

Streaming PCA for Markovian Data

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Abstract

Since its inception in 1982, Oja’s algorithm has become an established method for streaming principle component analysis (PCA). We study the problem of streaming PCA, where the data-points are sampled from an irreducible, aperiodic, and reversible Markov chain. Our goal is to estimate the top eigenvector of the unknown covariance matrix of the stationary distribution. This setting has implications in scenarios where data can solely be sampled from a Markov Chain Monte Carlo (MCMC) type algorithm, and the objective is to perform inference on parameters of the stationary distribution. Most convergence guarantees for Oja’s algorithm in the literature assume that the data-points are sampled IID. For data streams with Markovian dependence, one typically downsamples the data to get a "nearly" independent data stream. In this paper, we obtain the first sharp rate for Oja’s algorithm on the entire data, where we remove the logarithmic dependence on the sample size, n , resulting from throwing data away in downsampling strategies.

1 Introduction

Streaming Principal Component Analysis (PCA) is an important and well studied problem where the principal eigenvector of the sample covariance matrix of a dataset is computed one data-point at a time. One of the most popular algorithms for streaming PCA was introduced by Erikki Oja in 1982 [28, 29]. Most existing analyses of Oja’s algorithm are done when the data is sampled IID.

However, in many practical applications, the data-points are dependent and are sampled from an MCMC process converging to a target stationary distribution. This naturally arises in the context of token algorithms for Federated PCA settings [10, 11, 12] with multiple machines communicating via a fixed and connected graph topology. Each machine contains an arbitrary fraction of data-points and the goal is to design a streaming algorithm that respects this topology and returns the principal component of the whole dataset. This is typically achieved using a Metropolis-Hastings scheme that uses local information to design the transition matrix of a Markov chain with any desired stationary distribution. The stationary distribution, π , of the random walk is chosen so that the distribution of the samples under π matches the uniform distribution over data-points. Governed by this Markov chain, a random walker then travels the network of machines and samples one data-point at a time from the current machine, and computes the update. However, even under the stationary distribution, the data-points are dependent, which deviates from the IID setup. *Our goal is to obtain a sharp analysis of the $\sin^2$ error of the estimated vector w.r.t true top eigenvector of the unknown covariance matrix in the Markovian setting.*

Estimating the first principal component with streaming PCA : Let X_t be a mean zero d dimensional vector with covariance matrix Σ , and let η_t be a decaying learning rate. The update rule of Oja’s algorithm is given as -

$$w_t \leftarrow (I + \eta_t X_t X_t^T) w_{t-1}, \quad w_t \leftarrow \frac{w_t}{\|w_t\|_2} \quad (1)$$

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where w_t is the estimate of v_1 and η_t is the step-size at timestep t . We aim to analyse the $\sin^2$ error of Oja's iterate at timestep t , defined as $1 - \langle w_t, v_1 \rangle^2$, where v_1 is the top eigenvector of Σ .

Streaming PCA in the IID setting: For an IID data stream with $\mathbb{E}[X_i] = 0$ and $\mathbb{E}[X_i X_i^T] = \Sigma$, there has been a lot of work on determining the non-asymptotic convergence rates for Oja's algorithm and its various adaptations [15, 1, 3, 37, 13, 14, 25, 20, 24]. Amongst these, [15], [1] and [14] match the optimal offline sample complexity bound, suggested by the independent and identically distributed (IID) version of Theorem 1 (See Theorem 1.1 in [15]).

We consider Oja's algorithm in the setting where the data is generated from a reversible, irreducible, and aperiodic Markov chain with stationary distribution π . We denote by $\mathbb{E}_\pi[\cdot]$ the expectation under the stationary distribution. In this setting our goal is to estimate the principal eigenvector of $\mathbb{E}_\pi[X_i X_i^T]$. As in the IID setting, $\mathbb{E}_\pi[X_i] = 0$. The challenge is that the data, even when it reaches stationarity, is dependent. Here the degree of dependence is captured by the second eigenvalue in the magnitude of the transition matrix P (denoted as $|\lambda_2(P)|$) of the Markov chain. This is closely related to the mixing time of a Markov chain [19], denoted as τ_{mix} , which is the time after which the conditional distribution of a state is close in total variational distance to its stationary distribution, π (See Section 2.1).

Our contribution: Using a series of approximations, we obtain an optimal error rate for the $\sin^2$ error, which is worse by a factor of $1/(1 - |\lambda_2(P)|)$ from the corresponding error rate of the IID case. Previous work [3] has established rates worse by a poly-logarithmic factor by using downsampling, i.e. applying the update on every k^{th} datapoint. In Figure 1, we compare Oja's algorithm with its downsampled and offline variants (see Section 6 for more details on setup). We see that Oja's algorithm performs significantly better than the downsampled variant, and similarly to the offline variant where for the i^{th} data point we compute the eigenvector of the sample covariance matrix of all data-points up-to i . Our work provides a concrete and novel result that explains these observations. In Table 1, we compare our bounds with related analyses of Oja's algorithm. The last row shows that we are the first to obtain an error whose main term is *free of logarithmic dependence* on n or d for *streaming PCA in the Markovian case*.

We *break the logarithmic barrier* in previous work by considering a series of approximations of finer granularity which uses reversibility of the Markov chain and standard mixing conditions of irreducible and aperiodic Markov chains. Our rates are comparable to the recent work of [27] (Proposition 1) that establishes an offline error analysis for estimating the principal component of the empirical covariance matrix of Markovian data by using a Matrix Bernstein inequality. Our results also imply a linearly convergent decentralized algorithm for streaming PCA in a distributed setting. As a simple byproduct of our theoretical result, we also obtain a rate for Oja's algorithm applied on downsampled data, which is worse by a factor of $\log n$, as shown in Figure 1. To our knowledge, this is the first work that analyzes the Markovian streaming PCA problem without any downsampling that matches the error of the offline algorithm.

The crux of our analysis uses the mixing properties of the Markov chain. Strong mixing intuitively says that the conditional distribution of a state s in timestep k given the starting state is exponentially close to the stationary distribution of s , the closeness being measured using the total variation distance. All previous work on Markovian data exploits this property by conditioning on states many time steps before. However, it is crucial to a) adaptively find how far to look back and b) bound the error of the sequence of matrices we ignore between the current state and the state we are conditioning on. Observe that these two components are related. Looking back too far makes the dependence very small but increases the error resulting from approximating a larger matrix product of intermediate matrices. We present a fine analysis that balances these two parts and then uses spectral theory to bound the second part within a factor of a variance parameter that characterizes the variability of the matrices and shows up in the analysis of [15, 27].

Related work on streaming PCA and online matrix decomposition on Markovian data: Amongst recent work, [3] is very relevant to our setting, since it analyzes Oja's algorithm with Markovian Data samples. Inspired by the ideas of [8], the authors propose a downsampled version of Oja's algorithm to reduce dependence amongst samples and provide a Stochastic Differential

Paper	Markov?	Online?	Log-free main-term	$\sin^2$ error rate	Sample-Complexity
Jain et al. [15]	N	Y N	Y N	$O\left(\frac{\mathcal{V}}{\text{gap}^2} \frac{1}{n}\right)$ $O\left(\frac{\mathcal{V} \log(d)}{\text{gap}^2} \frac{1}{n}\right)$	$O\left(\frac{\mathcal{V}}{\text{gap}^2} \frac{1}{\epsilon}\right)$ $O\left(\frac{\mathcal{V} \log(d)}{\text{gap}^2} \frac{1}{\epsilon}\right)$
Chen et al. [3]	Y	Y	N	-	$O\left(\frac{G}{\text{gap}^2} \frac{1}{\epsilon} \log^2\left(\frac{G}{\text{gap}^2} \frac{1}{\epsilon}\right)\right)$
Neeman et al. [27]	Y	N	N	$O\left(\frac{\mathcal{V} \log\left(d^2 - \frac{\pi}{4}\right)}{(1- \lambda_2(P)) \text{gap}^2} \frac{1}{n}\right)$	$O\left(\frac{\mathcal{V} \log\left(d^2 - \frac{\pi}{4}\right)}{(1- \lambda_2(P)) \text{gap}^2} \frac{1}{\epsilon}\right)$
Theorem 1	Y	Y	Y	$O\left(\frac{\mathcal{V}}{(1- \lambda_2(P)) \text{gap}^2} \frac{1}{n}\right)$	$O\left(\frac{\mathcal{V}}{(1- \lambda_2(P)) \text{gap}^2} \frac{1}{\epsilon}\right)$

Table 1: Comparison of $\sin^2$ error rates and sample complexities. Here $\text{gap} := (\lambda_1 - \lambda_2)$, where λ_1, λ_2 are the top 2 eigenvalues of Σ and the sample complexities represent the number of samples required to achieve $\sin^2$ error at most ϵ . We note that [1] and [14] also match the online sample complexity bound provided in [15]. Further, for the offline algorithm with IID data, [16] removes the $\log(d)$ factor in exchange for a constant probability of success for large enough n .

Equation (SDE) based analysis to achieve a sample complexity of $O\left(\frac{G}{(\lambda_1 - \lambda_2)^2} \frac{1}{\epsilon} \log^2\left(\frac{G}{(\lambda_1 - \lambda_2)^2} \frac{1}{\epsilon}\right)\right)$ for $\sin^2$ error smaller than ϵ , where G is a variance parameter. We obtain a similar rate in Corollary 1 through our techniques. However, comparing with Theorem 1, we observe that downsampling leads to an extra $O(\log(n))$ factor. It is important to point out that [3] provides an analysis for estimating top k principal components, whereas this paper focuses on obtaining a sharp rate for the first principal component. [21] consider the harder problem of online non-negative matrix factorization for Markovian data. Their analysis establishes asymptotic convergence of error, but does not provide a rate.

Stochastic Optimization with Markovian

Data : Markovian models are often considered in Reinforcement Learning and Linear Dynamic Systems[2, 5, 9, 30, 4, 34, 18, 23]. There have been many notable nonasymptotic bounds for stochastic gradient descent (SGD) methods for general convex and nonconvex functions with Markovian data [8, 31, 6, 7, 10, 38, 33]. The convergence rates (sample complexities) obtained in these works apply to more general problems but do not exploit the matrix product structure inherent to Oja’s algorithm. In this work, we develop novel techniques to show that a sharper analysis is possible for the PCA objective. The paper is organized as follows. Section 2 contains the problem setup and preliminaries about Markov Chains. Section 3 contains Theorem 1. We present a sketch of the main technical tools in Section 4, intermediate theorems needed for the main theorem in Section 5 and conclude with simulations in Section 6.

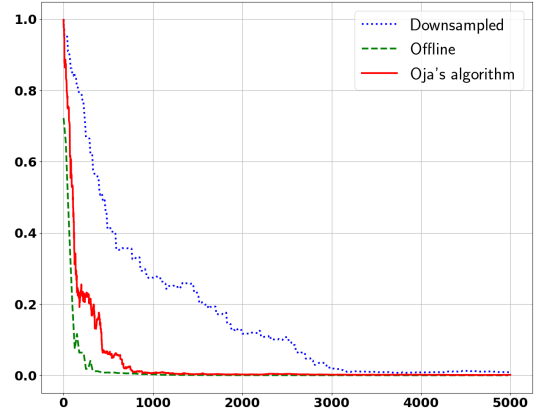


Figure 1: Comparison of Oja’s algorithm with downsampled and offline variants. The X-axis represents the sample size and the Y axis represents the $\sin^2$ error of each algorithm’s estimate of the leading eigenvector. The experimental setup is available in Section 6.

2 Problem Setup and Preliminaries

This section presents the problem setup and outlines important properties of the Markov chain that will be utilized subsequently. We assume that:

Assumption 1. *The Markov chain is irreducible, aperiodic, reversible, and starts in stationarity, with state distribution π ¹.*

Such a Markov chain can arise in various situations, for e.g., while performing random walks on expander graphs which are used extensively in fields such as computer networks, error-correcting codes, and pseudorandom generators. Each state s of the Markov chain is associated with a distribution $D(s)$ over d -dimensional vectors with mean $\mu_s \in \mathbb{R}^d$ and covariance matrix $\Sigma_s \in \mathbb{R}^{d \times d}$.

For a random walk $s_1, s_2, \dots, s_t$ on C , we define the sequence of random variables $X_1, X_2, \dots, X_t$, where conditioned on the state s_i , $X_i \sim D(s_i)$. We represent the mean as $\mu := \mathbb{E}_{s \sim \pi} [\mu_s] = \sum_s \pi_s \mu_s$ and the covariance matrix as $\Sigma \in \mathbb{R}^{d \times d}$, which, for $i \in [n]$ can be expressed as:

$$\Sigma := \mathbb{E}_{s_i \sim \pi} \mathbb{E}_{D(s_i)} [(X_i - \mu)(X_i - \mu)^T] = \mathbb{E}_{s \sim \pi} [\Sigma_s] + \mathbb{E}_{s \sim \pi} [\mu_s \mu_s^T] - \mu \mu^T$$

In this work, we assume $\mu = 0$, which is a common assumption in the IID setting (see [15, 1]). While it may be possible to extend our analysis to the non-zero mean case, it is out of the scope of this paper. Therefore, $\Sigma = \mathbb{E}_{s_i \sim \pi} \mathbb{E}_{D(s_i)} [X_i X_i^T]$ for $i \in [n]$

Let the eigenvalues of Σ be denoted as $\lambda_1 > \lambda_2 \geq \lambda_3 \dots \lambda_d$. Let v_1 denote the leading eigenvector of Σ and $V_\perp$ denote the $\mathbb{R}^{d \times (d-1)}$ matrix with the remaining eigenvectors as columns. We proceed under the following standard assumptions for $i \in [n]$, (see for eg. [14]).

Assumption 2. $\|\mathbb{E}_{s_i \sim \pi} \mathbb{E}_{D(s_i)} [(X_i X_i^T - \Sigma)^2]\|_2 \leq \mathcal{V}$.

Assumption 3. $\|X_i X_i^T - \Sigma\|_2 \leq \mathcal{M}$ with probability 1.

Assumption 3 also implies $\|\Sigma_s + \mu_s \mu_s^T - \Sigma\|_2 \leq \mathcal{M}$ with probability 1. WLOG, we assume $\mathcal{M} + \lambda_1 \geq 1$. We use $\mathbb{E}[\cdot] := \mathbb{E}_{s \sim \pi} \mathbb{E}_{D(s)}[\cdot]$ to denote the expectation over state $s \sim \pi$ and over the state-specific distributions $D(\cdot)$, unless otherwise specified. Define the matrix product

$$B_t := (I + \eta_t X_t X_t^T) (I + \eta_t X_{t-1} X_{t-1}^T) \dots (I + \eta_1 X_1 X_1^T) \quad (2)$$

Unrolling the recursion in 1, the output of Oja's algorithm at timestep t is given as $w_t = B_t w_0 / \|B_t w_0\|_2$. In this work, $\|\cdot\|_2$ denotes the Euclidean L_2 norm for vectors and the operator norm for matrices unless otherwise specified. I denotes the identity matrix.

2.1 Markov chain mixing times

Now we will discuss some well-known properties of an irreducible, aperiodic, and reversible Markov chain (also see [19]). Let $|\lambda_2(P)|$ denote the second largest absolute eigenvalue of the Markov chain; let the state-distribution of the Markov chain at timestep t with $s_1 = x$ be $P^t(x, \cdot)$. For any two probability distributions ν_1 and ν_2 , recall that the total variational distance is $TV(\nu_1, \nu_2) := \|\nu_1 - \nu_2\|_{TV} := \frac{1}{2} \sum_{x \in \Omega} |\nu_1(x) - \nu_2(x)|$. The distance from π at the t^{th} timestep is defined as $d_{\text{mix}}(t) := \sup_{x \in \Omega} TV(P^t(x, \cdot), \pi)$. For irreducible and aperiodic Markov chains, by Theorem 4.9 in [19], we have $d_{\text{mix}}(t) \leq C \exp(-ct)$ for some $C, c > 0$. The mixing time of the Markov chain is defined as:

$$\tau_{\text{mix}}(\epsilon) := \inf\{t : d_{\text{mix}}(t) \leq \epsilon\} \quad (3)$$

As in [19], we will denote $\tau_{\text{mix}} := \tau_{\text{mix}}(\frac{1}{4})$. Then, we have $\tau_{\text{mix}}(\epsilon) \leq \lceil \log_2(1/\epsilon) \rceil \tau_{\text{mix}}$. It is worth mentioning the useful relationship between $d_{\text{mix}}(\cdot)$ and τ_{mix} , given as $d_{\text{mix}}(l\tau_{\text{mix}}) \leq 2^{-l} \quad \forall l \in \mathbb{N}_0$. These results about mixing time are valid for general irreducible and aperiodic Markov chains. A reversible Markov chain satisfies $\forall x, y \in \Omega, \pi(x)P(x, y) = \pi(y)P(y, x)$. For a reversible, irreducible, and aperiodic Markov chain, the gap $1 - |\lambda_2(P)|$, is inversely proportional to τ_{mix} [19].

¹The last assumption may be eliminated by observing an initial burn-in period of τ_{mix} .

3 Main Results

In this section, we present our main result, a near-optimal convergence rate for Oja's algorithm on Markovian data. As a corollary, we also establish a rate of convergence for Oja's algorithm applied on downsampled data, where every k^{th} data-point is considered. Supplement S.5 contains comprehensive proofs of Theorem 1 and Corollary 1 while the proof of Proposition 1 can be found in Supplement Section S.2.

Theorem 1. Fix a $\delta \in (0, 1)$ and let the step-sizes be $\eta_i := \frac{\alpha}{(\lambda_1 - \lambda_2)(\beta + i)}$ with $\eta_0 \leq \frac{1}{e}$, $\alpha > 2$. Under assumptions 1, 2 and 3, for sufficiently large number of samples n such that $\frac{n}{\log(\frac{1}{\eta_n})} > \frac{\beta}{\log(\frac{1}{\eta_0})}$,

$$\beta := \frac{1000\alpha^2 \max \left\{ \tau_{\text{mix}} \log \left(\frac{1}{\eta_0} \right) (\mathcal{M} + \lambda_1)^2, \left(\frac{\mathcal{V}}{1 - |\lambda_2(P)|} + \lambda_1^2 \right) \right\}}{(\lambda_1 - \lambda_2)^2 \log \left(1 + \frac{\delta}{200} \right)}$$

the output w_n of Oja's algorithm (1) satisfies

$$1 - (w_n^T v_1)^2 \leq \frac{C \log \left(\frac{1}{\delta} \right)}{\delta^2} \left[d \left(\frac{2\beta}{n} \right)^{2\alpha} + \frac{C_1 \mathcal{V}}{(\lambda_1 - \lambda_2)^2 (1 - |\lambda_2(P)|)} \frac{1}{n} + \frac{C_2 \mathcal{M} (\mathcal{M} + \lambda_1)^2 \tau_{\text{mix}} (\eta_n^2)^2}{(\lambda_1 - \lambda_2)^3 n^2} \right]$$

with probability atleast $(1 - \delta)$. Here C is an absolute constant and

$$C_1 := \frac{\alpha^2 (3 + 7|\lambda_2(P)|)}{2\alpha - 1}, \quad C_2 := \frac{35\alpha^3}{\alpha - 1}$$

Next, we compare the rate of convergence proposed in Theorem 1 with the offline algorithm having access to the entire dataset $\{X_i\}_{i=1}^n$ using a recent result from [27]. Here, the authors extend the Matrix Bernstein inequality [35, 32], to Markovian random matrices. Their setup is much like ours except that the matrix at any state is fixed, i.e., there is no data distribution $D(s)$ as in our setup. However, it is easy to extend their result to our setting by observing that conditioned on the state sequence, the matrices $X_i X_i^T$, $i \in [n]$ are independent under our model, and we can push in the expectation over the state-specific distributions, $D(s)$, whenever required. Therefore, we have the following result -

Proposition 1 (Theorem 2.2 of [27]+Wedin's theorem). Fix $\delta \in (0, 1)$. Consider an irreducible and aperiodic Markov chain. Under assumptions 2 and 3, with probability $1 - \delta$, the leading eigenvector $\hat{v}$ of $\sum_{i=1}^n X_i X_i^T / n$ satisfies

$$1 - (\hat{v}^T v_1)^2 \leq C'_1 \frac{\mathcal{V} \log \left(\frac{d^{2-\frac{\pi}{4}}}{\delta} \right)}{(\lambda_1 - \lambda_2)^2} \left(\frac{1 + |\lambda_2(P)|}{1 - |\lambda_2(P)|} \right) \cdot \frac{1}{n} + C'_2 \left(\frac{\mathcal{M} \log \left(\frac{d^{2-\frac{\pi}{4}}}{\delta} \right)}{(\lambda_1 - \lambda_2) (1 - |\lambda_2(P)|)} \right)^2 \cdot \frac{1}{n^2} \quad (4)$$

for absolute constants C'_1 and C'_2 .

Observe that Theorem 1 matches the leading term $\frac{\mathcal{V}}{(\lambda_1 - \lambda_2)^2 (1 - |\lambda_2(P)|)}$ in Eq 1 except the $\log(d)$ term. We believe, much like the IID case (also see the remark in [15]), this logarithmic term in [27]'s result is removable for large n and a constant probability of success.

Remark 1. (Comparison with IID algorithm) Fix a $\delta \in (0, 1)$. If the data-points $\{X_i\}_{i=1}^n$ are sampled IID from the stationary distribution π , then under assumptions 2 and 3, using Theorem 4.1 from [15], we have that the output w_n of Oja's algorithm 1 satisfies -

$$1 - (w_n^T v_1)^2 \leq \frac{C \log \left(\frac{1}{\delta} \right)}{\delta^2} \left[d \left(\frac{\beta'}{n} \right)^{2\alpha} + \frac{\alpha'^2 \mathcal{V}}{(2\alpha' - 1) (\lambda_1 - \lambda_2)^2} \frac{1}{n} \right] \quad (5)$$

The leading term of Theorem 1 is worse by a factor of $\frac{1}{1-|\lambda_2(P)|}$. Further, it has an additive lower order term $O\left(\frac{\log^2(n)}{n^2}\right)$ due to the covariance between data-points in the Markovian case.

Corollary 1. (*Downsampled Oja’s algorithm*) Fix a $\delta \in (0, 1)$. If Oja’s algorithm is applied on the downsampled data-stream with every k^{th} data-point, where $k := \tau_{\text{mix}}(\eta_n^2)$ then under the conditions of Theorem 1 with appropriately modified α and β , the output w_n satisfies

$$1 - (w_n^T v_1)^2 \leq \frac{C \log\left(\frac{1}{\delta}\right)}{\delta^2} \left[d \left(\frac{2\beta\tau_{\text{mix}} \log(n)}{n} \right)^{2\alpha} + \frac{C_1 \mathcal{V}\tau_{\text{mix}} \log(n)}{(\lambda_1 - \lambda_2)^2 n} + \frac{C_2 \mathcal{M}(\mathcal{M} + \lambda_1)^2 \log^2(n) \tau_{\text{mix}}(\eta_n^2)^2}{(\lambda_1 - \lambda_2)^3 n^2} \right]$$

with probability atleast $(1 - \delta)$. Here C is an absolute constant and $C_1 := \frac{30\alpha^2}{2\alpha-1}$, $C_2 := \frac{35\alpha^3}{\alpha-1}$.

Remark 2. Data downsampling to reduce dependence amongst samples has been suggested in recent work [26, 22, 3]. In Corollary 1, we establish that the rate obtained is sub-optimal compared to Theorem 1 by a $\log(n)$ factor. We prove this by a simple yet elegant observation: the downsampled data stream can be considered to be drawn from a Markov chain with transition kernel $P^k(\cdot, \cdot)$ since each data-point is k steps away from the previous one. For sufficiently large k , this implies that the mixing time of this chain is $\Theta(1)$. These new parameters are used to select the modified values of α, β according to Lemma S.12 in the Supplement.

The proof of Theorem 1 follows the same general recipe as in [15] for obtaining a bound on the $\sin^2$ error. However, the original proof techniques heavily rely on the IID setting. We carry out a refined analysis for each step under the Markovian data model by a careful control of error terms arising out of dependence. The first step involves obtaining a high-probability bound on the $\sin^2$ error, by noting that Oja’s algorithm on n data-points can be viewed as a single iteration of the power method on B_n . Therefore, fixing a $\delta \in (0, 1)$ using Lemma 3.1 from [15], we have with probability at least $(1 - \delta)$,

$$\sin^2(w_n, v_1) \leq \frac{C \log\left(\frac{1}{\delta}\right)}{\delta} \frac{\text{Tr}(V_\perp^T B_n B_n^T V_\perp)}{v_1^T B_n B_n^T v_1}, \quad (6)$$

where C is an absolute constant. The numerator is bounded by first bounding its expectation (see Theorem 3) and then using Markov’s inequality. To bound the denominator, similar to [15], we will use Chebyshev’s inequality. Theorem 4 provides a lower bound for the expectation $\mathbb{E}[v_1^T B_n B_n^T v_1]$. Chebyshev’s inequality also requires upper-bounding the variance of $\mathbb{E}[v_1^T B_n B_n^T v_1]$, which requires us to bound $\mathbb{E}[(v_1^T B_n B_n^T v_1)^2]$ (see Theorem 5).

4 Main Technical Tools

In this section, we provide a sketch of the main arguments used in our proof.

Warm-up with downsampled Oja’s algorithm: We start with the simple downsampled Oja’s algorithm to build intuition. Here, one applies Oja’s update rule (Eq 1) to every k^{th} data-point, for a suitably chosen k . For $k = \lceil L\tau_{\text{mix}} \log n \rceil$, the total variation distance between any consecutive data-points in the downsampled data stream is $O(n^{-L})$. As we show in Corollary 1, the error of this algorithm is similar to the error of Oja’s algorithm applied to n/k data-points in the IID setting, i.e., $O(\mathcal{V}\tau_{\text{mix}} \log n/n)$.

We will take $\mathbb{E}[v_1^T B_n B_n^T v_1]$ as an example. Let us introduce some notation.

$$B_{j,i} := (I + \eta_j X_j X_j^T) (I + \eta_{j-1} X_{j-1} X_{j-1}^T) \dots (I + \eta_i X_i X_i^T) \quad (7)$$

We peel this quantity one matrix at a time from the inside. Note that for a reversible Markov chain, standard results imply (see Lemma 1) that the mixing conditions apply to the conditional distribution of a state given another state k steps in the “future” (see Supplement section S.3 for a proof). Recall $d_{\text{mix}}(k)$ from Section 2.1.

Lemma 1. Under Assumption 1, $\frac{1}{2} \sup_{t \in \Omega} \sum_s |\mathbb{P}(Z_t = s | Z_{t+k} = t) - \pi(s)| = d_{\text{mix}}(k)$.

It will be helpful to explain our analysis by comparing it with the IID setting. For this reason, we will use $\mathbb{E}_{\text{IID}}[\cdot]$ to denote the expectation under the IID data model.

$$\begin{aligned} \alpha_{n,1} &:= \mathbb{E}[v_1^T B_n B_n^T v_1] = \mathbb{E}\left[v_1^T B_{n,2} (I + \eta_1 \Sigma + \eta_1 (X_1 X_1^T - \Sigma)) (I + \eta_1 \Sigma + \eta_1 (X_1 X_1^T - \Sigma))^T B_{n,2}^T v_1\right] \\ &= \mathbb{E}\left[v_1^T B_{n,2} (I + \eta_1 \Sigma)^2 B_{n,2}^T v_1\right] + 2\eta_1 T_1 + \eta_1^2 T_2, \end{aligned} \quad (8)$$

where the first term is smaller than $(1 + \eta_1 \lambda_1)^2 \alpha_{n,2}$. We define T_1 and T_2 as follows. $T_1 := \mathbb{E}[v_1^T B_{n,2} (I + \eta_1 \Sigma) (X_1 X_1^T - \Sigma) B_{n,2}^T v_1]$, and $T_2 := \mathbb{E}[v_1^T B_{n,2} (X_1 X_1^T - \Sigma)^2 B_{n,2}^T v_1]$.

For the IID setting, the *second term is zero*, and the third term can be bounded as follows:

$$\mathbb{E}_{\text{IID}}[v_1^T B_{n,2} (X_1 X_1^T - \Sigma)^2 B_{n,2}^T v_1] = \mathbb{E}_{\text{IID}}[v_1^T B_{n,2} \mathbb{E}[(X_1 X_1^T - \Sigma)^2] B_{n,2}^T v_1] \leq \mathcal{V} \mathbb{E}_{\text{IID}}[v_1^T B_{n,2} B_{n,2}^T v_1]$$

Let us denote the IID version of $\alpha_{n,i}$ by $\alpha_{n,i}^{\text{IID}} = \mathbb{E}_{\text{IID}}[v_1^T B_{n,i} B_{n,i}^T v_1]$. The final recursion for the IID case becomes: $\alpha_{n,1}^{\text{IID}} \leq (1 + 2\eta_1 \lambda_1 + \eta_1^2 (\lambda_1^2 + \mathcal{V})) \alpha_{n,1}^{\text{IID}}$. So, for our Markovian data model, the hope is that the cross term T_1 (which has a multiplicative factor of η_1) is $O(\eta_1)$ and T_2 is $O(\eta_1^2)$. We will start with the T_1 term, which is zero in the IID setting.

We hope to reduce the product $B_{n,2} (X_1 X_1^T - \Sigma)$ into a product of nearly independent matrices. One hope is that if instead of $B_{n,2}$, we had $B_{n,2+k}$ for some suitably large integer k , then using (reverse) mixing properties of the Markov chain, we could argue using Lemma 1 that $\mathbb{E}[X_1 X_1^T - \Sigma | s_{1+k}, \dots, s_n]$ is very close to zero. The following lemma formally bounds the deviation of the length- k matrix product from identity.

Lemma 2. Let Assumption 3 hold. If $\forall i \in [n], \eta_i k_i (\mathcal{M} + \lambda_1) \leq \epsilon, \epsilon \in (0, 1)$ and η_i forms a non-increasing sequence then $\forall m \leq n - k_n$,

$$\|B_{m+k_m-1,m} - I\|_2 \leq (1 + \epsilon) k_m \eta_m (\mathcal{M} + \lambda_1) \text{ and} \quad (9)$$

$$\left\| B_{m+k_m-1,m} - I - \sum_{t=m}^{m+k_m-1} \eta_t X_t X_t^T \right\|_2 \leq k_m^2 \eta_m^2 (\mathcal{M} + \lambda_1)^2 \quad (10)$$

Lemma 2 bounds the norm of the matrix product $B_{t+k_t-1,t}$ at two levels. The first result provides a coarse bound, approximating linear and higher-order terms. The second result provides a finer bound, preserving the linear term and approximating quadratic and higher-order terms. The proofs involve a straightforward combinatorial expansion of $B_{t+k_t-1,t}$ and are deferred to the Supplement section S.3.

Approximating $\prod_{i=2}^{k+1} (I + \eta_i X_i X_i^T)$ requires $\eta_1 k$ to be small. Since this is a recursive argument, we would need $\eta_i k$ to be small for $i = 1, \dots, n$, which is satisfied by the strong condition $\eta_1 k$ is small. To obtain a tight analysis, we choose k adaptively. We set $k_i = \tau_{\text{mix}}(\eta_i^2)$ (see definition in Eq 3).

As we will show in detail in the Supplement, Lemma 2 Eq 10 along with the adaptive choice of k_i gives us a sharp error bound. Using it, we can bound T_1 (see Eq 8) as:

$$T_1 \leq \sum_{j=2}^{k+1} \eta_j \mathbb{E} \left[v_1^T B_{n,k+2} \underbrace{\mathbb{E}[(X_j X_j^T) (I + \eta_1 \Sigma) (X_1 X_1 - \Sigma) | X_{k+2}, \dots, X_n]}_{T_{1,j}} B_{n,k+2}^T v_1 \right] + O(\eta_1^2 k_1^2) \alpha_{n,k+2}$$

Naively bounding the $T_{1,j}$ term by $O(1)$ leads to the same rate as downsampled Oja's algorithm.

In the following lemma, we will establish that, indeed, $T_{1,j}$ has a much smaller norm. The novelty of our bound is not just in using the mixing properties of the Markov chain but also in teasing out the variance parameter $\mathcal{V}$. We will state the lemma, in a slightly more general form as -

Lemma 3. *Under Assumptions 1, 2 and 3, for $i < j \leq i + k_i$,*

$$\|\mathbb{E}[(X_i X_i^T - \Sigma) S X_j X_j^T | s_{i+k_i}, \dots, s_n]\|_2 \leq \left(|\lambda_2(P)|^{j-i} \mathcal{V} + 8\eta_i^2 \mathcal{M}(\mathcal{M} + \lambda_1)\right) \|S\|_2$$

where k_i is as defined in Lemma S.12 and S is a constant symmetric positive semi-definite matrix.

Lemma 3 bounds the norm of the covariance between matrices $(X_i X_i^T - \Sigma) S$ and $X_j X_j^T$. In particular, this implies that the norm of $T_{1,j}$ decays as $|\lambda_2(P)|^{j-1}$. The proof uses a spectral argument that replaces a coarse approximation by a sum of k_i $O(1)$ terms to sum of k exponentially decaying terms, thereby removing the dependence on k_i , which can be as large as $\log(n)$. The proof is deferred to the Supplement section S.4. The details can be found in Supplement section S.4.

Let $\{c_1, c_2, c_3, c_4\}$ be positive constants for ease of notation. Coming back to Eq 8, we can bound T_1 as follows: $T_1 \leq \alpha_{n,k+2} \left(\eta_1 \frac{c_1 |\lambda_2(P)| \mathcal{V}}{1 - |\lambda_2(P)|} + c_2 \eta_1^2 k_1^2 \right)$. A similar argument can be applied to bound T_2 as: $T_2 \leq \alpha_{n,k+2} (\mathcal{V} + c_3 \eta_1 k_1^2)$. Putting everything together in 8, we have

$$\alpha_{n,1} \leq \underbrace{\left((1 + \eta_1 \lambda_1)^2 + \mathcal{V} \right) \alpha_{n,2}}_{\text{Recursion for IID setting}} + \underbrace{\left(\frac{c_1 |\lambda_2(P)|}{1 - |\lambda_2(P)|} \right) \mathcal{V} \eta_1^2 \alpha_{n,k+2}}_{\text{Error due to Markovian dependence}} + \underbrace{c_4 \eta_1^3 k_1^2 \alpha_{n,k+2}}_{\text{Error due to approximation of matrix product}}$$

Recurring on this inequality gives us our bound on $\mathbb{E}[v_1^T B_n B_n^T v_1]$ (Theorem 2). We are now ready to present all our accompanying theorems.

5 Intermediate Theorems for Convergence Analysis

In this section, we present our accompanying theorems which are used to obtain the main result in Theorem 1. But before doing so, we will need to establish some notation. Let $k_i := \tau_{\text{mix}}(\eta_i^2)$, and the step-sizes be set as $\eta_i := \frac{\alpha}{(\lambda_1 - \lambda_2)(\beta + i)}$ with α, β as defined in Theorem 1. Let $\epsilon := \frac{1}{100}$. As shown in Lemma S.12 in Supplement Section S.3 our choice of step-sizes satisfy, $\forall i \in [n]$,

$$\text{C.1 } \eta_i k_i (\mathcal{M} + \lambda_1) \leq \epsilon \quad \text{C.2 (Slow decay) } \eta_i \leq \eta_{i-k_i} \leq (1 + 2\epsilon) \eta_i \leq 2\eta_i$$

Further, we define scalar variables -

$$\begin{aligned} r &:= 2(1 + \epsilon) k_n \eta_n (\mathcal{M} + \lambda_1), & \zeta_{k,t} &:= 40k_{t+1} (\mathcal{M} + \lambda_1)^2 \\ \psi_{k,t} &:= 6\mathcal{M} \left[1 + 3k_{t+1}^2 (\mathcal{M} + \lambda_1)^2 \right], & \mathcal{V}' &:= \frac{1 + (3 + 4\epsilon) |\lambda_2(P)|}{1 - |\lambda_2(P)|} \mathcal{V} \end{aligned} \quad (11)$$

and recall the definitions of B_t and $B_{j,i}$ in Eqs 2 and 7, respectively. We are now ready to present the theoretical results needed to prove our main result. For simplicity of notation, we present versions of the results by using $\eta_i := \frac{\alpha}{(\lambda_1 - \lambda_2)(\beta + i)}$ with α, β as defined in Theorem 1. However, these theorems are in fact valid under more general step-size schedules. We state and prove the more general versions in the Supplement Section S.4.

Theorem 2. *Under Assumptions 1, 2 and 3, for all $n > k_n$, and η_i satisfying C.1 and C.2, we have:*

$$\mathbb{E}[v_1^T B_n B_n^T v_1] \leq (1 + r)^2 \exp \left(\sum_{t=1}^{n-k_n} (2\eta_t \lambda_1 + \eta_t^2 (\mathcal{V}' + \lambda_1^2) + \eta_t^3 \psi_{k,t}) \right).$$

The three primary differences with the IID case are a) the $(1 + r)^2$ term, which arises since the recursion sketched in Section 4 leaves out the last k_n terms which are bounded by $(1 + r)^2$; (b) the new factor of $\frac{1}{1 - |\lambda_2(P)|}$ with $\mathcal{V}$ due to the Markovian dependence between terms; and c) the extra lower order term $\eta_t^3 \psi_{k,t}$ arising from the use of Lemmas 2 and 3.

Theorem 3. Let $u := \min \{t : t \in [n], t - k_t \geq 0\}$. Under Assumptions 1, 2 and 3, for all $n > u$, and η_i satisfying C.1 and C.2,

$$\mathbb{E} [\text{Tr} (V_{\perp}^T B_n B_n^T V_{\perp})] \leq (1 + 5\epsilon) \exp \left(\sum_{t=u+1}^n 2\eta_t \lambda_2 + \eta_{t-k_t}^2 (\mathcal{V}' + \lambda_1^2) + \eta_{t-k_t}^3 \psi_{k,t} \right) \\ \times \left(d + \sum_{t=u+1}^n (\mathcal{V}' + \eta_t \psi_{k,t}) C'_{k,t} \eta_{t-k_t}^2 \exp \left(\sum_{i=u+1}^t 2\eta_i (\lambda_1 - \lambda_2) \right) \right)$$

where $C'_{k,t} := (1 + \frac{\delta}{200}) \exp (2\lambda_1 \sum_{i=1}^u \eta_j)$.

Here, the difference is mainly in the new variable u , arising since the recursion stops at u , not 1. $(1 + 5\epsilon)$ represents the approximation of the first u terms.

Theorem 4. Under Assumptions 1, 2 and 3, for all $n > k_n$, η_i satisfying C.1 and C.2, and $s := 2r + \frac{\delta}{1000}$, we have: $\mathbb{E} [v_1^T B_n B_n^T v_1] \geq (1 - s) \exp \left(\sum_{t=1}^{n-k_n} 2\eta_t \lambda_1 - \sum_{t=1}^{n-k_n} 4\eta_t^2 \lambda_1^2 \right)$.

This differs from its IID counterpart by a multiplicative factor of $(1 - s)$ for the same reason as before, which also makes the sums go up to $(n - k_n)$ instead of n . Note that for sufficiently large n (Lemma S.13), $r = O \left(\frac{\log(n)}{n} \right)$ is very small and $\delta \in (0, 1)$. Therefore, $(1 - s) \approx 1$ as large n .

Theorem 5. Under Assumptions 1, 2 and 3, for all $n > k_n$, and η_i satisfying C.1 and C.2, we have:

$$\mathbb{E} \left[(v_1^T B_n B_n^T v_1)^2 \right] \leq (1 + r)^4 \exp \left(\sum_{t=1}^{n-k_n} 4\eta_t \lambda_1 + \sum_{t=1}^{n-k_n} \eta_t^2 \zeta_{k,t} \right).$$

The differences are similar to the last theorems involving v_1 . Surprisingly, for this, the coarse approximation suffices, leading to an absence of the $\mathcal{V}$ term in the bound. Having established these results, the final step is to substitute them into Eq 6 and follow the proof recipe described earlier. This requires significant calculations and is deferred to the Supplement Section S.5.

6 Experimental Validation

In this section, we present some simple experiments to validate our theoretical results. For more detailed experiments, see the Supplement. We design a Markov chain with $|\Omega| = 10$ states, where the transition matrix entries P_{ij} equal $\rho/(|\Omega|-1)$ for $i \neq j$ and $1 - \rho$ for $i = j$. Smaller values of ρ lead to larger mixing times. It can be verified that the stationary distribution $\pi = \mathcal{U}(\Omega)$ is uniform over the state-space and $|\lambda_2(P)| \approx (1 - \rho)$. We set $\rho = 0.2$ for Figures 1 and 2a, and vary it in Figure 2b. Each point in the plot is averaged over 20 random runs over different Markov chains, datasets, and initialization.

Each state $s \in \Omega$ is associated with $D(s) := \text{Bernoulli}(p_s)$ distribution. We set $d = 1000$ and select $p_s \sim \mathcal{U}(0, 0.05)$ at the start of each random run. The covariance matrix, Σ_s , for each state is set as $\Sigma_s(i, j) = \exp(-|i - j|c_s) \sigma_i \sigma_j$ where $c_s := 1 + 9 \left(\frac{s-1}{|\Omega|-1} \right)$, $\sigma_i := 5i^{-\beta}$. We start with the stationary distribution π , and for each state s_i , we draw IID samples $Z_i \sim D(s_i)$. We standardize Z_i such that all components have zero mean and unit variance under the state distribution, $D(s_i)$. We then generate the sample data-point for PCA as $X_i = \Sigma_i^{\frac{1}{2}} Z_i$. By construction, $\mathbb{E}_{D(s_i)} [X_i X_i^T] = \Sigma_i$ and $\mathbb{E}[X_i] = 0^d$. The step sizes for Oja's algorithm are set as $\eta_i = \frac{\alpha}{(\beta+i)(\lambda_1-\lambda_2)}$ for $\alpha = 5, \beta = \frac{5}{1-|\lambda_2(P)|}$. For the downsampled variant, every 10^{th} data-point is considered, and β is accordingly divided by 10. For the offline algorithm, we recompute the leading eigenvector of the sample covariance matrix of data-points seen so far.

Figure 1 compares the performance of different algorithms for the Bernoulli distribution. Here, we are checking if the results obtained in Theorem 1, Proposition 1, and Corollary 1 are reflected in the experiments. The experimental results demonstrate that Oja's algorithm performs significantly

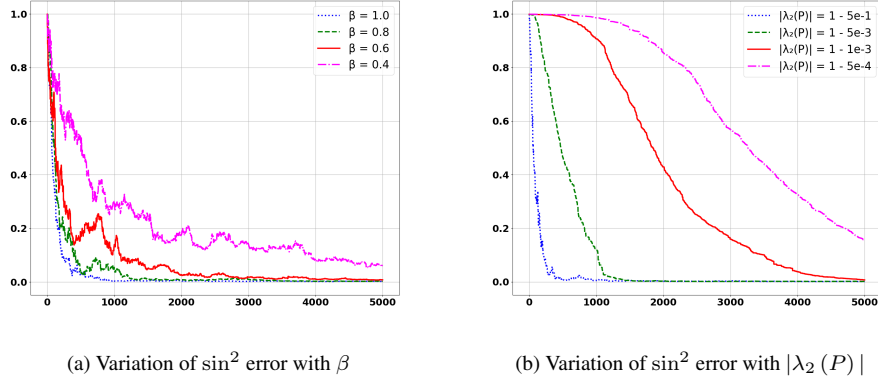


Figure 2: X axis represents the sample size, and Y axis represents the $\sin^2$ error.

better than the downsampled version, consistent with the theoretical results. It also shows that Oja’s algorithm performs similarly to the offline algorithm, which is also confirmed by our theoretical results and that of [27]. Figure 2a compares the performance of Oja’s algorithm for different covariance matrices. Smaller values of β decrease the eigengap $\lambda_1 - \lambda_2$, and hence lead to a slower convergence. Figure 2b confirms that smaller values of ρ (larger values of $|\lambda_2(P)|$) also worsen the rate, which matches with our theoretical results.

7 Conclusion

We have considered the problem of streaming PCA for Markovian data, which has implications in various settings like decentralized optimization, reinforcement learning, etc. The analysis of streaming algorithms in such settings has seen a renewed surge of interest in recent years. However, the dependence between data-points makes it difficult to obtain sharp bounds. We provide, to our knowledge, the first sharp bound for obtaining the first principal component from a Markovian data stream that breaks the logarithmic barrier present in the analysis done for downsampled data. We believe that the theoretical tools that we have developed in this paper would enable one to obtain sharp bounds for other dependent data settings, learning top k principal components, and online inference algorithms with updates involving products of matrices.

8 Acknowledgements

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Supplement

The Supplement is organized as follows -

- Section S.1 introduces notation that will be useful for concise representation.
- Section S.2 provides the proof of Proposition 1.
- Section S.3 contains useful intermediate results which are used in subsequent proofs of our main results.
- Section S.4 proves bounds on $v_1 B_n B_n^T v_1^T$ and $V_\perp B_n B_n^T V_\perp^T$ (Theorems 2, 3, 4 and 5).
- Section S.5 puts everything together and provides proofs of our main result - Theorem 1, along with Corollary 1.
- Section S.6 provides additional experiments to further support our claims.

S.1 Notation and assumptions

For conciseness, we define the stochastic function $A : \Omega \rightarrow \mathbb{R}^{d \times d}$ which maps each state variable of the Markov chain to a $(d \times d)$ positive semi-definite symmetric matrix as

$$A(s_t) := X_t X_t^T$$

Where $X_t \sim D(s_t)$ is drawn from the distribution corresponding to the state at timestep s_t . All the theoretical results are derived under Assumptions 1, 2 and 3.

S.2 Offline PCA with Markovian Data

In this section, we prove Proposition 1. We note that [27] considers $F_j(s_j)$ to be random only with respect to the states. Therefore, we first show that their results generalize to our setting as well, using $F_j(s_j) := A(s_j) - \Sigma$. From Eq (5) in [27], we have

$$\begin{aligned} \left\| \prod_{j=1}^n \exp \left(\frac{\theta}{2} (A(s_j) - \Sigma) \right) \right\|_F^2 &= \text{Tr} \left(\prod_{j=1}^n \exp \left(\frac{\theta}{2} (A(s_j) - \Sigma) \right) \prod_{j=n}^1 \exp \left(\frac{\theta}{2} (A(s_j) - \Sigma) \right) \right) \\ &= \text{vec}(I_d)^T \left(\prod_{j=1}^n \exp(\theta H(s_j)) \right) \text{vec}(I_d) \end{aligned}$$

where $H(s_j) := \frac{1}{2} [(A(s_j) - \Sigma) \otimes I_d + I_d \otimes (A(s_j) - \Sigma)]$. Noting that conditioned on the state sequence, the matrices $A(s_i), i \in [n]$ are independent under our model, we can push in the expectation over the state-specific distributions inside. Let $\mathbb{E}_\pi$ denote the expectation over the stationary state-sequence of the Markov chain, and $\mathbb{E}_D$ denote the distribution over states. Therefore,

$$\mathbb{E}_\pi \mathbb{E}_D \left[\left\| \prod_{j=1}^n \exp \left(\frac{\theta}{2} (A(s_j) - \Sigma) \right) \right\|_F^2 \right] = \mathbb{E}_\pi \left[\text{vec}(I_d)^T \left(\prod_{j=1}^n \mathbb{E}_{D(s_j)} [\exp(\theta H(s_j))] \right) \text{vec}(I_d) \right]$$

Defining the multiplication operator $(E_j^\theta \mathbf{h})(x) = \mathbb{E}_{D(x)} [\exp(\theta H_j(x))] \mathbf{h}(x)$ for any vector-valued function $\mathbf{h}$, we note that Eq (8) from [27] holds for our case as well.

Next, we adapt Proposition 5.3 from [27] for our setting. Specifically, we have the following lemma -

Lemma S.1. Consider the operator $H(x) := \frac{1}{2} [(A(x) - \Sigma) \otimes I_d + I_d \otimes (A(x) - \Sigma)]$. Then, under assumptions 3 and 2 and the definition of Σ , we have,

1. $\mathbb{E}_\pi \mathbb{E}_{D(x)} [H(x)] = 0$
2. $H(x) \preceq \mathcal{M}I$
3. $\left\| \mathbb{E}_\pi \mathbb{E}_{D(x)} [H(x)^2] \right\|_2 \leq \mathcal{V}$

Proof. The proof follows by using the same arguments as Proposition 5.3 from [27] and using the expectation $\mathbb{E}_\pi \mathbb{E}_{D(x)}$ over both the state sequence and the distribution over states, along with assumptions 3 and 2. $\square$

Finally, to prove Bernstein's inequality, we prove that Lemma 6.7 from [27] holds for our case. To note this, we start with equation (57) in their work. We have, using Lemma S.1,

$$\begin{aligned} |\langle v_2, \mathbb{E}_\pi \mathbb{E}_{D(x)} [\exp(\theta H(x))] v_1 \rangle| &= |\langle v_2, \mathbb{E}_\pi \mathbb{E}_{D(x)} [\exp(\theta H(x))] v_1 \rangle| \\ &= \left| \left\langle v_2, \left(I + \mathbb{E}_\pi \mathbb{E}_{D(x)} [H(x)] + \sum_{k=2}^{\infty} \frac{\theta^k}{k!} \mathbb{E}_\pi \mathbb{E}_{D(x)} [H(x)^k] \right) v_1 \right\rangle \right| \\ &= \left| \langle v_2, v_1 \rangle + \left\langle v_2, \left(\sum_{k=2}^{\infty} \frac{\theta^k}{k!} \mathbb{E}_\pi \mathbb{E}_{D(x)} [H(x)^k] \right) v_1 \right\rangle \right| \\ &\leq |\langle v_2, v_1 \rangle| \left(1 + \mathcal{V} \left(\sum_{k=2}^{\infty} \frac{\theta^k}{k!} \mathcal{M}^{k-2} \right) \right) \end{aligned}$$

Therefore, Eq (60) from [27] follows. The other bounds in the proof of Lemma 6.7 from [27] follow similarly. Therefore, we have the following version of Theorem 2.2 from [27] -

Proposition S.1. Under assumptions 2 and 3, we have

$$P \left(\left\| \frac{1}{n} \sum_{j=1}^n A(s_j) - \Sigma \right\|_2 \geq t \right) \leq d^{2-\frac{\pi}{4}} \exp \left(\frac{t^2 / \frac{32}{\pi^2}}{\frac{1+|\lambda_2(P)|}{1-|\lambda_2(P)|} n \mathcal{V} + \frac{8/\pi}{1-|\lambda_2(P)|} \mathcal{M} t} \right)$$

The proof of Proposition 1 now follows by converting the tail bound into a high probability bound and using Wedin's theorem [36]. See proof of Theorem 1.1 in [15] for details.

S.3 Useful Results

This section presents some useful lemmas and their proofs that are subsequently used in our proofs.

Lemma S.2. (Reverse mixing) Consider a reversible, irreducible, and aperiodic Markov chain started from the stationary distribution. Then,

$$\frac{1}{2} \sup_{t \in \Omega} \sum_s |\mathbb{P}(Z_t = s | Z_{t+k} = t) - \pi(s)| = d_{\text{mix}}(k)$$

Proof. Let the transition probabilities of the Markov chain be represented as $P(x|y) := P(Z_{t+1} = x | Z_t = y)$. Consider the time-reversed chain $Y_i := Z_{n-i+1}$ for $i = 1, 2, \dots, n$. Then,

$$\begin{aligned} \mathbb{P}(Y_l = s_l | Y_{l-1} = s_{l-1}, Y_{l-2} = s_{l-2} \dots Y_1 = s_1) \\ &= \mathbb{P}(Z_{n-l+1} = s_l | Z_{n-l+2} = s_{l-1}, Z_{n-l+3} = s_{l-2}, \dots Z_n = s_1) \\ &= \mathbb{P}(Z_{n-l+1} = s_l | Z_{n-l+2} = s_{l-1}) \quad \text{using Lemma S.6} \\ &= \frac{\mathbb{P}(Z_{n-l+1} = s_l, Z_{n-l+2} = s_{l-1})}{\mathbb{P}(Z_{n-l+2} = s_{l-1})} \\ &= \frac{\pi(s_l) P(s_{l-1} | s_l)}{\pi(s_{l-1})} \\ &= P(s_l | s_{l-1}) \quad \text{using reversibility} \end{aligned}$$

This proves that Y_n is an irreducible Markov chain with the same transition probabilities as the original Markov chain. The irreducibility of Y_n follows from the original Markov chain being irreducible. Therefore,

$$\mathbb{P}(Z_t = s_1 | Z_{t+k} = s_2) = \mathbb{P}(Y_{n+1-t} = s_1 | Y_{n+1-t-k} = s_2) \quad (\text{S.12})$$

Then,

$$\frac{1}{2} \sup_{t \in \Omega} \sum_s |\mathbb{P}(Z_t = s | Z_{t+k} = t) - \pi(s)| = \frac{1}{2} \sup_{t \in \Omega} \sum_s |\mathbb{P}(Y_{n+1-t} = s | Y_{n+1-t-k} = t) - \pi(s)| = d_{\text{mix}}(k)$$

where the last inequality follows from the forward mixing properties of the Markov chain. $\square$

Lemma S.3. Let $C_{j,i} = \prod_{t=j}^i (I + Z_t)$ for $i \leq j \leq n$, where $Z_t \in \mathbb{R}^{d \times d}$ are symmetric PSD matrices. Let $U \in \mathbb{R}^{d \times d'}$. Then,

$$\text{Tr}(U^T C_{j,i+1} C_{j,i+1}^T U) \leq \text{Tr}(U^T C_{j,i} C_{j,i}^T U)$$

Proof.

$$\begin{aligned} \text{Tr}(U^T C_{j,i} C_{j,i}^T U) &= \text{Tr}(U^T C_{j,i+1} (I + 2Z_i + Z_i^2) C_{j,i+1}^T U) \\ &= \text{Tr}(U^T C_{j,i+1} C_{j,i+1}^T U) + \text{Tr}(U^T C_{j,i+1} (2Z_i + Z_i^2) C_{j,i+1}^T U) \end{aligned}$$

Since Z_i and Z_i^2 are both PSD, the second term on the RHS is always positive. This yields the proof. $\square$

Lemma S.4. Let $B_t = \prod_{i=t}^1 (I + Z_i)$, where $Z_i \in \mathbb{R}^{d \times d}$ are symmetric PSD matrices.

$$\text{Tr}(B_{n-1} B_{n-1}^T) \leq \text{Tr}(B_n B_n^T)$$

Proof.

$$\begin{aligned} \text{Tr}(B_n B_n^T) &= \text{Tr}((I + Z_n) B_{n-1} B_{n-1}^T (I + Z_n)) \\ &= \text{Tr}(B_{n-1} B_{n-1}^T) + \text{Tr}(Z_n B_{n-1} B_{n-1}^T) + \text{Tr}(B_{n-1} B_{n-1}^T Z_n) + \text{Tr}(Z_n B_{n-1} B_{n-1}^T Z_n) \\ &= \text{Tr}(B_{n-1} B_{n-1}^T) + 2 \text{Tr}(B_{n-1}^T Z_n B_{n-1}) + \text{Tr}(B_{n-1}^T Z_n^2 B_{n-1}) \end{aligned}$$

Since Z_n and Z_n^2 are both PSD, the last two terms on the RHS are always positive. This yields the proof. $\square$

Lemma S.5. Consider matrices $X \in \mathbb{R}^{d \times d'}$ and $A \in \mathbb{R}^{d \times d}$. Then,

$$|\text{Tr}(X^T A X)| \leq \|A\|_2 \text{Tr}(X^T X)$$

Proof. For a matrix $Z \in \mathbb{R}^{d \times d}$, let the singular values be denoted as :

$$\sigma_{\max}(Z) = \sigma_1(Z) \geq \sigma_2(Z) \dots \geq \sigma_d(Z)$$

Using Von-Neumann's trace inequality, we have

$$\begin{aligned} |\text{Tr}(X^T A X)| &= |\text{Tr}(A X X^T)| \\ &\leq \sum_{i=1}^d \sigma_i(A) \sigma_i(X X^T) \\ &\leq \sigma_{\max}(A) \sum_{i=1}^d \sigma_i(X X^T) \\ &= \|A\|_2 \text{Tr}(X X^T) \\ &= \|A\|_2 \text{Tr}(X^T X) \end{aligned}$$

$\square$

Lemma S.6. *Given the Markov property in a Markov chain, the reverse Markov property holds, i.e*

$$P(Z_t = s | Z_{t+1} = w, Z_{t+2} = s_{t+2} \dots Z_n = s_n) = P(Z_t = s | Z_{t+1} = w)$$

Proof.

$$\begin{aligned} & P(Z_t = s | Z_{t+1} = w, Z_{t+2} = s_{t+2} \dots Z_n = s_n) \\ &= \frac{P(Z_t = s, Z_{t+1} = w, Z_{t+2} = s_{t+2} \dots Z_n = s_n)}{P(Z_{t+1} = t, Z_{t+2} = s_{t+2} \dots Z_n = s_n)} \\ &= \frac{P(Z_t = s, Z_{t+1} = w) P(Z_{t+2} = s_{t+2} \dots Z_n = s_n | Z_t = s, Z_{t+1} = w)}{P(Z_{t+1} = w) P(Z_{t+2} = s_{t+2} \dots Z_n = s_n | Z_{t+1} = w)} \\ &= \frac{P(Z_t = s, Z_{t+1} = w) P(Z_{t+2} = s_{t+2} \dots Z_n = s_n | Z_{t+1} = w)}{P(Z_{t+1} = w) P(Z_{t+2} = s_{t+2} \dots Z_n = s_n | Z_{t+1} = w)} \\ &= \frac{P(Z_t = s, Z_{t+1} = w)}{P(Z_{t+1} = w)} \\ &= P(Z_t = s | Z_{t+1} = w) \end{aligned}$$

□

S.3.1 Proof of Lemma 2

Now we are ready to provide a proof of Lemma 2.

Proof of Lemma 2. Without loss of generality, we prove the statement for $m = 1$. For convenience of notation, we denote $k := k_1$. Note that,

$$B_{k,1} = \sum_{r=0}^k \sum_{(i_1, i_2 \dots i_r) \in G_r} \prod_{j=1}^r \eta_{i_j} A(s_{i_j}), \quad G_r = \{(i_1, \dots, i_r) \in \{1, \dots, N\}^r : i_1 < \dots < i_r\}$$

with the convention that $\prod_{\phi} = I$. Therefore, since η_i forms a non-increasing sequence and $|G_r| = \binom{k}{r}$, we have,

$$\begin{aligned} \|B_{k,1} - I\|_2 &= \left\| \sum_{r=1}^k \sum_{(i_1, i_2 \dots i_r) \in G_r} \prod_{j=1}^r \eta_{i_j} A(s_{i_j}) \right\|_2 \\ &\leq \sum_{r=1}^k \sum_{(i_1, i_2 \dots i_r) \in G_r} \left\| \prod_{j=1}^r \eta_{i_j} A(s_{i_j}) \right\|_2 \\ &\leq \sum_{r=1}^k \binom{k}{r} \left(\prod_{i=1}^r \eta_i \right) (\mathcal{M} + \lambda_1)^r \\ &\leq \sum_{r=1}^k \frac{k^r}{r!} \left(\prod_{i=1}^r \eta_i \right) (\mathcal{M} + \lambda_1)^r \\ &\leq \sum_{r=1}^k \frac{k^r}{r!} \eta_1^r (\mathcal{M} + \lambda_1)^r \\ &\leq \exp(k\eta_1 (\mathcal{M} + \lambda_1)) - 1 \\ &\leq k\eta_1 (\mathcal{M} + \lambda_1) (1 + k\eta_1 (\mathcal{M} + \lambda_1)) \text{ using S.14} \\ &\leq (1 + \epsilon) k\eta_1 (\mathcal{M} + \lambda_1) \end{aligned} \tag{S.13}$$

where we have used the assumptions that $\|A(s)\|_2 \leq \|A(s) - \Sigma\| + \|\Sigma\|_2 = (\mathcal{M} + \lambda_1)$, $k\eta_1 (\mathcal{M} + \lambda_1) < 1$ and the useful result that

$$e^x \leq 1 + x + x^2, \quad x \in [0, 1.79] \tag{S.14}$$

This completes the proof for (a).

For part (b), we have

$$\begin{aligned}
\left\| B_{k,1} - I - \sum_{t=1}^k \eta_t A(s_t) \right\|_2 &= \left\| \sum_{r=2}^k \sum_{(i_1, i_2, \dots, i_r) \in G_r} \prod_{j=1}^r \eta_{i_j} A(s_{i_j}) \right\|_2 \\
&\leq \sum_{r=2}^k \sum_{(i_1, i_2, \dots, i_r) \in G_r} \left\| \prod_{j=1}^r \eta_{i_j} A(s_{i_j}) \right\|_2 \\
&\leq \sum_{r=2}^k \binom{k}{r} \left(\prod_{i=2}^r \eta_i \right) (\mathcal{M} + \lambda_1)^r \\
&\leq \sum_{r=2}^k \frac{k^r}{r!} \left(\prod_{i=2}^r \eta_i \right) (\mathcal{M} + \lambda_1)^r \\
&\leq \sum_{r=2}^k \frac{k^r}{r!} \eta_1^r (\mathcal{M} + \lambda_1)^r \\
&\leq \exp(k \eta_1 (\mathcal{M} + \lambda_1)) - 1 - k \eta_1 (\mathcal{M} + \lambda_1) \\
&\leq k^2 \eta_1^2 (\mathcal{M} + \lambda_1)^2 \text{ using S.14 along with } k \eta_1 (\mathcal{M} + \lambda_1) < 1
\end{aligned} \tag{S.15}$$

which completes the proof. $\square$

S.3.2 Proof of Lemma 3

Before proving Lemma 3, we will need the following lemma.

Lemma S.7. For arbitrary matrices $M_i \in \mathbb{R}^{d \times d}$, $i \in [n]$ and $Q \in \mathbb{R}^{n \times n}$, we have

$$\left\| \sum_{x,y \in [n]} Q(x,y) M_x M_y^T \right\|_2 \leq \|Q\|_2 \left\| \sum_{x \in [n]} M_x M_x^T \right\|_2$$

where $\|\cdot\|_2$ denotes the spectral norm.

Proof. Define matrix $X \in \mathbb{R}^{d \times nd}$ as $X := [M_1 \ M_2 \ \dots \ M_n]$. We note that

$$\begin{aligned}
\|X\|_2 &= \sqrt{\lambda_{\max}(X X^T)} \\
&= \sqrt{\lambda_{\max} \left(\sum_{x \in [n]} M_x M_x^T \right)} \\
&= \sqrt{\left\| \sum_{x \in [n]} M_x M_x^T \right\|_2} \text{ since } \sum_{x \in [n]} M_x M_x^T \text{ is a symmetric matrix}
\end{aligned}$$

Then, we have,

$$\begin{aligned}
\sum_{x,y \in [n]} Q(x,y) M_x M_y^T &= X (Q \otimes I_{d \times d}) X^T, \text{ where } \otimes \text{ denotes the kronecker product} \\
&\leq \|X\|_2^2 \|Q \otimes I_{d \times d}\|_2 \text{ using submultiplicativity of the spectral norm} \\
&= \|X\|_2^2 \|Q\|_2 \text{ since } \|A \otimes B\|_2 = \|A\|_2 \|B\|_2
\end{aligned}$$

which completes our proof. $\square$

Proof of Lemma 3. We denote $k_i := k$ for convenience of notation. By using reversibility (see S.12), we know that the time-reversed process is also a Markov chain with the same transition probabilities. Then, for $i < j \leq i + k$ and any m ,

$$\begin{aligned}
P(s_i = s, s_j = t | s_{i+k} = u) &= P(s_i = s | s_j = t) P(s_j = t | s_{i+k} = u) \\
&\stackrel{(i)}{=} P^{j-i}(t, s) P^{i+k-j}(u, t) \\
&= P(s_m = s | s_{m-j+i} = t) P(s_{m-j+i} = t | s_{m-k} = u) \\
&= P(s_m = s, s_{m-j+i} = t | s_{m-k} = u) \tag{S.16}
\end{aligned}$$

Step (i) uses reversibility. Therefore,

$$\begin{aligned}
\mathbb{E}[(A(s_i) - \Sigma) SA(s_j) | s_{i+k}, \dots, s_n] &= \sum_{s,t} (\Sigma_s + \mu_s \mu_s^T - \Sigma) S (\Sigma_t + \mu_t \mu_t^T) P(s_i = s, s_j = t | s_{i+k}, \dots, s_n) \\
\text{using Lemma S.6} &= \sum_{s,t} (\Sigma_s + \mu_s \mu_s^T - \Sigma) S (\Sigma_t + \mu_t \mu_t^T) P(s_i = s, s_j = t | s_{i+k}) \\
\text{using Eq S.16} &= \sum_{s,t} (\Sigma_s + \mu_s \mu_s^T - \Sigma) S (\Sigma_t + \mu_t \mu_t^T) P(s_m = s, s_{m-j+i} = t | s_{m-k} = u) \\
&= \mathbb{E}[(A(s_m) - \Sigma) SA(s_{m-j+i}) | s_{m-k}] \\
&= \mathbb{E}[(A(s_j) - \Sigma) SA(s_i) | s_{j-k}] \text{ setting } m := j
\end{aligned}$$

Therefore, without loss of generality, we proceed with the second form.

$$\begin{aligned}
&\|\mathbb{E}[(A(s_j) - \Sigma) SA(s_i) | s_{j-k} = x_0]\|_2 \\
&\leq \underbrace{\|\mathbb{E}[(A(s_j) - \Sigma) S \Sigma | s_{j-k} = x_0]\|_2}_{T_1} + \underbrace{\|\mathbb{E}[(A(s_j) - \Sigma) S (A(s_i) - \Sigma) | s_{j-k} = x_0]\|_2}_{T_2}
\end{aligned}$$

$$\begin{aligned}
T_1 &:= \|\mathbb{E}[(A(s_j) - \Sigma) S \Sigma | s_{j-k} = x_0]\|_2 \\
&= \|\mathbb{E}[\mathbb{E}_{D(s_j)}[(A(s_j) - \Sigma) | s_{j-k} = x_0] S \Sigma]\|_2 \\
&= \left\| \mathbb{E} \left[\left(\Sigma_{s_j} + \mu_{s_j} \mu_{s_j}^T - \Sigma \right) | s_{j-k} = x_0 \right] S \Sigma \right\|_2 \\
&= \left\| \sum_{s \in \Omega} P^k(s_{j-k}, s) (\Sigma_s + \mu_s \mu_s^T - \Sigma) S \Sigma \right\|_2 \\
&\leq \left\| \sum_{s \in \Omega} (P^k(s_{j-k}, s) - \pi(s)) (\Sigma_s + \mu_s \mu_s^T - \Sigma) + \underbrace{\mathbb{E}_\pi[(\Sigma_s + \mu_s \mu_s^T - \Sigma)]}_{=0} \right\|_2 \|S\|_2 \|\Sigma\|_2 \\
&= \lambda_1 \|S\|_2 \left(\left\| \sum_{s \in \Omega} (P^k(s_{j-k}, s) - \pi(s)) (\Sigma_s + \mu_s \mu_s^T - \Sigma) \right\|_2 \right) \\
&\leq \lambda_1 \|S\|_2 \mathcal{M} \sum_{s \in \Omega} |P^k(s_{j-k}, s) - \pi(s)| \\
&\leq 2\lambda_1 \|S\|_2 \mathcal{M} d_{\text{mix}}(k_{i+1}) \\
&\leq 2\eta_i^2 \mathcal{M} \lambda_1 \|S\|_2 \tag{S.17}
\end{aligned}$$

$$\begin{aligned}
T_2 &= \|\mathbb{E}[(A(s_j) - \Sigma) S(A(s_i) - \Sigma) | s_{j-k} = x_0]\|_2 \\
&= \left\| \sum_{x,y \in \Omega} \mathbb{P}(s_j = x, s_i = y | s_{j-k} = x_0) \mathbb{E}_{D(x)}[A(x) - \Sigma] S \mathbb{E}_{D(y)}[A(y) - \Sigma] \right\|_2 \quad \text{using independence of} \\
&\quad D(x) \text{ and } D(y) \text{ conditioned on } x \text{ and } y \\
&= \left\| \sum_{x,y \in \Omega} \mathbb{P}(s_j = x, s_i = y | s_{j-k} = x_0) \underbrace{(\Sigma_x + \mu_x \mu_x^T - \Sigma)}_{W_x} S^{\frac{1}{2}} S^{\frac{1}{2}} \underbrace{(\Sigma_y + \mu_y \mu_y^T - \Sigma)}_{W_y^T} \right\|_2 \\
&= \left\| \sum_{x,y \in \Omega} \mathbb{P}(s_j = x | s_i = y) \mathbb{P}(s_i = y | s_{j-k} = x_0) W_x W_y^T \right\|_2 \quad \text{using the Markov property} \\
&= \left\| \sum_{x,y \in \Omega} P^{j-i}(y, x) P^{i-j+k}(x_0, y) W_x W_y^T \right\|_2 \\
&= \left\| \sum_{x,y \in \Omega} (P^{j-i}(y, x) - \pi(x)) P^{i-j+k}(x_0, y) W_x W_y^T + \sum_{x,y \in \Omega} \pi(x) P^{i-j+k}(x_0, y) W_x W_y^T \right\|_2 \\
&= \left\| \sum_{x,y \in \Omega} (P^{j-i}(y, x) - \pi(x)) P^{i-j+k}(x_0, y) W_x W_y^T + \underbrace{\sum_{x \in \Omega} \pi(x) W_x}_{=0} \sum_{y \in \Omega} P^{i-j+k}(x_0, y) W_y^T \right\|_2 \\
&= \left\| \sum_{x,y \in \Omega} (P^{j-i}(y, x) - \pi(x)) P^{i-j+k}(x_0, y) W_x W_y^T \right\|_2 \\
&\leq \underbrace{\left\| \sum_{x,y \in \Omega} (P^{j-i}(y, x) - \pi(x)) (P^{i-j+k}(x_0, y) - \pi(y)) W_x W_y^T \right\|_2}_{T_{21}} + \underbrace{\left\| \sum_{x,y \in \Omega} (P^{j-i}(y, x) - \pi(x)) \pi(y) W_x W_y^T \right\|_2}_{T_{22}} \\
&\hspace{15cm} \text{(S.18)}
\end{aligned}$$

For T_{21} , we have,

$$\begin{aligned}
T_{21} &\leq \sum_{x,y \in \Omega} |P^{j-i}(y, x) - \pi(x)| |P^{i-j+k}(x_0, y) - \pi(y)| \|W_x W_y^T\|_2 \\
&\leq \|S\|_2 \mathcal{M}^2 \sum_{y \in \Omega} |P^{i-j+k}(x_0, y) - \pi(y)| \sum_{x \in \Omega} |P^{j-i}(y, x) - \pi(x)| \\
&\leq 2 \|S\|_2 \mathcal{M}^2 d_{\text{mix}}(j-i) \sum_{y \in \Omega} |P^{i-j+k}(x_0, y) - \pi(y)| \\
&\leq 4 \|S\|_2 \mathcal{M}^2 d_{\text{mix}}(j-i) d_{\text{mix}}(i-j+k) \\
&\leq 4 \|S\|_2 \mathcal{M}^2 2^{-\lfloor \frac{j-i}{\tau_{\text{mix}}} \rfloor} 2^{-\lfloor \frac{i-j+k}{\tau_{\text{mix}}} \rfloor} \\
&\leq 8 \|S\|_2 \mathcal{M}^2 2^{-\lfloor \frac{j-i+i-j+k}{\tau_{\text{mix}}} \rfloor} \quad \text{since } \forall a, b \quad \lfloor a \rfloor + \lfloor b \rfloor \geq \lfloor a+b \rfloor - 1 \\
&\leq 8 \|S\|_2 \mathcal{M}^2 2^{-\lfloor \frac{k}{\tau_{\text{mix}}} \rfloor} \leq 8 \|S\|_2 \mathcal{M}^2 d_{\text{mix}}(k) \leq 8 \eta_i^2 \mathcal{M}^2 \|S\|_2 \quad \text{(S.19)}
\end{aligned}$$

For T_{22} , we have,

$$\begin{aligned}
T_{22} &= \left\| \sum_{x,y \in \Omega} (P^{j-i}(y,x) - \pi(x)) \pi(y) W_x W_y^T \right\|_2 \\
&= \left\| \sum_{x,y \in \Omega} \frac{(P^{j-i}(y,x) - \pi(x))}{\sqrt{\pi(x)}} \sqrt{\pi(y)} \left(\sqrt{\pi(x)} W_x \right) \left(\sqrt{\pi(y)} W_y^T \right) \right\|_2 \\
&= \left\| \sum_{x,y \in \Omega} \frac{(P^{j-i}(y,x) - \pi(x))}{\sqrt{\pi(x)}} \sqrt{\pi(y)} \left(\sqrt{\pi(x)} (\Sigma_x + \mu_x \mu_x^T - \Sigma) S^{\frac{1}{2}} \right) \left(\sqrt{\pi(y)} S^{\frac{1}{2}} (\Sigma_y + \mu_y \mu_y^T - \Sigma) \right) \right\|_2 \\
&\stackrel{(i)}{\leq} \|Q\|_2 \left\| \sum_{x \in \Omega} \pi(x) (\Sigma_x + \mu_x \mu_x^T - \Sigma) S (\Sigma_x + \mu_x \mu_x^T - \Sigma) \right\|_2 \\
&= \|Q\|_2 \left\| \mathbb{E}_\pi [(\Sigma_x + \mu_x \mu_x^T - \Sigma) S (\Sigma_x + \mu_x \mu_x^T - \Sigma)] \right\|_2 \\
&\leq \|Q\|_2 \|S\|_2 \left\| \mathbb{E}_\pi [(\Sigma_x + \mu_x \mu_x^T - \Sigma)^2] \right\|_2 \\
&\leq \mathcal{V} \|Q\|_2 \|S\|_2
\end{aligned} \tag{S.20}$$

Step (i) uses Lemma S.7 with $Q(y,x) := \frac{(P^{j-i}(y,x) - \pi(x))}{\sqrt{\pi(x)}} \sqrt{\pi(y)}$ and $M_x = \sqrt{\pi(x)} (\Sigma_x + \mu_x \mu_x^T - \Sigma) S^{\frac{1}{2}}$.

Let's now bound $\|Q\|_2$. Let $\Pi := \text{diag}(\pi) \in \mathbb{R}^{\Omega \times \Omega}$ and $t := j - i$. Then, we have

$$\begin{aligned}
Q &= \Pi^{\frac{1}{2}} (P^t - \mathbb{1} \mathbb{1}^T \Pi) \Pi^{-\frac{1}{2}} \\
&= \Pi^{\frac{1}{2}} P^t \Pi^{-\frac{1}{2}} - \Pi^{\frac{1}{2}} \mathbb{1} \mathbb{1}^T \Pi^{\frac{1}{2}}
\end{aligned}$$

Now, since we have a reversible Markov chain, $\Pi P = P^T \Pi$. Therefore,

$$\begin{aligned}
\Pi^{\frac{1}{2}} P \Pi^{-\frac{1}{2}} &= \Pi^{\frac{1}{2}} \Pi^{-1} P^T \Pi \Pi^{-\frac{1}{2}} \\
&= \Pi^{-\frac{1}{2}} P^T \Pi^{\frac{1}{2}}
\end{aligned}$$

Therefore, P is similar to the self-adjoint matrix $\Pi^{\frac{1}{2}} P \Pi^{-\frac{1}{2}}$ and their eigenvalues are real and the same. Further note that $\Pi^{\frac{1}{2}} \mathbb{1}$ is the leading eigenvector of $\Pi^{\frac{1}{2}} P \Pi^{-\frac{1}{2}}$ with eigenvalue 1 since

$$\begin{aligned}
\Pi^{\frac{1}{2}} P \Pi^{-\frac{1}{2}} \Pi^{\frac{1}{2}} \mathbb{1} &= \Pi^{\frac{1}{2}} P \mathbb{1} \\
&= \Pi^{\frac{1}{2}} \mathbb{1} \text{ since } P \text{ is a stochastic matrix}
\end{aligned}$$

Now,

$$\begin{aligned}
\|Q\|_2 &= \left\| \Pi^{\frac{1}{2}} P^t \Pi^{-\frac{1}{2}} - \Pi^{\frac{1}{2}} \mathbb{1} \mathbb{1}^T \Pi^{\frac{1}{2}} \right\|_2 \\
&= \left\| \left(\Pi^{\frac{1}{2}} P \Pi^{-\frac{1}{2}} \right)^t - \Pi^{\frac{1}{2}} \mathbb{1} \mathbb{1}^T \Pi^{\frac{1}{2}} \right\|_2 \\
&\leq \left| \lambda_2 \left(\Pi^{\frac{1}{2}} P \Pi^{-\frac{1}{2}} \right) \right|^t \\
&= |\lambda_2(P)|^t
\end{aligned}$$

where $|\lambda_2(\cdot)|$ denotes the second-largest eigenvalue in magnitude. Therefore, using S.17, S.19 and S.20, we have

$$\begin{aligned}
\mathbb{E}[(A(s_i) - \Sigma) S A(s_j) | s_{i+k}, \dots, s_n] &\leq \left(|\lambda_2(P)|^{j-i} \mathcal{V} + 8\eta_i^2 \mathcal{M}^2 + 2\eta_i^2 \mathcal{M} \lambda_1 \right) \|S\|_2 \\
&\leq \left(|\lambda_2(P)|^{j-i} \mathcal{V} + 8\eta_i^2 \mathcal{M} (\mathcal{M} + \lambda_1) \right) \|S\|_2
\end{aligned}$$

Hence proved. $\square$

Lemma S.8. Let $\forall i \in [n], \eta_i k_i (\mathcal{M} + \lambda_1) \leq \epsilon, \epsilon \in (0, 1)$ and η_i forms a non-increasing sequence. Set $k_i := \tau_{\text{mix}}(\gamma \eta_i^2), \gamma \in (0, 1]$. Then for constant matrix $U \in \mathbb{R}^{d \times d'}$, and constant positive semi-definite matrix $G \in \mathbb{R}^{d \times d}, i \leq j \leq n, j - i \geq k_i$, we have

$$\begin{aligned} & |\mathbb{E} [\text{Tr} (U^T B_{j,i+1} G (A_i - \Sigma) B_{j,i+1}^T U)]| \\ & \leq \eta_{i+1} \|G\|_2 \left(\frac{2\mathcal{V} |\lambda_2(P)|}{1 - |\lambda_2(P)|} + \eta_{i+1} \mathcal{M} \left(2\gamma (1 + 8\epsilon) + \left(2 + (1 + \epsilon)^2 \right) k_{i+1}^2 (\mathcal{M} + \lambda_1)^2 \right) \right) \\ & \quad \times \mathbb{E} [\text{Tr} (U^T B_{j,i+k_{i+1}} B_{j,i+k_{i+1}}^T U)] \end{aligned}$$

where $B_{j,i}$ is defined in 7.

Proof. For the convenience of notation, we denote $k_{i+1} := k$. Let $B_{j,i+1} = B_{j,i+k} (I + R)$, then

$$\begin{aligned} \mathbb{E} [\text{Tr} (U^T B_{j,i+1} G (A_i - \Sigma) B_{j,i+1}^T U)] &= \\ \mathbb{E} \left[\underbrace{\text{Tr} (U^T B_{j,i+k} G (A_i - \Sigma) B_{j,i+k}^T U)}_{T_1} \right] &+ \mathbb{E} \left[\underbrace{\text{Tr} (U^T B_{j,i+k} G (A_i - \Sigma) R^T B_{j,i+k}^T U)}_{T_2} \right] + \\ \mathbb{E} \left[\underbrace{\text{Tr} (U^T B_{j,i+k} R G (A_i - \Sigma) B_{j,i+k}^T U)}_{T_3} \right] &+ \mathbb{E} \left[\underbrace{\text{Tr} (U^T B_{j,i+k} R G (A_i - \Sigma) R^T B_{j,i+k}^T U)}_{T_4} \right] \end{aligned} \quad (\text{S.21})$$

We will now bound each of the terms $\mathbb{E} [T_1], \mathbb{E} [T_2], \mathbb{E} [T_3]$ and $\mathbb{E} [T_4]$.

$$\begin{aligned} \mathbb{E} [T_1] &= \mathbb{E} [\text{Tr} (U^T B_{j,i+k} G (A_i - \Sigma) B_{j,i+k}^T U)] \\ &= \mathbb{E} \left[\mathbb{E} [\text{Tr} (U^T B_{j,i+k} G (A_i - \Sigma) B_{j,i+k}^T U) \mid s_{i+k}, \dots, s_{j-1}, s_j] \right] \\ &= \mathbb{E} \left[\text{Tr} \left(U^T B_{j,i+k} G \mathbb{E} [(A_i - \Sigma) \mid s_{i+k}, \dots, s_{j-1}, s_j] B_{j,i+k}^T U \right) \right] \\ &= \mathbb{E} \left[\text{Tr} \left(U^T B_{j,i+k} G \mathbb{E} [(A_i - \Sigma) \mid s_{i+k}] B_{j,i+k}^T U \right) \right] \text{ using Lemma S.6} \end{aligned}$$

Now, using Lemma 1, we have,

$$\begin{aligned} \left\| \mathbb{E} [(A_i - \Sigma) \mid s_{i+k}] \right\|_2 &= \left\| \sum_{s \in \Omega} P^k(s_{i+k}, s) (A_i - \Sigma) \right\|_2 \\ &= \left\| \sum_{s \in \Omega} (P^k(s_{i+k}, s) - \pi(s)) (A_i - \Sigma) + \underbrace{\mathbb{E}_\pi [(A_i - \Sigma)]}_{=0} \right\|_2 \\ &= \left\| \sum_{s \in \Omega} (P^k(s_{i+k}, s) - \pi(s)) (A_i - \Sigma) \right\|_2 \\ &\leq \mathcal{M} \sum_{s \in \Omega} |P^k(s_{i+k}, s) - \pi(s)| \\ &\leq 2\mathcal{M} d_{\text{mix}}(k_{i+1}) \\ &\leq 2\gamma \eta_{i+1}^2 \mathcal{M} \end{aligned} \quad (\text{S.22})$$

where we have used Lemma S.5. Therefore,

$$|\mathbb{E} [T_1]| \leq \gamma \eta_{i+1}^2 \mathcal{M} \|G\|_2 \mathbb{E} [\text{Tr} (U^T B_{j,i+k} B_{j,i+k}^T U)] \quad (\text{S.23})$$

We will now bound $\mathbb{E}[T_2]$. Let $R_0 := \sum_{\ell=i+1}^{i+k-1} \eta_\ell A_\ell$. Using Lemma 2 we have

$$\|R - R_0\|_2 \leq \eta_{i+1}^2 k_{i+1}^2 (\mathcal{M} + \lambda_1)^2$$

Then,

$$\begin{aligned} \mathbb{E}[T_2] &= \mathbb{E}[\text{Tr}(U^T B_{j,i+k} G(A_i - \Sigma) R^T B_{j,i+k}^T U)] \\ &= \mathbb{E}[\text{Tr}(U^T B_{j,i+k} G(A_i - \Sigma) R_0^T B_{j,i+k}^T U)] + \mathbb{E}[\text{Tr}(U^T B_{j,i+k} G(A_i - \Sigma) (R - R_0)^T B_{j,i+k}^T U)] \\ &= \mathbb{E}[\text{Tr}(U^T B_{j,i+k} G \mathbb{E}[(A_i - \Sigma) R_0^T | s_{i+k}, \dots, s_j] B_{j,i+k}^T U)] + \\ &\quad \mathbb{E}[\text{Tr}(U^T B_{j,i+k} G(A_i - \Sigma) (R - R_0)^T B_{j,i+k}^T U)] \end{aligned}$$

Using Lemma 3 with $S := I$ we have,

$$\begin{aligned} \|\mathbb{E}[(A_i - \Sigma) R_0^T | s_{i+k}, \dots, s_j]\|_2 &\leq \sum_{\ell=i+1}^{i+k-1} \eta_\ell \left(|\lambda_2(P)|^{\ell-i} \mathcal{V} + 8\gamma \eta_{i+1}^2 \mathcal{M} (\mathcal{M} + \lambda_1) \right) \\ &\leq \eta_{i+1} \mathcal{V} \frac{|\lambda_2(P)|}{1 - |\lambda_2(P)|} + 8\gamma \eta_{i+1}^3 k_{i+1} \mathcal{M} (\mathcal{M} + \lambda_1) \quad (\text{S.24}) \end{aligned}$$

Therefore,

$$\begin{aligned} |\mathbb{E}[T_2]| &\leq \|G\|_2 \left(\eta_{i+1} \frac{\mathcal{V} |\lambda_2(P)|}{1 - |\lambda_2(P)|} + 8\gamma \eta_{i+1}^3 k_{i+1} \mathcal{M} (\mathcal{M} + \lambda_1) + \eta_{i+1}^2 k_{i+1}^2 \mathcal{M} (\mathcal{M} + \lambda_1)^2 \right) \mathbb{E}[\text{Tr}(U^T B_{j,i+k} B_{j,i+k}^T U)] \\ &= \eta_{i+1} \|G\|_2 \left(\frac{\mathcal{V} |\lambda_2(P)|}{1 - |\lambda_2(P)|} + 8\gamma \eta_{i+1}^2 k_{i+1} \mathcal{M} (\mathcal{M} + \lambda_1) + \eta_{i+1} k_{i+1}^2 \mathcal{M} (\mathcal{M} + \lambda_1)^2 \right) \mathbb{E}[\text{Tr}(U^T B_{j,i+k} B_{j,i+k}^T U)] \quad (\text{S.25}) \end{aligned}$$

Similarly using Lemma 3 with $S := G$,

$$|\mathbb{E}[T_3]| \leq \eta_{i+1} \|G\|_2 \left(\frac{\mathcal{V} |\lambda_2(P)|}{1 - |\lambda_2(P)|} + 8\gamma \eta_{i+1}^2 k_{i+1} \mathcal{M} (\mathcal{M} + \lambda_1) + \eta_{i+1} k_{i+1}^2 \mathcal{M} (\mathcal{M} + \lambda_1)^2 \right) \mathbb{E}[\text{Tr}(U^T B_{j,i+k} B_{j,i+k}^T U)] \quad (\text{S.26})$$

Finally,

$$\begin{aligned} |\mathbb{E}[T_4]| &\leq \mathcal{M} \|G\|_2 \|R\|_2^2 \mathbb{E}[\text{Tr}(U^T B_{j,i+k} B_{j,i+k}^T U)] \\ &\leq (1 + \epsilon)^2 \eta_{i+1}^2 k_{i+1}^2 \mathcal{M} (\mathcal{M} + \lambda_1)^2 \|G\|_2 \mathbb{E}[\text{Tr}(U^T B_{j,i+k} B_{j,i+k}^T U)] \quad \text{using Lemma 2} \quad (\text{S.27}) \end{aligned}$$

Therefore, using Eqs S.23, S.25, S.26, S.27 along with S.21, we have

$$\begin{aligned} &|\mathbb{E}[\text{Tr}(U^T B_{j,i+1} G(A_i - \Sigma) B_{j,i+1}^T U)]| \\ &\leq \eta_{i+1} \|G\|_2 \left(\frac{2\mathcal{V} |\lambda_2(P)|}{1 - |\lambda_2(P)|} + \eta_{i+1} \mathcal{M} \left(2\gamma + 16\gamma \eta_{i+1} k_{i+1} (\mathcal{M} + \lambda_1) + \left(2 + (1 + \epsilon)^2 \right) k_{i+1}^2 (\mathcal{M} + \lambda_1)^2 \right) \right) \\ &\quad \times \mathbb{E}[\text{Tr}(U^T B_{j,i+k} B_{j,i+k}^T U)] \\ &\leq \eta_{i+1} \|G\|_2 \left(\frac{2\mathcal{V} |\lambda_2(P)|}{1 - |\lambda_2(P)|} + \eta_{i+1} \mathcal{M} \left(2\gamma (1 + 8\epsilon) + \left(2 + (1 + \epsilon)^2 \right) k_{i+1}^2 (\mathcal{M} + \lambda_1)^2 \right) \right) \\ &\quad \times \mathbb{E}[\text{Tr}(U^T B_{j,i+k} B_{j,i+k}^T U)] \end{aligned}$$

where in the last line we used $\eta_{i+1} k_{i+1} (\mathcal{M} + \lambda_1) \leq \epsilon$. Hence proved. $\square$

Lemma S.9. Let $\forall i \in [n], \eta_i k_i (\mathcal{M} + \lambda_1) \leq \epsilon, \epsilon \in (0, 1)$ and η_i forms a non-increasing sequence. Set $k_i := \tau_{\text{mix}}(\gamma \eta_i^2), \gamma \in (0, 1]$. Then for constant matrices $U \in \mathbb{R}^{d \times d'}, G \in \mathbb{R}^{d \times d}, i \leq j \leq n, j - i \geq k_i$, we have

$$\begin{aligned} & \left| \mathbb{E} \left[\text{Tr} \left(U^T B_{j,i+1} G (A_i - \Sigma)^2 B_{j,i+1}^T U \right) \right] \right| \\ & \leq (\mathcal{V} + \eta_{i+1} \mathcal{M}^2 (2\gamma \eta_{i+1} + (1 + \epsilon)(2 + \epsilon(1 + \epsilon)) k_{i+1} (\mathcal{M} + \lambda_1))) \|G\|_2 \mathbb{E} \left[\text{Tr} \left(U^T B_{j,i+k_{i+1}} B_{j,i+k_{i+1}}^T U \right) \right] \end{aligned}$$

where $B_{j,i}$ is defined in 7.

Proof. For convenience of notation, we denote $k_{i+1} := k$. Let $B_{j,i+1} = B_{j,i+k} (I + R)$, then

$$\begin{aligned} \mathbb{E} \left[\text{Tr} \left(U^T B_{j,i+1} G (A_i - \Sigma)^2 B_{j,i+1}^T U \right) \right] &= \\ & \mathbb{E} \left[\underbrace{\text{Tr} \left(U^T B_{j,i+k} G (A_i - \Sigma)^2 B_{j,i+k}^T U \right)}_{T_1} \right] + \mathbb{E} \left[\underbrace{\text{Tr} \left(U^T B_{j,i+k} G (A_i - \Sigma)^2 R^T B_{j,i+k}^T U \right)}_{T_2} \right] + \\ & \mathbb{E} \left[\underbrace{\text{Tr} \left(U^T B_{j,i+k} R G (A_i - \Sigma)^2 B_{j,i+k}^T U \right)}_{T_3} \right] + \mathbb{E} \left[\underbrace{\text{Tr} \left(U^T B_{j,i+k} R G (A_i - \Sigma)^2 R^T B_{j,i+k}^T U \right)}_{T_4} \right] \end{aligned}$$

We will now bound each of the terms $\mathbb{E}[T_1], \mathbb{E}[T_2], \mathbb{E}[T_3]$ and $\mathbb{E}[T_4]$.

Since $\left\| \mathbb{E}_\pi \left[(A_t - \Sigma)^2 \right] \right\|_2 \leq \mathcal{V}$, therefore

$$\begin{aligned} \mathbb{E}[T_1] &= \mathbb{E} \left[\text{Tr} \left(U^T B_{j,i+k} G (A_i - \Sigma)^2 B_{j,i+k}^T U \right) \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[\text{Tr} \left(U^T B_{j,i+k} G (A_i - \Sigma)^2 B_{j,i+k}^T U \right) \middle| s_{i+k}, \dots, s_{j-1}, s_j \right] \right] \\ &= \mathbb{E} \left[\text{Tr} \left(U^T B_{j,i+k} G \mathbb{E} \left[(A_i - \Sigma)^2 \middle| s_{i+k}, \dots, s_{j-1}, s_j \right] B_{j,i+k}^T U \right) \right] \\ &= \mathbb{E} \left[\text{Tr} \left(U^T B_{j,i+k} G \mathbb{E} \left[(A_i - \Sigma)^2 \middle| s_{i+k} \right] B_{j,i+k}^T U \right) \right] \text{ using Lemma S.6} \\ &\stackrel{(i)}{\leq} (\mathcal{V} + 2d_{\text{mix}}(k) \mathcal{M}^2) \|G\|_2 \mathbb{E} \left[\text{Tr} \left(U^T B_{j,i+k} B_{j,i+k}^T U \right) \right] \end{aligned}$$

where in (i), we used similar steps as S.22 to get

$$\left\| \mathbb{E} \left[(A_i - \Sigma)^2 \middle| s_{i+k} \right] \right\|_2 \leq \left\| \mathbb{E}_\pi \left[(A_i - \Sigma)^2 \right] \right\|_2 + 2d_{\text{mix}}(k) \mathcal{M}^2 \quad (\text{S.28})$$

Next, using Lemma 2 we have that

$$\|R\|_2 \leq (1 + \epsilon) k_{i+1} \eta_{i+1} (\mathcal{M} + \lambda_1). \quad (\text{S.29})$$

Therefore,

$$\begin{aligned} \mathbb{E}[T_2] &= \mathbb{E} \left[\text{Tr} \left(U^T B_{j,i+k} G (A_i - \Sigma)^2 R^T B_{j,i+k}^T U \right) \right] \\ &\leq (1 + \epsilon) k_{i+1} \eta_{i+1} \mathcal{M}^2 (\mathcal{M} + \lambda_1) \|G\|_2 \mathbb{E} \left[\text{Tr} \left(U^T B_{j,i+k} B_{j,i+k}^T U \right) \right] \end{aligned}$$

Similarly,

$$\begin{aligned} \mathbb{E}[T_3] &= \mathbb{E} \left[\text{Tr} \left(U^T B_{j,i+k} R G (A_i - \Sigma)^2 B_{j,i+k}^T U \right) \right] \\ &\leq (1 + \epsilon) k_{i+1} \eta_{i+1} \mathcal{M}^2 (\mathcal{M} + \lambda_1) \|G\|_2 \mathbb{E} \left[\text{Tr} \left(U^T B_{j,i+k} B_{j,i+k}^T U \right) \right] \end{aligned}$$

Finally, using the bound on $\|R\|_2$ from Eq S.29, we have:

$$\begin{aligned}\mathbb{E}[T_4] &= \mathbb{E}\left[\text{Tr}\left(U^T B_{j,i+k} R G(A_i - \Sigma)^2 R^T B_{j,i+k}^T U\right)\right] \\ &\leq (1 + \epsilon)^2 k_{i+1}^2 \eta_{i+1}^2 \mathcal{M}^2 (\mathcal{M} + \lambda_1)^2 \|G\|_2 \mathbb{E}\left[\text{Tr}\left(U^T B_{j,i+k} B_{j,i+k}^T U\right)\right] \\ &\leq \epsilon (1 + \epsilon)^2 k_{i+1} \eta_{i+1} \mathcal{M}^2 (\mathcal{M} + \lambda_1) \|G\|_2 \mathbb{E}\left[\text{Tr}\left(U^T B_{j,i+k} B_{j,i+k}^T U\right)\right] \text{ using } \forall i, \eta_i k_i (\mathcal{M} + \lambda_1) \leq c\end{aligned}$$

Therefore,

$$\begin{aligned}&\left|\mathbb{E}\left[\text{Tr}\left(U^T B_{j,i+1} G(A_i - \Sigma)^2 B_{j,i+1}^T U\right)\right]\right| \\ &\stackrel{(i)}{\leq} (\mathcal{V} + \eta_{i+1} (2\gamma\eta_{i+1}\mathcal{M}^2 + (1 + \epsilon)(2 + \epsilon(1 + \epsilon))k_{i+1}\mathcal{M}^2(\mathcal{M} + \lambda_1))) \|G\|_2 \mathbb{E}\left[\text{Tr}\left(U^T B_{j,i+k} B_{j,i+k}^T U\right)\right] \\ &= (\mathcal{V} + \eta_{i+1}\mathcal{M}^2 (2\gamma\eta_{i+1} + (1 + \epsilon)(2 + \epsilon(1 + \epsilon))k_{i+1}(\mathcal{M} + \lambda_1))) \|G\|_2 \mathbb{E}\left[\text{Tr}\left(U^T B_{j,i+k} B_{j,i+k}^T U\right)\right]\end{aligned}$$

where in (i), we used $d_{\text{mix}}(k) = d_{\text{mix}}(k_{i+1}) \leq \gamma\eta_{i+1}^2$. Hence proved. $\square$

Lemma S.10. Let $\forall i \in [n], \eta_i k_i (\mathcal{M} + \lambda_1) \leq \epsilon, \epsilon \in (0, 1)$ and step-sizes η_i forms a non-increasing sequence. Further, let the step-sizes follow a slow-decay property, i.e, $\forall i, \eta_i \leq \eta_{i-k_i} \leq 2\eta_i$. Set $k_i := \tau_{\text{mix}}(\gamma\eta_i^2), \gamma \in (0, 1]$. Let $G \in \mathbb{R}^{d \times d}$ be a constant positive semi-definite matrix, and $P_t := \text{Tr}(B_{t-1} B_{t-1}^T G(A_t - \Sigma))$, then,

$$\mathbb{E}[P_t] \leq \eta_{t-k_t} \left(\frac{2\mathcal{V}|\lambda_2(P)|}{1 - |\lambda_2(P)|} + \eta_{t-k_t} \mathcal{M} \left(2\gamma(1 + 8\epsilon) + \left(2 + (1 + \epsilon)^2\right) k_t^2 (\mathcal{M} + \lambda_1)^2 \right) \right) \|G\|_2 \mathbb{E}[\text{Tr}(B_{t-k_t} B_{t-k_t}^T)]$$

where B_t is defined in 2.

Proof. Let $B_t = (I + R) B_{t-k_t}$ with $\|R\|_2 \leq r$. Then,

$$\begin{aligned}\mathbb{E}[P_t] &= \mathbb{E}\left[\underbrace{\text{Tr}(B_{t-k_t} B_{t-k_t}^T G(A_t - \Sigma))}_{P_{t,1}}\right] + \mathbb{E}\left[\underbrace{\text{Tr}(B_{t-k_t} B_{t-k_t}^T R^T G(A_t - \Sigma))}_{P_{t,2}}\right] \\ &\quad + \mathbb{E}\left[\underbrace{\text{Tr}(B_{t-k_t} B_{t-k_t}^T G(A_t - \Sigma) R)}_{P_{t,3}}\right] + \mathbb{E}\left[\underbrace{\text{Tr}(B_{t-k_t} B_{t-k_t}^T R^T G(A_t - \Sigma) R)}_{P_{t,4}}\right]\end{aligned}$$

Let's consider each of the terms above. Using Von-Neumann's trace inequality and S.25, we have,

$$\begin{aligned}\mathbb{E}[P_{t,1}] &= \mathbb{E}\left[\text{Tr}(B_{t-k_t} B_{t-k_t}^T \mathbb{E}[G(A_t - \Sigma) | s_1, s_2, \dots, s_{t-k_t}])\right] \\ &\leq \mathbb{E}\left[\text{Tr}(B_{t-k_t} B_{t-k_t}^T G \mathbb{E}[(A_t - \Sigma) | s_{t-k_t}])\right] \\ &\leq \|G \mathbb{E}[(A_t - \Sigma) | s_{t-k_t}]\|_2 \mathbb{E}[\text{Tr}(B_{t-k_t} B_{t-k_t}^T)] \\ &\leq 2\mathcal{M} d_{\text{mix}}(k_t) \|G\|_2 \mathbb{E}[\text{Tr}(B_{t-k_t} B_{t-k_t}^T)] \text{ using S.22} \\ &\leq 2\gamma\eta_t^2 \mathcal{M} \|G\|_2 \mathbb{E}[\text{Tr}(B_{t-k_t} B_{t-k_t}^T)]\end{aligned}$$

$$\begin{aligned}\mathbb{E}[P_{t,2}] &= \mathbb{E}\left[\text{Tr}(B_{t-k_t} B_{t-k_t}^T, \mathbb{E}[R^T G(A_t - \Sigma) U | s_1, s_2, \dots, s_{t-k_t}])\right] \\ &\leq \|\mathbb{E}[R^T G(A_t - \Sigma) | s_1, s_2, \dots, s_{t-k_t}]\|_2 \mathbb{E}[\text{Tr}(B_{t-k_t} B_{t-k_t}^T)] \\ &= \|\mathbb{E}[R^T G(A_t - \Sigma) | s_{t-k_t}]\|_2 \mathbb{E}[\text{Tr}(B_{t-k_t} B_{t-k_t}^T)] \\ &\leq \eta_{t-k_t} \|G\|_2 \left(\frac{\mathcal{V}|\lambda_2(P)|}{1 - |\lambda_2(P)|} + 8\gamma\eta_{t-k_t}^2 k_t \mathcal{M} (\mathcal{M} + \lambda_1) + \eta_{t-k_t} k_t^2 \mathcal{M} (\mathcal{M} + \lambda_1)^2 \right) \mathbb{E}[\text{Tr}(B_{t-k_t} B_{t-k_t}^T)] \text{ using S.2}\end{aligned}$$

$$\mathbb{E}[P_{t,3}] \leq \eta_{t-k_t} \|G\|_2 \left(\frac{\mathcal{V} |\lambda_2(P)|}{1 - |\lambda_2(P)|} + 8\gamma\eta_{t-k_t}^2 k_t \mathcal{M}(\mathcal{M} + \lambda_1) + \eta_{t-k_t} k_t^2 \mathcal{M}(\mathcal{M} + \lambda_1)^2 \right) \mathbb{E}[\text{Tr}(B_{t-k_t} B_{t-k_t}^T)]$$

using similar steps as $\mathbb{E}[P_{t,2}]$

$$\begin{aligned} \mathbb{E}[P_{t,4}] &= \mathbb{E}[\text{Tr}(B_{t-k_t} B_{t-k_t}^T R^T G(A_t - \Sigma) R)] \\ &\leq r^2 \mathcal{M} \|G\|_2 \mathbb{E}[\text{Tr}(B_{t-k_t} B_{t-k_t}^T)] \\ &\leq (1 + \epsilon)^2 \eta_{t-k_t+1}^2 k_t^2 \mathcal{M}(\mathcal{M} + \lambda_1)^2 \|G\|_2 \mathbb{E}[\text{Tr}(B_{t-k_t} B_{t-k_t}^T)] \text{ using Lemma 2} \\ &\leq (1 + \epsilon)^2 \eta_{t-k_t}^2 k_t^2 \mathcal{M}(\mathcal{M} + \lambda_1)^2 \|G\|_2 \mathbb{E}[\text{Tr}(B_{t-k_t} B_{t-k_t}^T)] \end{aligned}$$

Therefore we have,

$$\begin{aligned} \mathbb{E}[P_t] &\leq \eta_{t-k_t} \left(\frac{2\mathcal{V} |\lambda_2(P)|}{1 - |\lambda_2(P)|} + \mathcal{M} \left(2\gamma\eta_t + 16\gamma\eta_{t-k_t}^2 k_t (\mathcal{M} + \lambda_1) + \left(2 + (1 + \epsilon)^2 \right) \eta_{t-k_t} k_t^2 (\mathcal{M} + \lambda_1)^2 \right) \right) \|G\|_2 \\ &\quad \times \mathbb{E}[\text{Tr}(B_{t-k_t} B_{t-k_t}^T)] \\ &\stackrel{(i)}{\leq} \eta_{t-k_t} \left(\frac{2\mathcal{V} |\lambda_2(P)|}{1 - |\lambda_2(P)|} + \eta_{t-k_t} \mathcal{M} \left(2\gamma + 16\gamma\eta_t k_t (\mathcal{M} + \lambda_1) + \left(2 + (1 + \epsilon)^2 \right) k_t^2 (\mathcal{M} + \lambda_1)^2 \right) \right) \|G\|_2 \\ &\quad \times \mathbb{E}[\text{Tr}(B_{t-k_t} B_{t-k_t}^T)] \\ &\stackrel{(ii)}{\leq} \eta_{t-k_t} \left(\frac{2\mathcal{V} |\lambda_2(P)|}{1 - |\lambda_2(P)|} + \eta_{t-k_t} \mathcal{M} \left(2\gamma(1 + 8\epsilon) + \left(2 + (1 + \epsilon)^2 \right) k_t^2 (\mathcal{M} + \lambda_1)^2 \right) \right) \|G\|_2 \mathbb{E}[\text{Tr}(B_{t-k_t} B_{t-k_t}^T)] \end{aligned}$$

where in (i) we used $2\eta_{t-k_t} \leq \eta_t \leq \eta_{t-k_t}$ along with $\eta_t k_t (\mathcal{M} + \lambda_1) \leq \epsilon$ in (ii). Hence proved. $\square$

Lemma S.11. Let $\forall i \in [n], \eta_i k_i (\mathcal{M} + \lambda_1) \leq \epsilon, \epsilon \in (0, 1)$ and η_i forms a non-increasing sequence. Set $k_i := \tau_{\text{mix}}(\gamma\eta_i^2), \gamma \in (0, 1]$. Let $U \in \mathbb{R}^{d \times d}$ be a constant matrix and $Q_t := \text{Tr}(B_{t-1} B_{t-1}^T (A_t - \Sigma) U (A_t - \Sigma))$. Further, let the decay of the step-sizes be slow such that $\forall i, \eta_i \leq \eta_{i-k_i} \leq 2\eta_i$. Then

$$\mathbb{E}[Q_t] \leq (\mathcal{V} + \eta_{t-k_t+1} \mathcal{M}^2 (2\gamma\eta_t + 2(1 + \epsilon)(1 + \epsilon(1 + \epsilon)) k_t (\mathcal{M} + \lambda_1))) \|U\|_2 \mathbb{E}[\text{Tr}(B_{t-k_t} B_{t-k_t}^T)]$$

where B_t is defined in 2.

Proof. Let $B_t = (I + R) B_{t-k_t}$ with $\|R\|_2 \leq r$. Then,

$$\begin{aligned} \mathbb{E}[Q_t] &= \mathbb{E} \left[\underbrace{\text{Tr}(B_{t-k_t} B_{t-k_t}^T (A_t - \Sigma) U (A_t - \Sigma))}_{Q_{t,1}} \right] + \mathbb{E} \left[\underbrace{\text{Tr}(B_{t-k_t} B_{t-k_t}^T R^T (A_t - \Sigma) U (A_t - \Sigma))}_{Q_{t,2}} \right] \\ &\quad + \mathbb{E} \left[\underbrace{\text{Tr}(R B_{t-k_t} B_{t-k_t}^T (A_t - \Sigma) U (A_t - \Sigma))}_{Q_{t,3}} \right] + \mathbb{E} \left[\underbrace{\text{Tr}(R B_{t-k_t} B_{t-k_t}^T R^T (A_t - \Sigma) U (A_t - \Sigma))}_{Q_{t,4}} \right] \end{aligned}$$

Let's consider each of the terms above. Using Von-Neumann's trace inequality and noting that

$$\left\| \mathbb{E}_\pi [(A_t - \Sigma)^2] \right\|_2 \leq \mathcal{V}, \text{ we have}$$

$$\begin{aligned} \mathbb{E}[Q_{t,1}] &= \mathbb{E}[\text{Tr}(B_{t-k_t} B_{t-k_t}^T \mathbb{E}[(A_t - \Sigma) U (A_t - \Sigma) | s_1, s_2, \dots, s_{t-k_t}])] \\ &= \mathbb{E}[\text{Tr}(B_{t-k_t} B_{t-k_t}^T \mathbb{E}[(A_t - \Sigma) U (A_t - \Sigma) | s_{t-k_t}])] \\ &\leq \|\mathbb{E}[(A_t - \Sigma) U (A_t - \Sigma) | s_{t-k_t}]\|_2 \mathbb{E}[\text{Tr}(B_{t-k_t} B_{t-k_t}^T)] \\ &\leq \|U\|_2 \|\mathbb{E}[(A_t - \Sigma)^2 | s_{t-k_t}]\|_2 \mathbb{E}[\text{Tr}(B_{t-k_t} B_{t-k_t}^T)] \text{ using S.28} \\ &\leq \|U\|_2 (\mathcal{V} + 2d_{\text{mix}}(k_t) \mathcal{M}^2) \mathbb{E}[\text{Tr}(B_{t-k_t} B_{t-k_t}^T)] \\ &\leq \|U\|_2 (\mathcal{V} + 2\gamma\eta_t^2 \mathcal{M}^2) \mathbb{E}[\text{Tr}(B_{t-k_t} B_{t-k_t}^T)] \end{aligned}$$

$$\begin{aligned}
\mathbb{E}[Q_{t,2}] &= \mathbb{E}[\text{Tr}(B_{t-k_t} B_{t-k_t}^T \mathbb{E}[R^T(A_t - \Sigma)U(A_t - \Sigma)|s_1, s_2, \dots, s_{t-k_t}])] \\
&\leq \|\mathbb{E}[R^T(A_t - \Sigma)U(A_t - \Sigma)|s_{t-k_t}]\|_2 \mathbb{E}[\text{Tr}(B_{t-k_t} B_{t-k_t}^T)] \\
&\leq (1 + \epsilon) \eta_{t-k_t+1} k_t \mathcal{M}^2 (\mathcal{M} + \lambda_1) \|U\|_2 \mathbb{E}[\text{Tr}(B_{t-k_t} B_{t-k_t}^T)] \quad \text{using Lemma 2}
\end{aligned}$$

$$\mathbb{E}[Q_{t,3}] \leq (1 + \epsilon) \eta_{t-k_t+1} k_t \mathcal{M}^2 (\mathcal{M} + \lambda_1) \|U\|_2 \mathbb{E}[\text{Tr}(B_{t-k_t} B_{t-k_t}^T)] \quad \text{using a similar argument as } Q_{t,2}$$

$$\begin{aligned}
\mathbb{E}[Q_{t,4}] &= \mathbb{E}[\text{Tr}(R B_{t-k_t} B_{t-k_t}^T R^T (A_t - \Sigma) U (A_t - \Sigma))] \\
&= \mathbb{E}[\text{Tr}(B_{t-k_t} B_{t-k_t}^T R^T (A_t - \Sigma) U (A_t - \Sigma) R)] \\
&\leq r^2 \|U\|_2 \mathcal{M}^2 \mathbb{E}[\text{Tr}(B_{t-k_t} B_{t-k_t}^T)] \\
&\leq (1 + \epsilon)^2 \eta_{t-k_t+1}^2 k_t^2 \mathcal{M}^2 (\mathcal{M} + \lambda_1)^2 \|U\|_2 \mathbb{E}[\text{Tr}(B_{t-k_t} B_{t-k_t}^T)] \quad \text{using Lemma 2}
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\mathbb{E}[Q_t] &\leq \left(\mathcal{V} + \eta_{t-k_t+1} \left(2\gamma \eta_t \mathcal{M}^2 + 2(1 + \epsilon) k_t \mathcal{M}^2 (\mathcal{M} + \lambda_1) + (1 + \epsilon)^2 \eta_{t-k_t+1} k_t^2 \mathcal{M}^2 (\mathcal{M} + \lambda_1)^2 \right) \right) \|U\|_2 \mathbb{E}[\text{Tr}(B_{t-k_t} B_{t-k_t}^T)] \\
&\stackrel{(i)}{\leq} \left(\mathcal{V} + \eta_{t-k_t+1} \mathcal{M}^2 \left(2\gamma \eta_t + 2(1 + \epsilon) k_t (\mathcal{M} + \lambda_1) + 2\epsilon (1 + \epsilon)^2 k_t (\mathcal{M} + \lambda_1) \right) \right) \|U\|_2 \mathbb{E}[\text{Tr}(B_{t-k_t} B_{t-k_t}^T)] \\
&= (\mathcal{V} + \eta_{t-k_t+1} \mathcal{M}^2 (2\gamma \eta_t + 2(1 + \epsilon) (1 + \epsilon (1 + \epsilon)) k_t (\mathcal{M} + \lambda_1))) \|U\|_2 \mathbb{E}[\text{Tr}(B_{t-k_t} B_{t-k_t}^T)]
\end{aligned}$$

In (i), we used the slow-decay assumption on η_i mentioned in the lemma statement along with $\eta_i k_i (\mathcal{M} + \lambda_1) \leq \epsilon$. Hence proved. $\square$

Lemma S.12. (*Learning Rate Schedule*) Fix any $\delta \in (0, 1)$. Set $k_i := \tau_{\text{mix}}(\eta_i^2)$. Suppose the step sizes are set such that

$$\eta_i = \frac{\alpha}{(\lambda_1 - \lambda_2)(\beta + i)}$$

Define the linear function

$$\forall i \in [n], f(i) := \frac{1}{\eta_i} = \frac{(\lambda_1 - \lambda_2)(\beta + i)}{\alpha},$$

With $\epsilon := \frac{1}{100}$ and $\xi_{k,t}, \zeta_{k,t}, \mathcal{V}', \overline{\mathcal{V}_{k,t}}$ defined in S.49, set $\alpha > 2$, $f(0) \geq e$, $m := 200$ and

$$\beta := 600 \max \left\{ \frac{\tau_{\text{mix}} \log(f(0)) (\mathcal{M} + \lambda_1) \alpha}{\lambda_1 - \lambda_2}, \frac{5\tau_{\text{mix}} \log(f(0)) (\mathcal{M} + \lambda_1)^2 \alpha^2}{3(\lambda_1 - \lambda_2)^2 \log(1 + \frac{\delta}{m})}, \frac{(\mathcal{V}' + 5\lambda_1^2) \alpha^2}{300(\lambda_1 - \lambda_2)^2 \log(1 + \frac{\delta}{m})} \right\}$$

then we have

1. $\eta_i k_i (\mathcal{M} + \lambda_1) \leq \epsilon$
2. $\forall i, \eta_i \leq \eta_{i-k_i} \leq (1 + 2\epsilon) \eta_i \leq 2\eta_i$ (slow-decay)
3. $\sum_{i=1}^n (\overline{\mathcal{V}_{k,i}} + \zeta_{k,i} + 4\lambda_1^2) \eta_i^2 \leq \log(1 + \frac{\delta}{m})$
4. $\sum_{i=1}^n (\mathcal{V}' + \xi_{k,i}) \eta_{i-k_i}^2 \exp \left(- \sum_{j=i+1}^n 2\eta_j (\lambda_1 - \lambda_2) \right) \leq$

$$\left(\frac{2(1 + 10\epsilon) \alpha^2}{2\alpha - 1} \right) \frac{\mathcal{V}'}{(\lambda_1 - \lambda_2)^2} \frac{1}{n} + \left(\frac{24(1 + 10\epsilon) \alpha^3}{(\alpha - 1)} \right) \frac{\mathcal{M} (\mathcal{M} + \lambda_1)^2 k_n^2}{(\lambda_1 - \lambda_2)^3 n^2}$$

Proof. We use the following inequalities -

$$\sum_{j=i}^t \eta_j^2 \leq \frac{\alpha^2}{(\lambda_1 - \lambda_2)^2 (\beta + i - 1)} \quad \left(\text{Using } \frac{1}{x+1} \leq \sum_{i=1}^{\infty} \frac{1}{(x+i)^2} \leq \frac{1}{x} \right) \quad (\text{S.30})$$

$$\sum_{j=i}^t \eta_j \geq \frac{\alpha}{(\lambda_1 - \lambda_2)} \log \left(\frac{t + \beta + 1}{i + \beta} \right) \quad (\text{S.31})$$

$$\sum_{j=i}^t \eta_j \leq \frac{\alpha}{(\lambda_1 - \lambda_2)} \log \left(\frac{t + \beta}{i + \beta - 1} \right) \quad (\text{S.32})$$

$$\sum_{j=i}^t (j + \beta)^\ell \leq \frac{(t + \beta + 1)^{\ell+1} - (i + \beta)^{\ell+1}}{\ell + 1} \leq \frac{(t + \beta + 1)^{\ell+1}}{\ell + 1} \quad \forall \ell > 0 \quad (\text{S.33})$$

For the first result, we observe that $f(x) = \frac{\log(x)}{x}$ is a decreasing function of x for $x \geq e$. Using properties of the mixing time (see Section 2.1 in the manuscript), we have

$$k_i := \tau_{\text{mix}}(\eta_i^2) \leq \frac{2\tau_{\text{mix}}}{\log(2)} \log \left(\frac{1}{\eta_i^2} \right) = \frac{4\tau_{\text{mix}}}{\log(2)} \log \left(\frac{(\beta + i)(\lambda_1 - \lambda_2)}{\alpha} \right) = \frac{4\tau_{\text{mix}}}{\log(2)} \log(f(i)) \quad (\text{S.34})$$

for $\eta_i < 1$. For $i \geq 0$

$$f(i) \geq f(0) = \frac{\beta(\lambda_1 - \lambda_2)}{\alpha} \geq e$$

Therefore,

$$\begin{aligned} \eta_i k_i (\mathcal{M} + \lambda_1) &\leq \frac{4\tau_{\text{mix}}(\mathcal{M} + \lambda_1)}{\log(2)} \frac{\alpha}{(\beta + i)(\lambda_1 - \lambda_2)} \log \left(\frac{(\beta + i)(\lambda_1 - \lambda_2)}{\alpha} \right) \\ &= \frac{4\tau_{\text{mix}}(\mathcal{M} + \lambda_1)}{\log(2)} \frac{\log(f(i))}{f(i)} \\ &\leq \frac{4\tau_{\text{mix}}(\mathcal{M} + \lambda_1)}{\log(2)} \frac{\log(f(0))}{f(0)} \end{aligned}$$

From the assumptions mentioned in the Lemma statement, we have

$$\frac{\log(f(0))}{f(0)} < \frac{\epsilon \log(2)}{4\tau_{\text{mix}}(\mathcal{M} + \lambda_1)} = \frac{\log(2)}{400\tau_{\text{mix}}(\mathcal{M} + \lambda_1)} \quad (\text{S.35})$$

Therefore,

$$\forall i, \eta_i k_i (\mathcal{M} + \lambda_1) \leq \epsilon \quad (\text{S.36})$$

For the second result, we note that $\forall i \in [n]$,

$$\begin{aligned} \frac{\eta_{i-k_i}}{\eta_i} &= \frac{\beta + i}{\beta + i - k_i} \\ &= 1 + \frac{k_i}{\beta + i - k_i} \\ &= 1 + \frac{1}{\frac{\beta + i}{k_i} - 1} \end{aligned}$$

Consider the fraction $\frac{\beta+i}{k_i}$. We can simplify it as :

$$\begin{aligned}
\frac{\beta+i}{k_i} &\geq \frac{\log(2)}{4\tau_{\text{mix}}} \frac{\beta+i}{\log\left(\frac{(\beta+i)(\lambda_1-\lambda_2)}{\alpha}\right)} \\
&= \frac{\alpha \log(2)}{4\tau_{\text{mix}}(\lambda_1-\lambda_2)} \frac{f(i)}{\log(f(i))} \\
&\geq \frac{\alpha \log(2)}{4\tau_{\text{mix}}(\lambda_1-\lambda_2)} \frac{f(0)}{\log(f(0))} \\
&\geq \frac{1}{\epsilon} \text{ from S.35}
\end{aligned}$$

where we used the fact that $\frac{x}{\log(x)}$ is an increasing function for $x \geq e$. Therefore, we have that

$$\begin{aligned}
\frac{\eta_{i-k_i}}{\eta_i} &\leq 1 + \frac{1}{\frac{1}{\epsilon} - 1} \\
&= \frac{1}{1 - \epsilon} \\
&\leq 1 + 2\epsilon \text{ for } \epsilon \in (0, 0.1)
\end{aligned}$$

For the third result, we note that

$$\begin{aligned}
\zeta_{k,t} &:= 40k_{t+1}(\mathcal{M} + \lambda_1)^2, \\
\xi_{k,t} &:= 2\eta_t \mathcal{M} \left[3 + 9k_{t+1}^2(\mathcal{M} + \lambda_1)^2 \right] \\
&\leq 24\eta_t \mathcal{M} \left[k_{t+1}^2(\mathcal{M} + \lambda_1)^2 \right] \text{ since } (\mathcal{M} + \lambda_1) \geq 1 \text{ WLOG} \\
&\leq 24\epsilon(1+\epsilon)k_{t+1}(\mathcal{M} + \lambda_1)^2 \text{ since } \eta_t \leq (1+2\epsilon)\eta_{t+1} \text{ and } \eta_{t+1}k_{t+1}(\mathcal{M} + \lambda_1) \leq \epsilon
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{i=1}^n (\overline{\mathcal{V}_{k,i}} + \zeta_{k,i}) \eta_i^2 &= (\mathcal{V}' + 5\lambda_1^2) \sum_{i=1}^n \eta_i^2 + 41(\mathcal{M} + \lambda_1)^2 \sum_{i=1}^n \eta_i^2 k_{i+1} \\
&\stackrel{(i)}{\leq} (\mathcal{V}' + 5\lambda_1^2) \underbrace{\sum_{i=1}^n \eta_i^2}_{T_1} + 45(\mathcal{M} + \lambda_1)^2 \underbrace{\sum_{i=1}^n \eta_{i+1}^2 k_{i+1}}_{T_2} \quad (\text{S.37})
\end{aligned}$$

where (i) follows from the slow decay property of η_i .

For T_1 , using S.30 we have,

$$T_1 \leq \frac{\alpha^2}{(\lambda_1 - \lambda_2)^2 \beta} \quad (\text{S.38})$$

For T_2 , substituting the value of k_i from S.34 for $\eta_i < 1$ we have,

$$T_2 := \sum_{i=1}^n \eta_{i+1}^2 k_{i+1} \leq \frac{4\tau_{\text{mix}}}{\log(2)} \sum_{i=1}^n \left(\frac{\alpha}{(\lambda_1 - \lambda_2)(\beta + i + 1)} \right)^2 \log \left(\frac{(\lambda_1 - \lambda_2)(\beta + i + 1)}{\alpha} \right) \quad (\text{S.39})$$

$$= \frac{4\tau_{\text{mix}}}{\log(2)} \sum_{i=1}^n \frac{\log(f(i+1))}{f(i+1)^2} \quad (\text{S.40})$$

Note that $f(i)$ is a linear function of i and $\forall i, f(i+1) - f(i) = \frac{\lambda_1 - \lambda_2}{\alpha}$. We observe that $g(x) = \frac{\log(x)}{x^2}$ is a decreasing function of x for $x \geq e^{\frac{1}{2}} \sim 1.65$. Therefore,

$$\left(\frac{\lambda_1 - \lambda_2}{\alpha} \right) \sum_{i=1}^n \frac{\log(f(i+1))}{f(i+1)^2} \leq \int_{f(1)}^{f(n+1)} \frac{\log(x)}{x^2} dx$$

Substituting in S.40 we have,

$$\begin{aligned} T_2 &\leq \frac{4\tau_{\text{mix}}}{\log(2)} \left(\frac{\alpha}{\lambda_1 - \lambda_2} \right) \int_{f(1)}^{f(n+1)} \frac{\log(x)}{x^2} dx \\ &= \frac{4\tau_{\text{mix}}}{\log(2)} \left(\frac{\alpha}{\lambda_1 - \lambda_2} \right) \left(- \left(\frac{\log(x)}{x} + \frac{1}{x} \right) \Big|_{f(1)}^{f(n)} \right) \\ &\leq \frac{4\tau_{\text{mix}}}{\log(2)} \left(\frac{\alpha}{\lambda_1 - \lambda_2} \right) \left(\frac{\log(f(1))}{f(1)} + \frac{1}{f(1)} \right) \\ &\leq \frac{8\tau_{\text{mix}}}{\log(2)} \left(\frac{\alpha}{\lambda_1 - \lambda_2} \right) \left(\frac{\log(f(1))}{f(1)} \right) \\ &\leq \frac{8\tau_{\text{mix}}}{\log(2)} \left(\frac{\alpha}{\lambda_1 - \lambda_2} \right) \left(\frac{\log(f(0))}{f(0)} \right) \text{ since } \frac{\log(x)}{x} \text{ is a decreasing function of } x \text{ for } x \geq e \end{aligned}$$

Putting everything together in S.37 and using the bounds on $\beta, f(0)$ mentioned in the lemma statement, we have,

$$\begin{aligned} \sum_{i=1}^n (\overline{\mathcal{V}_{k,i}} + \zeta_{k,i}) \eta_i^2 &\leq 460 (\mathcal{M} + \lambda_1)^2 \tau_{\text{mix}} \left(\frac{\alpha}{\lambda_1 - \lambda_2} \right) \frac{\log(f(0))}{f(0)} + \frac{\alpha^2}{(\lambda_1 - \lambda_2)^2 \beta} (\mathcal{V}' + 5\lambda_1^2) \\ &= 460 \tau_{\text{mix}} \log(f(0)) \frac{\alpha^2}{(\lambda_1 - \lambda_2)^2 \beta} (\mathcal{M} + \lambda_1)^2 + \frac{\alpha^2}{(\lambda_1 - \lambda_2)^2 \beta} (\mathcal{V}' + 5\lambda_1^2) \\ &\leq \log \left(1 + \frac{\delta}{m} \right) \end{aligned}$$

Finally, for the last result we first note that

$$\begin{aligned} \xi_{k,t} &:= 2\eta_t \mathcal{M} \left[3 + 9k_{t+1}^2 (\mathcal{M} + \lambda_1)^2 \right] \\ &\leq 24\eta_t \mathcal{M} \left[k_{t+1}^2 (\mathcal{M} + \lambda_1)^2 \right] \text{ since } (\mathcal{M} + \lambda_1) \geq 1 \text{ WLOG} \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{i=1}^n (\mathcal{V}' + \xi_{k,i}) \eta_{i-k_i}^2 \exp \left(- \sum_{j=i+1}^n 2\eta_j (\lambda_1 - \lambda_2) \right) \\ \leq (1 + 2\epsilon)^2 \sum_{i=1}^n (\mathcal{V}' + \xi_{k,i}) \eta_i^2 \exp \left(- \sum_{j=i+1}^n 2\eta_j (\lambda_1 - \lambda_2) \right) \\ \leq (1 + 5\epsilon) \sum_{i=1}^n (\mathcal{V}' + \xi_{k,i}) \eta_i^2 \exp \left(- \sum_{j=i+1}^n 2\eta_j (\lambda_1 - \lambda_2) \right) \text{ since } \epsilon \in (0, 0.1) \\ = (1 + 5\epsilon) \left[\sum_{i=1}^n \mathcal{V}' \eta_i^2 \exp \left(- \sum_{j=i+1}^n 2\eta_j (\lambda_1 - \lambda_2) \right) + \sum_{i=1}^n \xi_{k,i} \eta_i^2 \exp \left(- \sum_{j=i+1}^n 2\eta_j (\lambda_1 - \lambda_2) \right) \right] \end{aligned} \tag{S.41}$$

Let's define

$$g(i) := \exp \left(- \sum_{j=i+1}^n 2\eta_j (\lambda_1 - \lambda_2) \right), \quad T_3 := \sum_{i=1}^n \eta_i^2 g(i), \quad T_4 := \sum_{i=1}^n \eta_i^3 g(i), \quad T_5 := \sum_{i=1}^n \eta_i^3 k_i^2 g(i),$$

Note that since $k_n \geq k_i$,

$$T_5 = \sum_{i=1}^n \eta_i^3 k_i^2 g(i) \leq k_n^2 \sum_{i=1}^n \eta_i^3 g(i) = k_n^2 T_4$$

Then,

$$\begin{aligned} \sum_{i=1}^n (\mathcal{V}' + \xi_{k,i}) \eta_{i-k_i}^2 \exp \left(- \sum_{j=i+1}^n 2\eta_j (\lambda_1 - \lambda_2) \right) &\leq (1 + 5\epsilon) \left[\mathcal{V}' T_3 + 24\mathcal{M} (\mathcal{M} + \lambda_1)^2 T_5 \right] \\ &\leq (1 + 5\epsilon) \left[\mathcal{V}' T_3 + 24\mathcal{M} (\mathcal{M} + \lambda_1)^2 k_n^2 T_4 \right] \end{aligned} \quad (\text{S.42})$$

Using S.31, $g(i) \leq \left(\frac{i+\beta+1}{n+\beta+1} \right)^{2\alpha}$. Noting that $\left(\frac{\beta+1}{\beta} \right)^2 \leq \left(\frac{\beta+1}{\beta} \right)^3 \leq 2$, we have

$$\begin{aligned} T_3 &:= \sum_{i=1}^n \eta_i^2 \exp \left(-2 \sum_{j=i+1}^n \eta_j (\lambda_1 - \lambda_2) \right) \\ &= \left(\frac{\alpha}{\lambda_1 - \lambda_2} \right)^2 \sum_{i=1}^n \frac{1}{(\beta+i)^2} \left(\frac{i+\beta+1}{n+\beta+1} \right)^{2\alpha} \\ &\leq \left(\frac{\alpha}{\lambda_1 - \lambda_2} \right)^2 \left(\frac{\beta+1}{\beta} \right)^2 \sum_{i=1}^n \frac{1}{(\beta+i+1)^2} \left(\frac{i+\beta+1}{n+\beta+1} \right)^{2\alpha} \\ &= \left(\frac{\alpha}{\lambda_1 - \lambda_2} \right)^2 \left(\frac{\beta+1}{\beta} \right)^2 \sum_{i=1}^n \frac{1}{(\beta+i+1)^2} \left(\frac{i+\beta+1}{n+\beta+1} \right)^{2\alpha} \\ &\leq 2 \left(\frac{\alpha}{\lambda_1 - \lambda_2} \right)^2 \frac{1}{(n+\beta+1)^{2\alpha}} \sum_{i=1}^n (i+\beta+1)^{2\alpha-2} \\ &\leq \frac{2}{2\alpha-1} \left(\frac{\alpha}{\lambda_1 - \lambda_2} \right)^2 \frac{1}{(n+\beta+2)} \left(\frac{n+\beta+2}{n+\beta+1} \right)^{2\alpha} \quad \text{using S.33} \\ &= \frac{2}{2\alpha-1} \left(\frac{\alpha}{\lambda_1 - \lambda_2} \right)^2 \frac{1}{(n+\beta+2)} \left(1 + \frac{1}{n+\beta+1} \right)^{2\alpha} \end{aligned} \quad (\text{S.43})$$

and similarly,

$$\begin{aligned} T_4 &:= \sum_{i=1}^n \eta_i^3 \exp \left(-2 \sum_{j=i+1}^n \eta_j (\lambda_1 - \lambda_2) \right) \\ &= \left(\frac{\alpha}{\lambda_1 - \lambda_2} \right)^3 \sum_{i=1}^n \frac{1}{(\beta+i)^3} \left(\frac{i+\beta+1}{n+\beta+1} \right)^{2\alpha} \\ &\leq \left(\frac{\alpha}{\lambda_1 - \lambda_2} \right)^3 \left(\frac{\beta+1}{\beta} \right)^3 \sum_{i=1}^n \frac{1}{(\beta+i+1)^3} \left(\frac{i+\beta+1}{n+\beta+1} \right)^{2\alpha} \\ &= \left(\frac{\alpha}{\lambda_1 - \lambda_2} \right)^3 \left(\frac{\beta+1}{\beta} \right)^3 \sum_{i=1}^n \frac{1}{(\beta+i+1)^2} \left(\frac{i+\beta+1}{n+\beta+1} \right)^{2\alpha} \\ &\leq 2 \left(\frac{\alpha}{\lambda_1 - \lambda_2} \right)^3 \frac{1}{(n+\beta+1)^{2\alpha}} \sum_{i=1}^n (i+\beta+1)^{2\alpha-3} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\alpha-1} \left(\frac{\alpha}{\lambda_1 - \lambda_2} \right)^3 \frac{1}{(n+\beta+2)^2} \left(\frac{n+\beta+2}{n+\beta+1} \right)^{2\alpha} \text{ using S.33} \\
&= \frac{1}{\alpha-1} \left(\frac{\alpha}{\lambda_1 - \lambda_2} \right)^3 \frac{1}{(n+\beta+2)^2} \left(1 + \frac{1}{n+\beta+1} \right)^{2\alpha}
\end{aligned} \tag{S.44}$$

Using S.36, we have

$$\frac{\alpha}{n+\beta+1} = \eta_n (\lambda_1 - \lambda_2) \leq \eta_n \lambda_1 \leq \eta_n k_n \lambda_1 \leq \epsilon \leq 0.1 \tag{S.45}$$

Therefore, using [17]

$$\left(1 + \frac{1}{n+\beta+1} \right)^{2\alpha} \stackrel{(i)}{\leq} \frac{1}{1 - \frac{2\alpha}{n+\beta+1}} \stackrel{(ii)}{\leq} 1 + \frac{4\alpha}{n+\beta+1} \leq 1 + 4\epsilon \tag{S.46}$$

where (i) follows since $\frac{2\alpha}{n+\beta+1} < 1$ by S.45 and (ii) follows since $\frac{1}{1-x} \leq 1 + 2x$ for $x \in [0, \frac{1}{2}]$.

Using S.46 with S.43, we have

$$\begin{aligned}
T_3 &\leq \frac{2}{2\alpha-1} \left(\frac{\alpha}{\lambda_1 - \lambda_2} \right)^2 \frac{1}{(n+\beta+2)} \left(1 + \frac{4\alpha}{n+\beta+1} \right) \\
&\leq \frac{2(1+4\epsilon)}{2\alpha-1} \left(\frac{\alpha}{\lambda_1 - \lambda_2} \right)^2 \frac{1}{(n+\beta+2)}
\end{aligned} \tag{S.47}$$

Using S.46 with S.44, we have

$$T_4 \leq \frac{1+4\epsilon}{\alpha-1} \left(\frac{\alpha}{\lambda_1 - \lambda_2} \right)^3 \frac{1}{(n+\beta+2)^2} \tag{S.48}$$

Let

$$C_1 := \frac{2(1+10\epsilon)\alpha^2}{2\alpha-1}, C_2 := \frac{24(1+10\epsilon)\alpha^3}{(\alpha-1)},$$

Putting together S.47, S.48 in S.42 and using the definition of k_i in S.34 we have

$$\begin{aligned}
(1+5\epsilon) \mathcal{V}' T_3 &\leq \frac{2(1+5\epsilon)(1+4\epsilon)}{2\alpha-1} \left(\frac{\alpha}{\lambda_1 - \lambda_2} \right)^2 \frac{\mathcal{V}'}{(n+\beta+2)} \\
&\leq \frac{2(1+10\epsilon)\alpha^2}{2\alpha-1} \frac{\mathcal{V}'}{(\lambda_1 - \lambda_2)^2} \frac{1}{n} \text{ since } \epsilon \leq 0.05
\end{aligned}$$

and similarly,

$$24(1+5\epsilon) \mathcal{M} (\mathcal{M} + \lambda_1)^2 k_n^2 T_4 \leq \frac{24(1+5\epsilon)(1+4\epsilon)\alpha^3}{\alpha-1} \frac{\mathcal{M} (\mathcal{M} + \lambda_1)^2}{(\lambda_1 - \lambda_2)^3} \frac{k_n^2}{n^2}$$

Therefore from S.42, we have

$$\sum_{i=1}^n (\mathcal{V}' + \xi_{k,i}) \eta_{i-k_i}^2 \exp \left(- \sum_{j=i+1}^n 2\eta_j (\lambda_1 - \lambda_2) \right) \leq C_1 \frac{\mathcal{V}'}{(\lambda_1 - \lambda_2)^2} \frac{1}{n} + C_2 \frac{\mathcal{M} (\mathcal{M} + \lambda_1)^2}{(\lambda_1 - \lambda_2)^3} \frac{k_n^2}{n^2}$$

Hence proved. $\square$

S.4 Proofs : Convergence Analysis of Oja's Algorithm for Markovian Data

In this section, we present proofs of Theorems 2, 3, 4 and 5. We state versions of these theorems that are valid under more general conditions on the step sizes. Specifically, for the following, we only require a sequence of non-increasing step-sizes which satisfy, for $\epsilon := \frac{1}{100}, \forall i \in [n]$ -

$$\text{C.1 } \eta_i k_i (\mathcal{M} + \lambda_1) \leq \epsilon \quad \text{C.2 (Slow decay) } \eta_i \leq \eta_{i-k_i} \leq (1 + 2\epsilon) \eta_i \leq 2\eta_i$$

The version of these theorems stated in the main manuscript are obtained by plugging in the step-sizes as $\eta_i := \frac{\alpha}{(\lambda_1 - \lambda_2)(\beta + i)}$ for the values of α, β provided in Lemma S.12. Before starting with the proofs, we define the following scalar variables -

$$\begin{aligned} r &:= 2(1 + \epsilon) k_n \eta_n (\mathcal{M} + \lambda_1), & \zeta_{k,t} &:= 40k_{t+1} (\mathcal{M} + \lambda_1)^2 \\ \psi_{k,t} &:= 6\mathcal{M} \left[1 + 3k_{t+1}^2 (\mathcal{M} + \lambda_1)^2 \right], & \xi_{k,t} &:= \eta_{t-k_t} \psi_{k,t} \\ \mathcal{V}' &:= \frac{1 + (3 + 4\epsilon) |\lambda_2(P)|}{1 - |\lambda_2(P)|} \mathcal{V}, & \overline{\mathcal{V}}_{k,t} &:= \mathcal{V}' + \lambda_1^2 + \xi_{k,t} \end{aligned} \quad (\text{S.49})$$

The basic idea behind these proofs is illustrated in Figure S.1, where we are trying to approximate the matrix product by conditioning back in time just the right amount, to balance the tradeoff between the advantage of the mixing decay and the norm of the product of matrices.

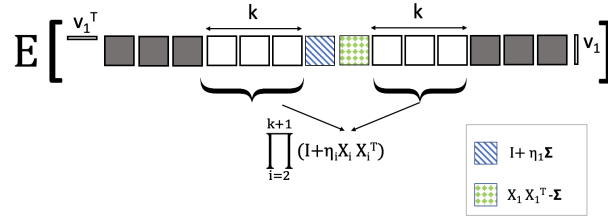


Figure S.1: If we could replace the intermediate products (white matrices) by I , the conditional expectation of the noise matrix $X_1 X_1^T - \Sigma$ conditioned on the grey matrices would be nearly zero.

Theorem 2. (General Version) Under Assumptions 1, 2 and 3, for all $n > k_n$, and any decaying step-size schedule η_i satisfying C.1 and C.2, we have:

$$\mathbb{E} [v_1^T B_{n,1} B_{n,1}^T v_1] \leq (1 + r)^2 \exp \left(\sum_{t=1}^{n-k_n} (2\eta_t \lambda_1 + \eta_t^2 (\mathcal{V}' + \lambda_1^2 + \xi_{k,t})) \right)$$

where $B_{j,i}$ is defined in 7.

Proof. Define $\alpha_{n,t} := \mathbb{E} [\text{Tr} (v_1^T B_{n,t} B_{n,t}^T v_1)] = \mathbb{E} [v_1^T B_{n,t} B_{n,t}^T v_1], i \leq t \leq n$. Then, we have

$$\begin{aligned} v_1^T B_{n,t} B_{n,t}^T v_1 &= v_1^T B_{n,t+1} (I + \eta_t \Sigma)^2 B_{n,t+1}^T v_1 + 2\eta_t \underbrace{(v_1^T B_{n,t+1} (I + \eta_t \Sigma) (A_t - \Sigma) B_{n,t+1}^T v_1)}_{P_{n,t}} \\ &\quad + \underbrace{\eta_t^2 (v_1^T B_{n,t+1} (A_t - \Sigma)^2 B_{n,t+1}^T v_1)}_{Q_{n,t}} \\ &\leq v_1^T B_{j,t+1} B_{j,t+1}^T v_1 ((1 + \eta_t \lambda_1)^2) + \eta_t^2 Q_{n,t} + 2\eta_t P_{n,t} \end{aligned} \quad (\text{S.50})$$

Using Lemma S.8 with $U = v_1, G = (I + \eta_t \Sigma), \gamma = 1$ and noting that $\mathbb{E}_\pi [A_t - \Sigma] = 0$, along with observing that $\alpha_{n,t+k_{t+1}} \leq \alpha_{n,t+k_t}$ from Lemma S.3, we have

$$|\mathbb{E} [P_{n,t}]| \leq \eta_{t+1} (1 + \eta_t \lambda_1) \left(\frac{2\mathcal{V} |\lambda_2(P)|}{1 - |\lambda_2(P)|} + \eta_{t+1} \mathcal{M} \left(2 + 16\epsilon + \left(2 + (1 + \epsilon)^2 \right) k_{t+1}^2 (\mathcal{M} + \lambda_1)^2 \right) \right) \alpha_{n,t+k_t}$$

We note that $\forall i, k_i \geq 1$, therefore, using the assumption in S.49, $1 + \eta_t \lambda_1 \leq 1 + \eta_t k_t (\mathcal{M} + \lambda_1) \leq 1 + \epsilon$.

Next, using Lemma S.9 with $U = v_1, G = I, \gamma = 1$ and noting that $\left\| \mathbb{E}_\pi \left[(A_t - \Sigma)^2 \right] \right\|_2 \leq \mathcal{V}$ along with observing that $\alpha_{n,t+k_{t+1}} \leq \alpha_{n,t+k_t}$ using Lemma S.3, we have

$$\begin{aligned} |\mathbb{E}[Q_{n,t}]| &\leq (\mathcal{V} + \eta_{t+1} \mathcal{M}^2 (2\eta_{t+1} + (1 + \epsilon)(2 + \epsilon(1 + \epsilon))k_{t+1}(\mathcal{M} + \lambda_1))) \alpha_{n,t+k_t} \\ &\leq (\mathcal{V} + 2\epsilon\eta_{t+1} \mathcal{M} + \eta_{t+1} \mathcal{M}^2 ((1 + \epsilon)(2 + \epsilon(1 + \epsilon))k_{t+1}(\mathcal{M} + \lambda_1))) \alpha_{n,t+k_t} \end{aligned}$$

where in the last line, we used $\eta_{t+1} \mathcal{M} \leq \eta_{t+1}(\mathcal{M} + \lambda_1) \leq \eta_{t+1} k_{t+1}(\mathcal{M} + \lambda_1) \leq \epsilon$.

Then from S.50 for $n - k_t \geq t \geq 1$,

$$\alpha_{n,t} \leq (1 + \eta_t \lambda_1)^2 \alpha_{n,t+1} + \left(\frac{1 + (3 + 4\epsilon)|\lambda_2(P)|}{1 - |\lambda_2(P)|} \right) \mathcal{V} \eta_t^2 \alpha_{n,t+k_t} + C_{k,t} \eta_t^3 \alpha_{n,t+k_t} \quad (\text{S.51})$$

where $C_{k,t}$ is defined as

$$\begin{aligned} C_{k,t} &:= \mathcal{M} \left[4(1 + \epsilon)(1 + 8\epsilon) + 2\epsilon + k_{t+1}(\mathcal{M} + \lambda_1) \left((1 + \epsilon)(2 + \epsilon(1 + \epsilon))\mathcal{M} + 2(2 + (1 + \epsilon)^2)k_{t+1}(\mathcal{M} + \lambda_1) \right) \right] \\ &\stackrel{(i)}{\leq} \mathcal{M} \left[4(1 + \epsilon)(1 + 8\epsilon) + 2\epsilon + \left((1 + \epsilon)(2 + \epsilon(1 + \epsilon)) + 2(2 + (1 + \epsilon)^2) \right) k_{t+1}^2(\mathcal{M} + \lambda_1)^2 \right] \\ &= \mathcal{M} \left[4 + 38\epsilon + 32\epsilon^2 + \left(6 + 2\epsilon + (1 + \epsilon)^2(1 + 2\epsilon) \right) k_{t+1}^2(\mathcal{M} + \lambda_1)^2 \right] \end{aligned}$$

where in (i) we used $\mathcal{M} \leq k_{t+1}(\mathcal{M} + \lambda_1)$.

Then recalling the definition of $\xi_{k,t}$ in S.49, and noting that $\alpha_{n,t+k_t} \leq \alpha_{n,t+1}$ using Lemma S.3 we have from S.51,

$$\begin{aligned} \alpha_{n,t} &\leq (1 + \eta_t \lambda_1)^2 \alpha_{n,t+1} + \left(\left(\frac{1 + (3 + 4\epsilon)|\lambda_2(P)|}{1 - |\lambda_2(P)|} \right) \mathcal{V} + \xi_{k,t} \right) \eta_t^2 \alpha_{n,t+k_t} \\ &= \left(1 + 2\eta_t \lambda_1 + \eta_t^2 \left(\left(\frac{1 + (3 + 4\epsilon)|\lambda_2(P)|}{1 - |\lambda_2(P)|} \right) \mathcal{V} + \lambda_1^2 + \xi_{k,t} \right) \right) \alpha_{n,t+1} \end{aligned}$$

Therefore using this recursion, we have,

$$\alpha_{n,1} \leq \alpha_{n,n-k_n+1} \exp \left(2\lambda_1 \sum_{t=1}^{n-k_n} \eta_t + \sum_{t=1}^{n-k_n} \eta_t^2 \left(\left(\frac{1 + (3 + 4\epsilon)|\lambda_2(P)|}{1 - |\lambda_2(P)|} \right) \mathcal{V} + \lambda_1^2 + \xi_{k,t} \right) \right)$$

Let $B_{n,n-k_n+1} = I + R'$, where $\|R'\| \leq r$ a.s.

$$\begin{aligned} \alpha_{n,n-k_n+1} &= \mathbb{E} \left[v_1^T B_{n,n-k_n+1} B_{n,n-k_n+1}^T v_1 \right] \\ &= \mathbb{E} \left[v_1^T v_1 \right] + \mathbb{E} \left[v_1^T (R' + R'^T) v_1 \right] + \mathbb{E} \left[v_1^T R' R'^T v_1 \right] \\ &\leq 1 + 2r + r^2 \end{aligned}$$

Using Lemma 2 we have

$$\begin{aligned} r &\leq (1 + \epsilon) k_n \eta_{n-k_n+1} (\mathcal{M} + \lambda_1) \\ &\leq (1 + \epsilon) k_n \eta_{n-k_n} (\mathcal{M} + \lambda_1) \\ &\leq 2(1 + \epsilon) k_n \eta_n (\mathcal{M} + \lambda_1) \quad \text{since } \eta_{n-k_n} \leq 2\eta_n \end{aligned}$$

Therefore,

$$\alpha_{n,1} \leq (1 + 2r + r^2) \exp \left(2\lambda_1 \sum_{t=1}^{n-k_n} \eta_t + \sum_{t=1}^{n-k_n} \eta_t^2 \left(\left(\frac{1 + (3 + 4\epsilon)|\lambda_2(P)|}{1 - |\lambda_2(P)|} \right) \mathcal{V} + \lambda_1^2 + \xi_{k,t} \right) \right)$$

Hence proved. $\square$

Theorem 3. (General Version) Let $u := \min \{t : t \in [n], t - k_t \geq 0\}$. Under Assumptions 1, 2 and 3, for all $n > u$, and any decaying step-size η_i satisfying **C.1** and **C.2**, we have,

$$\begin{aligned} \mathbb{E} [\text{Tr} (V_{\perp}^T B_n B_n^T V_{\perp})] &\leq (1 + 5\epsilon) \exp \left(\sum_{i=u+1}^n 2\eta_i \lambda_2 + (\mathcal{V}' + \lambda_1^2 + \xi_{k,i}) \eta_{i-k_i}^2 \right) \\ &\quad \times \left(d + \sum_{i=u+1}^n (\mathcal{V}' + \xi_{k,i}) C'_{k,i} \eta_{i-k_i}^2 \exp \left(\sum_{j=u+1}^i 2\eta_j (\lambda_1 - \lambda_2) \right) \right) \end{aligned}$$

where $C'_{k,t} := \exp \left(2\lambda_1 \sum_{j=1}^u (\eta_j - \eta_{t-u+j}) + \sum_{j=1}^{t-u} \eta_j^2 (\overline{\mathcal{V}_{k,j}} - \overline{\mathcal{V}_{k,j+u}}) \right)$ and B_t is defined in 2.

Proof. For $t \leq n$, let

$$\begin{aligned} \alpha_t &:= \alpha_{t,1} = \mathbb{E} [v_1^T B_t B_t^T v_1] = \mathbb{E} [\text{Tr} (v_1^T B_t B_t^T v_1)], \text{ as defined in Theorem 2} \\ \beta_t &:= \mathbb{E} [\text{Tr} (V_{\perp}^T B_t B_t^T V_{\perp})] \end{aligned}$$

Note that $\alpha_t + \beta_t = \text{Tr} (B_t B_t^T)$ by definition. Then,

$$\begin{aligned} \text{Tr} (B_t B_t^T V_{\perp} V_{\perp}^T) &= \text{Tr} (B_{t-1} B_{t-1}^T (I + \eta_t \Sigma) V_{\perp} V_{\perp}^T (I + \eta_t \Sigma)) + \eta_t \text{Tr} (B_{t-1}^T (I + \eta_t \Sigma) V_{\perp} V_{\perp}^T (A_t - \Sigma) B_{t-1}) \\ &\quad + \eta_t \text{Tr} (B_{t-1}^T (A_t - \Sigma) V_{\perp} V_{\perp}^T (I + \eta_t \Sigma) B_{t-1}) + \eta_t^2 \text{Tr} (B_{t-1} B_{t-1}^T (A_t - \Sigma) V_{\perp} V_{\perp}^T (A_t - \Sigma)) \\ &\leq (1 + \eta_t \lambda_2)^2 \text{Tr} (B_{t-1} B_{t-1}^T V_{\perp} V_{\perp}^T) + 2\eta_t \underbrace{\text{Tr} (B_{t-1} B_{t-1}^T (I + \eta_t \Sigma) V_{\perp} V_{\perp}^T (A_t - \Sigma))}_{P_t} \\ &\quad + \eta_t^2 \underbrace{\text{Tr} (B_{t-1} B_{t-1}^T (A_t - \Sigma) V_{\perp} V_{\perp}^T (A_t - \Sigma))}_{Q_t} \end{aligned}$$

Let $B_{t-1} = (I + R) B_{t-k_t}$ with $\|R\|_2 \leq r$. Using Lemma S.10 with $G = (I + \eta_t \Sigma) V_{\perp} V_{\perp}^T = V_{\perp} (I + \eta_t \Lambda_{\perp}) V_{\perp}^T$, $\gamma = 1$, where $\Lambda_{\perp}$ is a $d-1 \times d-1$ diagonal matrix of eigenvalues $\lambda_2, \dots, \lambda_d$ of Σ , and noting that $\|V_{\perp} V_{\perp}^T\|_2 = 1$,

$$\begin{aligned} \mathbb{E} [P_t] &\leq (1 + \eta_t \lambda_1) \eta_{t-k_t} \left(\frac{2\mathcal{V} |\lambda_2(P)|}{1 - |\lambda_2(P)|} + \eta_{t-k_t} \mathcal{M} \left(2(1 + 8\epsilon) + \left(2 + (1 + \epsilon)^2 \right) k_t^2 (\mathcal{M} + \lambda_1)^2 \right) \right) (\alpha_{t-k_t} + \beta_{t-k_t}) \\ &\leq (1 + \epsilon) \eta_{t-k_t} \left(\frac{2\mathcal{V} |\lambda_2(P)|}{1 - |\lambda_2(P)|} + \eta_{t-k_t} \mathcal{M} \left(2(1 + 8\epsilon) + \left(2 + (1 + \epsilon)^2 \right) k_t^2 (\mathcal{M} + \lambda_1)^2 \right) \right) (\alpha_{t-k_t} + \beta_{t-k_t}) \end{aligned}$$

where in the last line, we used $\eta_t \lambda_1 \leq \eta_t k_t (\mathcal{M} + \lambda_1) \leq \epsilon$.

Using Lemma S.11 with $U = V_{\perp} V_{\perp}^T$, $\gamma = 1$,

$$\begin{aligned} \mathbb{E} [Q_t] &\leq (\mathcal{V} + \eta_{t-k_t+1} \mathcal{M}^2 (2\eta_t + 2(1 + \epsilon)(1 + \epsilon(1 + \epsilon)) k_t (\mathcal{M} + \lambda_1))) (\alpha_{t-k_t} + \beta_{t-k_t}) \\ &\stackrel{(i)}{\leq} (\mathcal{V} + 2\epsilon \eta_t \mathcal{M} + 2\eta_{t-k_t+1} \mathcal{M}^2 ((1 + \epsilon)(1 + \epsilon(1 + \epsilon)) k_t (\mathcal{M} + \lambda_1))) (\alpha_{t-k_t} + \beta_{t-k_t}) \\ &\stackrel{(ii)}{\leq} (\mathcal{V} + 2\epsilon \eta_t \mathcal{M} + 2\eta_{t-k_t+1} \mathcal{M} ((1 + \epsilon)(1 + \epsilon(1 + \epsilon)) k_t^2 (\mathcal{M} + \lambda_1)^2)) (\alpha_{t-k_t} + \beta_{t-k_t}) \end{aligned}$$

where in (i) we used $\forall i, \eta_i \mathcal{M} \leq \eta_i k_i (\mathcal{M} + \lambda_1) \leq \epsilon$ and in (ii) we used $\mathcal{M} \leq k_t (\mathcal{M} + \lambda_1)$.

Putting everything together, we have,

$$\begin{aligned}
& \mathbb{E} [\text{Tr} (B_t B_t^T V_\perp V_\perp^T)] \\
& \leq (1 + \eta_t \lambda_2)^2 \beta_{t-1} \\
& \quad + 2(1 + \epsilon) \eta_t \eta_{t-k_t} \left(\frac{2\mathcal{V} |\lambda_2(P)|}{1 - |\lambda_2(P)|} + \eta_{t-k_t} \mathcal{M} \left(2(1 + 8\epsilon) + \left(2 + (1 + \epsilon)^2 \right) k_t^2 (\mathcal{M} + \lambda_1)^2 \right) \right) (\alpha_{t-k_t} + \beta_{t-k_t}) \\
& \quad + \eta_t^2 \left(\mathcal{V} + 2\epsilon \eta_t \mathcal{M} + 2\eta_{t-k_t+1} \mathcal{M} \left((1 + \epsilon) (1 + \epsilon (1 + \epsilon)) k_t^2 (\mathcal{M} + \lambda_1)^2 \right) \right) (\alpha_{t-k_t} + \beta_{t-k_t}) \\
& \leq (1 + \eta_t \lambda_2)^2 \beta_{t-1} \\
& \quad + 2(1 + \epsilon) \eta_{t-k_t}^2 \left(\frac{2\mathcal{V} |\lambda_2(P)|}{1 - |\lambda_2(P)|} + \eta_{t-k_t} \mathcal{M} \left(2(1 + 8\epsilon) + \left(2 + (1 + \epsilon)^2 \right) k_t^2 (\mathcal{M} + \lambda_1)^2 \right) \right) (\alpha_{t-k_t} + \beta_{t-k_t}) \\
& \quad + \eta_{t-k_t}^2 \left(\mathcal{V} + 2\epsilon \eta_t \mathcal{M} + 2\eta_{t-k_t+1} \mathcal{M} \left((1 + \epsilon) (1 + \epsilon (1 + \epsilon)) k_t^2 (\mathcal{M} + \lambda_1)^2 \right) \right) (\alpha_{t-k_t} + \beta_{t-k_t}) \\
& \leq (1 + \eta_t \lambda_2)^2 \beta_{t-1} + \eta_{t-k_t}^2 \left(\left(\frac{1 + (3 + 4\epsilon) |\lambda_2(P)|}{1 - |\lambda_2(P)|} \right) \mathcal{V} + \xi_{k,t} \right) (\alpha_{t-k_t} + \beta_{t-k_t})
\end{aligned}$$

where $\xi_{k,t}$ is as defined in S.49. Therefore using Lemma S.4,

$$\begin{aligned}
\mathbb{E} [\text{Tr} (B_t B_t^T V_\perp V_\perp^T)] & \leq \left(1 + 2\eta_t \lambda_2 + \eta_{t-k_t}^2 \left(\left(\frac{1 + (3 + 4\epsilon) |\lambda_2(P)|}{1 - |\lambda_2(P)|} \right) \mathcal{V} + \lambda_2^2 + \xi_{k,t} \right) \right) \beta_{t-1} \\
& \quad + \eta_{t-k_t}^2 \left(\left(\frac{1 + (3 + 4\epsilon) |\lambda_2(P)|}{1 - |\lambda_2(P)|} \right) \mathcal{V} + \xi_{k,t} \right) \alpha_{t-1} \tag{S.52}
\end{aligned}$$

Let $\chi_\epsilon := 1 + 4\epsilon(1 + \epsilon)(1 + \epsilon + \epsilon^2) \leq 1.05$. From Theorem 2 denoting

$$r_{k,t} := 1 + 4(1 + \epsilon) \eta_{t-1} k_{t-1} (\mathcal{M} + \lambda_1) + 4(1 + \epsilon)^2 \eta_{t-1}^2 k_{t-1}^2 (\mathcal{M} + \lambda_1)^2 \leq 1 + 4\epsilon(1 + \epsilon)(1 + \epsilon + \epsilon^2) = \chi_\epsilon, \tag{S.53}$$

we have,

$$\alpha_{t-1} \leq r_{k,t} \exp \left(2\lambda_1 \sum_{i=1}^{t-k_t-1} \eta_i + \sum_{i=1}^{t-k_t-1} \eta_i^2 \left(\left(\frac{1 + (3 + 4\epsilon) |\lambda_2(P)|}{1 - |\lambda_2(P)|} \right) \mathcal{V} + \lambda_1^2 + \xi_{k,i} \right) \right)$$

Now, we note the definition of $\overline{\mathcal{V}_{k,t}}$ and $\mathcal{V}'$ as mentioned in S.49 -

$$\begin{aligned}
\overline{\mathcal{V}_{k,t}} & := \left(\frac{1 + (3 + 4\epsilon) |\lambda_2(P)|}{1 - |\lambda_2(P)|} \right) \mathcal{V} + \lambda_1^2 + \xi_{k,t} \\
& = \mathcal{V}' + \lambda_1^2 + \xi_{k,t}
\end{aligned}$$

Therefore using S.52,

$$\beta_t \leq (1 + 2\eta_t \lambda_2 + \eta_{t-k_t}^2 \overline{\mathcal{V}_{k,t}}) \beta_{t-1} + \eta_{t-k_t}^2 r_{k,t} (\mathcal{V}' + \xi_{k,t}) \exp \left(2\lambda_1 \sum_{i=1}^{t-k_t-1} \eta_i + \sum_{i=1}^{t-k_t-1} \eta_i^2 \overline{\mathcal{V}_{k,i}} \right)$$

Recurring on the above inequality for $u < t \leq n$ where $u = \min \{i : i \in [n], i - k_i \geq 0\}$, we have,

$$\begin{aligned}
\beta_n & \leq \beta_u \exp \left(2 \sum_{i=u+1}^n \eta_i \lambda_2 + \sum_{i=u+1}^n \overline{\mathcal{V}_{k,i}} \eta_{i-k_i}^2 \right) \\
& \quad + \sum_{i=u+1}^n r_{k,i} (\mathcal{V}' + \xi_{k,i}) \eta_{i-k_i}^2 \exp \left(\sum_{j=i+1}^n (2\eta_j \lambda_2 + \overline{\mathcal{V}_{k,j}} \eta_{j-k_j}^2) \right) \exp \left(\sum_{j=1}^{i-k_i} 2\eta_j \lambda_1 + \overline{\mathcal{V}_{k,j}} \eta_j^2 \right) \\
& \leq \exp \left(\sum_{i=u+1}^n 2\eta_i \lambda_2 + \overline{\mathcal{V}_{k,i}} \eta_{i-k_i}^2 \right) \\
& \quad \times \left(\beta_u + \sum_{i=u+1}^n r_{k,i} (\mathcal{V}' + \xi_{k,i}) \eta_{i-k_i}^2 \exp \left(\sum_{j=1}^{i-k_i} (2\eta_j \lambda_1 + \overline{\mathcal{V}_{k,j}} \eta_j^2) - \sum_{j=u+1}^i (2\eta_j \lambda_2 + \overline{\mathcal{V}_{k,j}} \eta_{j-k_j}^2) \right) \right)
\end{aligned}$$

Now, since $k_i, k_j \geq k_u = u$, therefore, we have

$$\beta_n \leq \exp \left(\sum_{i=u+1}^n 2\eta_i \lambda_2 + \overline{\mathcal{V}_{k,i}} \eta_{i-k_i}^2 \right) \times \left(\beta_u + \sum_{i=u+1}^n r_{k,i} (\mathcal{V}' + \xi_{k,i}) \eta_{i-k_i}^2 \exp \left(\sum_{j=1}^{i-u} (2\eta_j \lambda_1 + \overline{\mathcal{V}_{k,j}} \eta_j^2) - \sum_{j=u+1}^i (2\eta_j \lambda_2 + \overline{\mathcal{V}_{k,j}} \eta_{j-u}^2) \right) \right)$$

Recall that $C'_{k,i} := \exp \left(2\lambda_1 \sum_{j=1}^u (\eta_j - \eta_{i-u+j}) + \sum_{j=1}^{i-u} \eta_j^2 (\overline{\mathcal{V}_{k,j}} - \overline{\mathcal{V}_{k,j+u}}) \right)$ as defined in S.49. Therefore,

$$\beta_n \leq \exp \left(\sum_{i=u+1}^n 2\eta_i \lambda_2 + \overline{\mathcal{V}_{k,i}} \eta_{i-k_i}^2 \right) \times \left(\beta_u + \sum_{i=u+1}^n r_{k,i} (\mathcal{V}' + \xi_{k,i}) C'_{k,i} \eta_{i-k_i}^2 \exp \left(\sum_{j=u+1}^i 2\eta_j (\lambda_1 - \lambda_2) \right) \right)$$

Let $B_u = I + R'$ with $\|R'\| \leq r'$ a.s. Using Lemma 2 we have

$$\begin{aligned} r' &\leq (1 + \epsilon) k_u \eta_1 (\mathcal{M} + \lambda_1) \\ &\leq (1 + \epsilon) k_u \eta_0 (\mathcal{M} + \lambda_1) \\ &\leq 2(1 + \epsilon) k_u \eta_u (\mathcal{M} + \lambda_1) \quad \text{since } \eta_0 = \eta_{u-k_u} \leq 2\eta_u \\ &< 2\epsilon(1 + \epsilon) \end{aligned}$$

Therefore,

$$\begin{aligned} \beta_u &= \mathbb{E} [\text{Tr} (V_\perp^T B_u B_u^T V_\perp)] \\ &= \mathbb{E} [\text{Tr} (V_\perp^T V_\perp)] + \mathbb{E} [\text{Tr} (V_\perp^T (R' + R'^T) V_\perp)] + \mathbb{E} [\text{Tr} (V_\perp^T R' R'^T V_\perp)] \\ &\leq d(1 + 2r' + r'^2) \\ &\leq d(1 + 4\epsilon(1 + \epsilon) + 4\epsilon^2(1 + \epsilon)^2) \\ &= d(1 + 4\epsilon(1 + \epsilon)(1 + \epsilon + \epsilon^2)) \\ &= \chi_\epsilon d \end{aligned}$$

The proof follows by noting that $r_{k,t} \leq \chi_\epsilon$ as shown in S.53. $\square$

Theorem 4. (General Version) Under Assumptions 1, 2 and 3, for all $n > k_n$, any decaying step-size η_i satisfying C.1 and C.2, we have:

$$\mathbb{E} [v_1^T B_{n,1} B_{n,1}^T v_1] \geq (1 - t) \exp \left(\sum_{i=1}^{n-k_n} 2\eta_i \lambda_1 - \sum_{i=1}^{n-k_n} 4\eta_i^2 \lambda_1^2 \right)$$

where $t := 2r + s$, $s := 3(1 + r)^2 \exp(2\lambda_1^2 \sum_{i=1}^n \eta_i^2) \sum_{t=1}^{n-k_n} W_{k,t} \eta_t^2 \exp(\sum_{i=t+1}^{n-k_n} \eta_i^2)$, $W_{k,t} := \mathcal{V}' + \xi_{k,t}$ and $B_{j,i}$ has been defined in 7.

Proof. We will start will expanding the quantity of interest using Eq S.50.

$$\alpha_{n,t} = \mathbb{E} [v_1^T B_{n,t} B_{n,t}^T v_1] \geq \mathbb{E} [v_1^T B_{n,t+1} (I + \eta_t \Sigma)^2 B_{n,t+1}^T v_1 + 2\eta_t P_{n,t}] \quad (\text{S.54})$$

where $P_{n,t}$ has been defined in Theorem 2. Let's define

$$\begin{aligned} \mathcal{S}_t &:= \prod_{i=t}^1 (I + \eta_i \Sigma) \prod_{i=1}^t (I + \eta_i \Sigma), \quad \mathcal{S}_0 = I \quad \text{and} \\ \delta_{n,t} &:= \mathbb{E} [v_1^T B_{n,t+1} \mathcal{S}_t B_{n,t+1}^T v_1] \end{aligned}$$

Note that $\delta_{n,0} = \alpha_{n,1}$. First we bound $\delta_{n,n-k_n}$. Let $B_{n,n-k_n} = I + R'$. By Lemma 2 along with the slow-decay assumption on the step-sizes, we know that $\|R'\|_2 \leq r := 2(1+\epsilon)\eta_n k_n (\mathcal{M} + \lambda_1)$ a.s. Then,

$$\delta_{n,n-k_n} - \prod_{i=1}^{n-k_n} (1 + \eta_i \lambda_1)^2 \geq -2 |E[v_1^T R' \mathcal{S}_{n-k_n} v_1]| \geq -2r \prod_{i=1}^{n-k_n} (1 + \eta_i \lambda_1)^2$$

Therefore,

$$\begin{aligned} \delta_{n,n-k_n} &\geq \prod_{i=1}^{n-k_n} (1 + \eta_i \lambda_1)^2 (1 - 2r) \\ &= (1 - 2r) \|\mathcal{S}_{n-k_n}\|_2 \end{aligned} \tag{S.55}$$

Now using S.54, we have

$$\delta_{n,t-1} \geq \delta_{n,t} + 2\eta_t \mathbb{E} \left[\underbrace{v_1^T B_{n,t+1} (I + \eta_t \Sigma) \mathcal{S}_{t-1} (A_t - \Sigma) B_{n,t+1}^T v_1}_{U_t} \right]$$

First, observe that $\mathcal{S}_{t-1} = U \Lambda U^T$, where U denotes a matrix of eigenvectors of Σ , and Λ is a PSD diagonal matrix. Since $I + \eta_t \Sigma = U \Lambda' U^T$ for some other PSD diagonal matrix Λ' , the product will also be PSD.

By using Lemma S.8 with $U = v_1$, $G = (I + \eta_t \Sigma) \mathcal{S}_{t-1}$, $\gamma = 1$ and noting that $\mathbb{E}_\pi [A_t - \Sigma] = 0$, we have

$$\begin{aligned} |\mathbb{E}[U_t]| &\leq (1 + \eta_t \lambda_1) \eta_{t+1} \|\mathcal{S}_{t-1}\|_2 \left(\frac{2\mathcal{V} |\lambda_2(P)|}{1 - |\lambda_2(P)|} \right. \\ &\quad \left. + \eta_{t+1} \mathcal{M} \left(2(1 + 8\epsilon) + \left(2 + (1 + \epsilon)^2 \right) k_{t+1}^2 (\mathcal{M} + \lambda_1)^2 \right) \right) \alpha_{n,t+k_{t+1}} \\ &\leq (1 + \epsilon) \eta_{t+1} \|\mathcal{S}_{t-1}\|_2 W_{k,t} \alpha_{n,t+1} \end{aligned}$$

where $W_{k,t} = \mathcal{V}' + \xi_{k,t}$. Therefore,

$$\delta_{n,t-1} \geq \delta_{n,t} - 2(1 + \epsilon) W_{k,t} \eta_t^2 \alpha_{n,t+1} \|\mathcal{S}_{t-1}\|_2 \text{ for } t \leq n - k_n$$

Let

$$\mathcal{V}' := \left(\frac{1 + (3 + 4\epsilon) |\lambda_2(P)|}{1 - |\lambda_2(P)|} \right) \mathcal{V}$$

as defined in S.49. Unwinding the recursion for $t \leq n - k_n$, we have,

$$\begin{aligned} \delta_{n,0} &\geq \delta_{n,n-k_n} - 2(1 + \epsilon) \sum_{t=1}^{n-k_n} W_{k,t} \eta_t^2 \alpha_{n,t+1} \|\mathcal{S}_{t-1}\|_2 \\ &\geq (1 - 2r) \|\mathcal{S}_{n-k_n}\|_2 \\ &\quad - 2(1 + \epsilon) (1 + r)^2 \sum_{t=1}^{n-k_n} W_{k,t} \eta_t^2 \exp \left(2\lambda_1 \sum_{i=t+1}^{n-k_n} \eta_i + \sum_{i=t+1}^{n-k_n} \eta_i^2 (\mathcal{V}' + \lambda_1^2 + C_{k,i}) \right) \|\mathcal{S}_{t-1}\|_2 \end{aligned}$$

where second step followed from Theorem 2 and S.55.

Using the inequalities $\forall x \in \mathbb{R}, 1 + x \leq e^x$ and $\forall x \in \mathbb{R}, x \geq 0, 1 + x \geq e^{x-x^2}$, $\forall t$ we

have,

$$\begin{aligned}\|\mathcal{S}_t\|_2 &= \prod_{i=1}^t (1 + \eta_i \lambda_1)^2 \leq \exp \left(2\lambda_1 \sum_{i=1}^t \eta_i \right), \text{ and} \\ \|\mathcal{S}_t\|_2 &= \prod_{i=1}^t (1 + \eta_i \lambda_1)^2 \geq \exp \left(2\lambda_1 \sum_{i=1}^t \eta_i - 4\lambda_1^2 \sum_{i=1}^t \eta_i^2 \right)\end{aligned}$$

Therefore denoting $\theta_\epsilon := 2(1 + \epsilon) \exp(2\lambda_1^2 \sum_{i=1}^n \eta_i^2)$, we have

$$\begin{aligned}\delta_{n,0} &\geq \exp \left(2\lambda_1 \sum_{i=1}^{n-k_n} \eta_i - 4\lambda_1^2 \sum_{i=1}^{n-k_n} \eta_i^2 \right) \left[(1 - 2r) - \theta_\epsilon (1 + r)^2 \sum_{t=1}^{n-k_n} W_{k,t} \eta_t^2 \exp \left(\sum_{i=t+1}^{n-k_n} \eta_i^2 (\mathcal{V}' + \lambda_1^2 + C_{k,i}) \right) \right] \\ &\geq \exp \left(2\lambda_1 \sum_{i=1}^{n-k_n} \eta_i - 4\lambda_1^2 \sum_{i=1}^{n-k_n} \eta_i^2 \right) \left[(1 - 2r) - \theta_\epsilon (1 + r)^2 \sum_{t=1}^{n-k_n} W_{k,t} \eta_t^2 \exp \left(\sum_{i=t+1}^{n-k_n} \eta_i^2 (\mathcal{V}' + \lambda_1^2 + C_{k,i}) \right) \right] \\ &\geq \exp \left(2\lambda_1 \sum_{i=1}^{n-k_n} \eta_i - 4\lambda_1^2 \sum_{i=1}^{n-k_n} \eta_i^2 \right) \left[1 - \left(2r + \theta_\epsilon (1 + r)^2 \sum_{t=1}^{n-k_n} W_{k,t} \eta_t^2 \exp \left(\sum_{i=t+1}^{n-k_n} \eta_i^2 \overline{\mathcal{V}_{k,i}} \right) \right) \right]\end{aligned}$$

where $\overline{\mathcal{V}_{k,i}}$ is defined in S.49. Hence proved. $\square$

Theorem 5. (General Version) Under Assumptions 1, 2 and 3, for all $n > k_n$, and decaying step-size η_i satisfying **C.1** and **C.2**, we have:

$$\mathbb{E} \left[(v_1^T B_{n,1} B_{n,1}^T v_1)^2 \right] \leq (1 + r)^4 \exp \left(\sum_{i=1}^{n-k_n} 4\eta_i \lambda_1 + \sum_{i=1}^{n-k_n} \eta_i^2 \zeta_{k,i} \right)$$

where $B_{j,i}$ has been defined in 7.

Proof. Define $Q_{n,t} := v_1^T B_{n,t+1} (A_t - \Sigma)^2 B_{n,t+1}^T v_1$, and $P_{n,t} := v_1^T B_{n,t+1} (I + \eta_t \Sigma) (A_t - \Sigma) B_{n,t+1}^T v_1$. Using S.50, we have, for $n \geq t \geq 1$,

$$\begin{aligned}0 &\leq v_1^T B_{n,t} B_{n,t}^T v_1 = v_1^T B_{n,t+1} (I + \eta_t \Sigma)^2 B_{n,t+1}^T v_1 + \eta_t^2 Q_{n,t} + 2\eta_t P_{n,t} \\ &\leq v_1^T B_{j,t+1} B_{j,t+1}^T v_1 (1 + \eta_t \lambda_1)^2 + \eta_t^2 \mathcal{M}^2 (v_1^T B_{n,t+1} B_{n,t+1}^T v_1) + 2\eta_t P_{n,t} \\ &\leq v_1^T B_{j,t+1} B_{j,t+1}^T v_1 \underbrace{((1 + \eta_t \lambda_1)^2 + \eta_t^2 \mathcal{M}^2)}_{c_t} + 2\eta_t P_{n,t}\end{aligned}$$

Thus, we have -

$$\begin{aligned}\kappa_{n,t} &:= \mathbb{E} \left[(v_1^T B_{n,t} B_{n,t}^T v_1)^2 \right] \leq \mathbb{E} \left[(c_t v_1^T B_{n,t+1} B_{n,t+1}^T v_1 + 2\eta_t P_{n,t})^2 \right] \\ &\leq c_t^2 \kappa_{n,t+1} + 4\eta_t^2 \mathbb{E} [P_{n,t}^2] + 4c_t \eta_t \mathbb{E} \left[(v_1^T B_{n,t+1} B_{n,t+1}^T v_1) P_{n,t} \right] \quad (\text{S.56})\end{aligned}$$

Note that,

$$\begin{aligned}\mathbb{E} [P_{n,t}^2] &\leq \mathbb{E} \left[(v_1^T B_{n,t+1} (I + \eta_t \Sigma) (A_t - \Sigma) B_{n,t+1}^T v_1)^2 \right] \\ &\leq (1 + \eta_t \lambda_1)^2 \mathcal{M}^2 \mathbb{E} \left[(v_1^T B_{n,t+1} B_{n,t+1}^T v_1)^2 \right] \\ &= (1 + \eta_t \lambda_1)^2 \mathcal{M}^2 \kappa_{n,t+1}\end{aligned}$$

Now we work on the cross-term. For the convenience of notation, let's denote $k := k_{t+1}$ unless otherwise specified. Let $B_{n,t+1} = B_{n,t+k} (I + R)$ with,

$$\|R\|_2 \leq (1 + c) \eta_{t+1} k (\mathcal{M} + \lambda_1) =: r_t \leq \epsilon (1 + \epsilon)$$

Using Lemma 2, we have

$$\begin{aligned} & \underbrace{|v_1^T B_{n,t+1} B_{n,t+1}^T v_1 - v_1^T B_{n,t+k} B_{n,t+k}^T v_1|}_{Y_1} = |v_1^T B_{n,t+k} (R + R^T + R R^T) B_{n,t+k}^T v_1| \\ & \leq |v_1^T B_{n,t+k} B_{n,t+k}^T v_1| (2r_t + r_t^2) \end{aligned} \quad (\text{S.57})$$

We will also bound

$$\begin{aligned} & \underbrace{|v_1^T B_{n,t+1} (I + \eta_t \Sigma) (A_t - \Sigma) B_{n,t+1}^T v_1 - v_1^T B_{n,t+k} (I + \eta_t \Sigma) (A_t - \Sigma) B_{n,t+k}^T v_1|}_{Y_2} \\ & = |v_1^T B_{n,t+k} R (I + \eta_t \Sigma) (A_t - \Sigma) (I + R^T) B_{n,t+k}^T v_1 + v_1^T B_{n,t+k} (I + \eta_t \Sigma) (A_t - \Sigma) R^T B_{n,t+k}^T v_1| \\ & \leq (2r_t + r_t^2) (1 + \eta_t \lambda_1) \mathcal{M} |v_1^T B_{n,t+k} B_{n,t+k}^T v_1| \end{aligned} \quad (\text{S.58})$$

So, now we have:

$$\begin{aligned} & \mathbb{E} [(v_1^T B_{n,t+1} B_{n,t+1}^T v_1 P_{n,t})] \\ & = \mathbb{E} [(v_1^T B_{n,t+1} B_{n,t+1}^T v_1) (v_1^T B_{n,t+1} (I + \eta_t \Sigma) (A_t - \Sigma) B_{n,t+1}^T v_1)] \\ & = \mathbb{E} [(Y_1 + v_1^T B_{n,t+k} B_{n,t+k}^T v_1) (Y_2 + v_1^T B_{n,t+k} (I + \eta_t \Sigma) (A_t - \Sigma) B_{n,t+k}^T v_1)] \\ & = \underbrace{\mathbb{E} [Y_1 Y_2]}_{T_1} + \underbrace{\mathbb{E} [Y_1 v_1^T B_{n,t+k} (I + \eta_t \Sigma) (A_t - \Sigma) B_{n,t+k}^T v_1]}_{T_2} + \underbrace{\mathbb{E} [Y_2 v_1^T B_{n,t+k} B_{n,t+k}^T v_1]}_{T_3} \\ & \quad + \underbrace{\mathbb{E} [(v_1^T B_{n,t+k} B_{n,t+k}^T v_1) (v_1^T B_{n,t+k} (I + \eta_t \Sigma) (A_t - \Sigma) B_{n,t+k}^T v_1)]}_{T_4} \end{aligned}$$

Lets start with the last term, T_4 . Using Lemma S.3 we have,

$$\begin{aligned} |T_4| & \leq |\mathbb{E} [(v_1^T B_{n,t+k} B_{n,t+k}^T v_1) (v_1^T B_{n,t+k} (I + \eta_t \Sigma) \mathbb{E} [(A_t - \Sigma) | s_{t+k}] B_{n,t+k}^T v_1)]| \\ & \leq 2(1 + \eta_t \lambda_1) \mathcal{M} d_{\text{mix}}(k) \kappa_{n,t+k} \\ & \leq 2\eta_{t+1}^2 (1 + \eta_t \lambda_1) \mathcal{M} \kappa_{n,t+k} \\ & \leq 2\eta_{t+1}^2 (1 + \eta_t \lambda_1) \mathcal{M} \kappa_{n,t+1} \end{aligned}$$

Using Eqs S.57 and S.58 the first three terms can be bounded as:

$$\begin{aligned} |T_1| & \leq \mathbb{E} [|Y_1 Y_2|] \leq (2r_t + r_t^2)^2 (1 + \eta_t \lambda_1) \mathcal{M} \kappa_{n,t+k} \\ & \leq (2r_t + r_t^2)^2 (1 + \eta_t \lambda_1) \mathcal{M} \kappa_{n,t+1} \text{ using Lemma S.3} \\ & = (2 + r_t)^2 r_t^2 (1 + \eta_t \lambda_1) \mathcal{M} \kappa_{n,t+1} \\ & \leq (1 + \epsilon)^2 (2 + \epsilon (1 + \epsilon))^2 (1 + \eta_t \lambda_1) \eta_{t+1}^2 k_{t+1}^2 \mathcal{M} (\mathcal{M} + \lambda_1)^2 \kappa_{n,t+1} \\ & \leq (1 + \epsilon)^3 (2 + \epsilon + \epsilon^2)^2 \eta_{t+1}^2 k_{t+1}^2 \mathcal{M} (\mathcal{M} + \lambda_1)^2 \kappa_{n,t+1} \text{ since } \eta_t \lambda_1 \leq \epsilon \end{aligned}$$

$$\begin{aligned} |T_2| & \leq \mathbb{E} [|Y_1 v_1^T B_{n,t+k} (I + \eta_t \Sigma) (A_t - \Sigma) B_{n,t+k}^T v_1|] \\ & \leq (2 + r_t) r_t (1 + \eta_t \lambda_1) \mathcal{M} \kappa_{n,t+k} \\ & \leq (2 + r_t) r_t (1 + \eta_t \lambda_1) \mathcal{M} \kappa_{n,t+1} \text{ using Lemma S.3} \\ & \leq (2 + \epsilon + \epsilon^2) (1 + \epsilon) (1 + \eta_t \lambda_1) \eta_{t+1} k_{t+1} (\mathcal{M} + \lambda_1) \mathcal{M} \kappa_{n,t+1} \\ & \leq (1 + \epsilon)^2 (2 + \epsilon + \epsilon^2) \eta_{t+1} k_{t+1} \mathcal{M} (\mathcal{M} + \lambda_1) \kappa_{n,t+1} \end{aligned}$$

and similarly,

$$\begin{aligned}
|T_3| &\leq \mathbb{E} [Y_2 v_1^T B_{n,t+k} B_{n,t+k}^T v_1] \\
&\leq r_t (2 + r_t) (1 + \eta_t \lambda_1) \mathcal{M} \kappa_{n,t+k} \\
&\leq (1 + \epsilon) (2 + \epsilon + \epsilon^2) (1 + \eta_t \lambda_1) \eta_{t+1} k_{t+1} \mathcal{M} (\mathcal{M} + \lambda_1) \kappa_{n,t+k} \\
&\leq (1 + \epsilon)^2 (2 + \epsilon + \epsilon^2) \eta_{t+1} k_{t+1} \mathcal{M} (\mathcal{M} + \lambda_1) \kappa_{n,t+k} \\
&\leq (1 + \epsilon)^2 (2 + \epsilon + \epsilon^2) \eta_{t+1} k_{t+1} \mathcal{M} (\mathcal{M} + \lambda_1) \kappa_{n,t+1} \text{ using Lemma S.3}
\end{aligned}$$

Note that

$$\begin{aligned}
c_t &:= (1 + \eta_t \lambda_1)^2 + \eta_t^2 \mathcal{M}^2 \leq 1 + 2\epsilon + 2\epsilon^2, \text{ and} \\
c_t^2 &= (1 + 2\eta_t \lambda_1 + \eta_t^2 (\mathcal{M}^2 + \lambda_1^2))^2 \\
&= 1 + 4\eta_t^2 \lambda_1^2 + \eta_t^4 (\mathcal{M}^2 + \lambda_1^2)^2 + 4\eta_t \lambda_1 + 4\eta_t^3 \lambda_1 (\mathcal{M}^2 + \lambda_1^2) + 2\eta_t^2 (\mathcal{M}^2 + \lambda_1^2) \\
&\leq 1 + 4\eta_t \lambda_1 + \eta_t^2 (2\mathcal{M}^2 + 6\lambda_1^2 + \epsilon \mathcal{M} + \epsilon \lambda_1) + 4\eta_t^3 \lambda_1 (\mathcal{M}^2 + \lambda_1^2) \\
&\leq 1 + 4\eta_t \lambda_1 + 6\eta_t^2 (\mathcal{M} + \lambda_1)^2 + 4\eta_t^3 \lambda_1 (\mathcal{M} + \lambda_1)^2
\end{aligned}$$

Define

$$\begin{aligned}
\phi_\epsilon &:= (1 + \epsilon) (2 + \epsilon + \epsilon^2) \\
\omega_\epsilon &:= 1 + 2\epsilon + 2\epsilon^2 \\
\zeta_{k,t} &:= (10 + 8(1 + \epsilon) + 4(1 + 2\epsilon) \phi_\epsilon) \phi_\epsilon c_t k_{t+1} (\mathcal{M} + \lambda_1)^2
\end{aligned}$$

Putting everything together in Eq S.56, for $t \leq n - k_{t+1}$ we have,

$$\begin{aligned}
&\frac{\kappa_{n,t}}{\kappa_{n,t+1}} \\
&\leq c_t^2 + 4\eta_t^2 (1 + \eta_t \lambda_1)^2 \mathcal{M}^2 + 4(1 + \epsilon) c_t \eta_t \mathcal{M} \left(2\phi_\epsilon \eta_{t+1} k_{t+1} (\mathcal{M} + \lambda_1) + \left(2 + \phi_\epsilon^2 k_{t+1}^2 (\mathcal{M} + \lambda_1)^2 \right) \eta_{t+1}^2 \right) \\
&\leq c_t^2 + 4\eta_t^2 (1 + \eta_t \lambda_1)^2 \mathcal{M}^2 + 4(1 + \epsilon) c_t \eta_t \mathcal{M} \left(2\phi_\epsilon \eta_t k_{t+1} (\mathcal{M} + \lambda_1) + \left(2 + \phi_\epsilon^2 k_{t+1}^2 (\mathcal{M} + \lambda_1)^2 \right) \eta_t^2 \right) \\
&= c_t^2 + 4\eta_t^2 [\mathcal{M}^2 + 2\phi_\epsilon (1 + \epsilon) c_t \mathcal{M} (\mathcal{M} + \lambda_1) k_{t+1}] + 4\eta_t^3 \left[(1 + 2\epsilon) \lambda_1 + (1 + \epsilon) c_t \mathcal{M} \left(2 + \phi_\epsilon^2 k_{t+1}^2 (\mathcal{M} + \lambda_1)^2 \right) \right] \\
&\leq c_t^2 + 4\eta_t^2 [2 + 2\phi_\epsilon (1 + \epsilon) c_t k_{t+1}] \mathcal{M} (\mathcal{M} + \lambda_1) + 4(1 + 2\epsilon) \eta_t^3 \left[\lambda_1 + c_t \mathcal{M} \left(2 + \phi_\epsilon^2 k_{t+1}^2 (\mathcal{M} + \lambda_1)^2 \right) \right] \\
&\leq 1 + 4\eta_t \lambda_1 + \eta_t^2 [10 + 8\phi_\epsilon (2 + \epsilon) c_t k_{t+1}] (\mathcal{M} + \lambda_1)^2 + 4(1 + 2\epsilon) \eta_t^3 \left[\lambda_1 + 2c_t \mathcal{M} + c_t \phi_\epsilon^2 k_{t+1}^2 (\mathcal{M} + \lambda_1)^3 \right] \\
&\leq \exp \left(4\eta_t \lambda_1 + \eta_t^2 (10 + 8\phi_\epsilon (1 + \epsilon) c_t k_{t+1}) (\mathcal{M} + \lambda_1)^2 + 4(1 + 2\epsilon) \eta_t^3 \left(\lambda_1 + c_t \mathcal{M} + 2c_t \phi_\epsilon^2 k_{t+1}^2 (\mathcal{M} + \lambda_1)^3 \right) \right) \\
&\leq \exp \left(4\eta_t \lambda_1 + \eta_t^2 (10 + 8\phi_\epsilon (1 + \epsilon) c_t k_{t+1}) (\mathcal{M} + \lambda_1)^2 + 4\epsilon (1 + 2\epsilon) \eta_t^2 \left(2c_t + c_t \phi_\epsilon^2 k_{t+1}^2 (\mathcal{M} + \lambda_1)^2 \right) \right) \\
&\leq \exp \left(4\eta_t \lambda_1 + \eta_t^2 \left(8\epsilon (1 + 2\epsilon) \omega_\epsilon + (10 + 8(1 + \epsilon) + 4\epsilon (1 + 2\epsilon) \phi_\epsilon) \phi_\epsilon \omega_\epsilon k_{t+1} \right) (\mathcal{M} + \lambda_1)^2 \right) \\
&\leq \exp \left(4\eta_t \lambda_1 + \eta_t^2 \left(1 + (10 + 20k_{t+1}) (\mathcal{M} + \lambda_1)^2 \right) \right) \\
&\leq \exp \left(4\eta_t \lambda_1 + \eta_t^2 \left(1 + (10 + 20k_{t+1}) (\mathcal{M} + \lambda_1)^2 \right) \right) \\
&\leq \exp \left(4\eta_t \lambda_1 + 40\eta_t^2 k_{t+1} (\mathcal{M} + \lambda_1)^2 \right) \text{ since } (\mathcal{M} + \lambda_1), k_{t+1} \geq 1
\end{aligned}$$

Recall our definition of $k := k_{t+1}$. We can use the above recursion for $1 \leq t \leq n - k_{t+1}$. We note that $t = n - k_n$ satisfies the conditions. Therefore,

$$\kappa_{n,1} \leq \exp \left(\sum_{i=1}^{n-k_n} 4\eta_i \lambda_1 + \sum_{i=1}^{n-k_n} \eta_i^2 \zeta_{k,i} \right) \kappa_{n,n-k_n+1}$$

Let $B_{n,n-k_n+1} = I + R'$, with $\|R'\|_2 \leq r$ a.s.

$$\begin{aligned}\kappa_{n,n-k_n+1} &= \mathbb{E} \left[\left(v_1^T B_{n,n-k_n+1} B_{n,n-k_n+1}^T v_1 \right)^2 \right] \\ &= \mathbb{E} \left[\left(v_1^T v_1 + v_1^T (R' + R'^T) v_1 + v_1^T R' R'^T v_1 \right)^2 \right] \\ &\leq (1 + 2r + r^2)^2 \mathbb{E} \left[\left(v_1^T v_1 \right)^2 \right]\end{aligned}$$

Using Lemma 2, we have

$$\begin{aligned}r &\leq (1 + \epsilon) k_{n+1} \eta_{n-k_{n+1}} (\mathcal{M} + \lambda_1) \\ &\leq (1 + \epsilon) k_n \eta_{n-k_n} (\mathcal{M} + \lambda_1) \\ &\leq 2(1 + \epsilon) k_n \eta_n (\mathcal{M} + \lambda_1) \quad \text{since } \eta_{n-k_n} \leq 2\eta_n\end{aligned}$$

which completes our proof. $\square$

S.5 Main Results : Details and Proofs

S.5.1 Proof of Theorem 1

Lemma S.13. *This lemma proves conditions required later in the proof. Let the step-sizes be set according to Lemma S.12 and $m := 200$. Define*

$$\begin{aligned}r &:= 2(1 + \epsilon) \eta_n k_n (\mathcal{M} + \lambda_1), \\ s &:= 3(1 + r)^2 \sum_{t=1}^{n-k_n-1} W_{k,t} \eta_t^2 \exp \left(\sum_{i=t+1}^{n-k_n-1} \overline{\mathcal{V}_{k,i}} \eta_i^2 \right)\end{aligned}$$

where $W_{k,t}$ is defined in Theorem 4, $\overline{\mathcal{V}_{k,i}}$ is defined in S.49 and $\alpha, \beta, f(\cdot), \delta$ are defined in Lemma S.12. Then for sufficiently large number of samples n , such that

$$\frac{n}{\log(f(n))} > \frac{\beta}{\log(f(0))}$$

we have

$$1. \quad 2r + s \leq \frac{1}{2} \quad (\text{S.63})$$

$$2. \quad r = 2(1 + \epsilon) \eta_n k_n (\mathcal{M} + \lambda_1) < \frac{1}{50} \frac{\delta/m}{1+\delta/m} \quad (\text{S.66})$$

Proof. For (1), using Lemma S.12-(3), we note that

$$\begin{aligned}s &\leq 3(1 + r)^2 \sum_{t=1}^{n-k_n-1} W_{k,t} \eta_t^2 \exp \left(\sum_{i=t+1}^{n-k_n-1} \overline{\mathcal{V}_{k,i}} \eta_i^2 \right) \\ &\leq 3(1 + r)^2 \sum_{t=1}^{n-k_n-1} W_{k,t} \eta_t^2 \left(1 + \frac{\delta}{m} \right) \\ &\leq \frac{3(1 + r)^2}{100} \left(1 + \frac{\delta}{m} \right) \log \left(1 + \frac{\delta}{m} \right) \\ &\leq \frac{3(1 + r)^2 \log(2)}{50} \quad \text{since } \frac{\delta}{m} < 1\end{aligned} \tag{S.59}$$

Therefore,

$$\begin{aligned}2r + s &\leq 2r + \frac{3(1 + r)^2}{25} \\ &= \frac{3}{25} + \frac{56}{25}r + \frac{3}{25}r^2\end{aligned} \tag{S.60}$$

Setting $\frac{3}{25} + \frac{56}{25}r + \frac{3}{25}r^2 \leq \frac{1}{2}$, we have,

$$\begin{aligned} \frac{3}{25} + \frac{56}{25}r + \frac{3}{25}r^2 &\leq \frac{1}{2} \\ \implies 6r^2 + 112r - 19 &\leq 0 \end{aligned}$$

which holds for $r \in [0, \frac{1}{10}]$.

For (2), using Lemma S.12 and substituting the value of $k_i := \tau_{\text{mix}}(\eta_i^2) \leq \frac{2\tau_{\text{mix}}}{\log(2)} \log\left(\frac{1}{\eta_i^2}\right)$ for $\eta_i < 1$, we note that

$$\begin{aligned} r &\leq \frac{8(1+\epsilon)\tau_{\text{mix}}(\mathcal{M} + \lambda_1)}{\log(2)} \frac{\alpha}{(\lambda_1 - \lambda_2)(\beta + n)} \log\left(\frac{(\lambda_1 - \lambda_2)(\beta + n)}{\alpha}\right) \\ &= \frac{8(1+\epsilon)\tau_{\text{mix}}(\mathcal{M} + \lambda_1)}{\log(2)} \frac{\log\left(\frac{(\lambda_1 - \lambda_2)(\beta + n)}{\alpha}\right)}{\frac{(\lambda_1 - \lambda_2)(\beta + n)}{\alpha}} \\ &= \frac{8(1+\epsilon)\tau_{\text{mix}}(\mathcal{M} + \lambda_1)}{\log(2)} \frac{\log(f(n))}{f(n)} \end{aligned}$$

Therefore (2) holds for sufficiently large n , i.e.,

$$\frac{f(n)}{\log(f(n))} \geq \frac{400\left(1 + \frac{\delta}{m}\right)(1+\epsilon)\tau_{\text{mix}}(\mathcal{M} + \lambda_1)}{\log(2) \frac{\delta}{m}}$$

This is satisfied if

$$\frac{n}{\log(f(n))} \geq \frac{400\tau_{\text{mix}}\left(1 + \frac{\delta}{m}\right)(1+\epsilon)(\mathcal{M} + \lambda_1)\alpha}{\log(2)(\lambda_1 - \lambda_2) \frac{\delta}{m}} \quad (\text{S.61})$$

From Lemma S.12, we have

$$\frac{\beta}{\log(f(0))} \geq \frac{600\tau_{\text{mix}}(1+2\epsilon)^2(\mathcal{M} + \lambda_1)^2\alpha^2}{(\lambda_1 - \lambda_2)^2 \log\left(1 + \frac{\delta}{m}\right)} \stackrel{(i)}{\geq} \frac{400\tau_{\text{mix}}\left(1 + \frac{\delta}{m}\right)(1+\epsilon)(\mathcal{M} + \lambda_1)\alpha}{\log(2)(\lambda_1 - \lambda_2) \frac{\delta}{m}}$$

where (i) follows since $\frac{\mathcal{M} + \lambda_1}{\lambda_1 - \lambda_2} > 1$, $\alpha > 2$ and $\log(1+x) \leq x \forall x$. Therefore, $\frac{n}{\log(f(n))} > \frac{\beta}{\log(f(0))}$ suffices. Further, we note that (2) implies (1) for $m = 200$, $\delta \leq 1$. Therefore, the condition on n is sufficient for both results. Hence proved. $\square$

Lemma S.14. *Let*

$$u := \min \{i : i \in [n], i - k_i \geq 0\}$$

where k_i is defined in Lemma S.12. Then,

$$u \leq \lfloor \beta \rfloor \leq \beta$$

Proof. Using the definition of k_i mentioned in Lemma S.12, we have

$$\begin{aligned} k_i &:= \tau_{\text{mix}}(\eta_i^2) \leq \frac{2\tau_{\text{mix}}}{\log(2)} \log\left(\frac{1}{\eta_i^2}\right) \\ &= \frac{4\tau_{\text{mix}}}{\log(2)} \log\left(\frac{(\lambda_1 - \lambda_2)(\beta + i)}{\alpha}\right) \end{aligned}$$

Therefore,

$$\begin{aligned}
\lfloor \beta \rfloor - k_{\lfloor \beta \rfloor} &\geq \lfloor \beta \rfloor - \frac{4\tau_{\text{mix}}}{\log(2)} \log \left(\frac{\beta + \lfloor \beta \rfloor}{\frac{\alpha}{\lambda_1 - \lambda_2}} \right) \\
&\geq \frac{\beta}{2} - \frac{4\tau_{\text{mix}}}{\log(2)} \log \left(\frac{2\beta}{\frac{\alpha}{\lambda_1 - \lambda_2}} \right) \text{ since } \beta > 1 \\
&= \beta \left[\frac{1}{2} - \frac{4\tau_{\text{mix}}}{\log(2)} \frac{\log(2f(0))}{\beta} \right], \text{ where } f(\cdot) \text{ is defined in Lemma S.12}
\end{aligned}$$

Now, from Lemma S.12, we know that $f(0) > e$. Therefore, $\log(2f(0)) \leq 2\log(f(0))$. Then,

$$\lfloor \beta \rfloor - k_{\lfloor \beta \rfloor} \geq \beta \left[\frac{1}{2} - \frac{8\tau_{\text{mix}}}{\log(2)} \frac{\log(f(0))}{\beta} \right]$$

Again, from the conditions in Lemma S.12, we know that

$$\frac{\log(f(0))}{\beta} \leq \frac{\epsilon}{6\tau_{\text{mix}}} \frac{\lambda_1 - \lambda_2}{(\mathcal{M} + \lambda_1)\alpha} \leq \frac{1}{120\tau_{\text{mix}}} \text{ since } \alpha > 2, \frac{\lambda_1 - \lambda_2}{\mathcal{M} + \lambda_1} \leq 1, \epsilon \leq \frac{1}{100}$$

Therefore,

$$\lfloor \beta \rfloor - k_{\lfloor \beta \rfloor} \geq \beta \left(\frac{1}{2} - \frac{8}{120\log(2)} \right) \geq 0$$

Hence proved. $\square$

S.5.1.1 Numerator

Using Theorem 3 and Markov's Inequality, we have with probability atleast $(1 - \delta)$

$$\begin{aligned}
&\text{Tr}(V_{\perp}^T B_n B_n^T V_{\perp}) \leq \\
&1.05 \frac{\exp\left(\sum_{i=u+1}^n 2\eta_i \lambda_2 + \overline{\mathcal{V}_{k,i}} \eta_{i-k_i}^2\right)}{\delta} \left(d + \sum_{i=u+1}^n (\mathcal{V}' + \xi_{k,i}) C'_{k,i} \eta_{i-k_i}^2 \exp\left(\sum_{j=u+1}^i 2\eta_j (\lambda_1 - \lambda_2)\right) \right)
\end{aligned}$$

S.5.1.2 Denominator

Using Chebyshev's Inequality we have, with probability atleast $(1 - \delta)$

$$v_1^T B_n B_n^T v_1 \geq \mathbb{E}[v_1^T B_n B_n^T v_1] \left(1 - \sqrt{\frac{1}{\delta}} \sqrt{\frac{\mathbb{E}\left[(v_1^T B_n B_n^T v_1)^2\right]}{\mathbb{E}[v_1^T B_n B_n^T v_1]^2}} - 1 \right) \quad (\text{S.62})$$

Let $r := 2(1 + \epsilon) \eta_n k_n (\mathcal{M} + \lambda_1) \leq \frac{1}{10}$. Using Theorem 3, we have

$$\mathbb{E}\left[(v_1^T B_n B_n^T v_1)^2\right] \leq (1 + r)^4 \exp\left(\sum_{i=1}^{n-k_n} 4\eta_i \lambda_1 + \sum_{i=1}^{n-k_n} \eta_i^2 \zeta_{k,i}\right)$$

Using Theorem 4, we have

$$\mathbb{E}[v_1^T B_{n,1} B_{n,1}^T v_1] \geq \exp\left(2\lambda_1 \sum_{i=1}^{n-k_n} \eta_i - 4\lambda_1^2 \sum_{i=1}^{n-k_n} \eta_i^2\right) \left[1 - \left(2r + 3(1 + r)^2 \sum_{t=1}^{n-k_n} W_{k,t} \eta_t^2 \exp\left(\sum_{i=t+1}^{n-k_n} \eta_i^2 \overline{\mathcal{V}_{k,i}}\right)\right) \right]$$

Let

$$s := 3(1+r)^2 \sum_{t=1}^{n-k_n} W_{k,t} \eta_t^2 \exp \left(\sum_{i=t+1}^{n-k_n} \eta_i^2 \overline{\mathcal{V}_{k,i}} \right)$$

Then,

$$\frac{\mathbb{E} \left[(v_1^T B_n B_n^T v_1)^2 \right]}{\mathbb{E} [v_1^T B_n B_n^T v_1]^2} \leq \frac{(1+r)^4}{(1-2r-s)^2} \exp \left(\sum_{i=1}^{n-k_n} \eta_i^2 (\zeta_{k,i} + 4\lambda_1^2) \right)$$

By Lemma S.13, we have that

$$2r + s \leq \frac{1}{2}. \quad (\text{S.63})$$

Then, using

$$\frac{1}{(1-x)^2} \leq 1 + 6x \text{ for } x \in \left[0, \frac{1}{2}\right] \text{ and, } (1+x)^4 \leq 1 + 5x \text{ for } x \in \left[0, \frac{1}{10}\right]$$

we have,

$$\begin{aligned} \frac{\mathbb{E} \left[(v_1^T B_n B_n^T v_1)^2 \right]}{\mathbb{E} [v_1^T B_n B_n^T v_1]^2} &\leq (1+5r)(1+12r+6s) \exp \left(\sum_{i=1}^{n-k_n} \eta_i^2 (\zeta_{k,i} + 4\lambda_1^2) \right) \\ &\leq (1+17r+6s+60r^2+30rs) \exp \left(\sum_{i=1}^{n-k_n} \eta_i^2 (\zeta_{k,i} + 4\lambda_1^2) \right) \\ &\leq (1+22r+12s) \exp \left(\sum_{i=1}^{n-k_n} \eta_i^2 (\zeta_{k,i} + 4\lambda_1^2) \right) \text{ since } r \leq \frac{1}{10} \end{aligned}$$

By Lemma S.12-(3), we have that

$$\exp \left(\sum_{i=1}^{n-k_n} \eta_i^2 (\zeta_{k,i} + 4\lambda_1^2) \right) \leq 1 + \frac{\delta}{m} \quad (\text{S.64})$$

By S.59, we have that

$$\begin{aligned} 12s &\leq \frac{48(1+r)^2}{100} \left(1 + \frac{\delta}{m}\right)^2 \log \left(1 + \frac{\delta}{m}\right) \\ &\leq \frac{3}{5} \left(1 + \frac{\delta}{m}\right)^2 \log \left(1 + \frac{\delta}{m}\right) \text{ since } r \leq \frac{1}{10} \end{aligned} \quad (\text{S.65})$$

By Lemma S.13, we have that

$$r = 2(1+\epsilon) \eta_n k_n (\mathcal{M} + \lambda_1) < \frac{1}{50} \frac{\delta/m}{1 + \delta/m} \quad (\text{S.66})$$

Then,

$$\begin{aligned}
\frac{\mathbb{E} \left[(v_1^T B_n B_n^T v_1)^2 \right]}{\mathbb{E} \left[v_1^T B_n B_n^T v_1 \right]^2} &\leq (1 + 22r + 12s) \left(1 + \frac{\delta}{m} \right) \\
&= 1 + \frac{\delta}{m} + 22r \left(1 + \frac{\delta}{m} \right) + 12s \left(1 + \frac{\delta}{m} \right) \\
&\leq 1 + \frac{\delta}{m} + \frac{22}{50} \frac{\delta}{m} + \frac{3}{5} \left(1 + \frac{\delta}{m} \right)^3 \log \left(1 + \frac{\delta}{m} \right) \\
&\leq 1 + \frac{\delta}{m} + \frac{22}{50} \frac{\delta}{m} + \frac{7}{10} \log \left(1 + \frac{\delta}{m} \right) \text{ since } \delta \leq 1, m = 200 \\
&\leq 1 + \frac{\delta}{m} + \frac{22}{50} \frac{\delta}{m} + \frac{7}{10} \frac{\delta}{m} \text{ since } \forall x, \log(1+x) \leq x \\
&\leq 1 + 3 \frac{\delta}{m}
\end{aligned}$$

Then setting $m = 200$, from S.62 we have

$$\begin{aligned}
v_1^T B_n B_n^T v_1 &\geq \exp \left(\sum_{i=1}^{n-k_n} 2\eta_i \lambda_1 - 4\eta_i^2 \lambda_1^2 \right) (1 - 2r - s) \left(1 - \sqrt{\frac{1}{\delta}} \sqrt{\frac{3\delta}{m}} \right) \\
&\geq \exp \left(\sum_{i=1}^{n-k_n} 2\eta_i \lambda_1 - 4\eta_i^2 \lambda_1^2 \right) \left(1 - \frac{1}{25} \frac{\delta/m}{1 + \delta/m} - \frac{1}{20} \left(1 + \frac{\delta}{m} \right)^2 \log \left(1 + \frac{\delta}{m} \right) \right) \left(1 - \sqrt{\frac{3}{m}} \right) \\
&\geq \frac{5}{6} \exp \left(\sum_{i=1}^{n-k_n} 2\eta_i \lambda_1 - 4\eta_i^2 \lambda_1^2 \right) \text{ since } \delta \leq 1 \text{ and } m = 200
\end{aligned}$$

The second inequality uses Eqs S.65, S.66.

S.5.1.3 Fraction

Now that we have established this result let's calculate the fraction. Let the step-sizes be set according to Lemma S.12. Define

$$\begin{aligned}
\mathcal{S} &:= \exp \left(\sum_{i=u+1}^n \overline{\mathcal{V}_{k,i}} \eta_{i-k_i}^2 + \sum_{i=1}^{n-k_n} 4\lambda_1^2 \eta_i^2 \right) \\
Q_u &:= \exp \left(2\lambda_1 \left(\sum_{j=1}^u \eta_j - \sum_{j=n-k_n+1}^n \eta_j \right) \right) \\
\mathcal{R}_{k,t} &:= \frac{\exp \left(\sum_{j=1}^{t-u} \eta_j^2 (\overline{\mathcal{V}_{k,j}} - \overline{\mathcal{V}_{k,j+u}}) \right) \exp \left(2\lambda_1 \sum_{j=n-k_n+1}^n \eta_j \right)}{\exp \left(2\lambda_1 \sum_{j=1}^u \eta_{t-u+j} \right)}
\end{aligned}$$

Then, recall that

$$\begin{aligned}
u &:= \min \{ i : i \in [n], i - k_i \geq 0 \} \\
\xi_{k,t} &:= 6\eta_{t-k_t} \mathcal{M} \left[1 + 3k_{t+1}^2 (\mathcal{M} + \lambda_1)^2 \right] \\
\mathcal{V}' &:= \left(\frac{1 + (3 + 4\epsilon) |\lambda_2(P)|}{1 - |\lambda_2(P)|} \right) \mathcal{V} \\
\overline{\mathcal{V}_{k,t}} &:= \mathcal{V}' + \lambda_1^2 + \xi_{k,t} \\
C'_{k,t} &:= \exp \left(2\lambda_1 \sum_{j=1}^u (\eta_j - \eta_{t-u+j}) + \sum_{j=1}^{t-u} \eta_j^2 (\overline{\mathcal{V}_{k,j}} - \overline{\mathcal{V}_{k,j+u}}) \right) = Q_u \mathcal{R}_{k,t}
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \frac{\text{Tr} (V_{\perp}^T B_n B_n^T V_{\perp})}{v_1^T B_n B_n^T v_1} \\
& \leq \frac{1.3}{\delta} \frac{\exp(\sum_{i=u+1}^n 2\eta_i \lambda_2 + \overline{\mathcal{V}_{k,i}} \eta_{i-k_i}^2)}{\exp(\sum_{i=1}^{n-k_n} 2\eta_i \lambda_1 - 4\eta_i^2 \lambda_1^2)} \left(d + \sum_{i=u+1}^n (\mathcal{V}' + \xi_{k,i}) C'_{k,i} \eta_{i-k_i}^2 \exp\left(\sum_{j=u+1}^i 2\eta_j (\lambda_1 - \lambda_2)\right) \right) \\
& \leq \frac{1.3}{\delta} \frac{\mathcal{S}}{Q_u} \exp\left(\sum_{i=u+1}^n 2\eta_i (\lambda_2 - \lambda_1)\right) \left(d + \sum_{i=u+1}^n (\mathcal{V}' + \xi_{k,i}) C'_{k,i} \eta_{i-k_i}^2 \exp\left(\sum_{j=u+1}^i 2\eta_j (\lambda_1 - \lambda_2)\right) \right) \\
& \leq \frac{1.3}{\delta} \mathcal{S} \left(\underbrace{\frac{d \exp(\sum_{i=u+1}^n 2\eta_i (\lambda_2 - \lambda_1))}{Q_u}}_{X_1} + \underbrace{\sum_{i=u+1}^n (\mathcal{V}' + \xi_{k,i}) \mathcal{R}_{k,i} \eta_{i-k_i}^2 \exp\left(-\sum_{j=i+1}^n 2\eta_j (\lambda_1 - \lambda_2)\right)}_{X_2} \right) \tag{S.67}
\end{aligned}$$

For X_1 , we have

$$\begin{aligned}
X_1 & \leq \frac{d \exp(\sum_{i=u+1}^n 2\eta_i (\lambda_2 - \lambda_1))}{Q_u} \\
& = \frac{d \exp(\sum_{i=u+1}^n 2\eta_i (\lambda_2 - \lambda_1))}{\exp\left(2\lambda_1 \left(\sum_{j=1}^u \eta_j - \sum_{j=n-k_n+1}^n \eta_j\right)\right)} \\
& \leq \frac{d \exp(\sum_{i=u+1}^n 2\eta_i (\lambda_2 - \lambda_1))}{\exp\left(-2\lambda_1 \left(\sum_{j=n-k_n+1}^n \eta_j\right)\right)} \\
& \leq d \exp\left(\sum_{i=u+1}^n 2\eta_i (\lambda_2 - \lambda_1)\right) \exp\left(2\lambda_1 \left(\sum_{j=n-k_n+1}^n \eta_j\right)\right)
\end{aligned}$$

Note that

$$\begin{aligned}
\exp\left(2\lambda_1 \sum_{j=n-k_n+1}^n \eta_j\right) & \leq \exp(2(1+2\epsilon)\lambda_1 k_n \eta_{n-k_n+1}) \text{ using monotonicity of } \eta_i \\
& \leq \exp(4(1+2\epsilon)\lambda_1 k_n \eta_n) \text{ using slow-decay of } \eta_i \\
& \leq 1 + 2\frac{\delta}{m} \text{ using Lemma S.13 along with } e^x \leq 1 + x + x^2 \text{ for } x \in (0, 1)
\end{aligned}$$

Therefore, using S.31

$$X_1 \leq d \left(1 + \frac{2\delta}{m}\right) \left(\frac{\beta + u}{n}\right)^{2\alpha}$$

Next, for X_2 , we first have

$$\begin{aligned}
\mathcal{R}_{k,t} & := \frac{\exp\left(\sum_{j=1}^{t-u} \eta_j^2 (\overline{\mathcal{V}_{k,j}} - \overline{\mathcal{V}_{k,j+u}})\right) \exp\left(2\lambda_1 \sum_{j=n-k_n+1}^n \eta_j\right)}{\exp\left(2\lambda_1 \sum_{j=1}^u \eta_{t-u+j}\right)} \\
& \leq \exp\left(\sum_{j=1}^{t-u} \eta_j^2 \overline{\mathcal{V}_{k,j}}\right) \exp\left(2\lambda_1 \sum_{j=n-k_n+1}^n \eta_j\right) \\
& \leq \left(1 + \frac{2\delta}{m}\right)^2 \text{ using Lemmas S.12 - (3), S.13 and } e^x \leq 1 + x + x^2 \text{ for } x \in (0, 1)
\end{aligned}$$

Now, using S.12-(4) we have,

$$\sum_{i=1}^n \overline{\mathcal{V}_{k,i} \eta_{i-k_i}^2} \exp \left(- \sum_{j=i+1}^n 2\eta_j (\lambda_1 - \lambda_2) \right) \leq$$

$$\left(\frac{2(1+10\epsilon)\alpha^2}{2\alpha-1} \right) \frac{\mathcal{V}'}{(\lambda_1 - \lambda_2)^2} \frac{1}{n} + \left(\frac{800(1+10\epsilon)\alpha^3}{(\alpha-1)} \right) \frac{\mathcal{M}(\mathcal{M} + \lambda_1)^2 \tau_{\text{mix}}^2}{(\lambda_1 - \lambda_2)^3} \frac{\log^2 \left(\frac{(\beta+n)(\lambda_1 - \lambda_2)}{\alpha} \right)}{n^2}$$

Then,

$$X_2 \leq \left(1 + \frac{2\delta}{m} \right)^2 \left[\underbrace{\left(\frac{2(1+10\epsilon)\alpha^2}{2\alpha-1} \right)}_{C_1} \frac{\mathcal{V}'}{(\lambda_1 - \lambda_2)^2} \frac{1}{n} + \underbrace{\left(\frac{24(1+10\epsilon)\alpha^3}{(\alpha-1)} \right)}_{C_2} \frac{\mathcal{M}(\mathcal{M} + \lambda_1)^2}{(\lambda_1 - \lambda_2)^3} \frac{k_n^2}{n^2} \right]$$

Therefore substituting in S.67,

$$\frac{\text{Tr}(V_{\perp}^T B_n B_n^T V_{\perp})}{v_1^T B_n B_n^T v_1} \leq \frac{1.3\mathcal{S}}{\delta} \left(1 + \frac{2\delta}{m} \right)^2 \left[d \left(\frac{\beta+u}{n} \right)^{2\alpha} + \frac{C_1 \mathcal{V}'}{(\lambda_1 - \lambda_2)^2} \frac{1}{n} + \frac{C_2 \mathcal{M}(\mathcal{M} + \lambda_1)^2}{(\lambda_1 - \lambda_2)^3} \frac{k_n^2}{n^2} \right] \quad (\text{S.68})$$

Proof of Theorem 1. To complete our proof, we bound $\mathcal{S}$ to simplify S.68. We note that under the learning rate schedule presented in Lemma S.12-(3),

$$\mathcal{S} \leq \left(1 + \frac{\delta}{m} \right)$$

Therefore,

$$\begin{aligned} \frac{\text{Tr}(V_{\perp}^T B_n B_n^T V_{\perp})}{v_1^T B_n B_n^T v_1} &\leq \frac{1.3}{\delta} \left(1 + \frac{2\delta}{m} \right)^3 \left[d \left(\frac{\beta+u}{n} \right)^{2\alpha} + \frac{C_1 \mathcal{V}'}{(\lambda_1 - \lambda_2)^2} \frac{1}{n} + \frac{C_2 \mathcal{M}(\mathcal{M} + \lambda_1)^2}{(\lambda_1 - \lambda_2)^3} \frac{k_n^2}{n^2} \right] \\ &\leq \frac{1.4}{\delta} \left[d \left(\frac{\beta+u}{n} \right)^{2\alpha} + \frac{C_1 \mathcal{V}'}{(\lambda_1 - \lambda_2)^2} \frac{1}{n} + \frac{C_2 \mathcal{M}(\mathcal{M} + \lambda_1)^2}{(\lambda_1 - \lambda_2)^3} \frac{k_n^2}{n^2} \right] \end{aligned}$$

Using lemma S.14, we have that $u \leq \beta$. Then, using Lemma 3.1 from [15] completes our proof. $\square$

S.5.2 Proof of Corollary 1

Proof of Corollary 1. We note that the downsampled data stream can be considered to be drawn from a Markov chain with transition kernel $P^k(\cdot, \cdot)$ since each data-point is k steps away from the previous one. We will denote the parameters of this transformed chain by $\tilde{y}$ when the corresponding parameter is y under the original chain. For example, $\tilde{\tau}_{\text{mix}}$ is the mixing time of the new chain.

Note that this modified transition matrix has the same stationary distribution π . It is also reversible. This can be seen by considering the diagonal matrix of stationary distribution probabilities Π , where $\Pi_{ii} = \pi_i$. For a reversible Markov Chain, we have $\Pi P = P \Pi$. However, that also implies $\Pi P^2 = (\Pi P)P = (P \Pi)P = P(\Pi P) = P^2 \Pi$. This same technique works for P^k yielding $\Pi P^k = P^k \Pi$.

Using standard results on Markov chains [19],

$$\frac{|\lambda_2(P)|}{1 - |\lambda_2(P)|} \log \left(\frac{1}{2\epsilon} \right) \leq \tau_{\text{mix}}(\epsilon) \leq \frac{1}{1 - |\lambda_2(P)|} \log \left(\frac{1}{\epsilon \pi_{\min}} \right), \quad (\text{S.69})$$

where $\pi_{\min} := \min_i \pi_i$. Therefore, as noted in the theorem statement, we substitute the modified parameters in the bound we have proven for Theorem 1.

First, we will show that the mixing time for this new chain is $\Theta(1)$. We will use $k := \tau_{\text{mix}}(\eta_n^2)$. So by definition the $d_{\text{mix}}(k) \leq \eta_n^2$ using the definition of d_{mix} in Section 2.1. Hence $d_{\text{mix}}(k) \leq 1/4$ using conditions on the learning rate schedule imposed in Theorem 1. Therefore, in the transformed chain, the “new” $\tilde{\tau}_{\text{mix}}$ is $\Theta(1)$.

We also have:

$$k \leq \frac{2\tau_{\text{mix}}}{\log(2)} \log\left(\frac{1}{\eta_n^2}\right) \leq \underbrace{\frac{2\log(4/\pi_{\min})}{\log 2}}_C \frac{1}{1 - |\lambda_2(P)|} \log(n)$$

We see that $C > 1$. Next, we note that for the transition kernel $P^k(\cdot, \cdot)$, the second-largest absolute eigenvalue is given as $|\lambda_2(P)|^k$. Consider the function $f(x) := x^{\frac{1}{1-x}}$ for $x \in (0, 1)$. Then,

$$f'(x) = f(x) \left(\frac{1 - x - x \log(x)}{x(1-x)^2} \right) > 0$$

Therefore, $f(x) < \lim_{x \rightarrow 1} f(x) = \frac{1}{e} < 1$, which implies $|\lambda_2(P)|^k \stackrel{(i)}{\leq} \left(\frac{1}{e}\right)^{C \log(n)} < \frac{1}{e}$. Here (i) follows if $C > 1, n > 3$, which is true. Therefore,

$$\tilde{\mathcal{V}}' := \left(\frac{1 + (3 + 4\epsilon) |\lambda_2(P)|^k}{1 - |\lambda_2(P)|^k} \right) \mathcal{V} \leq 5\mathcal{V}$$

This also implies that the mixing time for the new Markov chain for sub-sampled data is $\Theta(1)$. The bound then follows by substituting n to be $\frac{n}{k} = n_k = \Theta\left(\frac{n}{C\tau_{\text{mix}} \log(n)}\right)$ and setting the τ_{mix} in the original expression of Theorem 1 to a constant. $\square$

S.6 Additional Experiments

In this section, we provide additional experiments to support the results established in Section 3 of the manuscript. We present experiments with distributions that have nonzero mean vectors at each state, but zero mean with respect to the stationary distribution. This means that the Z_i ’s are not necessarily zero-mean with respect to each state distribution $D(s)$. To normalize the data-points, we estimate the mean μ and covariance matrix Σ empirically from a much larger independently generated dataset.

We experiment with two different settings here - Figure S.2 contains the results for each state distribution being $D(s) := \text{Bernoulli}(p_s)$ with $p_s \sim \mathcal{U}(0, 0.05)$ being fixed for each dataset. Figure S.3 provides results for each state distribution being $D(s) := \mathcal{U}(0, \ell_s)$ with $\ell_s \sim \mathcal{U}(0, 10)$ being selected at the start of each random run. We observe that these experiments depict similar trends to those shown in the main manuscript, which validates our results for the case of non-zero state means. Furthermore, the Bernoulli data, being sparse compared to the Uniform one, seems to exhibit a clearer difference between data downsampling and the traditional Oja’s algorithm. To provide clear plots demonstrating the relative behavior of the algorithms considered in this paper, we have shown the averaged $\sin^2$ errors in Figures S.2 and S.3. In Figure S.4 we show six random runs where we fixed the $p_s, s \in \Omega$ for each state for all runs. These figures clearly show that in general, Downsampled Oja has a worse performance than Oja’s algorithm, which has a similar performance as the offline algorithm. It also shows that the Downsampled algorithm has the most variability, whereas Oja’s algorithm on the whole dataset has much less variability, and finally, and not surprisingly, the offline algorithm has the least variability. Similar qualitative trends can be observed for the other settings.

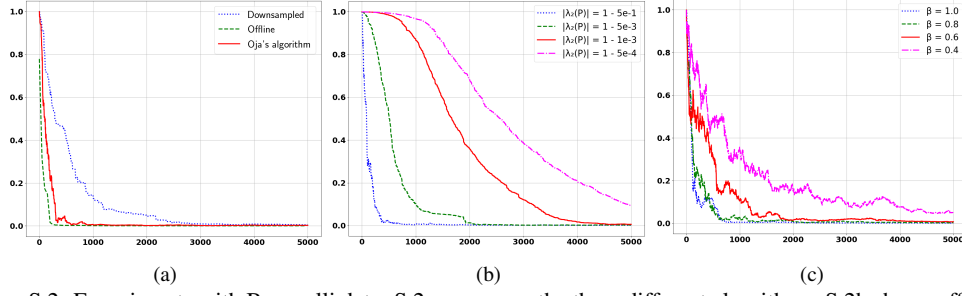


Figure S.2: Experiments with Bernoulli data. S.2a compares the three different algorithms, S.2b shows effect of changing the eigengap of the transition_matrix and S.2c records the variation in performance on changing the eigengap of the data covariance matrix.

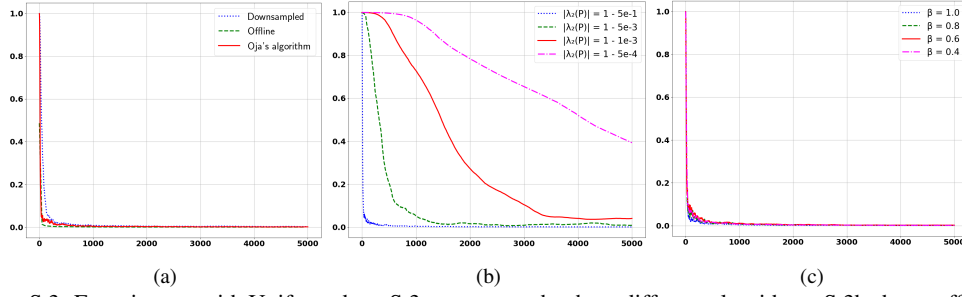


Figure S.3: Experiments with Uniform data. S.3a compares the three different algorithms, S.3b shows effect of changing the eigengap of the transition matrix and S.3c records the variation in performance on changing the eigengap of the data covariance matrix.

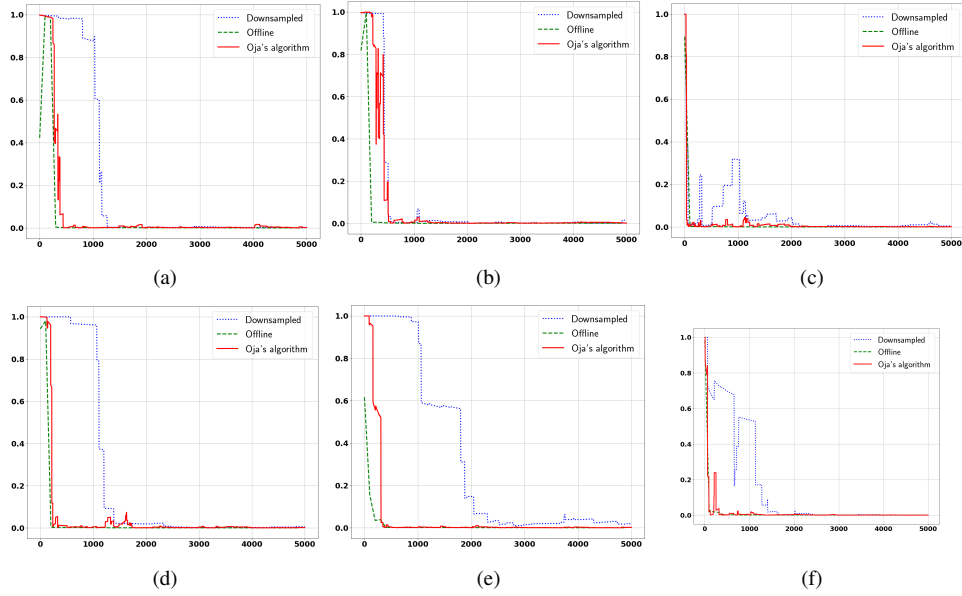


Figure S.4: Randomly chosen runs for the Bernoulli case