Abstract Dynamics: A Progressive Linearization of Nonlinear Dynamics

Amir Shahhosseini[1] and Kiran D'Souza[1]

The Ohio State University, Columbus, Ohio 43235, USA,
Shahhosseini.3@osu.edu

Abstract. This work proposes a computational approach that has its roots in the early ideas of local Lyapunov exponents, yet, it offers new perspectives toward analyzing these problems. The method of interest, namely abstract dynamics, is an indirect quantitative measure of the variations of the governing vector fields based on the principles of linear systems. The examples in this work, ranging from simple limit cycles to chaotic attractors, are indicative of the new interpretation that this new perspective can offer. The presented results can be exploited in the structure of algorithms (most prominently machine learning algorithms) that are designed to estimate the complex behavior of nonlinear systems, even chaotic attractors, within their horizon of predictability.

Keywords: Nonlinear Dynamics, Abstract Dynamics, Cardinal Lines, Fictitious Coordinates

1 Introduction

The existing literature on the analysis of dynamical systems, both linear and nonlinear, can be divided into two categories. The first category attempts to use analytical approaches for studying the governing equations and thus, either directly obtains responses and their corresponding properties or alternatively, definitive characteristics of the response. The evaluations of the eigenvalues in the case of linear time-invariant systems and obtaining the analytical solutions of nonlinear systems, if possible, can be considered examples of this category. The second category uses numerical approaches to examine the behavior of the systems [1]. Computational methods that use time responses obtained by numerical integration, spectral frequency analysis, and Poincare maps can be counted as the most common methods in this category [2, 3]. There are also numerous approaches that are used for all types of specialized cases (e.g., periodic responses [4], piecewise linear systems [5] etc.).

An interesting idea in the analysis of nonlinear dynamics that stems from the case of linear systems is the notion of Lyapunov exponents. Lyapunov exponents are a measure of the exponential divergence of the trajectory from its original path upon experiencing an infinitesimal disturbance [6]. The Lyapunov exponents are characteristics of the attractor and determine the general expansion or attraction of the motion in the long run and are global properties that are independent of the trajectory if the dynamics is ergodic [7]. In sharp contrast, the idea of local Lyapunov exponents (LLEs) attempts to truncate the global approach of the Lyapunov exponents to the case of the divergence of infinitesimally perturbed trajectories in a finite (and mostly short-term) time interval [8]. This idea is of significant importance when the short-term behavior of nonlinear dynamics is of interest. For many chaotic systems, such as the models of climate and economic dynamics, long-term prediction is proven to be impossible and frankly unnecessary but short-term predictions play a critical part [9]. The accuracy and validity of the short-term predictions can then be estimated using LLEs. In fact, LLEs are introduced to be a measure of the local unpredictability of the motion.

As previously discussed, LLEs have been proposed as a localization of the Lyapunov exponents and thus, both their notion and computation rely on the fundamental idea of Lyapunov exponents. In his pioneering studies [9], Abarbanel uses the multiplicative ergodic theorem of Oseledec [10] to propose a computational framework for the calculation of the LLEs that observe a trajectory and its corresponding variation after applying an infinitesimal perturbation a few steps ahead. In his work, Abarbanel studies the LLEs induced by the two-step-ahead and fifty-step-ahead settings. Nevertheless, Abarbanel was focused on the time evolution of the individual LLEs and also neglected the analysis of one-step-ahead LLEs. As will be discussed later, the LLEs that are computed in these works are obtained by the truncation of the Oseledec matrix and suffer from conceptual shortcomings. In contrast, this work argues that such attempts are flawed and thus, focuses on the variational equation [6] and evaluates and examines the eigenvalues of the local Jacobian matrix as a measure of unpredictability. The contribution of this work is the observation of patterns, and more specifically, dependencies in the eigenvalues of the Jacobian matrix when evaluated along the trajectory of the system. This observation stems from the fact that contrary to previous works, the time evolution of the eigenvalues is not the primary target but their relative behavior is examined. It is then observed that in nearly all the examples, there are areas, in the space of these eigenvalues, called the *abstract dynamics* space, where the eigenvalues vary linearly. Additionally, the one-to-one mappings of the eigenvalues to the trajectory of the state variables indicate an area-to-area mapping.

Apart from the intriguing mathematical implications of these dependencies, these relations can also be used within the structure of estimators to improve their prediction capabilities. Most notably, it is possible to estimate and reconstruct all eigenvalues from one in specific regions, and by adding abstract dynamics as a feature of machine learning algorithms, it would be possible to offer local predictions that are more accurate.

In this work, the concept of abstract dynamics is implemented via the progressive linearization of the nonlinear dynamics along its response. The proposed methodology can accurately identify the level of complexity of the motion (e.g., its level of aperiodicity and the randomness of the motion), the inter-dependency of the variables of the system, and nonlinear phenomena such as period doubling.

Moreover, the results of this work demonstrate the fading of the transient response of certain dynamical entities when the system is observed through this methodology. Inexorably, this can advance and simplify the identification process of numerous systems using raw numerical data.

2 Methodology

A discussion on preliminary concepts of Lyapunov exponents is offered in this section that highlights the fundamental idea and the existing shortcomings in the definition of LLEs. Additionally, the methodology for obtaining abstract dynamics is presented in this section.

2.1 Preliminaries

Lyapunov exponents are a quantitative measure of the average rate of exponential divergence of trajectories that are perturbed infinitesimally [7]. To provide intuition, consider the trajectory $\mathbf{x}_1(t)$ of a nonlinear dynamical system and an alternative trajectory $\mathbf{x}_2(t)$ that is infinitesimally ($\Delta\mathbf{x}(0)$) perturbed in its initial condition at time zero. So mathematically speaking,

$$\mathbf{x}_2(0) = \mathbf{x}_1(0) + \Delta\mathbf{x}(0) \tag{1}$$

As the trajectories evolve in time, the distance between the two trajectories increases and it is possible to denote the divergence as ($\Delta\mathbf{x}(t)$) and track it in time. It can be shown that the divergence can be approximated by

$$||\Delta\mathbf{x}(t)|| = ||\Delta\mathbf{x}(0)||e^{\lambda t}, \tag{2}$$

where λ is the local rate of expansiveness. More rigorously, the maximum Lyapunov exponent is defined through a double limit as

$$\lambda_{max} = \lim_{t\to\infty}(\lim_{\Delta\mathbf{x}(0)\to 0}\frac{1}{t}ln(\frac{\Delta\mathbf{x}(t)}{\Delta\mathbf{x}(0)})). \tag{3}$$

However, to calculate the entire Lyapunov spectrum which consists of all the Lyapunov exponents, alternative approaches must be considered [11]. It is noteworthy to state that the extension of this notion to discrete-time is also very simple. Owing to the Oseledec multiplicative theorem [10], it is possible to obtain the Lyapunov exponents of dynamical mappings by constructing the Oseledec matrix. This idea has been widely used within the literature and is among the most principal approaches toward this problem. Oseledec proved that the natural logarithm of the eigenvalues of this matrix is the Lyapunov exponents of the dynamical mappings of the interest.

This idea has a firm intuitive notion behind it and thus, the Lyapunov exponents can be used for the characterization of the global behavior of the attractors. A list of correspondences between the Lyapunov exponents and the type of attractor is provided in the study [12]. Nevertheless, the definition of the local

Lyapunov exponents is provided via the truncation of Oseledec's matrix to provide a local sense of the sensitivity of the behavior to perturbations [9]. This localization is trivially contingent upon both the initial condition of the system and the number of steps used for the evaluation of the Oseledec matrix. To address the initial condition dependency, Abarbanel used an invariant density and obtained LLEs with two steps and fifty steps in their evaluation. Nevertheless, the validity of the truncated Oseledec matrix's eigenvalues as a measure of predictability is not simply justifiable. Additionally, due to the alteration of the strength of the exponential divergence along the attractor [1], the value of the LLE becomes heavily dependent on the number of steps used for its computation and this dependency defeats the purpose of a localized measure.

An alternative view of the effects of the perturbation, which is in fact the core of Oseledec's matrix is the variational equation. The variational equation describes the development of the perturbation dynamics and is as follows

$$\delta\dot{\mathbf{x}} = J(x_0)\delta\mathbf{x}, \tag{4}$$

Where $J(x_0)$ is the Jacobian matrix of the flow at the point of initial infinitesimal perturbation. Note that the local behavior of the perturbation is governed by the eigenvalues and eigenvectors of the Jacobian matrix. In fact, and contrary to the artificial definition of the LLEs, these eigenvalues are the real measures of the variability and unpredictability of the motion!

Trivially, the importance of the eigenvalues of the Jacobian matrix along the response of the system is not something that is the novel contribution of this work. Nevertheless, the general perspective in the analysis of these eigenvalues, in the existing literature, is to examine their evolution versus time. However, in this work, the evolution of the eigenvalues of these local Jacobian matrices are visualized and interpreted versus each other and the abstract dynamics space is introduced. The rest of this section revolves around the computation and analysis of this space.

2.2 Overall Structure

The process of computing abstract dynamics begins with the calculation of the response, using numerical methods [7] (such as numerical integration) or measurement of the response if studying a system experimentally. Agorithm 1 demonstrates the procedure of obtaining the abstract dynamics.

It might seem counterintuitive to mention that the process of obtaining the eigenvalues, even in the case of high dimensional systems, is computationally inexpensive. This is due to the continuous behavior of the dynamics and consequently, the response. Inevitably, the eigenvalues at each step are close to the eigenvalues of the previous step (except in a few exceptional cases) since they are solutions to the continuous polynomialcharacteristics equation of the instantaneous Jacobian matrix. Therefore, by exploiting methodologies with their performance tied to the accuracy of the provided initial guesses, such as the nonlinear least square method, it would be possible to obtain these eigenvalues with minimal computational costs.

Algorithm 1 Computing Abstract Dynamics

1: **Input:** The Governing Vector Field $\mathbf{f}(\underline{\mathbf{x}})$, Physical State Variables $\underline{\mathbf{x}}(\mathrm{t})$
2: **Output:** Abstract Dynamics Vector $\underline{\Upsilon}$ with the same dimensionality of the system
3: Compute the Jacobian matrix $J(\underline{\mathbf{x}})$ symbolically as a function of $\underline{\mathbf{x}}$ using $\mathbf{f}(\underline{\mathbf{x}})$,
4: **for all** steps of the response $\underline{\mathbf{x}}(t)$ **do**
5: Obtain the response of the system via numerical methods using $\underline{\mathbf{f}}(\mathbf{x})$,
6: Update the symbolic Jacobian matrix $J(\underline{\mathbf{x}})$ with instantanous values of $\underline{\mathbf{x}}(t)$ at every step to derive the local Jacobian matrix,
7: Calculate the local eigenvalues $\underline{\Upsilon}$ and eigenvectors,
8: **end for**
9: Abstract Dynamics can be illustrated by the graphical depiction of $\underline{\Upsilon}$ for two- and three- dimensional systems.

3 Results

To demonstrate the insight provided by abstract dynamics, a number of classical nonlinear systems are examined in this section.

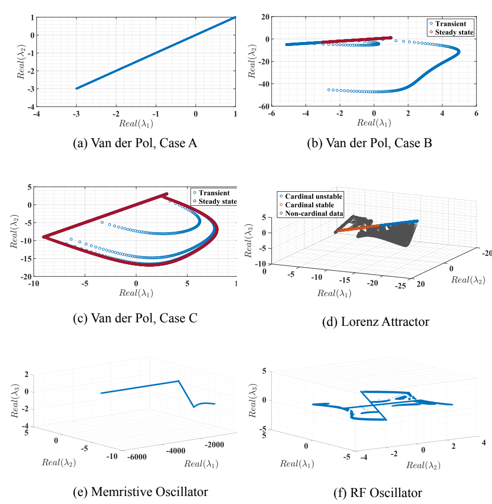

Fig. 1. The abstract dynamics of the discussed systems. (a) The abstract dynamics of the Van der Pol system corresponding to case A; (b) The abstract dynamics of the Van der Pol system corresponding to case B with highlighted steady-state response; (c) The abstract dynamics of the Van der Pol system corresponding to case C with highlighted steady-state response; (d) The abstract dynamics of the Lorenz system with highlighted cardinal line; (e) The abstract dynamics of the memristive oscillator; (f) The abstract dynamics of the Rabinovich-Fabrikant attractor.

3.1 Simple Periodic Motion of a Limit Cycle: The Van der Pol Oscillator

The first system of interest will be the famous limit cycle of the Van der Pol oscillator. The governing equation of motion is

$$\ddot{x} + c(x^2 - 1)\dot{x} + kx = 0, \tag{5}$$

and the constant parameters are $k = 100$ and $c = 2$ for case A.

The results are insightful yet exceptionally peculiar. Primarily, the transient part of the response is dissolved in abstract dynamics and cannot be observed as a separate part of motion. In fact, abstract dynamics illustrates the general qualitative behavior of the system. Case (A) and Case (B) are the abstract dynamics of the same system with initial conditions inside and outside the limit cycle.

The possibility of the dissolution of the transient response in abstract dynamics can be used toward the development of plausible and practical methods for the efficient identification of various nonlinear systems using raw data only. Thus, by solely using the transient part of the response, the abstract dynamics could predict the existence of an upcoming limit cycle.

Finally, it is beneficial to disambiguate a possible misconception. Abstract dynamics is not necessarily simplistic and is a reflection of the complexity of the motion. In fact, by changing the parameter c from 2 to 6 in Eq. (1), the new abstract dynamics of Fig. 1(c) will be obtained which represents case (C).

In Fig. 1(c), contrary to Fig. 1(a-b), there exists a curve in addition to the main line in the steady state part of the abstract dynamics as well. Conclusively, there is no necessity for the motion to reside solely on the straight line and can have a different behavior. Nevertheless, it will remain a closed loop (and simple) for periodic dynamics.

3.2 Scrutinizing Abstract Dynamics by Chaos: The Lorenz Attractor

The behavior of the Lorenz attractor is sophisticated, aperiodic, and chaotic. It does not follow any specific patterns and its only constraint is its boundedness. Since the Lorenz attractor is a 3D system, its trajectory also occupies a 3D space and ergo, it is logical to expect its abstract dynamics to be a 3D entity as well. Additionally, the non-repetitive nature of the motion suggests a high degree of aperiodicity and therefore, a rather complex structure for the Lorenz's abstract dynamics. Figure 1(d) illustrates the abstract dynamics of the Lorenz attractor with parameters $\rho = 28$, $\beta = \frac{8}{3}$ and $\sigma = 10$.

More peculiarities can be observed. The first peculiarity is the 2D nature of the abstract dynamics in its entirety. As it can be seen from Fig. 1(d), abstract dynamics is the graphical representation of the *real* part of the local eigenvalues plotted versus each other. For certain parts of the Lorenz system, this 2D nature of the abstract dynamics can be easily justified via basic linear algebra. However, for the case where all the eigenvalues are real, there exists no such constraint and this raises several questions. A captivating conjecture can be the existence of a hidden constraint in the dynamics of the Lorenz attractor that explains the dependency of the local eigenvalues.

Alternatively, there seems to be a very important straight line in the abstract dynamic that forms a considerable portion of it. In fact, the said line, highlighted in Fig. 1(d), is named the "cardinal line" in this work due to its significant importance. Nevertheless, the conclusion of a heavier hidden constraint in this part is superficial since the provided abstract dynamics solely accounts for the real part. When the motion is on the cardinal line, the local eigenvalues have imaginary parts as well and therefore, their time evolution also effects the behavior of the system.

A more rigorous explanation of this conjecture, namely the existence of a hidden constraint, requires one to delve a bit deeper into the mathematics of abstract dynamics. For the sake of simplicity, the case of motion on the cardinal line is considered. The local linearized decoupled governing equation, which is accurate in its very limited neighborhood, can be written as

$$\dot{z}_i = \lambda_{z_i}(z_i - z_i^0) + f_{z_i}^0, \tag{6}$$

where λ_{z_i} describes the eigenvalue that corresponds to the i^{th} fictitious coordinate and $f_{z_i}^0$ and z_i^0 are the corresponding fictitious vector field and the i^{th} fictitious coordinate at that instant, respectively.

It is possible to obtain the analytical solution to this ordinary differential equation as

$$z_i = z_i^0 + \frac{f_{z_i}^0}{\lambda_{z_i}}(e^{\lambda_{z_i} t} - 1). \tag{7}$$

Now, it is possible to consider a discrete approximation in the evaluation of the response. Then, the expression of Eq. (5) can be written as

$$z_i^1 = z_i(t = 1 \times h) = z_i^0 + \frac{f_{z_i}^0}{\lambda_{z_i}}(e^{\lambda_{z_i} h} - 1), \tag{8}$$

similarly

$$z_i^2 = z_i(t = 2 \times h) = z_i^1 + \frac{f_{z_i}^1}{\lambda_{z_i}}(e^{\lambda_{z_i} h} - 1). \tag{9}$$

By doing so and writing the answer recursively, it is possible to obtain the response for an extended amount of time as

$$z_i^n = z_i(t = n \times h) = z_i^0 + \sum_{j=0}^{n-1} \frac{f_{z_i}^j}{\lambda_{z_i}^j}(e^{\lambda_{z_i}^j h} - 1), \tag{10}$$

where superscripts indicate the step in time and subscripts indicate the index of the coordinate (which corresponds to the decoupled differential equation).

As it can be concluded, when the eigenvalues are connected, the solutions are not independent of each other. In a limited neighborhood, the variation of the vector field can be considered negligible and therefore, the solutions depend solely on the eigenvalues. The dependency of the eigenvalues results in the dependency of the solutions in the decoupled coordinates. This means that the fictitious coordinates of the Lorenz system only have two truly independent variables.

If the points on the cardinal line of the abstract dynamics are identified and their corresponding state space variables, in trajectory space, highlighted, the trajectory of the Lorenz system can be seen to have a region-to-region mapping with the abstract dynamics as illustrated in Fig. 2. Additionally, the distinction between these two segments in the trajectory space is also very clear. Consequently, it is possible to predict the motion to a great extent when it lies on the cardinal line.

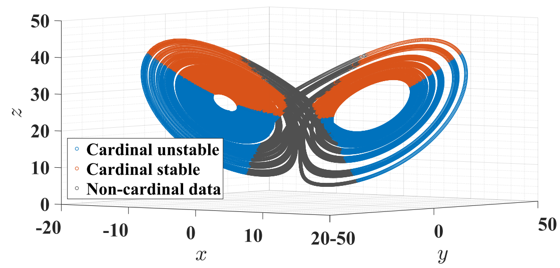

Fig. 2. The trajectory of the Lorenz attractor and the existing region-to-region mapping with the abstract dynamics.

3.3 Chaotic or Nonchaotic, Periodic or Aperiodic

The next two systems of interest are the memristive oscillator of study [13] ($a = 0.0895$, $b = 2.4082 \times 1e - 5$ and $k = 3.2$) and the Rabinovich–Fabrikant attractor ($a = 0.1$ and $b = 0.2875$) examined in study [14]. The memristive system is chaotic since it has a positive Lyapunov exponent but its motion is very close to periodic. In contrast, the Rabinovich–Fabrikant attractor is nonchaotic yet its behavior is comparatively more aperiodic if meticulously observed in the trajectory space.

4 Conclusion

This paper provides a novel perspective for the analysis of continuous smooth nonlinear dynamical systems. The presented idea of *abstract dynamics* facilitates the understanding of the behavior of dynamical systems by offering acute simplified contexts. It has also been argued that abstract dynamics can be used within the structure of alternative methodologies for the prediction of motion

Nonlinear System	Abstract Dynamics	Notes and Discussions
Van der Pol - Case A	The abstract dynamics is a straight line that includes both the transient portion of the response and the steady-state portion of the response.	The initial condition lies within the limit cycle. The simple geometry of the abstract dynamics verifies the simplicity of the motion. The dissolution of the transient portion of the response is particularly interesting.
Van der Pol - Case B	The abstract dynamics consists of a straight line with certain additional curvatures. The straight lines account for the steady-state response and the curvatures represent the transient portion of the response.	The initial condition lies outside the limit cycle. The simple geometry of the abstract dynamics verifies the simplicity of the motion. Contrary to the previous case, there exists no dissolution of the transient portion of the response.
Van der Pol - Case C	The abstract dynamics consists of a straight line with certain additional curvatures. Contrary to case B, part of the curvature accounts for the steady-state motion and verifies the relatively more complicated motion.	The initial condition lies within the limit cycle. The simple geometry of the abstract dynamics verifies the simplicity of the motion. The increased complexity of the abstract dynamics, compared to Cases A-B, stems from the increased complexity of the motion.
Lorenz Attractor	The abstract dynamics consists of a complex geometry that fills a 2D area. A significant portion of the motion also lies on a line that represents the contingency of the fictitious coordinates and consequently, the physical variables.	The existence of the “Cardinal Line” and the 2D nature of the abstract dynamics indicates the existence of a hidden constraint acting upon the dynamics of the Lorenz attractor. The correspondence of the motion while on the cardinal line with the trajectory space can facilitate the development of predictive ML algorithms.
Memristive Oscillator	The abstract dynamics consists of two straight lines and a small limited curvature with a very minor area where the two straight lines meet. The simplicity of abstract dynamics indicates the simplicity and predictability of the motion.	The system, despite being chaotic due to having a positive Lyapunov exponent, has a very close to periodic behavior. This example elucidates that the concept of abstract dynamics revolves around the degree of the complexity of the motion and not necessarily its regime.
Rabinovich–Fabrikant Attractor	The abstract dynamics includes a cardinal line similar to all the other systems yet due to the considerable variations of the vector fields along the trajectory, the abstract dynamics also fills certain 2D areas.	The filled areas are comparatively smaller than that of the Lorenz system that indicates the much simpler motion. The comparison of the Memristive oscillator and this attractor highlights the tight connection of abstract dynamics with the complexity of motion.

even in the case of chaotic attractors. The elimination of the transient portion of the response in certain dynamics, when examined using abstract dynamics, can also be a promising property that can be utilized in the structure of data-driven system identification methodologies.

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