

TRAVELING WAVES FOR THE KELLER-SEGEL-FKPP EQUATION WITH STRONG CHEMOTAXIS

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ABSTRACT. We show that there exist traveling wave solutions of the Keller-Segel-FKPP equation, which models a diffusing and logistically growing population subject to chemotaxis. In contrast to previous results, our result is in the strong aggregation regime; that is, we make no smallness assumption on the parameters. The lack of a smallness condition makes L^∞ -estimates difficult to obtain as the comparison principle no longer gives them “for free.” Instead, our proof is based on suitable energy estimates in a carefully tailored uniformly local L^p -space. Interestingly, our uniformly local space involves a scaling parameter, the choice of which is a crux of the argument. Numerical experiments exploring the stability, qualitative properties, and speeds of these waves are presented as well.

1. INTRODUCTION

The main goal of this work is to analyze front propagation phenomena in an FKPP-Keller-Segel system:

$$(1.1) \quad \begin{cases} u_t + \chi(uv_x)_x = u_{xx} + u(1-u) & \text{in } (0, \infty) \times \mathbb{R}, \\ -dv_{xx} = u - v & \text{in } (0, \infty) \times \mathbb{R}, \end{cases}$$

where $d > 0$ and $\chi \in \mathbb{R}$. We point out that the unique bounded solution of the second equation is given in terms of a convolution:

$$(1.2) \quad v = K_d * u \quad \text{where} \quad K_d(x) = \frac{1}{2\sqrt{d}} e^{-\frac{|x|}{\sqrt{d}}}.$$

The interpretation of equation (1.1) is the following: u represents the population density of a species that diffuses, reproduces and competes logistically, as well as interacts intraspecifically via a “chemical signal” v . In this model, that interaction is chemotaxis: each individual both secretes a chemical signal and moves in response to the chemical signal of its “neighbors.” The sensitivity constant, $\chi \in \mathbb{R}$, describes each individual’s perception and response to the chemical signal, with its magnitude encoding the strength of the chemotaxis, and the sign of χ determines if the chemotaxis is aggregative ($\chi > 0$) or dispersive ($\chi < 0$). Roughly, the diffusion coefficient, $d > 0$, is the length-scale on which chemotaxis acts. Chemotaxis is a well-studied phenomenon that is often seen in slime molds and bacteria [25, 30, 33].

We are interested in traveling wave solutions to (1.1) and their speed (we define these terms in Definition 1.1). Traveling waves have previously been studied for similar reaction-diffusion equations including the Fisher-KPP equation, which arises when $\chi = 0$ in (1.1). In this case, the speed of the slowest traveling wave is 2, and this minimal speed wave describes the long-time behavior of solutions with “localized” initial data. This is typical of reaction-diffusion models: the (minimal speed) traveling waves are stable in a suitable sense, meaning that their speed and profile generically describes the invasion of a species into a new environment. We are interested in establishing a more complete understanding of the effects chemotaxis has on traveling waves.

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The spreading properties of (1.1) have been the subject of great interest recently. At the level of traveling waves, their behavior is fairly well understood as long as χ is not too positive. Indeed, traveling waves have been shown to exist under the condition

$$(1.3) \quad \chi < \min\{1, d\}.$$

This is contained in [31] when $\chi > 0$ and [20] when $\chi < 0$ (see also [37, 39]). We explain the technical importance of (1.3) in Section 1.2. In this regime, it is known that, roughly, if $|\chi|$ and d are not “too big,” any minimal speed traveling wave has speed 2, while the minimal speed tends to infinity if $\chi \rightarrow -\infty$ [20, 23, 31, 40]. One expects, and sees numerically [3], a transition at a critical curve in the (χ, d) parameter space. The Cauchy problem is more difficult, but certain aspects of spreading have been established [22, 36, 37]. We note that there is an enormous PDE literature on chemotaxis focused on aspects like blow-up [6, 8, 10, 11, 13, 15, 44], as well as front propagation-like questions for the many related models [5, 7–9, 17–19, 21, 26–29, 34, 35, 38, 47]. The literature is truly vast, and this is only a small sampling of it.

Our interest in this paper is to investigate what happens when condition (1.3) is not satisfied. Specifically, we construct traveling waves for any $\chi, d > 0$, which requires a new approach to *a priori* estimates for the traveling wave problem associated to (1.1). We complement this with a numerical investigation of the behavior of the Cauchy problem that reveals (1) the minimal speed appears to always be 2 in contrast to what happens when $\chi \rightarrow -\infty$, and (2) the possible existence of a bifurcation: when χ is large relative to d , pulsating fronts (roughly, traveling waves with patterns in the back) appear.

1.1. Main result. Let us begin by defining the notion of a traveling wave in our context.

Definition 1.1. A *traveling wave solution* to (1.1) is a triple (c, U, V) such that

- (1) $0 < U \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$,
- (2) $c > 0$,
- (3) $u(t, x) = U(x - ct)$ and $v(t, x) = V(x - ct)$ solve (1.1), and
- (4) $\liminf_{x \rightarrow -\infty} U(x) > 0$, $\lim_{x \rightarrow \infty} U(x) = 0$.

We refer to c and U as the *speed* and *profile*, respectively.

Applying the coordinate change in Definition 1.1, the existence of a traveling wave solution to (1.1) is equivalent to finding a triple (c, U, V) that solves the system:

$$(1.4) \quad \begin{cases} -cU' + \chi(UV')' = U'' + U(1 - U) & \text{in } \mathbb{R}, \\ -dV'' = U - V & \text{in } \mathbb{R}. \end{cases}$$

We now state our main theorem.

Theorem 1.2. For all $\chi, d > 0$, there exists a traveling wave solution to (1.1). Moreover, the speed c of any traveling wave solution satisfies $c \geq 2$.

We make a few comments about the theorem. First, note that this result actually holds for any $\chi \in \mathbb{R}$. As mentioned above, the case when χ is nonpositive has been considered in [20, 23], and the case when χ is small relative to d was considered in, e.g., [31, 40]; however, the smallness condition (1.3) was crucial to the argument in these works.

Second, it is natural to ask if our choice of K_d in (1.2) is necessary for our argument or if they would hold for a similar model but with nonlocal advection $v = K_d * u$ defined by a different K_d . This type of model was considered in [23], where the interested reader can find natural conditions on K_d . Our proof seems quite flexible in this regard. The form of K_d is mainly used in relating the L^∞ -norm of V to a certain “uniformly local” norm of U (see Definition 1.3 and Lemma 3.6). If,

e.g., $K_d \sim (1+|x|)^{-1-\alpha}$ for $\alpha > 0$, it seems one could replace φ in Definition 1.3 by $(1+|\sigma_{\text{ul}}x|)^{-1-\alpha}$ and reproduce the same estimate.

Finally, let us discuss the role of “minimal speed” traveling waves. We say that a traveling wave (c^*, U^*, V^*) is a minimal speed traveling wave if $c^* \leq c$ for all traveling wave solutions (c, U, V) . Often, for equations with a logistic reaction term such as (1.1), there is a traveling wave for every speed $c \geq c_*$ but the minimal speed one is stable with regard to “localized” initial data [2]. One might suspect that there is an infinite half-line of speeds here as well; however, we do not address this, nor do we address whether the wave we construct is the minimal speed one. On the other hand, the procedure we implement “should” return the minimal speed wave, as it does when $\chi = 0$. The stability of the wave that we construct is a subtle issue. As we discuss in Section 2, it seems solutions starting from Heaviside initial data form a speed 2 traveling wave when $\chi \leq (1 + \sqrt{d})^2$; however, it appears they form a “pulsating front” otherwise (see [46, Section 2.2] for a definition, as well as [41] for the first work introducing the concept). We leave the discussion of this to the sequel.

1.2. Discussion of the proof. One standard procedure for establishing the existence of traveling wave solutions for similar reaction-diffusion systems is to first consider the “slab problem,” i.e., (1.4) but on a finite interval $[-a, a]$ for $a \gg 1$ with the boundary conditions $U_a(-a) = 1$ and $U_a(a) = 0$. One then obtains existence of solutions on sufficiently large slabs via the Leray-Schauder topological degree theory, the key step of which is to establish suitable bounds on c_a and U_a . In order to take the limit as $a \rightarrow \infty$ to obtain the traveling wave solution, it is crucial that the established bounds depend only on d and χ and not on a .

Main difficulties. For the particular model considered in this article, the difficult part of this approach is establishing the uniform bounds on c_a and U_a . Previous works on traveling wave solutions of (1.1), assume a smallness condition on d and χ when $\chi > 0$, whereas we do not. To illustrate why such a condition is helpful, we briefly outline the argument that establishes an L^∞ -bound on U_a . Suppose $\chi/d < 1$ and U_a achieves its maximum M at $x_M \in (-a, a)$. By combining the two equations in (1.4), we have

$$(1.5) \quad -c_a U'_a + \chi U'_a V'_a = U''_a + U_a \left(1 - U_a - \frac{\chi}{d} (U_a - V_a) \right).$$

Thus, at x_M , we have

$$(1.6) \quad \begin{aligned} 0 &= -c_a U'_a + \chi U'_a V'_a = U''_a + M \left(1 - M - \frac{\chi}{d} (M - V_a) \right) \\ &\leq M \left(1 - M + \frac{\chi}{d} (M - V_a) \right) \leq M \left(1 - M + \frac{\chi}{d} M \right), \end{aligned}$$

which immediately yields the bound

$$(1.7) \quad \|U_a\|_{L^\infty} = M \leq \frac{1}{1 - \chi/d}.$$

We note that a naïve phase plane argument using “trapping region” arguments leads to the same obstruction.

Tello-Winkler approach. From (1.6), we see that the only hope to make the above argument work without the smallness condition is to use the V_a term that is dropped in (1.6). This would require a lower bound on V_a ; however, any lower bound on V_a will necessarily require regularity estimates of U_a , which, in turn depend on c_a due to (1.4). On the other hand, the standard approach for estimating the speed c_a for the slab problem yields

$$(1.8) \quad c_a \leq 2 + \left(\frac{\chi}{\sqrt{d}} + \frac{\chi}{d} \right) \|V_a\|_{L^\infty}$$

(see Lemma 3.8). Alas, we find ourselves in a ‘loop’ of inequalities, which is the main difficulty in proving Theorem 1.2.

Our approach to bounding c_a and U_a is based on the method employed in [42] with some modifications. The authors in [42] consider a slightly more general model than (1.1) on a bounded domain $\Omega \subseteq \mathbb{R}^n$ supplemented with Neumann boundary conditions for u and v .

We outline their argument used to obtain a uniform bound on $\|u(t, \cdot)\|_{L^\infty(\Omega)}$ now. They first show $\|u(t, \cdot)\|_{L^p(\Omega)}$ is uniformly bounded for values of $p > 1$ near 1. This is done by multiplying (1.1) by u^{p-1} , leveraging the negative quadratic term in $u(1-u)$, and carefully integrating by parts to obtain

$$(1.9) \quad \frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p dx + (p-1) \int_{\Omega} u^{p-2} |\nabla u|^2 dx \leq \left(\frac{\chi(p-1)}{d} - 1 \right) \int_{\Omega} u^{p+1} dx + \int_{\Omega} u^{p-1} dx.$$

The coefficient of the first integral on the right side of (1.9) is made negative by choosing p sufficiently close to 1. As a result, the first term on the right controls the second term via Hölder’s inequality:

$$(1.10) \quad \int_{\Omega} u^{p-1} dx \leq |\Omega|^{\frac{1}{p}} \left(\int_{\Omega} u^p dx \right)^{\frac{p-1}{p}} \quad \text{and} \quad \int_{\Omega} u^{p+1} dx \geq |\Omega|^{-p} \left(\int_{\Omega} u^p dx \right)^{\frac{p+1}{p}}.$$

From here, one can use a dynamical argument to rule out $\|u\|_{L^p}$ of ever becoming “too large.” As is clear from (1.10), the size of Ω affects the ultimate bound on $\|u(t, \cdot)\|_{L^p}$. This is natural as we expect u to equilibrate to an $O(1)$ steady solution, whence $\|u(t, \cdot)\|_{L^p} = O(|\Omega|^{1/p})$.

Afterwards, one can bootstrap the L^p -bound to higher L^r -estimates of u by using the gradient term in (1.9). The full regularity of u follows by standard parabolic regularity arguments once the high enough integrability of u is obtained.

Outline of the proof of Theorem 1.2. Two issues arise when applying this argument in our context: we require a bound that is independent of a (the size of the domain), and we have no time dependence (recall we have made the traveling wave change of variables that turns the u_t term into $-cU'$). We discuss the former issue at greater length, and simply mention that the complication of the latter is to induce additional c dependence in each estimate.

To overcome the first issue, we introduce a new variant of *uniformly local L^p -spaces*. Although we only use this with $p = 2$, we state the general definition.

Definition 1.3. Fix $p \in [1, \infty)$, $\sigma_{\text{ul}} > 0$, and $\psi \in \{C_c^\infty(\mathbb{R}) : \|\psi\|_{L^1} = 1 \text{ and } 0 \leq \psi \leq 1\}$. Let $\varphi = e^{-\sigma_{\text{ul}}|x|} * \psi$, and, for any $s \in \mathbb{R}$, let $\varphi_s(x) = \varphi(x-s)$. We define the **uniformly local L^p -norm** of a measurable function f as

$$\|f\|_{L_{\text{ul}}^p} = \sup_{s \in \mathbb{R}} \|\varphi_s^{1/p} f\|_{L^p} = \sup_{s \in \mathbb{R}} \left(\int_{-\infty}^{\infty} \varphi_s(x) |f(x)|^p dx \right)^{\frac{1}{p}}.$$

The original uniformly local L^p -spaces were introduced by Kato [24] to construct solutions to a hyperbolic system of equations and involves a compactly supported φ . These spaces have been used in various contexts, e.g., the Boltzmann equation [1], in the last half century.

As we describe below, the uniformly local L_{ul}^2 -space presents many advantages for us. For one, it localizes our estimates to a domain of “size” $O(1/\sigma_{\text{ul}})$, which, to a degree, solves the problem of not working on a finite domain. An important feature of the uniformly local norm is that, when $\sigma_{\text{ul}} < 1/\sqrt{d}$, we have

$$(1.11) \quad \|V_a\|_{L^\infty} \leq C \|U_a\|_{L_{\text{ul}}^2},$$

which is a consequence of the fact that $K_d \leq C\varphi$. (See Lemma 3.5 and Lemma 3.6.)

With the observations above and some care, one can prove an energy estimate in this uniformly local space of the form (Lemma 3.7):

$$(1.12) \quad \|U_a\|_{L^2_{\text{ul}}}^2 \leq C \left(\frac{1}{\sigma_{\text{ul}}} + \sigma_{\text{ul}} c_a^2 \right).$$

Actually, boundary condition at $x = -a$ causes some issues, so what we really prove is that, for each s ,

$$(1.13) \quad \int_{x_1}^a \varphi_s U_a^2 \leq C \left(\frac{1}{\sigma_{\text{ul}}} + \sigma_{\text{ul}} c_a^2 + \sqrt{\sigma_{\text{ul}}} \|U_a\|_{L^2_{\text{ul}}}^2 \right),$$

whenever $\|U_a\|_{L^\infty} > 2$. Here, x_1 is the leftmost element of the $3/2$ level set of U_a . This ensures that $U'_a(x_1) \geq 0$ and $U'_a(a) \leq 0$, which are “good” signs for all of the boundary terms that arise. Additionally, notice that the term on the left in (1.13) is comparable to the L^2_{ul} -norm of U_a up to σ_{ul} factors. This, and the “small” $\sqrt{\sigma_{\text{ul}}}$ factor in front of the L^2_{ul} -term, allow us to deduce (1.12).

Notice that we have not decoupled the dependence of the norms of U_a on c_a , yet; however, we have “won” a small parameter of σ_{ul} in front of the “bad” term c_a^2 . Indeed, standard arguments show that

$$c_a \leq 2 + \chi \|V_a''\|_{L^\infty} + \chi \|V_a'\|_{L^\infty},$$

and it is straightforward to check that $d \|V_a''\|_{L^\infty}, \sqrt{d} \|V_a'\|_{L^\infty} \leq \|V_a\|_{L^\infty}$ (see Lemma 3.8). Combining this with (1.12) and (1.11), we find

$$c_a^2 \leq C \left(\frac{1}{\sigma_{\text{ul}}} + \sigma_{\text{ul}} c_a^2 \right),$$

at which point we deduce the desired bound on c_a by further decreasing σ_{ul} . An L^2_{ul} -bound on U_a follows then from this bound on c_a and (1.12). At that point, we now have bounds on the coefficients of (1.5) (recall (1.11) for the bound on V_a and Lemma 3.1 for the bound on V'_a), so a standard argument to upgrade a local L^2 -estimate to an L^∞ -bound may be applied.

We point out that the main estimate (1.12) is, in fact, quite different from the estimate used in [42], although inspired by it.

Notation. Unless needed for clarity, we drop the integrand’s dependence on the variable of integration, e.g., we write

$$\int_0^1 U \quad \text{instead of} \quad \int_0^1 U(x) dx.$$

Also, we suppress the dependence on the domain for function spaces’ norms if the domain is $\mathbb{R}$. For example, we write $\|U\|_{L^p}$ instead of $\|U\|_{L^p(\mathbb{R})}$. Lastly, C represents a positive constant that may change line-by-line and depends only on d and χ . In general, it is assumed that all constants and conditions depend on d and χ .

2. NUMERICAL SIMULATIONS

The Cauchy problem of (1.1) is beyond the scope of this paper. We performed numerical experiments to investigate this question. We essentially use the “upwind” numerical scheme from [19]. Numerical simulations inherently take place on a finite domain, so we impose Neumann boundary conditions and take care to stop the simulation long before the front approaches the boundary.

Often (minimal speed) traveling waves are unique and stable, which means that solutions u to (1.1) converge in a $ct + o(t)$ moving frame to the minimal speed traveling wave solution. Further, in other simpler models, minimal speed traveling waves can be constructed by the process performed in this paper. This suggests that one might expect the solution we construct here to be the minimal speed traveling wave and that it is the limit of solutions to the Cauchy problem. We see, in the sequence, that this is not the case when χ is large.

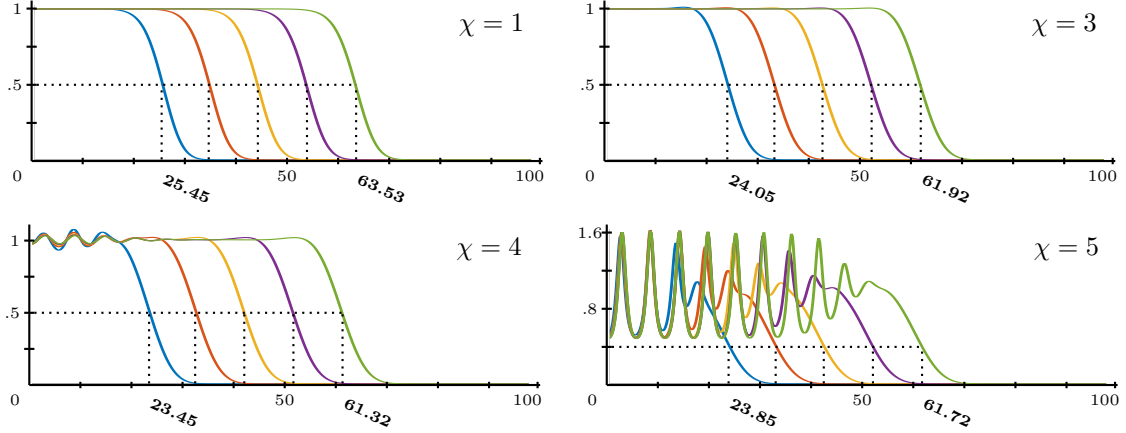


FIGURE 1. In all images above, $d = 1$ with initial data $u_{\text{in}} = \exp\{-2(x-10)_+^2/5\}$. We took spatial step size $\Delta x = 0.2$ and time step size $\Delta t = (\Delta x)^2/10$. In the first three plots above, we can approximately compute the speed by tracking the $1/2$ level set, and in the last one, we use $2/5$ level set (this avoids issues with the pattern making the $1/2$ level set have multiple elements). In a fixed plot, each curve is the profile u at five units of time beyond the profile to its left; indeed, the curves were sampled at $t = 10, 15, 20, 25, 30$.

The behavior of u changes greatly as one changes χ . It was already known that when χ is sufficiently negative, traveling waves are “sped up”, and that when χ is close enough to zero (with no constraint on the sign), the wave speed is the same as when $\chi = 0$, i.e., $c = 2$ [20, 23]. What we find here is that the wave speed seems to not change as χ is increased. On the other hand, a Hopf-type bifurcation occurs: the traveling wave seems to destabilize and a pulsating front appears and seems to be stable.

Let us look more closely at the speed first. Computing the approximate speed by subtracting location of the leftmost level set from the rightmost and dividing by 20 (4 intervals of 5 units of time), we find:

$$\begin{aligned} c_{\chi=1} &\approx \frac{63.53 - 25.45}{20} \approx 1.90, & c_{\chi=3} &\approx \frac{61.92 - 24.05}{20} \approx 1.89, \\ c_{\chi=4} &\approx \frac{61.32 - 23.45}{20} \approx 1.89, & \text{and} & c_{\chi=5} &\approx \frac{61.72 - 23.85}{20} \approx 1.89. \end{aligned}$$

We note that, despite the clear change in qualitative behavior in Figure 1 as χ increases, the speed remains essentially constant. Additionally, given the “large” Δx and Δt used in our simulations and the difficulties of computing the speed of waves numerically (see, e.g., the discussion in the introduction of [12]), it seems heuristically clear that $c = 2$ is within the margin of error. We do not investigate this more systematically; however, it is tempting to conjecture that $c = 2$ for $\chi > 0$.

We now turn our attention to the change in behavior that occurs when

$$(2.1) \quad \chi = 4 = (1 + \sqrt{d})^2$$

(recall that $d = 1$ in our simulations). In Figure 1, we see periodic behavior begin to appear here. For $\chi = 4$, the “wiggles” dissipate in time; however, they are permanent in the $\chi = 5$ plot. In a sense, this is not unexpected in that a linear stability analysis of the constant state $u \equiv 1$ reveals its stability if and only if $\chi < (1 + \sqrt{d})^2$ [31]. We note further work done on constructing patterns in similar models [14, 16]. Numerically, it appears that u converges (in a moving frame) to a *pulsating*

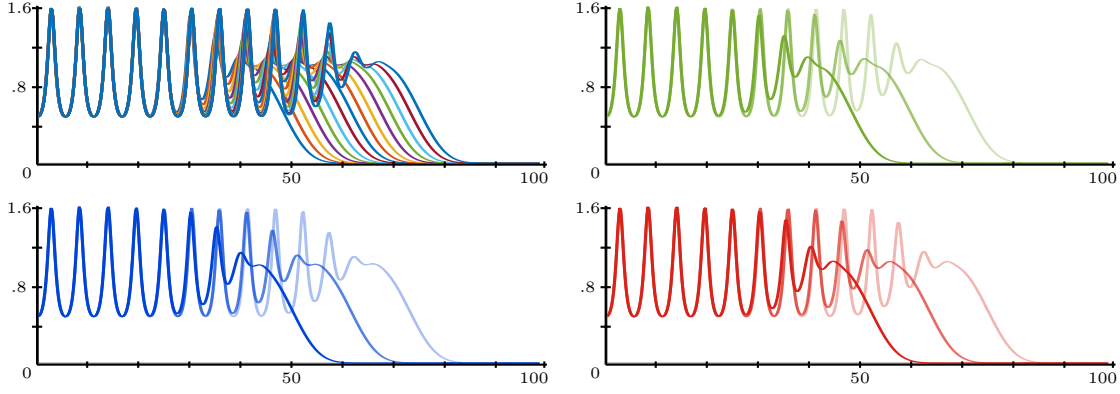


FIGURE 2. The above images show plots for $\chi = 5$ that reveal the time-periodic nature of the profile in the moving frame. The top left plot is of $u(t, \cdot)$ for all integer t between 24 and 38. It appears that the profile is 3 periodic in time. To show this, the top right plot is of $t = 24, 30, 36$, the bottom left plot is of $t = 25, 31, 37$, and the bottom right plot is of $t = 26, 32, 38$. Notice that, when the time difference is a multiple of 3 the profile appears to be an exact copy but shifted to the right. (We took time differences of 6 in order to minimize overcrowding in the plots, but the same features are present when the time difference is 3.)

front (see Figure 2); that is, a solution to (1.1) of the form $u(t, x) = U(x, x - ct)$, where U is periodic in the second variable [41, 46]. To our knowledge, it is an open question whether pulsating fronts exist for (1.1). Our numerics, however, suggest that these should exist and be stable when

$$(2.2) \quad \chi > (1 + \sqrt{d})^2.$$

To our knowledge, the existence of pulsating fronts that connect 0 to a nonconstant (periodic) steady states have not been proven for any reaction-diffusion models.

Let us note the gap in the parameters (χ, d) between where previous works had constructed traveling waves (under the assumption (1.3)) and where they are expected to no longer be stable (when (2.2) holds). Indeed, the interval

$$\min\{1, d\} \leq \chi \leq (1 + \sqrt{d})^2$$

can be arbitrarily large, depending on d . It is in this regime that we expect our result Theorem 1.2 to be relevant; however, proving the stability of the wave constructed here is beyond the scope of the current paper.

In view of recent results on finite domains [32, 43, 45], one might expect that the “peaks” behind the front in the pulsating front will grow to infinity as χ tends to infinity. Interestingly, we did not observe this in our numerical simulations. Further study is necessary to determine if this is a limitation of the simulations or if it reflects the true behavior of the model.

3. THE PROBLEM ON A FINITE SLAB

To establish the existence of a solution to (1.4), we first show there exists a solution on every finite “slab,” i.e., on every finite interval $(-a, a)$, where $a > 0$. In order to guarantee the positivity of U_a , we also slightly adjust the equation. To this end, consider (1.4) on $(-a, a)$ with added boundary conditions:

$$(3.1) \quad \begin{cases} -c_a U'_a + \tau \chi (U_a V'_a)' = U''_a + (U_a)_+(1 - U_a) & \text{in } (-a, a), \\ U_a(-a) = 1, \quad U_a(a) = 0, \quad \max_{x \geq 0} \tilde{U}_a(x) = \theta > 0, \end{cases}$$

where $x_+ := \max\{0, x\}$, $\theta \in (0, 1/4)$ is a small fixed parameter, $\tau \in [0, 1]$ is a parameter used when we apply Leray-Schauder degree theory in Proposition 3.10, and

$$(3.2) \quad V_a = \tilde{U}_a * K_d \quad \text{where} \quad \tilde{U}_a(x) = \begin{cases} 1 & \text{if } x \leq -a, \\ U_a(x) & \text{if } x \in (-a, a), \\ 0 & \text{if } x \geq a. \end{cases}$$

A direct computation with this definition of $\tilde{U}_a$ and V_a yields

$$(3.3) \quad -dV_a'' = \tilde{U}_a - V_a \quad \text{in } \mathbb{R}.$$

Note that, on the slab, we have

$$(3.4) \quad -dV_a'' = U_a - V_a \quad \text{in } [-a, a].$$

The subscripts indicate the solution's dependence on a . Note also that a solution (c_a, U_a, V_a) to (3.1) also depends on θ and τ , but, unless needed for clarity, we suppress this dependence in our notation.

Let us make two additional technical remarks. First, note that $\|\tilde{U}_a\|_{L^\infty} = \|U_a\|_{L^\infty([-a, a])}$. Therefore, for notational convenience, we use $\|U_a\|_{L^\infty}$ to denote both $\|\tilde{U}_a\|_{L^\infty}$ and $\|U_a\|_{L^\infty([-a, a])}$ whenever either term arises.

Second, note that first equation in (3.1) differs slightly from that of (1.1) because one of the U_a terms is changed to $(U_a)_+$. This allows us to immediately prove that $U_a > 0$ (see Lemma 3.2), at which point we see that the first equation in (3.1) and that of (1.1) agree.

3.1. Preliminaries. In this section, we collect a few useful inequalities that are deployed in the main argument. In particular, we establish the positivity of solutions to (3.1), as well as bounds on V_a (and its first two derivatives). The results in this subsection are either proved using standard arguments or are straightforward to deduce.

3.1.1. Estimates on U_a and V_a . First we point out some easy estimates on V_a coming from U_a .

Lemma 3.1 (Bounds on V_a , V_a' , and V_a''). *If (c_a, U_a, V_a) solves (3.1) and $U_a \geq 0$, then, for all $x \in \mathbb{R}$,*

$$(3.5) \quad |V_a'(x)| \leq \frac{1}{\sqrt{d}} V_a(x).$$

Proof. This follows from a simple computation using

$$|V_a'(x)| = |(K_d' * \tilde{U}_a)(x)| = \frac{1}{\sqrt{d}} \left| \int_{-\infty}^{\infty} -\text{sign}(y) K_d(y) \tilde{U}_a(x-y) dy \right| \leq \frac{1}{\sqrt{d}} V_a(x). \quad \blacksquare$$

Next, we check that U_a is, in fact, nonnegative.

Lemma 3.2 (Nonnegativity of U_a and V_a). *If (c_a, U_a, V_a) solves (3.1), then both U_a and V_a are nonnegative on $[-a, a]$ and positive in $(-a, a)$.*

Proof. Note that the result follows by the strong maximum principle once we show that

$$(3.6) \quad \min_{[-a, a]} U_a \geq 0.$$

We establish (3.6) by contradiction. Suppose that there exists x_0 such that

$$(3.7) \quad U_a(x_0) = \min_{[-a, a]} U_a < 0.$$

Then, due to the boundary conditions (3.1), $x_0 \in (-a, a)$. We see from (3.1) and (3.4) that, at x_0 ,

$$(3.8) \quad \begin{aligned} 0 &\leq U_a'' + 0 = U_a'' + (U_a)_+(1 - U_a) = -c_a U_a' + \tau \chi(U_a' V_a' + U_a V_a'') \\ &= \tau \chi U_a V_a'' = \tau \chi U_a (V_a - U_a). \end{aligned}$$

We note that $U_a(x_0) < 0$. Additionally, using (3.2), (3.7), and the fact that $\tilde{U}_a$ is nonconstant, we find

$$(3.9) \quad V_a(x_0) = K_d * \tilde{U}_a(x_0) > \min \tilde{U}_a = U_a(x_0).$$

It follows that, at x_0 ,

$$(3.10) \quad U_a(V_a - U_a) < 0,$$

which contradicts (3.8) and concludes the proof. $\blacksquare$

3.1.2. A lower bound on the speed. We must ensure that c_a remains uniformly positive. The first step to doing that is to show that V_a and U_a are not “too big” far on the right.

Lemma 3.3. *There exists $L > 0$, such that, for all $x \geq L$,*

$$(3.11) \quad V_a(x) \leq 2\theta \quad \text{and} \quad |V'_a(x)| \leq \frac{2}{\sqrt{d}}\theta.$$

The constant L is independent of a but does depend on $\|U_a\|_{L^\infty}$ and θ .

Proof. Since $\|K_d\|_{L^1} = 1$ (see (1.2)), we may choose $L > 0$ large enough so that

$$(3.12) \quad \int_L^\infty K_d \leq \|U_a\|_{L^\infty}^{-1}\theta.$$

Then, since $\max_{x \geq 0} \tilde{U}_a(x) = \theta$ (see (3.1)), we have, for all $x \geq L$,

$$\begin{aligned} V_a(x) &= \int_{-\infty}^\infty K_d(y) \tilde{U}_a(x-y) dy = \int_x^\infty K_d(y) \tilde{U}_a(x-y) dy + \int_{-\infty}^x K_d(y) \tilde{U}_a(x-y) dy \\ &\leq \int_x^\infty K_d(y) \|U_a\|_{L^\infty} dy + \int_{-\infty}^x K_d(y) \theta dy \leq \int_L^\infty K_d(y) \|U_a\|_{L^\infty} dy + \theta \leq 2\theta. \end{aligned}$$

In the last step, we used (3.12). The remaining inequality in (3.11) follows by applying the bound on $|V'_a|$ from (3.5). This concludes the proof. $\blacksquare$

Lemma 3.4 (Lower bound on the speed). *If (c_a, U_a, V_a) is a solution to (3.1) with $c_a \geq 0$, then, for all $\varepsilon > 0$, there exists $a_\varepsilon > 0$ and $\theta_\varepsilon > 0$ such that, for all $a > a_\varepsilon$ and for all $\theta \in (0, \theta_\varepsilon)$,*

$$(3.13) \quad c_a \geq 2 - \varepsilon.$$

The parameters a_ε and θ_ε depend only on $\|U_a\|_{L^\infty}$.

Proof. We argue by contradiction. Suppose $\varepsilon > 0$ and $c_a < 2 - \varepsilon$. Note that it is enough to prove the estimate under the assumption that $\varepsilon \in (0, 1)$.

Fix $\theta \in (0, \theta_\varepsilon)$ for θ_ε to be determined, and let L be as in Lemma 3.3. Let

$$(3.14) \quad R = \frac{a - 1 - L}{2}.$$

For $A, \lambda > 0$ to be chosen, let

$$(3.15) \quad \beta_A(x) = \frac{1}{A} e^{-\lambda x} \cos\left(\frac{\pi}{2R}(x - L - R)\right)^2 \quad \text{for } x \in [L, L + 2R] = [L, a - 1].$$

Since U_a is positive on $[L, L + 2R]$, we have, by continuity, that $\beta_A < U_a$ on $[L, L + 2R]$ if A is sufficiently large. Thus, the following quantity is well-defined:

$$A_0 = \inf\{A > 0 : \beta_A < U_a \text{ on } [L, L + 2R]\}.$$

Let $\beta = \beta_{A_0}$. By continuity, there exists $x_0 \in [L, L + 2R]$ such that $\beta(x_0) = U_a(x_0)$. Moreover, since $\beta(L) = \beta(L + 2R) = 0$ while both $U_a(L)$ and $U_a(L + 2R) = U_a(a - 1)$ are positive, it follows that $x_0 \in (L, L + 2R)$. Also, note that $U_a - \beta$ is nonnegative and attains a minimum of 0 at x_0 . As a result, we have

$$(3.16) \quad U_a(x_0) = \beta(x_0), \quad U'_a(x_0) = \beta'(x_0), \quad \text{and} \quad (U_a - \beta)''(x_0) \geq 0.$$

From (3.1), (3.3), and (3.16), we deduce that, at x_0 ,

$$\begin{aligned} 0 &\leq (U_a - \beta)'' = -c_a U_a' + \tau \chi V_a' U_a' + \tau \chi U_a \left(\frac{V_a - U_a}{d} \right) - U_a(1 - U_a) - \beta'' \\ &\leq -c_a \beta' + \tau \chi V_a' \beta' + \frac{\tau \chi}{d} V_a \beta - \beta(1 - U_a) - \beta'' \leq -c_a \beta' + \tau \chi V_a' \beta' + \frac{2\theta \tau \chi}{d} \beta - \beta(1 - \theta) - \beta''. \end{aligned}$$

In the last step, we used that $U_a \leq \theta$ on $[0, a]$ (see (3.1)) and $V_a \leq 2\theta$ on $[L, a]$. Then, using the explicit formula (3.15), multiplying by $A_0 e^{\lambda x_0}$, and using the shorthand $z_0 = (\pi/2R)(x_0 - L - R)$, we find

$$\begin{aligned} 0 &\leq (\tau \chi V_a' - c_a) \left(-\lambda \cos(z_0)^2 - \frac{\pi}{R} \cos(z_0) \sin(z_0) \right) + \left(\frac{2\theta \tau \chi}{d} + \theta - 1 \right) \cos(z_0)^2 \\ &\quad - \left(\lambda^2 \cos(z_0)^2 + \frac{2\lambda \pi}{R} \cos(z_0) \sin(z_0) + \frac{\pi^2}{2R^2} \sin(z_0)^2 - \frac{\pi^2}{2R^2} \cos(z_0)^2 \right) \\ &= -\frac{\pi^2}{2R^2} \sin(z_0)^2 - \left(1 + \lambda^2 - \frac{\pi^2}{2R^2} - \theta \left(\frac{2\tau \chi}{d} + 1 \right) + \lambda(\tau \chi V_a' - c_a) \right) \cos(z_0)^2 \\ &\quad + \frac{\pi}{R} (c_a - \tau \chi V_a' - 2\lambda) \cos(z_0) \sin(z_0). \end{aligned}$$

The first two terms are negative, and this fact eventually leads to our contradiction. The third term's sign, however, is indeterminate, so we choose $\lambda = c_a/2$ in order to make this final term small (recall $\sqrt{d}|V_a'|, V_a \leq 2\theta$ due to Lemma 3.3). Then

$$\begin{aligned} 0 &\leq -\frac{\pi^2}{2R^2} \sin(z_0)^2 - \left(1 + \frac{c_a^2}{4} - \frac{\pi^2}{2R^2} - \theta \left(\frac{2\tau \chi}{d} + 1 \right) + \frac{c_a \tau \chi}{2} V_a' - \frac{c_a^2}{2} \right) \cos(z_0)^2 \\ &\quad - \frac{\pi}{R} \tau \chi V_a' \cos(z_0) \sin(z_0). \\ &= -\frac{\pi^2}{2R^2} \sin(z_0)^2 - \left(1 - \frac{c_a^2}{4} - \frac{\pi^2}{2R^2} - \theta \left(\frac{2\tau \chi}{d} + 1 \right) - \frac{c_a \tau \chi}{2} V_a' \right) \cos(z_0)^2 - \frac{\pi}{R} \tau \chi V_a' \cos(z_0) \sin(z_0). \end{aligned}$$

Using first the bounds that $0 \leq c_a < 2 - \varepsilon$ and the bounds $\sqrt{d}|V_a'|, V_a \leq 2\theta$ and then Young's inequality, we find

$$\begin{aligned} 0 &\leq -\frac{\pi^2}{2R^2} \sin(z_0)^2 - \left(1 - \left(1 - \frac{\varepsilon}{2} \right)^2 - \frac{\pi^2}{2R^2} - \theta \left(\frac{2\tau \chi}{d} + 1 \right) - \frac{\tau \chi \theta}{\sqrt{d}} \right) \cos(z_0)^2 \\ &\quad - \frac{\pi \tau \chi \theta}{R \sqrt{d}} |\cos(z_0) \sin(z_0)| \\ &\leq -\left(1 - \left(1 - \frac{\varepsilon}{2} \right)^2 - \frac{\pi^2}{2R^2} - \theta \left(\frac{2\tau \chi}{d} + 1 \right) - \frac{\tau \chi \theta}{\sqrt{d}} - \frac{\tau^2 \chi^2 \theta^2}{2d} \right) \cos(z_0)^2. \end{aligned}$$

Recall $z_0 \in (-\pi/2, \pi/2)$ since $x_0 \in (L, L + 2R)$. Then, $\cos(z_0) > 0$, and we deduce that the above is negative if R is sufficiently large (which necessitates the largeness of a) and θ is sufficiently small. This is a contradiction and concludes the proof. $\blacksquare$

3.1.3. Preliminaries on uniformly local L^p -spaces. We present some preliminary estimates on V_a and φ involving uniformly local L^p -spaces. The next two estimates rely on the usefulness of our version of the uniformly local spaces.

First, we compile the behavior of φ . Importantly, φ' and φ'' are “like” φ but much smaller (recall that we eventually choose $\sigma_{ul} \ll 1$). Also, crucially, K_d and φ can be compared (3.18). The proof is omitted as it is elementary.

Lemma 3.5. *Suppose that $\sigma_{\text{ul}} \in (0, 1/\sqrt{d})$. The function φ , defined in Definition 1.3, satisfies the following: there exists $C > 0$ such that, for all $x \in \mathbb{R}$,*

$$(3.17) \quad \begin{aligned} Ce^{-\sigma_{\text{ul}}|x|} &\leq \varphi(x) \leq Ce^{-\sigma_{\text{ul}}|x|}, & |\varphi(x)| &\leq 1, \\ |\varphi'(x)| &\leq \sigma_{\text{ul}}\varphi(x), & \text{and} & \quad |\varphi''(x)| \leq \sigma_{\text{ul}}^2\varphi(x). \end{aligned}$$

Additionally, for any s and x ,

$$(3.18) \quad K_d(x-s) \leq \frac{C}{\sqrt{d}}\varphi_s(x).$$

Finally, we compile one last estimate that we require often in the sequel. This is one major advantage of our particular form of the uniformly local L^p -spaces. A naïve estimate that follows directly from the definition of K_d (1.2) and of V_a (3.2) is

$$\|V_a\|_{L^\infty} \leq \|U_a\|_{L^\infty}.$$

Lemma 3.6 replaces this, which involves the same regularity of V_a and U_a , with an estimate that connects a higher regularity norm of V_a (the L^∞ -norm) with a lower regularity norm of U_a (the L_{ul}^p -norm).

Lemma 3.6. *Suppose that $U_a \geq 0$ is related to V_a by (3.2). Then, for $p \in [1, \infty)$, there is $C > 0$ such that, for all x ,*

$$(3.19) \quad V_a(x) \leq Cd^{-\frac{1}{2p}}\|U_a\|_{L_{\text{ul}}^p}.$$

Proof. By (3.18) and Hölder's inequality, we have

$$\begin{aligned} V_a(x) &= \int_{-\infty}^{\infty} \tilde{U}_a(y) K_d(x-y)^{\frac{1}{p}} K_d(x-y)^{\frac{p-1}{p}} dy \\ &\leq Cd^{-\frac{1}{2p}} \int_{-\infty}^{\infty} \tilde{U}_a(y) \varphi^{\frac{1}{p}}(x-y) K_d(x-y)^{\frac{p-1}{p}} dy \\ &\leq Cd^{-\frac{1}{2p}} \left(\int_{-\infty}^{\infty} \tilde{U}_a(y)^p \varphi(x-y) dy \right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} K_d(x-y) dy \right)^{\frac{p-1}{p}} \leq Cd^{-\frac{1}{2p}} \|U_a\|_{L_{\text{ul}}^p}. \quad \blacksquare \end{aligned}$$

3.2. Uniform bounds on solutions to the slab problem. In this section, the key estimate is to obtain a -independent bounds on c_a and $\|U_a\|_{L^\infty}$. The main difficulty is that these two quantities are coupled.

3.2.1. The main estimate. We obtain the bound on U_a via a main “energy estimate” paired with a standard bound on the size of c_a in terms of the L^∞ -norm of the advection.

We state these two results here, although defer their proofs until Section 5. This allows us to show exactly how they piece together to obtain the crucial bound on c_a and $\|U_a\|_{L^\infty}$ (Proposition 3.9).

We begin by stating the “energy estimate.” Note that we can choose σ_{ul} to be small and this allows us to temper the dependence on c_a . It is the crucial estimate on which our construction hinges.

Lemma 3.7. *Suppose that (c_a, U_a, V_a) solves (3.1). If the positive parameter σ_{ul} is sufficiently small, depending only on d and χ , then*

$$(3.20) \quad \|U_a\|_{L_{\text{ul}}^2}^2 \leq C \left(\sigma_{\text{ul}} c_a^2 + \frac{1}{\sigma_{\text{ul}}} \right).$$

We now obtain our second ingredient, which is the bound on c_a in terms of $\|V_a\|_{L^\infty}$. The main steps in its proof, which is found in Section 5.2, is somewhat standard and dates at least back to [4].

Lemma 3.8 (Upper bound on the speed). *If (c_a, U_a, V_a) solves (3.1) and $a > \ln(1/\theta)$, then*

$$(3.21) \quad c_a \leq 2 + \left(\frac{\chi}{\sqrt{d}} + \frac{\chi}{d} \right) \|V_a\|_{L^\infty}.$$

3.2.2. *Uniform estimates on c_a and $\|U_a\|_{L^\infty}$.* We now put together our estimates from Section 3.2.1 in order to establish uniform bounds on c_a and $\|U_a\|_{L^\infty}$.

Proposition 3.9. *If (c_a, U_a, V_a) solves (3.1), then c_a and U_a are bounded uniformly in terms of d and χ .*

Proof. We first obtain a bound on the speed c_a . By combining Lemma 3.6 and Lemma 3.8, we have

$$c_a \leq C(1 + \|U_a\|_{L^2_{\text{ul}}}).$$

Then, squaring the above and applying the bound in Lemma 3.7, we find

$$c_a^2 \leq C + C\sigma_{\text{ul}}c_a^2 + \frac{C}{\sigma_{\text{ul}}}.$$

The bound for c_a follows after taking σ_{ul} sufficiently small and absorbing the c_a^2 term on the right into the left.

Using this uniform bound on c_a in Lemma 3.7, we deduce that

$$(3.22) \quad \|U_a\|_{L^2_{\text{ul}}} \leq C,$$

which, by Lemma 3.1 and Lemma 3.6, yields

$$\|V_a\|_{W^{1,\infty}} \leq C.$$

The rest of the proof essentially follows by classical elliptic regularity theory since U_a satisfies an elliptic equation with bounded coefficients and enjoys a (local) L^2 -bound. We are not, however, able to find a reference that we can quote “out-of-the-box,” so we prove it directly.

Let us show how to conclude a bound in $(-a+1, a-1)$. The modifications to handle the boundary behavior are immediate so we omit that case. Fix any $x_0 \in (-a+1, a-1)$ and let ψ be any nonnegative cut-off function that is 1 on $(x_0 - 1/2, x_0 + 1/2)$ and 0 outside of $(x_0 - 1, x_0 + 1)$. Then, multiplying (3.1) by $\psi^2 U_a$ and integrating by parts, we find

$$(3.23) \quad \begin{aligned} \int_{-\infty}^{\infty} \psi^2 |U'_a|^2 &= \int_{-\infty}^{\infty} (-2\psi\psi' U_a U'_a + c_a \psi^2 U_a U'_a - \chi(V'_a U_a)' \psi^2 U_a + \psi^2 U_a^2 (1 - U_a)) \\ &\leq \frac{1}{2} \int_{-\infty}^{\infty} \psi^2 |U'_a|^2 + C \int_{-\infty}^{\infty} (|\psi'|^2 + \psi^2) (|U_a|^2) \\ &\leq \frac{1}{2} \int_{-\infty}^{\infty} \psi^2 |U'_a|^2 + C \|U_a\|_{L^2_{\text{ul}}}^2 \leq \frac{1}{2} \int_{-\infty}^{\infty} \psi^2 |U'_a|^2 + C. \end{aligned}$$

Above we used the bounds on c_a and V_a , Young’s inequality, and the fact that

$$\|\psi U_a\|_{L^2} \leq C \|U_a\|_{L^2_{\text{ul}}} \leq C.$$

The first inequality above is follows from the compact support of ψ , and the second follows from (3.22).

After absorbing the gradient term on the right hand side of (3.23) into the left, we obtain

$$\|U'_a\|_{L^2([x_0-1/2, x_0+1/2])} \leq C.$$

The proof is finished after an application of the Sobolev embedding theorem. ■

3.3. Existence of a solution on the slab. Having established bounds on c_a and U_a that are independent of a , we are ready to show the existence of solutions to the slab problem using Leray-Schauder degree theory.

Proposition 3.10. *There exists $\theta_0 > 0$ such that, for all $\theta \in (0, \theta_0)$ and for all $a > 0$ sufficiently large, there exists a solution (c_a, U_a, V_a) of (3.1) with $\tau = 1$.*

Proof. We first note that any solution to (3.1) (c_a, U_a, V_a) must satisfy “nice” bounds. Indeed, from Lemma 3.2, we know that U_a is positive on $(-a, a)$. Additionally, from Proposition 3.9, c_a and U_a are, respectively, bounded in $\mathbb{R}$ and L^∞ independently of a . Applying elliptic regularity theory and noting the bounds on V_a in Lemma 3.1, we find that U_a is bounded in $C^{2,\alpha}$. It then follows from Lemma 3.4 that $c_a \geq 1$ for all a sufficiently large (up to decreasing θ). Although not necessary here, we note that all bounds are independent of a as long as a is sufficiently large. We summarize this as:

$$(3.24) \quad \frac{1}{C_0} \leq c_a + \|U_a\|_{C^{2,\alpha}([-a,a])} \leq C_0,$$

where C_0 is independent of a .

We now seek to apply a Leray-Schauder degree theory argument (see [48] for the general theory). For $R > C_0$ to be chosen, let

$$\mathcal{B} := \left\{ (c_a, U_a) \in [0, R] \times C^{1,\alpha}([-a, a]) : \|U_a\|_{C^{1,\alpha}([-a,a])} \leq R, U_a \geq 0 \right\},$$

and define the operator $S_\tau : \mathcal{B} \rightarrow \mathbb{R} \times C^{1,\alpha}([-a, a])$ by

$$S_\tau(c_a, U_a) = \left(c_a + \theta - \max_{x \geq 0} \bar{U}_a(x), \bar{U}_a \right),$$

where, for fixed U_a , we define $\bar{U}_a$ to be the unique solution of the linear problem

$$\begin{cases} -c_a \bar{U}_a' + \tau \chi(\bar{U}_a V_a')' = \bar{U}_a'' + U_a(1 - U_a) & \text{in } (-a, a), \\ \bar{U}_a(-a) = 1, \quad \bar{U}_a(a) = 0 \end{cases}$$

with $V_a = K_d * \tilde{U}_a$. Note that the existence of a solution to (3.1) is equivalent to S_τ .

Standard results in elliptic regularity theory provide bounds on the $C^{2,\alpha}([-a, a])$ norm of $\bar{U}_a$ which depend solely on a and the $C^{1,\alpha}([-a, a])$ norm of U_a . Thus, S_τ is a compact operator. Moreover, if a is sufficiently large, any fixed point of S_τ is an element of the interior of $\mathcal{B}$ by (3.24). Hence, we have

$$(3.25) \quad \deg(\text{Id} - S_1, \mathcal{B}, 0) = \deg(\text{Id} - S_0, \mathcal{B}, 0).$$

Thus, it suffices to show that $\deg(\text{Id} - S_0, \mathcal{B}, 0) \neq 0$. While this is likely known in the literature as fixed points of S_0 are solutions of the Fisher-KPP equation, we are unable to find a clear reference and so outline the details below.

In order to show, this we construct a second homotopy $\mathcal{F}_\tau : \mathcal{B} \rightarrow \mathbb{R} \times C^{1,\alpha}([-a, a])$. Let

$$\mathcal{F}_\tau(c, U) = \left(c + \theta - \max_{x \geq 0} \bar{U}(x), \bar{U} \right)$$

where $\bar{U}_a$ is the unique solution of

$$(3.26) \quad -c\bar{U}' - \bar{U}'' = \tau U(1 - U).$$

with the boundary conditions $U(-a) = 1$ and $U(a) = 0$. We notice that $S_0 = \mathcal{F}_1$ and $\mathcal{F}_\tau$ is a homotopy connecting $\mathcal{F}_0$ and $\mathcal{F}_1$; Thus,

$$(3.27) \quad \deg(\text{Id} - S_0, \mathcal{B}, 0) = \deg(\text{Id} - \mathcal{F}_1, \mathcal{B}, 0) = \deg(\text{Id} - \mathcal{F}_0, \mathcal{B}, 0)$$

as long as any fixed point $\mathcal{F}_\tau(c_\tau, U_\tau) = (c_\tau, U_\tau)$ avoids the boundary of $\mathcal{B}$. When $\tau = 0$, there is nothing to show because the solution is explicit (see (3.28)). Otherwise, letting (c, U) be the fixed point, we immediately see, as before, that

$$\max_{x \geq 0} U_\tau(x) = \theta \quad \text{and} \quad \|U_\tau\|_{L^\infty} = 1.$$

After proving a bound on c_τ , which we do below, this immediately yields a $C^{1,\alpha}$ -bound on U_τ . The first follows from the definition of $\mathcal{F}_\tau$, while the second can be deduce from an easy comparison principle argument. Further, arguing exactly as in Lemma 3.8, we see that

$$c_\tau \leq 2 < R.$$

To see that $c > 0$, let

$$(3.28) \quad U_0(x) = \frac{e^{-c_\tau x} - e^{-c_\tau a}}{e^{c_\tau a} - e^{-c_\tau a}}.$$

Then AU_0 is the unique solution to (3.26) with $\tau = 0$ and the boundary conditions $AU_0(-a) = A$ and $AU_0(a) = 0$. Arguing as in Lemma 3.4 we can “raise” A up until AU_0 “touches” U_τ from below. It is clear that AU_0 is a strict subsolution of (3.26) so AU_0 cannot “touch” U_τ from below in the interior or at $x = +a$ (by the Hopf lemma). By analyzing the boundary at $x = -a$, we see that $A = 1$, whence we conclude that

$$\frac{1 - e^{-c_\tau a}}{e^{c_\tau a} - e^{-c_\tau a}} = U_0(0) \leq U_\tau(0) \leq \theta.$$

As $c_\tau a \rightarrow 0$, we see that the left hand side tends to $1/2$, which yields a contradiction (recall that $\theta < 1/2$, see below (3.1)). It follows that $c_\tau > 0$.

To finish, we need now show that the rightmost term in (3.27) is non-zero. Since a fixed point (c_0, U_0) of $\mathcal{F}_0$ corresponds to the unique solution of

$$-c_0 U'_0 - U''_0 = 0 \quad \text{where } U_0(-a) = 1, \quad U_0(0) = \theta, \quad \text{and } U_0(a) = 0,$$

we have that

$$(3.29) \quad \deg(\text{Id} - \mathcal{F}_0, \mathcal{B}, 0) = \pm 1$$

as long as we show that the Fréchet derivative of $\mathcal{F}_0$ at the fixed point (c_0, U_0) does not have 1 as an eigenvalue (see [48, Proposition 14.5]).

We establish (3.29) now. For each c , let u_c be the unique solution of

$$-cu'_c - u''_c = 0 \quad \text{where } u_c(-a) = 1 \quad \text{and } u_c(a) = 0$$

Then,

$$\mathcal{F}_0(c, U) = \left(c + \theta - \max_{x \geq 0} u_c(x), u_c \right) = (c + \theta - u_c(0), u_c).$$

Notice that the right hand side has no dependence on U . Observe that

$$u_{c_0} = U_0 \quad \text{and} \quad \partial_c u_c(0)|_{c=c_0} = a \frac{2 - (e^{c_0 a} + e^{-c_0 a})}{(e^{c_0 a} - e^{-c_0 a})^2}.$$

It follows that, for any $h \in \mathbb{R}$ and $w \in C^{1,\alpha}([-a, a])$,

$$(3.30) \quad (D\mathcal{F}_0(c_0, U_0))(h, w) = \left(\left(1 + a \frac{e^{c_0 a} + e^{-c_0 a} - 2}{(e^{c_0 a} - e^{-c_0 a})^2} \right), \partial_c u_c|_{c=c_0} \right) h,$$

which clearly does not have 1 as an eigenvalue. Thus, (3.29) is justified and the proof is complete. $\blacksquare$

4. OBTAINING A TRAVELING WAVE SOLUTION: TAKING $a \rightarrow \infty$

We now use the solution constructed on the slab to obtain a traveling wave solution on the whole space by applying our uniform estimates and a compactness argument. As usual, a difficulty here is to ensure that, when passing to the limit $a \rightarrow \infty$, we do not end up with a trivial traveling wave; i.e., one that is everywhere 1 or everywhere 0.

Proposition 4.1. *There exists a traveling wave solution (c, U, V) in the sense of Definition 1.1.*

Proof. Applying Proposition 3.10, we obtain a solution (c_a, U_a) to (3.1). From the estimates established in the first paragraph of the proof of Proposition 3.10 as well as elliptic regularity theory, there is C_0 such that

$$(4.1) \quad \frac{1}{C_0} \leq c_a, \|U_a\|_{C^{2,\alpha}}, \|V_a\|_{C^{2,\alpha}} \leq C_0.$$

Moreover $U_a > 0$ on $(-a, a)$.

By compactness, there exists a sequence $a_n \rightarrow \infty$ and $(c, U) \in [1/C_0, C_0] \times C^{2,\alpha}(\mathbb{R})$ such that, as n tends to infinity, $c_{a_n} \rightarrow c$ in $\mathbb{R}$ and $U_{a_n} \rightarrow U$ locally uniformly in $C^2(\mathbb{R})$. Thus, it follows from (3.1) that (c, U, V) satisfies

$$-cU' + \chi(UV')' = U'' + U(1 - U) \quad \text{in } \mathbb{R},$$

where $V = K_d * U$.

Next, we prove that U tends to zero as $x \rightarrow +\infty$. To do this, we show that U is bounded in $L^1([0, \infty))$. Then, since $U \in C^{0,\alpha}(\mathbb{R})$, it follows that U vanishes at infinity.

To establish our $L^1([0, \infty))$ bound on U , we first integrate (3.1) over $[0, a]$:

$$(4.2) \quad \int_0^a U_a(1 - U_a) = -c_a U_a(0) - U'_a(a) + U'_a(0) - \chi U_a(0) V'_a(0).$$

It follows from (3.24) that the right-hand side of (4.2) is bounded only in terms of d and χ . Also, due to the normalization $\max_{x \in [0, a]} U_a(x) = \theta < 1$ and the positivity of U_a on $[0, a]$, we have

$$(4.3) \quad \int_0^a U_a(1 - \theta) \leq \int_0^a U_a(1 - U_a) \leq C.$$

Taking the limit as $a \rightarrow \infty$ and using Fatou's lemma, we find

$$\int_0^\infty U \leq \liminf_{a \rightarrow \infty} \int_0^a U_a \leq C.$$

as desired. The fact that $U(+\infty) = 0$ follows immediately.

Lastly, we show $L := \liminf_{x \rightarrow -\infty} U(x)$ is positive. Choose a sequence (x_n) such that, as $n \rightarrow \infty$, $x_n \rightarrow -\infty$ and $U(x_n) \rightarrow L$, and define

$$U_n(x) := \frac{U(x + x_n)}{U(x_n)} \quad \text{and} \quad V_n(x) := V(x + x_n).$$

The Harnack inequality shows that U_n is bounded on any compact set containing $x = 0$. Since (c, U, V) satisfies (1.4), it follows that U_n and V_n satisfy

$$(4.4) \quad \begin{cases} -cU'_n + \chi(U_n V'_n)' = U''_n + U_n(1 - U(x_n)U_n) & \text{in } \mathbb{R}, \\ -dV''_n = U(x_n)U_n - V_n & \text{in } \mathbb{R}. \end{cases}$$

The right side of the first equation in (4.4) is bounded by U_n . Therefore, by elliptic regularity theory, it follows that, up to a subsequence, $U_n \rightarrow U_\infty$ and $V_n \rightarrow V_\infty$ in $C^2_{\text{loc}}(\mathbb{R})$, and these functions satisfy

$$(4.5) \quad \begin{cases} -cU'_\infty + \chi(U_\infty V'_\infty)' = U''_\infty + U_\infty(1 - LU_\infty) & \text{in } \mathbb{R}, \\ -dV''_\infty = LU_\infty - V_\infty & \text{in } \mathbb{R}. \end{cases}$$

Suppose, by way of contradiction, that $L = 0$. Then, (4.5) becomes

$$(4.6) \quad \begin{cases} -cU'_\infty + \chi(U_\infty V'_\infty)' = U''_\infty + U_\infty & \text{in } \mathbb{R}, \\ -dV''_\infty = -V_\infty & \text{in } \mathbb{R}. \end{cases}$$

Recall that

$$(4.7) \quad V_n(x) = \int K_d(x + x_n - y)U(y)dy = \int K_d(x - y)U(y + x_n)dy.$$

Since $U(x_n) \rightarrow 0$ as $n \rightarrow \infty$, we conclude that $U(y + x_n) \rightarrow 0$ locally uniformly in y by the Harnack inequality. It follows from this and (4.7) that $V_n \rightarrow 0$ locally uniformly in x ; that is, $V_\infty = 0$. Thus, the first equation in (4.5) becomes

$$(4.8) \quad -cU'_\infty = U''_\infty + U_\infty.$$

By construction, U_∞ achieves its minimum of 1 at $x = 0$. From (4.8), we see that

$$0 = -cU'_\infty = U''_\infty + 1 > 0.$$

This is a clearly a contradiction. It follows that $L > 0$. This completes the proof. $\blacksquare$

We also have a lower bound on the speed of the traveling wave in Proposition 4.1.

Proposition 4.2. *If (c, U, V) is a traveling wave solution to (1.1), then $c \geq 2$.*

Proof. The proof of this claim follows almost exactly like the proof of Lemma 3.4, except we replace each θ with $\theta_L := \min_{x \in [L, \infty)} U(x)$. The proof follows since $\theta_L \rightarrow 0$ as $L \rightarrow \infty$ due to Definition 1.1. $\blacksquare$

The combination of Proposition 4.1 and Proposition 4.2 yields Theorem 1.2.

5. PROOFS OF THE TECHNICAL LEMMAS

5.1. The key energy estimate: Lemma 3.7. As we saw above, the estimate in Lemma 3.7 is the centerpiece of the construction of traveling wave solutions. It is inspired by the proof in [42]; however, being time-independent, it differs in several key aspects.

Before we begin, we prove a state and a prove a small technical estimate that shows we can reduce the L^2_{ul} norm to integrals on a well-chosen domain.

Lemma 5.1. *Suppose that $\|U_a\|_{L^\infty} > 1$. Define*

$$(5.1) \quad x_1 = \min \left\{ x : U_a(x) = \frac{1}{2}(1 + \min\{\|U_a\|_{L^\infty}, 2\}) \right\}.$$

Then

$$(5.2) \quad \|\tilde{U}_a\|_{L^2_{\text{ul}}}^2 \leq \sup_{s \in \mathbb{R}} \int_{x_1}^a \varphi_s U_a^2 + \frac{C}{\sigma_{\text{ul}}}.$$

Proof. Fix any $\varepsilon > 0$. Choose s such that

$$(5.3) \quad \|\tilde{U}_a\|_{L^2_{\text{ul}}}^2 \leq (1 + \varepsilon) \int_{-\infty}^{\infty} \varphi_s \tilde{U}_a^2.$$

Then, using that $\tilde{U}_a \leq 2$ on $(-\infty, x_1)$ and $\tilde{U}_a = 0$ on (a, ∞) , we find

$$(5.4) \quad \begin{aligned} \frac{1}{1 + \varepsilon} \|\tilde{U}_a\|_{L^2_{\text{ul}}}^2 &\leq \int_{-\infty}^{\infty} \varphi_s \tilde{U}_a^2 \leq 2 \int_{-\infty}^{x_1} \varphi_s + \int_{x_1}^a \varphi_s U_a^2 \\ &\leq \frac{C}{\sigma_{\text{ul}}} + \int_{x_1}^a \varphi_s U_a^2. \end{aligned}$$

The proof is concluded by taking $\varepsilon \rightarrow 0$. $\blacksquare$

Let us comment on the usefulness of x_1 . As we see above, the portion of the L^2_{ul} -norm occurring on $[x_1, a]^c$ is nicely bounded. Hence, we do not “lose anything” by only integrating on $[x_1, a]$. On the other hand, in our energy estimate below, we obtain, through integration by parts, boundary terms involving U'_a . By defining x_1 as above (and choosing a as the other boundary), we are assured that these have a “good sign.”

We are now in a position to prove the energy estimate. A key insight, inspired by the work in [42], is to obtain the estimate via the quadratic term $-U_a^2$ in (3.1).

Proof of Lemma 3.7. We first observe that if $\|U_a\|_{L^\infty} \leq 1$, the proof follows immediately. Hence, we proceed assuming that $\|U_a\|_{L^\infty} > 1$ so that x_1 is well-defined (recall (5.1)) and we have access to Lemma 5.1.

We claim that, for any $N \geq 1$,

$$(5.5) \quad \sup_{s \in \mathbb{R}} \int_{x_1}^a \varphi_s U_a^2 \leq \frac{CN}{\sqrt{\sigma_{\text{ul}}}} + CN\sigma_{\text{ul}}c_a^2 + C\left(\sqrt{\sigma_{\text{ul}}} + \frac{1}{N}\right)\|U_a\|_{L^2_{\text{ul}}}^2.$$

Let us postpone the proof of (5.5) momentarily and, first, show how to conclude the proof of the lemma with it.

After applying Lemma 5.1, we find that $\|U_a\|_{L^2_{\text{ul}}}^2$ satisfies the same inequality:

$$(5.6) \quad \|U_a\|_{L^2_{\text{ul}}}^2 \leq \frac{CN}{\sqrt{\sigma_{\text{ul}}}} + CN\sigma_{\text{ul}}c_a^2 + C\left(\sqrt{\sigma_{\text{ul}}} + \frac{1}{N}\right)\|U_a\|_{L^2_{\text{ul}}}^2.$$

Indeed, the extra C/σ_{ul} term may be absorbed into the $CN/\sqrt{\sigma_{\text{ul}}}$ term. Sufficiently decreasing σ_{ul} and increasing N , we see that the last term on the right hand side of (5.6) may be absorbed into the left hand side of (5.6) to yield the desired inequality:

$$\frac{1}{2}\|U_a\|_{L^2_{\text{ul}}}^2 \leq \left(1 - C\sqrt{\sigma_{\text{ul}}} - \frac{C}{N}\right)\|U_a\|_{L^2_{\text{ul}}}^2 \leq \frac{CN}{\sqrt{\sigma_{\text{ul}}}} + CN\sigma_{\text{ul}}c_a^2.$$

We now show the proof of (5.5). It is enough to establish the result for any fixed s . We begin by rewriting (3.1) using (3.3):

$$(5.7) \quad \left(1 - \frac{\chi}{d}\right)U_a^2 = c_a U'_a - \chi U'_a V'_a - \frac{\chi}{d}U_a V_a + U''_a + U_a.$$

Define x_1 as in the statement of Lemma 5.1. Multiplying (5.7) by φ_s and integrating over $[x_1, a]$ gives

$$(5.8) \quad \left(1 - \frac{\chi}{d}\right) \int_{x_1}^a \varphi_s U_a^2 = c_a \int_{x_1}^a \varphi_s U'_a - \chi \int_{x_1}^a \varphi_s U'_a V'_a - \frac{\chi}{d} \int_{x_1}^a \varphi_s U_a V_a + \int_{x_1}^a \varphi_s U''_a + \int_{x_1}^a \varphi_s U_a.$$

Let $I_1, \dots, I_5$ denote, respectively, the five terms on the right of (5.8).

The integral I_1 . Integrating by parts gives

$$(5.9) \quad I_1 = c_a \varphi_s U_a \Big|_{x_1}^a - c_a \int_{x_1}^a \varphi'_s U_a = -c_a \varphi_s(x_1) U_a(x_1) - c_a \int_{x_1}^a \varphi'_s U_a \leq -c_a \int_{x_1}^a \varphi'_s U_a.$$

By the properties of φ_s in (3.17), as well as the choice of x_1 , we have

$$I_1 \leq C c_a \sigma_{\text{ul}} \int_{x_1}^a \varphi_s U_a = C \int_{x_1}^a \varphi_s (\sigma_{\text{ul}} c_a) (U_a).$$

Young’s inequality then gives

$$I_1 \leq CN \int_{x_1}^a \varphi_s \sigma_{\text{ul}}^2 c_a^2 + \frac{1}{N} \int_{x_1}^a \varphi_s U_a^2 \leq CN \sigma_{\text{ul}} c_a^2 + \frac{\|U_a\|_{L^2_{\text{ul}}}^2}{N}.$$

Notice that the nonlinear term c_a appears with a smaller parameter σ_{ul} in front, which is what makes this a useful estimate. On the other hand, the integral term of U_a appears with a small $1/N$ factor, so it can eventually be absorbed.

The integral I_2 . Integrating by parts and using (3.3) gives

$$\begin{aligned} I_2 &= -\chi \int_{x_1}^a \varphi_s U_a' V_a' = -\chi \varphi_s U_a V_a' \Big|_{x_1}^a + \chi \int_{x_1}^a \varphi_s' U_a V_a' + \chi \int_{x_1}^a \varphi_s U_a V_a'' \\ (5.10) \quad &= -\chi \varphi_s U_a V_a' \Big|_{x_1}^a + \chi \int_{x_1}^a \varphi_s' U_a V_a' + \frac{\chi}{d} \int_{x_1}^a \varphi_s U_a V_a - \frac{\chi}{d} \int_{x_1}^a \varphi_s U_a^2. \end{aligned}$$

Consider the first term in the right hand side of (5.10). We use the choice of x_1 in (5.1) to control $\varphi_s U_a$ and Lemma 3.1 to control V_a' in order to obtain

$$-\chi \varphi_s U_a V_a' \Big|_{x_1}^a = \chi \varphi_s(x_1) U_a(x_1) V_a'(x_1) \leq C \|V_a\|_{L^\infty} \leq C \|U_a\|_{L^2_{\text{ul}}} \leq C \sqrt{\sigma_{\text{ul}}} \|U_a\|_{L^2_{\text{ul}}}^2 + \frac{C}{\sqrt{\sigma_{\text{ul}}}}.$$

In the second-to-last inequality, we used Lemma 3.6, and in the last one, we used Young's inequality. Note that there is no importance to the square root above; it is chosen for convenience as the square root naturally shows up in the next estimate (5.11).

Arguing using Lemmas 3.1 and 3.5, we control the second term on the right in (5.10) as follows:

$$\begin{aligned} \chi \int_{x_1}^a \varphi_s' U_a V_a' &\leq C \sigma_{\text{ul}} \int_{x_1}^a \varphi_s U_a V_a \leq C \sigma_{\text{ul}} \int_{x_1}^a \varphi_s \left(\frac{1}{\sqrt{\sigma_{\text{ul}}}} U_a^2 + \sqrt{\sigma_{\text{ul}}} \|V_a\|_{L^\infty}^2 \right) \\ (5.11) \quad &= C \sqrt{\sigma_{\text{ul}}} \|U_a\|_{L^2_{\text{ul}}}^2 + C \sigma_{\text{ul}} \sqrt{\sigma_{\text{ul}}} \|V_a\|_{L^\infty}^2 \int_{x_1}^a \varphi_s. \end{aligned}$$

The last inequality followed by Young's inequality. Then, recalling Lemma 3.6, we find

$$\begin{aligned} \chi \int_{x_1}^a \varphi_s' U_a V_a' &\leq C \sqrt{\sigma_{\text{ul}}} \int_{x_1}^a \varphi_s U_a^2 + C \sigma_{\text{ul}} \sqrt{\sigma_{\text{ul}}} \|U_a\|_{L^2_{\text{ul}}}^2 \int_{x_1}^a \varphi_s \\ &\leq C \sqrt{\sigma_{\text{ul}}} \int_{x_1}^a \varphi_s U_a^2 + C \sqrt{\sigma_{\text{ul}}} \|U_a\|_{L^2_{\text{ul}}}^2 \leq C \sqrt{\sigma_{\text{ul}}} \|U_a\|_{L^2_{\text{ul}}}^2. \end{aligned}$$

Therefore, we have the following estimate for I_2 :

$$I_2 \leq \frac{C}{\sqrt{\sigma_{\text{ul}}}} + C \sqrt{\sigma_{\text{ul}}} \|U_a\|_{L^2_{\text{ul}}}^2 + \frac{\chi}{d} \int_{x_1}^a \varphi_s U_a V_a - \frac{\chi}{d} \int_{x_1}^a \varphi_s U_a^2.$$

The integral I_3 . We do not estimate I_3 . It will be used to cancel one of the bad terms from the estimate of I_2 .

The integral I_4 . Integrating by parts gives

$$(5.12) \quad I_4 = \int_{x_1}^a \varphi_s U_a^{p-1} U_a'' = \varphi_s U_a' \Big|_{x_1}^a - \int_{x_1}^a \varphi_s' \leq - \int_{x_1}^a \varphi_s' = \varphi_s(x_1) - \varphi_s(a) \leq 1.$$

The first inequality above follows from the choice of x_1 , which ensures that $U_a'(a) \leq 0$ and $U_a'(x_1) \geq 0$. We note that we have no control of $\|U_a\|_{L^\infty}$ at this point in the proof, so it is crucial to our argument that U_a' satisfies these inequalities at the endpoints. This is the motivation for our definition (5.1) of x_1 .

The integral I_5 . By the properties of φ_s in (3.17) and Young's inequality, we have

$$I_5 = \int_{x_1}^a \varphi_s U_a \leq \int_{x_1}^a \varphi_s \left(N + \frac{U_a^2}{N} \right) \leq C \frac{N}{\sigma_{\text{ul}}} + \frac{1}{N} \int_{x_1}^a \varphi_s U_a^2 \leq \frac{CN}{\sigma_{\text{ul}}} + \frac{1}{N} \|U_a\|_{L^2_{\text{ul}}}^2.$$

Combining all above with (5.8) yields (5.5), which completes the proof of the lemma. $\blacksquare$

5.2. An upper bound on c_a by $\|V_a\|_{L^\infty}$. As we noted above, the proof in the sequel is quite standard.

Proof of Lemma 3.8. Let $\mathcal{A} = \{A \in \mathbb{R} : \alpha_A \geq U_a\}$, where we define

$$\alpha_A(x) = Ae^{-x}.$$

Observe that $\mathcal{A}$ is nonempty as $A = e^a \|U_a\|_{L^\infty}$ is in $\mathcal{A}$. Indeed, with this choice for A , we have, for all $x \in [-a, a]$,

$$\alpha_A(x) \geq \alpha_A(a) = \|U_a\|_{L^\infty} \geq U_a(x),$$

where the first inequality follows since α_A is decreasing. Also, notice that the value $A = \frac{1}{2}e^{-a}$ is positive, but $A \notin \mathcal{A}$ since

$$\alpha_A(-a) = \frac{1}{2} < 1 = U_a(-a).$$

Hence, $\mathcal{A}$ has a lower bound that is positive. Therefore, by the continuity of U_a and α_A , it follows that

$$A_0 := \min \mathcal{A}$$

is a well-defined, positive constant.

For simplicity, denote $\alpha = \alpha_{A_0}$. By continuity, there exists $x_0 \in [-a, a]$ such that $\alpha(x_0) = U_a(x_0)$. Since $\alpha(a) > 0$ but $U_a(a) = 0$, we have $x_0 \neq a$. Now, suppose $x_0 = -a$. Then,

$$1 = U_a(-a) = \alpha(-a) = A_0 e^a,$$

implying $A_0 = e^{-a}$. Let x_θ be a point in $[0, a]$ in which U_a achieves its maximum on this interval, i.e., $U_a(x_\theta) = \max_{x \geq 0} U_a(x) = \theta$. Thus,

$$e^{-a} = A_0 = \alpha(0) \geq \alpha(x_\theta) \geq U_a(x_\theta) = \theta.$$

This contradicts our assumptions that $a > \ln(1/\theta)$. Hence, $x_0 \neq -a$.

We deduce that $x_0 \in (-a, a)$. The function $\alpha - U_a$ attains its minimum of zero at $x_0 \in (-a, a)$, which yields the following:

$$U_a(x_0) = \alpha(x_0), \quad U'_a(x_0) = \alpha'(x_0) = -\alpha(x_0), \quad \text{and} \quad U''_a(x_0) \leq \alpha''(x_0) = \alpha(x_0).$$

Using these relations in (3.1), along with the bounds on V'_a and V''_a from Lemma 3.1 and the positivity of U_a from Lemma 3.2, we have, at x_0 ,

$$\begin{aligned} 0 &= U''_a + U_a(1 - U_a) - \tau\chi U_a V''_a - \tau\chi U_a V'_a + c_a U'_a \\ &\leq \alpha + \alpha + \tau\chi\alpha \frac{1}{d} \|V_a\|_{L^\infty} + \tau\chi\alpha \frac{1}{\sqrt{d}} \|V_a\|_{L^\infty} - c_a \alpha \\ (5.13) \quad &\leq \alpha \left(2 + \frac{\chi}{d} \|V_a\|_{L^\infty} + \frac{\chi}{\sqrt{d}} \|V_a\|_{L^\infty} - c_a \right). \end{aligned}$$

Since $\alpha(x_0)$ is positive, its coefficient in (5.13) must be nonnegative. Hence, (3.21) follows. $\blacksquare$

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