Elasticity in Apéry Sets

Abstract. A numerical semigroup S is an additive subsemigroup of the non-negative integers, containing zero, with finite complement. Its multiplicity m is its smallest nonzero element. The Apéry set of S is the set of elements $\mathrm{Ap}(S) = \{n \in S : n - m \notin S\}$. Fixing a numerical semigroup, we ask how many elements of its Apéry set have nonunique factorization, and define several new invariants.

1. INTRODUCTION Every child's first semigroup is the natural numbers, and their first factorization theorem is the Fundamental Theorem of Arithmetic, which gives unique factorization as a product of primes. The other operation, addition, is not addressed. Much attention has been given to factorization in various semigroups; for a general introduction, see [**10**]. Often, the operation is multiplication [**4, 7, 12**], but addition is worth studying as well [**19**]; it will be our operation here.

A *numerical semigroup* S is a subset of $\mathbb{Z}_{\geq 0}$ with finite complement that is closed under $+$ and contains 0. Numerical semigroups have been the subject of considerable recent study [**8, 11, 14, 15, 16, 17**]. Many applications are known, such as in coding theory [**6**]. For a general introduction to numerical semigroups, see [**3**] or [**18**].

The atoms of a numerical semigroup S are the nonzero elements that cannot be expressed as the combination of two nonzero elements. The set $\mathcal{A}(S)$ of atoms of S is finite; we call $e(S) = |\mathcal{A}(S)|$ the *embedding dimension* of S. We write $\langle a_1, a_2, \ldots, a_k \rangle$, with a_i listed in ascending order, to denote the numerical semigroup with atoms $a_1, \ldots, a_k$. The smallest atom a_1 is also the smallest nonzero element of S; we call it $m(S)$, the *multiplicity* of S.

An important tool for the study of numerical semigroups, from [**2**], is the Apéry set

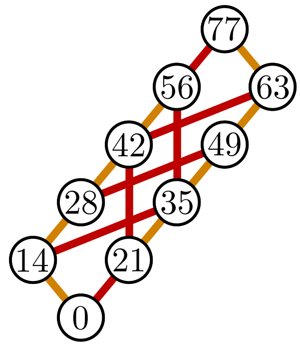
$$\mathrm{Ap}(S) = \{x \in S : x - m(S) \notin S\},$$

which contains the smallest element of S in each congruence class modulo $m(S)$. It is easy to show that $|\mathrm{Ap}(S)| = m(S)$ and $\mathcal{A}(S) \setminus \{m(S)\} \subseteq \mathrm{Ap}(S)$. If we want to express elements of S as a free combination of atoms, $\mathcal{A}(S)$ is what we study. However, if we want to use as many copies of $m(S)$ as possible and other atoms as little as possible, we look to $\mathrm{Ap}(S)$ instead.

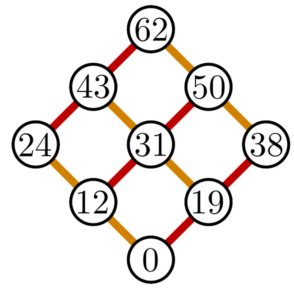
We study properties about factorization into atoms. The most famous factorization invariant is elasticity. Given a semigroup S and some $x \in S$, we write x as the combination of atoms in every possible way. The *elasticity* of x, denoted $\rho(x)$, is the the largest number of atoms that can be used, divided by the smallest number. Clearly $\rho(x) \geq 1$; if equality holds, we say x is *half-factorial*. Conventionally we say that the unit 0 is half-factorial.

In this note, we consider the elasticity function restricted to elements of $\mathrm{Ap}(S)$. We write $\rho(\mathrm{Ap}(S))$ for the maximum elasticity over the elements of $\mathrm{Ap}(S)$; if $\rho(\mathrm{Ap}(S)) = 1$, we say that S is *Apéry half factorial*, or AHF.

We can visualize the factorization structure of $\mathrm{Ap}(S)$ using a partially ordered set $(\mathrm{Ap}(S), \preceq)$ with $x \preceq y$ whenever $y - x \in S$, called the *Apéry poset* of S. The Hasse diagrams of two Apéry posets are depicted in Figure 1. The atoms of the Apéry poset (i.e., the elements directly above the unique minimal element 0) are precisely the elements of $\mathcal{A}(S)$ apart from $m(S)$, and an edge connects x up to y in the Hasse diagram exactly when $y - x \in \mathcal{A}(S)$. This leads to the following interesting observation.



(a) Poset for $S = \langle 10, 14, 21\rangle$, as in Theorem 4 with $n = 2$ and $p = 7$.



(b) Poset for $S = \langle 9, 12, 19\rangle$, as in Theorem 5 with $n = 3$.

Figure 1. Examples of Apéry posets.

Theorem 1. *A numerical semigroup S has graded Apéry poset if and only if S is AHF.*

Proof. By the above discussion, each length ℓ chain (set of mutually comparable elements) from 0 to an element $n \in S$ corresponds to an ordered factorization of n with length ℓ. As such, two different chain lengths are present if and only if n is not half-factorial. ∎

2. APÉRY ELASTICITIES We begin by observing that if $e(S) = 2$, then we can write $S = \langle m, a\rangle$ and $\mathrm{Ap}(S) = \{0, a, 2a, \ldots, (m-1)a\}$. Each element of $\mathrm{Ap}(S)$ is then not only half-factorial, but has unique factorization. On the other hand, if S has *maximal embedding dimension* (i.e., $e(S) = m(S)$), then $S = \langle m, a_1, a_2, \ldots, a_{m-1}\rangle$ and $\mathrm{Ap}(S) = \{0, a_1, a_2, \ldots, a_{m-1}\}$. Again each element of $\mathrm{Ap}(S)$ has unique factorization. These observations are extended slightly in the following.

Theorem 2. *Let S be a numerical semigroup with $e(S) = 2$ or $e(S) \geq m(S) - 1$. Then S is Apéry half-factorial.*

Proof. If $e(S) = m(S) - 1$, then the only element of $\mathrm{Ap}(S)$ that is not an atom only has length 2 factorizations. ∎

Theorem 2 can't be extended in general to smaller embedding dimension than $m(S) - 1$. Consider $S = \langle 5, 6, 9\rangle$, where $m = 5$, $a_1 = 6$, $a_2 = 9$. Now

$$\mathrm{Ap}(S) = \{0, a_1, a_2, 2a_1, 3a_1 = 2a_2\},$$

so $\rho(3a_1) = \frac{3}{2}$.

Given a subset $T \subset S$, define the *set of elasticities* of T as

$$R(T) = \{\rho(n) : n \in T\}.$$

This invariant has been studied for numerical semigroups in [**5**], wherein $R(S)$ is characterized for all but finitely many elasticities coming from "small" elements of S. As such, $R(\mathrm{Ap}(S))$ is a natural starting place for studying the remainder of $R(S)$.

It is easy to see that if $e(S) = m(S) - 2$, either S is AHF or $R(\mathrm{Ap}(S)) = \{1, \frac{3}{2}\}$. Determining $R(\mathrm{Ap}(S))$ for other near-maximal embedding dimensions remains open.

We now present a family of semigroups in Theorem 3 which demonstrate several extremal behaviors, as discussed thereafter.

Theorem 3. *Fix $a > b \geq 1$ with $\gcd(a, b) = 1$. There is a numerical semigroup S with (i) $R(\mathrm{Ap}(S)) = \{1, \frac{a}{b}\}$ and (ii) only one element of $\mathrm{Ap}(S)$ has elasticity $\frac{a}{b}$.*

Proof. Fix a prime $p \nmid (a+b)$ with $a + b < pb$, and let $S = \langle a+b, pa, pb\rangle$. We have

$$\mathrm{Ap}(S) = \{0, pb, 2pb, \ldots, (a-1)pb, pa, 2pa, \ldots, (b-1)pa, pab\},$$

wherein each element has unique factorization except pab, which has elasticity $\frac{a}{b}$. ■

One natural question to ask is: which subsets of $\mathbb{Q}_{\geq 1}$ can occur as $R(\mathrm{Ap}(S))$ for some numerical semigroup S? Certainly we must have $1 \in R(\mathrm{Ap}(S))$, and the sole singleton subset, $\{1\}$, is achieved for all Apéry half-factorial S. All subsets of size two are realizable by Corollary 1. Larger subsets of $\mathbb{Q}_{\geq 1}$ remain unresolved.

Corollary 1. *Given $r \in \mathbb{Q}_{>1}$, some numerical semigroup S has $R(\mathrm{Ap}(S)) = \{1, r\}$.*

Proof. Write $r = \frac{a}{b}$ in reduced form, and apply Theorem 3. ■

Since $\mathrm{Ap}(S)$ is a finite set, we can consider the full distribution of elasticity over its elements, and not just its maximum $\rho(\mathrm{Ap}(S))$. We call the *Apéry half-factorial fraction*, or AHFF, the ratio of the number of half-factorial elements of $\mathrm{Ap}(S)$, to $|\mathrm{Ap}(S)|$. If S is AHF, then its AHFF is 1.

Theorem 3 produced a single non-half-factorial element of $\mathrm{Ap}(S)$; hence S had AHFF close to 1. Certainly the AHFF cannot be zero, as each element of $\mathcal{A}(S)$ is half-factorial. One wonders how small the AHFF can be. Theorem 4, illustrated in Figure 1(a), displays the smallest possible AHFF while maintaining $e(S) = 3$.

Theorem 4. *The fraction of Apéry set elements of a numerical semigroup that are half-factorial can be arbitrarily close to 0.*

Proof. Let $p, n \in \mathbb{Z}_{\geq 1}$ with p prime, $p \neq 5$, and $2p > 5n > 5$. Set $m = 5n$, $a_1 = 2p$, $a_2 = 3p$, and take $S = \langle m, a_1, a_2\rangle$. We have

$$\mathrm{Ap}(S) = \{0, 2p, 3p, \ldots, (5n-1)p, (5n+1)p\}.$$

Since $6p = 3a_1 = 2a_2$, only 0, $2p$, $3p$, $4p$, $5p$ and $7p$ are half-factorial in $\mathrm{Ap}(S)$. As such, the AHFF of S is $\frac{6}{5n}$. ■

With the generality of the family in Theorem 3, one might wonder if any S with $e(S) = 3$ can be AHF. One such family is provided in Theorem 5, an example of which is illustrated in Figure 1(b).

Theorem 5. *For each $n \in \mathbb{Z}_{\geq 2}$, the semigroup $S = \langle n^2, n^2 + n, 2n^2 + 1\rangle$ is AHF.*

Proof. $\mathrm{Ap}(S) = \{a(n^2 + n) + b(2n^2 + 1) : 0 \leq a, b \leq n - 1\}$. ■

Theorem 5 also demonstrates that the width of the Apéry poset, which is always bounded below by $e(S)$, can be larger.

Corollary 2. *The width of an Apéry poset can be arbitrarily large, even for $e(S) = 3$.*

3. MEAN APÉRY ELASTICITY Motivated in part by recent investigations into "average" factorization lengths in numerical semigroups [**9**], we next consider the *mean Apéry elasticity*, i.e.,

$$MAE(S) = \frac{1}{|\operatorname{Ap}(S)|} \sum_{n \in \operatorname{Ap}(S)} \rho(n).$$

If S is half-factorial, of course $MAE(S) = 1$. The family from Corollary 1 has mean Apéry elasticity $1 + \frac{1}{b} - \frac{2}{a+b}$. Theorem 6 will show that mean Apéry elasticity may be arbitrarily large, though one may still wonder which elements of $\mathbb{Q}_{\geq 1}$ occur as $MAE(S)$ for some numerical semigroup S.

Theorem 6. *The values of $MAE(S)$, with $\mathrm{e}(S) = 3$, can be arbitrarily large.*

Proof. Let p, q be odd primes with $p > 2q + 4$. Set $m = 4q + 8$, $a_1 = 2p$, $a_2 = qp$, and take $S = \langle m, a_1, a_2 \rangle$. We have

$$Ap(S) = \{0, 2p, 4p, \ldots, (q-1)p, qp, (q+1)p, \ldots, \tfrac{1}{2}(9q+17)p\},$$

where all multiples of p are present after qp except $(4q + 8)p$. Now, consider the set

$$T = \{(2q + 2i)p : 0 \leq i < q\} \subset \operatorname{Ap}(S).$$

We calculate elasticity of the elements of T as

$$\rho((2q + 2i)p) = \rho((q + i)a_1) = \rho(2a_2 + ia_1) = \frac{q + i}{2 + i} \geq \frac{q}{2 + i},$$

and consequently

$$MAE(S) = \frac{1}{m} \sum_{n \in \operatorname{Ap}(S)} \rho(n) \geq \frac{3q + 8}{m} + \frac{1}{m} \sum_{n \in T} \rho(n) \geq \frac{3q + 8}{m} + \frac{q}{m} \sum_{i=0}^{q-1} \frac{1}{2 + i}$$

grows arbitrarily large as $q \to \infty$. ∎

4. ASYMPTOTIC DISTRIBUTIONS Given a numerical semigroup S, denote by $g(S) = |\mathbb{Z}_{\geq 0} \setminus S|$ the *genus* of S. Let n_g denote the number of numerical semigroups with genus g, and let $n_{m,g}$ denote the number of numerical semigroups with multiplicity m and genus g. For example, letting f_g denote the g'th Fibonacci number, it was recently proven that n_g/f_g approaches a constant as $g \to \infty$ [**20**], although it is still open whether $n_{g+1} \geq n_g$ for every $g \geq 0$. On the other hand, for fixed m, the ratio $n_{m,g}/g^{m-1}$ approaches a constant as $g \to \infty$.

There has been a recent push to understand the distribution of numerical semigroups with a given genus across different special families. For example, if M_g and $M_{m,g}$ denote, respectively, the number of maximal embedding dimension numerical semigroups with genus g and the number with both multiplicity m and genus g, then $M_g/n_g \to 0$ as $g \to \infty$, while $M_{m,g}/n_{m,g} \to 1$ as $g \to \infty$; see [**1, 13**].

Continuing in this vein, let h_g denote the number of AHF numerical semigroups with genus g, and let $h_{m,g}$ denote the number of AHF numerical semigroups with multiplicity m and genus g. Theorems 7 and 8 below demonstrate that AHF numerical semigroups form a much larger class than those with maximum embedding dimension.

Theorem 7. *For each fixed $m \geq 2$, we have*

$$\lim_{g \to \infty} \frac{h_{m,g}}{n_{m,g}} = 1.$$

Proof. Apply [**1**, Corollary 1]. ■

Identifying the precise value of the limit below will likely be challenging, considering the long and technical nature of the proof of [**20**, Theorem 1]. Out of the 1179593 numerical semigroups with genus at most 25, we find 1032971 (about 88%) are AHF.

Theorem 8. *We have*

$$0 < \lim_{g \to \infty} \frac{h_g}{n_g} < 1.$$

Proof. Let f_n denote the n'th Fibonacci number. By [**20**, Theorem 1], we have

$$\lim_{g \to \infty} \frac{f_{g+1}}{n_g} > 0.$$

As such, for the first inequality, it suffices to show that $f_{g+1} \leq h_g$. Fix a multiplicity $m \leq g + 1$. For each subset $T \subset \{1, \ldots, m-1\}$, consider the numerical semigroup S with Apéry set given by $\operatorname{Ap}(S) = \{0, a_1, \ldots, a_{m-1}\}$, where

$$a_i = \begin{cases} 2m + i & \text{if } i \in T; \\ m + i & \text{if } i \notin T. \end{cases}$$

It is clear S has multiplicity m and genus $m + |T|$, and is AHF. As such,

$$h_g = \sum_{m=2}^{g+1} h_{m,g} \geq \sum_{m=2}^{g+1} \binom{m-1}{g-(m-1)} = f_{g+1}.$$

For the other inequality, we use a similar construction, where we first let $a_1 = m + 1$, $a_2 = 2m + 2$, $a_3 = 3m + 3$, $a_4 = m + 4$, and $a_{m-1} = m + (m-1)$, and then choose the remaining a_i as above. In each resulting semigroup,

$$a_3 = 3a_1 = a_4 + a_{m-1}$$

is not half-factorial, and by similar reasoning to above, this family of semigroups also comprises a positive asymptotic proportion of those with genus g. ■

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