

KINETIC SCHAUDER ESTIMATES WITH TIME-IRREGULAR COEFFICIENTS AND UNIQUENESS FOR THE LANDAU EQUATION

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ABSTRACT. We prove a Schauder estimate for kinetic Fokker-Planck equations that requires only Hölder regularity in space and velocity but not in time; we require only measurability in time. This allows us sidestep a major technical issue for kinetic equations by decoupling the time, space, and velocity variables, which are intertwined by the transport operator. As an application, we consider the spatially inhomogeneous Landau equation. Leveraging the convolutional nature of the coefficients (which yields extra v -regularity) and applying our new estimates, we deduce a weak-strong uniqueness result of classical solutions beginning from initial data having Hölder regularity in x and only a logarithmic modulus of continuity in v . This replaces an earlier result requiring Hölder continuity in both variables and indicates that well-posedness requires less regularity than previously thought.

1. Introduction. This paper is concerned with the regularity of kinetic Fokker-Planck type equations of the form

$$(\partial_t + v \cdot \nabla_x) f = \text{tr}(\bar{a} D_v^2 f) + \bar{b} \cdot \nabla_v f + \bar{c} f + g \quad \text{in } (0, T) \times \mathbb{R}^d \times \mathbb{R}^d \quad (1)$$

and the applications of this regularity theory to the Landau equation, which, roughly, is a fundamental model from gas dynamics for the evolution of a density of colliding particles [51, 59].

Interest in the regularity of equations of the form (1) dates back to Kolmogorov [46], who studied it with the choices $\bar{a} = \text{Id}$, $\bar{b} = v$, and $\bar{c} = d$. Kolmogorov explicitly computed the fundamental solution, which readily yields smoothing¹ of f in all variables despite only being elliptic in the v -variable. We note two other computations of the fundamental solution in more general settings by Il'in [40] and Weber [63]. Eventually the observation that, in the setting of (1), regularity in v transfers to

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¹It does not seem to be explicitly stated in [46] that Kolmogorov noticed the smoothing effect. As a result, it is not clear when this “hypoelliptic” behavior was first identified in the simple setting Kolmogorov considered.

regularity in x due to the transport operator $\partial_t + v \cdot \nabla_x$ led to Hörmander's development of the theory of hypoellipticity [39].

Over the past few decades, a robust understanding of the role of the transport operator $\partial_t + v \cdot \nabla_x$ in regularity theory has been developed in the setting of Sobolev spaces. The literature is truly vast, so we only cite a few prominent examples [8, 26, 60]; although, we encourage the reader to explore the references therein and the work that developed as a result.

More recently, there has been interest in precise quantitative estimates of regularity of solutions to kinetic equations in analogy with the regularity theory for parabolic equations. In particular, the interest has been in the development of estimates in *continuity spaces*, such as Hölder spaces. A suitable Harnack inequality has been proven [25, 31, 61, 62, 65], which yields the Hölder regularity of solutions to *divergence form* kinetic Fokker-Planck equations when the coefficients are merely bounded and elliptic-in- v (note that (1) is in *non-divergence form*). A Harnack inequality for non-divergence form kinetic operators remains elusive [56].

Additionally, Schauder estimates have been deduced and applied to various kinetic models [9, 17, 36, 41, 50] (see also [35, 42, 43] for estimates in the kinetic integro-differential setting). These estimates yield bounds on higher Hölder regularity of the solution as long as the coefficients \bar{a} , b , and \bar{c} are Hölder continuous in all variables.

Our interest here is to investigate the minimal assumptions on the coefficients in (1) for proving the Schauder estimates. As we detail below, this is inspired by the connection between this question and the conditions needed to prove uniqueness of solutions to the Landau equation. Indeed, despite being nonlinear, the coefficients of the highest order terms in the Landau equation enjoy *better* regularity in v than f does. It is, thus, natural to hope that only regularity in v is necessary to prove the Schauder estimates. We note that there are a number of related equations with similar structure to which the methods developed here may be applied: e.g., isotropic Landau [27–30], the Imbert-Mouhot toy model [6, 41], and the Vlasov-Poisson-Landau equation [33].

For parabolic equations, minimal assumptions for the Schauder estimates similar to those considered here were first investigated by Brandt [11], who showed that Hölder regularity in t is not necessary to establish partial Schauder estimates. More precisely, one need only have boundedness in t and Hölder continuity in the spatial variables x in order to show that $D_x^2 f$ is Hölder continuous in x . Knerr [45] later strengthened this to deduce time regularity of f under the same assumptions. These two papers are the main inspiration for the present manuscript. Their strategies are based on the comparison principle and are quite different from that used here, as we detail below. There has been a large body of literature on this over the ensuing decades, see, e.g., [18, 19, 47, 57].

In this paper, we show that Hölder regularity in the time variable t is not necessary to establish partial Schauder estimates for kinetic Fokker-Planck equations. As an application of this, we deduce a weak-strong uniqueness result for classical solutions of the Landau equation starting from initial data that is C^α in x and has a logarithmic modulus of continuity in v . This improves upon an earlier uniqueness result in which Hölder regularity was required in both variables, and it indicates that the role of regularity in the uniqueness theory may be more technical than fundamental (although probably not nonexistent). Below, we expand on this in detail and formalize a conjecture on less restrictive assumptions for uniqueness to hold.

We now make our main results more precise.

1.1. Schauder estimates. For simplicity, we consider the slightly less general equation

$$(\partial_t + v \cdot \nabla_x) f = \text{tr}(\bar{a}(t, x, v) D_v^2 f) + \bar{c}(t, x, v) f + g(t, x, v). \quad (2)$$

We note, however, that this essentially does not lose any generality. This is discussed after the statement of the main result of this section Theorem 1.

We assume that \bar{a} is uniformly elliptic and \bar{c} and g are bounded: there is $\Lambda > 1$ such that

$$\frac{1}{\Lambda} \text{Id} \leq \bar{a}(t, x, v) \leq \Lambda \text{Id} \quad \text{and} \quad |\bar{c}(t, x, v)|, |g(t, x, v)| \leq \Lambda. \quad (3)$$

We also assume that \bar{a} , \bar{c} , and g are Hölder continuous in (x, v) : $\bar{a}, \bar{c}, g \in C_x^{\alpha/3} C_v^\alpha$. The notation for this Hölder space is defined in Section 1.4.

Our first result is a general Schauder estimate that does not require the t -regularity of the coefficients. Its proof is found in Section 2.

Theorem 1. *Fix $\alpha \in (0, 1)$. Assume that (3) holds and $f, D_v^2 f, \bar{a}, \bar{c}, g \in C_x^{\alpha/3} C_v^\alpha(Q_1)$. Then*

$$\begin{aligned} & [f]_{C_x^{(2+\alpha)/3}(Q_{1/2})} + [D_v^2 f]_{C_x^{\alpha/3} C_v^\alpha(Q_{1/2})} \\ & \lesssim \left(1 + [\bar{c}]_{C_x^{\alpha/3} C_v^\alpha(Q_1)} + [\bar{a}]_{C_x^{\alpha/3} C_v^\alpha(Q_1)}^{1+\frac{2}{\alpha}} \right) \|f\|_{L^\infty(Q_1)} \\ & \quad + \left(1 + [\bar{a}]_{C_x^{\alpha/3} C_v^\alpha(Q_1)} \right) [g]_{C_x^{\alpha/3} C_v^\alpha(Q_1)}. \end{aligned} \quad (4)$$

The implied constant depends only on d , α , and Λ .

We note that a simple consequence of Theorem 1 and (2) is that

$$\begin{aligned} & [(\partial_t + v \cdot \nabla_x) f]_{C_x^{\alpha/3} C_v^\alpha(Q_{1/2})} \lesssim \left(1 + [c]_{C_x^{\alpha/3} C_v^\alpha(Q_1)} + [a]_{C_x^{\alpha/3} C_v^\alpha(Q_1)}^{1+\frac{2}{\alpha}} \right) \|f\|_{L^\infty(Q_1)} \\ & \quad + \left(1 + [a]_{C_x^{\alpha/3} C_v^\alpha(Q_1)} \right) [g]_{C_x^{\alpha/3} C_v^\alpha(Q_1)}. \end{aligned} \quad (5)$$

Before commenting on the proof, we note that regularity in t can easily be obtained at this point by two different methods. The first is the hypoelliptic approach of [41, Lemma 2.8]. The technique of the authors shows that shifts in t decompose into a shifts in v as well as shifts in transport (roughly, shifts according to the operator $\partial_t + v \cdot \nabla_x$). The v -regularity is provided by Theorem 1 and the transport regularity is provided by (5). The second approach is to notice that time regularity can easily be obtained in the course of establishing Theorem 1 with the same methods. We did not opt for this due to (i) the desire for simplicity, (ii) the fact that time regularity does not play a role in our application (Theorem 2), and (iii) the fact that the hypoelliptic approach of [41, Lemma 2.8] yields it in a simple manner as a consequence of Theorem 1. Unfortunately, both approaches only provide Hölder continuity in t of f and do not yield regularity of $\partial_t f$. For greater regularity in t , it appears that one needs more regularity of the coefficients. We refer to [36]. If one focuses instead on transport regularity, the optimal result was recently established in [49].

Note that the partial Hölder regularity considered above is a “time slice” of kinetic Hölder regularity. In this way, these arise from the Galilean invariance of the equation and respect its scaling. They are, thus, the natural spaces to consider

for our setting. In order to ease the barrier to entry for reading this work, we opt to avoid the notation for the kinetic Hölder spaces and the associated Galilean translations. Indeed, as we are often taking time slices, these play almost no role in our arguments besides scaling.

In fact, this is an advantage to our approach. The Galilean invariance is often a barrier to the application of parabolic techniques in the kinetic setting. It ties the t , x , and v regularity together, and many parabolic techniques implicitly use the ‘‘independence’’ of the time and space regularity. By restricting to time slices here and considering only shifts in x and v , we do not see this dependence, thus sidestepping many complications.

Let us finally note that global estimates can easily be deduced from Theorem 1. Indeed, one simply notes that the global Hölder norm is comparable to the supremum over all cylinders Q_r (with r fixed) of the Hölder norms on Q_r . See [21] for a presentation of global estimates as well as a quite different approach than the one considered here. In particular, note that Dong and Yastrzhembskiy’s approach is kernel free. Additional L^p estimates have been investigated in similar settings in, e.g., [20, 53, 54]

1.1.1. *Strategy of the proof.* Our approach is along the lines of [36]. The proof proceeds with the same main two steps of every proof of Schauder estimates – direct estimates for an equation with coefficients that are constant in (x, v) and then perturbing off of this constant-in- (x, v) coefficients equation using the regularity of the coefficients.

The first step is slightly different from that of [36]. For us, the relevant ‘constant coefficients’ equation is the one where $\bar{a}(t, x, v) = \bar{a}(t)$. Roughly, this allows us to perturb off of this case by using that, for (x_0, v_0) fixed and $(x, v) \approx (x_0, v_0)$,

$$|\bar{a}(t, x, v) - \bar{a}(t, x_0, v_0)| \leq (|x - x_0|^{\alpha/3} + |v - v_0|^\alpha) [\bar{a}]_{C_x^{\alpha/3} C_v^\alpha} \ll 1.$$

Notice that this depends only on the regularity of \bar{a} in (x, v) and not in t .

It is worth discussing the constant-in- (x, v) coefficients equation further. Many proofs of the Schauder estimates for parabolic or kinetic equations (Brandt’s [11] being a notable exception), hinge on the scaling in t of moments of the fundamental solution $\Gamma_{\bar{a}}$, that is, integrals in (x, v) of $\Gamma_{\bar{a}}$ with polynomial weights. To obtain the estimates here, we compute the fundamental solution $\Gamma_{\bar{a}}$ explicitly (see Proposition 3). In [36], where the relevant constant-in- (x, v) coefficients equation is $\bar{a} \equiv \text{Id}$, it is essentially a basic calculus exercise to go from the explicit form of Γ_{Id} to the correct moment estimates. In our setting, however, it is more difficult and requires a somewhat involved proof based on the dynamics of some matrix valued terms (see Lemma 6). Indeed, from Proposition 3 it is not even obvious that $\Gamma_{\bar{a}}$ is integrable in (x, v) .

The second step, that is, the procedure of perturbing off of the constant-in- (x, v) coefficients equation, proceeds as usual.

1.1.2. *Estimates for (1) versus (2).* As we mentioned above, there is essentially no loss in generality in considering (2) in place of (1). The reason for this is that, one can obtain (2) from (1) by letting

$$g_{(2)} = \bar{b} \cdot \nabla_v f + g_{(1)}.$$

Here, to differentiate between the forcing term g in (1) and the forcing term g in (2), we use the equation number as the subscript. In this case, after applying Theorem 1,

one has a $C_x^{\alpha/3}C_v^\alpha$ -norm of $\nabla_v f$ on the right hand side of (4). By interpolating, one can “absorb” this lower order term into the left hand side of (4). While this is complicated by the different domains on which the norms are based, with $Q_{1/2}$ appearing on the left-hand side of (4) and Q_1 appearing on the right-hand side of (4), it is generally possible to do, depending on the application. The reader will surely have noticed that the same procedure should apply to the \bar{c} term as well, and so the simplest presentation would consider only the case of (2) with $\bar{c} \equiv 0$. This is true; however, for the application we have in mind (Theorem 2), it streamlines future computations to already have the explicit dependence on $[\bar{c}]_{C_x^{\alpha/3}C_v^\alpha}$.

1.1.3. Further comments on related time irregular Schauder estimates. As mentioned above, to our knowledge, the first result in this direction is due to Brandt [11], whose approach is entirely based on the comparison principle. Indeed, in a very simple short paper, Brandt establishes Schauder estimates with precise dependence on the coefficients via the construction of an upper barrier for an appropriate finite difference of the solution to the parabolic equation under consideration. Later, Knerr [45] improved on the regularity obtained by Brandt by showing that, surprisingly, solutions had time regularity as well. Knerr’s strategy was also based on the comparison principle.

Unfortunately, despite (2) enjoying a comparison principle, we were unable to adapt Brandt’s strategy to the kinetic case. We give a heuristic description of the obstruction. One expects $\partial_t + v \cdot \nabla_x$ in the kinetic case to act analogously to ∂_t in the parabolic case. A major difference, however, is that shifts in t (the appropriate shifts related to ∂_t regularity) have a directionality: time is one dimensional and there is a preferred direction, often called the “arrow of time.” Unfortunately, it is not clear what the analogue to this is in the kinetic setting with the operator $\partial_t + v \cdot \nabla_x$. Very roughly, this is the roadblock to adapting Brandt’s argument. We note that this seems to be related to the impediment to proving a Harnack inequality for (1) using the methods of Krylov and Safonov; see [56, Section 8.2] for further discussion.

A few days prior to posting this manuscript to the arxiv, another very interesting paper was also posted by Biagi and Bramanti [7] that investigates a similar problem to Theorem 1. The authors consider ultraparabolic equations, a general class of equations that includes kinetic equations as a particular example, and they prove a Schauder estimate for time irregular coefficients. Their proof proceeds along the same lines as ours; that is, they derive an explicit formula for the fundamental solution and use it to deduce the Schauder estimates. Their paper is focused entirely on the question of Schauder estimates of the form Theorem 1 for a general family of ultraparabolic equations, and, as such, they do not consider applications of their theorem, as we do in Section 1.2. Their work builds upon an earlier work of Bramanti and Polidoro [10] in which the fundamental solution of a class of ultraparabolic operators was studied in depth. In particular, the authors construct it and establish that it has the appropriate regularity and Gaussian bounds. We also mention connections to the other very recent preprint by Lucertini, Pagliarani, and Pascucci [48] in which the authors deduce optimal bounds on the higher regularity of the fundamental solution. The estimates in [48] are strong enough to replace Lemma 4 in our proof of Theorem 1. It seems likely that one could establish Theorem 1 via an alternative approach to the Schauder estimates using the results of [48] directly.

1.2. **The Landau equation.** The Landau equation has the form:

$$(\partial_t + v \cdot \nabla_x) f = \text{tr}(\bar{a}^f D_v^2 f) + \bar{c}^f f \quad \text{in } (0, T) \times \mathbb{R}^3 \times \mathbb{R}^3, \quad (6)$$

where, for any function $h : \mathbb{R}^3 \rightarrow \mathbb{R}$,

$$\begin{aligned} \bar{a}^h(t, x, v) &= a_\gamma \int_{\mathbb{R}^3} \left(\text{Id} - \frac{w \otimes w}{|w|^2} \right) |w|^{2+\gamma} h(v-w) dw \\ \bar{c}^h(t, x, v) &= \begin{cases} c_\gamma \int_{\mathbb{R}^3} |w|^\gamma h(v-w) dw & \text{for } \gamma > -3, \\ c_\gamma h & \text{for } \gamma = -3. \end{cases} \end{aligned} \quad (7)$$

Here, a_γ and c_γ are positive constants whose exact value plays no role in the analysis and $\gamma \in [-3, 0)$. The physically relevant case is $\gamma = -3$. We note that the regime $\gamma < 0$ is often called the *soft potentials* case and that the case $\gamma \in [0, 1]$ is considered in many works, but we do not address it here. We also note that (6) is more often written in an equivalent divergence form, although that is not convenient for our work below.

We refer the reader to [51, 59] for a general discussion of the Landau equation, its physical relevance, and its mathematical history. We mention only that (6) is nonlocal (that is, its coefficients at a point (t, x, v) depend on the f at other points (t, x, v')) and quasilinear (that is, the coefficient of the highest order term \bar{a}^f depends on f). As a result, the unconditional global well-posedness of classical solutions to (6) is an extremely difficult problem that appears to be out of reach for the time being.

A new approach to this problem was initiated by Silvestre [55], who proposed to study (6) with methods coming from parabolic regularity theory under certain physically reasonable boundedness assumptions on the mass, energy, and entropy densities (see [42] for a discussion of a similar program for the related Boltzmann equation). We do not discuss these assumptions further, and we refer to this program as the *conditional regularity* program in the sequel. These ideas have led to many new results, see, e.g. [13, 25, 36, 41]. The most relevant work to the present setting coming out of this program is [38] that leveraged the ideas and theorems of the previous works [13, 25, 36] to obtain local well-posedness with fairly “rough” initial data. In particular, the existence result in [38] is in a weighted L^∞ space, while the uniqueness result supposes, additionally, that the initial data is C^α for some $\alpha > 0$.

One of the key insights used in the conditional regularity program is that, as previously mentioned, \bar{a}^f enjoys better regularity in v than f does. Indeed, $\bar{a}^f \in C_v^\alpha$ for any $\alpha \in (0, 1)$ as long as f is bounded and decays sufficiently quickly in v . The gap between the existence and uniqueness results of [38], described above, partially reflects the fact that the authors were not able to leverage this insight. Our next result, a new uniqueness result for the Landau equation (6), provides a path in this direction.

1.2.1. *Uniqueness for the Landau equation.* We require the following non-degeneracy condition on f_{in} : there exist r , δ , and $R > 0$ so that

$$\text{for every } x \in \mathbb{R}^3, \text{ there is } v_x \in \mathbb{R}^3 \text{ such that } f_{\text{in}}(x, \cdot) \geq \delta \mathbb{1}_{B_r(x, v_x)}. \quad (8)$$

The reason for (8) is that, from it, one can obtain a pointwise lower bound for f . This, in turn, yields the local-in- v uniform ellipticity of \bar{a}^f . This was originally shown in [37, Theorem 1.3]; see also [38, Lemma 2.5] for the connection between the lower bound on f and the ellipticity of \bar{a}^f . Let us note that this rules out

“localized” initial data such as $f_0 \lesssim \exp\{-|v|^2 - |x|^2\}$. It is, however, often easier to prove uniqueness in such localized settings as integral estimates can be used.

We are now ready to state the second main result. It is proved in Section 3.

Theorem 2. *Fix $k, \theta > 0$ and $\alpha \in (0, 1)$. Assume that $f_{in} \in C_x^{\alpha/3} \log(1/C_v)^{-\theta} \cap L^{\infty, k}$ and satisfies (8). Let $f \in L^{\infty, k}([0, T] \times \mathbb{R}^6)$ be any solution of (6) constructed in [38, Theorem 1.2] starting from initial data f_{in} .*

Fix any uniformly continuous function $g \in L^{\infty, 5+\gamma+\eta}([0, T] \times \mathbb{R}^6)$, where $\eta > 0$, such that g solves equation (6) weakly (in the sense of [38]) and $g(t, x, v) \rightarrow f_{in}$ as $t \searrow 0$.

Then, if k is sufficiently large, depending on θ, α , and γ , and

$$\frac{\theta}{2} \frac{\alpha}{2 + \alpha} > 1,$$

there is $T_1 \in (0, T]$, depending only on f_{in} , α , θ , k and γ , such that $f = g$ in $[0, T_1] \times \mathbb{R}^6$. If $k = \infty$ then $T_1 = T$.

We note that the nonstandard continuity space $\log(1/C_v)^{-\theta}$ is defined in Section 1.4 below, as are all of the notational conventions we use. We also note that the particular type of weak solution plays almost no explicit role in our analysis since we immediately deduce various regularity properties of g from previous results. Hence, our choice of weak solution is made simply so that it is compatible with the previous results in [25, 38]. Roughly, though, g is in an appropriate kinetic H^1 -space and solves (6) in the sense of integration against other kinetic H^1 test functions with compact support.

We make no effort to optimize the condition on k in our estimates. When presented the opportunity, we always opt for more simply stated results in intermediate lemmas as opposed to sharper results. That said, a clearly non-optimal lower bound for k , found by tracking all computations, appears to be

$$k \geq \max \left\{ \max \left\{ 2 + \frac{4}{\mu' \alpha}, 9 + \frac{\mu' \alpha}{1 - \frac{\mu' \alpha}{2}}, \frac{5}{1 - \mu}, 5 + \frac{4}{1 - \frac{\mu'}{\mu}} \right\} : 1 > \mu > \mu' > \frac{1}{\theta} + \sqrt{\frac{1}{\theta^2} + \frac{4}{\theta \alpha}} \right\}. \quad (9)$$

The μ and μ' terms above arise when passing decay-in- v to higher regularity norms of f via interpolation such as Lemma 17. As a result, one can obtain a better condition on k by instead assuming that $\langle v \rangle^m f \in C_x^{\alpha/3} \log(1/C_v)^{-\theta}$ for some m . The cost, however, is a lower bound condition on m . Let us lastly note that γ does not appear in the condition (9). Greater care in the estimates will lessen the restrictions above in a way depending on γ .

1.2.2. The strategy of the proof. We give a rough outline of the uniqueness argument used to prove Theorem 2. For simplicity, we ignore all complications due to “weights” in this discussion, although these are required in the proof due to the fact that \bar{a}^f is only defined when f decays sufficiently quickly as $|v| \rightarrow \infty$. The proof follows the standard outline – find an equation for the difference $f - g$ and use a Gronwall-type argument. This, however, is complicated by the fact that (6) has nonlocal coefficients and is quasilinear, that is, the highest order coefficient is nonlinear in f . As a result, we require an L^∞ bound on $D_v^2 f$.

We obtain such a $W_v^{2,\infty}$ -bound by applying a scaled version of Theorem 1. Were we to only assume that $f \in L^\infty$, such an estimate would degenerate like $1/t$ as $t \rightarrow 0$ (recall that $f_{\text{in}} \notin W_v^{2,\infty}$). This can be seen easily by scaling arguments. However, by propagating forward bounds on the $C_x^{\alpha/3} \log(1/C_v)^{-\theta}$ -norm, we are able to, via interpolation, obtain a bound that degenerates like

$$\|D_v^2 f(t)\|_{L^\infty(\mathbb{R}^6)} \lesssim \frac{1}{t (\log \frac{1}{t})^{\frac{\theta}{2} \frac{\alpha}{2+\alpha}}}.$$

Crucially, this is integrable in t near $t = 0$, which allows the Gronwall argument to close.

One key step above, scaling the Schauder estimates, was developed in [36]. The other key step above, in which we propagate the $C_x^{\alpha/3} \log(1/C_v)^{-\theta}$ -norm, relies on the general ideas of [38, Proposition 4.4], in which the $C_x^{\alpha/3} C_v^\alpha$ -norm was propagated. It is, however, significantly more complicated in our case. The reason being that, while \bar{a}^f is v -Hölder continuous, regardless of the regularity of f , \bar{c}^f does not enjoy this property. In particular, when $\gamma = -3$, which is the physically relevant case, $\bar{c}^f = c_\gamma f$. As such, it is exactly as irregular as f .

Roughly, we overcome this by obtaining a bound on $\|D_v^2 f(t_0)\|_{L^\infty(\mathbb{R}^6)}$ that depends on the $C_x^{\alpha/3} \log(1/C_v)^{-\theta}$ -norm of f as well as on $\|D_v^2 f\|_{L^\infty([t_0/4, t_0] \times \mathbb{R}^6)}$. The appearance of this second term is exactly due to the (potential) irregularity of \bar{c}^f . In this bound, the coefficient of $\|D_v^2 f\|_{L^\infty([t_0/4, t_0] \times \mathbb{R}^6)}$ is small. Hence, by a careful argument, we are able to absorb it back into the $W_v^{2,\infty}$ -term at t_0 , despite the difference in time domains. This step is contained in Proposition 11.

The reason Theorem 1 is useful in this application is the following. In [38], (x, v) -Hölder regularity of f is propagated from initial (x, v) -Hölder regularity. An additional argument shows that regularity is passed to t as well. Stated imprecisely, if $f \in C_x^{\alpha/3} C_v^\alpha$ then $f \in C_t^{\alpha/2} C_x^{\alpha/3} C_v^\alpha$. The coefficients are then Hölder regular and the full Schauder estimates can be applied. As we only have $C_x^{\alpha/3} \log(1/C_v)^{-\theta}$ -regularity of f , the only regularity that could potentially be passed to t is that with a log modulus. At best, then, \bar{a}^f will be Hölder in (x, v) , due to the Hölder regularity of f in x and the fact that \bar{a}^f is defined by convolution with a “nice” kernel in v , but with only a log modulus of continuity in t . Thus, the full Schauder estimates could not be applied.

We point out a subtle additional benefit to the application of Theorem 1 in place of the Schauder estimates of [36]. When the estimates of [36] are applied in [38], there is a loss of regularity between \bar{a}^f and f due to how time shifts interact with the appropriate notion of “kinetic distance”²; see [38, Lemma 2.7]. This is avoided here due to our not considering time shifts. As such, we achieve *sharper estimates* on the various quantities, such as $\|D_v^2 f(t)\|_{L^\infty(\mathbb{R}^6)}$, as $t \searrow 0$.

1.2.3. Related work. As mentioned above, Theorem 2 supercedes the earlier work [38, Theorem 1.4], which required (x, v) -Hölder regularity of f_{in} for uniqueness the hold. We also mention the work of Anceschi and Zhu in [6] on a similar model.

²We have largely avoided discussing the kinetic distance since it plays no role in our analysis. Indeed, without shifts in time, which we need not consider due to our not considering time regularity, the kinetic distance collapses to $C_x^{\alpha/3} C_v^\alpha$. We point the interested reader to a clear discussion of the kinetic distance in [43, Section 2.1].

To our knowledge, the local well-posedness theory of Landau is relatively un-studied, with more interest directed toward a related kinetic integro-differential equation, the Boltzmann equation. There, the first local well-posedness results are due to the AMUXY group [2, 3, 5]. In particular, a general uniqueness result in an appropriate Sobolev space (of order $2s$, that is, twice the order of the differential operator in the equation) was proven in [4] for a restricted class of parameters.

While [4] is an extremely nice result, we describe in slightly more detail its limitations in order to highlight the difficulties in our setting. Their result requires H_v^{2s} -regularity of solutions. The Landau equation essentially corresponds to the $s = 1$ case. Were their result to apply, it would require H_v^2 -regularity of f , which corresponds to $C_v^{1/2}$ -regularity and is significantly more than we require here. Indeed, for reasons related to this, we note that uniqueness is, in some ways, more difficult for Landau than Boltzmann as the differential operator is of higher order. Additionally, their uniqueness result requires regularity of both solutions in contrast to our result that has only mild conditions on the other potential solution g . On the other hand, their result only requires boundedness in x .

The close-to-equilibrium and homogeneous setting for (6) have seen more focus. This is probably due to the fact that one is often able to establish strong results such as global well-posedness and convergence to equilibrium. The state-of-the-art, techniques, and types of questions asked in these settings are quite different from those raised in the current manuscript, so we do not go into much detail here. We simply mention a few landmark results in each case. The story in the homogeneous setting (that is, when f is independent of x) is somewhat complicated by the functional setting one works in, but we mention the works of [1, 15, 16, 23, 24, 64]. In the close-to-equilibrium setting (that is, when f_{in} is “close” to a Maxwellian of the form $\alpha e^{-|v|^2/\beta}$ for some $\alpha, \beta > 0$), we refer to [22, 32, 52, 58]. Both settings are extremely well studied and, as a result, we are only able to reference a small selection of the work completed over the past several decades.

Outside of these settings, little is known about the global well-posedness of classical solutions. To our knowledge, the conditional result of [38], which yields global well-posedness as long as the mass and energy densities remain bounded in t and x in the case $\gamma > -2$ (or, in the case of $\gamma \leq -2$, if certain L^p -norms remain bounded), is currently the sharpest condition ruling out “blow-up.”

1.3. Two conjectures. We now formulate two conjectures regarding ways in which the results above might be strengthened.

First, if we trust the analogy discussed above, that $\partial_t + v \cdot \nabla_x$ in the kinetic setting is similar to ∂_t in the parabolic one, we are led to the following conjecture:

Conjecture 1.1. *Fix any $\alpha \in (0, 1)$. Assume that (3) holds and that $f, D_v^2 f, \bar{a}, \bar{c}, g \in C_v^\alpha(Q_1)$. Then*

$$[D_v^2 f]_{C_v^\alpha(Q_{1/2})} \lesssim \left(1 + [\bar{c}]_{C_v^\alpha(Q_1)} + [\bar{a}]_{C_v^\alpha(Q_1)}^{1+\frac{2}{\alpha}}\right) \|f\|_{L^\infty(Q_1)} + (1 + [\bar{a}]_{C_v^\alpha(Q_1)}) [g]_{C_v^\alpha(Q_1)}.$$

The implied constant depends only on d, α , and Λ .

Notice that the conjectured result above does not require any x -regularity. It seems that a uniqueness result for the Landau equation is an immediate consequence of this. We state this roughly here:

Conjecture 1.2. *In the setting of Theorem 2, although assuming only that $f_{\text{in}} \in \log(1/C_v)^{-\theta} \cap L^{\infty,k}$ (that is, we drop the Hölder regularity in x), the same weak-strong uniqueness conclusion holds as long as*

$$\theta > 2.$$

It is not clear that the above conjecture, were it true, would be sharp. There is a strong connection between regularity and uniqueness results. Indeed, recent work has established the nonuniqueness of irregular (weak) solutions of fluid equations, see, e.g., [12, 14]. We also note the work of Kiselev, Nazarov, and Shterenberg, who, in the critical case of the fractal Burgers equation studied in [44], see a situation similar to that of the Landau equation: rough solutions immediately become smooth but uniqueness is unknown without further regularity assumptions. In fact, despite the intense interest in [44], uniqueness of these rough solutions remains open as far as we know.

On the other hand, in the homogeneous (x -independent) case for the Landau equation, where the Landau equation has more structure, uniqueness has been established through a probabilistic approach that yields bounds on the Wasserstein distance between two solutions [23, 24]. This result requires essentially no regularity of f , although it is only applicable in the homogeneous case.

We expect the conjectures above to be difficult to establish for reasons related to the fundamental difference between ∂_t in the parabolic setting and $\partial_t + v \cdot \nabla_x$ in the kinetic setting that were discussed in Section 1.1.3.

1.4. Notation and continuity spaces.

1.4.1. *Points and kinetic cylinders.* For succinctness, we often write

$$z = (t, x, v), \quad \tilde{z} = (\tilde{t}, \tilde{x}, \tilde{v}), \quad \text{and} \quad z' = (t', x', v').$$

For any $r > 0$ and z_0

$$Q_r(z_0) = \{(t, x, v) : t_0 - r^2 < t \leq t_0, |x - x_0 - (t - t_0)v_0| < r^3, |v - v_0| < r\}.$$

When no base point is specified, we let

$$Q_r = (-r^2, 0] \times B_{r^3} \times B_r,$$

where we use the convention that $B_r = B_r(0)$. The reason for the choice of Q_r is the natural scaling $(t, x, v) \mapsto (r^2 t, r^3 x, r v)$ associated to (2).

1.4.2. *Continuity spaces.* Throughout we work with some anisotropic continuity spaces, i.e., those in which different ‘‘amounts’’ of regularity are required in each variable. In particular, for any set $Q \subset \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ and parameters $\alpha_1, \alpha_2 \in (0, 1]$, we let

$$C_x^{\alpha_1} C_v^{\alpha_2}(Q) := \{f : Q \rightarrow \mathbb{R} : f \in L^\infty(Q), [f]_{C_x^{\alpha_1} C_v^{\alpha_2}(Q)} < \infty\},$$

where

$$[f]_{C_x^{\alpha_1} C_v^{\alpha_2}(Q)} := \sup_{\substack{(t, x, v) \neq (t', x', v') \in Q, \\ |x - x'|, |v - v'| < 1/2}} \frac{|f(t, x, v) - f(t', x', v')|}{|x - x'|^{\alpha_1} + |v - v'|^{\alpha_2}}.$$

Finally, for the uniqueness result for the Landau equation, we define a space of functions whose modulus of continuity is logarithmic. Indeed, for $Q \subset \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ and parameters $\alpha \in (0, 1)$ and $\theta > 0$, we let

$$C_x^{\alpha/3} \log(1/C_v)^{-\theta}(Q) := \{f : Q \rightarrow \mathbb{R} : f \in L^\infty(Q), [f]_{C_x^{\alpha/3} \log(1/C_v)^{-\theta}(Q)} < \infty\},$$

where

$$[f]_{C_x^{\alpha/3} \log(1/C_v)^{-\theta}(Q)} = \sup_{\substack{(t,x,v) \neq (t,x',v') \in Q, \\ |x-x'|, |v-v'| < 1/2}} \frac{|f(t,x,v) - f(t,x',v')|}{|x-x'|^{\alpha/3} + \log(1/|v-v'|)^{-\theta}}. \quad (10)$$

Abusing notation, we also use the $C_x^{\alpha/3} \log(1/C_v)^{-\theta}$ notation for functions f that are independent of t but for which the supremum in (10), without the t terms, is finite.

When Q is not specified in the norms above, it is taken to be either \mathbb{R}^6 or $\mathbb{R}_+ \times \mathbb{R}^6$, depending on the setting. For example, if $f : \mathbb{R}^6 \rightarrow \mathbb{R}$, then we say $f \in C_x^{\alpha/3} C_v^\alpha$ to mean $f \in C_x^{\alpha/3} C_v^\alpha(\mathbb{R}^6)$.

1.4.3. *Multi-indices.* Given a multi-index $\alpha \in (\mathbb{N} \cup \{0\})^d$, we write

$$\partial_v^\alpha = \partial_{v_1}^{\alpha_1} \cdots \partial_{v_d}^{\alpha_d}.$$

The object ∂_x^α is defined analogously.

1.4.4. *Other notation.* Throughout the work, constants are assumed to change line-by-line and depend on various parameters such as the dimension d , the ellipticity constant Λ , and the regularity parameter α . In the statement of each result, we make clear the dependencies and in its proof, we simply write $A \lesssim B$ when $A \leq CB$, where C is a constant depending on those parameters. We use $A \approx B$ when $A \lesssim B$ and $B \lesssim A$.

In the uniqueness result for the Landau equation, we must work with weighted spaces. To this end, we recall the Japanese bracket: for any $v \in \mathbb{R}^d$,

$$\langle v \rangle = \sqrt{1 + |v|^2}.$$

Then we define the associated weighted L^∞ -spaces: for any $n \in \mathbb{R}$,

$$L^{\infty,n} := \{f : \langle v \rangle^n f \in L^\infty\} \quad \text{with norm } \|f\|_{L^{\infty,n}} := \|\langle \cdot \rangle^n f\|_{L^\infty}.$$

2. **The Schauder estimates.** In this section, we prove our first main result Theorem 1, which is the Schauder estimates for (2). As usual, the proof proceeds in two steps. The first step is an estimate for solutions of a relevant “constant coefficients” equation. For us, as discussed above, this “constant coefficients” equation is the one where the coefficients depend only on time t . The second step (Section 2.2) is to bootstrap to the general case by perturbing off of this “constant coefficients” equation.

2.1. **The first step: Schauder estimates when the coefficients are constant in (x, v) .** Consider the basic kinetic Fokker-Planck equation involving only transport in x and diffusion in v which the diffusion has a coefficient depending only on t :

$$(\partial_t + v \cdot \nabla_x) f = \text{tr}(\bar{a}(t) D_v^2 f) + g. \quad (11)$$

Our assumption on \bar{a} is the following: \bar{a} is symmetric and there is $\Lambda \geq 1$ such that

$$\bar{a} : \mathbb{R} \rightarrow \mathbb{R}^3 \times \mathbb{R}^3 \text{ is measurable and } \frac{1}{\Lambda} \text{Id} \leq \bar{a}(t) \leq \Lambda \text{Id} \quad \text{for all } t. \quad (12)$$

We stress that \bar{a} does not satisfy any further regularity assumptions.

We begin by studying the fundamental solution of this problem; that is, the function $\Gamma_{\bar{a}}$ for which the solution f of (11) is given by

$$f(z) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Gamma_{\bar{a}}(t, x - \tilde{x} - (t - \tilde{t})\tilde{v}, v - \tilde{v}; \tilde{t}) g(\tilde{t}, \tilde{x}, \tilde{v}) d\tilde{z}. \quad (13)$$

In the simple case $\bar{a} \equiv \text{Id}$, it is well-known [39] that Γ_{Id} is given by

$$\Gamma_{\text{Id}}(t, x, v) = \begin{cases} \left(\frac{\sqrt{3}}{2\pi t^2}\right)^d \exp\left\{-\frac{3|x-vt/2|^2}{t^3} - \frac{|v|^2}{4t}\right\} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases} \quad (14)$$

We point out two features: (1) integrating Γ_{Id} in x recovers the standard heat kernel, and (2) the ‘‘kinetic convolution’’ involved in (13) respects the Galilean invariance induced by the transport operator.

By a somewhat complicated, but nonetheless straightforward, Fourier transform-based computation we can compute the fundamental solution associated to a general (x, v) -independent \bar{a} . Indeed, we find the following:

Proposition 3. *Under the assumption (14), solutions of (11) are given by (13) with the fundamental solution*

$$\Gamma_{\bar{a}}(t, x, v; s) = \begin{cases} \frac{\exp\left\{-\frac{v \cdot A_0(t; s)^{-1} v}{4} - (x - M(t; s)v) \cdot P(t; s)^{-1}(x - M(t; s)v)\right\}}{(4\pi)^d \sqrt{\det[A_0(t; s)P(t; s)]}} & \text{if } t > s, \\ 0 & \text{if } t \leq s, \end{cases} \quad (15)$$

where

$$\begin{aligned} A_i(t; s) &= \int_s^t (r - s)^i \bar{a}(r) dr, \quad \text{for } i = 0, 1, 2, \\ P(t; s) &= A_2(t; s) - A_1(t; s) A_0(t; s)^{-1} A_1(t; s), \quad \text{and} \\ M(t; s) &= (t - s) \text{Id} - A_0(t; s)^{-1} A_1(t; s). \end{aligned} \quad (16)$$

We postpone the proof of Proposition 3 to Appendix A.

It is not obvious that (13) is well-defined from (15). Indeed, while the positive-definiteness of A_0 and its having the same scaling in time as the analogous term in (14) are immediately obvious, the same cannot be said for $P(t)$. In fact, even the positive definiteness of P is not clear. However, we need a stronger estimate than merely positive definiteness of P as the crucial step in most proofs of the Schauder estimates is in understanding the scaling in t of $\Gamma_{\bar{a}}$ and its integrals in x and v .

We now state this scaling property. Notice that it is, up to constants, the same as one would obtain using Γ_{Id} defined in (14). Its proof is contained in Section 2.1.1.

Lemma 4. *Let $\Gamma_{\bar{a}}$ be as in (15), with \bar{a} under the assumptions given by (12). Fix any multi-indices $\alpha, \beta \in (\mathbb{N} \cup \{0\})^d$, any natural number $j \geq 0$, and any real numbers $r, s \geq 0$. For $t > \tilde{t}$,*

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \max_{(0, \xi_2, \xi_3) \in Q_{\sqrt{(t-\tilde{t})/2}}} |\partial_t^j \partial_x^\beta \partial_v^\alpha \Gamma_{\bar{a}}(t, x + \xi_2, v + \xi_3; \tilde{t})| |x|^r |v|^s dx dv \lesssim (t - \tilde{t})^{-\frac{2j+|\alpha|+3|\beta|}{2} + \frac{3r+s}{2}}.$$

Using this estimate, Lemma 4, we are now able to establish the main result in the constant-in- (x, v) coefficients setting that will be the basis of the main Schauder estimate.

Proposition 5. *Fix $\alpha \in (0, 1)$. Suppose that f , $(\partial_t + v \cdot \nabla_x)f$, D_v^2f , and $g \in C_x^{\alpha/3}C_v^\alpha(Q_1)$. Assume that f and g have compact support in Q_1 and satisfy (11) with coefficient \bar{a} that satisfies (12). Then*

$$[f]_{C_x^{(2+\alpha)/3}(Q_1)} + [D_v^2f]_{C_x^{\alpha/3}C_v^\alpha(Q_1)} \lesssim [g]_{C_x^{\alpha/3}C_v^\alpha(Q_1)},$$

where the implied constants depend only on α , Λ , and the dimension d .

Proof. We begin by estimating $[D_v^2f]_{C_x^{\alpha/3}C_v^\alpha(Q_1)}$. Recalling (13), for any $z = (t, x, v) \in Q_1$ and any $1 \leq i, j \leq d$,

$$\partial_{v_i v_j} f(z) = \int_{-1}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_{v_i v_j} \Gamma_{\bar{a}}(t, x - \tilde{x} - (t - \tilde{t})\tilde{v}, v - \tilde{v}; \tilde{t}) g(\tilde{z}) d\tilde{z}.$$

Fix another point $z' \in Q_1$ of the form

$$z' = (t, x', v').$$

Notice that z and z' have the same t -coordinate. This is due to the fact that we do not prove any regularity of D_v^2f in t . Let

$$\begin{aligned} h &= |x - x'|^{1/3} + |v - v'| \quad \text{and} \\ \delta g(\tilde{z}) &= g(t - \tilde{t}, x - \tilde{x} - \tilde{t}(v - \tilde{v}), v - \tilde{v}) - g(t - \tilde{t}, x' - \tilde{x} - \tilde{t}(v' - \tilde{v}), v' - \tilde{v}). \end{aligned}$$

Then, after making the change of variables

$$\tilde{z} \mapsto (t - \tilde{t}, x - \tilde{x} - (t - \tilde{t})\tilde{v}, v - \tilde{v}),$$

we find

$$\begin{aligned} \partial_{v_i v_j} f(z) - \partial_{v_i v_j} f(z') &= \left(\int_0^{2h^2} + \int_{2h^2}^{1+t} \right) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_{v_i v_j} \Gamma_{\bar{a}}(t, \tilde{x}, \tilde{v}; t - \tilde{t}) \delta g(\tilde{z}) d\tilde{z} \\ &= I_1 + I_2. \end{aligned}$$

We now estimate each of I_1 and I_2 in turn.

Estimating I_1 : Here, integrating $\partial_{v_i v_j} \Gamma_{\bar{a}}$ over (\tilde{x}, \tilde{v}) leaves us with an $O(1/\tilde{t})$ term. This means that our approach needs to use the regularity of g to obtain \tilde{t} -terms, either directly or via Lemma 4. Using the regularity of $\partial_{v_i v_j} \Gamma_{\bar{a}}$ will only exacerbate this issue, so we do not use it, but obtain extra smallness instead by working on a small interval $[0, 2h^2]$.

To this end, we smuggle in a new term. Setting $\tilde{z}_0 = (\tilde{t}, \tilde{x}, 0)$, we see that

$$\int_0^{2h^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_{v_i v_j} \Gamma_{\bar{a}}(t, \tilde{x}, \tilde{v}; t - \tilde{t}) \delta g(\tilde{z}_0) d\tilde{z} = 0.$$

Hence, we obtain

$$|I_1| = \left| \int_0^{2h^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_{v_i v_j} \Gamma_{\bar{a}}(t, \tilde{x}, \tilde{v}; t - \tilde{t}) (\delta g(\tilde{z}) - \delta g(\tilde{z}_0)) d\tilde{z} \right|.$$

Next, we point out that

$$\begin{aligned}
|\delta g(\tilde{z}) - \delta g(\tilde{z}_0)| &= \left| (g(t - \tilde{t}, x - \tilde{x} - \tilde{t}(v - \tilde{v}), v - \tilde{v}) - g(t - \tilde{t}, x' - \tilde{x} - \tilde{t}(v' - \tilde{v}), v' - \tilde{v})) \right. \\
&\quad \left. - (g(t - \tilde{t}, x - \tilde{x} - \tilde{t}v, v) - g(t - \tilde{t}, x' - \tilde{x} - \tilde{t}v', v')) \right| \\
&= \left| (g(t - \tilde{t}, x - \tilde{x} - \tilde{t}(v - \tilde{v}), v - \tilde{v}) - g(t - \tilde{t}, x - \tilde{x} - \tilde{t}v, v)) \right. \\
&\quad \left. - (g(t - \tilde{t}, x' - \tilde{x} - \tilde{t}(v' - \tilde{v}), v' - \tilde{v}) - g(t - \tilde{t}, x' - \tilde{x} - \tilde{t}v', v')) \right| \\
&\leq 2[g]_{C_x^{\alpha/3} C_v^\alpha(Q_1)} \left((\tilde{t}|\tilde{v}|)^{\alpha/3} + |\tilde{v}|^\alpha \right), \tag{17}
\end{aligned}$$

where, to get the last inequality, we swapped the places of the second and third terms in the absolute values. This has the advantage of avoiding a $t - t'$ term that would require time regularity of g .

Using (17) and then Lemma 4, we find

$$\begin{aligned}
|I_1| &\lesssim [g]_{C_x^{\alpha/3} C_v^\alpha(Q_1)} \int_0^{2h^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\partial_{v_i v_j} \Gamma_{\bar{a}}(t, \tilde{x}, \tilde{v}; t - \tilde{t})| ((\tilde{t}|\tilde{v}|)^{\alpha/3} + |\tilde{v}|^\alpha) d\tilde{z} \\
&\lesssim [g]_{C_x^{\alpha/3} C_v^\alpha(Q_1)} \int_0^{2h^2} s^{\alpha/2-1} ds \lesssim [g]_{C_x^{\alpha/3} C_v^\alpha(Q_1)} h^\alpha. \tag{18}
\end{aligned}$$

Estimating I_2 : In this case, we are insulated from $\tilde{t} = 0$ so we may (and do) use the regularity of $\partial_{v_i v_j} \Gamma_{\bar{a}}$ here. The first step is to separate the two integrals in I_2 (recall that δg is a difference of two terms) and then change variables $\tilde{z} \mapsto z - \tilde{z}$ and $\tilde{z} \mapsto z' - \tilde{z}$, respectively. This yields:

$$I_2 = \int_{-1}^{t-2h^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\partial_{v_i v_j} \Gamma_{\bar{a}}(t, x - \tilde{x}, v - \tilde{v}; \tilde{t}) - \partial_{v_i v_j} \Gamma_{\bar{a}}(t, x' - \tilde{x}, v' - \tilde{v}; \tilde{t})) g(\tilde{t}, \tilde{x} - (t - \tilde{t})\tilde{v}, \tilde{v}) d\tilde{z}.$$

The key reason for doing this is so that the resulting terms, $\partial_{v_i v_j} \Gamma_{\bar{a}}$, are a full \tilde{v} -derivative. Hence,

$$\int_{\mathbb{R}^d} \partial_{v_i v_j} \Gamma_{\bar{a}}(t, x - \tilde{x}, v - \tilde{v}; \tilde{t}) d\tilde{v} = \int_{\mathbb{R}^d} \partial_{v_i v_j} \Gamma_{\bar{a}}(t, x' - \tilde{x}, v' - \tilde{v}; \tilde{t}) d\tilde{v} = 0.$$

Therefore, we rewrite I_2 as

$$\begin{aligned}
I_2 &= \int_{-1}^{t-2h^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\partial_{v_i v_j} \Gamma_{\bar{a}}(t, x - \tilde{x}, v - \tilde{v}; \tilde{t}) - \partial_{v_i v_j} \Gamma_{\bar{a}}(t, x' - \tilde{x}, v' - \tilde{v}; \tilde{t})) \\
&\quad \times (g(\tilde{t}, \tilde{x} - (t - \tilde{t})\tilde{v}, \tilde{v}) - g(\tilde{t}, \tilde{x} - (t - \tilde{t})v, v)) d\tilde{z}.
\end{aligned}$$

Notice that $z - z' \in Q_h$. By a Taylor approximation, we see that

$$\begin{aligned}
&|\partial_{v_i v_j} \Gamma_{\bar{a}}(t, x - \tilde{x}, v - \tilde{v}; \tilde{t}) - \partial_{v_i v_j} \Gamma_{\bar{a}}(t, x' - \tilde{x}, v' - \tilde{v}; \tilde{t})| \\
&\leq \max_{\xi \in Q_h, \xi_1=0} (h^3 |\nabla_x \partial_{v_i v_j} \Gamma_{\bar{a}}(t, x - \tilde{x} + \xi_2, v - \tilde{v} + \xi_3; \tilde{t})| + h |\nabla_v \partial_{v_i v_j} \Gamma_{\bar{a}}(t, x - \tilde{x} + \xi_2, v - \tilde{v} + \xi_3; \tilde{t})|).
\end{aligned}$$

Additionally, we have

$$|g(\tilde{t}, \tilde{x} - (t - \tilde{t})\tilde{v}, \tilde{v}) - g(\tilde{t}, \tilde{x} - (t - \tilde{t})v, v)| \leq [g]_{C_x^{\alpha/3} C_v^\alpha(Q_1)} [(|t - \tilde{t}| |\tilde{v} - v|)^{\alpha/3} + |\tilde{v} - v|^\alpha].$$

Therefore, by a shifting back in all variables, $(\tilde{t}, \tilde{x}, \tilde{v}) \mapsto (t - \tilde{t}, x - \tilde{x}, v - \tilde{v})$, we see

$$|I_2| \leq [g]_{C_x^{\alpha/3} C_v^\alpha(Q_1)} \int_{2h^2}^{t+1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \max_{\xi \in Q_h, \xi_1=0} \left(h^3 |\nabla_x \partial_{v_i v_j} \Gamma_{\bar{a}}(t, \tilde{x} + \xi_2, \tilde{v} + \xi_3; t - \tilde{t})| + h |\nabla_v \partial_{v_i v_j} \Gamma_{\bar{a}}(t, \tilde{x} + \xi_2, \tilde{v} + \xi_3; t - \tilde{t})| \right) (|\tilde{t}|^{\alpha/3} |\tilde{v}|^{\alpha/3} + |\tilde{v}|^\alpha) d\tilde{z}.$$

Using then Lemma 4, which, effectively turns \tilde{x} and \tilde{v} into $\tilde{t}^{3/2}$ and $\tilde{t}^{1/2}$, respectively, and ∂_t , ∇_x and ∇_v into \tilde{t}^{-1} , $\tilde{t}^{-3/2}$, and $\tilde{t}^{-1/2}$, respectively, we find

$$|I_2| \lesssim [g]_{C_x^{\alpha/3} C_v^\alpha(Q_1)} \int_{2h^2}^{t+1} \left(\frac{h^3}{\tilde{t}^{5/2}} + \frac{h}{\tilde{t}^{3/2}} \right) \tilde{t}^{\alpha/2} d\tilde{t} \lesssim [g]_{C_x^{\alpha/3} C_v^\alpha(Q_1)} h^\alpha. \quad (19)$$

Combining this, (18), and (19) completes the estimate of $[D_v^2 f]_{C_x^{\alpha/3} C_v^\alpha(Q_1)}$ as claimed in the statement.

The estimate of $[f]_{C_x^{\frac{2+\alpha}{3}}(Q_1)}$ essentially proceeds along the same lines, but is significantly simpler as there is no difference in the Galilean terms $-\tilde{t}(v - \tilde{v})$ in δg . Additionally, the details are exactly the same as in [36, Lemma 2.5]. As such we omit the proof. \square

2.1.1. Proof of Lemma 4: integrals of $\Gamma_{\bar{a}}$ and their scaling in t . Our first observation in service of establishing Lemma 4 is that the integral is well-defined due to the positivity of the exponential terms and that it satisfies the appropriate scaling laws.

Lemma 6. *The matrix $P(t)$ is invertible. Additionally, we have the following bounds: for all $t > \tilde{t}$, any $i = 0, 1, 2$, and any vector $w \in \mathbb{R}^d$,*

$$(i) \quad w \cdot P(t; \tilde{t})w \approx (t - \tilde{t})^3 |w|^2 \quad (ii) \quad |A_i(t; \tilde{t})w| \approx (t - \tilde{t})^{i+1} |w| \\ (iii) \quad |M(t; \tilde{t})w| \lesssim (t - \tilde{t}) |w|.$$

where the constants depends only on Λ .

We note that the lower bound in (ii) and all of the upper bounds are straightforward, but the lower bound in (i) is not obvious and nontrivial to prove. As the proof is somewhat long, we postpone it to Section 2.1.2.

Next, we observe that the partial derivative of Γ appearing in Lemma 4 has a particular form.

Lemma 7. *Fix any multi-indices α and β as in Lemma 4. Then there exist a homogeneous polynomial $\mathcal{P}_{\alpha, \beta}$ of order $|\alpha| + 3|\beta|$ such that*

$$\frac{\partial_x^\beta \partial_v^\alpha \Gamma_{\bar{a}}(t, x, v; \tilde{t})}{\Gamma_{\bar{a}}(t, x, v; \tilde{t})} = \mathcal{P}_{\alpha, \beta}((A_0^{-1/2})_{ij}, (P^{-1/2}M)_{ij}, (A_0^{-1}v)_j, (M^T P^{-1}Mv)_j, (M^T P^{-1}x)_j, P^{-1/6}, (M^T P^{-1})^{1/3}), \quad (20)$$

where the last two terms in the polynomial, $(M^T P^{-1})^{1/3}$ and $P^{-1/6}$, are understood to only appear in the polynomial in powers that are multiples of three.

As its proof is somewhat short, we give it here. Before doing so, however, we make two observations. First, the above is essentially obvious when $\bar{a}(t) = \text{Id}$. Second (setting $\tilde{t} = 0$ for ease), using Lemma 7 and the kinetic scaling, in which we think of $v \sim \sqrt{t}$ and $x \sim t^{3/2}$, every input in the polynomial $\mathcal{P}_{\alpha, \beta}$ is $\sim t^{-1/2}$, making the entire polynomial $\sim t^{-|\alpha|-3|\beta|}$. This is precisely the reason that Lemma 4 holds.

Proof. Our proof proceeds by induction, first on the magnitude of α and then on the magnitude of β . The case $|\alpha| = |\beta| = 0$ is obvious.

We now consider the case where $\partial_v^\alpha = \partial_{v_i} \partial_v^{\tilde{\alpha}}$ for some $i \in \{1, \dots, d\}$ and $|\tilde{\alpha}| \geq 0$, and we assume that (20) holds for $\partial_v^{\tilde{\alpha}} \Gamma$. The derivative ∂_{v_i} can, by the product rule, either fall on $\mathcal{P}_{\tilde{\alpha}}$ or Γ . We consider each case in turn.

First, consider the former case; that is, consider the term that arises when the ∂_{v_i} falls on $\mathcal{P}_{\tilde{\alpha}}$. Observe that $\partial_{v_i} \mathcal{P}_{\tilde{\alpha}}$ yields a linear combination of terms that are a $|\tilde{\alpha}| - 1$ homogeneous polynomial multiplied by either

$$(A_0^{-1})_{ji} = (A_0^{-1/2})_{jk} (A_0^{-1/2})_{ki} \quad (21)$$

or

$$(M^T P^{-1} M)_{ji} = (M^T P^{-1/2})_{jk} (M^T P^{-1/2})_{ik}. \quad (22)$$

Each of (21) and (22) are 2-homogeneous in the variables of \mathcal{P}_α , making the resulting terms $|\tilde{\alpha}| - 1 + 2 = |\tilde{\alpha}| + 1$ homogeneous polynomials, as desired.

We now consider the latter case; that is, when ∂_{v_i} falls on Γ . The conclusion is then clear as

$$\mathcal{P}_{\tilde{\alpha}} \partial_{v_i} \Gamma = \mathcal{P}_{\tilde{\alpha}} \left(-\frac{1}{2} (A_0^{-1} v)_i + 2(M^T P x)_i - 2(M^T P M v)_i \right) \Gamma.$$

Hence, we are finished with the proof when $|\beta| = 0$.

The proof of the induction on β is essentially the same; hence, we omit it. \square

We are now in a position to prove Lemma 4.

Proof of Lemma 4. For ease, set $\tilde{t} = 0$. We discuss first the case when $(\xi_1, \xi_2, \xi_3) = 0$ and $j = 0$.

First, for notational ease, let

$$I = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\partial_v^\alpha \partial_x^\beta \Gamma_{\bar{a}}(t, x, v; 0)| |x|^r |v|^s dv dx.$$

By Lemma 7, we have

$$I = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{P}_{\alpha, \beta} \Gamma_{\bar{a}}(t, x, v; 0)| |x|^r |v|^s dv dx.$$

Using Lemma 6, we notice that

$$|\mathcal{P}_{\alpha, \beta}| \lesssim \tilde{\mathcal{P}}_{\alpha, \beta} \left(\frac{1}{\sqrt{t}}, \frac{x}{t^2}, \frac{v}{t} \right)$$

for some positive $|\alpha| + 3|\beta|$ -homogeneous polynomial $\tilde{\mathcal{P}}_{\alpha, \beta}$. Therefore, we get

$$\begin{aligned} I &\lesssim \frac{1}{\sqrt{\det(A_0 P)}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{\mathcal{P}}_{\alpha, \beta} \left(\frac{1}{\sqrt{t}}, \frac{x}{t^2}, \frac{v}{t} \right) e^{-\frac{v^T A_0^{-1} v}{4}} e^{-(x - Mv) \cdot P^{-1}(x - Mv)} |x|^r |v|^s dv dx \\ &= \frac{t^{\frac{3r}{2} + \frac{s}{2}}}{\sqrt{\det(A_0 P)}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{\mathcal{P}}_{\alpha, \beta} \left(\frac{1}{\sqrt{t}}, \frac{x}{t^2}, \frac{v}{t} \right) e^{-\frac{v^T A_0^{-1} v}{4}} e^{-(x - Mv) \cdot P^{-1}(x - Mv)} \frac{|x|^r}{t^{\frac{3r}{2}}} \frac{|v|^s}{t^{\frac{s}{2}}} dv dx. \end{aligned}$$

Next, we change variables to find

$$I \lesssim \frac{t^{\frac{3r}{2} + \frac{s}{2} - 2d}}{\sqrt{\det(A_0 P)}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{\mathcal{P}}_{\alpha, \beta} \left(\frac{1}{\sqrt{t}}, \frac{\bar{x}}{\sqrt{t}}, \frac{\bar{v}}{\sqrt{t}} \right) e^{-t \frac{\bar{v}^T A_0^{-1} \bar{v}}{4}} e^{-t^3 (\bar{x} - t^{-1} M \bar{v}) \cdot P^{-1} (\bar{x} - t^{-1} M \bar{v})} |\bar{x}|^r |\bar{v}|^s d\bar{v} d\bar{x}.$$

Notice that

$$\tilde{\mathcal{P}}_{\alpha, \beta} \left(\frac{1}{\sqrt{t}}, \frac{\bar{x}}{\sqrt{t}}, \frac{\bar{v}}{\sqrt{t}} \right) = t^{-\frac{|\alpha|+3|\beta|}{2}} \tilde{\mathcal{P}}_{\alpha, \beta}(1, \bar{x}, \bar{v})$$

due to the homogeneity of $\tilde{\mathcal{P}}$. Hence,

$$I \lesssim \frac{t^{\frac{3r+s-|\alpha|-3|\beta|}{2}-2d}}{\sqrt{\det(A_0 P)}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{\mathcal{P}}_{\alpha,\beta}(1, \bar{v}, \bar{x}) e^{-t^{\frac{\bar{v} \cdot A_0^{-1} \bar{v}}{4}}} e^{-t^3(\bar{x} - t^{-1} M \bar{v}) \cdot P^{-1}(\bar{x} - t^{-1} M \bar{v})} |\bar{x}|^r |\bar{v}|^s d\bar{v} d\bar{x}.$$

We change variables one final time with $\bar{y} = \bar{x} - t^{-1} M \bar{v}$ to find

$$\begin{aligned} I &\lesssim \frac{t^{\frac{3r+s-|\alpha|-3|\beta|}{2}-2d}}{\sqrt{\det(A_0 P)}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{\mathcal{P}}_{\alpha,\beta}(1, \bar{v}, \bar{y} + t^{-1} M \bar{v}) e^{-t^{\frac{\bar{v} \cdot A_0^{-1} \bar{v}}{4}}} e^{-t^3 \bar{y} \cdot P^{-1} \bar{y}} |\bar{y} + t^{-1} M \bar{v}|^r |\bar{v}|^s d\bar{v} d\bar{y} \\ &\lesssim \frac{t^{\frac{3r+s-|\alpha|-3|\beta|}{2}-2d}}{\sqrt{\det(A_0 P)}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{\mathcal{P}}_{\alpha,\beta}(1, \bar{v}, \bar{y} + t^{-1} M \bar{v}) e^{-t^{\frac{\bar{v} \cdot A_0^{-1} \bar{v}}{4}}} e^{-t^3 \bar{y} \cdot P^{-1} \bar{y}} (|\bar{y}|^r + |\bar{v}|^r) |\bar{v}|^s d\bar{v} d\bar{y}. \end{aligned}$$

In the last inequality we used Lemma 6 to bound $t^{-1}|M| \lesssim 1$.

At this point, it follows from Lemma 6 that the quadratic terms in the exponential are bounded below as

$$t\bar{v} \cdot A_0^{-1} \bar{v} + t^3 \bar{y} \cdot P^{-1} \bar{y} \gtrsim |\bar{v}|^2 + |\bar{y}|^2.$$

The conclusion follows then from a simple calculation:

$$I \lesssim \frac{t^{\frac{3r+s-|\alpha|-3|\beta|}{2}-2d}}{\sqrt{\det(A_0 P)}}.$$

The proof of this case is concluded after applying Lemma 6 again in order to bound the determinant.

The case where $j > 0$ reduces to the case above via the identity:

$$\partial_t \Gamma_{\bar{a}} + v \cdot \nabla_x \Gamma_{\bar{a}} = \text{tr}(a(t) D_v^2 \Gamma_{\bar{a}}).$$

This concludes the proof of all cases where $(\xi_1, \xi_2, \xi_3) = 0$.

The general case can easily be handled as follows. First change variables:

$$\begin{aligned} &\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \max_{(0, \xi_1, \xi_2) \in Q_{\sqrt{t/2}}} |\partial_t^j \partial_x^\beta \partial_v^\alpha \Gamma_{\bar{a}}(t, x + \xi_1, v + \xi_2; 0)| |x|^r |v|^s dx dv \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \max_{(0, \xi_1, \xi_2) \in Q_{\sqrt{t/2}}} |\partial_t^j \partial_x^\beta \partial_v^\alpha \Gamma_{\bar{a}}(t, x, v; 0)| |x - \xi_1|^r |v - \xi_2|^s dx dv. \end{aligned}$$

Next, using the inequalities

$$|x - \xi_1|^r \lesssim |x|^r + |\xi_1|^r \quad \text{and} \quad |v - \xi_2|^s \lesssim |v|^s + |\xi_2|^s.$$

At this point, the four resulting integrals may be estimated using the case above (keeping in mind the conditions $|\xi_1| \leq t^{3/2}$ and $|\xi_2| \leq \sqrt{t}$). This concludes the proof. \square

2.1.2. The proof of Lemma 6: understanding the matrices A_i , P , and M .

Proof. We note that the upper bounds in all cases (i), (ii), and (iii) are obvious from the assumptions on \bar{a} (12) and the definition of the matrices (16). The lower bounds of A_i in (ii) are also obvious. Hence, we need only prove the lower bound for P in (i).

For ease, we set $\tilde{t} = 0$ and simply drop the “; 0” notation from all quantities.

To obtain this lower bound, notice that it suffices to establish a uniform bound of the form

$$w \cdot (P(t)w) \gtrsim t^3 \tag{23}$$

for any vector $w \in \mathbb{R}^d$ with $|w| = 1$. We proceed by analyzing the time derivative of P . First,

$$\begin{aligned} P'(t) &= (A_2 - A_1 A_0^{-1} A_1)'(t) \\ &= t^2 \bar{a} - t \bar{a} A_0^{-1} A_1 + A_1 (A_0^{-1} \bar{a} A_0^{-1}) A_1 - A_1 A_0^{-1} t \bar{a} \\ &= (t \sqrt{\bar{a}} - A_1 A_0^{-1} \sqrt{\bar{a}})(t \sqrt{\bar{a}} - \sqrt{\bar{a}} A_0^{-1} A_1) = M^T \bar{a} M \geq 0. \end{aligned}$$

Thus,

$$w \cdot P(t)w = \int_0^t (M(s)w) \cdot \bar{a} M(s)w \, ds \geq \frac{1}{\Lambda} \int_0^t |M(s)w|^2 \, ds. \quad (24)$$

In order to establish (23), it is enough to show

$$\int_0^t |M(s)w|^2 \, ds \gtrsim t^3. \quad (25)$$

This is our focus for the remainder of the proof.

To obtain this lower bound, we use the following intuition. Recall in equation (16)

$$A_i(s) = \int_0^s \tau^i \bar{a}(\tau) \, d\tau, \quad \text{and} \quad M(s) = s \text{Id} - A_0^{-1}(s) A_1(s).$$

The time derivative of M is

$$\begin{aligned} M'(s) &= \text{Id} - (A_0^{-1} A_1)'(s) = \text{Id} + A_0(s)^{-1} \bar{a}(s) A_0^{-1}(s) A_1(s) - s A_0(s)^{-1} \bar{a}(s) \\ &= \text{Id} - A_0(s)^{-1} \bar{a}(s) M(s). \end{aligned} \quad (26)$$

From (26), we see that when $M(s)w$ is “small,” that is $o(t)$, $M'(s)w$ is approximately w . That is $M(s)w$ moves radially (in the direction of w) with a velocity ≈ 1 away from the origin. This means that, eventually, $M(s)w$ will move radially across a distance $O(t)$ at a bounded velocity. This would yield the desired bound (25).

In order to make this rigorous, we proceed in three steps. Fix $\varepsilon > 0$ sufficiently small in a way to be determined. The first step is to note that either there is an interval $[\varepsilon t, 2\varepsilon t]$ where $M(s)w$ always has magnitude $O(t)$ or not. If so, we are done. If not, we proceed to the second step. The second step takes a time t_0 in the interval $[\varepsilon t, 2\varepsilon t]$ in which $M(t_0)w$ is “small” and shows that it gets ‘big.’ The third step is to show that $M(t_0)w$ remains “big.” The second and third steps are dependent on the time derivative of M .

Step one: Notice that $|M(0)w| = 0$. If

$$|M(s)w| \geq \varepsilon^3 t \quad \text{for all } s \in [\varepsilon t, 2\varepsilon t]$$

then we are finished with the proof. Hence, assume that there is

$$t_0 \in [\varepsilon t, 2\varepsilon t] \quad \text{such that } |M(t_0)w| < \varepsilon^3 t.$$

Step two: We claim that

$$t_1 := \inf\{s > t_0 : |M(s)w| \geq \varepsilon^3 t\} \leq 3\varepsilon t. \quad (27)$$

Roughly, t_1 is the first time after t_0 that $|M(s)w|$ becomes “big,” that is, has norm $\varepsilon^3 t$.

Using the time derivative of M (26), we obtain the identity

$$M(t_1)w = M(t_0)w + \int_{t_0}^{t_1} (w - A_0(s)^{-1} a(s) M(s)w) \, ds. \quad (28)$$

Combining (28) with the definition of t_1 (27) and the bound Lemma 6.(ii), we find

$$\begin{aligned}
t_1 - t_0 &= (t_1 - t_0)|w| = \left| \int_{t_0}^{t_1} w \, ds \right| = \left| M(t_1)w - M(t_0)w + \int_{t_0}^{t_1} A_0(s)^{-1} \bar{a}(s)M(s)w \, ds \right| \\
&\leq |M(t_1)w| + |M(t_0)w| + \int_{t_0}^{t_1} |A_0(s)^{-1} \bar{a}(s)M(s)w| \, ds \\
&\leq 2\varepsilon^3 t + \int_{t_0}^{t_1} \frac{C|M(s)w|}{s} \, ds \leq 2\varepsilon^3 t + \int_{t_0}^{t_1} \frac{C\varepsilon^3 t}{s} \, ds \leq 2\varepsilon^3 t + \frac{C\varepsilon^3 t(t_1 - t_0)}{\varepsilon t}.
\end{aligned} \tag{29}$$

where C is a universal constant depending only on d and Λ . The last inequality uses that $s \geq t_0 \geq \varepsilon t$. Before continuing, we note that the last integral in (29) reveals the necessity of Step One, above. Indeed, the final integral above is not bounded for t_0 near 0. Step One allows us to avoid this singularity.

Returning to (29), notice that, if ε is sufficiently small then $C\varepsilon^2 < 1/2$. Thus, after rearranging (29), we find

$$\frac{t_1 - t_0}{2} \leq 2\varepsilon^3 t.$$

Rearranging this, recalling that $t_0 \leq 2\varepsilon t$, and further decreasing ε , we obtain

$$t_1 \leq t_0 + 4\varepsilon^2 t < 2\varepsilon t + \varepsilon t.$$

Hence (27) is established.

Step Three: We claim that

$$t_2 := \sup\{s \in (t_1, t] : |M(s)w| \geq \varepsilon^4 t\} \geq t_1 + \varepsilon^4 t. \tag{30}$$

Roughly, t_2 is the first time after t_1 (at which time $|M(s)w|$ is “big”) that $|M(s)w|$ becomes “small,” that is $\varepsilon^4 t$.

Before showing this, we claim this allows us to conclude. Indeed,

$$|M(s)w| \geq \varepsilon^4 t \quad \text{for } s \in (t_1, t_2) \quad \text{and} \quad t_2 - t_1 \geq \varepsilon^4 t.$$

Hence,

$$\int_0^t |M(s)w|^2 \, ds \geq \int_{t_1}^{t_2} |M(s)w|^2 \, ds \geq (t_2 - t_1)(\varepsilon^4 t)^2 \geq \varepsilon^{12} t^3.$$

In view of (24), this establishes the claim (25), which concludes the proof. Thus, it is enough to prove (30), which is our focus now.

If $t_2 = t$, we are finished. Hence, we assume that $t_2 < t$, which implies that

$$|M(t_2)w| = \varepsilon^4 t.$$

Also, using (26) once again, we find

$$M(t_1)w - M(t_2)w = -(t_2 - t_1)w + \int_{t_1}^{t_2} A_0^{-1}(s) \bar{a}(s)M(s)w \, ds.$$

Combining the two identities above, and recalling from (27) that $|M(t_1)w| = \varepsilon^3 t$, we find

$$\varepsilon^3 t(1 - \varepsilon) \leq |M(t_1)w - M(t_2)w| \leq (t_2 - t_1) + \int_{t_1}^{t_2} C \, ds = (C + 1)(t_2 - t_1).$$

Rearranging this and decreasing ε if necessary, we find (30). This concludes the proof. \square

2.2. The second step: full Schauder estimates by perturbing off the constant-in- (x, v) coefficients problem. By a careful procedure taking into account the natural scalings and available interpolations, we can perturb off the constant-in- (x, v) coefficients problem in order to obtain the full Schauder estimates. In short, we finish the proof of Theorem 1 by leveraging Proposition 5.

We begin by stating two important technical lemmas. The proof of the first is given in [34] and the second is standard (it can be seen easily by scaling, for example), but a proof can be found in [43, Lemma 2.10] up to an application of Young's inequality.

Lemma 8 (Lemma 4.3 in [34]). *Let $\omega(r) > 0$ be bounded in $[r_0, r_1]$ with $r_0 \geq 0$. Suppose that there is $\mu \in (0, 1)$ and constants $A, B, p \geq 0$ so that, for all $r_0 \leq r < R \leq r_1$,*

$$\omega(r) \leq \mu\omega(R) + \frac{A}{(R-r)^p} + B.$$

Then for any $r_0 \leq r < R \leq r_1$, there holds

$$\omega(r) \lesssim \frac{A}{(R-r)^p} + B,$$

where the implied constant depends only on μ and p .

Lemma 9 (Interpolation inequalities). *Fix any $Q = Q_r$ for any $r \geq 1/2$ and any $\alpha \in (0, 1)$. For any $\epsilon \in (0, 1)$, the following hold:*

$$\begin{aligned} [u]_{C_x^{\alpha/3}C_v^\alpha(Q)} &\lesssim \epsilon^2 \left([u]_{C_x^{(2+\alpha)/3}(Q)} + [D_v^2 u]_{C_x^{\alpha/3}C_v^\alpha(Q)} \right) + \epsilon^{-\alpha} \|u\|_{L^\infty(Q)}, \\ [D_v u]_{C_x^{\alpha/3}C_v^\alpha(Q)} &\lesssim \epsilon \left([u]_{C_x^{(2+\alpha)/3}(Q)} + [D_v^2 u]_{C_x^{\alpha/3}C_v^\alpha(Q)} \right) + \epsilon^{-\alpha-1} \|u\|_{L^\infty(Q)}, \\ \|D_v u\|_{L^\infty(Q)} &\lesssim \epsilon^{\alpha+1} [D_v^2 u]_{C_x^{\alpha/3}C_v^\alpha(Q)} + \epsilon^{-1} \|u\|_{L^\infty(Q)}, \quad \text{and} \\ \|D_v^2 u\|_{L^\infty(Q)} &\lesssim \epsilon^\alpha [D_v^2 u]_{C_x^{\alpha/3}C_v^\alpha(Q)} + \epsilon^{-2} \|u\|_{L^\infty(Q)}. \end{aligned}$$

With these in hand, we now prove the full Schauder estimates.

Proof of Theorem 1. We estimate $[D_v^2 f]_{C_x^{\alpha/3}C_v^\alpha(Q_{1/2})}$ and omit the proof of the other terms as they are similar. For succinctness, in this proof, we use the following notation:

$$[f]_{2+\alpha,r}' := [D_v^2 f]_{C_x^{\alpha/3}C_v^\alpha(Q_r)} + [f]_{C_x^{(\alpha+2)/3}(Q_r)}.$$

The key estimate that we establish is the following. There is $\varepsilon_0 > 0$ sufficiently small so that, with

$$\theta_0 := \min \left\{ \frac{1}{8}, \varepsilon_0 [\bar{a}]_{C_x^{\alpha/3}C_v^\alpha(Q_1)}^{-1/\alpha} \right\} \quad (31)$$

then

$$[f]_{2+\alpha,r}' \leq \frac{1}{2} [f]_{2+\alpha,r+2\theta}' + C([\bar{c}]_{C_x^{\alpha/3}C_v^\alpha(Q_1)} + \theta^{-2-\alpha}) \|f\|_{L^\infty(Q_1)} + C\theta^{-\alpha} \|g\|_{C_x^{\alpha/3}C_v^\alpha(Q_1)}, \quad (32)$$

for some $C > 0$ and all $\theta \in (0, \theta_0]$ and $r \in [1/4, 3/4]$. The proof of (32) is complicated, so we postpone it until after we show how Theorem 1 follows from it.

In order to prove Theorem 1 from (32), we first rewrite (32) in a manner more adapted to Lemma 8. Indeed, applying Lemma 8 with, in its notation, the choices

$$\begin{aligned} r_1 &= 1/4, \quad r_2 = 3/4, \quad \omega(r) = [f]_{2+\alpha,r}', \quad R = r + 2\theta_0, \quad \mu = \frac{1}{2}, \\ A &= \|f\|_{L^\infty(Q_1)} + \theta_0^2 [g]_{C_x^{\alpha/3} C_v^\alpha(Q_1)}, \quad B = C[\bar{c}]_{\alpha(Q_1)} \|f\|_{L^\infty(Q_1)}, \quad \text{and} \quad p = 2 + \alpha, \end{aligned}$$

yields

$$\begin{aligned} [f]_{2+\alpha,1/2}' &\lesssim \theta_0^{-2-\alpha} \left(\|f\|_{L^\infty(Q_1)} + \theta_0^2 [g]_{C_x^{\alpha/3} C_v^\alpha(Q_1)} \right) + [\bar{c}]_{C_x^{\alpha/3} C_v^\alpha(Q_1)} \|f\|_{L^\infty(Q_1)} \\ &\lesssim \left(1 + [\bar{c}]_{C_x^{\alpha/3} C_v^\alpha(Q_1)} + [\bar{a}]_{C_x^{\alpha/3} C_v^\alpha(Q_1)}^{1+\frac{2}{\alpha}} \right) \|f\|_{L^\infty(Q_1)} + \left(1 + [\bar{a}]_{C_x^{\alpha/3} C_v^\alpha(Q_1)} \right) [g]_{C_x^{\alpha/3} C_v^\alpha(Q_1)}. \end{aligned}$$

Thus, Theorem 1 is proved, up to establishing (32).

We now prove (32). We argue under the assumption that

$$[D_v^2 f]_{C_x^{\alpha/3} C_v^\alpha(Q_r)} \geq [f]_{C_x^{(2+\alpha)/3}(Q_r)}$$

so that

$$[f]_{2+\alpha,r}' \leq 2[D_v^2 f]_{C_x^{\alpha/3} C_v^\alpha(Q_r)},$$

although the proof is similar in the opposite case.

Fix $z_0, z_1 \in Q_r$ with $t_0 = t_1$ so that

$$\frac{|D_v^2 f(z_0) - D_v^2 f(z_1)|}{|x_0 - x_1|^{\alpha/3} + |v_0 - v_1|^\alpha} \geq \frac{1}{2} [D_v^2 f]_{C_x^{\alpha/3} C_v^\alpha(Q_r)} \geq \frac{1}{4} [f]_{2+\alpha,Q_r}'. \quad (33)$$

Up to a change of variables, we may assume that $z_1 = 0$, which make the expressions in the sequel simpler.

Fix $\theta \in (0, \theta_0]$. If

$$|x_0|^{\alpha/3} + |v_0|^\alpha > \theta^\alpha, \quad (34)$$

then we have, using Lemma 9,

$$\begin{aligned} [f]_{2+\alpha,Q_r}' &\lesssim \frac{|D_v^2 f(z_0) - D_v^2 f(0)|}{|x_0|^{\alpha/3} + |v_0|^\alpha} \lesssim \theta^{-\alpha} \|D_v^2 f\|_{L^\infty(Q_r)} \\ &\leq \frac{1}{2} [D_v^2 f]_{C_x^{\alpha/3} C_v^\alpha(Q_r)} + C\theta^{-2-\alpha} \|f\|_{L^\infty(Q_r)}, \end{aligned}$$

and (32) is proved.

Next we consider the case when (34) does not hold. We introduce a cut-off function $0 \leq \chi \leq 1$ such that

$$\chi(t, x, v) = \begin{cases} 1 & \text{if } |t|^{1/2} + |x|^{1/3} + |v| \leq \theta, \\ 0 & \text{if } |t|^{1/2} + |x|^{1/3} + |v| \geq 2\theta, \end{cases}$$

that satisfies the bounds

$$[(\partial_t + v \cdot \nabla_x) \chi]_{C_x^{\alpha/3} C_v^\alpha(Q_1)} + [D_v^2 \chi]_{C_x^{\alpha/3} C_v^\alpha(Q_1)} \lesssim \theta^{-2-\alpha}. \quad (35)$$

and

$$\|(\partial_t + v \cdot \nabla_x) \chi\|_{L^\infty(Q)} + \|D_v^2 \chi\|_{L^\infty(Q)} \lesssim \theta^{-2}. \quad (36)$$

We note that estimates on the other norms and semi-norms of χ can be obtained easily via Lemma 9. Additionally, to make the notation simpler, we define

$$\tilde{a}(t) = \bar{a}(t, 0, 0) \quad \text{and} \quad L = \partial_t + v \cdot \nabla_x - \text{tr}(\bar{a} D_v^2 \cdot).$$

First, using (33) and the fact that, due to (34) does not hold, $\chi \equiv 1$ between $(0, 0)$ and (x_0, v_0) , we find

$$\begin{aligned} [D_v^2 f]_{C_x^{\alpha/3} C_v^\alpha(Q_r)} &\leq 2 \frac{|D_v^2 f(z_0) - D_v^2 f(0, 0)|}{|x_0|^{\alpha/3} + |v_0|^\alpha} = 2 \frac{|D_v^2(\chi f)(z_0) - D_v^2(\chi f)(0, 0)|}{|x_0|^{\alpha/3} + |v_0|^\alpha} \\ &\leq 2[D_v^2(\chi f)]_{C_x^{\alpha/3} C_v^\alpha(Q_1)}. \end{aligned}$$

Then, applying Proposition 5, we have

$$\begin{aligned} [D_v^2 f]_{C_x^{\alpha/3} C_v^\alpha(Q_r)} &\lesssim [D_v^2(\chi f)]_{C_x^{\alpha/3} C_v^\alpha(Q_1)} \lesssim [\partial_t(\chi f) + v \cdot \nabla_x(\chi f) - \text{tr}(\tilde{a} D_v^2(\chi f))]_{C_x^{\alpha/3} C_v^\alpha(Q_1)} \\ &\lesssim [L(\chi f)]_{C_x^{\alpha/3} C_v^\alpha(Q_1)} + [\text{tr}((\bar{a} - \tilde{a}) D_v^2(\chi f))]_{C_x^{\alpha/3} C_v^\alpha(Q_1)}. \end{aligned} \quad (37)$$

We consider the first term on the right hand side of (37). Using the equation for f , we see

$$L(\chi f) = f L \chi - 2\bar{a} \nabla_v f \nabla_v \chi + \bar{c} \chi f + \chi g.$$

Thus, by the triangle inequality

$$\begin{aligned} &[L(\chi f)]_{C_x^{\alpha/3} C_v^\alpha(Q_1)} \\ &\leq [\chi g]_{C_x^{\alpha/3} C_v^\alpha(Q_1)} + [f L \chi]_{C_x^{\alpha/3} C_v^\alpha(Q_1)} + [\bar{c} \chi f]_{C_x^{\alpha/3} C_v^\alpha(Q_1)} + 2[\bar{a} \nabla_v f \nabla_v \chi]_{C_x^{\alpha/3} C_v^\alpha(Q_1)} \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

For I_1 , we use the boundedness of the cut-off function and have

$$I_1 \lesssim \theta^{-\alpha} \|g\|_{C_x^{\alpha/3} C_v^\alpha(Q_1)}.$$

Next we consider I_2 . Keeping in mind the support of χ and using the interpolation inequality Lemma 9 and (35) yields

$$\begin{aligned} I_2 &\lesssim [f]_{C_x^{\alpha/3} C_v^\alpha(Q_{r+2\theta})} \|L \chi\|_{L^\infty(Q_{r+2\theta})} + [L \chi]_{C_x^{\alpha/3} C_v^\alpha(Q_{r+2\theta})} \|f\|_{L^\infty(Q_{r+2\theta})} \\ &\lesssim \theta^{-2} [f]_{C_x^{\alpha/3} C_v^\alpha(Q_{r+2\theta})} + (\theta^{-2-\alpha} + [\bar{a} D_v^2 \chi]_{C_x^{\alpha/3} C_v^\alpha(Q_{r+2\theta})}) \|f\|_{L^\infty(Q_{r+2\theta})}. \end{aligned}$$

Using (35)-(36), we have

$$\begin{aligned} [\bar{a} D_v^2 \chi]_{C_x^{\alpha/3} C_v^\alpha(Q_{r+2\theta})} &\leq [\bar{a}]_{C_x^{\alpha/3} C_v^\alpha(Q_{r+2\theta})} \|D_v^2 \chi\|_{L^\infty(Q_{r+2\theta})} + \|\bar{a}\|_{L^\infty(Q_{r+2\theta})} [D_v^2 \chi]_{C_x^{\alpha/3} C_v^\alpha(Q_{r+2\theta})} \\ &\lesssim \theta^{-2} [\bar{a}]_{C_x^{\alpha/3} C_v^\alpha(Q_{r+2\theta})} + \theta^{-2-\alpha}. \end{aligned}$$

and, for $\varepsilon > 0$ to be chosen depending only on d , α , and Λ (recall (3)),

$$\theta^{-2} [f]_{C_x^{\alpha/3} C_v^\alpha(Q_{r+2\theta})} \lesssim \varepsilon [f]_{2+\alpha, Q_{r+2\theta}}' + \theta^{-2-\alpha} \|f\|_{L^\infty(Q_1)}.$$

Note that, as ε will not be chosen to depend on $[\bar{a}]_{C_x^{\alpha/3} C_v^\alpha(Q_1)}$ or θ , we omit all negative powers of ε .

Therefore, we conclude that

$$I_2 \lesssim \varepsilon [f]_{2+\alpha, Q_{r+2\theta}}' + (\theta^{-2} [\bar{a}]_{C_x^{\alpha/3} C_v^\alpha(Q_{r+2\theta})} + \theta^{-2-\alpha}) \|f\|_{L^\infty(Q_1)}.$$

The terms I_3 and I_4 may be handled similarly to obtain

$$I_3 \lesssim ([\bar{c}]_{C_x^{\alpha/3} C_v^\alpha(Q_1)} + \theta^{-\alpha}) \|f\|_{L^\infty(Q_1)} + \varepsilon [f]_{2+\alpha, Q_{r+2\theta}}'$$

and

$$\begin{aligned}
I_4 &\lesssim ([\bar{a}]_{C_x^{\alpha/3}C_v^\alpha(Q_1)}\theta^{-1} + \theta^{-1-\alpha})\|\nabla_v f\|_{L^\infty(Q_{r+2\theta})} + \theta^{-1}[\nabla_v f]_{C_x^{\alpha/3}C_v^\alpha(Q_{r+2\theta})} \\
&\lesssim ([\bar{a}]_{C_x^{\alpha/3}C_v^\alpha(Q_1)}\theta^{-1} + \theta^{-1-\alpha})(\varepsilon\theta^{1+\alpha}[D_v^2 f]_{C_x^{\alpha/3}C_v^\alpha(Q_{r+2\theta})} + \theta^{-1}\|f\|_{L^\infty(Q_{r+2\theta})}) \\
&\quad + \theta^{-1}(\varepsilon\theta[D_v^2 f]_{C_x^{\alpha/3}C_v^\alpha(Q_{r+2\theta})} + \theta^{-1-\alpha}\|f\|_{C_x^{\alpha/3}C_v^\alpha(Q_{r+2\theta})}) \\
&\lesssim (\varepsilon + \theta^\alpha[\bar{a}]_{C_x^{\alpha/3}C_v^\alpha(Q_1)})[D_v^2 f]_{C_x^{\alpha/3}C_v^\alpha(Q_{r+2\theta})} + (\theta^{-2}[\bar{a}]_{C_x^{\alpha/3}C_v^\alpha(Q_1)} + \theta^{-2-\alpha})\|f\|_{L^\infty(Q_1)}.
\end{aligned}$$

Hence, we conclude that

$$\begin{aligned}
[L(\chi f)]_{C_x^{\alpha/3}C_v^\alpha(Q_1)} &\lesssim (\varepsilon + \theta^\alpha[\bar{a}]_{C_x^{\alpha/3}C_v^\alpha(Q_1)})[f]_{2+\alpha,Q_{r+2\theta}}' \\
&\quad + ([c]_{C_x^{\alpha/3}C_v^\alpha(Q_1)} + \theta^{-2}[\bar{a}]_{C_x^{\alpha/3}C_v^\alpha(Q_1)} + \theta^{-2-\alpha})\|f\|_{L^\infty(Q_1)} + \theta^{-\alpha}\|g\|_{C_x^{\alpha/3}C_v^\alpha(Q_1)}.
\end{aligned}$$

We now consider the second term on the right hand side of (37). We begin with the usual splitting:

$$\begin{aligned}
[\text{tr}(\bar{a} - \tilde{a})D_v^2(\chi f)]_{C_x^{\alpha/3}C_v^\alpha(Q_1)} &\lesssim \|\bar{a} - \tilde{a}\|_{L^\infty(\text{supp}(\chi))}[D_v^2(\chi f)]_{C_x^{\alpha/3}C_v^\alpha(Q_1)} \\
&\quad + [\text{tr}(\bar{a} - \tilde{a})]_{C_x^{\alpha/3}C_v^\alpha(Q_1)}\|D_v^2(\chi f)\|_{L^\infty(Q_1)}.
\end{aligned} \tag{38}$$

The second term in (38) can be handled easily using the methods above (recall Lemma 9 and (35)):

$$[\text{tr}(\bar{a} - \tilde{a})]_{C_x^{\alpha/3}C_v^\alpha(Q_1)}\|D_v^2(\chi f)\|_{L^\infty(Q_1)} \lesssim [\bar{a}]_{C_x^{\alpha/3}C_v^\alpha(Q_1)} \left(\theta^\alpha[f]_{2+\alpha,Q_{r+2\theta}}' + \theta^{-2}\|f\|_{L^\infty(Q_1)} \right).$$

Estimating the first term in (38) uses the fact that χ has support of size θ :

$$\|\bar{a} - \tilde{a}\|_{L^\infty(\text{supp}(\chi))} \lesssim \theta^\alpha[\bar{a}]_{C_x^{\alpha/3}C_v^\alpha(Q_1)}.$$

After applying Lemma 9 and (35), we arrive at

$$\|\bar{a} - \tilde{a}\|_{L^\infty(\text{supp}(\chi))}[D_v^2(\chi f)]_{C_x^{\alpha/3}C_v^\alpha(Q_1)} \lesssim \theta^\alpha[\bar{a}]_{C_x^{\alpha/3}C_v^\alpha(Q_1)} \left([f]_{2+\alpha,Q_{r+2\theta}}' + \theta^{-2}\|f\|_{L^\infty(Q_1)} \right).$$

Therefore, we obtain the following bound on the second term on the right hand side of (37):

$$[\text{tr}(\bar{a} - \tilde{a})D_v^2(\chi f)]_{C_x^{\alpha/3}C_v^\alpha(Q_1)} \lesssim \theta^\alpha[\bar{a}]_{C_x^{\alpha/3}C_v^\alpha(Q_1)} \left([f]_{2+\alpha,Q_{r+2\theta}}' + \theta^{-2}\|f\|_{L^\infty(Q_1)} \right).$$

Combining all above estimates, we have

$$\begin{aligned}
[f]_{2+\alpha,r}' &\leq C(\varepsilon + \theta^\alpha[\bar{a}]_{C_x^{\alpha/3}C_v^\alpha(Q_1)})[f]_{2+\alpha,Q_{r+2\theta}}' \\
&\quad + C([\bar{c}]_{C_x^{\alpha/3}C_v^\alpha(Q_1)} + \theta^{-2}[\bar{a}]_{C_x^{\alpha/3}C_v^\alpha(Q_1)} + \theta^{-2-\alpha})\|f\|_{L^\infty(Q_1)} + C\theta^{-\alpha}\|g\|_{C_x^{\alpha/3}C_v^\alpha(Q_1)},
\end{aligned} \tag{39}$$

where C is some universal constant. Choosing ε and ε_0 sufficiently small and recalling the definition of θ_0 (31) and that $\theta \leq \theta_0$, we have

$$C(\varepsilon + \theta^\alpha)[\bar{a}]_{C_x^{\alpha/3}C_v^\alpha(Q_1)} \leq \frac{1}{2} \quad \text{and} \quad \theta^{-2}[\bar{a}]_{C_x^{\alpha/3}C_v^\alpha(Q_1)} \lesssim \theta^{-2-\alpha}.$$

Using this in (39), we obtain (32), which concludes the proof. \square

3. Uniqueness for the Landau equation: Theorem 2. Before beginning the proof, we review a few useful bounds that follow from our assumptions. For any $h \in L^{\infty, 5+\gamma+\eta}$ for any $\eta > 0$, we have

$$e \cdot \bar{a}^h(t, x, v) e \lesssim \|h\|_{L^{\infty, 5+\gamma+\eta}} \begin{cases} \langle v \rangle^\gamma & \text{if } e \parallel v, \\ \langle v \rangle^{2+\gamma} & \text{otherwise,} \end{cases} \quad (40)$$

for any $e \in \mathbb{S}^2$. The lower order coefficient \bar{c}^h satisfies a similar bound

$$0 \leq \bar{c}^h(t, x, v) \lesssim \langle v \rangle^\gamma \|h\|_{L^{\infty, 3}}. \quad (41)$$

These bounds are not optimal in the weight; it is clear that 3 can be replaced by any $k > 3 + \gamma$. The proofs of (40) and (41) are straightforward but can be seen in [38, Lemma 2.1].

Finally, due to the assumption (8), the solution f that is the subject of Theorem 2 satisfies a matching lower bound to (40):

$$e \cdot \bar{a}^f(t, x, v) e \gtrsim \begin{cases} \langle v \rangle^{2+\gamma} & \text{if } e \perp v, \\ \langle v \rangle^\gamma & \text{otherwise.} \end{cases} \quad (42)$$

Here we suppress the explicit dependence on f as it depends in a complicated way on δ, r, R , and $\|f\|_{L^{\infty, k}}$. This inequality (42) follows from [38, Theorem 1.3 and Lemma 4.3].

We now state the main quantitative estimate that allows us to deduce uniqueness (Theorem 2), which is postponed until Section 3.1. This estimate requires Theorem 1 in a crucial way.

Proposition 10. *Under the assumptions of Theorem 2, there are $\alpha' \in (0, \alpha)$ and $\theta' \in (0, \theta)$ so that*

$$\frac{\theta'}{2} \frac{\alpha'}{2 + \alpha'} > 1,$$

and $T_0 < 1/2$ such that, for any $t \in [0, T_0]$,

$$\|\langle v \rangle^7 D_v^2 f(t)\|_{L^\infty(\mathbb{R}^6)} \lesssim \frac{1}{t \log(\frac{1}{t})^{\frac{\theta'}{2} \frac{\alpha'}{2 + \alpha'}}}.$$

The final time T_0 depends only on $\alpha, \theta, k, \|f_0\|_{L^{\infty, k}}$, and $\|f_0\|_{C_x^{\alpha/3} \log(1/C_v)^{-\theta}}$.

With this in hand, we are in position to prove the second main theorem, the uniqueness of solutions to the Landau equation with initial data having Hölder regularity in x and log-Hölder regularity in v . While the estimate of Proposition 10 is different from its analogue in [38, Lemma 4.3 and Proposition 4.4], its application in deducing uniqueness is quite similar to the proof of uniqueness in [38]. However, we provide the proof as some technical details must be altered.

Proof of Theorem 2. For succinctness, we set

$$\ell = 5 + \gamma + \eta,$$

and, without loss of generality, we may assume that

$$\ell \leq 5. \quad (43)$$

Let $r \in C(0, T_0] \cap L^1[0, T_0]$ be a positive function to be determined and define

$$w = e^{-\int_0^t r(s) ds} (g - f) \quad \text{and} \quad W = \langle v \rangle^{2\ell} w^2.$$

Our goal is to show that $W \equiv 0$ as this immediately implies that $f \equiv g$. We proceed by contradiction, assuming that there is $\epsilon > 0$ such that

$$\sup_{[0, T_0] \times \mathbb{R}^6} W(t, x, v) > \epsilon.$$

Following the work in [38, Proposition 5.2], we may find a point $z_\epsilon = (t_\epsilon, x_\epsilon, z_\epsilon)$ with $t_\epsilon > 0$ such that

$$W(z_\epsilon) = \epsilon \quad \text{and} \quad \sup_{[0, t_\epsilon] \times \mathbb{R}^6} W(t, x, v) \leq \epsilon. \quad (44)$$

Next, a direct, albeit lengthy, computation from equation (6) yields an equation for W . Indeed:

$$\begin{aligned} (\partial_t + v \cdot \nabla_x) W &= 2\langle v \rangle^{2\ell} w e^{-\int_0^t r} (\partial_t + v \cdot \nabla_x)(g - f) - 2rW \\ &= 2\langle v \rangle^{2\ell} w e^{-\int_0^t r} ((\text{tr } \bar{a}^g D_v^2 g) + \bar{c}^g g - (\text{tr } (\bar{a}^f D_v^2 f) + \bar{c}^f f)) - 2rW \\ &= 2\langle v \rangle^{2\ell} w e^{-\int_0^t r} (\text{tr } (\bar{a}^g D_v^2 g) + \bar{c}^g(g - f) - (\text{tr } (\bar{a}^f D_v^2 f) - \bar{c}^{g-f} f)) - 2rW \\ &= 2\langle v \rangle^{2\ell} w e^{-\int_0^t r} (\text{tr } (\bar{a}^g D_v^2 g) - \text{tr } (\bar{a}^f D_v^2 f)) + 2\bar{c}^g W + 2\langle v \rangle^{2\ell} w \bar{c}^w f - 2rW. \end{aligned} \quad (45)$$

We point out three computations the allow us to write the right hand side above in terms of w and W . First:

$$e^{-\int_0^t r} (\bar{a}^g D_v^2 g - \bar{a}^g D_v^2 f) = e^{-\int_0^t r} (\bar{a}^g D_v^2(g - f) + \bar{a}^{g-f} D_v^2 f) = \bar{a}^g D_v^2 w + \bar{a}^w D_v^2 f$$

Second:

$$2\langle v \rangle^{2\ell} w \nabla_v w = \nabla_v W - 2\ell \frac{v}{\langle v \rangle^2} W. \quad (46)$$

Third (using (46)):

$$\begin{aligned} D_v^2 W &= \langle v \rangle^{2\ell} D_v^2 w^2 + 2\nabla_v \langle v \rangle^{2\ell} \otimes \nabla_v w^2 + w^2 D_v^2 \langle v \rangle^{2\ell} \\ &= 2\langle v \rangle^{2\ell} w D_v^2 w + 2\langle v \rangle^{2\ell} \nabla_v w \otimes \nabla_v w + 8\ell \langle v \rangle^{2\ell-2} w v \otimes \nabla_v w + w^2 D_v^2 \langle v \rangle^{2\ell} \\ &= 2\langle v \rangle^{2\ell} w D_v^2 w + \frac{1}{2W} \left(\nabla_v W - 2\ell \frac{v}{\langle v \rangle^2} W \right) \otimes \left(\nabla_v W - 2\ell \frac{v}{\langle v \rangle^2} W \right) \\ &\quad + \frac{4\ell}{\langle v \rangle^2} v \otimes \left(\nabla_v W - 2\ell \frac{v}{\langle v \rangle^2} W \right) + \frac{2\ell W}{\langle v \rangle^2} \left(\text{Id} + \frac{2\ell-1}{\langle v \rangle^2} v \otimes v \right) \\ &= 2\langle v \rangle^{2\ell} w D_v^2 w + \frac{1}{2W} \nabla_v W \otimes \nabla_v W + \frac{2\ell}{\langle v \rangle^2} v \otimes \nabla_v W - \frac{2\ell(\ell+1)W}{\langle v \rangle^4} v \otimes v + \frac{2\ell W}{\langle v \rangle^2} \text{Id}. \end{aligned}$$

Plugging these three equalities into (45), we deduce the equation for W :

$$\begin{aligned} (\partial_t + v \cdot \nabla_x) W &= 2\langle v \rangle^{2\ell} w (\text{tr } (\bar{a}^g D_v^2 w) + \text{tr } (\bar{a}^w D_v^2 f)) + 2\bar{c}^g W + 2\langle v \rangle^{2\ell} w \bar{c}^w f - 2rW \\ &= \text{tr} \left(\bar{a}^g \left(D_v^2 W - \frac{1}{2W} \nabla_v W \otimes \nabla_v W - \frac{2\ell}{\langle v \rangle^2} v \otimes \nabla_v W + \frac{2\ell(\ell+1)W}{\langle v \rangle^4} v \otimes v \right. \right. \\ &\quad \left. \left. - \frac{2\ell W}{\langle v \rangle^2} \text{Id} \right) \right) + 2\langle v \rangle^{2\ell} w \text{tr} (\bar{a}^w D_v^2 f) + 2\bar{c}^g W + 2\langle v \rangle^{2\ell} w \bar{c}^w f - 2rW \\ &= \text{tr} (\bar{a}^g D_v^2 W) - \frac{1}{2W} \nabla_v W \cdot (\bar{a}^g \nabla_v W) - 2\ell \langle v \rangle^{-2} v \cdot (\bar{a}^g \nabla_v W) \\ &\quad + \left[\frac{2\ell(\ell+1)}{\langle v \rangle^4} v \cdot (\bar{a}^g v) - \frac{2\ell}{\langle v \rangle^2} \text{tr} (\bar{a}^g) + 2\bar{c}^g \right] W \\ &\quad + 2\langle v \rangle^{2\ell} w \text{tr} (\bar{a}^w D_v^2 f) + 2\langle v \rangle^{2\ell} w \bar{c}^w f - 2rW. \end{aligned}$$

It is in this step that we need the technical condition $W(z_\varepsilon) > 0$; indeed, otherwise the second term on the right hand side would not be well-defined.

We notice three things. First, as z_ε is a maximum point (recall (44)), it follows that, at z_ε ,

$$\nabla_v W = 0, \quad D_v^2 W \leq 0, \quad \text{and} \quad (\partial_t + v \cdot \nabla_x) W \geq 0. \quad (47)$$

Second, we have that,

$$\|w\|_{L^{\infty,\ell}([0,t_\varepsilon] \times \mathbb{R}^6)}^2 = W(z_\varepsilon). \quad (48)$$

At this point, we drop the indication of the domain from the L^∞ -norms as it will always be $[0, t_\varepsilon] \times \mathbb{R}^6$.

Next, after using (40) and (41) to bound the \bar{a}^g and \bar{c}^g terms in (45) and using (47) to remove several other terms, we obtain, at z_ε ,

$$2rW \lesssim W + \langle v \rangle^{2\ell} |w| |\bar{a}^w| |D_v^2 f| + \langle v \rangle^{2\ell} |w| |\bar{c}^w| f. \quad (49)$$

Recalling (48) and (40), we have, at z_ε ,

$$|\bar{a}^w| \lesssim \langle v \rangle^{(2+\gamma)_+} \|w\|_{L^{\infty,\ell}} = \langle v \rangle^{(2+\gamma)_+} \sqrt{W} \quad (50)$$

and, by (41),

$$|\bar{c}^w| \lesssim \langle v \rangle^\gamma \|w\|_{L^{\infty,\ell}} = \langle v \rangle^\gamma \sqrt{W}. \quad (51)$$

Plugging the estimates (50) and (51) into (49) yields

$$rW \lesssim W + \langle v \rangle^{\ell+(2+\gamma)_+} W |D_v^2 f| + \langle v \rangle^\ell W f \lesssim \left(1 + \langle v \rangle^{\ell+(2+\gamma)_+} |D_v^2 f|\right) W. \quad (52)$$

Above we used that $\|f\|_{L^{\infty,\ell}}$ is bounded. Applying Proposition 10 and using (43), this becomes

$$rW \leq C_0 \left(1 + \frac{1}{t_\varepsilon \log(\frac{1}{t_\varepsilon})^{\frac{\theta'}{2} \frac{\alpha'}{2+\alpha'}}}\right) W$$

for some $C_0 > 0$. We note that this is where the restriction to $[0, T_0]$ is inherited from Proposition 10.

Choosing

$$r(t) = 2C_0 \left(1 + \frac{1}{t \log(\frac{1}{t})^{\frac{\theta'}{2} \frac{\alpha'}{2+\alpha'}}}\right)$$

contradicts (52). The condition that

$$\frac{\theta'}{2} \frac{\alpha'}{2+\alpha'} > 1$$

ensures that $r \in L^1([0, T_0])$, as desired.

Therefore, this rules out the existence of z_ε . We conclude that

$$\sup W < \varepsilon.$$

As ε is arbitrary in the above argument, we deduce that $W \equiv 0$. Thus, $g = f$.

It remains to address the case when $f_{\text{in}} \in L^{\infty,k}$ for all k . Here, however, the arguments of [38, Theorem 1.4] directly apply. Indeed, these arguments are based on showing that $f(T_1) \in C_x^{\alpha/3} C_v^\alpha(\mathbb{R}^6)$ and lies in $L^{\infty,k}$ for all k , which do not require the stronger smoothness assumptions of [38, Theorem 1.2]. The idea is to then re-apply the uniqueness argument on an interval starting at T_1 . Hence, we deduce that uniqueness on the entire time interval $[0, T]$. \square

3.1. A t -integrable bound on $\|D_v^2 f(t)\|_{L_{x,v}^\infty}$: proof of Proposition 10. We now state a more precise estimate that immediately yields Proposition 10. It establishes a bound on $D_v^2 f$ at the same time as one on the $C_x^{\alpha/3} \log(1/C_v)^{-\theta}$ -norm of f . In the sequel we refer to these as a Hessian bound and as propagation of regularity, respectively. It is interesting to note that, although the latter is a “hyperbolic” estimate (that is, it does not involve a *gain of regularity*), it is dependent on the Schauder estimates in an essential way.

Proposition 11. *Let $f \in L^{\infty,k}([0, T] \times \mathbb{R}^6)$ be the solution constructed in [38, Theorem 1.2] with the nondegeneracy condition (8). Fix any $\theta > 0$, $\alpha \in (0, 1)$, and $\mu < 1$ such that*

$$\frac{\mu\theta}{2} \frac{\mu\alpha}{2 + \mu\alpha} > 1$$

Then, for any $m > 5 + \gamma$ and any k sufficiently large depending on m , α , θ , and μ , there exists a time $T_0 \leq \min\{1/2, T\}$ such that

$$\sup_{t \in [0, T_0]} \left(t \left(\log \frac{1}{t} \right)^{\frac{\mu\theta}{2} \frac{\mu\alpha}{2 + \mu\alpha}} \|D_v^2 f\|_{L^{\infty,m+(2+\gamma)_+}([t/2,t] \times \mathbb{R}^6)} \right)^{\frac{\mu\alpha}{2 + \mu\alpha}},$$

$$\|\langle v \rangle^m f\|_{C_x^{\mu\alpha/3} \log(1/C_v)^{-\mu\theta}([0, T_0] \times \mathbb{R}^6)} \lesssim 1 + \|f_{\text{in}}\|_{C_x^{\alpha/3} \log(1/C_v)^{-\theta}(\mathbb{R}^6)}.$$

The implied constant depends on $\|f\|_{L^{\infty,k}([0, T_0] \times \mathbb{R}^6)}$. The final time T_0 depends only on α , θ , k , $\|f_0\|_{L^{\infty,k}}$, and $\|f_0\|_{C_x^{\alpha/3} \log(1/C_v)^{-\theta}}$.

We observe that the restriction $T_0 \leq 1/2$ is a technical one. Indeed, one can iterate Proposition 11 starting at time $t = 0$, T_0 , T_1, \dots to obtain the bound at some (potentially) large time. As we see in its proof, and as is already hinted at by the exponent of the first term in the left hand side of the main inequality in Proposition 11, it may be that the weighted Hölder norm blows up at a finite time. We do not address this further here, as it was already handled at the conclusion of the proof of Theorem 2.

In [38], the analogue to Proposition 11 was broken up into two separate steps [38, Proposition 4.4 and Lemma 4.6], one for each of the two inequalities. Here, however, we must deduce the Hessian bound and the propagation of regularity simultaneously. We discuss this in further detail after stating the next lemma, which plays a key role in the proof of Proposition 11

The next lemma is an estimate on $D_v^2 f$ in terms of the $C_x^{\alpha/3} \log(1/C_v)^{-\theta}$ -norm of f . This is obtained by rescaling the equation, applying the Schauder estimates (Theorem 1), and then interpolating between the resulting $C_x^{\frac{2+\alpha}{3}} C_v^{2+\alpha}$ -estimate and the $C_x^{\alpha/3} \log(1/C_v)^{-\theta}$ -seminorm of f .

We note that, in order to do this, it is crucial that our Schauder estimates do not require t -Hölder regularity of the coefficients. Indeed, the coefficient \bar{a}^f is a v -convolution of f and a kernel, and, hence, will have no more t -regularity than that of f . *A priori* we do not have any bounds on the t -Hölder regularity of f . One might attempt to obtain apply known estimates (e.g. the De Giorgi estimates [25, Theorem 12]) to obtain a t -Hölder bound; however, these estimates will scale poorly in t , leading to a non-integrable bound in t . This is overcome in [38, Proposition A.1] by a lemma showing that f obtains t Hölder continuity from (x, v) Hölder regularity. This is clearly not useful in our setting as we do not yet have “nice” v Hölder regularity of f .

Lemma 12. *Under the assumptions of Proposition 11,*

$$\begin{aligned} \|D_v^2 f\|_{L^{\infty,m-2}([t_0/2,t_0]\times\mathbb{R}^6)} &\lesssim \frac{1}{t_0 \log(\frac{1}{t_0})^{\frac{\theta}{2} \frac{\alpha}{2+\alpha}}} (1 + \|\langle v \rangle^m f\|_{C_x^{\alpha/3} \log(1/C_v)^{-\theta}([t_0/4,t_0]\times\mathbb{R}^6)})^{1+\frac{2}{\alpha}} \\ &\quad + \frac{t_0^{\alpha/2}}{\left(\log \frac{1}{t_0}\right)^{\frac{\alpha}{2(2+\alpha)} \frac{\theta}{2}}} \|D_v^2 f\|_{L^{\infty,m-2}([t_0/4,t_0]\times\mathbb{R}^6)}^{\alpha/2}, \end{aligned}$$

for any $t_0 \in (0, \min\{1/2, T\}]$. The implied constant in the above estimate depends additionally on $\|f\|_{L^{\infty,k}([0,T]\times\mathbb{R}^6)}$.

Again, we note that the fact that t_0 is restricted to be less than $1/2$ is only so that the log in the denominator does not take the value zero.

We now briefly comment that the necessity of proving both the Hessian bound and the propagation of regularity simultaneously is related to the fact that in Lemma 12, one obtains both the $C_x^{\alpha/3} \log(1/C_v)^{-\theta}$ -norm and a $W_v^{2,\infty}$ -norm on the right hand side. Hence, a dynamic argument is required in order to “absorb” the $W_v^{2,\infty}$ -norm. The reason that both terms appear in our setting (in contrast to the work in [38]) is that we cannot bound the $C_x^{\alpha/3} C_v^\alpha$ -norm of \bar{c}^f , which is required for the Schauder estimates, by the $C_x^{\alpha/3} \log(1/C_v)^{-\theta}$ -norm of f .

The proof of Lemma 12 is contained in Section 3.3.

3.2. The Hessian bound and propagation of regularity: the proof of Proposition 11. In this section, we prove the main estimate. Before that, we need to recast our notion of regularity. For any point $(t, x, v, \chi, \nu) \in \mathbb{R}_+ \times \mathbb{R}^6 \times B_{1/2}(0)^2$ and any real number $m > 0$, we define

$$\begin{aligned} \tau f(t, x, v, \chi, \nu) &:= f(t, x + \chi, v + \nu), \quad \delta f(t, x, v, \chi, \nu) := f(t, x + \chi, v + \nu) - f(t, x, v), \\ \text{and } g(t, x, v, \chi, \nu) &:= \frac{|\delta f(t, x, v, \chi, \nu)|^2}{(|\chi|^2 + |\log|\nu||^{-2\theta/\alpha})^{\mu\alpha}} \langle v \rangle^{2m}. \end{aligned}$$

Then we have the following obvious equivalence between bounds on g and the regularity of f . We omit the proof.

Lemma 13. *We have*

$$\begin{aligned} \|g\|_{L_{x,v}^\infty} + \|\langle v \rangle^m f\|_{L^\infty(\mathbb{R}^6)}^2 &\approx \|\langle v \rangle^m f\|_{C_x^{\mu\alpha/3} \log(1/C_v)^{-\mu\theta}(\mathbb{R}^6)}^2 \\ &\approx \sup_{(x_0, v_0)} \langle v_0 \rangle^m \|f\|_{C_x^{\mu\alpha/3} \log(1/C_v)^{-\mu\theta}(B_{1/2}(x_0, v_0))}^2, \end{aligned}$$

where the implied constant depend only on m , θ , and α .

With Lemma 13 in hand, we are now able to prove our main estimate Proposition 11 using the strategy of [38, Proposition 4.4]. When the details are the same as in [38, Proposition 4.4] we note this and omit them.

Proof of Proposition 11. Before beginning we note two things. The first is that, since we are proving a statement regarding a solution constructed in [38], we may assume without loss of generality that f is smooth. Indeed, in [38], the solution f is approximated by smooth solutions of (6). Were we to prove the claim for the approximating solution, it holds for f in the limit.

Next, we note that $f \in L^{\infty,k}$, by assumption. Hence, we ignore this norm throughout and absorb all instances of it into the \lesssim notation.

As the proof is somewhat complicated, we break it up into a number of steps.

Step 1: an equation for g and straightforward estimates. Using (6), we find

$$\begin{aligned} \partial_t g + v \cdot \nabla_x g + \nu \cdot \nabla_\chi g + & \frac{2\alpha\mu\chi \cdot \nu}{|\chi|^2 + |\log|\nu||^{-2\theta/\alpha}} g \\ & = 2 \frac{\text{tr}(\bar{a}^{\delta f} D_v^2 \tau f + \bar{a}^f D_v^2 \delta f) + \bar{c}^{\delta f} \tau f + \bar{c}^f \delta f}{(|\chi|^2 + |\log|\nu||^{-2\theta/\alpha})^{\mu\alpha}} \delta f \langle v \rangle^{2m}. \end{aligned}$$

Three terms are estimated exactly³ as in [38, Proposition 4.4]:

$$-\frac{2\alpha\mu\chi \cdot \nu}{|\chi|^2 + |\log|\nu||^{-2\theta/\alpha}} g + \frac{\bar{c}^{\delta f} \tau f + \bar{c}^f \delta f}{(|\chi|^2 + |\log|\nu||^{-2\theta/\alpha})^{\mu\alpha}} \delta f \langle v \rangle^{2m} \lesssim g + \sqrt{g \|g(t)\|_{L^\infty(\mathbb{R}^6 \times B_{1/2}^2)}}.$$

Here we used (41), the condition that $m > 5 + \gamma$, and that the $L^{\infty,k}$ -norm of f bounds $\langle v \rangle^m \tau f$.

Additionally, arguing as in [38, Proposition 4.4], one sees

$$\frac{|\bar{a}^{\delta f}|}{(|\chi|^2 + |\log|\nu||^{-2\theta/\alpha})^{\mu\alpha/2}} \lesssim \langle v \rangle^{(2+\gamma)_+} \sqrt{\|g(t)\|_{L^\infty(\mathbb{R}^6 \times B_1^2)}}.$$

The argument for this uses the definition of g in terms of δf and (40).

Hence, we have

$$\begin{aligned} \partial_t g + v \cdot \nabla_x g + \nu \cdot \nabla_\chi g - 2 & \frac{\text{tr}(\bar{a}^f D_v^2 \delta f)}{(|\chi|^2 + |\log|\nu||^{-2\theta/\alpha})^{\mu\alpha}} \delta f \langle v \rangle^{2m} \\ & \lesssim g + \left(1 + \|D_v^2 \tau f(t)\|_{L^{\infty,m+(2+\gamma)_+}(\mathbb{R}^6)}\right) \sqrt{g \|g(t)\|_{L^\infty(\mathbb{R}^6 \times B_1^2)}}. \end{aligned} \quad (53)$$

This concludes the first step.

To briefly comment on how we proceed from here, note that, roughly the terms on the left hand side should have a good sign at a maximum of g (if we think of δf as, approximately \sqrt{g}). On the other hand, the pure g term on the right hand side lend itself to the construction of a barrier. The most complicated term is the Hessian term in τf . For this, we use Lemma 12 and the fact that the Hessian term on the right has a small parameter in front (when $t_0 \ll 1$), which, through a somewhat complicated process, allows us to absorb this into Hessian in the left hand side of Lemma 12.

Step 2: an upper barrier. With $N > 1$ to be chosen later and fixing any $0 < \mu' < \mu$ such that

$$\frac{\mu'\theta}{2} \frac{\mu'\alpha}{2 + \mu'\alpha} > 1,$$

define \bar{G} to be the unique solution to

$$\begin{cases} \frac{d}{dt} \bar{G}(t) = \frac{N^2}{t(\log \frac{1}{t})^{\frac{\mu\theta}{2} - \frac{\mu\alpha}{2 + \mu\alpha}}} (1 + \bar{G})^{\frac{1}{2} + \frac{1}{\mu'\alpha}} \bar{G}, \\ \bar{G}(0) = \|g(0)\|_{L^\infty} + N \|f\|_{L^{\infty,m}}^2 + 1. \end{cases} \quad (54)$$

We note that $\|g(0)\|_{L^\infty}$ is finite due an application of Lemma 17, after increasing k if necessary. Let T_1 be the largest time in $[0, 1/2]$ that $\bar{G}(T_1) \leq 2\bar{G}(0)$. Let

$$T_0 = \min\{1, T_1, T_2\}$$

for T_2 to be chosen in the sequel. Clearly T_1 depends on N , but N will be chosen to depend only on μ, α, θ, m , and k . We note that

$$\bar{G}(t) \geq 1 \quad \text{for all } t \in [0, T_0]. \quad (55)$$

³This corresponds to the estimates of J_1 , J_4 , and J_5 in [38, Proposition 4.4].

We define the auxiliary function

$$G_2(t) = t \left(\log \frac{1}{t} \right)^{\frac{\mu' \theta}{2} \frac{\mu' \alpha}{2 + \mu' \alpha}} \|D_v^2 f\|_{L^{\infty, m+(2+\gamma)_+}([t/2, t] \times \mathbb{R}^6)},$$

and then let

$$G(t, x, v, \chi, \nu) = \max \left\{ g(t, x, v, \chi, \nu), \left(\frac{1}{N} G_2(t) \right)^{\frac{2\mu' \alpha}{2 + \mu' \alpha}} \right\}.$$

Our goal is to show that, for $t \in [0, T_0]$,

$$G(t, x, v, \chi, \nu) < \bar{G}(t).$$

This is true at $t = 0$ by construction (recall that, without loss of generality, our f is smooth, so that $G_2(0) = 0$). Hence, we may define

$$t_0 = \sup \{ \bar{t} \in [0, T_0] : \|G(s)\|_{L^\infty(\mathbb{R}^6)} < \bar{G}(s) \text{ for all } s \in [0, \bar{t}] \}.$$

If $t_0 = T_0$, we are finished. Hence, we argue by contradiction, assuming that

$$t_0 < T_0. \quad (56)$$

Step 3: The case where g is not the dominant term in G . Clearly $\|G(t_0)\|_{L^\infty(\mathbb{R}^6 \times B_1^2)} = \bar{G}(t_0)$. Consider the case where

$$\|g(t_0)\|_{L^\infty(\mathbb{R}^6 \times B_1^2)} < \left(\frac{1}{N} G_2(t_0) \right)^{\frac{2\mu' \alpha}{2 + \mu' \alpha}} \quad \text{so that} \quad \left(\frac{1}{N} G_2(t_0) \right)^{\frac{2\mu' \alpha}{2 + \mu' \alpha}} = \bar{G}(t_0). \quad (57)$$

Then, using Lemma 12 and that

$$t_0 \left(\log \frac{1}{t_0} \right)^{\frac{\mu' \theta}{2} \frac{\mu' \alpha}{2 + \mu' \alpha}} \|D_v^2 f\|_{L^{\infty, m+(2+\gamma)_+}([t_0/4, t_0] \times \mathbb{R}^6)} \lesssim G_2(t_0/2) + G_2(t_0),$$

we find

$$\begin{aligned} G_2(t_0) &\lesssim \left(1 + \|\langle v \rangle^{m+2+(2+\gamma)_+} f\|_{C_x^{\mu' \alpha/3} \log(1/C_v)^{-\mu' \theta}([t_0/4, t_0] \times \mathbb{R}^6)} \right)^{1 + \frac{2}{\mu' \alpha}} \\ &\quad + t_0^{\frac{\mu' \alpha}{2}} (G_2(t_0/2) + G_2(t_0))^{\frac{\mu' \alpha}{2}}. \end{aligned}$$

Actually, the last term above involves a more complicated expression involving a product of t_0 and $\log(1/t_0)$ to positive powers; however, for t_0 bounded, this can be absorbed into the implied constant. Using the interpolation lemma (Lemma 17), Lemma 13, and increasing k if necessary, we find

$$G_2(t_0) \lesssim \left(1 + \|g\|_{L^\infty([t_0/4, t_0] \times \mathbb{R}^6 \times B_1^2)}^{1/2} \right)^{1 + \frac{2}{\mu' \alpha}} + t_0^{\frac{\mu' \alpha}{2}} \left(G_2(t_0/2)^{\frac{\mu' \alpha}{2}} + G_2(t_0)^{\frac{\mu' \alpha}{2}} \right). \quad (58)$$

We recall that we are not tracking the $L^{\infty, k}$ -norm of f as it is bounded by assumption. We also note that it is in this step that we used that $\mu' < \mu$.

By the choice of t_0 and the fact that \bar{G} is increasing, we see that

$$G_2(t_0/2) \leq N \bar{G}(t_0/2)^{\frac{2+\mu' \alpha}{2\mu' \alpha}} \leq N \bar{G}(t_0)^{\frac{2+\mu' \alpha}{2\mu' \alpha}} = G_2(t_0).$$

Also, by the definition of t_0 and (57),

$$\|g\|_{L^\infty([t_0/4, t_0] \times \mathbb{R}^6 \times B_1^2)} \leq \sup_{t \in [t_0/4, t_0]} \bar{G}(t) = \bar{G}(t_0) = \left(\frac{1}{N} G_2(t_0) \right)^{\frac{2\mu' \alpha}{2 + \mu' \alpha}}.$$

Then (58) becomes:

$$G_2(t_0) \lesssim \left(1 + \left(\frac{1}{N} G_2(t_0)\right)^{\frac{\mu' \alpha}{2+\mu' \alpha}}\right)^{\frac{2+\mu' \alpha}{\mu' \alpha}} + t_0^{\frac{\mu' \alpha}{2}} G_2(t_0)^{\frac{\mu' \alpha}{2}} \lesssim 1 + \frac{1}{N} G_2(t_0) + t_0^{\frac{\mu' \alpha}{2}} G_2(t_0)^{\frac{\mu' \alpha}{2}}.$$

Since $G_2(t_0) = N\bar{G}(t_0)^{\frac{2+\mu' \alpha}{2\mu' \alpha}} > 1$, we have that

$$G_2(t_0) \lesssim 1 + \frac{1}{N} G_2(t_0) + t_0^{\frac{\mu' \alpha}{2}} G_2(t_0).$$

After increasing N and decreasing T_2 , we may absorb the last two terms on the right into the left hand side. After this and recalling (57), we find

$$(N\bar{G}(t_0))^{\frac{2+\mu' \alpha}{2\mu' \alpha}} = G_2(t_0) \lesssim 1.$$

After further increasing N and recalling (55), this is clearly a contradiction. It follows that (57) cannot hold. We conclude that

$$\|g(t_0)\|_{L^\infty(\mathbb{R}^6 \times B_1^2)} = \bar{G}(t_0). \quad (59)$$

An important consequence of this is that, for all $t \leq t_0$,

$$\|D_v^2 f\|_{L^{\infty, m+(2+\gamma)_+}([t/2, t] \times \mathbb{R}^6)} \leq \frac{N}{t \left(\log \frac{1}{t}\right)^{\frac{\mu' \theta}{2} \frac{\mu' \alpha}{2+\mu' \alpha}}} \|g(t)\|_{L^\infty(\mathbb{R}^6 \times B_1^2)}^{\frac{1}{2} + \frac{1}{\mu' \alpha}}. \quad (60)$$

Step 4: The bad Hessian term in (53) and an interpolation. We now use (60) in (53) to bound the norm of the Hessian that arises there.

We require one additional fact. By the choice of t_0 and by (59), we have

$$\|g\|_{L^\infty([0, t_0] \times \mathbb{R}^6 \times B_1^2)} = \|g(t_0)\|_{L^\infty(\mathbb{R}^6 \times B_1^2)}. \quad (61)$$

Thus, at $(t_0, x_0, v_0, \chi_0, \nu_0)$, the combination of (60) and (61) in (53) yields

$$\begin{aligned} \partial_t g + v \cdot \nabla_x g + \nu \cdot \nabla_\chi g - 2 \frac{\text{tr}(\bar{a}^f D_v^2 \delta f)}{(|\chi|^2 + |\log |\nu||^{-2\theta/\alpha})^{\mu\alpha}} \delta f \langle v \rangle^{2m} \\ \lesssim g + \frac{N}{t_0 \left(\log \frac{1}{t_0}\right)^{\frac{\mu' \theta}{2} \frac{\mu' \alpha}{2+\mu' \alpha}}} g^{\frac{3}{2} + \frac{1}{\mu' \alpha}} \lesssim \frac{N}{t_0 \left(\log \frac{1}{t_0}\right)^{\frac{\mu' \theta}{2} \frac{\mu' \alpha}{2+\mu' \alpha}}} (1 + g^{\frac{1}{2} + \frac{1}{\mu' \alpha}}) g. \end{aligned} \quad (62)$$

Step 5: finding a touching point. Using (59) and arguing exactly as in the proof of [38, Proposition 4.4], we may assume without loss of generality that there exists $(x_0, v_0, \chi_0, \nu_0) \in \mathbb{R}^6 \times \bar{B}_1(0)^2$ such that

$$g(t_0, x_0, v_0, \chi_0, \nu_0) = \bar{G}(t_0). \quad (63)$$

We omit the argument.

Step 6: the touching point must be in $B_1(0)^2$. If χ_0 or ν_0 were on the boundary, that is, either $\chi_0 \in \partial B_1(0)$ or $\nu_0 \in \partial B_1(0)$, we deduce from the definition of g that

$$\begin{aligned} g(t_0, x_0, v_0, \chi_0, \nu_0) &\lesssim |\delta f(t_0, x_0, v_0, \chi_0, \nu_0)|^2 \langle v_0 \rangle^{2m} \\ &\lesssim (f(t_0, x_0 + \chi_0, v_0 + \nu_0)^2 + f(t_0, x_0, v_0)^2) \langle v_0 \rangle^{2m} \lesssim \|f\|_{L^{\infty, m}}^2. \end{aligned} \quad (64)$$

In particular, this implies that, up to enlarging N large enough depending only on the implied constant in (64),

$$g(t_0, x_0, v_0, \chi_0, \nu_0) \leq N \|f\|_{L^{\infty, m}}^2.$$

We see from (54) that \bar{G} increases with time t . Thus,

$$\bar{G}(t_0) \geq \bar{G}(0) > N\|f\|_{L^{\infty,m}}^2,$$

which contradicts (63).

Step 7: estimating the remaining term in (62). We begin by expanding the last term on the left hand side of (62) at the point $(t_0, x_0, v_0, \chi_0, \nu_0)$. This is a simple multivariable calculus computation that is exactly as in [38, Proposition 4.4], so we omit it and simply state that:

$$\frac{\text{tr}(\bar{a}^f D_v^2 \delta f)}{(|\chi|^2 + |\log |\nu||^{-2\theta/\alpha})^{\mu\alpha}} \delta f \langle v \rangle^{2m} = \text{tr}(\bar{a}^f D_v^2 g) + \frac{2mg}{\langle v_0 \rangle^4} ((m+2)v_0 \cdot \bar{a}^f v_0 - \langle v_0 \rangle^2 \text{tr} \bar{a}^f).$$

This argument occurs at and below (4.7) in [38].

Since g is at a maximum, we further obtain

$$\frac{\text{tr}(\bar{a}^f D_v^2 \delta f)}{(|\chi|^2 + |\log |\nu||^{-2\theta/\alpha})^{\mu\alpha}} \delta f \langle v \rangle^{2m} \leq \frac{2mg}{\langle v_0 \rangle^4} ((m+2)v_0 \cdot \bar{a}^f v_0 - \langle v_0 \rangle^2 \text{tr} \bar{a}^f).$$

Hence, arguing as in [38, Proposition 4.4] to bound the terms on the right hand side above, we find⁴

$$\frac{\text{tr}(\bar{a}^f D_v^2 \delta f)}{(|\chi|^2 + |\log |\nu||^{-2\theta/\alpha})^{\mu\alpha}} \delta f \langle v \rangle^{2m} \lesssim g.$$

Combining the above with (62), we have, at $(t_0, x_0, v_0, \chi_0, \nu_0)$,

$$\partial_t g + v \cdot \nabla_x + \nu \nabla_\chi g \lesssim \frac{N}{t \left(\log \frac{1}{t} \right)^{\frac{\mu'\theta}{2} - \frac{\mu'\alpha}{2+\mu'\alpha}}} (1+g)^{\frac{1}{2} + \frac{1}{\mu'\alpha}} g. \quad (65)$$

Step 8: concluding the proof. By the construction of $(t_0, x_0, v_0, \chi_0, \nu_0)$, it is a interior⁵ minimum of $\bar{G} - g$ on $[0, t_0] \times \mathbb{R}^6 \times B_1^2$. Hence,

$$\partial_t(\bar{G} - g) + v \cdot \nabla_x(\bar{G} - g) + \nu \cdot \nabla_\chi(\bar{G} - g) \leq 0.$$

Using (54) and (65) and recalling that $\bar{G}(t_0) = g(t_0, x_0, v_0, \chi_0, \nu_0)$, this implies that, at $(t_0, x_0, v_0, \chi_0, \nu_0)$,

$$\begin{aligned} \frac{N^2}{t_0 \left(\log \frac{1}{t_0} \right)^{\frac{\mu'\theta}{2} - \frac{\mu'\alpha}{2+\mu'\alpha}}} (1 + \bar{G})^{\frac{1}{2} + \frac{1}{\mu'\alpha}} \bar{G} &\lesssim \frac{N}{t \left(\log \frac{1}{t} \right)^{\frac{\mu'\theta}{2} - \frac{\mu'\alpha}{2+\mu'\alpha}}} (1+g)^{\frac{1}{2} + \frac{1}{\mu'\alpha}} g \\ &= \frac{N}{t \left(\log \frac{1}{t} \right)^{\frac{\mu'\theta}{2} - \frac{\mu'\alpha}{2+\mu'\alpha}}} (1 + \bar{G})^{\frac{1}{2} + \frac{1}{\mu'\alpha}} \bar{G}. \end{aligned}$$

This is a contradiction if N is sufficiently large. Hence, it must be that (56) does not hold, implying that $t_0 = T_0$ and, thus,

$$\sup_{(x,v,\chi,\nu) \in \mathbb{R}^6 \times B_1^2} g(t, x, v, \chi, \nu) \leq \bar{G}(t) \quad \text{for all } t \in [0, T_0].$$

Recalling Lemma 13, this concludes the proof of the bound of

$$\|\langle v \rangle^m f\|_{C_x^{\mu\alpha/3} \log(1/C_v)^{-\mu\theta}([0, T_0] \times \mathbb{R}^6)}.$$

The proof of the bound on the Hessian term in Proposition 11 follows from (60), the arbitrariness of μ and μ' , and the fact that $t_0 = T_0$. \square

⁴This is the estimate of J_3 in [38]. It is somewhat obvious from (40).

⁵That is, it is not on the parabolic boundary of $[0, t_0] \times \mathbb{R}^6 \times B_1^2$.

3.3. Scaling the Schauder estimates: proof of Lemma 12. Due to the degeneracy of the ellipticity constants of \bar{a}^f as $|v| \rightarrow \infty$ and the fact that $Q_1(t_0, x_0, v_0)$ may involve negative times, we must change of variables. We begin by defining this change of variables. It is the one used in [13, 36, 38].

Fix $z_0 \in \mathbb{R}_+ \times \mathbb{R}^6$. Let S be the linear transformation such that

$$Se = \begin{cases} \langle v_0 \rangle^{1+\gamma/2} e, & e \cdot v_0 = 0 \\ \langle v_0 \rangle^{\gamma/2} e, & e \cdot v_0 = |v_0|, \end{cases} \quad (66)$$

and let

$$r_0 = \langle v_0 \rangle^{-(1+\gamma/2)+} \min(1, \sqrt{t_0/2}). \quad (67)$$

Then we have the rescaled function

$$f_{z_0}(z) := f(r_0^2 t + t_0, r_0^3 Sx + x_0 + r_0^2 tv_0, r_0 S v + v_0), \quad (68)$$

which satisfies the rescaled equation

$$(\partial_t + v \cdot \nabla_x) f_{z_0} = \text{tr}(\bar{A} D_v^2 f_{z_0}) + \bar{C} f_{z_0}$$

with coefficients

$$\begin{aligned} \bar{A}(z) &= S^{-1} \bar{a}^f(r_0^2 t + t_0, r_0^3 Sx + x_0 + r_0^2 tv_0, r_0 S v + v_0) S^{-1} \quad \text{and} \\ \bar{C}(z) &= r_0^2 \bar{c}^f(r_0^2 t + t_0, r_0^3 Sx + x_0 + r_0^2 tv_0, r_0 S v + v_0). \end{aligned} \quad (69)$$

Roughly, the input of f in the definition of f_{z_0} can be written as $z_0 \circ (Sz)_{r_0}$ where $z_r = (r^2 t, r^3 x, rv)$ is the kinetic scaling by a factor r and

$$z' \circ z = (t' + t, x' + x + tv', v' + v)$$

is the related to the Galilean Lie group structure associated to $\partial_t + v \cdot \nabla_x$. For simplicity, we opt not to use this further, although it is common in the literature.

It is immediate from (40), (42), and [36, Proposition 3.1] that

$$\bar{A} \approx \text{Id} \quad \text{on } Q_1,$$

and, by an easy computation (see [38, eqn (2.15)]),

$$\bar{C}(z) \lesssim \langle v_0 \rangle^{-2} \min\{1, t_0\} \|f\|_{L^{\infty, m}}$$

for any $m > 3$. Additionally, one can observe that

$$\|f_{z_0}\|_{L^\infty(Q_1)} \lesssim \langle v_0 \rangle^{-k} \|f\|_{L^{\infty, k}}. \quad (70)$$

We omit the proof of the above inequalities as they are straightforward and already contained in [36, 38].

We note that the coefficients have the following regularity:

Lemma 14. *For $m, k > 5 + \gamma$ and $\alpha \in (0, 1)$, we have*

$$\begin{aligned} [\bar{A}]_{C_x^{\alpha/3} C_v^\alpha(Q_{3/4})} &\lesssim t_0^{\frac{\alpha}{2}} \left(\langle v_0 \rangle^{(3-k)+2+\gamma-\alpha} \|f\|_{L^{\infty, k}([t_0/4, t_0] \times \mathbb{R}^6)} \right. \\ &\quad \left. + \langle v_0 \rangle^{2+\gamma+(3-m)+} \|\langle v \rangle^m f\|_{C_x^{\alpha/3}([t_0/4, t_0] \times \mathbb{R}^6)} \right) \end{aligned}$$

and

$$[\bar{C}]_{C_x^{\alpha/3} C_v^\alpha(Q_{3/4})} \lesssim t_0^{1+\frac{\alpha}{2}} \langle v_0 \rangle^{(3-m)++\gamma} \|\langle v \rangle^m f\|_{C_x^{\alpha/3} C_v^\alpha([t_0/4, t_0] \times \mathbb{R}^6)}.$$

We note that Lemma 14 is stronger than its analogue [38, Lemma 2.7] as we leverage the convolutional nature of \bar{a}^f to obtain additional regularity in v even when f lacks regularity in v . Additionally, the fact that we do not require t -regularity allows us to avoid the slight loss of regularity seen in [38, Lemma 2.7]. On the other hand, we note that we make no effort to optimize the v_0 -weights in Lemma 14. We prove Lemma 14 in Section 3.4.

Moreover, we immediately see that the regularity of f_{z_0} and f are related by:

$$\|f\|_{C_x^{\alpha/3} C_v^\alpha(Q_{r_0/2}(z_0))} \lesssim \min\{1, t_0\}^{-\alpha/2} \langle v_0 \rangle^{\alpha((1+\gamma/2)-\gamma/2)} \|f_{z_0}\|_{C_x^{\alpha/3} C_v^\alpha(Q_{1/2})}. \quad (71)$$

Analogous statements hold for higher regularity seminorms of f as well. Let us recall, from Section 1.4 the notation

$$Q_r(z_0) = \{(t, x, v) : t_0 - r^2 < t \leq t_0, |x - x_0 - (t - t_0)v_0| < r^3, |v - v_0| < r\},$$

with the convention that $Q_r = Q_r(0)$.

Finally, before proving Lemma 12, we state two final technical results related to scaling.

Lemma 15 (log-Hölder interpolation inequality). *Fix any $u : \mathbb{R}^d \rightarrow \mathbb{R}$ and $r > 0$. For $\alpha \in (0, 1)$, $\theta > 0$, and any $\varepsilon \in (0, \min\{1/2, r\})$,*

$$\|D_v^2 u\|_{L^\infty(Q_r)} \lesssim \frac{\log(1/\varepsilon)^{-\theta}}{\varepsilon^2} [u]_{\log(\frac{1}{C})^{-\theta}(Q_r)} + \varepsilon^\alpha [u]_{C^{2,\alpha}(Q_r)}.$$

The implied constant depends only on θ and α .

Let us note that results like Lemma 15 are well-known for standard Hölder spaces (see, e.g., [43, Proposition 2.10]), where they take the form: for all $\varepsilon \in (0, 1)$,

$$\|D_v^2 u\|_{L^\infty} \lesssim \frac{1}{\varepsilon^{2-\alpha}} [u]_{C^\alpha} + \varepsilon^\alpha [u]_{C^{2,\alpha}}.$$

Roughly this holds because C^2 is α derivatives away from $C^{2,\alpha}$ and $(2 - \alpha)$ away from C^α . The adaptation to our nonstandard spaces has the same intuition.

Lemma 16. *We have, for $t_0 < 1/2$,*

$$[f_{z_0}]_{\log(\frac{1}{C})^{-\theta/2}(Q_1)} \lesssim [f]_{\log(\frac{1}{C})^{-\theta}(Q_{t_0/2}(z_0))} \log\left(\frac{1}{t_0}\right)^{-\theta/2}.$$

The proofs of these two lemmas are also postponed to Section 3.4. We now prove the lemma on the scaling of the Schauder estimates.

Proof of Lemma 12. Throughout the proof we assume that

$$[\langle v \rangle^m f]_{C_x^{\alpha/3} \log(1/C_v)^{-\theta}([t_0/4, t_0] \times \mathbb{R}^6)} < \infty.$$

If this were not true, then the claim in Lemma 12 follows immediately.

Fix $\varepsilon \in (0, 1/2)$ to be determined. Applying our log-Hölder interpolation lemma (Lemma 15), we see

$$\|D_v^2 f_{z_0}\|_{L^\infty(Q_{1/2})} \lesssim \frac{\log(1/\varepsilon)^{-\theta/2}}{\varepsilon^2} [f_{z_0}]_{\log(\frac{1}{C_v})^{-\theta/2}(Q_{1/2})} + \varepsilon^\alpha [D_v^2 f_{z_0}]_{C_x^{\alpha/3} C_v^\alpha(Q_{1/2})}. \quad (72)$$

Clearly the first term in (72) can be bounded by simply removing the scaling. Indeed, applying Lemma 16, we find

$$[f_{z_0}]_{\log(\frac{1}{C})^{-\theta/2}} \lesssim \left(\log \frac{1}{t_0}\right)^{-\theta/2} [f]_{C_x^{\alpha/3} \log(1/C_v)^{-\theta}(Q_{t_0/2}(z_0))}. \quad (73)$$

For the second term in (72), we require our Schauder estimates Theorem 1. Applying this yields

$$[D_v^2 f_{z_0}]_{C_x^{\alpha/3} C_v^\alpha(Q_{1/2})} \lesssim \left(1 + [\bar{C}]_{C_x^{\alpha/3} C_v^\alpha(Q_{3/4})} + [\bar{A}]_{C_x^{\alpha/3} C_v^\alpha(Q_{3/4})}^{1+\frac{2}{\alpha}} \right) \|f_{z_0}\|_{L^\infty(Q_{3/4})}.$$

We note that the statement of Theorem 1 involves a cylinder Q_1 on the right hand side instead of $Q_{3/4}$; however, it is a simple scaling argument to obtain the above, so we omit the details. We use this cylinder in order to obtain an estimate below insulated from $t = 0$ by $t_0/4$.

Using (70) and Lemma 14, we obtain

$$\begin{aligned} [D_v^2 f_{z_0}]_{C_x^{\alpha/3} C_v^\alpha(Q_{1/2})} &\lesssim \langle v_0 \rangle^{-k} \left(1 + t_0^{1+\frac{\alpha}{2}} \langle v_0 \rangle^{(3-m)_+ + \gamma} \|\langle v \rangle^m f\|_{C_v^\alpha([t_0/4, t_0] \times \mathbb{R}^6)} + \right. \\ &\quad \left. \left(t_0^{\frac{\alpha}{2}} \langle v_0 \rangle^{2-\alpha} \|f\|_{L^{\infty,k}([t_0/4, t_0] \times \mathbb{R}^6)} + t_0^{\frac{\alpha}{2}} \langle v_0 \rangle^2 \|\langle v \rangle^m f\|_{C_x^{\alpha/3}([t_0/4, t_0] \times \mathbb{R}^6)} \right)^{1+\frac{2}{\alpha}} \right) \|f\|_{L^{\infty,k}}. \end{aligned}$$

We recall, by assumption, $\|f\|_{L^{\infty,k}}$ is finite. This is inherited from [38, Theorem 1.2]. Hence,

$$\begin{aligned} [D_v^2 f_{z_0}]_{C_x^{\alpha/3} C_v^\alpha(Q_{1/2})} &\lesssim \langle v_0 \rangle^{(3-m)_+ - k + \gamma} t_0^{1+\frac{\alpha}{2}} [\langle v \rangle^m f]_{C_v^\alpha([t_0/4, t_0] \times \mathbb{R}^6)} \\ &\quad + \langle v_0 \rangle^{-k+2+\frac{4}{\alpha}} \left(1 + \|\langle v \rangle^m f\|_{C_x^{\alpha/3}([t_0/4, t_0] \times \mathbb{R}^6)} \right)^{1+\frac{2}{\alpha}}. \end{aligned} \quad (74)$$

Using (73) and (74) in (72), we find

$$\begin{aligned} \|D_v^2 f_{z_0}\|_{L^\infty(Q_{1/2})} &\lesssim \frac{\log(1/\varepsilon)^{-\theta/2}}{\varepsilon^2} \left(\log \frac{1}{t_0} \right)^{-\frac{\theta}{2}} [f]_{C_x^{\alpha/3} \log(1/C_v)^{-\theta}(Q_{t_0/4}(z_0))} \\ &\quad + \varepsilon^\alpha \langle v_0 \rangle^{-k+2+\frac{4}{\alpha}} \left(1 + \|\langle v \rangle^m f\|_{C_x^{\alpha/3}([t_0/4, t_0] \times \mathbb{R}^6)} \right)^{1+\frac{2}{\alpha}} \\ &\quad + \varepsilon^\alpha \langle v_0 \rangle^{(3-m)_+ - k + \gamma} t_0^{1+\frac{\alpha}{2}} [\langle v \rangle^m f]_{C_v^\alpha([t_0/4, t_0] \times \mathbb{R}^6)}. \end{aligned}$$

Undoing the change of variables (similar to (71)) and combining terms yields

$$\begin{aligned} \frac{t_0}{\langle v_0 \rangle^2} \|D_v^2 f\|_{L^\infty(Q_{t_0/2}(z_0))} &\lesssim \frac{\log(1/\varepsilon)^{-\theta/2}}{\varepsilon^2} \left(\log \frac{1}{t_0} \right)^{-\frac{\theta}{2}} [f]_{C_x^{\alpha/3} \log(1/C_v)^{-\theta}(Q_{t_0/4}(z_0))} \\ &\quad + \varepsilon^\alpha \langle v_0 \rangle^{-k+2+\frac{4}{\alpha}} \left(1 + \|\langle v \rangle^m f\|_{C_x^{\alpha/3}([t_0/4, t_0] \times \mathbb{R}^6)} \right)^{1+\frac{2}{\alpha}} \\ &\quad + \varepsilon^\alpha \langle v_0 \rangle^{(3-m)_+ - k + \gamma} t_0^{1+\frac{\alpha}{2}} [\langle v \rangle^m f]_{C_v^\alpha([t_0/4, t_0] \times \mathbb{R}^6)}. \end{aligned} \quad (75)$$

Next, we take

$$\varepsilon = \min \left\{ 1/4, \log(1/t_0)^{-\frac{\theta}{2(2+\alpha)}} \right\}$$

so that (75) becomes

$$\begin{aligned} \frac{t_0}{\langle v_0 \rangle^2} \|D_v^2 f\|_{L^\infty(Q_{t_0/2}(z_0))} &\lesssim \left(\log \frac{1}{t_0} \right)^{-\frac{\alpha}{2(2+\alpha)} \frac{\theta}{2}} [f]_{C_x^{\alpha/3} \log(1/C_v)^{-\theta}(Q_{t_0/4}(z_0))} \\ &\quad + \left(\log \frac{1}{t_0} \right)^{-\frac{\alpha}{2(2+\alpha)} \frac{\theta}{2}} \langle v_0 \rangle^{-k+2+\frac{4}{\alpha}} \left(1 + \|\langle v \rangle^m f\|_{C_x^{\alpha/3}([t_0/4, t_0] \times \mathbb{R}^6)} \right)^{1+\frac{2}{\alpha}} \\ &\quad + \left(\log \frac{1}{t_0} \right)^{-\frac{\alpha}{2(2+\alpha)} \frac{\theta}{2}} \langle v_0 \rangle^{(3-m)_+ - k + \gamma} t_0^{1+\frac{\alpha}{2}} [\langle v \rangle^m f]_{C_v^\alpha([t_0/4, t_0] \times \mathbb{R}^6)}. \end{aligned}$$

Dividing by t_0 , multiplying by $\langle v_0 \rangle^m$, increasing k if necessary, and taking the supremum over all choices of (x_0, v_0) , we find

$$\begin{aligned} \|D_v^2 f\|_{L^{\infty, m-2}([t_0/2, t_0])} &\lesssim \frac{1}{t_0 \left(\log \frac{1}{t_0}\right)^{\frac{\alpha}{2(2+\alpha)} \frac{\theta}{2}}} \left(1 + \|\langle v \rangle^m f\|_{C_x^{\alpha/3} \log(1/C_v)^{-\theta}([t_0/4, t_0] \times \mathbb{R}^6)}\right)^{1+\frac{2}{\alpha}} \\ &\quad + \left(\log \frac{1}{t_0}\right)^{-\frac{\alpha}{2(2+\alpha)} \frac{\theta}{2}} t_0^{\frac{\alpha}{2}} [\langle v \rangle^m f]_{C_v^\alpha([t_0/4, t_0] \times \mathbb{R}^6)}. \end{aligned}$$

In order to remove the last term above, it suffices to apply [38, Lemma B.2] (which is analogous to Lemma 17 but stated for standard Hölder spaces) to obtain

$$\left(\log \frac{1}{t_0}\right)^{-\frac{\alpha}{2(2+\alpha)} \frac{\theta}{2}} t_0^{\frac{\alpha}{2}} [\langle v \rangle^m f]_{C_v^\alpha([t_0/4, t_0] \times \mathbb{R}^6)} \lesssim \left(\log \frac{1}{t_0}\right)^{-\frac{\alpha}{2(2+\alpha)} \frac{\theta}{2}} t_0^{\frac{\alpha}{2}} \left(\|D_v^2 f\|_{L^{\infty, m-2}([t_0/4, t_0] \times \mathbb{R}^6)}^{\frac{\alpha}{2}} + 1\right)$$

We remind the reader that $\|f\|_{L^{\infty, k}} \lesssim 1$. This concludes the proof. \square

3.4. Proof of technical lemmas. We begin by establishing the Hölder regularity of the transformed coefficients \bar{A} and \bar{C} . In order to make the notation more compact, we define, for any z ,

$$\tilde{z} := z_0 \circ (Sz)_{r_0} = (r_0^2 t + t_0, r_0^3 Sx + x_0 + r_0^2 t v_0, r_0 S v + v_0). \quad (76)$$

As r_0 and z_0 remain fixed in the following proof, there is no risk of confusion.

Let us point out a useful fact about the Japanese bracket $\langle \cdot \rangle$: when $|b| \leq C$, for a fixed constant C , we have

$$\langle a + b \rangle \approx \langle a \rangle. \quad (77)$$

The implied constants above depend on C . Indeed, if $|a| \geq 2C$, we have $3|a|/2 \geq |a + b| \geq |a|/2$ so that (77) follows. If $|a| \leq 2C$, we have $\langle a \rangle \approx 1 \approx \langle a + b \rangle$, which yields (77).

We use this in two situations in the sequel. First, recalling the definitions of r_0 and S in (66)-(67), if $|v| \leq 1$ then $|r_0 S v| \leq 1$. Hence

$$\langle \tilde{v} - w \rangle = \langle r_0 S v + v_0 - w \rangle \approx \langle v_0 - w \rangle \quad (78)$$

and

$$\langle \tilde{v} \rangle = \langle r_0 S v + v_0 \rangle \approx \langle v_0 \rangle. \quad (79)$$

Further, if $|w| \leq C$, we have

$$\langle v_0 - w \rangle \approx \langle v_0 \rangle.$$

With this in hand, we now prove Lemma 14.

Proof of Lemma 14. We begin by establishing the v -regularity of \bar{C} and \bar{A} . Notice that the estimates are slightly different: the estimate of \bar{C} uses the regularity of f while that of \bar{A} does not. We show both proofs, noting that they have substantial similarities.

Let us begin with the v -regularity of \bar{C} . This is obvious in the case $\gamma = -3$, given the special form of \bar{c} in this case (recall (7)). We, thus, consider only the case $\gamma > -3$.

Next we establish the v -regularity of \bar{C} . Let $z, z' \in Q_1$ with $t = t'$ and $x = x'$. Changing variables and using the regularity of f , we have

$$\begin{aligned} |\bar{C}(z) - \bar{C}(z')| &= \left| r_0^2 c_\gamma \int |w|^\gamma (f(\tilde{t}, \tilde{x}, \tilde{v} - w) - f(\tilde{t}, \tilde{x}, \tilde{v}' - w)) dw \right| \\ &\lesssim t_0 \langle v_0 \rangle^{-(2+\gamma)_+} \left(\int |w|^\gamma \langle \tilde{v} - w \rangle^{-m} |\langle \tilde{v} - w \rangle^m f(\tilde{t}, \tilde{x}, \tilde{v} - w) - \langle \tilde{v}' - w \rangle^m f(\tilde{t}, \tilde{x}, \tilde{v}' - w)| dw \right. \\ &\quad \left. + \int |w|^\gamma \langle \tilde{v}' - w \rangle^m f(\tilde{t}, \tilde{x}, \tilde{v} - w) |\langle \tilde{v} - w \rangle^{-m} - \langle \tilde{v}' - w \rangle^{-m}| dw \right) \\ &\lesssim t_0 \langle v_0 \rangle^{-(2+\gamma)_+} \left(\int |w|^\gamma \langle \tilde{v} - w \rangle^{-m} |\tilde{v} - \tilde{v}'|^\alpha \|\langle v \rangle^m f\|_{C_v^\alpha} dw \right. \\ &\quad \left. + \int |w|^\gamma \|f\|_{L^{\infty, m}} |\tilde{v} - \tilde{v}'|^\alpha \langle \tilde{v} - w \rangle^{-m-\alpha} dw \right) \\ &\lesssim t_0 \langle v_0 \rangle^{-(2+\gamma)_+} |\tilde{v} - \tilde{v}'|^\alpha \|\langle v \rangle^m f\|_{C_v^\alpha} \left(\langle \tilde{v} \rangle^{(3-m)_++\gamma} + \langle \tilde{v}' \rangle^{(3-m)_++\gamma} \right). \end{aligned}$$

The step uses standard estimates of convolutions of algebraic functions as well as the condition on m , which ensures integrability. Recalling (77), we deduce that $\langle \tilde{v} \rangle \approx \langle \tilde{v}' \rangle$. Then, from (79), we deduce that

$$|\bar{C}(z) - \bar{C}(z')| \lesssim t_0 \langle v_0 \rangle^{-(2+\gamma)_+ + (3-m)_++\gamma} |\tilde{v} - \tilde{v}'|^\alpha \|\langle v \rangle^m f\|_{C_v^\alpha}.$$

Finally, returning to the original variables using the definition (67) of r_0 and the bounds (66) of S , we conclude the proof:

$$\begin{aligned} |\bar{C}(z) - \bar{C}(z')| &\lesssim t_0 \langle v_0 \rangle^{-(2+\gamma)_+ + (3-m)_++\gamma} |r_0 S(v - v')|^\alpha \|\langle v \rangle^m f\|_{C_v^\alpha} \\ &\lesssim t_0^{1+\frac{\alpha}{2}} \langle v_0 \rangle^{-(2+\gamma)_+ + (3-m)_++\gamma} |v - v'|^\alpha \|\langle v \rangle^m f\|_{C_v^\alpha}. \end{aligned}$$

We now consider the v -regularity of \bar{A} . Since we avoid using the regularity of f , this is a bit more involved. Again, We now prove the v -regularity of \bar{A} . Let $z, z' \in Q_1$ with $t = t'$ and $x = x'$. Using the form (69) of \bar{A} , as well as the bounds (66) on S , we find

$$\begin{aligned} |\bar{A}(z) - \bar{A}(z')| &\lesssim \langle v_0 \rangle^{-\gamma} \int |w|^{2+\gamma} - |w + \tilde{v}' - \tilde{v}|^{2+\gamma} |f(\tilde{t}, \tilde{x}, \tilde{v} - w)| dw \\ &\lesssim \langle v_0 \rangle^{-\gamma} \left(\int_{B_R} + \int_{B_R^c} \right) |w|^{2+\gamma} - |w + \tilde{v}' - \tilde{v}|^{2+\gamma} |f(\tilde{t}, \tilde{x}, \tilde{v} - w)| dw = I_1 + I_2, \end{aligned}$$

where we have chosen $R = 2|\tilde{v} - \tilde{v}'|$. For I_1 , the integrability of the kernel $|\cdot|^{2+\gamma}$, the approximation (78), and then the choice of R yields

$$\begin{aligned} I_1 &\lesssim \langle v_0 \rangle^{-\gamma} \int_{B_R} (|w|^{2+\gamma} + |w + \tilde{v}' - \tilde{v}|^{2+\gamma}) \langle \tilde{v} - w \rangle^{-k} \|f\|_{L^{\infty, k}} dw \\ &\lesssim \langle v_0 \rangle^{-\gamma} \langle \tilde{v} \rangle^{-k} \|f\|_{L^{\infty, k}} \int_{B_R} (|w|^{2+\gamma} + |w + \tilde{v}' - \tilde{v}|^{2+\gamma}) dw \lesssim \langle v_0 \rangle^{-k-\gamma} \|f\|_{L^{\infty, k}} |\tilde{v}' - \tilde{v}|^{5+\gamma}. \end{aligned}$$

Using (76), the bounds (66) on S , and the fact that $5 + \gamma \geq 2 > \alpha$, we find

$$I_1 \lesssim \langle v_0 \rangle^{-k-\gamma} \langle \tilde{v} \rangle^{-k} \|f\|_{L^{\infty, k}} |r_0 S(v' - v)|^{5+\gamma} \lesssim t_0 \langle v_0 \rangle^{-k-\gamma} \|f\|_{L^{\infty, k}} |v' - v|^\alpha.$$

For I_2 , w is larger than $2(\tilde{v}' - \tilde{v})$. Hence, we have

$$|w|^{2+\gamma} - |w + \tilde{v}' - \tilde{v}|^{2+\gamma} = |w|^{2+\gamma} \left| \left| \frac{w}{|w|} \right|^{2+\gamma} - \left| \frac{w + \tilde{v}' - \tilde{v}}{|w|} \right|^{2+\gamma} \right| \lesssim |w|^{2+\gamma} \left| \frac{\tilde{v} - \tilde{v}'}{|w|} \right|^\alpha.$$

Using this, along with many of the same inequalities as above, we find

$$\begin{aligned} I_2 &\lesssim \int_{B_R^c} |w|^{2+\gamma-\alpha} |\tilde{v} - \tilde{v}'|^\alpha \langle v_0 - w \rangle^{-k} \|f\|_{L^{\infty,k}} dw \\ &\lesssim \|f\|_{L^{\infty,k}} |\tilde{v} - \tilde{v}'|^\alpha \int |w|^{2+\gamma-\alpha} \langle v_0 - w \rangle^{-k} dw \approx \|f\|_{L^{\infty,k}} |\tilde{v} - \tilde{v}'|^\alpha \langle v_0 \rangle^{(3-k)_++2+\gamma-\alpha} \\ &\lesssim t_0^{\frac{\alpha}{2}} \|f\|_{L^{\infty,k}} |v - v'|^\alpha \langle v_0 \rangle^{(3-k)_++2+\gamma-\alpha}. \end{aligned}$$

Putting together the bounds on I_1 and I_2 , we arrive at

$$|\bar{A}(z) - \bar{A}(z')| \lesssim t_0^{\frac{\alpha}{2}} \|f\|_{L^{\infty,k}} |v - v'|^\alpha \langle v_0 \rangle^{(3-k)_++2+\gamma-\alpha},$$

which concludes the proof of v -regularity.

Next we consider x -regularity. We note that the proofs for \bar{C} and \bar{A} are essentially the same. Indeed, at the level of \bar{c} and \bar{a} , both coefficients are essentially of the same general form

$$\int f(t, x, v - w) |w|^\theta dw$$

for $\theta > -3$. Because \bar{C} involves the more singular integration kernel, it is the more difficult term to bound. We, thus, only include the proof of the x -regularity of \bar{C} .

As above, we consider only the case where $\gamma > -3$. We begin by fixing any $z, z' \in Q_1$ with $t = t'$ and $v = v'$, but $x \neq x'$. Then, recalling the notation (76), we find

$$\begin{aligned} |\bar{C}(z) - \bar{C}(z')| &\lesssim \int |w|^\gamma |f(\tilde{t}, \tilde{x}, \tilde{v} - w) - f(\tilde{t}, \tilde{x}', \tilde{v} - w)| dw \\ &= \int |w|^\gamma \frac{|\tilde{x} - \tilde{x}'|^{\alpha/3}}{\langle \tilde{v} - w \rangle^m} \frac{|\langle \tilde{v} - w \rangle^m f(\tilde{t}, \tilde{x}, \tilde{v} - w) - \langle \tilde{v} - w \rangle^m f(\tilde{t}, \tilde{x}', \tilde{v} - w)|}{|\tilde{x} - \tilde{x}'|^{\alpha/3}} dw \\ &\lesssim \int |w|^\gamma \frac{|\tilde{x} - \tilde{x}'|^{\alpha/3}}{\langle \tilde{v} - w \rangle^m} \|\langle v \rangle^m f\|_{C_x^{\alpha/3}([t_0/2, t_0] \times \mathbb{R}^6)} dw. \end{aligned}$$

Recalling the definitions of r_0 and S in (66)-(67), we have

$$|\tilde{x} - \tilde{x}'|^{\alpha/3} = |r_0^3 (Sx - Sx')|^{\alpha/3} \lesssim t_0^{\alpha/2} |x - x'|^{\alpha/3}.$$

Thus, applying (78), we find

$$\begin{aligned} |\bar{C}(z) - \bar{C}(z')| &\lesssim t_0^{\alpha/2} |x - x'|^{\alpha/3} \|\langle v \rangle^m f\|_{C_x^{\alpha/3}([t_0/2, t_0] \times \mathbb{R}^6)} \int |w|^\gamma \langle v_0 - w \rangle^{-m} dw \\ &\approx t_0^{\alpha/2} |x - x'|^{\alpha/3} \|\langle v \rangle^m f\|_{C_x^{\alpha/3}([t_0/2, t_0] \times \mathbb{R}^6)} \langle v_0 \rangle^{(3-m)_++\gamma}. \end{aligned}$$

Again, the last line is a simple computation for the decay rate of the convolution of algebraically decaying functions. This concludes the proof of x -regularity for \bar{C} . The proof of Lemma 14 is finished. \square

We next prove the log-Hölder interpolation lemma.

Proof of Lemma 15. We begin by obtaining a bound on $\|Du\|_{L^\infty(Q_r)}$. Let $v_0 \in Q_r$ be a point such that

$$\|Du\|_{L^\infty(Q_r)} \leq 2|Du(v_0)|. \quad (80)$$

We claim that there is \bar{v} so that

$$v_0 + \varepsilon \bar{v} \in Q_r, \quad |\bar{v}| = 1, \quad \text{and} \quad |\bar{v} \cdot Du(v_0)| \gtrsim |Du(v_0)|. \quad (81)$$

This is a basic (though somewhat complicated) plane geometry exercise that we postpone to the end of the proof.

A Taylor expansion at $\epsilon \approx 0$ yields, for some $\theta \in [0, 1]$,

$$u(v_0 + \epsilon \bar{v}) - u(v_0) = \epsilon \bar{v} \cdot Du(v_0) + \frac{\epsilon^2}{2} \bar{v} \cdot D^2u(v_0 + \theta \epsilon \bar{v}) \bar{v}.$$

Rearranging this, recalling (80) and (81), and dividing by ϵ , we arrive at (recalling the definition 10)

$$\begin{aligned} \|Du\|_{L^\infty(Q_r)} &\leq 2|Du(v_0)| \lesssim \frac{|u(v_0 + \epsilon \bar{v}) - u(v_0)|}{\epsilon} + \epsilon |D^2u(v_0 + \theta \epsilon \bar{v})| \\ &\leq \frac{\log(1/\epsilon)^{-\theta}}{\epsilon} [u]_{\log(\frac{1}{C})^{-\theta}(Q_r)} + \epsilon \|D^2u\|_{L^\infty(Q_r)}. \end{aligned} \quad (82)$$

With (82) in hand, we now use interpolation to obtain a bound on D_v^2u . Indeed, using standard interpolation estimates (see, e.g., using Young's inequality in [43, Proposition 2.10]), we have

$$\|D^2u\|_{L^\infty(Q_r)} \lesssim \left(\frac{\epsilon}{\delta}\right)^\alpha [D^2u]_{C^\alpha(Q_r)} + \frac{\delta}{\epsilon} \|Du\|_{L^\infty(Q_r)},$$

where $\delta > \epsilon$ is a parameter to be chosen. Combining this with (82), we find

$$\|D^2u\|_{L^\infty(Q_r)} \lesssim \left(\frac{\epsilon}{\delta}\right)^\alpha [D^2u]_{C^\alpha(Q_r)} + \frac{\delta}{\epsilon} \left(\frac{\log(1/\epsilon)^{-\theta}}{\epsilon} [u]_{\log(\frac{1}{C})^{-\theta}(Q_r)} + \epsilon \|D^2u\|_{L^\infty(Q_r)} \right).$$

After choosing δ and ϵ sufficiently small, depending only on the implied constant, we may absorb the $\|D^2u\|_{L^\infty}$ term from the right hand side into the left hand side. This yields

$$\|D^2u\|_{L^\infty(Q_r)} \lesssim \epsilon^\alpha [D^2u]_{C^\alpha(Q_r)} + \frac{\log(1/\epsilon)^{-\theta}}{\epsilon^2} [u]_{\log(\frac{1}{C})^{-\theta}(Q_r)},$$

which concludes the proof up to establishing (81).

We now prove (81). At the expense of a multiplicative constant, we may assume that $\epsilon < r/10$. Without loss of generality, we may assume that

$$\frac{Du(v_0)}{|Du(v_0)|} \cdot v_0 \leq 0. \quad (83)$$

Were this not the case, we work with $-Du(v_0)/|Du(v_0)|$ instead. Then, we let

$$\bar{v} = \frac{1}{10} \frac{Du(v_0)}{|Du(v_0)|} - \mu v_0,$$

where μ is chosen so that $|\bar{v}| = 1$. Clearly, due to (83),

$$|v_0| \mu \in [9/10, 1]. \quad (84)$$

Notice that

$$\bar{v} \cdot Du(v_0) = \frac{1}{10} |Du(v_0)| - \mu v_0 \cdot \frac{Du(v_0)}{|Du(v_0)|} \geq \frac{1}{10} |Du(v_0)|,$$

where the second inequality holds due to (83). Next, using (83) again as well as the fact that $\epsilon < r/10$,

$$|v_0 + \epsilon \bar{v}| = \left| (1 - \epsilon \mu)v_0 + \frac{\epsilon}{10} \frac{Du(v_0)}{|Du(v_0)|} \right| \leq |1 - \epsilon \mu| |v_0| + \frac{\epsilon}{10}.$$

Consider the case when $\epsilon \mu \geq 1$, then, using (84)

$$|v_0 + \epsilon \bar{v}| \leq \epsilon \mu |v_0| + \frac{\epsilon}{10} \leq \frac{11\epsilon}{10} < r.$$

which implies that $v_0 + \epsilon \bar{v} \in Q_r$.

Next consider the case when $\varepsilon\mu < 1$. Then

$$|v_0 + \varepsilon\bar{v}| \leq |v_0| - \frac{9\varepsilon}{10} + \frac{\varepsilon}{10} < |v_0| < r,$$

which again implies that $v_0 + \varepsilon\bar{v} \in Q_r$. Thus, we have established (81), which concludes the proof. \square

We now prove the final technical lemma, Lemma 16, which involves the time scaling of the log-Hölder norm of f_{z_0} , defined in (68).

Proof of Lemma 16. Fix any $z \neq \tilde{z} \in Q_1$, with $t = \tilde{t}$, and notice that

$$(r_0^2 t + t_0, r_0^3 Sx + x_0, r_0 S v + v_0), (r_0^2 \tilde{t} + t_0, r_0^3 S\tilde{x} + x_0, r_0 S\tilde{v} + v_0) \in Q_{t_0/2}(z_0).$$

Hence,

$$|f_{z_0}(z) - f_{z_0}(\tilde{z})| \leq (r_0^\alpha |Sx - S\tilde{x}|^{\alpha/3} + \log(1/|r_0(Sv - S\tilde{v})|))^{-\theta} [f]_{C_x^{\alpha/3} \log(1/C_v)^{-\theta}(Q_{t_0/2}(z_0))}.$$

From the definition of S , it is clear that

$$r_0^3 |S(x - \tilde{x})| \lesssim t_0^{3/2} |x - \tilde{x}| \quad \text{and} \quad r_0 |S(v - \tilde{v})| \leq \sqrt{t_0} |v - \tilde{v}|.$$

Hence,

$$\frac{|f_{z_0}(z) - f_{z_0}(\tilde{z})|}{[f]_{C_x^{\alpha/3} \log(1/C_v)^{-\theta}(Q_{t_0/2}(z_0))}} \lesssim t_0^{\alpha/2} |x - \tilde{x}|^{\alpha/3} + \left(\log \frac{1}{\sqrt{t_0}} + \log \frac{1}{|v - \tilde{v}|} \right)^{-\theta}. \quad (85)$$

Young's inequality yields

$$\left(\log \frac{1}{\sqrt{t_0}} + \log \frac{1}{|v - \tilde{v}|} \right)^{-\theta} \lesssim \left(\log \frac{1}{\sqrt{t_0}} \right)^{-\theta/2} \left(\log \frac{1}{|v - \tilde{v}|} \right)^{-\theta/2}$$

and, it is straightforward to see that

$$t_0^{\alpha/2} \lesssim \left(\log \frac{1}{t_0} \right)^{-\theta/2}.$$

Returning to (85), we find

$$\frac{|f_{z_0}(z) - f_{z_0}(\tilde{z})|}{[f]_{C_x^{\alpha/3} \log(1/C_v)^{-\theta}(Q_{t_0/2}(z_0))}} \lesssim \left(\log \frac{1}{t_0} \right)^{-\theta/2} \left(|x - \tilde{x}|^{\alpha/3} + \left(\log \frac{1}{|v - \tilde{v}|} \right)^{-\theta/2} \right),$$

which concludes the proof. \square

Appendix A. Computation of the fundamental solution (15). In this section, we establish the form of the fundamental solution $\Gamma_{\bar{a}}$ for the constant-in- (x, v) coefficients kinetic Fokker-Planck equation; that is, we prove Proposition 3.

Proof of Proposition 3. We first notice that it is enough to find $\Gamma_{\bar{a}}$ such that the solution to the initial value problem

$$(\partial_t + v \cdot \nabla_x) f = \text{tr}(\bar{a}(t) D_v^2 f), \quad (86)$$

with suitably decaying initial data at $t = \tilde{t}$ is given by

$$f(t, x, v) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Gamma_{\bar{a}}(t, x - \tilde{x} - (t - \tilde{t})\tilde{v}, v - \tilde{v}; \tilde{t}) f(\tilde{t}, \tilde{x}, \tilde{v}) d\tilde{x} d\tilde{v}. \quad (87)$$

Indeed, it is simply an application of Duhamel's principle to go from (87) to (15). As \tilde{t} plays essentially no role in the computations below, we simply set $\tilde{t} = 0$ and drop the "; 0" notation.

Next, we notice that (87) is equivalent to

$$\hat{f}(t, \xi, \omega) = (2\pi)^d \hat{f}(0, \xi, \omega + \xi t) \hat{\Gamma}_{\bar{a}}(t, \xi, \omega). \quad (88)$$

Indeed, taking the Fourier transform of (87) and computing, we find

$$\begin{aligned} \hat{f}(t, \xi, \omega) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Gamma_{\bar{a}}(t, x - \tilde{x} - t\tilde{v}, v - \tilde{v}) f(0, \tilde{x}, \tilde{v}) d\tilde{x} d\tilde{v} \right) \exp \{-ix \cdot \xi - iv \cdot \omega\} dx dv \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Gamma_{\bar{a}}(t, x - \tilde{x} - t\tilde{v}, v - \tilde{v}) f(0, \tilde{x}, \tilde{v}) \exp \{-ix \cdot \xi - iv \cdot \omega\} d\tilde{x} d\tilde{v} dx dv. \end{aligned}$$

As shifts in “physical space” correspond to multiplication in “Fourier space,” we have

$$\hat{\Gamma}_{\bar{a}}(t, \xi, \omega) = \frac{e^{i(\tilde{x} + t\tilde{v}) \cdot \xi + i\tilde{v} \cdot \omega}}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Gamma_{\bar{a}}(t, x - \tilde{x} - t\tilde{v}, v - \tilde{v}) \exp \{-ix \cdot \xi - iv \cdot \omega\} dx dv.$$

Thus,

$$\begin{aligned} \hat{f}(t, \xi, \omega) &= \hat{\Gamma}_{\bar{a}}(t, \xi, \omega) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(0, \tilde{x}, \tilde{v}) \exp \{-i(\tilde{x} + t\tilde{v}) \cdot \xi - i\tilde{v} \cdot \omega\} d\tilde{x} d\tilde{v} \\ &= \hat{\Gamma}_{\bar{a}}(t, \xi, \omega) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(0, \tilde{x}, \tilde{v}) \exp \{-i\tilde{x} \cdot \xi - i\tilde{v} \cdot (\omega + \xi t)\} d\tilde{x} d\tilde{v} \\ &= (2\pi)^d \hat{\Gamma}_{\bar{a}}(t, \xi, \omega) \hat{f}(0, \xi, \omega + \xi t). \end{aligned}$$

We now find $\Gamma_{\bar{a}}$ through the identity (88). The first step is to take the Fourier transform of (86) in x and v to obtain:

$$\partial_t \hat{f} - \xi \cdot \nabla_{\omega} \hat{f} = -\omega^T \bar{a}(t) \omega \hat{f},$$

Next, letting $\hat{F}(t, \xi, \omega) = \hat{f}(t, \xi, \omega - \xi t)$, we have

$$\partial_t \hat{F} = -(\omega - \xi t)^T \bar{a}(t) (\omega - \xi t) \hat{F}.$$

Integrating this in time, we find

$$\begin{aligned} \hat{F}(t, \xi, \omega) &= \exp \left\{ - \int_0^t (\omega - \xi s) \cdot \bar{a}(s) (\omega - \xi s) ds \right\} \hat{F}(0, \xi, \omega) \\ &= \exp \left\{ - \int_0^t (\omega - \xi s) \cdot \bar{a}(s) (\omega - \xi s) ds \right\} \hat{f}(0, \xi, \omega). \end{aligned}$$

Therefore,

$$\hat{f}(t, \xi, \omega) = \hat{f}(0, \xi, \omega + \xi t) \exp \left\{ - \int_0^t (\omega - \xi(s-t)) \cdot \bar{a}(s) (\omega - \xi(s-t)) ds \right\}.$$

It follows from (88) that

$$\hat{\Gamma}_{\bar{a}}(t, \xi, \omega) = \frac{1}{(2\pi)^d} \exp \left\{ - \int_0^t (\omega - \xi(s-t)) \cdot \bar{a}(s) (\omega - \xi(s-t)) ds \right\}. \quad (89)$$

The remainder of the proof is in computing the inverse Fourier transform of (89). We begin by computing that:

$$\begin{aligned}
\Gamma_{\bar{a}}(t, x, v) &= \frac{1}{(2\pi)^{2d}} \int \int e^{-\int_0^t (\omega - \xi(s-t)) \cdot \bar{a}(\omega - \xi(s-t)) ds + ix \cdot \xi + iv \cdot \omega} d\omega d\xi \\
&= \frac{1}{(2\pi)^{2d}} \int e^{-\xi \cdot (A_2 - 2tA_1 + t^2 A_0) \xi + ix \cdot \xi} \left(\int e^{-\omega \cdot A_0 \omega + (2A_1 \xi - 2tA_0 \xi + iv) \omega} d\omega \right) d\xi \\
&= \frac{1}{(2\pi)^{2d}} \int e^{-\xi \cdot N_2(t) \xi + ix \cdot \xi} \bar{\Gamma}_{\bar{a}} d\xi,
\end{aligned} \tag{90}$$

where we have introduced the notation

$$\begin{aligned}
N_1(t, \xi, v) &= 2A_1(t)\xi - 2tA_0(t)\xi + iv, \quad N_2(t) = A_2(t) - 2tA_1(t) + t^2 A_0(t), \\
\text{and} \quad \bar{\Gamma}_{\bar{a}}(t, \xi, v) &= \int e^{-\omega \cdot A_0(t)\omega + N_1(t)\cdot \omega} d\omega.
\end{aligned}$$

We simplify $\bar{\Gamma}_{\bar{a}}$ by completing the square:

$$\begin{aligned}
\bar{\Gamma}_{\bar{a}}(t, \xi, v) &= \int e^{-(\omega - \frac{1}{2}A_0^{-1}N_1) \cdot A_0(\omega - \frac{1}{2}A_0^{-1}N_1) + \frac{1}{4}N_1 \cdot A_0^{-1}N_1} d\omega \\
&= e^{\frac{1}{4}N_1 \cdot A_0^{-1}N_1} \int e^{-(\omega - \frac{1}{2}A_0^{-1}N_1) \cdot A_0(\omega - \frac{1}{2}A_0^{-1}N_1)} d\omega = e^{\frac{1}{4}N_1 \cdot A_0^{-1}N_1} \frac{\pi^{d/2}}{\sqrt{\det A_0}}.
\end{aligned}$$

Plugging this into (90) and then completing the square for the ξ -integral, we find

$$\begin{aligned}
\Gamma_{\bar{a}}(t, x, v) &= \frac{1}{2^{2d}\pi^{\frac{3d}{2}}\sqrt{\det A_0}} \int e^{-\xi \cdot N_2 \xi + ix \cdot \xi + \frac{1}{4}N_1 \cdot A_0^{-1}N_1} d\xi \\
&= \frac{1}{2^{2d}\pi^{\frac{3d}{2}}\sqrt{\det A_0}} e^{-\frac{v \cdot A_0^{-1}v}{4} - \frac{1}{4}q \cdot P^{-1}q} \int e^{-(\xi - \frac{i}{2}P^{-1}q) \cdot P(\xi - \frac{i}{2}P^{-1}q)} d\xi,
\end{aligned}$$

where (recall M from (16))

$$\begin{aligned}
P &= N_2 - (tA_0 - A_1)A_0^{-1}(tA_0 - A_1) = A_2 - A_1A_0^{-1}A_1 \\
\text{and} \quad q &= x - vt + A_1A_0^{-1}v = x - Mv.
\end{aligned}$$

Computing the the integral and simplifying, we find

$$\Gamma_{\bar{a}}(t, x, v) = \frac{1}{2^{2d}\pi^d\sqrt{\det(A_0)\det(P)}} e^{-\frac{v \cdot A_0^{-1}v}{4} - \frac{1}{4}q \cdot P^{-1}q}.$$

This concludes the proof. \square

Appendix B. Interpolation of weights between $L^{\infty, k}$ and $C_x^{\alpha/3} \log(1/C_v)^{-\theta}$.

Lemma 17. *Fix any $\alpha, \mu \in (0, 1)$ and any $\theta, k > 0$. Suppose that*

$$\varphi \in L^{\infty, k}(\mathbb{R}^3) \cap \log(1/C_v)^{-\theta}(\mathbb{R}^3).$$

Then $\langle v \rangle^{(1-\mu)k} \varphi \in \log(1/C_v)^{-\theta\mu}$ and

$$[\langle v \rangle^{(1-\mu)k} \varphi]_{\log(1/C_v)^{-\theta\mu}} \lesssim \|\varphi\|_{L^{\infty, k}}^{1-\mu} [\varphi]_{\log(1/C_v)^{-\theta}}^{\mu} + \|\varphi\|_{L^{\infty, ((1-\mu)k-1)_+}}.$$

Proof. First, for $(t, v) \neq (t, v') \in \mathbb{R}_+ \times \mathbb{R}^3$ with $|v - v'| < 1/2$, we let

$$R = \langle v \rangle^{-k} \|\phi\|_{L^{\infty, k}} [\varphi]_{\log(1/C_v)^{-\theta}}^{-1}.$$

Then, we obtain

$$\begin{aligned}
& |\langle v \rangle^{(1-\mu)k} \varphi(t, v) - \langle v' \rangle^{(1-\mu)k} \varphi(t, v')| \\
& \lesssim \langle v \rangle^{(1-\mu)k} |\varphi(t, v) - \varphi(t, v')| + |\varphi(t, v')| |\langle v \rangle^{(1-\mu)k} - \langle v' \rangle^{(1-\mu)k}| \\
& \lesssim \langle v \rangle^{(1-\mu)k} |\varphi(t, v) - \varphi(t, v')| + |\varphi(t, v')| \langle v \rangle^{((1-\mu)k-1)+} |v - v'| \\
& \lesssim \langle v \rangle^{(1-\mu)k} |\varphi(t, v) - \varphi(t, v')| + \|\varphi\|_{L^{\infty,((1-\mu)k-1)+}} |v - v'|.
\end{aligned}$$

Notice that

$$\frac{|v - v'|}{\log(1/|v - v'|)^{-\mu\theta}} \lesssim 1.$$

Hence, we need only bound

$$H := \frac{\langle v \rangle^{(1-\mu)k} |\varphi(t, v) - \varphi(t, v')|}{\log(1/|v - v'|)^{-\mu\theta}}.$$

If $\log(1/|v - v'|)^{-\theta} \geq R$, we have

$$H \lesssim 2\langle v \rangle^{-\mu k} \frac{\|\varphi\|_{L^{\infty,k}}}{R^\mu} = 2\|\varphi\|_{L^{\infty,k}}^{1-\mu} [\varphi]_{\log(1/C_v)^{-\theta}}^\mu,$$

which is the desired inequality.

On the other hand, if $\log(1/|v - v'|)^{-\theta} \leq R$, we see

$$\begin{aligned}
H &= \langle v \rangle^{(1-\mu)k} \frac{|\varphi(t, v) - \varphi(t, v')|}{\log(1/|v - v'|)^{-\theta}} (\log(1/|v - v'|)^{-\theta})^{1-\mu} \leq \langle v \rangle^{(1-\mu)k} [\varphi]_{\log(1/C_v)^{-\theta}} R^{1-\mu} \\
&\lesssim [\varphi]_{\log(1/C_v)^{-\theta}} \|\phi\|_{L^{\infty,k}}^{1-\mu} [\varphi]_{\log(1/C_v)^{-\theta}}^{\mu-1} = [\varphi]_{\log(1/C_v)^{-\theta}}^\mu \|\phi\|_{L^{\infty,k}}^{1-\mu},
\end{aligned}$$

which is, again, the desired inequality. This concludes the proof. \square

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