\|_{\dot H^1\cap L^2}+\|a_0\|_{L^2})\le \frac{C_2}{2}.$$

The bootstrapping argument then yields the desired bound on $[0, T_1]$. Repeating this process on the time intervals $[T_1, 2T_1]$, $[2T_1, 3T_1]$, $\cdots$, $[(k_0-1)T_1, k_0T_1]$ and $[T_2,\infty)$ allows us to obtain a global bound on $[0,\infty)$. Combining with the global bound for $\|(u,b)\|_{L^4(0,T_1;\dot H^{\frac12})}$ yields the desired global bound for $\|(u_\gamma,b_\gamma)\|_{L^4(0,T_1;\dot H^{\frac12})}$.

Next we explain that

$$u_\gamma\in C([0,\infty);L^2)\cap L^2(0,\infty;\dot H^1),\quad b_\gamma\in C([0,\infty);L^2)\cap L^2(0,\infty;\dot H^1).$$

Due to the global bound for $\|(u_\gamma,b_\gamma)\|_{L^4(0,\infty;\dot H^{\frac12})}$, (u_γ,b_γ) satisfies the integral equation in $L^4(0,\infty;\dot H^{\frac12})$,

$$\begin{aligned} u_\gamma &= e^{\nu t\Delta}u_0-\int_0^t e^{\nu(t-s)\Delta}\mathbb{P}\nabla\cdot(u_\gamma\otimes u_\gamma)(s)\,ds\\ &\quad+\int_0^t e^{\nu(t-s)\Delta}\mathbb{P}\nabla\cdot(b_\gamma\otimes b_\gamma)(s)\,ds,\\ b_\gamma &= (K_0^\gamma+\frac12K_1^\gamma)b_0+K_1^\gamma(\gamma\,a_0)\\ &\quad-\int_0^t K_1^\gamma(t-s)(\nabla\cdot(u_\gamma\otimes b_\gamma)-\nabla\cdot(b_\gamma\otimes u_\gamma))(s)\,ds. \end{aligned}$$

According to Proposition 4.1, for $u_0\in L^2$,

$$e^{\nu t\Delta}u_0\in\bigcap_{p=2}^{\infty}L^p(0,\infty;\dot H^{\frac2p})\cap C([0,\infty);L^2).$$

For $(u_\gamma,b_\gamma)\in L^4(0,\infty;\dot H^{\frac12})$,

$$\nabla\cdot(u_\gamma\otimes u_\gamma),\quad \nabla\cdot(b_\gamma\otimes b_\gamma)\in L^2(0,\infty;\dot H^{-1})$$

where we have used the bound

$$\|\partial(f\,g)\|_{\dot H^{\frac d2-2}(\mathbb{R}^d)}\le C\,\|f\|_{\dot H^{\frac{d-1}{2}}(\mathbb{R}^d)}\,\|g\|_{\dot H^{\frac{d-1}{2}}(\mathbb{R}^d)}.$$

By Proposition 4.1 again,

$$\int_0^t e^{\nu(t-s)\Delta}\mathbb{P}\nabla\cdot(u_\gamma\otimes u_\gamma)(s)\,ds\in\bigcap_{p=2}^{\infty}L^p(0,\infty;\dot H^{\frac2p})\cap C([0,\infty);L^2),$$

$$\int_0^t e^{\nu(t-s)\Delta}\mathbb{P}\nabla\cdot(b_\gamma\otimes b_\gamma)(s)\,ds\in\bigcap_{p=2}^{\infty}L^p(0,\infty;\dot H^{\frac2p})\cap C([0,\infty);L^2).$$

Thus, we have shown that

$$u_\gamma\in\bigcap_{p=2}^{\infty}L^p(0,\infty;\dot H^{\frac2p})\cap C([0,\infty);L^2).$$

The proof is similar for

$$b_\gamma \in \bigcap_{p=2}^{\infty} L^p(0,\infty;\dot H^{\frac{2}{p}}) \cap C([0,\infty);L^2).$$

This completes the proof of the global existence part in Theorem 1.1. □

7. Uniqueness

This section proves the uniqueness part of Theorem 1.1. As explained below, the proof can not be achieved via energy estimates. Instead we need to make use of the integral representation.

Proof of the Uniqueness Part of Theorem 1.1. We first explain why the method of energy estimates would not work here. The main reason is that some of the terms can not be bounded suitably. Assume that

$$(u_0^{(1)}, b_0^{(1)}, a_0^{(1)}) \in L^2\times H^1\times L^2 \quad \text{and} \quad (u_0^{(2)}, b_0^{(2)}, a_0^{(2)}) \in L^2\times H^1\times L^2$$

are two initial data. Let $(u_\gamma^{(1)}, b_\gamma^{(1)}) \in L^\infty(0,\infty;L^2)\cap L^2(0,\infty;\dot H^1)$ and $(u_\gamma^{(2)}, b_\gamma^{(2)}) \in L^\infty(0,\infty;L^2)\cap L^2(0,\infty;\dot H^1)$ be the corresponding solutions. We estimate their difference

$$(\widetilde u, \widetilde b) = ((u_\gamma^{(1)}, b_\gamma^{(1)}) - (u_\gamma^{(2)}, b_\gamma^{(2)}).$$

We explain that the energy method of estimating the L^2-norm of the difference would not work! In fact, if we proceed with this approach, we would encounter a term that can not bounded. The difference $(\widetilde u, \widetilde b)$ satisfies

$$\begin{cases} \partial_t \widetilde u + u_\gamma^{(1)}\cdot\nabla \widetilde u = -\nabla \widetilde P + \nu\Delta\widetilde u + b_\gamma^{(1)}\cdot\nabla\widetilde b + \widetilde b\cdot\nabla b_\gamma^{(2)} - \widetilde u\cdot\nabla u_\gamma^{(2)},\\ \gamma\partial_{tt}\widetilde b + \partial_t\widetilde b + u_\gamma^{(1)}\cdot\nabla\widetilde b = \eta\Delta\widetilde b + b_\gamma^{(1)}\cdot\nabla\widetilde u + \widetilde b\cdot\nabla u_\gamma^{(2)} - \widetilde u\cdot\nabla b_\gamma^{(2)}. \end{cases}$$

The L^2-norm obeys

$$\frac12\frac{d}{dt}\|(\widetilde u,\widetilde b)\|_{L^2}^2 + \nu\|\nabla\widetilde u\|_{L^2}^2 + \eta\|\nabla\widetilde b\|_{L^2}^2 = -\gamma\int \widetilde b\cdot\partial_{tt}\widetilde b\,dx + I,$$

where

$$I = \int(\widetilde b\cdot\nabla b_\gamma^{(2)} - \widetilde u\cdot\nabla u_\gamma^{(2)})\cdot\widetilde u\,dx + \int(\widetilde b\cdot\nabla u_\gamma^{(2)} - \widetilde u\cdot\nabla b_\gamma^{(2)})\cdot\widetilde b\,dx.$$

Integration by parts yields

$$\int \widetilde b\cdot\partial_{tt}\widetilde b\,dx = \frac{d}{dt}\int\widetilde b\cdot\partial_t\widetilde b\,dx - \|\partial_t\widetilde b\|_{L^2}^2.$$

Therefore,

$$\frac12\frac{d}{dt}\left(\|(\widetilde u,\widetilde b)\|_{L^2}^2 + 2\gamma\int\widetilde b\cdot\partial_t\widetilde b\,dx\right) + \nu\|\nabla\widetilde u\|_{L^2}^2 + \eta\|\nabla\widetilde b\|_{L^2}^2 = \gamma\|\partial_t\widetilde b\|_{L^2}^2 + I. \tag{7.1}$$

Dotting the equation of $\widetilde b$ by $\partial_t\widetilde b$, we have

$$\frac12\frac{d}{dt}\left(\gamma\|\partial_t\widetilde b\|_{L^2}^2 + \eta\|\nabla\widetilde b\|_{L^2}^2\right) + \|\partial_t\widetilde b\|_{L^2}^2 = J, \tag{7.2}$$

where

$$J=\int \partial_t\widetilde{b}\cdot\left(-u_\gamma^{(1)}\cdot\nabla\widetilde{b}+b_\gamma^{(1)}\cdot\nabla\widetilde{u}+\widetilde{b}\cdot\nabla u_\gamma^{(2)}-\widetilde{u}\cdot\nabla b_\gamma^{(2)}\right)\,dx.$$

(7.1)+$2\gamma\times$ (7.2) yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|(\widetilde{u},\widetilde{b})\|_{L^2}^2+2\gamma\int\widetilde{b}\cdot\partial_t\widetilde{b}\,dx+2\gamma^2\|\partial_t\widetilde{b}\|_{L^2}^2+2\gamma\eta\|\nabla\widetilde{b}\|_{L^2}^2\right)\\&+\nu\|\nabla\widetilde{u}\|_{L^2}^2+\eta\|\nabla\widetilde{b}\|_{L^2}^2+\gamma\|\partial_t\widetilde{b}\|_{L^2}^2=I+J.\end{aligned}$$

I can be bounded by

$$\begin{aligned}|I|&\le(\|\nabla u_\gamma^{(2)}\|_{L^2}+\|\nabla b_\gamma^{(2)}\|_{L^2})(\|\widetilde{u}\|_{L^4}+\|\widetilde{b}\|_{L^4})^2\\&\le\frac{\nu}{4}\|\nabla\widetilde{u}\|_{L^2}^2+\frac{\eta}{4}\|\nabla\widetilde{b}\|_{L^2}^2\\&\quad+C\,(\nu^{-1}+\eta^{-1})(\|\nabla u_\gamma^{(2)}\|_{L^2}^2+\|\nabla b_\gamma^{(2)}\|_{L^2}^2)\|(\widetilde{u},\widetilde{b})\|_{L^2}^2.\end{aligned}$$

But unfortunately J can not be bounded suitably. For example, the term in J

$$-\int\partial_t\widetilde{b}\cdot u_\gamma^{(1)}\cdot\nabla\widetilde{b}\,dx$$

can not be bounded. We can only use L^2 on $\partial_t\widetilde{b}$ and $\nabla\widetilde{b}$, and then we have to take L^∞-norm of $u_\gamma^{(1)}$. But then we need more than one-derivative since

$$\|u_\gamma^{(1)}\|_{L^\infty}\le C\,\|u_\gamma^{(1)}\|_{H^s},\ \ s>1.$$

which is beyond the regularity of $u_\gamma^{(1)}$,

$$(u_\gamma^{(1)},b_\gamma^{(1)})\in L^\infty(0,\infty;L^2)\cap L^2(0,\infty;\dot{H}^1).$$

This explains why the energy method fails.

In order to prove the uniqueness, we use the integral form, which has an advantage. $(\widetilde{u},\widetilde{b})$ satisfies

$$\begin{aligned}\widetilde{u}&=e^{\nu t\Delta}(u_0^{(1)}-u_0^{(2)})\\&\quad-\int_0^t e^{\nu(t-s)\Delta}\mathbb{P}\nabla\cdot(u_\gamma^{(1)}\otimes\widetilde{u}+\widetilde{u}\otimes u_\gamma^{(2)})(s)\,ds\\&\quad+\int_0^t e^{\nu(t-s)\Delta}\mathbb{P}\nabla\cdot(b_\gamma^{(1)}\otimes\widetilde{b}+\widetilde{b}\otimes b_\gamma^{(2)})(s)\,ds,\\\widetilde{b}&=(K_0^\gamma+\frac{1}{2}K_1^\gamma)(b_0^{(1)}-b_0^{(2)})+K_1^\gamma(\gamma\,(a_0^{(1)}-a_0^{(2)}))\\&\quad+\int_0^t K_1^\gamma(t-s)(-u_\gamma^{(1)}\cdot\nabla\widetilde{b}+b_\gamma^{(1)}\cdot\nabla\widetilde{u}\\&\qquad\qquad+\widetilde{b}\cdot\nabla u_\gamma^{(2)}-\widetilde{u}\cdot\nabla b_\gamma^{(2)})(s)\,ds.\end{aligned}$$

Taking the $X_T:=L^4(0,T;\dot{H}^{\frac{1}{2}})$-norm and estimating the terms via the propositions in Section 4, we have

$$\|\widetilde{u}\|_{X_T}\le\nu^{-\frac{1}{4}}\|u_0^{(1)}-u_0^{(2)}\|_{L^2}$$

$$+ C\,\nu^{-\frac{3}{4}}((\|u_\gamma^{(1)}\|_{X_T} + \|u_\gamma^{(2)}\|_{X_T})\|\widetilde{u}\|_{X_T} + (\|b_\gamma^{(1)}\|_{X_T} + \|b_\gamma^{(2)}\|_{X_T})\|\widetilde{b}\|_{X_T}),$$
$$\|\widetilde{b}\|_{X_T} \le C\,(\eta^{-\frac{1}{4}} + \gamma^{\frac{1}{4}})\|b_0^{(1)} - b_0^{(2)}\|_{H^1} + C\,\eta^{-\frac{1}{4}}\|a_0^{(1)} - a_0^{(2)}\|_{L^2}$$
$$+ C\,\eta^{-\frac{3}{4}}((\|u_\gamma^{(1)}\|_{X_T} + \|u_\gamma^{(2)}\|_{X_T})\|\widetilde{b}\|_{X_T} + (\|b_\gamma^{(1)}\|_{X_T} + \|b_\gamma^{(2)}\|_{X_T})\|\widetilde{u}\|_{X_T}).$$

Adding the inequalities and taking $T > 0$ to be sufficiently small such that

$$C\,(\nu^{-\frac{3}{4}} + \eta^{-\frac{3}{4}})(\|u_\gamma^{(1)}\|_{X_T} + \|u_\gamma^{(2)}\|_{X_T} + \|b_\gamma^{(1)}\|_{X_T} + \|b_\gamma^{(2)}\|_{X_T}) \le \frac{1}{2},$$

we obtain

$$\|\widetilde{u}\|_{X_T} + \|\widetilde{b}\|_{X_T} \le C\,(\nu^{-\frac{1}{4}} + \gamma^{\frac{1}{4}})\|u_0^{(1)} - u_0^{(2)}\|_{L^2} + C\,(\eta^{-\frac{1}{4}} + \gamma^{\frac{1}{4}})\|b_0^{(1)} - b_0^{(2)}\|_{H^1}$$
$$+ C\,\eta^{-\frac{1}{4}}\|a_0^{(1)} - a_0^{(2)}\|_{L^2} + \frac{1}{2}(\|\widetilde{u}\|_{X_T} + \|\widetilde{b}\|_{X_T}).$$

Therefore,

$$\|\widetilde{u}\|_{X_T} + \|\widetilde{b}\|_{X_T} \le 2C\,(\nu^{-\frac{1}{4}} + \gamma^{\frac{1}{4}})\|u_0^{(1)} - u_0^{(2)}\|_{L^2} + 2C\,(\eta^{-\frac{1}{4}} + \gamma^{\frac{1}{4}})\|b_0^{(1)} - b_0^{(2)}\|_{H^1}$$
$$+ 2C\,\eta^{-\frac{1}{4}}\|a_0^{(1)} - a_0^{(2)}\|_{L^2}.$$

In particular, if

$$(u_0^{(1)}, b_0^{(1)}, a_0^{(1)}) = (u_0^{(2)}, b_0^{(2)}, a_0^{(2)}),$$

then, on $[0, T]$,

$$\widetilde{u} = 0, \quad \widetilde{b} = 0.$$

Repeating this process on the time intervals $[T, 2T]$, $[2T, 3T]$ and so on yields the desired uniqueness on any time interval. This finishes the proof for the uniqueness part. □

8. High regularity

This section establishes the higher regularity part for b_γ in Theorem 1.1, namely for any $0 < s < 1$,

$$b_\gamma \in C(0, \infty; \dot{H}^s) \cap L^2(0, \infty, \dot{H}^{s+1}). \tag{8.1}$$

Proof of the Higher Regularity Part for b_γ in Theorem 1.1. The regularity in (8.1) can be verified using the integral form of b_γ. Recall that

$$\begin{aligned} b_\gamma \;&=\; (K_0^\gamma + \frac{1}{2}K_1^\gamma)b_0 + K_1^\gamma(\gamma\, a_0) \\ &\quad - \int_0^t K_1^\gamma(t-s)(\nabla\cdot(u_\gamma \otimes b_\gamma) - \nabla\cdot(b_\gamma \otimes u_\gamma))(s)\,ds. \end{aligned}$$

Applying Λ^s and taking the L^2-norm yield

$$\|\Lambda^s b_\gamma\|_{L^2} \le \|\Lambda^s(K_0^\gamma + \frac{1}{2}K_1^\gamma)b_0\|_{L^2} + \|\Lambda^s K_1^\gamma(\gamma\, a_0)\|_{L^2}$$
$$+ \left\|\Lambda^s \int_0^t K_1^\gamma(t-s)(\nabla\cdot(u_\gamma \otimes b_\gamma) - \nabla\cdot(b_\gamma \otimes u_\gamma))(s)\,ds\right\|_{L^2}.$$

Invoking the upper bounds on K_0^γ and K_1^γ in Lemma 3.1, we have

$$\|\Lambda^s(K_0^\gamma + \frac{1}{2}K_1^\gamma)b_0\|_{L^2} \le C\left(\int |\xi|^{2s}|\widehat{b}_0(\xi)|^2\,d\xi\right)^{\frac12} \le C\,\|b_0\|_{H^1},$$

where we have used the fact that $|\widehat{K}_0^\gamma|, |\widehat{K}_0^\gamma| \le C$. Using the facts from Lemma 3.1 that $\xi \in S_1$, or $4\gamma\eta|\xi|^2 \ge \frac34$,

$$|\widehat{K}_1^\gamma| \le C\,\gamma^{-s/2}\eta^{-s/2}|\xi|^{-s}e^{-\frac{1}{8\gamma}t} \quad \text{for any} \quad 0 \le s \le 1,$$

and for $\xi \in S_2$, or $4\gamma\eta|\xi|^2 < \frac34$,

$$|\widehat{K}_1^\gamma| \le C\,\gamma^{-s/2}\eta^{-s/2}|\xi|^{-s}e^{-\frac{3}{4\gamma}t} + C\,e^{-\eta|\xi|^2 t},$$

we obtain

$$\begin{aligned}
&\gamma^2\|\Lambda^s K_1^\gamma a_0\|_{L^2}^2 \le C\,\gamma^2\int_{S_1}|\xi|^{2s}\gamma^{-s}\eta^{-s}|\xi|^{-2s}e^{-\frac{1}{4\gamma}t}|\widehat{a}_0(\xi)|^2\,d\xi\\
&+ C\,\gamma^2\int_{S_2}|\xi|^{2s}\gamma^{-s}\eta^{-s}|\xi|^{-2s}e^{-\frac{3}{2\gamma}}t|\widehat{a}_0(\xi)|^2\,d\xi\\
&+ C\,\gamma^2\int_{S_2}|\xi|^{2s}e^{-2\eta|\xi|^2}|\widehat{a}_0(\xi)|^2\,d\xi\\
&\le C\gamma^{2-s}\eta^{-s}\|a_0\|_{L^2}^2 + C\,\gamma^2\int_{S_2}\gamma^{-s}\eta^{-s}|\widehat{a}_0(\xi)|^2\,d\xi \le C\,\gamma^{2-s}\eta^{-s}\|a_0\|_{L^2}^2.
\end{aligned}$$

That is,

$$\gamma\|\Lambda^s K_1^\gamma a_0\|_{L^2} \le C\,\gamma^{1-\frac{s}{2}}\eta^{-\frac{s}{2}}\|a_0\|_{L^2}.$$

We turn to the nonlinear terms. We start with the first one

$$\begin{aligned}
&\left\|\Lambda^s\int_0^t K_1^\gamma(t-s)\nabla\cdot(u_\gamma\otimes b_\gamma)(s)\,ds\right\|_{L^2}\\
&\le \left\|\left\|\Lambda^s\int_0^t K_1^\gamma(t-s)\nabla\cdot(u_\gamma\otimes b_\gamma)(s)\,ds\right\|_{L^2}\right\|_{L^\infty(0,\infty)}\\
&\le \left\|\left\|\int_0^t |\xi|^s|\widehat{K}_1^\gamma(t-s)||\xi||\widehat{u_\gamma\otimes b_\gamma}|(s)\,ds\right\|_{L^2}\right\|_{L^\infty(0,\infty)}\\
&\le C\,\gamma^{-\frac12}\eta^{-\frac12}\left\|\left\|\int_0^t |\xi|^s|\xi|^{-1}e^{-\frac{1}{8\gamma}(t-s)}|\xi||\widehat{u_\gamma\otimes b_\gamma}|(s)\,ds\right\|_{L^2(S_1)}\right\|_{L^\infty(0,\infty)}\\
&\quad + C\,\gamma^{-\frac12}\eta^{-\frac12}\left\|\left\|\int_0^t |\xi|^s|\xi|^{-1}e^{-\frac{3}{4\gamma}(t-s)}|\xi||\widehat{u_\gamma\otimes b_\gamma}|(s)\,ds\right\|_{L^2(S_2)}\right\|_{L^\infty(0,\infty)}\\
&\quad + C\left\|\left\|\int_0^t |\xi|^s e^{-\eta|\xi|^2(t-s)}|\xi||\widehat{u_\gamma\otimes b_\gamma}|(s)\,ds\right\|_{L^2(S_2)}\right\|_{L^\infty(0,\infty)}\\
&\le C\,\gamma^{-\frac12}\eta^{-\frac12}\left\|\left\|\int_0^t |\xi|^s e^{-\frac{1}{8\gamma}(t-s)}|\widehat{u_\gamma\otimes b_\gamma}|(s)\,ds\right\|_{L^\infty(0,\infty)}\right\|_{L^2(\mathbb{R}^2)}
\end{aligned}$$

$$
\begin{aligned}
&+ C\,\Big\|\Big\|\int_0^t |\xi|^s e^{-\eta|\xi|^2(t-s)}|\xi||\widehat{u_\gamma\otimes b_\gamma}|(s)\,ds\Big\|_{L^\infty(0,\infty)}\Big\|_{L^2(S_2)}\\
&\le C\,\gamma^{-\frac12}\eta^{-\frac12}\Big\|\Big\|e^{-\frac{1}{8\gamma}t}\Big\|_{L^{\frac{2}{1-s}}(0,\infty)}\|\Lambda^s\widehat{(u_\gamma\otimes b_\gamma)}\|_{L^{\frac{2}{1+s}}(0,\infty)}\Big\|_{L^2(\mathbb{R}^2)}\\
&+ C\,\Big\||\xi|\,\Big\|e^{-\eta|\xi|^2 t}\Big\|_{L^{\frac{2}{1-s}}(0,\infty)}\|\Lambda^s\widehat{(u_\gamma\otimes b_\gamma)}\|_{L^{\frac{2}{1+s}}(0,\infty)}\Big\|_{L^2(S_2)}\\
&\le C\,\gamma^{-\frac{s}{2}}\eta^{-\frac12}\Big\|\|\Lambda^s\widehat{(u_\gamma\otimes b_\gamma)}\|_{L^{\frac{2}{1+s}}(0,\infty)}\Big\|_{L^2(\mathbb{R}^2)}\\
&\le C\,\gamma^{-\frac{s}{2}}\eta^{-\frac12}\|\|\Lambda^s(u_\gamma\otimes b_\gamma)\|_{L^2}\|_{L^{\frac{2}{1+s}}(0,\infty)}\\
&\le C\,\gamma^{-\frac{s}{2}}\eta^{-\frac12}\|\|\Lambda^s u_\gamma\|_{L^q}\|b_\gamma\|_{L^r}+\|u_\gamma\|_{L^r}\|\Lambda^s b_\gamma\|_{L^q}\|_{L^{\frac{2}{1+s}}(0,\infty)}\\
&\le C\,\gamma^{-\frac{s}{2}}\eta^{-\frac12}\|\Lambda^{\frac{1+s}{2}}u_\gamma\|_{L^{\frac{4}{1+s}}(0,\infty;L^2)}\|\Lambda^{\frac{1+s}{2}}b_\gamma\|_{L^{\frac{4}{1+s}}(0,\infty;L^2)}<\infty,
\end{aligned}
$$

where the last inequality holds due to the fact that $u_\gamma,\ b_\gamma\in L^p(0,\infty;\dot H^{\frac{2}{p}})$ for any $2\le p\le\infty$, and

$$
\frac1q+\frac1r=\frac12,\qquad \frac1q=\frac14+\frac{s}{4},\qquad \frac1r=\frac14-\frac{s}{4}.
$$

In summary, we have shown that, for any $0<s<1$,

$$
\begin{aligned}
\|b_\gamma\|_{L^\infty(0,\infty;\dot H^s)} &\le C\,\|b_0\|_{H^1}+C\,\gamma^{1-\frac{s}{2}}\eta^{-\frac{s}{2}}\|a_0\|_{L^2}\\
&\quad + C\,\gamma^{-\frac{s}{2}}\eta^{-\frac12}\|\Lambda^{\frac{1+s}{2}}u_\gamma\|_{L^{\frac{4}{1+s}}(0,\infty;L^2)}\|\Lambda^{\frac{1+s}{2}}b_\gamma\|_{L^{\frac{4}{1+s}}(0,\infty;L^2)}.
\end{aligned}
$$

Similarly, we can show that $\|b\|_{L^2(0,\infty;\dot H^{1+s})}$ admits the same bound. This completes the proof of the higher regularity part. □

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