On Numerical Semigroups with Almost-Maximal Genus

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Abstract - A numerical semigroup is a cofinite subset of $\mathbb{N}_0$, containing 0, that is closed under addition. Its genus is the number of nonnegative integers that it does not contain. A numerical set is a similar object, not necessarily closed under addition. If T is a numerical set, then $A(T) = \{n \in \mathbb{N}_0 : n+T \subseteq T\}$ is a numerical semigroup. Recently a paper appeared counting the number of numerical sets T where $A(T)$ is a numerical semigroup of maximal genus. We count the number of numerical sets T where $A(T)$ is a numerical semigroup of almost-maximal genus, i.e. genus one smaller than maximal.

Keywords : numerical semigroup; numerical set; genus; atom monoid

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1 Introduction

A numerical set is a cofinite subset of the nonnegative integers $\mathbb{N}_0$ containing 0. A numerical set closed under addition is called a numerical semigroup. The maximum integer missing from a numerical set or semigroup is called its Frobenius number. The number of positive integers that a numerical set or semigroup does not contain is called its genus. Numerical semigroups have been the subject of considerable study (e.g. [2, 4]); for a general reference see [1] or [6].

Let T be a numerical set. Set $A(T) = \{n \in \mathbb{N}_0 : n + T \subseteq T\}$. This is known to be a numerical semigroup, called its atom monoid, with $A(T) \subseteq T$. For a fixed numerical semigroup S, we write $N(S)$ to denote the number of numerical sets T satisfying $A(T) = S$. Numerical sets and their atom monoids have been of interest lately due to their connection with core partitions (see [3]).

Fairly recently [5] appeared, which fixed the Frobenius number f and considered all 2^{f-1} numerical sets with that Frobenius number. It focused on the numerical semigroup with Frobenius number f and maximal genus, i.e. $S_f = \{0, f{+}1, f{+}2, \ldots\} = \{0, f{+}1, \rightarrow\}$.

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It determined bounds on $N(S_f)$, and also found the asymptotic limit $\lim_{f\to\infty} \frac{N(S_f)}{2^{f-1}}$ to be approximately 0.48.

We wish to extend this work with Frobenius number f, from the maximum genus of f to the almost-maximum genus of $f-1$. Hence, we consider the semigroups $S_f(l) = \{0, f-l, f+1, \rightarrow\}$. We call a numerical set T with $A(T) = S_f(l)$ both (f,l)-good and f-good. We set $N(S_f(l))$ to denote the number of (f,l)-good numerical sets, and $N(S_f(\star))$ to denote the number of f-good numerical sets (over all l). We now look for bounds for $N(S_f(l))$ and $N(S_f(\star))$, as well as the asymptotic limit $\lim_{f\to\infty} \frac{N(S_f(\star))}{2^{f-1}}$. We first observe that if $l \geq \frac{f}{2}$, then $(f-l)+(f-l) \in S_f(l)$, as this is a semigroup and hence closed under addition; this will render the result no longer of the desired genus. Hence we must have $l < \frac{f}{2}$, and thus $N(S_f(\star)) = N(S_f(1)) + N(S_f(2)) + \cdots + N(S_f(\lfloor \frac{f-1}{2} \rfloor))$.

For a numerical set T and $x \in T$, we say that y is a witness to x if $y \in T$ and $x+y \notin T$. This leads to a simple characterization of $A(T)$, for all numerical sets.

Proposition 1.1 *Given numerical set T and $x \in T$, $x \notin A(T)$ if and only if there is some witness to x.*

Proof. If y is a witness to x, then $x+y \in x+T$ but $x+y \notin T$, so $x \notin A(T)$. If there is no witness to x, then for all $y \in \mathbb{Z}$, if $y \in T$ then $x+y \in T$; hence $x+T \subseteq T$ and thus $x \in A(T)$. □

Suppose that T is an (f,l)-good numerical set. For $x = f-l$ and for $x > f$, we must have $x \in T$ since $A(T) \subseteq T$. Also, $f \notin T$ since $T, A(T)$ share the same Frobenius number.

We now present a result specific to our $S_f(l)$ context.

Proposition 1.2 *Let T be an (f,l)-good numerical set, and $x \in \mathbb{Z}$. If $x \in T$ then $x+f-l \in T$.*

Proof. If $x+f-l \notin T$, then x would be a witness to $f-l$, and hence $f-l \notin T$. But this is impossible since T is (f,l)-good. □

2 Upper Bounds

In this section we provide some structural information about (f,l)-good sets, as well as an upper bound for their number.

Recall that if T is an (f,l)-good numerical set, then $f-l \in T$. Hence $l \notin T$, or else by Proposition 1.2 we would have $l+(f-l) = f \in T$. Set

$$Y = \{1, 2, \ldots, l-1\} \cup \{l+1, \ldots, f-l-1\} \cup \{f-l+1, \ldots, f-1\},$$

a union of three intervals of length $l-1$, $f-2l-1$, and $l-1$, respectively. All (f,l)-good numerical sets consist of a subset of Y, together with all of $S_f(l)$. Hence, naively we get an upper bound for $N(S_f(l))$ of $2^{|Y|} = 2^{f-3}$. We use Proposition 1.2 to improve this.

Theorem 2.1 *For fixed l, f, the number of (f,l)-good numerical sets $N(S_f(l))$ satisfies*

$$N(S_f(l)) \leq 3^{l-1} 2^{f-2l-1}.$$

Proof. For each $x \in \{1, 2, \ldots, l-1\}$, we have $x + f - l \in \{f - l + 1, \ldots, f - 1\}$. This yields $l - 1$ pairs $\{x, x + f - l\}$. By Proposition 1.2, if T is (f, l)-good and $x \in T$, then $x + f - l \in T$. Hence each pair gives three possibilities: neither element in T, both elements in T, or just $x + f - l \in T$. The fourth possibility, of just $x \in T$, is forbidden. This reduces the naive upper bound by a factor of $(3/4)^{l-1}$. $\square$

Corollary 2.2 *For a fixed f, the number of f-good numerical sets $N(S_f(\star))$ satisfies*

$$N(S_f(\star)) \leq 2^{f-1}\left(1 - \left(\frac{\sqrt{3}}{2}\right)^{f-1}\right).$$

Proof. Set $t = \lfloor (f-1)/2 \rfloor$, and we have

$$N(S_f(\star)) = \sum_{l=1}^{t} N(S_f(l)) \leq \sum_{l=1}^{t} 3^{l-1}2^{f-2l-1} = \frac{2^{f-1}}{3}\sum_{l=1}^{t}\left(\frac{3}{4}\right)^l = \frac{2^{f-1}}{3}\frac{\frac{3}{4} - \left(\frac{3}{4}\right)^{t+1}}{1 - \frac{3}{4}}$$

$$= 2^{f-1}\left(1 - \left(\frac{3}{4}\right)^t\right) \leq 2^{f-1}\left(1 - \left(\frac{3}{4}\right)^{\frac{f-1}{2}}\right)$$

$\square$

Corollary 2.2 bounds $N(S_f(\star))$ away from its maximum value of 2^{f-1}, proving that not all numerical sets are good[1]. Unfortunately, it is not sufficient to bound the asymptotic limit $\lim_{f\to\infty} \frac{N(S_f(\star))}{2^{f-1}}$ away from 1, much less away from 0.52.

3 Lower Bounds

We now turn to a lower bound for $N(S_f(l))$, which we provide in the following.

Theorem 3.1 *For fixed l, f, the number of (f, l)-good numerical sets $N(S_f(l))$ satisfies*

$$N(S_f(l)) \geq 2^{\lceil \frac{l-1}{2} \rceil + \lceil \frac{f-2l-1}{2} \rceil}.$$

Proof. We will define $\lceil \frac{l-1}{2} \rceil + \lceil \frac{f-2l-1}{2} \rceil$ subsets of Y, each of which may independently be included, or not, in an (f, l)-good numerical set.

First, for $x \in \{1, 2, \ldots, \lceil \frac{l-1}{2} \rceil\}$, we consider the set

$$Q_x = \{x, f - x, x + f - l, l - x\}.$$

Note that since $1 \leq x \leq \frac{l}{2}$, $f - \frac{l}{2} \leq f - x \leq f - 1$ and $f - l + 1 \leq x + f - l \leq f - \frac{l}{2}$ and $\frac{l}{2} \leq l - x \leq l - 1$. Consequently, $x \leq l - x < x + f - l \leq f - x$. In particular, $Q_x \neq Q_y$ for $x \neq y$, and $|Q_x| = 4$ (unless $x = \frac{l}{2}$, in which case $|Q_x| = 2$). Also, note that

$$\bigcup Q_x = \{1, 2, \ldots, l-1\} \cup \{f - l + 1, \ldots, f - 1\},$$

[1] Not a major observation, in light of the bound in [5].

leaving the subset $\{l+1, \dots, f-l-1\}$ of Y undisturbed. Note that for each $y \in Q_x$, also $f-y \in Q_x$, and these are witnesses for each other as their sum is $f \notin T$. Hence, if $Q_x \subseteq T$, then $Q_x \cap A(T) = \emptyset$.

Now, for $x \in \{l+1, \dots, \lceil \frac{f-1}{2} \rceil\}$, we consider the set $R_x = \{x, f-x\}$. Note that since $l+1 \leq x \leq \frac{f}{2}$, $\frac{f}{2} \leq f-x \leq f-l-1$. Consequently, $R_x \neq R_y$ for $x \neq y$, and $|R_x| = 2$ (unless $x = \frac{f}{2}$, in which case $|R_x| = 1$). Note that

$$\bigcup R_x = \{l+1, l+2, \dots, f-l-1\},$$

so $R_x \cap Q_y = \emptyset$ for all x, y. For each $y \in R_x$, also $f-y \in R_x$. These are witnesses for each other, and so if $R_x \subseteq T$, then $R_x \cap A(T) = \emptyset$.

Let T contain $S_f(l)$, together with an arbitrary collection of the subsets Q_x, R_x. In particular, $l, f \notin T$ and $f-l \in T$. By the above, $Y \cap A(T) = \emptyset$. It is easy to see that $0 \in A(T)$, $f \notin A(T)$, and $x \in A(T)$ for all $x > f$.

The only remaining concern is to prove that $f-l \in A(T)$. Suppose instead that $f-l \notin A(T)$. Then there would be some witness $y \in T$ with $y+f-l \notin T$. Note that if $y \geq l+1$, then $y+f-l \geq f+1$, and so $y+f-1 \in T$ and y cannot be a witness. In particular, it could not be among the R_x sets. If there is some x with $y \in Q_x$, then either $y = x$ or $y = l-x$ (else $y \geq l+1$ again). But for both of these choices, $y+f-l \in Q_x$ again, so y is again not a witness. Hence $f-l \in A(T)$. $\square$

Corollary 3.2 *For a fixed f, the number of f-good numerical sets $N(S_f(\star))$ satisfies*

$$N(S_f(\star)) \geq \frac{2^{\frac{f-3}{2}}}{\sqrt{2}-1}\left(1-2^{-\frac{f-2}{4}}\right).$$

Proof. We begin with $\lceil \frac{l-1}{2} \rceil + \lceil \frac{f-2l-1}{2} \rceil \geq \frac{f-l-2}{2}$. Set $t = \lfloor (f-1)/2 \rfloor$, and we have

$$N(S_f(\star)) = \sum_{l=1}^{t} N(S_f(l)) \geq 2^{(f-2)/2} \sum_{l=1}^{t} 2^{-l/2}$$

The sum is a geometric series, and thus

$$N(S_f(\star)) \geq 2^{(f-2)/2} \frac{2^{-\frac{1}{2}} - 2^{\frac{-t-1}{2}}}{1-2^{-\frac{1}{2}}} = \frac{2^{\frac{f-3}{2}}}{\sqrt{2}-1}\left(1-2^{-\frac{t}{2}}\right) \geq \frac{2^{\frac{f-3}{2}}}{\sqrt{2}-1}\left(1-2^{-\frac{f-2}{4}}\right)$$

$\square$

Although Corollary 3.2 provides a nontrivial lower bound for $N(S_f(\star))$, it is not sufficient to bound the asymptotic limit $\lim_{f\to\infty} \frac{N(S_f(\star))}{2^{f-1}}$ away from 0. We conjecture that this holds, and, more strongly, that for a fixed l, $\lim_{f\to\infty} \frac{N(l,f)}{2^{f-1}} \in (0,1)$.

We lastly observe that preprint [7] has very recently been made public, extending the above work, addressing our conjectures, and bounding the asymptotic limit $\lim_{f\to\infty} \frac{N(S_f(\star))}{2^{f-1}}$ away from 0.

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