Quantitative steepness, semi-FKPP reactions, and pushmi-pullyu fronts

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Abstract

We uncover a seemingly previously unnoticed algebraic structure of a large class of reaction-diffusion equations and use it, in particular, to study the long time behavior of the solutions and their convergence to traveling waves in the pulled and pushed regimes, as well as at the pushmi-pullyu boundary. One such new object introduced in this paper is the shape defect function, which, indirectly, measures the difference between the profiles of the solution and the traveling wave. While one can recast the classical notion of 'steepness' in terms of the positivity of the shape defect function, its positivity can, surprisingly, be used in numerous quantitative ways. In particular, the positivity is used in a new weighted Hopf-Cole transform and in a relative entropy approach that play a key role in the stability arguments. The shape defect function also gives a new connection between reaction-diffusion equations and reaction conservation laws at the pulled-pushed transition. Other simple but seemingly new algebraic constructions in the present paper supply various unexpected inequalities sprinkled throughout the paper. Of note is a new variational formulation that applies equally to pulled and pushed fronts, opening the door to an as-yet-elusive variational analysis in the pulled case.

1 Introduction

We consider the long time behavior of the solutions to reaction-diffusion equations of the form

$$u_t = u_{xx} + f(u). \tag{1.1}$$

Throughout the paper, we assume that the nonlinearity $f(u)$ is non-negative on $[0, 1]$ and satisfies

$$f(0) = f(1) = 0, \quad f'(0) > 0, \quad f(u) > 0 \text{ for all } u \in (0, 1). \tag{1.2}$$

For simplicity, we also assume that $f(u)$ is as smooth as needed.

Reaction-diffusion equations of this type have been extensively studied, dating back nearly a century to the pioneering works of Fisher [27] and Kolmogorov, Petrovskii and Piskunov [35]. The main goal of the present paper is to uncover novel algebraic structure and properties of these equations that we found to be quite surprising. Indeed, they are somewhat 'concealed' and seem to be of an independent interest. As an application of this theory, we analyze the long time asymptotics of the solutions to (1.1); that is, we show that the convergence of $u(t, \cdot)$ to the minimal speed traveling wave and identify the precise moving frame in which this convergence occurs. To make this more precise, we first recall some well-established notions.

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Traveling waves and their speed

Under the positivity assumption (1.2), there exists a minimal speed $c_* > 0$ so that for all $c \geq c_*$ the reaction-diffusion equation (1.1) admits traveling wave solutions of the form $u(t,x) = U_c(x - ct)$. The profiles $U_c(x)$ satisfy

$$-cU_c' = U_c'' + f(U_c), \quad U_c(-\infty) = 1, \quad \text{and} \quad U_c(+\infty) = 0. \tag{1.3}$$

The minimal speed c_* is characterized by the variational formula of [33]:

$$c_*[f] = \inf_{p\in\mathcal{K}} \sup_{u\in[0,1]} \left(p'(u) + \frac{f(u)}{p(u)} \right). \tag{1.4}$$

Here, $\mathcal{K}$ is the class of continuously differentiable functions $p(u)$ such that

$$p(0) = 0, \quad p'(0) > 0, \quad \text{and} \quad p(u) > 0 \text{ for } u \in (0,1).$$

Existence of traveling waves was first established in the original papers [27, 35] for the Fisher-KPP type nonlinearities. Recall that $f(u)$ is of the Fisher-KPP type if it satisfies, in addition to (1.2),

$$f(u) \leq f'(0)u, \quad \text{for all } 0 \leq u \leq 1. \tag{1.5}$$

For the Fisher-KPP nonlinearities, the minimal speed given by (1.4) is explicit:

$$c_* = 2\sqrt{f'(0)}. \tag{1.6}$$

Such nonlinearities have attracted enormous attention throughout the past few decades. This is perhaps due to the fact that, under this assumption, the linearized model (around $u = 0$) acts as a reasonably faithful approximation of (1.1) under the Fisher-KPP type assumption and, hence, many computations can be done semi-explicitly. This allows one to study (1.1) in impressive details with highly precise results. We refer to a very recent paper [53] for an elegant formulation of this linearizability property of Fisher-KPP reactions. Moreover, there is a large class of nonlinearities, the 'McKean nonlinearities', all satisfying the Fisher-KPP condition, that connect solutions to (1.1) to branching Brownian motion, as discovered by McKean in [41]. We explain in Section 2 below how this connection can be extended to a much larger class of nonlinearities $f(u)$, not only of the Fisher-KPP type.

Convergence in shape

Another seminal result of the original KPP paper [35] is that the solution $u(t,x)$ to (1.1) with the step-function initial condition $u(0,x) = \mathbb{1}(x \leq 0)$ 'converges in shape' to a minimal speed traveling wave. That is, there exists a reference frame $m(t)$ such that

$$|u(t, x + m(t)) - U_*(x)| \to 0 \qquad \text{as } t \to +\infty, \text{ uniformly in } x \in \mathbb{R}. \tag{1.7}$$

Here, $U_*(x)$ is the traveling wave solution to (1.3) with speed $c = c_*$, normalized, for example, so that $U_*(0) = 1/2$. While (1.7) was stated in [35] only for the Fisher-KPP type nonlinearities, with an extra assumption $f'(u) \leq f'(0)$ for $u \in [0,1]$, the original proof can be easily adapted to a large class of nonlinearities $f(u)$. The KPP paper also showed that

$$m(t) = c_* t + o(t), \quad \text{as } t \to +\infty. \tag{1.8}$$

Fisher formally argued in [27] that

$$m(t) = c_* t + O(\log t), \quad \text{as } t \to +\infty,$$

although his prediction for the coefficient in front of the logarithmic correction turned out to be incorrect.

Pushed and pulled fronts in reaction-diffusion equations

As noted above, convergence in shape to a traveling wave in (1.7) holds for a much larger class of nonlinearities $f(u)$ than, say, the Fisher-KPP nonlinearities. On the other hand, the rate of convergence and the asymptotics of the front location $m(t)$ depend strongly on the nature of the spreading. One needs to distinguish between the so-called 'pulled' regime in which propagation is dominated by the behavior far ahead of the front where $u \approx 0$ and the 'pushed' regime in which spreading is governed by the behavior of the solution near the front where u is close neither to 0 nor 1. In a sense, this is a question of whether the propagation is linearly determined (pulled fronts) or nonlinearly determined (pushed fronts).

In the pushed case, convergence in (1.7) is exponential in time. This is, roughly, due to the fact that the important behavior occurs in the *compact* region around the front. In addition, in the pushed case the front location $m(t)$ has the asymptotics

$$m(t) = c_* t + x_0 + o(1) \quad \text{as } t \to +\infty.$$

These results go back to the classical paper [26] by Fife and McLeod.

In the pulled case, the convergence rate in (1.7) is algebraic in time, roughly due to the fact that the important behavior occurs on the *non-compact* half-line to the right of the front. Moreover, when the Fisher-KPP condition (1.5), which guarantees the pulled nature of the front, is satisfied, $m(t)$ has the asymptotics

$$m(t) = c_* t - \frac{3}{2\lambda_*} \log t + x_0 + o(1) \quad \text{as } t \to +\infty. \tag{1.9}$$

It is notable for the unbounded 'delay' between $m(t)$ and the moving frame $m_{TW}(t) = c_* t + x_0$ of the traveling wave. The formula (1.9) was first established by Bramson in [17, 18] with probabilisitic tools for the aforementioned 'McKean sub-class' of the Fisher-KPP nonlinearities $f(u)$ for which the reaction-diffusion equation (1.1) is directly connected to branching Brownian motion. More recent analytical proofs allowed for general Fisher-KPP type nonlinearities [32, 46, 47]. The algebraic rates of convergence for pulled fronts have been investigated in [6, 7, 10, 11, 21, 32, 47, 51].

Another distinction between the pushed and pulled cases is in the shape of the traveling wave. Pushed traveling waves have purely exponential asymptotics:

$$U_*(x) \sim B e^{-\lambda_* x}, \quad \text{as } x \to +\infty, \tag{1.10}$$

while pulled traveling waves have an extra linear factor:

$$U_*(x) \sim (Dx + B) e^{-\lambda_* x}, \quad \text{as } x \to +\infty. \tag{1.11}$$

Note that the exponential decay rate λ_* appears both in (1.11) and in the pulled front location asymptoptics (1.9).

It is well known that the fronts associated to Fisher-KPP type nonlinearities, that is, those satisfying (1.5), are pulled. However, the Fisher-KPP criterion does not describe the boundary of the pushed-pulled transition. While the transition from the pulled to pushed behavior has been extensively studied in the applied literature, see [21, 37, 51] and references therein, including the recent study of the stochastic effects [14, 15], a true mathematical understanding is still lacking. We mention [28] for a general criterion for pulled fronts, [30] for a preliminary investigation into the asymptotics of m, the very recent papers [6, 7, 8] for a spectral approach to this question, and [2] for the study of the pushed-pulled transition in the context of the Burgers-FKPP equation. We also mention [20] for an investigation into some explicitly solvable cases similar to our setting. Shedding new light on this old problem is one motivation in this work.

1.1 New objects and main results

The shape defect function and an energy functional

The proof of (1.7) in [35] and many of the later references relies on a 'steepness comparison' between the solution $u(t,x)$ to (1.1) with the initial condition $u(0,x) = \mathbb{1}(x \le 0)$ and the minimal speed traveling wave $U_*(x)$. Roughly, if u starts 'steeper' than U_*, then it remains so. We refer to [31] for a recent beautiful exposition of these ideas and to [9] for its adaptation to a time-discrete setting. We simply note that $u(t,\cdot)$ being 'steeper' than U_* means that any shift of U_* intersects the profile of $u(t,\cdot)$ precisely once.

We would like to introduce a seemingly new way to quantify these elegant arguments. It is well known that, for all $c \ge c_*$, the traveling wave profiles $U_c(x)$ are strictly decreasing in x. This can be seen, for instance, by the sliding method [12]. Thus, there exists a 'traveling wave profile function' $\eta_c(u)$, so that $U_c(x)$ satisfies a *first order* ordinary differential equation

$$-U_c' = \eta_c(U_c), \quad U_c(-\infty) = 1, \quad U_c(+\infty) = 0, \tag{1.12}$$

in addition to (1.3). Thus, to quantify the difference between the shape of the solution $u(t,x)$ to (1.1), we propose the following object:

Definition 1.1. *The shape defect function is*

$$w(t,x) = -u_x(t,x) - \eta_c(u(t,x)). \tag{1.13}$$

To the best of our knowledge, this notion is new. Additionally, introduce the energy associated to the shape defect function:

$$\mathcal{E}_c(t) = \int e^{cx} |w(t,x+ct)|^2 dx. \tag{1.14}$$

We should stress that the traveling wave profile function is rarely explicit, except in the pushmi-pullyu and some pushed cases pointed out below, but its explicit form is not needed. A generalization to higher dimensions is discussed in Section 4.1 below.

We mention two related concepts that have been considered in the past. Recently, Matano and Poláčik leveraged the trajectories $\tau_t = \{(u(t,x), u_x(t,x)) : x \in \mathbb{R}\} \subset \mathbb{R}^2$ to access phase plane methods and deduce strong results in the context of propagating terraces [40, 48]. Even earlier, Fife and McLeod [25] studied the convergence to pushed traveling waves by deriving a degenerate partial differential equation for $p(t,z) = u_x(t, u(t)^{-1}(z))$. In our notation, these approaches have, as goals, to prove the convergence of $\tau_t \to \{(z, -\eta_*(z))\}$ and $p(t,z) \to -\eta_*(z)$, respectively, as $t \to \infty$.

The early method of the original KPP paper [35] and some subsequent work used the intersection number to compare the steepness of $u(t,\cdot)$ and U_* and relied on the fact that it is decreasing [1, 39]. The shape defect function provides a new way to capture and *quantify* this insightful notion. Indeed, a simple calculation yields:

Proposition 1.2. *For any time $t > 0$, the profile $u(t,\cdot)$ is steeper than U_* if and only if the shape defect function satisfies:*

$$w(t,x) = -u_x(t,x) - \eta_*(u(t,x)) > 0, \qquad \text{for all } x \in \mathbb{R}.$$

Further, the larger w is, roughly, the more severe the difference in steepness.

The key property noticed in the original KPP paper [35] is that steepness ordering is preserved; that is, if u_0 is steeper than U_*, then so is $u(t,\cdot)$. Again, their argument is based on the intersection number. By recasting this in terms of the shape defect function, this property is:

$$\text{if} \quad w(0,\cdot) \ge 0, \quad \text{then} \quad w(t,\cdot) \ge 0 \quad \text{for all } t \ge 0. \tag{1.15}$$

This can be established, for instance, using the parabolic equation (4.1) below satisfied by w and applying the maximum principle (see (4.2)). Surprisingly, the positivity of the shape defect has numerous other, much more quantitative consequences than (1.15), and this is one of the main points of this paper. In particular, the equation (4.1) for w can be analyzed directly, for example, in order to obtain bounds on w, as in Lemma 6.6. In addition, the positivity of w leads to, for example, the new weighted Hopf-Cole transform and entropy inequalities discussed below. Such bounds and inequalities play a key role in the arguments throughout the paper, and while they use crucially the positivity of w, they are of a different nature than the intersection number arguments.

Importantly, the shape defect function also measures, in a sense, the distance between u and the 'nearest' traveling wave. Indeed, if $u(t,x) = U_c(x-ct-x_0)$ with some shift $x_0 \in \mathbb{R}$, then $w(t,x) \equiv 0$ because of (1.12). An elementary computation in Section 4.1 shows that for any $c \geq c_*$ the reaction-diffusion equation (1.1) is a gradient flow for $\mathcal{E}_c$ and

$$\frac{d\mathcal{E}_c(t)}{dt} \leq 0. \tag{1.16}$$

This is a natural and, as far as we are aware, new way to quantify the convergence in shape result in (1.7). Alternative variational formulations for reaction-diffusion equations have been previously introduced, for instance, in [29, 38, 43, 44, 49]. However, again, to the best of our knowledge, this is the first kind of energy that can be used for equations of the Fisher-KPP type at the minimal speed $c = c_*$ or, more broadly, for pulled type reaction-diffusion equations. Previous variational formulations were restricted to pushed type reaction-diffusion equations. Moreover, (1.16) gives a quantitative reason behind the convergence to traveling waves in reaction-diffusion equations.

Let us stress that the definition of the shape defect function implicitly depends on the choice of the speed $c \geq c_*$ via the function $\eta_c(u)$. Unless otherwise specified, we use it with $c = c_*$ and also use the notation

$$\eta_*(u) = \eta_{c_*}(u).$$

The front location asymptotics

To be concrete, in this paper we consider a family of nonlinearities of the form

$$f(u) = f'(0)(u - A(u))(1 + \chi A'(u)). \tag{1.17}$$

We assume that

$$A(0) = 0 = A'(0), \quad A(1) = 1, \quad \text{and} \quad \alpha(u) := \frac{A(u)}{u} \in C^2 \text{ is increasing and convex.} \tag{1.18}$$

As a consequence of (1.18), we know that $A(u)$ itself is increasing and convex. This class of nonlinearities appears to be fairly general: for example, it includes the nonlinearities suggested in [21, 33, 45]

$$f(u) = u(1 - u^{n-1})(1 + n\chi u^{n-1}) \qquad n \geq 2. \tag{1.19}$$

While the Fisher-KPP nonlinearity has an extremely elegant interpretation in terms of the position of the maximal particle of branching Brownian motion, due to McKean [41], this is not possible with other simple models like (1.19), even when $\chi = 0$. In Section 2 we investigate a connection to branching Brownian motion "voting models," which applies to, e.g., (1.19) and which provides a motivation for the convexity assumption on $\alpha(u)$.

Let us comment on the form of $f(u)$ in (1.17) and the assumptions on $\alpha(u)$ in (1.18). The main advantage of (1.17)-(1.18) is that it helps highlight a concealed algebraic structure underlying (1.1).

As we discuss in Appendix A.3, for any $f(u)$ one can find χ and $A(u)$ satisfying (1.17), so the only assumption we are making is that $\alpha(u)$ is C^2 and convex. At the same time, writing $f(u)$ in the form of (1.17) helps to make certain aspects of the structure more apparent. For example, the form (1.17) allows one to see explicitly the connection between (1.1) and the reactive conservation law (1.24), as well as derive the equation satisfied by the weighted Hopf-Cole transform (1.26) of u (see (1.27)). On the other hand, the assumption (1.18) on the convexity of α is clearly not always satisfied for all $f(u)$. It, however, has the advantage of allowing for elegant proofs that bypass extra technicalities and highlight the role played by the algebraic structure. Indeed, the convexity of α allows us to discard several errors terms that have a "good sign." Otherwise, we believe that these terms could be estimated by the smallness of $w(t,x)$ at the price of loss of elegance (see below). Thus, the convexity assumption on $\alpha(u)$ can be relaxed. Nonetheless, those f satisfying (1.17)-(1.18) present a rich class of nonlinearities in which to perform our investigation.

Nonlinearities of the form (1.17)-(1.18) are of the Fisher-KPP type when $\chi = 0$ and also when χ is positive but sufficiently small. However, they are not of Fisher-KPP type for χ close to $\chi = 1$ (see (3.5) and Lemma A.1). We show in Section 3 that traveling waves for such nonlinearities have asymptotics (1.11) with $D \neq 0$ when $0 \leq \chi < 1$. We refer to such nonlinearities as **semi-FKPP** type, and one can show that they have speed $c_* = 2\sqrt{f'(0)}$ (see Proposition 3.3). Thus, these remain pulled waves, despite f not satisfying the Fisher-KPP condition.

When $\chi = 1$, the traveling wave decay becomes purely exponential (1.10), as in the pushed case. Nevertheless, the speed remains linearly determined $c_* = 2\sqrt{f'(0)}$, as in the pulled case. This is the boundary between the pulled and pushed regimes. As such, we call this the **pushmi-pullyu** case, following [2]. Below, we discuss various remarkable algebraic properties of the solutions to (1.1) in the pushmi-pullyu case. Here, we simply note that the traveling wave profile function $\eta_*(u)$ is explicit when $\chi = 1$ (see Proposition 3.1):

$$\eta_*(u) = u - A(u). \tag{1.20}$$

An immediate consequence is the purely exponential decay of the minimal speed traveling wave U_* mentioned above that follows from the regularity assumptions on $A(u)$. The identity (1.20), however, has many further consequences in the analysis of the pushmi-pullyu case. In the semi-FKPP case, the traveling wave profile $\eta_*(u)$ does not have the simple form (1.20) and is not explicit but satisfies the bounds

$$\sqrt{\chi}(z - A(z)) \leq \eta_*(z) \leq z - A(z).$$

proved in Lemma A.4.

Let us also comment that in this paper we do not treat the pushed regime $\chi > 1$ as it can be handled by the standard existing methods originating in [26]. For completeness, we provide a short proof that these fronts are pushed – see Corollary A.3.

We now state the main theorem of this paper on the asymptotics of the front location for the reaction-diffusion equation (1.1) with nonlinearities of the form (1.17)-(1.18). We also assume that the initial condition $u_0(x) = u(0,x)$ satisfies

$$0 \leq u_0(x) \leq 1, \quad \text{for all } x \in \mathbb{R},$$

and is compactly supported on the right: there is $L_0 \in \mathbb{R}$ so that

$$u_0(x) = 0, \quad \text{for all } x \geq L_0. \tag{1.21}$$

Theorem 1.3. *Under the above assumptions, suppose that $u(t,x)$ solves (1.1) and that initially the shape defect function is non-negative: $w(0,x) \geq 0$ for all $x \in \mathbb{R}$.*

(i) If $0 \le \chi < 1$ so that f is a semi-FKPP type nonlinearity, the front location has the asymptotics:

$$m(t) = 2t - \frac{3}{2}\log t + x_0 + o(1), \quad \text{as } t \to +\infty. \tag{1.22}$$

(ii) If $\chi = 1$ so that f is a pushmi-pullyu type nonlinearity, then $m(t)$ has the asymptotics

$$m(t) = 2t - \frac{1}{2}\log t + x_1 + o(1), \quad \text{as } t \to +\infty. \tag{1.23}$$

The constants x_0 and x_1 in (1.22) and (1.23) depend on the initial condition u_0 for (1.1) and the nonlinearity f.

The asymptotics (1.22) for semi-FKPP type nonlinearities is exactly the same as (1.9) for the Fisher-KPP nonlinearities. However, in the pushmi-pullyu case, the logarithmic correction in (1.23) is different. This change has been predicted in [21, 37, 51] using formal matched asymptotics for the situations when the minimal speed traveling wave has purely exponential decay as in (1.10). To the best of our knowledge, the only rigorous result in this direction is the expansion

$$m(t) = 2t - \frac{1}{2}\log t + o(\log t), \quad \text{as } t \to +\infty,$$

obtained in [30] by a careful gluing of sub- and supersolutions, a very different approach from the present paper.

Let us point out that a simple consequence of our analysis is a way to identify some pulled fronts: waves associated to (1.1) with a positive nonlinearity f are pulled if $f \le f_\chi$ for some f_χ of the form (1.17)-(1.18) with $\chi < 1$ and $f'(0) = f'_\chi(0)$. This is more general than the Fisher-KPP condition and, unlike the commonly used condition $c_* = 2\sqrt{f'(0)}$ for pulled fronts, allows one to distinguish between the pulled and pushmi-pullyu cases. Moreover, Corollary A.3 in Appendix A shows that if $f \ge f_\chi$ with $\chi > 1$ then the fronts for (1.1) are pushed. Actually, if $f(u) = f_\chi(u)$ and $\chi \ge 1$, then both the minimal speed c_* and the traveling wave profile function $\eta_*(u)$ are explicit, see Proposition A.2.

One technical note is that Theorem 1.3 does not explicitly mention the convergence of u to U_* as this follows from previous results (see the discussion surrounding (1.8)). Due to the type of soft arguments used, these previous convergence results are not quantitative, and, thus, it is not possible to use them in any way in the present proof of Theorem 1.3. As a result, we provide an alternative proof of convergence as a part of establishing the precise asymptotics for $m(t)$.

Connection to the reactive conservation laws and the precise front asymptotics

A first example of the special structure in the pushmi-pullyu case is the connection to the reactive conservation law:

$$\mu_t + (A(\mu))_x = \mu_{xx} + \mu - A(\mu). \tag{1.24}$$

In particular, we show in Section 4.2 that the reaction-diffusion equation (1.1) with $f(u)$ given by

$$f(u) = (u - A(u))(1 + A'(u))$$

and the reactive conservation law (1.24) have exactly the same minimal speed traveling wave solutions. Thus, the traveling wave profile function $\eta_*(u)$ is the same for the two equations, and the shape defect function for (1.24) is still defined as[1] in Definition 1.1:

$$w_{\rm rcl}(t,x) = -\mu_x(t,x) - \eta_*(\mu(t,x)).$$

[1] Although with a notational change from w to $w_{\rm rcl}$ in order to not confuse it with the shape defect function for (1.1).

The connection between the two equations (1.1) and (1.24) extends further: if the shape defect function $w(t,x) \geq 0$, defined by (1.13) then the solution to (1.1) is a subsolution to (1.24):

$$u_t + (A(u))_x \leq u_{xx} + u - A(u).$$

This algebraic miracle allows us to bound the solutions to the pushmi-pullyu reaction-diffusion equation and the conservation law (1.24) in terms of each other:

$$u \leq \mu.$$

Further novel algebraic properties of the semi-FKPP and pushmi-pullyu nonlinearities are discussed in Section 1.2 below, as well as in Sections 3 and 4.

We also obtain the analogous result of Theorem 1.3.(ii) for the reactive conservation law (1.24).

Theorem 1.4. *Let μ be a solution to (1.24), with an initial condition such that $w_{\rm rcl}(0,x) \geq 0$ for all $x \in \mathbb{R}$. Then there is $x_2 \in \mathbb{R}$ so that, for all $L > 0$,*

$$\lim_{t\to\infty} \sup_{|x|\leq L} |\mu(t, x + m(t)) - U_*(x)| = 0 \qquad \textit{with} \;\; m(t) = 2t - \frac{1}{2}\log t + x_2. \tag{1.25}$$

It seems that the positivity assumption of the initial shape defect function for Theorems 1.3 and 1.4 can be relaxed. Indeed, the lack of positivity of the shape defect function can likely be compensated by its smallness that, as we have mentioned, can be proved independently by directly analyzing its dynamics in (4.1) (see Lemma 6.6 and the forthcoming work [4]). We have opted to use this positivity assumption on w, not simply because it shortens the proof but also because it makes many steps in the proofs elegant rather than technical and reveals a number of algebraic properties that are not seen without this assumption.

1.2 Key elements of the proofs of Theorem 1.3 and Theorem 1.4

As the proofs are quite intricate, a detailed summary would be too long to contain here. Instead we discuss the major aspects of the proof and, in particular, the new tools that become available due to the positivity of the shape defect function. A more detailed outline of the proofs can be found in Section 5.

The weighted Hopf-Cole transform

The crucial tool to compensate for the lack of the Fisher-KPP condition for f is the weighted Hopf-Cole transform: letting $\hat{u}(t,x) = u(t, x+2t)$, we define

$$v(t,x) = \exp\Big(x + \sqrt{\chi}\int_x^\infty \alpha(\hat{u}(t,y))dy\Big)\hat{u}(t,x). \tag{1.26}$$

We recall that $\alpha(u) \geq 0$. One should stress a key difference between the semi-FKPP ($\chi \in [0,1)$) and pushmi-pullyu ($\chi = 1$) cases:

$$v(t,-\infty) = 0 \quad \text{when } \chi < 1 \qquad \text{and} \qquad v(t,-\infty) > 0 \quad \text{when } \chi = 1.$$

Due to the remarkable algebraic structure discussed in Section 4, whenever u solves (1.1) and the shape defect function is nonnegative, we obtain the differential inequality

$$v_t \leq v_{xx}. \tag{1.27}$$

This is slightly easier to see in the pushmi-pullyu case ($\chi = 1$) where, after an intricate computation in the proof of Proposition 4.1, we obtain

$$v_t - v_{xx} \leq -2\alpha(\hat{u})w \leq 0.$$

Intuitively, the importance of (1.27) is the following. When $\chi < 1$, it is clear that $v(t,x)$ is 'small' for $x < 0$. We can then, roughly, think of v as solving the heat equation on the half-line with *Dirichlet* boundary conditions, which implies that v decays like $O(t^{-3/2})$. This 3/2 is the same as the one in (1.22). Indeed,

$$O(t^{-3/2}) = v(t, m(t) - 2t) = e^{m(t)-2t+O(1)}u(t, m(t)) = O(e^{m(t)-2t}). \tag{1.28}$$

When $\chi = 1$, we have, on the other hand,

$$v_x(t,x) = -\exp\Big(x + \int_x^\infty \alpha(\hat{u}(t,y))dy\Big)w(t,x),$$

which is 'small' for $x < 0$. We can then, roughly, think of v as solving the heat equation on the half-line with *Neumann* boundary conditions, which implies that v decays like $O(t^{-1/2})$. This 1/2 is the same as the one in (1.23) by a similar computation to (1.28).

We note that this intuition can only be turned into a proof in the semi-FKPP case. For technical reasons, it does not go through in the pushmi-pullyu case, and one must first obtain the 'rough' front asymptotics

$$m(t) = 2t - (1/2)\log t + O(1), \tag{1.29}$$

without the use of the weighted Hopf-Cole transform. The transform, however, does play a crucial role in upgrading the front asymptotics to the precision of (1.23) (resp. (1.25)) in the pushmi-pullyu case.

Relative entropy and a weighted Nash inequality in the pushmi-pullyu case

Let us now explain how the 'rough' front asymptotics (1.29) is obtained. We focus on the conservation law (1.24). Along the lines of (1.28), this asymptotics can be recast as the L^∞ upper and lower estimates

$$\|p(t,x)\|_\infty = O(1/\sqrt{t}) \tag{1.30}$$

for the function

$$p(t,x) = e^x \hat{\mu}(t,x)),$$

where $\hat{\mu}(t,x) = \mu(t, x + 2t)$. It turns out that the upper bound in (1.30) is much more difficult to establish, so we discuss that now.

The intuitive reason behind (1.30) is that $p(t,x)$ satisfies an inhomogeneous conservation law

$$p_t + (\alpha(\hat{\mu})p)_x = p_{xx}. \tag{1.31}$$

Heuristically, $\alpha(\hat{\mu})$ pushes all of the mass of p to the right, where (1.31) is essentially the heat equation as $\alpha(\hat{\mu}) \approx 0$ as $x \to +\infty$. This indicates that, after a boundary layer in time, we should have heat-equation-like decay $O(1/\sqrt{t})$, as desired.

In order to create a proof out of this simple idea, we use a relative entropy approach. Another surprising consequence of the steepness comparison of the solution and the traveling wave is that, if $w_{\rm rcl}(0,x) \geq 0$ for all $x \in \mathbb{R}$, then the function

$$\rho(t,x) = \exp\Big(-\int_0^{\hat{\mu}(t,x)} \frac{A(u')}{u'(u' - A(u'))}\,du'\Big) \tag{1.32}$$

is a supersolution to (1.31)

$$\rho_t + (\alpha(\hat{\mu})\rho)_x \geq \rho_{xx}.$$

This follows from a rather involved computation in the proof of Lemma 6.2. An observation coming from [2] is that if $\rho(t,x)$ is a supersolution and $p(t,x)$ is a solution to (1.31) then the function

$$\varphi(t,x) = \frac{p(t,x)}{\rho(t,x)} = \frac{e^x \hat{\mu}(t,x)}{\rho(t,x)},$$

obeys a dissipation inequality

$$\frac{1}{2}\frac{d}{dt}\int \varphi(t,x)^2 \rho(t,x) dx \leq -\int \varphi_x(t,x)^2 \rho(t,x) dx.$$

Were $\rho \equiv 1$, this would be the key differential inequality for the heat equation that, along with the Nash inequality and a self-adjointness 'trick,' yields the $O(1/\sqrt{t})$-decay desired in (1.30). Unfortunately, $\rho \not\equiv 1$ (notice that $\rho(t,-\infty) = 0$ and $\rho(t,+\infty) = 1$) and our problem is not self-adjoint. By developing a suitable Nash-type inequality with time-dependent dynamic weights to yield suitable weighted $L^1 \to L^2$ estimates and an additional bootstrapping procedure to yield $L^2 \to L^\infty$ estimates, we obtain the analogous bound $\|\varphi\|_\infty \leq O(1/\sqrt{t})$ that is the key step in obtaining (1.30) and, thus, the asymptotics (1.23) and (1.25).

Decay of w and the reactive conservation law

As we discussed above, the weighted Hopf-Cole transform is not used in the pushmi-pullyu case to prove the 'rough' $2t - (1/2)\log t + O(1)$ front asymptotics. It is, however, needed to upgrade this estimate to (1.23) and (1.25); that is, that the $O(1)$ term is actually of the form $x_0 + o(1)$ for some $x_0 \in \mathbb{R}$. In the reaction-diffusion case (1.1), we use (1.27) and the arguments of [2] to obtain this improved precision.

Unfortunately, the Hopf-Cole transform does not yield (1.27) in the reactive conservation law case. Instead, analogous computations yield

$$v_t - v_{xx} \leq e^{\gamma}(\hat{\mu}\alpha'(\hat{\mu}) - \alpha(\hat{\mu}))w_{\rm rcl},$$

where we abuse notation and use v for the same quantity (1.26) with $\hat{\mu}$ replacing $\hat{u}$. The right hand side is *positive* for convex α except in the special case $\alpha(u) = u$ considered in [2].

As we do not have the miraculous Hopf-Cole cancelation so we require an additional argument. In particular, we need smallness of $w_{\rm rcl}$. As can be seen by a change to self-similar coordinates, it turns out that the required bounds are of the type $\|w_{\rm rcl}\|_\infty = O(1/t)$ and suitable integrability. To obtain these bounds, we leverage the fact that $w_{\rm rcl}$ satisfies the equation

$$\partial_t w_{\rm rcl} + A'(\hat{\mu})\partial_x w_{\rm rcl} = \partial_{xx} w_{\rm rcl} + w_{\rm rcl}(1 - A'(\hat{\mu})). \tag{1.33}$$

It is not immediately obvious why $w_{\rm rcl}$ should decay like $O(1/t)$ from the above. Indeed, $w_{\rm rcl}$ has a positive growth rate for $x \geq 2t - (1/2)\log t + O(1)$.

Roughly, $w_{\rm rcl}$ decays because the region where it has a positive growth rate 'moves too fast' to the right. Indeed, the production rate $1 - A'(\hat{\mu})$ in (1.33) is only positive when $\hat{\mu} \approx 1$, which, by the work outlined above, corresponds to $x \geq 2t - (1/2)\log t + O(1)$. On the other hand, the linearization of (1.33) is the same as that of Fisher-KPP, indicating that $w_{\rm rcl}$ 'wants' to have a 'front' at $2t - (3/2)\log t + O(1)$. The $\log t$ discrepancy between this and where $w_{\rm rcl}$ has a positive growth rate, along with the natural $O(e^{-x})$ decay of the problem (see (1.10)) indicates that

$$w_{\rm rcl} \leq O(e^{-\log t}) = O\Big(\frac{1}{t}\Big).$$

Interestingly, it seems that analyzing the equations for w and $w_{\rm rcl}$ yields sharp bounds for convergence rates of u to the traveling wave by using the heuristic ideas outlined above. We plan to explore this in a future work.

A note on the relationship between the present work and [2]. Let us note that some parts of the analysis here are motivated by the considerations in [2] for the Burgers-FKPP equation, which is the reactive conservation law (1.24) in the special case $A(u) = u^2$. We take care to present the work here to highlight *only* the novel elements of the proofs as compared to [2]. We point out that [2] examined only the reactive conservation law equation and, moreover, only the particular case $\alpha(u) = u$. In this simple setting, there are additional cancellations and many explicit calculations are possible. Hence, a key aspect of our work here is to explore the algebraic structure of the reactive conservation laws (1.24) and the reaction diffusion equations (1.1) and to show that the seemingly *ad hoc* techniques in [2] are actually a part of this larger structure. One example of this is the choice of ρ: in [2] a 'lucky guess' led to the choice $\rho = 1 - \hat{u}$, but it is now clear that this follows from (1.32) where, in that particular case, $A(u) = u^2$.

1.3 Organization of the paper

In Section 2, we discuss a connection between branching Brownian motion and reaction-diffusion equations (1.1) using probabilistic voting models. While some examples of Fisher-KPP type nonlinearities are McKean nonlinearities, those that, roughly, give the statistics of the maximal particle in a branching Brownian motion, many simple choices fall outside this class; for example, $f(u) = u - u^n$ for any $n \geq 3$ is *not* McKean type. We present a class of voting models that allows to go beyond the McKean class of nonlinearities to a much larger class of nonlinearities that includes Fisher-KPP type nonlinearities such as $u - u^n$ and many non-Fisher-KPP type nonlinearities. This provides an additional motivation for the setting of Theorem 1.3.

We then discuss the basic properties of the semi-FKPP and pushmi-pullyu nonlinearities in Section 3. In particular, we discuss the properties of the traveling wave profile function $\eta_c(u)$ and explain why the minimal traveling wave speed is given by (1.6) not only for the Fisher-KPP but also for semi-FKPP nonlinearities.

Section 4 discusses some of remarkable algebraic properties of reaction-diffusion equations that are elucidated by the use of the shape defect and traveling wave profile functions. There, we consider in greater detail the shape defect function, establish the variational formulation (1.14) for reaction-diffusion equations, and explain the aforementioned natural connection between reaction-diffusion equations and reactive conservation laws. We also introduce the weighted Hopf-Cole transform (see (1.26)) and establish the key differential inequality (1.27).

Section 5 gives the outlines of the proofs of Theorems 1.3 and 1.4. These results are quite intricate, and so we provide a high level discussion of its major difficulties and how the elements of the proof introduced above fit together to overcome these obstructions. The full proofs are contained in Section 6 (pushmi-pullyu case) and Section 7 (semi-FKPP case).

Finally, Appendix A contains some auxiliary results used in the paper as well as a discussion of the generality of the model (1.17).

2 Probabilistic interpretations of reaction-diffusion equations

2.1 The McKean nonlinearities

A special class of the Fisher-KPP nonlinearities arises naturally in the context of the branching Brownian motion (BBM), as originally discovered by McKean in [41]. They have the form

$$f(u) = \gamma\Big(1 - u - \sum_{k=2}^{\infty} p_k (1-u)^k\Big), \tag{2.1}$$

with $\gamma > 0$ and $p_k \geq 0$ such that

$$\sum_{k=2}^{\infty} p_k = 1.$$

Here, p_k is the probability that a BBM particle branches into k offspring particles at a branching event, and $\gamma > 0$ is the exponential rate of branching. Specifically, when $f(u)$ has the form (2.1), the solution to (1.1) is given by

$$u(t,x) = 1 - \mathbb{E}_x\Big(\prod_{k=1}^{N_t}(1 - u_0(X_k(t))\Big).$$

Here, $X_1(t), \ldots, X_{N_t}(t)$ are the locations of the BBM particles at the time t. In the absence of branching, this is simply the standard interpretation of the solutions of the heat equation in terms of a Brownian motion. We refer to [13, 16] for excellent introductions to BBM and the McKean connection between BBM and reaction-diffusion equations.

2.2 Voting nonlinearities

However, the nonlinearities that admit the McKean interpretation form just a sub-class of the Fisher-KPP nonlinearities. Maybe the most basic example of an FKPP nonlinearity not in the McKean class is $f(u) = u - u^n$ with $n \geq 3$, as it can be easily checked that it both satisfies (1.5) and can not be written in the form (2.1).

Let us now briefly describe the construction in [3], originating in the beautiful and insightful ideas of [22], that provides a connection between a much larger class of semilinear parabolic equations and BBM than that of McKean. For the sake of concreteness and simplicity, we consider $f(u) = u - u^n$ and refer to [3] for a description of this connection in more generality. Let us start with a BBM running at an exponential rate $\beta > 0$: the branching times have the law

$$\mathbb{P}(\tau > t) = e^{-\beta t}.$$

We assume that at each branching event, the parent particle produces exactly n offspring. There is a natural way to associate a random genealogical tree $\mathcal{T}$ to each realization of the BBM, with each vertex of the tree corresponding to a branching event. This is illustrated in Figure 1. Each of the edges coming out of a vertex represents an offspring particle born at that branching event. The root of the tree $\mathcal{T}$ is the original particle that started at the time $t = 0$ at the position x.

We now describe the voting procedure. Let us run the above BBM until a final time $t > 0$. At that time, each of the particles $X_1(t), \ldots, X_{N_t}(t)$ that are present at the time t votes 1 or 0, with the probability

$$\mathbb{P}(\text{Vote}(X_k(t)) = 1) = 1 - \mathbb{P}(\text{Vote}(X_k(t)) = 0) = g(X_k(t)). \tag{2.2}$$

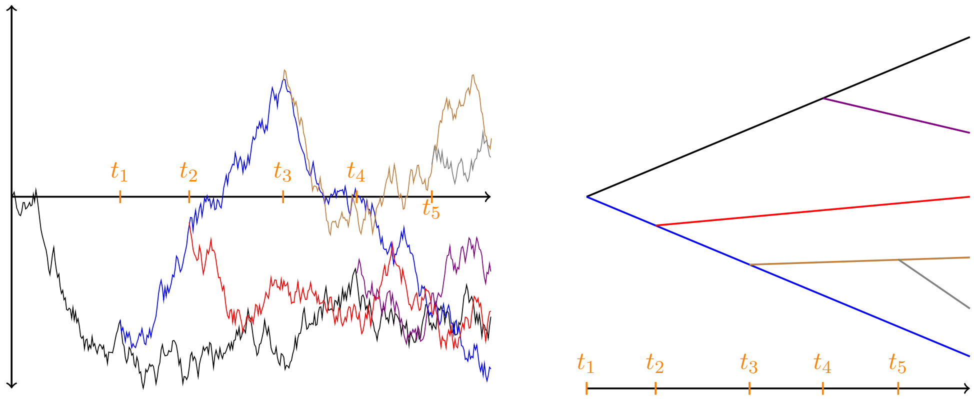

Figure 1: On the left, a sample (binary) branching Brownian motion, which branches at times t_i. On the right, the associated tree.

Here, the function $g(x)$ is fixed and takes values in $[0,1]$.

Given the votes of the last generation of particles that are present at the time t, we propagate the vote up the genealogical tree $\mathcal{T}$ as follows. Let us fix $\gamma > 0$ sufficiently small and define the probabilities μ_{kn}, $k = 0, \dots, n$, as

$$\mu_{0n} = 0, \qquad \mu_{nn} = 1, \qquad \text{and} \qquad \mu_{kn} = \frac{(1+\gamma)k}{n} \text{ when } 0 < k < n. \tag{2.3}$$

As we need to have $0 \le \mu_{kn} \le 1$ for all k, the parameter $\gamma > 0$ must satisfy

$$0 < \gamma \le \frac{1}{n-1}. \tag{2.4}$$

With the probabilities μ_{kn} in hand, given a parent particle on the genealogical tree $\mathcal{T}$, if k out of its n children voted 1, then the parent particle votes 1 with the probability μ_k given by (2.3). Using this rule iteratively to go up the tree all the way to the root produces the random vote $\mathrm{Vote}_{\mathrm{orig}}$ of the original ancestor particle, and we can define

$$u(t,x) = \mathbb{P}_x(\mathrm{Vote}_{\mathrm{orig}} = 1). \tag{2.5}$$

Here, the probability is taken both with respect to the randomness in the original voting in (2.2), and with respect to the randomness in the vote of each parent. If there was no branching event until the time t, so that $N_t = 1$, then the vote of the original particle is 1 with the probability $g(X_1(t))$.

An elementary computation in [3] shows that the function $u(t,x)$ defined in (2.5) satisfies the initial value problem

$$\begin{aligned} u_t &= \Delta u + f(u), \\ u(0,x) &= g(x), \end{aligned} \tag{2.6}$$

with the nonlinearity

$$f(u) = \beta\gamma(u - u^n). \tag{2.7}$$

Note that the range of γ is restricted by (2.4) but $\beta > 0$ can be arbitrary.

We may consider the above voting model with a more general branching Brownian motion, with the probability α_n to branch into n children at each branching event. If we keep the probabilities μ_{kn}

for the parent with n total children to vote 1 if k of its n children voted 1, we would obtain a convex combinations of the nonlinearities in (2.7):

$$f(u) = \beta\gamma \sum_{k=1}^{N} \alpha_k (u - u^k), \quad \sum_{k=2}^{N} \alpha_k = 1. \tag{2.8}$$

They are still within the Fisher-KPP class. We refer to nonlinearities of the form (2.8) as voting-FKPP nonlinearities. Such nonlinearities have the form

$$f(u) = \lambda(u - A(u)), \qquad \text{with } \lambda = f'(0) > 0. \tag{2.9}$$

The functions $A(u)$ are non-negative, convex on $[0, 1]$, and satisfy (1.18):

$$A(0) = 0, \quad A(1) = 1, \quad \text{and} \quad A'(0) = 0. \tag{2.10}$$

It follows that the function $A(u)$ is increasing since $A'(0) = 0$ and $A(u)$ is convex on $[0, 1]$.

It is sometimes convenient for us to write $f(u)$ in the form

$$f(u) = \lambda u(1 - \alpha(u)),$$

with

$$\alpha(u) = \frac{A(u)}{u}. \tag{2.11}$$

For nonlinearities of the form (2.8), the functions $A(u)$ and $\alpha(u)$ have the forms

$$A(u) = \sum_{k=2}^{N} \alpha_k u^k \quad \text{and} \quad \alpha(u) = \sum_{k=2}^{\infty} \alpha_k u^{k-1}.$$

Therefore, in these examples both $A(u)$ and $\alpha(u)$ are increasing and convex. We should mention that for the McKean nonlinearities (2.1) if we were to write them in the form (2.9), the function $A(u)$ is increasing and convex but $\alpha(u)$ is necessarily concave. That is, the voting Fisher-KPP nonlinearities represent a complementary class to the McKean type.

Let us also briefly comment that the simple voting procedure described above can be generalized in many ways, and, unlike the McKean interpretation, voting models can lead to reaction-diffusion equations (1.1) with nonlinearities $f(u)$ that need not be of the Fisher-KPP type. This is done simply by changing the voting rules, without changing the underlying branching Brownian motion. That is, one considers the same BBM, with exactly the same genealogical tree as above. However, we modify the probabilities μ_{kn} in (2.3) for the parent to vote 1 if k out of its n children voted 1. Let us assume for simplicity that the BBM has a fixed number n of children at each branching event. Then, the function $u(t, x)$ defined by (2.5), which is the probability for the original ancestor particle to vote 1, satisfies the reaction-diffusion equation (2.6), although with the nonlinearity

$$f(u) = \beta\Big(\sum_{k=0}^{2n-1} \binom{2n-1}{k} \mu_{kn} u^k (1 - u)^{2n-1-k} - u \Big).$$

As discussed in [3], with a suitable choice of μ_{kn} one can obtain nonlinearities not of the Fisher-KPP type. Indeed, the original example in [22] is the Allen-Cahn equation

$$u_t = u_{xx} + f(u), \tag{2.12}$$

with the nonlinearity

$$f(u) = u(1 - u)(2u - 1), \tag{2.13}$$

that is not of the Fisher-KPP type. To obtain (2.12)-(2.13), one considers ternary BBM, $n = 3$, and the voting probabilities $\mu_{03} = \mu_{13} = 0$ and $\mu_{23} = \mu_{33} = 1$ that come from the simple majority voting rule. More examples can be found in [3].

3 The algebra of semi-FKPP and pushmi-pullyu traveling waves

In this section, we explore the identities and properties of the traveling wave profile function $\eta_c(u)$ (see Propositions 3.1 and 3.2) as well as the relationship between its regularity and the decay of the traveling wave $U_*(x)$ as $x \to +\infty$ (see Proposition 3.2). We also show in Proposition 3.3 that the minimal speed for traveling waves is still given by the FKPP formula $c_* = 2\sqrt{f'(0)}$ for the nonlinearities of the semi-FKPP type.

3.1 Semi-FKPP and pushmi-pullyu nonlinearities

We first introduce the (seemingly) new types of nonlinearities: semi-FKPP and pushmi-pullyu. Let us start with a Fisher-KPP nonlinearity of the form

$$\zeta(u) = \lambda(u - A(u)),$$

with some $\lambda > 0$ and an increasing convex function $A(u)$ that satisfies (2.10). Note that

$$\lambda = \zeta'(0), \tag{3.1}$$

since $A'(0) = 0$ by (2.10). This class includes both the McKean and voting-FKPP nonlinearities.

We say that a function $f(u)$ is a pushmi-pullyu nonlinearity if it has the form

$$f(u) = \zeta(u)(2\lambda - \zeta'(u)) = \lambda^2(u - A(u))(1 + A'(u)), \tag{3.2}$$

with $\zeta(u)$ as above. As in (3.1), since $A'(0) = 0$, it follows that $f'(0) = \lambda^2$. Thus, we may represent a pushmi-pullyu nonlinearity in the form

$$f(u) = f'(0)(u - A(u))(1 + A'(u)), \tag{3.3}$$

As we shall see, this class of nonlinearities represents the boundary between those of pushed and pulled type. Any positive nonlinearity $g(u)$ smaller than $f(u)$ and such that $g'(0) = f'(0)$ should be pulled; however, they may not be Fisher-KPP type. It is, thus, natural to say that to say that a function $f(u)$ is of the semi-FKPP type if, in contrast to (3.3), it satisfies

$$f(u) \le f'(0)(u - A(u))(1 + \chi A'(u)), \tag{3.4}$$

with $0 \le \chi < 1$ and an increasing convex function $A(u)$ that satisfies (2.10).

Let us note that a semi-FKPP nonlinearity satisfies the Fisher-KPP condition (1.5) if

$$(u - A(u))(1 + \chi A'(u)) \le u, \quad \text{for all } 0 \le u \le 1,$$

or, equivalently,

$$0 \le \chi \le \chi_{FKPP} = \min_{u\in[0,1]} \frac{A(u)}{A'(u)[u - A(u)]}. \tag{3.5}$$

We are mostly interested in the range $\chi_{FKPP} \le \chi \le 1$, where $f(u)$ is not of the Fisher-KPP type. As Theorem 1.3 shows, the solutions to (1.1) still exhibit the Fisher-KPP type behavior. Let us comment that Lemma A.1, below, shows that, as long as $A(u)$ satisfies (2.10), we have $\chi_{FKPP} \le 1/2$, so that the semi-FKPP range is 'uniformly nontrivial.' However, in the pushmi-pullyu case $\chi = 1$ the behavior of the solutions changes drastically, as seen from the second statement in Theorem 1.3.

A well-known example from [33], also discussed in detail in [45], is the nonlinearity

$$f(u) = u(1 - u)(1 + au), \tag{3.6}$$

with $a > 0$. This nonlinearity satisfies the FKPP property for all $0 \le a \le 1$. In the terminology of the present paper, with $\zeta(u) = u - u^2$, it is semi-FKPP type in the larger range $0 \le a < 2$ and pushmi-pullyu type when $a = 2$. A generalization of this example:

$$f(u) = u(1 - u^n)(1 + au^n), \tag{3.7}$$

was considered in [21]. This corresponds to (3.4) with $\zeta(u) = u - u^n$. To put this in the form of (3.4), we set $A(u) = u^{n+1}$ and $\chi = a/(n+1)$. This nonlinearity also has the Fisher-KPP property for all $0 \le a \le 1$, as can be seen from (3.5), but is of the semi-FKPP type in the range $0 \le a < n+1$. It is pushmi-pullyu type when $a = n+1$.

3.2 The traveling wave profile and the nonlinearity

In order to further explain the particular form of the nonlinearity (1.17), we need to discuss some basic facts about traveling waves for semi-linear parabolic equations. The main result of this section is an expression for the nonlinearity $f(u)$ in terms of the traveling wave profile function $\eta_c(u)$ with speeds $c \ge c_*$, defined by (1.12). It is elementary, but is quite interesting in its own right and used frequently in the sequel, so we state this connection as a standalone result.

Proposition 3.1. *For each $c \ge c_*$, the function $\eta_c(u)$ is continuously differentiable for $u \in [0,1]$ and satisfies*

$$\eta_c(0) = \eta_c(1) = 0, \quad \eta_c'(0) > 0, \quad \eta_c(u) > 0 \quad \text{and} \quad \eta_c'(u) < c,$$

for all $u \in (0,1)$. Moreover, for each $c \ge c_*$, the function $f(u)$ can be expressed in terms of $\eta_c(u)$ by

$$f(u) = \eta_c(u)(c - \eta_c'(u)), \quad \text{for all } u \in (0,1). \tag{3.8}$$

Proof. The positivity of $\eta_c(u)$ for $u \in (0,1)$ follows from the aforementioned strict negativity of U_c', which also implies that $\eta_c(0) = \eta_c(1) = 0$. To check the continuous differentiability of $\eta_c(u)$ we only need to analyze the behavior near $u = 0$ and $u = 1$; indeed, the case $u \in (0,1)$ follows by the negativity of U_c' and the inverse function theorem. We consider only the behavior near $u = 0$ as the other case can be handled similarly. Let $U_c(x)$ be a solution to (1.3) with some $c \ge c_*$. Recall that traveling waves have the asymptotics (1.11)

$$U_c(x) \sim (D_c x + B_c)e^{-\lambda_c x}, \tag{3.9}$$

with λ_c being a positive root of

$$\lambda_c^2 - c\lambda_c + f'(0) = 0, \tag{3.10}$$

given by

$$\lambda_c = \frac{c \pm \sqrt{c^2 - 4f'(0)}}{2}. \tag{3.11}$$

Let us make two remarks about (3.9)-(3.11). First, the '+' sign in (3.11) appears only in the case $c = c_* > 2\sqrt{f'(0)}$, and the '−' sign corresponds to all the other cases: either $c = c_* = 2\sqrt{f'(0)}$, or $c > c_*$; see [5, Proposition 4.4]. Second, the coefficient D_c may be non-zero only if $c = 2\sqrt{f'(0)}$, so that λ_c is a double root of (3.10).

We deduce from (3.9) that

$$\lim_{x \to +\infty} \frac{U_c'(x)}{U_c(x)} = -\lambda_c. \tag{3.12}$$

Using (3.12) and that $\eta_c'(U_c(x))U_c'(x) = -U_c''(x)$, we obtain

$$\begin{aligned}\eta_c'(0) &= -\lim_{x\to+\infty}\frac{U_c''(x)}{U_c'(x)} = c + \lim_{x\to+\infty}\frac{f(U_c(x))}{U_c'(x)} = c + \lim_{x\to+\infty}\frac{f'(0)U_c(x)}{U_c'(x)} = c - \frac{f'(0)}{\lambda_c} \\ &= \frac{c\lambda_c - f'(0)}{\lambda_c} = \lambda_c.\end{aligned} \tag{3.13}$$

We used (3.10) in the last step. In particular, it follows from (3.13) that $\eta_c'(0) > 0$. To see that $\eta_c'(u) < c$, we use the monotonicity of $U_c(x)$ to write

$$\eta'(U_c(x)) = -\frac{U_c''(x)}{U_c'(x)} = c + \frac{f(U_c(x))}{U_c'(x)} < c,$$

In order to establish (3.8), insert (1.12) into (1.3) to find

$$c\eta(U_c) = U_c'' + f(U_c) = (-\eta(U_c))' + f(U_c) = -\eta'(U_c)U_c' + f(U_c) = \eta(U_c)\eta'(U_c) + f(U_c). \tag{3.14}$$

As (3.14) holds for all $x \in \mathbb{R}$ and $U_c(x)$ is monotonically decreasing and obeys the limits at infinity in (1.3), this gives expression (3.8) for $f(u)$:

$$f(u) = \eta_c(u)(c - \eta_c'(u)), \quad \text{for all } 0 \le u \le 1,$$

finishing the proof. □

One consequence of Proposition 3.1 is that $p(u) = \eta_{c_*}(x)$ is an admissible test function in the Hadeler-Rothe variational principle (1.4) as it is continuously differentiable. A simple observation, also going essentially back to [33] is that $\eta_*(u)$ is actually the optimizer in (1.4). This is a consequence of (3.8) with $c = c_*$: if $p(u) = \eta_*(u)$, the expression inside the inf sup in (1.4) becomes

$$\eta_*'(u) + \frac{f(u)}{\eta_*(u)} = \frac{\eta_*(u)\eta_*'(u) + f(u)}{\eta_*(u)} = \frac{\eta_*(u)\eta_*'(u) + \eta_*(u)(c_* - \eta_*'(u))}{\eta_*(u)} = c_*.$$

A kind of converse to Proposition 3.1 is also true. Let $f(u)$ be a $C^1([0,1])$ function of the form (3.8) with $\eta_c(u) \in C^1([0,1])$, and consider a traveling wave solution to

$$-cU_c' = U_c'' + \eta_c(U_c)(c - \eta_c'(U_c)), \quad U_c(-\infty) = 1, \; U_c(+\infty) = 0. \tag{3.15}$$

We claim that U_c solves the first order ODE (1.12):

$$-U_c' = \eta_c(U_c), \qquad U_c(-\infty) = 1, \quad U_c(+\infty) = 0. \tag{3.16}$$

To see this, let U be the unique (up to translation) solution of (3.16). Then, we have

$$-U'' - cU' = \eta'(U)U' - cU' = -\eta_c'(U)\eta_c(U) + c\eta_c(U) = \eta_c(U)(c - \eta_c'(U)),$$

which is (3.15). Since the traveling wave profiles for both (3.15) and (3.16) are unique, this shows that, up to translation, $U_c = U$. Hence, U_c satisfies (3.16).

Another simple but important comment is that if $f(u)$ has the pushmi-pullyu form (3.2), then the minimal speed traveling wave profile function is simply

$$\eta_*(u) = \zeta(u) = u - A(u). \tag{3.17}$$

This property is very convenient in the analysis of the pushmi-pullyu case.

3.3 Purely exponentially decaying waves

The next statement characterizes the nonlinearities $f(u)$ for which the decay is purely exponential, so that $D_c = 0$. It also gives the asymptotics of $\eta_c(u)$ as $u \to 0$ for waves that have an exponential decay with a linear pre-factor. In particular, it shows that the difference between pulled and pushmi-pullyu waves can be seen in the regularity of the traveling wave profile function η_c.

Proposition 3.2. *Let $c \geq c_*$ and $f \in C^1([0,1])$. If η_c is the traveling wave profile function associated to (1.1) then:*
(i) Suppose, for some $p > 1$, there exists $C > 0$ so that

$$\eta_c(u) \sim \lambda_c u + O\Big(\frac{u}{(1+|\log u|)^p}\Big) \qquad \text{as } u \searrow 0, \tag{3.18}$$

where λ_c is defined as in (3.11). In particular, this is true if $\eta_c \in C^{1,\delta}([0,1])$ for some $\delta > 0$. Then, the profile U_c has purely exponential decay:

$$U_c(x) \sim B_c e^{-\lambda_c x} \qquad \text{as } x \to \infty. \tag{3.19}$$

(ii) If U_c has exponential decay with the linear factor, that is, $D_c \neq 0$ in (3.9), then

$$\eta_c(u) \sim \lambda_c u\Big(1 + \frac{1}{\log u} + O\Big(\frac{\log(\log(u^{-1}))}{(\log u)^2}\Big)\Big) \qquad \text{as } u \searrow 0.$$

Proof of (i). Using directly (1.12) and then (3.18), we find

$$-1 = \frac{U_c'}{\eta_c(U_c)} \geq \frac{U_c'}{\lambda_c U_c(1 - C(1 + |\log U_c|)^{-p})}, \tag{3.20}$$

for some constant $C > 0$. After possibly shifting the traveling wave, we may assume without loss of generality that the denominator in (3.20) never vanishes for $x > 0$. Multiplying both sides by λ_c, integrating in x, and using the monotonicity of $U_c(x)$ to make a change of variables $z = -\log(U_c)$, yields, for $x > 0$:

$$\begin{aligned} -\lambda_c x &\geq \int_0^x \frac{U_c'}{U_c(1 - C(1+|\log U_c|)^{-p})} dx' = \int_{-\log U_c(0)}^{-\log U_c(x)} \frac{(-1)}{1 - C(1+z)^{-p}} dz \\ &\geq \int_{-\log U_c(0)}^{-\log U_c(x)} \Big(-1 + \frac{1}{C(1+z)^p}\Big) dz \geq \log U_c(x) - C. \end{aligned}$$

Hence, we have, for all $x > 0$,

$$U_c(x) \leq C e^{-\lambda_c x}. \tag{3.21}$$

To refine this bound to the asymptotics in (3.19), we let $\bar{U}_c(x) = e^{\lambda_c x} U_c(x)$. Using (1.12) and (3.18) again, we find

$$|\bar{U}_c'(x)| = e^{\lambda_c x} |\eta_c(U_c)(x) - \lambda_c U_c(x)| \leq \frac{C \bar{U}_c(x)}{(1 + |\log(U_c(x))|)^p}.$$

Using (3.21) to bound the numerator and the denominator, we obtain

$$|\bar{U}_c'(x)| \leq \frac{C}{(1+x)^p},$$

from which the claim (3.19) follows. □

Proof of (ii). First, by suitably shifting U_c, we may assume $B_c = 0$ without loss of generality. Then, for $x \gg 1$, we have

$$U_c(x) = A_c x e^{-\lambda_c x} + O(e^{-(\lambda_c+\delta)x}),$$

for some fixed $\delta > 0$. Next, fix any $u > 0$ sufficiently small and let x be such that $u = U_c(x)$. By (1.12), we have

$$\begin{aligned}\eta_c(u) = \eta_c(U_c(x)) &= -U_c'(x) = A_c(\lambda_c x - 1)e^{-\lambda_c x} + O(e^{-(\lambda_c+\delta)x}) \\ &= \lambda_c U_c(x) - \frac{U_c(x)}{x} + O\Big(U_c(x)^{1+\frac{\delta-\varepsilon}{\lambda_c}}\Big),\end{aligned} \tag{3.22}$$

for any $\varepsilon > 0$. Notice that

$$-\lambda_c x = \log U_c - \log(A_c x) + O(e^{-\delta x}) = \log U_c - \log(-\log(U_c)) + O(1).$$

Hence, (3.22) becomes

$$\begin{aligned}\eta_c(u) &= \lambda_c u + \frac{\lambda_c u}{\log u - \log(-\log u) + O(1)} + O(u^{1+\frac{\delta-\varepsilon}{\lambda_c}}) \\ &= \lambda_c u\Big(1 + \frac{1}{\log u} + O\Big(\frac{\log(-\log u)}{(\log u)^2}\Big)\Big).\end{aligned}$$

This concludes the proof. □

3.4 When the minimal speed is given by the Fisher-KPP formula

It is well known that for the Fisher-KPP type nonlinearities the minimal speed is given by the Fisher-KPP formula

$$c_*[f] = 2\sqrt{f'(0)}. \tag{3.23}$$

However, the Fisher-KPP condition (1.5) is not necessary for (3.23) to hold. A well-known example of a non-FKPP type nonlinearity that satisfies (3.23), is the nonlinearity of the form (3.6) in the range $1 \le a \le 2$, as discussed in detail in [33, 45]. This is also true for nonlinearities of the form (3.7) with $1 \le a \le n+1$, considered in [21]. A natural question is: for which other nonlinearities does the the Fisher-KPP formula for the speed (3.23) hold? Below, we give a sufficient condition for this.

Proposition 3.3. *Assume that $f(u)$ satisfies (1.2) and, in addition, that there is a $C^1([0,1])$-function $\zeta(u)$ that satisfies*

$$\zeta(0) = \zeta(1) = 0, \quad \zeta'(0) = 1, \quad \zeta(u) > 0, \ \zeta'(u) < 2, \quad \text{for } 0 < u < 1, \tag{3.24}$$

and such that

$$f(u) \le f'(0)\zeta(u)(2 - \zeta'(u)), \quad \text{for all } 0 \le u \le 1. \tag{3.25}$$

Then, the minimal speed $c_*[f]$ is

$$c_*[f] = 2\sqrt{f'(0)}.$$

Proof. The first observation is that if $f(u)$ satisfies (1.2), then we may find a Fisher-KPP type nonlinearity $f_1(u) \le f(u)$ such that $f_1'(0) = f'(0)$. As $f_1(u)$ is of the Fisher-KPP type, we have

$$c_*[f_1] = 2\sqrt{f_1'(0)} = 2\sqrt{f'(0)}.$$

The comparison principle implies that

$$c_*[f] \geq c_*[f_1] = 2\sqrt{f'(0)}.$$

To show that

$$c_*[f] \leq 2\sqrt{f'(0)},$$

we note that because of (3.24), the function $p(u) = \lambda_* \zeta(u)$ can be used as test function in the Hadeler-Rothe variational principle (1.4). Here, we have set

$$\lambda_* = \sqrt{f'(0)}.$$

Using assumption (3.25), this gives

$$c_*[f] \leq \sup_{u\in[0,1]} \Big(\lambda_* \zeta'(u) + \frac{f(u)}{\lambda_* \zeta(u)}\Big) \leq \sup_{u\in[0,1]} \Big(\lambda_* \zeta'(u) + 2\lambda_* - \lambda_* \zeta'(u)\Big) = 2\sqrt{f'(0)},$$

finishing the proof. □

An important consequence of Proposition 3.3 is that the Fisher-KPP formula holds for the semi-FKPP and pushmi-pullyu nonlinearities (that is, (1.17) with $\chi < 1$ and $\chi = 1$, respectively) as can be seen by taking $\zeta(u) = u - A(u)$. Notice that the pushmi-pullyu nonlinearities are at the boundary of the validity of our condition for the Fisher-KPP formula.

Corollary 3.4. *If $f(u)$ is a semi-FKPP or pushmi-pullyu nonlinearity then $c_*[f] = 2\sqrt{f'(0)}$.*

Corollary 3.4 shows that, at the level of the propagation speed, we do not see a difference between semi-FKPP and pushmi-pullyu nonlinearities – both behave similarly to the Fisher-KPP type.

Proposition 3.3 allows to use explicit functions $\zeta(u)$ to verify the validity of the Fisher-KPP formula. For example, if we take $\zeta(u) = u(1-u)$, then assumption (3.25) becomes

$$f(u) \leq u(1-u)(1+2u).$$

It holds for nonlinearities of the form (3.6) exactly in the semi-FKPP range $0 \leq a \leq 2$. On the other hand, for

$$\zeta(u) = u(1-u^n),$$

the assumption (3.25) becomes

$$f(u) \leq u(1-u^n)(1+(n+1)u^n). \tag{3.26}$$

Nonlinearities of the form (3.7) satisfy (3.26) in the range $0 \leq a \leq n+1$, in agreement with the aforementioned results in [21].

4 Algebraic properties of semi-FKPP and pushmi-pullyu nonlinearities: the Cauchy problem

In this section, we discuss some special properties of the semi-FKPP and pushmi-pullyu nonlinearities. In particular, we introduce the shape defect function, establish a new variational formulation for reaction-diffusion equations, and explain a natural connection between the reaction-diffusion equations and reactive conservation laws.

4.1 Convergence in shape and the shape defect function

As we have discussed in the introduction, it was proved already in the original KPP paper [35] that the solution $u(t,x)$ to (1.1) with the initial condition $u(0,x) = \mathbb{1}(x \le 0)$ converges in shape to a minimal speed traveling wave, in the sense that (1.7) holds: there exists a reference frame $m(t)$ such that

$$|u(t, x + m(t)) - U_*(x)| \to 0, \quad \text{as } t \to +\infty \text{, uniformly in } x \in \mathbb{R}.$$

Recall that we have defined in the introduction the traveling wave shape defect function (or shape defect function, for short) as

$$w(t,x) = -u_x(t,x) - \eta_c(u(t,x)).$$

In particular, if $u(t,x) = U_c(t,x)$ then $w(t,x) \equiv 0$ because the traveling wave $U_c(x)$ satisfies (1.12). However, we also have $w(t,x) \equiv 0$ if $u(t,x) \equiv 0$ or $u(t,x) \equiv 1$.

Let us note that if $w(t,x) > 0$ for all $x \in \mathbb{R}$ then $u(t,x)$ is steeper than the traveling wave profile $U_c(x)$. Here, we use the notion of steepness from [2, 31]: if $u_{1,2}(x)$ are two monotonically decreasing functions, then u_1 is steeper than u_2 if for every $u \in \mathbb{R}$ such that there exist $x_{1,2} \in \mathbb{R}$ such that $u_1(x_1) = u_2(x_2) = u$, we have $|u_1'(x)| > |u_2'(x)|$. Actually, a stronger relationship holds: among functions u connecting 1 at $x = -\infty$ and 0 at $x = +\infty$, u is as steep as U_* if and only if $w \ge 0$.

A direct computation, using (1.1) and the representation (3.8) for $f(u)$ shows that the shape defect function satisfies

$$\begin{aligned} w_t - w_{xx} &= -u_{xt} + u_{xxx} - \eta_c'(u)u_t + \eta_c'(u)u_{xx} + \eta_c''(u)u_x^2 \\ &= -(\eta_c(u)(c - \eta_c'(u)))_x - \eta_c'(u)(u_{xx} + \eta_c(u)(c - \eta_c'(u))) + \eta_c'(u)u_{xx} + \eta_c''(u)u_x^2 \\ &= -\eta_c'(u)(c - \eta_c'(u))u_x + \eta_c(u)\eta_c''(u)u_x - \eta_c(u)\eta_c'(u)(c - \eta_c'(u)) + \eta_c''(u)u_x^2 \\ &= \eta_c''(u)u_x(u_x + \eta_c(u)) - \eta_c'(u)(c - \eta_c'(u))(u_x + \eta_c(u)) = -(\eta_c''(u)u_x - \eta_c'(u)(c - \eta_c'(u)))w \\ &= \big(\eta_c''(u)(w + \eta_c(u)) + \eta_c'(u)(c - \eta_c'(u))\big)w. \end{aligned} \tag{4.1}$$

The maximum principle then implies that

$$\text{if } w(0,\cdot) \ge 0 \quad \text{then} \quad w(t,\cdot) \ge 0 \quad \text{for all } t > 0. \tag{4.2}$$

This is the preservation of steepness property of KPP: if the initial condition $u(0,x)$ is steeper than a traveling wave, it remains steeper than the wave for all $t > 0$. We use this property extensively throughout the paper.

A variational formulation in terms of the shape defect function

An interesting observation we have mentioned in the introduction is that the shape defect function provides an energy for the reaction-diffusion equation (1.1). Consider this equation in the moving frame in view of the identity (3.8):

$$u_t - cu_x = u_{xx} + \eta_c(u)(c - \eta_c'(u)), \tag{4.3}$$

and define the energy functional

$$\mathcal{E}_c(u) = \frac{1}{2}\int_{\mathbb{R}} e^{cx}(u_x + \eta_c(u))^2 dx. \tag{4.4}$$

Let us compute

$$\begin{aligned} \frac{\delta \mathcal{E}_c}{\delta u} &= -\frac{\partial}{\partial x}\Big(e^{cx}(u_x + \eta_c(u))\Big) + e^{cx}(u_x + \eta_c(u))\eta_c'(u) \\ &= e^{cx}\big(-u_{xx} - \eta_c'(u)u_x - cu_x - c\eta_c(u) + u_x\eta_c'(u) + \eta_c(u)\eta_c'(u)\big) \\ &= -e^{cx}\big(u_{xx} + cu_x + \eta_c(u)(c - \eta_c'(u))\big). \end{aligned} \tag{4.5}$$

Therefore, equation (4.3) has a variational formulation

$$\frac{\partial u}{\partial t} = -e^{-cx}\frac{\delta \mathcal{E}_c}{\delta u}.$$

As a consequence, it follows that if $u(t,x)$ is a solution to (4.3), then

$$\frac{d\mathcal{E}_c(t)}{dt} \le 0,$$

with a strict inequality unless $u = U_c(x)$, $u \equiv 0$ or $u \equiv 1$.

Other variational formulations for reaction-diffusion equations have been considered; see, for example, in [29, 38, 43, 44, 49]. The energy functional considered in those papers is

$$\tilde{\mathcal{E}}_c[u] = \int_{\mathbb{R}} e^{cx}\Big(\frac{1}{2}u_x^2 + F(u)\Big)dx. \tag{4.6}$$

Here, F is the anti-derivative of $-f$: $F'(u) = -f(u)$. One issue with the functional $\tilde{\mathcal{E}}_c[u]$ is that it is not defined for $u(t,x) = U_c(x)$ unless $2\lambda_c > c$. This essentially restricts its use to pushed fronts, that is, those that propagate at speed $c = c_*$ where $c_* > 2\sqrt{f'(0)}$, as these pushed waves have decay rate given by

$$\lambda_c = \frac{c + \sqrt{c^2 - 4f'(0)}}{2} > \frac{c}{2}.$$

On the other hand, the functional $\mathcal{E}_c[u]$ vanishes if $u(t,x) = U_c(x)$ and is thus well-defined, regardless of the pushed or pulled nature of the traveling wave. In addition, it coincides with $\tilde{\mathcal{E}}_c[u]$ for sufficiently rapidly decaying solutions. To see this, let us set

$$N_c(u) = \int_0^u \eta_c(u')du',$$

and write

$$\begin{aligned}
\mathcal{E}_c[u] &= \frac{1}{2}\int_{\mathbb{R}} e^{cx}(u_x + \eta_c(u))^2 dx = \frac{1}{2}\int_{\mathbb{R}} e^{cx}(u_x^2 + 2u_x\eta_c(u) + \eta_c^2(u))dx \\
&= \frac{1}{2}\int_{\mathbb{R}} e^{cx}(u_x^2 + 2(N_c(u))_x + \eta_c^2(u))dx = \frac{1}{2}\int_{\mathbb{R}} e^{cx}(u_x^2 - 2cN_c(u) + \eta_c^2(u))dx \\
&= \int_{\mathbb{R}} e^{2x}\Big(\frac{1}{2}u_x^2 + V_c(u)\Big)dx,
\end{aligned}$$

where, we have defined

$$V_c(u) = -cN_c(u) + \frac{1}{2}\eta_c^2(u).$$

However, $V_c(u)$ is an anti-derivative of $(-f(u))$ because

$$V_c'(u) = -c\eta_c(u) + \eta_c(u)\eta_c'(u) = -\eta_c(u)(c - \eta_c'(u)) = -f(u).$$

This agrees with (4.6), so that $\mathcal{E}_c[u]$ coincides with $\tilde{\mathcal{E}}_c[u]$ when both are defined.

Let us comment that the energy (4.4) can be generalized in a natural way to dimensions $d > 1$. Let $c \in \mathbb{R}$ be a speed such that a traveling wave solution to the one-dimensional problem (1.1) exists. Consider the corresponding reaction-diffusion equation in $\mathbb{R}^d$, in a moving frame, going in a direction $e \in \mathbb{S}^{n-1}$, with $\|e\| = 1$, at the speed c:

$$u_t - ce\cdot\nabla u = \Delta u + f(u), \quad t > 0,\ x \in \mathbb{R}^d. \tag{4.7}$$

Given $e \in \mathbb{S}^{n-1}$ and c, consider the energy

$$\mathcal{E}_{c,e}[u] = \frac{1}{2}\int e^{c(e\cdot x)}|\nabla u + e\eta_c(u)|^2 dx.$$

Then, an essentially verbatim computation to (4.5) shows that

$$\begin{aligned}
\frac{\delta \mathcal{E}_{c,e}}{\delta u} &= -\sum_{k=1}^{d} \frac{\partial}{\partial x_k}\Big(e^{c(e\cdot x)}\Big(\frac{\partial u}{\partial x_k} + e_k\eta_c(u)\Big)\Big) + e^{c(e\cdot x)}\sum_{k=1}^{d} e_k\Big(\frac{\partial u}{\partial x_k} + e_k\eta_c(u)\Big)\eta_c'(u)\\
&= e^{c(e\cdot x)}\big(- \Delta u - \eta_c'(u)(e\cdot\nabla u) - ce\cdot\nabla u - c\eta_c(u) + \eta_c'(u)(e\cdot\nabla u) + \eta_c(u)\eta_c'(u)\big)\\
&= -e^{c(e\cdot x)}\big(\Delta u + ce\cdot\nabla u + \eta_c(u)(c-\eta_c'(u))\big) = -e^{c(e\cdot x)}(\Delta u + ce\cdot\nabla u + f(u)).
\end{aligned}$$

It follows that (4.7) has a variational formulation

$$\frac{\partial u}{\partial t} = -e^{-c(e\cdot x)}\frac{\delta \mathcal{E}_{c,e}}{\delta u}.$$

In particular, we have

$$\frac{d\mathcal{E}_c(t)}{dt} \le 0. \tag{4.8}$$

A strict inequality holds in (4.8) unless $u(t,x) = U_c(x\cdot e - ct)$, $u \equiv 0$ or $u \equiv 1$.

We are not going to pursue this variational direction in the present paper, but this approach seems to make the variational tools of [29, 49] available for a larger class than the bistable equations considered in the aforementioned papers. In particular, it opens the door to a variational analysis of Fisher-KPP type equations.

4.2 Connection to reactive conservation laws for pushmi-pullyu nonlinearities

The common traveling wave profiles

We describe a new connection between reaction-diffusion equations and reactive conservation laws provided by the shape defect function. Let us assume that the nonlinearity $f(u)$ is of pushmi-pullyu type (recall (3.3)):

$$f(u) = \lambda_*^2(u - A(u))(1 + A'(u)),$$

with some $\lambda_* > 0$. Recall that in this case the wave profile function is $\eta_*(u) = \zeta(u)$, as in (3.17). To make the notation less heavy we assume without loss of generality that $\lambda_* = 1$ and $c_* = 2$. We also let $U(x)$ be the corresponding minimal speed traveling wave profile, the solution to (1.12):

$$-U' = U - A(U), \quad U(-\infty) = 1, \quad U(+\infty) = 0, \tag{4.9}$$

and

$$-2U' = U'' + f(U). \tag{4.10}$$

Let us write

$$-2U' + (A(U))' - U'' = -2U' + U' - (\zeta(U))' - U'' = -U' = U - A(U).$$

Thus, apart from (4.10), the solution to (4.9) is also a traveling wave solution to the reactive conservation law

$$u_t + (A(u))_x = u_{xx} + u - A(u). \tag{4.11}$$

In other words, if $f(u)$ is a pushmi-pullyu type nonlinearity, then the reactive conservation law (4.11) and the reaction-diffusion equation

$$u_t = u_{xx} + f(u), \tag{4.12}$$

with

$$f(u) = (u - A(u))(1 + A'(u)), \tag{4.13}$$

have exactly the same minimal speed traveling wave profiles.

Comparison to reactive conservation laws

The connection between the reactive conservation law (4.11) and the reaction-diffusion equation (4.12) goes beyond the common traveling wave profile. Let u be a solution to (4.12)-(4.13) and, as usual, assume that $A(u)$ satisfies (2.10) and is increasing:

$$A'(u) \geq 0 \quad \text{ for all } u \in [0, 1]. \tag{4.14}$$

We claim that if the shape defect function is non-negative:

$$w(t, x) = -u_x(t, x) - \eta_*(u(t, x)) \geq 0 \quad \text{ for all } x \in \mathbb{R} \text{ and } t \geq 0, \tag{4.15}$$

then $u(t, x)$ is a subsolution to the reactive conservation law (4.11):

$$u_t + (A(u))_x \leq u_{xx} + u - A(u). \tag{4.16}$$

Therefore, we can use the comparison principle to bound the solution to the reaction-diffusion equation (4.12) from above by the solution to the reactive conservation law (4.11). This is used extensively below. Let us recall that we have shown that (4.15) holds at all times $t > 0$ as long as it is satisfied at $t = 0$. Thus, (4.15) is at most a restriction on the initial condition.

To show that (4.16) holds, we use (3.17) to write

$$\begin{aligned} u_t + A'(u)u_x - u_{xx} - u + A(u) &= (u - A(u))(1 + A'(u)) + A'(u)u_x - u + A(u) \\ &= (u - A(u))(1 + A'(u)) + A'(u)(-w - \eta_*(u)) - u + A(u) \\ &= u - A(u) + A'(u)(u - A(u)) - wA'(u) - A'(u)(u - A(u)) - u + A(u) \\ &= -wA'(u) \leq 0, \end{aligned}$$

because of (4.14) and (4.15).

In the general case, if (4.15) does not hold, so that the shape defect function is not positive everywhere, a solution $u(t, x)$ to (4.12)-(4.13) satisfies the forced reactive conservation law

$$u_t + (A(u))_x = u_{xx} + u - A(u) - A'(u)w, \quad w = -u_x - u + A(u).$$

The shape defect function for reactive conservation laws

We note that the shape defect function can be defined for solutions of this reactive conservation law in the same manner. For any solution $u_{\rm rcl}$ to (4.11), let

$$w_{\rm rcl}(t, x) = -\partial_x u_{\rm rcl}(t, x) - \eta_*(u_{\rm rcl}(t, x)), \tag{4.17}$$

where we have used the 'rcl' subscript to distinguish $w_{\rm rcl}$ from the shape defect function for solutions to (4.12) and we have changed to the ∂ notation in order to avoid the awkward double subscript. Let us recall that if $f(u)$ is a pushmi-pullyu nonlinearity, then the minimal speed traveling wave profiles for the reaction-diffusion equation (4.12) and the reactive conservation law (4.11) are the same. This allows us to use the same wave profile function $\eta_*(u)$ both in the definition (4.15) of the shape defect function for the reaction-diffusion equation and in (4.17). In this case, we find

$$
\begin{aligned}
\partial_t w_{\rm rcl} - \partial_x^2 w_{\rm rcl} &= \big[-\partial_{xt} u_{\rm rcl} - \partial_t u_{\rm rcl}(1 - A'(u_{\rm rcl}))\big] + \big[\partial_x^3 u_{\rm rcl} + \partial_x^2 u_{\rm rcl} - A(u_{\rm rcl})_{xx}\big] \\
&= -\partial_x^3 u_{\rm rcl} + A(u_{\rm rcl})_{xx} - \partial_x u_{\rm rcl} + A(u_{\rm rcl})_x + (-\partial_x^2 u_{\rm rcl} + A(u_{\rm rcl})_x \\
&\qquad - u_{\rm rcl} + A(u_{\rm rcl}))(1 - A'(u_{\rm rcl})) + \partial_x^3 u_{\rm rcl} + \partial_x^2 u_{\rm rcl} - A(u_{\rm rcl})_{xx} \\
&= -\partial_x u_{\rm rcl} + A(u_{\rm rcl})_x + (-\partial_x u_{\rm rcl} + A(u_{\rm rcl})_x - u_{\rm rcl} + A(u_{\rm rcl}))(1 - A'(u_{\rm rcl})) + \partial_x^2 u_{\rm rcl} \\
&= -\partial_x u_{\rm rcl} + A(u_{\rm rcl})_x + \partial_x^2 u_{\rm rcl} A'(v) + A'(u_{\rm rcl})\partial_x u_{\rm rcl}(1 - A'(u_{\rm rcl})) \\
&\qquad - u_{\rm rcl}(1 - A'(u_{\rm rcl})) + A(u_{\rm rcl})(1 - A'(u_{\rm rcl})) \\
&= w_{\rm rcl}(1 - A'(u_{\rm rcl})) - A'(u_{\rm rcl})\partial_x w_{\rm rcl}.
\end{aligned}
\tag{4.18}
$$

While the form of the equation (4.18) for $w_{\rm rcl}$ is different from that of (4.1) for w, it still preserves positivity. We conclude that

$$
\text{if } w_{\rm rcl}(0,\cdot) \geq 0 \quad \text{then} \quad w_{\rm rcl}(t,\cdot) \geq 0 \qquad \text{for all } t \geq 0.
$$

4.3 The weighted Hopf-Cole transform

To finish this section, we introduce the weighted Hopf-Cole transform for pushmi-pullyu and semi-FKPP nonlinearities. It generalizes a similar transformation for the Burgers-FKPP equation considered in [2]. Let us consider a reaction-diffusion equation in the frame $x \to x - 2t$, with a semi-FKPP or pushmi-pullyu nonlinearity of the form

$$
u_t - 2u_x = u_{xx} + f(u), \tag{4.19}
$$

where f has the form

$$
f(u) = \zeta(u)(1 + \chi A'(u)),
$$

with $0 \leq \chi \leq 1$ and $\zeta(u)$ related to $A(u)$ by $\zeta(u) = u - A(u)$. The function $A(u)$ is convex and satisfies the familiar assumptions (2.10) and (4.14). Thus, the nonlinearity in (4.19) is of semi-FKPP type if $0 \leq \chi < 1$ and of pushi-pullyu type if $\chi = 1$.

As in (2.11), we set

$$
\alpha(u) = \frac{A(u)}{u},
$$

so that

$$
\zeta(u) = u(1 - \alpha(u)).
$$

We define the weighted Hopf-Cole transform via

$$
v(t,x) = \exp\Big(x + \sqrt{\chi}\int_x^\infty \alpha(u(t,y))dy\Big)u(t,x). \tag{4.20}
$$

Our goal is to show the following.

Proposition 4.1. *Let $u(t,x)$ be a solution to (4.19), with the above assumptions on the functions $\zeta(u)$ and $A(u)$, and $0 \leq \chi \leq 1$. Assume, in addition, that $\alpha(u)$ is convex and increasing on $[0,1]$. Suppose also that the shape defect function satisfies $w(0,x) \geq 0$ for all $x \in \mathbb{R}$. Then, the function $v(t,x)$ defined by (4.20) is a subsolution to the heat equation:*

$$
v_t - v_{xx} \leq 0.
$$

Proof. We use the notation

$$\Gamma(t,x) = x + \sqrt{\chi}\int_x^\infty \alpha(u(t,y))dy$$

for short, and utilize the following computations

$$v_x = e^\Gamma u_x + (1-\sqrt{\chi}\alpha(u))e^\Gamma u,$$

and

$$v_{xx} = e^\Gamma u_{xx} + 2(1-\sqrt{\chi}\alpha(u))e^\Gamma u_x - \sqrt{\chi}\alpha'(u)e^\Gamma u u_x + (1-\sqrt{\chi}\alpha(u))^2 e^\Gamma u,$$

as well as

$$\begin{aligned}
v_t &= e^\Gamma u_t + \Big(\sqrt{\chi}\int_x^\infty \alpha'(u(t,y))u_t(t,y)dy\Big)e^\Gamma u\\
&= e^\Gamma u_t + \Big(\sqrt{\chi}\int_x^\infty \alpha'(u(t,y))(u_{yy} + \zeta(u)(1+\chi A'(u)) + 2u_y)dy\Big)e^\Gamma u\\
&= e^\Gamma u_t + \sqrt{\chi}e^\Gamma u\Big(-\alpha'(u)u_x - \int_x^\infty \alpha''(u)u_y^2dy - 2\alpha(u) + \int_x^\infty \alpha'(u)\zeta(u)(1+\chi A'(u))dy\Big).
\end{aligned}$$

Here we used that $A \in C^2$. Next, we write

$$\begin{aligned}
e^{-\Gamma}\big(v_t - v_{xx}\big) &= u_t - u_{xx} - 2u_x + 2\sqrt{\chi}\alpha(u)u_x + \sqrt{\chi}\alpha'(u)uu_x - (1-\sqrt{\chi}\alpha(u))^2u\\
&\quad + \sqrt{\chi}u\Big(-\alpha'(u)u_x - \int_x^\infty \alpha''(u)u_y^2dy - 2\alpha(u) + \int_x^\infty \alpha'(u)\zeta(u)(1+\chi A'(u))dy\Big),
\end{aligned}$$

which is

$$\begin{aligned}
e^{-\Gamma}\big(v_t - v_{xx}\big) &= f(u) - u - \chi\alpha(u)^2u + 2\sqrt{\chi}\alpha(u)u_x\\
&\qquad + \sqrt{\chi}u\Big(-\int_x^\infty \alpha''(u)u_y^2dy + \int_x^\infty \alpha'(u)\zeta(u)(1+\chi A'(u))dy\Big).
\end{aligned} \tag{4.21}$$

By assumption, the function $\alpha(u)$ is increasing and convex. In addition, as $w(0,x) \ge 0$ for all $x \in \mathbb{R}$, we know that $w(t,x) \ge 0$ for all $t > 0$ and $x \in \mathbb{R}$. Positivity of $w(t,x)$ implies that $u_x(t,x) < 0$ for all $t > 0$ and $x \in \mathbb{R}$. We also note that, after using Lemma A.4,

$$\sqrt{\chi}\zeta(u) \le \eta_*(u) = -u_x - w \le -u_x. \tag{4.22}$$

With these ingredients in hand, we can estimate the first integral in the right side of (4.21) as

$$\begin{aligned}
-\sqrt{\chi}\int_x^\infty \alpha''(u)u_y^2dy &= -\sqrt{\chi}\int_x^\infty \alpha''(u)(-u_y)(-u_y)dy\\
&\le -\chi\int_x^\infty \alpha''(u)\zeta(u)(-u_y)dy.
\end{aligned} \tag{4.23}$$

Therefore, after integrating by parts, we have

$$-\sqrt{\chi}\int_x^\infty \alpha''(u)u_y^2dy \le -\chi\int_x^\infty \alpha''(u)\zeta(u)(-u_y)dy = -\chi\alpha'(u)\zeta(u) - \chi\int_x^\infty \alpha'(u)\zeta'(u)u_ydy.$$

To estimate the second integral in the right side of (4.21), we use (4.22) again:

$$\begin{aligned}
\sqrt{\chi}\int_x^\infty \alpha'(u)\zeta(u)(1+\chi A'(u))dy &\le -\int_x^\infty \alpha'(u)(1+\chi A'(u))u_ydy\\
&= \alpha(u) - \chi\int_x^\infty \alpha'(u)(1-\zeta'(u))u_ydy = \alpha(u) + \chi\alpha(u) + \chi\int_x^\infty \alpha'(u)\zeta'(u)u_ydy.
\end{aligned} \tag{4.24}$$

Combining (4.23) and (4.24) gives

$$\sqrt{\chi}u\Big(-\int_x^\infty \alpha''(u)u_y^2dy+\int_x^\infty \alpha'(u)\big(\zeta(u)(1+\chi A'(u))\big)dy\Big)\le \chi\alpha(u)u+\alpha(u)u-\chi\alpha'(u)\zeta(u)u.$$

Going back to (4.21) and recalling that $\zeta(u)=u(1-\alpha(u))$, and

$$f(u)=\zeta(u)(1+\chi(u\alpha'(u)+\alpha(u))),$$

we obtain

$$\begin{aligned}e^{-\Gamma}\big(v_t-v_{xx}\big)&\le f(u)-u-\chi\alpha(u)^2u+2\sqrt{\chi}\alpha(u)u_x+\chi\alpha(u)u+\alpha(u)u-\chi u\alpha'(u)\zeta(u)\\&=f(u)-u(1-\alpha(u))+\chi\alpha(u)u(1-\alpha(u))-\chi u\alpha'(u)\zeta(u)+2\sqrt{\chi}\alpha(u)u_x\\&=f(u)-\zeta(u)-\chi\alpha(u)\zeta(u)+2\chi\alpha(u)\zeta(u)-\chi u\alpha'(u)\zeta(u)+2\sqrt{\chi}\alpha(u)u_x\\&=2\sqrt{\chi}\alpha(u)(\sqrt{\chi}\zeta(u)+u_x)\le 0.\end{aligned}$$

We used (4.22) once again, as well as the positivity of $w(t,x)$ in the last line above. □

5 Discussion of the proofs of Theorems 1.3 and 1.4

5.1 Generalities

The first step in the proof of both of the convergence results in Theorem 1.3 is to identify the candidates for the coefficient r_χ and the shift x_0 in first three terms in the expansion

$$m(t)=2t-r_\chi\log t+x_0+o(1),\quad \text{as } t\to+\infty. \tag{5.1}$$

This is achieved by analyzing the precise tail behavior of the solution at the position $x=m(t)+t^\gamma$, with a small $\gamma>0$. The second step is to argue that, after shifting the traveling wave so that it matches the solution $u(t,x)$ at the position

$$x_\gamma(t)=2t-r_\chi\log t+t^\gamma, \tag{5.2}$$

the shifted wave and the solution are close not just at $x_\gamma(t)$ but everywhere to the left of it as well. This general strategy has been introduced for the Fisher-KPP nonlinearities in [32, 34, 46, 47], and is a manifestation of the pulled nature of the propagation: behavior at the front is controlled by the behavior far ahead of the front. In the Fisher-KPP case considered in the above references, the second step is relatively straightforward, and relies heavily on the Fisher-KPP property of the nonlinearity. More precisely, it allowed to control the sign of the principal eigenvalue for a certain half-line problem that ultimately induces the convergence. This gives a quantitative way to look at the pulled nature of the Fisher-KPP type equations.

As both the semi-FKPP and pushmu-pullyu nonlinearities do not satisfy the Fisher-KPP property, genuinely new ingredients are required in both steps of this approach. The types of difficulties that arise, and where they arise, are quite different in the semi-FKPP and pushmi-pullyu cases. We now briefly discuss the particulars of each case in greater detail.

5.2 Semi-FKPP fronts

First step: identifying a candidate expansion (5.1). The first step is to identify the coefficient $r_\chi=3/2$ in (5.1) and the constant x_0 for semi-FKPP nonlinearities. Here, the lack of the Fisher-KPP property of the nonlinearity is compensated by a use of the miracle of the weighted Hopf-Cole transform (4.20). While somewhat mysterious, it makes this step relatively straightforward, allowing to use the modification of the strategy of [34, 46, 47], introduced in [2], even though the nonlinearity is not of the Fisher-KPP type. That is, the weighted Hopf-Cole transform overcomes the absence of the Fisher-KPP property, to an extent, as it did in [2] for the Burgers-FKPP equation.

Second step: matching the traveling wave far ahead of the front and tracing back. Next, we use the precise information about $u(t,x)$ at the position $x_\gamma(t)$ given by (5.2) with $r_\chi = 3/2$ obtained in the first step, to match $u(t,x)$ to a shift of the traveling wave U_* at $x = x_\gamma(t)$. The argument of [34, 46, 47] proceeds by taking the difference $s = u - U_*$ and using the Fisher-KPP property of the nonlinearity to show that $s(t,x)$ satisfies a Dirichlet problem with a time-decaying solution. The time decay comes about because the zero-order term has a good sign and $x_\gamma(t)$ is not too large. In the Burgers-FKPP situation of [2], this was adapted to the difference $s = \tilde{u} - \psi$, where $\tilde{u}$ is the appropriate shift of the weighted Hopf-Cole transform of u and ψ is the weighted Hopf-Cole transform of U_*. This modification still relied heavily on the fact that the nonlinearity in the Burgers-FKPP equation is of the Fisher-KPP type. Here, this 'Hopf-Cole modified' argument fails, due to a term that does not have a sign because of the lack of the Fisher-KPP property. However, by a series of intricate manipulations, surprisingly, we manage to extract a term that is sufficiently positive, as long as the nonlinearity is precisely of the semi-FKPP type. With this, we obtain an upper bound for s that tends to zero. The main novelty here is in the proofs of Lemmas 7.2 and 7.3 that require extremely delicate cancellations. These computations quantify the pulled nature of semi-FKPP fronts.

The proof of Theorem 1.3.(i) is presented in Section 7 with the first step contained in Section 7.1 and the second step in Section 7.2.

5.3 Pushmi-pullyu fronts

First step: identifying the front location to precision $O(1)$. For the pushmi-pullyu fronts, we need to separate the identification of the constants r_χ and x_0 in (5.1) into two steps; that is, we first obtain the asymptotics of the front location as

$$m(t) = 2t - \tfrac{1}{2}\log t + O(1), \quad \text{as } t \to +\infty \tag{5.3}$$

and then we bootstrap (5.3) to find the precise $O(1)$ term.

For the proof of (5.3), the connection between the reaction-diffusion equation and the reactive conservation law is crucial. By (4.16), it is enough to obtain a lower bound on the reaction diffusion equation and an upper bound on the reactive conservation law. The lower bound in (5.3) can be obtained via a suitable estimate on an exponential moment, inspired by the Fabes-Stroock proof of the heat kernel bounds in [24].

For the upper bound, we apply the methods developed in [2], which we briefly outline. First, after passing to the moving frame $x \mapsto 2t + x$, a change of function $p = e^x u$ is made, resulting in a seemingly simple inhomogeneous conservation law

$$p_t + (\alpha(u)p)_x = p_{xx}, \tag{5.4}$$

where we recall that $\alpha(u) = A(u)/u$. At this point, it is enough to show that

$$\|p\|_\infty \le O(1/\sqrt{t})$$

in order to conclude the front is behind $2t - (1/2)\log t$ due to the definition of p. When $A(u) = u^2$, one can show that $\rho = 1 - u$ is a supersolution to (5.4) so that the relative entropy methods of [19, 42] can be extended and applied to find

$$\frac{d}{dt}\int \left(\frac{p}{\rho}\right)^2 \rho\, dx \le -C\int \left(\frac{p}{\rho}\right)_x^2 \rho\, dx.$$

A suitable Nash-type inequality with a dynamic weight is developed in [2] for the measure $d\mu_\rho = \rho\, dx$ allowing us to obtain $O(1/t^{1/4})$-decay of the $L^2_{\mu_\rho}$-norm of p/ρ. Bootstrapping this to $O(1/\sqrt{t})$-decay of the L^∞-norm of p requires an intricate argument in which one finds a "good" time $t_{\text{good}} \approx t$ where

$$\|p(t_{\text{good}}, \cdot)\|_\infty \le O(1/\sqrt{t}),$$

and then "trapping" that norm using that (5.4) conserves mass and enjoys a comparison principle.

In the context where $A(u) \neq u^2$, one must first determine an appropriate ρ that is a supersolution to (5.4) and satisfies the assumptions of the aforementioned Nash-type inequality. As we detail below, the appropriate choice, via yet another surprising piece of algebra, turns out to be

$$\rho(t,x) = \exp\Big(-\int_0^{u(t,x)} \frac{\alpha(u')}{u' - A(u')} du' \Big). \tag{5.5}$$

One can check that, in the special case

$$A(u) = u\alpha(u) = u^2,$$

the formula (5.5) simplifies to $\rho = 1 - u$, while in the case $A(u) = u^n$ (recall (3.7), which was introduced in [21]), it has the form

$$\rho = (1 - u^{n-1})^{1/(n-1)}.$$

However, it does not, in general, have such a simple form.

One can then check that such $\rho(t,x)$ satisfies the assumptions of the Nash-type inequality [2, Proposition 5.9] associated to the measure $d\mu_\rho = \rho\, dx$ in a uniform way (that is, independent of t). From here, the proof proceeds as in [2]. The proof of this step is contained in Section 6.2.

Second step: identifying the front location to precision $o(1)$. The approach of [2] to identifying the constant term x_0 in $m(t)$ is to apply the weighted Hopf-Cole transform, change to self-similar variables, and show convergence to a constant multiple of the principal eigenfunction of the resulting operator. This proceeds in a straightforward way for the solution to the reaction-diffusion equation (1.1) because the resulting function v is a subsolution to the heat equation (see Proposition 4.1). Unfortunately, this is not true for the solution to the reactive conservation law (1.24); indeed, passing to the moving frame $x \mapsto 2t + x$ and letting

$$v(t,x) = \exp\Big(x + \int_x^\infty \alpha(u(t,y))dy\Big) u(t,x),$$

we have

$$v_t - v_{xx} \le \frac{v}{u}(u\alpha'(u) - \alpha(u)) w_{\text{rcl}}. \tag{5.6}$$

We note that the Burgers-FKPP case considered in [2], in which $\alpha(u) = u$, is the unique case in which the right hand side vanishes. Changing to self-similar variables $t = e^\tau$, $y = e^{\tau/2}x$, the right hand side accumulates an e^τ multiplicative factor, making the right hand side not only positive but, potentially, large. The proof is saved, however, by showing that

$$w_{\text{rcl}} \le \frac{Cxe^{-x}}{t}, \tag{5.7}$$

in Lemma 6.6, controlling the right side of (5.6) by, in self-similar coordinates, by a bounded function that tends to zero in $L^p(0,\infty)$ for all $1 \leq p < +\infty$. The proof of this bound is achieved by the construction of a somewhat complicated supersolution (see Lemma 6.6).

These estimates provide a sharp estimate of the behavior of u at $x = O(t^\gamma)$ for $\gamma > 0$. The precise shift of the front x_0 is then identified via:

$$U_*(-x_0 + t^\gamma) = u\big(t, 2t - \tfrac{1}{2}\log t + t^\gamma\big) + o(e^{-t^\gamma}). \tag{5.8}$$

The step is treated in Section 6.3.

Third step: using the pulled nature of the problem. To upgrade the sharp estimate (5.8) at the position $x \sim t^\gamma$ to full convergence we can use the smallness and decay of w. This is done as follows. Let $\delta(x) = u(t, x + m(t)) - U(x)$ and notice that

$$-\delta_x - \delta\eta_*'(u) \approx -\delta_x - \delta\frac{\eta_*(u(t,\cdot + m(t))) - \eta_*(U)}{u(t,\cdot+m(t)) - U} = w.$$

Thus, ahead of the front we have

$$-\delta_x \approx \delta\eta'(0) + w = \delta + w,$$

which can be approximately solved:

$$e^x\delta(x) \approx \delta(t^\gamma)e^{t^\gamma} + \int_x^{t^\gamma} e^y w(y)dy.$$

The first term on the right is small due to (5.8). This reflects the pulled nature of the pushmi-pullyu case: u and U must be "very" close at far ahead of the front (at $x = m(t) + t^\gamma$) in order to be close everywhere. The second term is small due to (5.7). This step is contained in Section 6.4.

6 Pushmi-pullyu fronts: proofs of Theorems 1.3.(ii) and 1.4

6.1 Outline of the proofs and notation

In this section, we prove part (ii) of Theorems 1.3 and 1.4, both of which concern the pushmi-pullyu fronts. We work simultaneously with the solutions u_{rd} and u_{rcl} to the reaction-diffusion equation (1.1) and the reactive conservation law (1.24), respectively. Due to the comparison (4.16), we have immediately that

$$u_{\text{rd}} \leq u_{\text{rcl}},$$

as long as $u_{\text{rd}}(0,x) = u_{\text{rcl}}(0,x)$ for all $x \in \mathbb{R}$. There is, thus, some efficiency in considering both at the same time. The notation u_{rd} and u_{rcl} is somewhat overwrought, so we use simply u when it is possible to do so without risk of confusion or when the argument applies to both u_{rd} and u_{rcl}. We also use this subscript notation for derived quantities, such as the shape defect functions w_{rd} and w_{rcl}.

Additionally, we need to work in two different moving frames. We use $\hat{u}$ to denote u in the $x \mapsto 2t + x$ moving frame and $\tilde{u}$ for u in the $x \mapsto 2t - (1/2)\log(t+1) + x$ moving frame:

$$\hat{u}(t,x) = u(t, x + 2t) \quad \text{and} \quad \tilde{u}(t,x) = u\big(t, x + 2t - \tfrac{1}{2}\log(t+1)\big). \tag{6.1}$$

To formalize the idea of the 'location of the front', we define the position $(-\mu(t))$ by:

$$\alpha(\tilde{u}(t, -\mu(t))) = \frac{1}{2}. \tag{6.2}$$

This specific normalization is chosen just for technical convenience (indeed, as $w \geq 0$, any two level sets remain a bounded distance away from each other at all times). Since $u_{\rm rd} \leq u_{\rm rcl}$, and the function $\alpha(u)$ is increasing, we immediately get

$$\mu_{\rm rcl} \leq \mu_{\rm rd}.$$

Let us recall that Proposition 3.1 implies that in the pushmi-pullyu case we have

$$\eta_*(u) = \zeta(u) = u(1-\alpha(u)), \tag{6.3}$$

a relation that we use repeatedly in this section.

The proof proceeds as follows. First, we establish the location of the front with the precision $O(1)$, in Section 6.2. The main result in that section is Proposition 6.1. At the heart of the proof is a relative entropy computation with respect to a supersolution and a dynamically weighted Nash inequality. Both of them are originating in the analysis of [2] for the Burgers-FKPP equation but the details in the general case are very different and involve some additional fortunate pieces of algebra in the construction of the supersolution in Lemma 6.2. This proof occupies the bulk of this section.

The $O(1)$ term in the front location is identified in Section 6.3. Its main result is Proposition 6.5. Its proof for the reaction-diffusion case relies on the weighted Hopf-Cole transform in Proposition 4.1. For the reactive conservation law, the weighted Hopf-Cole transform gives not a subsolution to the heat equation but only an approximate subsolution far on the right. This requires additional estimates on the error term. This part of the argument is discussed in Section 6.3.2.

The final step in the proof of Theorems 1.3.(ii) and 1.4, convergence to a single wave, is presented in Section 6.4. It is very also different from the corresponding step in [2, 46, 47] and uses the algebra of the pushmi-pullyu case.

6.2 The first step: the expansion $m(t) = 2t - \frac{1}{2}\log(t) + O(1)$ of the front location

As a preliminary observation, we note two bounds on $\tilde{u}$ viewed from the location $(-\mu(t))$ that follow immediately from the positivity of the shape defect function w:

$$1 - Ce^{\alpha'(1)x} \leq \tilde{u}(t, x - \mu(t)) \leq Ce^{-x} \qquad \text{for all } x \in \mathbb{R}. \tag{6.4}$$

To see this, we argue as follows. Let us normalize a translate of the minimal speed wave so that $\alpha(U_*(0)) = 1/2$, and set

$$s(t,x) = \tilde{u}(t,x) - U_*(x + \mu(t)).$$

Notice that

$$-s_x = \xi s + w \quad \text{ and } \quad s(t, -\mu(t)) = 0, \tag{6.5}$$

with

$$\xi(t,x) = \frac{\eta_*(\tilde{u}(t,x)) - \eta_*(U_*(x+\mu(t)))}{\tilde{u}(t,x) - U_*(x+\mu(t))}.$$

In view of (6.5) and the positivity of w, we have

$$s > 0 \quad \text{for } x < -\mu(t) \qquad \text{and} \qquad s < 0 \quad \text{for } x > -\mu(t).$$

From here, (6.4) follows directly from the fact that

$$1 - Ce^{\alpha'(1)x} \leq U_*(x) \leq Ce^{-x} \qquad \text{for all } x \in \mathbb{R}, \tag{6.6}$$

because

$$-U_*' = \eta_*(U_*) = U_*(1-\alpha(U_*)),$$

as seen from (6.3). We note also that the inequalities in (6.6) are sharp in the sense that

$$\lim_{x\to\infty} e^x U_*(x) \quad \text{and} \quad \lim_{x\to-\infty} e^{-\alpha'(1)x}(1-U_*(x)) \qquad \text{exist and are positive.}$$

The goal of this section is to prove the following:

Proposition 6.1. *There exist $\underline{L} \le \bar{L}$ and $T_0 > 0$ sufficiently large, so that for all $t > T_0$ we have*

$$\tilde{u}_{\mathrm{rd}}(t,\bar{L}) \le \tilde{u}_{\mathrm{rcl}}(t,\bar{L}) \overset{(i)}{\le} \alpha^{-1}\Big(\frac{1}{2}\Big) \overset{(ii)}{\le} \tilde{u}_{\mathrm{rd}}(t,\underline{L}) \le \tilde{u}_{\mathrm{rcl}}(t,\underline{L}). \tag{6.7}$$

As a consequence, we have the following bounds on $\mu(t)$:

$$\underline{L} \le \mu_{\mathrm{rcl}}(t) \le \mu_{\mathrm{rd}}(t) \le \bar{L}, \quad \textit{for all } t > T_0.$$

We begin with the inequality (i), that is, the upper bound on $\tilde{u}_{\mathrm{rcl}}$, as it is required for the proof of (ii), the lower bound on $\tilde{u}_{\mathrm{rd}}$. Due to the assumption (1.21) on the initial condition and the comparison principle, we may assume, without loss of generality, that

$$u_0(x) = \mathbb{1}(x<0).$$

6.2.1 Proof of Proposition 6.1.(i): the upper bound on $\tilde{u}_{\mathrm{rcl}}$

In this section, we work only with u_{rcl}. As there is no possibility of confusion, we simply refer to u_{rcl} as u below.

Changing to the moving frame $x \to x + 2t$, yields

$$\hat{u}_t - 2\hat{u}_x + (\alpha(\hat{u})\hat{u})_x = \hat{u}_{xx} + \hat{u}(1-\alpha(\hat{u})). \tag{6.8}$$

Making the change of variable

$$p(t,x) = e^x \hat{u}(t,x),$$

we see that to prove part (i) of (6.7), it is enough to show that

$$\|p(t,\cdot)\|_\infty \le \frac{C}{\sqrt{t}}. \tag{6.9}$$

This estimate is the key for the proof of both the upper and the lower bound in Proposition 6.1.

The function p solves the (inhomogeneous) scalar conservation law

$$p_t + (\alpha(\hat{u})p)_x = p_{xx}, \tag{6.10}$$

with the initial condition $p(0,x) = e^x u_0(x)$. It is clear that (6.10) conserves mass:

$$\int p(t,x)dx = \int p(0,x)dx.$$

This is a general form of the structure used in [2] in order to prove (6.9) in the special case $A(u) = u^2$, $\alpha(u) = u$, which was considered there. While the outline of that proof applies here, there are substantial changes that need to be made in order to suitably generalize it to the case considered here. We outline the main steps, making note of the major differences. To make the outline easier to follow, we use the same notation as in [2] as much as possible.

Step one: the relative entropy calculation

The basis of the proof in [2] is a generalization of the relative entropy ideas introduced in [42] (see also [19]) to supersolutions. Suppose that a function ρ satisfies

$$\rho_t + (\alpha(\hat{u})\rho)_x \geq \rho_{xx}, \tag{6.11}$$

and let

$$\varphi(t,x) = \frac{p(t,x)}{\rho(t,x)}.$$

Then, [2, Proposition 5.8] gives a dissipation inequality

$$\frac{d}{dt}\int \varphi^2(t,x)\rho(t,x)dx \leq -2\int \varphi_x^2(t,x)\rho(t,x)dx, \tag{6.12}$$

that holds for any initial condition $p_0 = e^x u_0$ for which the quantities above are finite.

Let us point out that, in the special case of the heat equation, where $\alpha \equiv 0$, and we can take $\rho \equiv 1$, the inequality (6.12), paired with the Nash inequality, gives us the $O(t^{1/4})$-decay of the L^2-norm. Our approach is analogous, although we appeal to the weighted Nash inequality with time-dependent weights introduced in [2].

In order to use (6.12), we need to find a supersolution ρ. This is provided by the following lemma, whose proof is postponed to the end of this section. Let us define an auxiliary function

$$F(u) = \exp\Big(-\int_0^u \frac{\alpha(u')}{\eta_*(u')}\,du'\Big) = \exp\Big(-\int_0^u \frac{u'-\zeta(u')}{u'\eta_*(u')}\,du'\Big). \tag{6.13}$$

Lemma 6.2. *The function F defined in (6.13) is $C^2_{\text{loc}}([0,1))$, satisfies $F(0)=1$ and $F(1)=0$, and is decreasing and concave. Moreover, for any $\varepsilon > 0$, there exists C_ε such that*

$$\frac{1}{C_\varepsilon}(1-\alpha(u))^{\frac{1+\varepsilon}{\alpha'(1)}} \leq F(u) \leq C_\varepsilon(1-\alpha(u))^{\frac{1}{(1+\varepsilon)\alpha'(1)}} \qquad \text{for all } u \in [0,1]. \tag{6.14}$$

Moreover, $\rho(t,x) := F(\hat{u}(t,x))$ satisfies (6.11) as long as $\hat{w} \geq 0$ and $\hat{u}$ satisfies (6.8).

Step two: modification of the initial condition

An important issue with the step-function initial condition $u(0,x) = \mathbb{1}(x \leq 0)$, that was also present in [2], is that, at the time $t = 0$,

$$\rho(0,x) = F(u(0,x)) = \mathbb{1}(x > 0),$$

so that

$$\int \varphi^2(0,x)\rho(0,x)\,dx = \int \Big(\frac{p(0,x)}{\rho(0,x)}\Big)^2 \rho(0,x)\,dx = \int \frac{e^{2x}\mathbb{1}(x \leq 0)}{\mathbb{1}(x > 0)}\,dx = +\infty.$$

Hence, the differential inequality (6.12) cannot be used directly with such initial condition.

We address this by modifying the initial data $\hat{u}(0,\cdot)$ to not take the value 1 anywhere, so that $\rho(0,\cdot)$ does not vanish anywhere. It it is important that the new initial condition is steeper than the traveling wave, so that the corresponding shape defect function w remains non-negative. While in [2] it was possible to simply write an explicit approximation of $\mathbb{1}(x \leq 0)$, here, we use the traveling wave: for fixed $\gamma \in (1, 4/3)$ and any $a > 0$, we let

$$\hat{u}_a(0,x) = U_*(\gamma(x-a))\mathbb{1}(x \leq 0).$$

While γ remains fixed throughout the proof, we eventually take the limit $a \to \infty$. As a result, all constants depend on γ, but it is important to track the dependence on a throughout.

One can check that

$$\hat{w}_a(t,x) = -\partial_x \hat{u}_a(t,x) - \eta_*(\hat{u}_a(t,x))$$

is positive. Indeed, by (4.1), it is enough to show that $\hat{w}_a(0,x) \geq 0$, which follows by:

$$\begin{aligned}
\hat{w}_a(0,x) &= -\partial_x \hat{u}_a(0,x) - \eta_*(\hat{u}_a(0,x)) \\
&= \big(-\gamma U_*'(\gamma(x-a)) - \eta_*(U_*(\gamma(x-a)))\big)\,\mathbb{1}(x<0) + U_*(-\gamma a)\delta_0(x) \\
&\geq \big(-\gamma U_*'(\gamma(x-a)) - \eta_*(U_*(\gamma(x-a)))\big)\,\mathbb{1}(x<0) \\
&= -(\gamma-1)U_*'(\gamma(x-a))\mathbb{1}(x<0) \geq 0.
\end{aligned}$$

Above we used that $\gamma > 1$ and that U_* is decreasing, positive, and satisfies $-U_*' = \eta_*(U_*)$.

With $\hat{u}_a$, we now define

$$p_a(t,x) = e^x \hat{u}_a(t,x), \quad \rho_a(t,x) = F(\hat{u}_a(t,x)), \quad \text{and} \quad \varphi_a(t,x) = \frac{p_a(t,x)}{\rho_a(t,x)}.$$

Here, F is defined by (6.13). Using Lemma 6.2 with the choice $\varepsilon = 1/2$ and then (6.6) yields

$$\begin{aligned}
\int \varphi_a^2(0,x)\rho_a(0,x)\,dx &= \int \Big(\frac{p_a(0,x)}{\rho_a(0,x)}\Big)^2 \rho_a(0,x)\,dx \leq C\int_{-\infty}^0 \frac{e^{2x}\,U_*(\gamma(x-a))^2}{(1-\alpha(U_*(\gamma(x-a))))^{\frac{3}{2\alpha'(1)}}}\,dx \\
&\leq C\int_{-\infty}^0 \frac{e^{2x}}{(1-U_*(\gamma(x-a)))^{\frac{3}{2\alpha'(1)}}}\,dx \leq C\int_{-\infty}^0 e^{(2-\frac{3\gamma}{2})x+\frac{3\gamma a}{2}}\,dx \leq Ce^{\frac{3\gamma a}{2}}.
\end{aligned} \tag{6.15}$$

In the last step, we used that $3\gamma/2 < 2$, by assumption. Thus, due to the finiteness of the quantity in (6.15), we have the differential inequality (6.12) at our disposal. This is crucial in the following steps.

Before proceeding, we discuss how a bound on p_a yields a bound on p. Let us define

$$h = p - p_a \quad \text{and} \quad v = \alpha(\hat{u}) + \hat{u}_a \frac{\alpha(\hat{u}) - \alpha(\hat{u}_a)}{\hat{u} - \hat{u}_a}.$$

It is straightforward to check that h is nonnegative at $t = 0$ and satisfies, for all $t > 0$ and $x \in \mathbb{R}$,

$$h_t + (vh)_x = h_{xx}.$$

This equation conserves mass and preserves nonnegativity. Thus, $h \geq 0$ and

$$\begin{aligned}
\int h(t,x)\,dx &= \int h(0,x)\,dx = \int e^x\left(\hat{u}(0,x) - \hat{u}_a(0,x)\right)\,dx = \int_{-\infty}^0 e^x\left(1 - U_*(\gamma(x-a))\right)\,dx \\
&\leq C\int_{-\infty}^0 e^{(1+\gamma\alpha'(1))x - \gamma\alpha'(1)a}\,dx \leq Ce^{-\gamma\alpha'(1)a}.
\end{aligned}$$

In the second-to-last inequality, we applied (6.6) again. By parabolic regularity theory, it is easy to see that, for $t \geq 0$, we have a uniform approximation

$$0 \leq p(t,x) - p_a(t,x) \leq Ce^{-\gamma\alpha'(1)a}. \tag{6.16}$$

Step three: weighted L^2-decay of φ_a

From the work in the previous two steps, we have the dissipation inequality

$$\frac{d}{dt}\int \varphi_a(t,x)^2\rho_a(t,x)\,dx \le -2\int (\partial_x\varphi_a(t,x))^2\rho_a(t,x)\,dx. \tag{6.17}$$

We need a Nash-type inequality in order to proceed. This is given by [2, Proposition 5.9]: for any $\theta > 0$ and any smooth non-negative function $\varphi(x)$ that is sufficiently rapidly decaying as $x \to +\infty$ and bounded as $x \to -\infty$, we have

$$\int \varphi^2(x)\rho_a(t,x)\,dx \le \frac{2}{\theta}\Big(\int \varphi(x)\rho_a(t,x)\,dx\Big)^2 + 8C_1\max\{1,\theta^2\}\int |\varphi_x(x)|^2\rho_a(t,x)\,dx. \tag{6.18}$$

For this inequality to apply, the function $\rho_a(t,x)$ needs to be positive, bounded and increasing, with the left limit $\rho_a(t,-\infty) = 0$. In addition, it should satisfy

$$\overline{\overline{\rho}}_a(t,x) \le C_1\max\{1,\overline{\rho}_a^2(t,x)\}\rho_a(t,x). \tag{6.19}$$

Here, we use the notation that, for any $r : \mathbb{R} \to \mathbb{R}$,

$$\overline{r}(x) = \int_{-\infty}^{x} r(x)\,dx.$$

Lemma 6.3. *Under the setting above, $\rho_a(t,x)$ satisfies (6.19) with C_1 independent of a and t.*

We postpone the proof of this lemma for the moment. We may now proceed nearly verbatim as in the proof of [2, Lemma 5.7] and use the dissipation inequality (6.17) together with (6.18) to conclude that

$$\int \varphi_a(t,x)^2\rho_a(t,x)\,dx \le \frac{C}{\sqrt{t}}, \quad \text{for all } t \ge C\int \varphi_a(0,x)^2\rho_a(0,x)\,dx.$$

Then, in view of (6.15), we see that

$$\int \varphi_a(t,x)^2\rho_a(t,x)\,dx \le \frac{C}{\sqrt{t}} \qquad \text{for all } t \ge T_a = Ce^{\frac{3\gamma a}{2}}. \tag{6.20}$$

Step four: bootstrapping from L^2_ρ-decay to L^∞-decay

Fix any $T \ge 4T_a$ as in (6.20). The proof in [2] proceeds then by finding a good time $T_g \in [T/2, 3T/4]$ so that

$$(i)\ \max_{x\ge -\mu_a(T_g)} \varphi_a(T_g,x) \le \frac{C}{\sqrt{T}} \quad \text{and} \quad (ii)\ \mu_a(T_g) \ge \frac{1}{2}\log(T) - C. \tag{6.21}$$

To obtain (i), the arguments of [2, Lemma 5.10] require only the following ingredients: (1) the dissipation inequality (6.17); (2) the L^2_ρ-decay given by (6.20), (3) the mass conservation of (6.10); and (4) that $\rho_a(t,x) \in [C^{-1}, 1]$ for all $x > -\mu_a$. All four ingredients are present here, so the proof can be repeated nearly verbatim. We omit the details and assert (6.21).

We now argue that the bounds in (6.21) can be, essentially, preserved until time T. To this end, fix $K > 0$ to be chosen. An easy computation shows that

$$P(x) = e^x U_*\Big(x + \frac{1}{2}\log(K^2T)\Big)$$

is a steady solution to

$$P_t + (\alpha(e^{-x}P)P)_x = P_{xx}.$$

Recalling (6.6), that is, that the minimal speed traveling wave U_* has a purely exponential decay, we have

$$P(x) \leq \frac{C}{K\sqrt{T}}. \tag{6.22}$$

Let us decompose p_a into its P-part and its error part:

$$p_a(t,x) = \psi_P(t,x) + \psi_E(t,x).$$

Here, ψ_P is the solution to

$$\partial_t \psi_P(t,x) + (\alpha(e^{-x}\psi_P)\psi_P)_x = \partial_x^2 \psi_P, \quad \text{for all } t \in (T_g, T],$$

with initial condition

$$\psi_P(T_g, x) = \min\{p_a(T_g,x), P(x)\},$$

and ψ_E is the solution to

$$\partial_t \psi_E(t,x) + (v\psi_E)_x = \partial_x^2 \psi_E, \quad \text{for all } t \in (T_g, T], \tag{6.23}$$

with initial condition

$$\psi_E(T_g,x) = p_a(T_g,x) - \min\{p_a(T_g,x), P(x)\},$$

and drift term

$$v(t,x) = \alpha(\hat{u}_a) + \frac{\alpha(\hat{u}) - \alpha(e^{-x}\psi_P)}{\hat{u} - e^{-x}\psi_P} e^{-x}\psi_P.$$

We point out that $\psi_E \geq 0$, by the choice of initial condition and the maximum principle.

By the comparison principle and (6.22), we clearly have

$$\psi_P(T,x) \leq P(x) \leq \frac{C}{K\sqrt{T}}. \tag{6.24}$$

On the other hand, (6.23) conserves mass, so that

$$\int \psi_E(T,x)\,dx = \int \psi_E(T_g,x)\,dx.$$

Arguing exactly as in [2, Proof of Lemma 5.5], we see that, possibly after increasing K, we have

$$\psi_E(T_g,x) = 0, \quad \text{for } x \geq -\frac{1}{2}\log(T) + C,$$

due to (6.21) and (6.22). Hence, we have

$$\int \psi_E(T_g,x)\,dx \leq \int_{-\infty}^{-\frac{1}{2}\log T + C} p_a(t,x)\,dx \leq \int_{-\infty}^{-\frac{1}{2}\log T + C} e^x\,dx = \frac{C}{\sqrt{T}}.$$

By parabolic regularity theory, we have

$$\psi_E(T,x) \leq C\int \psi_E(T,x)\,dx = C\int \psi_E(T_g,x)\,dx \leq \frac{C}{\sqrt{T}}. \tag{6.25}$$

Thus, combining (6.24) and (6.25), we conclude

$$p_a(T,x) \leq \frac{C}{\sqrt{T}} \qquad \text{for any } T \geq 4Ce^{\frac{3\gamma a}{2}}. \tag{6.26}$$

Step five: conclusion of the proof of Proposition 6.1.(i)

Fix any T sufficiently large and let
$$a = \frac{1}{2\gamma\alpha'(1)} \log T.$$
Recall that γ is a fixed number in $(1, 4/3)$. By (6.16), we have
$$\sup_x p(T,x) \le \sup_x p_a(T,x) + \frac{C}{\sqrt{T}}.$$
Notice that
$$4Ce^{\frac{3\gamma a}{2}} = 4Ce^{\frac{3}{4\alpha'(1)}\log T} = 4CT^{\frac{3}{4\alpha'(1)}}.$$
Since $\alpha(0) = 0$, $\alpha(1) = 1$, and α is convex, we know that $\alpha'(1) \ge 1$. It follows that
$$4Ce^{\frac{3\gamma a}{2}} \le T$$
for T sufficiently large. We may thus apply (6.26) to deduce
$$\sup_x p(T,x) \le \sup_x p_a(T,x) + \frac{C}{\sqrt{T}} \le \frac{C}{\sqrt{T}},$$
which concludes the proof of Proposition 6.1.(i), except for the proof of Lemmas 6.2 and 6.3. □

Proof of Lemma 6.2

We first show how the function F in (6.13) can be obtained, and why it satisfies (6.11). We begin with the ansatz
$$\rho(t,x) = F(\hat{u}(t,x)),$$
leaving F as yet undetermined. Intuitively, as discussed in [2], we seek ρ that gives more weight to the right than to the left; hence, we wish it to be 1 at $x = +\infty$ and 0 at $x = -\infty$. This motivates the boundary conditions
$$1 = \rho(t,\infty) = F(0) \quad \text{and} \quad 0 = \rho(t,-\infty) = F(1). \tag{6.27}$$
Next, we compute:
$$\begin{aligned}
\rho_t + (\alpha(\hat{u})\rho)_x - \rho_{xx} &= F'(\hat{u})\hat{u}_t + [\alpha'(\hat{u})F(\hat{u}) + \alpha(\hat{u})F'(\hat{u})]\hat{u}_x - F'(\hat{u})\hat{u}_{xx} - F''(\hat{u})\hat{u}_x^2 \\
&= F'(\hat{u})[\hat{u}_{xx} + \hat{u} - A(\hat{u}) + 2\hat{u}_x - A'(\hat{u})\hat{u}_x] + [\alpha'(\hat{u})F(\hat{u}) + \alpha(\hat{u})F'(\hat{u})]\hat{u}_x \\
&\quad - F'(\hat{u})\hat{u}_{xx} - F''(\hat{u})\hat{u}_x^2 \\
&= F'(\hat{u})\big[\hat{u} - A(\hat{u})\big] + \hat{u}_x\big[2F'(\hat{u}) - A'(\hat{u})F'(\hat{u}) + \alpha'(\hat{u})F(\hat{u}) + \alpha(\hat{u})F'(\hat{u})\big] - F''(\hat{u})\hat{u}_x^2 \\
&= F'(\hat{u})\eta_*(\hat{u}) + \hat{u}_x\big[2F'(\hat{u}) - \hat{u}\alpha'(\hat{u})F'(\hat{u}) + \alpha'(\hat{u})F(\hat{u})\big] - F''(\hat{u})\hat{u}_x^2.
\end{aligned} \tag{6.28}$$
In the last step we used that in the pushmi-pullyu case we have $\eta_*(u) = u - A(u)$. The last term above makes it clear that we want F to be concave. It is also natural to expect F to be monotonic. In view of (6.27), this means F is decreasing. Hence, we require
$$F'(u), F''(u) \le 0 \quad \text{for all } u \in (0,1). \tag{6.29}$$
Then, recalling that
$$0 \le \hat{w} = -\hat{u}_x - \eta_*(\hat{u}), \tag{6.30}$$

we find

$$- F''(u)u_x^2 = (-F''(u))(-u_x)(-u_x) \geq (-F''(u))(-u_x)\eta_*(u). \tag{6.31}$$

Using (6.30) and (6.31) in (6.28) gives

$$\begin{aligned}
\rho_t + (\alpha(\hat{u})\rho)_x - \rho_{xx} &= F'(\hat{u})\eta_*(\hat{u}) + \hat{u}_x\big[2F'(\hat{u}) - \hat{u}\alpha'(\hat{u})F'(\hat{u}) + \alpha'(\hat{u})F(\hat{u})\big] - F''(\hat{u})\hat{u}_x^2 \\
&\geq F'(\hat{u})\eta_*(\hat{u}) + \hat{u}_x\big[2F'(\hat{u}) - \hat{u}\alpha'(\hat{u})F'(\hat{u}) + \alpha'(\hat{u})F(\hat{u})\big] + F''(\hat{u})\hat{u}_x\eta_*(\hat{u}) \\
&\geq F'(\hat{u})(-\hat{u}_x) + \hat{u}_x\big[2F'(\hat{u}) - \hat{u}\alpha'(\hat{u})F'(\hat{u}) + \alpha'(\hat{u})F(\hat{u}) + F''(\hat{u})\eta_*(\hat{u})\big] \\
&= \hat{u}_x\big[F'(\hat{u}) - \hat{u}\alpha'(\hat{u})F'(\hat{u}) + \alpha'(\hat{u})F(\hat{u}) + F''(\hat{u})\eta_*(\hat{u})\big].
\end{aligned}$$

Hence, we seek a concave function $F(u)$ such that

$$F'(u) - u\alpha'(u)F'(u) + \alpha'(u)F(u) + F''(u)\eta_*(u) = 0. \tag{6.32}$$

We construct such an F and then check that it verifies (6.27) and (6.29).

Notice that, using (6.3) once again, we have

$$\eta_*'(u) = 1 - \alpha(u) - u\alpha'(u).$$

Thus, (6.32) can be written as

$$\begin{aligned}
0 &= F''(u)\eta_*(u) + F'(u) - u\alpha'(u)F'(u) + \alpha'(u)F(u) \\
&= (F'(u)\eta_*(u))' - F'(u)(1 - \alpha(u) - u\alpha'(u)) + F'(u)[1 - u\alpha'(u)] + \alpha'(u)F(u) \\
&= (F'(u)\eta_*(u))' - F'(u)(-\alpha(u)) + \alpha'(u)F(u) = (F'(u)\eta_*(u) + \alpha(u)F(u))'.
\end{aligned} \tag{6.33}$$

Hence, we may take $F(u)$ such that

$$\frac{F'(u)}{F(u)} = -\frac{\alpha(u)}{\eta_*(u)}.$$

The boundary condition $F(0) = 1$ in (6.27) implies that

$$F(u) = \exp\Big(-\int_0^u \frac{\alpha(u')}{\eta_*(u')}du'\Big) = \exp\Big(-\int_0^u \frac{u' - \eta_*(u')}{u'\eta_*(u')}du'\Big),$$

which is exactly (6.13). Note that $\eta_*(1) = 0$ and $\eta_*'(1) = -\alpha'(1) < 0$ (recall that α is convex and increasing) so that, for some $u_0 \in (0,1)$,

$$\int_0^1 \frac{u' - \eta_*(u')}{u'\eta_*(u')}du' \geq \frac{1}{C}\int_{u_0}^1 \frac{1}{1-u'}du' = +\infty.$$

Hence, $F(1) = 0$. This completes the proof that $F(u)$ satisfies (6.27).

In addition, one can see directly that $F' \leq 0$. It remains to check the concavity condition in (6.29). We see from (6.33) that

$$\begin{aligned}
\eta_*(u)F''(u) = -F'(u) + u\alpha'(u)F'(u) - \alpha'(u)F(u) &= F(u)\Big[\frac{\alpha(u)}{\eta_*(u)} - \frac{\alpha(u)}{\eta_*(u)}u\alpha'(u) - \alpha'(u)\Big] \\
= \frac{F(u)}{\eta_*(u)}\big[\alpha(u) - u\alpha(u)\alpha'(u) - \alpha'(u)u(1-\alpha(u))\big] &= \frac{F(u)}{\eta_*(u)}\big[\alpha(u) - \alpha'(u)u\big] \leq 0,
\end{aligned}$$

where the last inequality holds by the assumed convexity of α. Thus (6.29) holds. Finally, we note that the fact that $F \in C^2_{\rm loc}$ follows directly from the computations above and the regularity of η_*.

It remains to prove (6.14). First notice that, for any $\varepsilon > 0$, there is u_ε so that

$$\frac{1}{(1+\varepsilon)}\frac{u\alpha'(u)}{\alpha'(1)} \le \alpha(u) \le \Big(\frac{1+\varepsilon}{\alpha'(1)}\Big)u\alpha'(u) \qquad \text{for all } u \in [u_\varepsilon, 1].$$

Indeed, to see this, notice that equality holds when $\varepsilon = 0$ and $u = 1$. By making $\varepsilon > 0$, we can then obtain the above inequalities for some range of u near 1 due to the regularity of α.

Hence, there is C_ε, depending on ε and changing line-by-line, so that, for all $u \in [u_\varepsilon, 1]$,

$$F(u) = \exp\Big(-\int_0^u \frac{\alpha(u')}{u'(1-\alpha(u'))}du'\Big) \ge \frac{1}{C_\varepsilon}\exp\Big(-\frac{1+\varepsilon}{\alpha'(1)}\int_{u_\varepsilon}^u \frac{\alpha'(u')}{1-\alpha(u')}\,du'\Big) = \frac{1}{C_\varepsilon}\big(1-\alpha(u)\big)^{\frac{1+\varepsilon}{\alpha'(1)}}.$$

The lower bound is clear for $u < u_\varepsilon$ by choosing $C_\varepsilon = 1/F(u_\varepsilon)$ and using the monotonicity of F. A similar argument yields the matching bound

$$F(u) \le C_\varepsilon(1-\alpha(u))^{\frac{1}{(1+\varepsilon)\alpha'(1)}} \qquad \text{for all } u \in [0,1].$$

This completes the proof. □

Proof of Lemma 6.3

From the definition of $\mu_a(t)$ (6.2) and the relationship between $\hat{u}$ and $\tilde{u}$ (6.1), we have

$$\alpha(\hat{u}_a(t, -\tfrac{1}{2}\log(t+1) - \mu_a(t))) = \frac{1}{2}.$$

Then, for $y \le x < -(1/2)\log(t+1) - \mu_a$, we have

$$\begin{aligned}\rho_a(t,y) = F(\hat{u}_a(t,y)) &= \exp\Big(-\int_0^{\hat{u}_a(t,y)} \frac{\alpha(u')}{\eta_*(u')}\,du'\Big) = \rho_a(t,x)\exp\Big(-\int_{\hat{u}_a(t,x)}^{\hat{u}_a(t,y)} \frac{\alpha(u')}{\eta_*(u')}\,du'\Big)\\ &= \rho_a(t,x)\exp\Big(\int_y^x \frac{\alpha(\hat{u}_a(t,z))}{\eta_*(\hat{u}_a(t,z))}\partial_z\hat{u}_a(t,z)\,dz\Big) \le \rho_a(t,x)\exp\Big(-\int_y^x \alpha(\hat{u}_a(t,z))\,dz\Big)\\ &\le \rho_a(t,x)e^{-\frac{1}{2}(x-y)}.\end{aligned}$$

In the first inequality, we used that

$$\partial_z\hat{u}_a = -\eta_*(\hat{u}_a) - \hat{w}_a \le -\eta_*(\hat{u}_a),$$

and in the second inequality, we used that $\alpha(\hat{u}_a) \ge 1/2$ due to the definition of μ_a and the monotonicity of $\hat{u}_a$. As an aside, we note that this is the motivation for the definition of μ_a as it guarantees α is positive on all of (y,x). This is not guaranteed (e.g., the case $\alpha(u) = 4^n(u-(3/4))_+^n$ for $n \ge 3$).

Thus we have, for all $x \le -(1/2)\log(t+1) - \mu_a$,

$$\overline{\rho}_a(t,x) = \int_{-\infty}^x \rho_a(t,y)\,dy \le \rho_a(t,x)\int_{-\infty}^x e^{\frac{y-x}{2}}\,dy = 2\rho_a(t,x), \tag{6.34}$$

and, consequentially,

$$\overline{\overline{\rho}}_a(t,x) \le 4\rho_a(t,x) \le 4\max\{1, \overline{\rho}_a^2(t,x)\}\rho_a(t,x). \tag{6.35}$$

This is exactly the desired conclusion when $x \le -(1/2)\log(t+1) - \mu_a$.

To conclude, we need to obtain the desired bound for $x > -(1/2)\log(t+1) - \mu_a$. By the definition of μ_a and the regularity of α, it is easy to see that

$$\rho_a(t,x) \ge \frac{1}{C} \qquad \text{for all } x > -\tfrac{1}{2}\log(t+1) - \mu_a.$$

Hence,

$$\overline{\rho}_a(t,x) \le C\overline{\rho}_a(t,x)\rho_a(t,x) \qquad \text{for all } x > -\tfrac{1}{2}\log(t+1) - \mu_a. \tag{6.36}$$

When $x > -(1/2)\log(t+1) - \mu_a$, we combine (6.34) with (6.36) and recall that ρ_a and $\overline{\rho}_a$ are increasing:

$$\begin{aligned}
\overline{\overline{\rho}}_a(t,x) &= \int_{-\infty}^{-\frac{1}{2}\log(t+1)-\mu_a} \overline{\rho}_a(t,y)\,dy + \int_{-\frac{1}{2}\log(t+1)-\mu_a}^{x} \overline{\rho}_a(t,y)\,dy \\
&\le 2\int_{-\infty}^{-\frac{1}{2}\log(t+1)-\mu_a} \rho_a(t,y)\,dy + \int_{-\frac{1}{2}\log(t+1)-\mu_a}^{x} C\overline{\rho}_a(t,y)\rho_a(t,y)\,dy \\
&\le 2\overline{\rho}_a\big(t, -\tfrac{1}{2}\log(t+1) - \mu_a\big) + C\overline{\rho}(t,x)\int_{-\frac{1}{2}\log(t+1)-\mu_a}^{x} \rho_a(t,y)\,dy \\
&\le 4\rho_a(t,-\mu_a) + C\overline{\rho}_a^2(t,x) \le 4\rho_a(t,x) + C^2\overline{\rho}_a^2(t,x)\rho_a(t,x) \\
&\le (4+C^2)\max\{1, \overline{\rho}_a^2(t,x)\}\rho_a(t,x).
\end{aligned}$$

This, in addition to (6.35), concludes the proof of (6.19). □

6.2.2 Proof of Proposition 6.1.(ii): the lower bound on $\tilde{u}_{\mathrm{rd}}$

In this section, we work only with u_{rd}. As there is no possibility of confusion, we simply referring to u_{rd} as u below. First, to shorten the proof, we use a preliminary bound obtained in Section 4.1 of [30]: for any $\varepsilon > 0$, we have

$$\tilde{u}(t,x) \ge \frac{C_\varepsilon}{t^\varepsilon} e^{-x-\frac{x^2}{Ct}}. \tag{6.37}$$

Note that this bound is off the optimal by a factor of t^ε but is useful as a first step.

The second ingredient is a preliminary upper bound on the shape defect function: for all $t \ge 1$ and any x, we have

$$\tilde{w}(t,x) \le \begin{cases} C & \text{if } x < -\log t, \\ \dfrac{C}{t}(x+1+\log t)e^{-x-\frac{(x+\log t)^2}{Ct}} & \text{if } x > -\log t. \end{cases} \tag{6.38}$$

Indeed, the uniform bound on w over $(-\infty, -\log t)$ follows by parabolic regularity theory. To obtain the second bound in (6.38), we note that, under the present assumptions, and since we are in the pushmi-pullyu case, the function $\eta_*(u) = u - A(u)$ is concave and

$$\eta_*'(u)(2 - \eta_*'(u)) = 1 - (1 - \eta_*'(u))^2 \le 1.$$

Therefore, it follows from (4.1) that the (non-negative) shape defect function w satisfies

$$w_t - w_{xx} \le w. \tag{6.39}$$

A lesson of [34] is that bounded functions starting from compactly supported (on the right) initial data that satisfy (6.39) must also satisfy

$$w\big(t, x + 2t - \frac{3}{2}\log t\big) \le C(x+1)e^{-x-\frac{x^2}{Ct}},$$

from which the second bound in (6.38) follows after changing variables.

The main estimate we need for the proof of Proposition 6.1.(ii) is a uniform bound on the exponential moment of $\tilde{u}$

$$I(t) := \int e^x \tilde{u}(t,x)dx.$$

This is provided by the following lemma:

Lemma 6.4. *There exists $C > 0$ such that, for sufficiently large t,*

$$\frac{1}{C} \leq \frac{I(t)}{\sqrt{t}} \leq C. \tag{6.40}$$

We note that the upper bound in Lemma 6.4 is not used in this paper; however, it comes 'for free' in the proof, so we state it as well. We postpone the proof of Lemma 6.4 momentarily and, first, show how to apply it to conclude the lower bound in Proposition 6.1.

Proof Proposition 6.1.(ii)

Fix $N > 0$ to be chosen. We argue by contradiction, assuming, for t fixed and sufficiently large, that

$$-\mu(t) < -N.$$

In this case (6.4) implies that

$$\tilde{u}(t,x) \leq Ce^{-x-N}. \tag{6.41}$$

Our goal is to deduce from (6.41) an upper bound on $I(t)$ that violates the lower bound of Lemma 6.4 if N is too large.

Define the left, middle, and right domains as

$$L = \{x : x < -N\}, \quad M = \{x : -N \leq x < N\sqrt{t}\}, \quad \text{and} \quad R = \{x : N\sqrt{t} \leq x\},$$

with $I_L(t)$, $I_M(t)$, and $I_R(t)$ the decomposition of $I(t)$ into integrals on each respective domain. To bound I_L, we simply use that $\tilde{u} \leq 1$:

$$I_L(t) = \int_{-\infty}^{-N} e^x \tilde{u}(t,x)\,dx \leq e^{-N}. \tag{6.42}$$

For I_M, we use (6.41) to find

$$I_M(t) = \int_{-N}^{N\sqrt{t}} e^x \tilde{u}(t,x)\,dx \leq \int_{-N}^{N\sqrt{t}} Ce^{-N}\,dx = CNe^{-N}(1+\sqrt{t}). \tag{6.43}$$

Finally, we estimate I_R from above. We integrate by parts and use the shape defect function to obtain

$$\begin{aligned} I_R(t) = \int_{N\sqrt{t}}^{\infty} e^x \tilde{u}\,dx &= -e^{N\sqrt{t}}\tilde{u}(t,N\sqrt{t}) - \int_{N\sqrt{t}}^{\infty} e^x \tilde{u}_x\,dx \leq -\int_{N\sqrt{t}}^{\infty} e^x \tilde{u}_x\,dx \\ &= \int_{N\sqrt{t}}^{\infty} e^x(\tilde{w} - \eta_*(\tilde{u}))\,dx \leq \int_{N\sqrt{t}}^{\infty} e^x \tilde{w}\,dx. \end{aligned}$$

Applying the second bound in (6.38) yields

$$I_R(t) \leq C\int_{N\sqrt{t}}^{\infty} \frac{(x+\log t)}{t} e^{-\frac{x^2}{Ct}}\,dx \leq Ce^{-\frac{N^2}{C}}. \tag{6.44}$$

Putting together (6.42), (6.43), and (6.44) with (6.48), we find

$$\frac{1}{C} \le \frac{I(t)}{\sqrt{t}} \le \frac{e^{-N}}{\sqrt{t}} + CNe^{-N} + \frac{C}{\sqrt{t}} e^{-\frac{N^2}{C}}.$$

This yields a contradiction for t and N sufficiently large. We conclude that there exists $\underline{\mathrm{L}}$ such that

$$\alpha(u(t, 2t - (1/2)\log(t) - \underline{\mathrm{L}})) \ge 1/2,$$

which finishes the proof. □

Proof of Lemma 6.4

First, we compute the time derivative of $I(t)$, using integration by parts repeatedly and the definition of the shape defect function:

$$\begin{aligned} \dot{I}(t) &= \int e^x \Big(\tilde{u}_{xx} + f(\tilde{u}) + \Big(2 - \frac{1}{2(t+1)}\Big) \tilde{u}_x \Big)\, dx \\ &= \int e^x \Big(-\tilde{u}_x + \eta_*(\tilde{u})(2 - \eta_*'(\tilde{u})) + 2\tilde{u}_x + \frac{1}{2(t+1)} \tilde{u} \Big)\, dx \\ &= \int e^x (\eta_*(\tilde{u}) + (\eta_*(\tilde{u}))_x - \tilde{w}(1 - \eta_*'(\tilde{u})))\, dx + \frac{I}{2(t+1)} \\ &= -\int e^x (1 - \eta_*'(\tilde{u}))\tilde{w}\, dx + \frac{I}{2(t+1)}. \end{aligned} \tag{6.45}$$

In the pushmi-pullyu case we have

$$u - \eta_*(u) = A(u) \quad \text{so that} \quad 1 - \eta_*'(u) = A'(u).$$

Recall that $A(u)$ is an increasing C^2 function with $A(0) = A'(0) = 0$. It follows that

$$0 \le \int e^x (1 - \eta_*'(\tilde{u}))\tilde{w}\, dx \le C \int e^x \tilde{u}\tilde{w}\, dx =: \mathcal{E}(t).$$

The upper bound of I is immediate from (6.45) and the fact that $\mathcal{E}(t) \ge 0$ due to the positivity of w. Next, we prove the lower bound of I. We claim that

$$\mathcal{E}(t) \le \frac{C \log^2(t+1)}{t+1}. \tag{6.46}$$

Before establishing (6.46), we show how to use it to conclude the lower bound of I. Using (6.46) in (6.45) yields

$$\frac{d}{dt}\left(\frac{I(t)}{\sqrt{t+1}} \right) \ge -\frac{C \log^2(t+1)}{(t+1)^{3/2}}.$$

Fix $t_0 > 0$ to be chosen. Integrating the above from t_0 to t, we find

$$\frac{I(t)}{\sqrt{t+1}} \ge \frac{I(t_0)}{\sqrt{t_0+1}} - C \int_{t_0}^{t} \frac{\log^2(s+1)}{(s+1)^{3/2}}\, ds \ge \frac{I(t_0)}{\sqrt{t_0+1}} - C \frac{\log^2(t_0+1)}{\sqrt{t_0+1}}. \tag{6.47}$$

On the other hand, for any $\varepsilon > 0$, by (6.37), we have

$$\frac{I(t_0)}{\sqrt{t_0+1}} \ge \frac{C}{(t_0+1)^{\varepsilon}}.$$

Hence, fixing t_0 sufficiently large, the right side of (6.47) is positive. It follows that

$$\frac{I(t)}{\sqrt{t+1}} \geq C_0 \qquad \text{for all } t > t_0, \tag{6.48}$$

which yields the claimed lower bound in (6.40).

To finish the proof, we establish (6.46). We decompose $\mathcal{E}(t)$ as

$$\mathcal{E}(t) = \int_{-\infty}^{-\log(t+1)} e^x \tilde{u}\tilde{w}\, dx + \int_{-\log(t+1)}^{0} e^x \tilde{u}\tilde{w}\, dx + \int_0^\infty e^x \tilde{u}\tilde{w}\, dx = \mathcal{E}_L(t) + \mathcal{E}_M(t) + \mathcal{E}_R(t).$$

For $\mathcal{E}_L(t)$, we note that $0 \leq \tilde{u}, \tilde{w} \leq C$ in order to find

$$\mathcal{E}_L \leq C \int_{-\infty}^{-\log(t+1)} e^x\, dx = Ce^{-\log(t+1)} \leq \frac{C}{t+1}. \tag{6.49}$$

Next, on the domain of integration of $\mathcal{E}_M(t)$, (6.38) implies that

$$e^x \tilde{w} \leq \frac{C\log(t+1)}{t+1}.$$

This gives the bound

$$\mathcal{E}_M(t) \leq \frac{C\log(t+1)^2}{t+1}. \tag{6.50}$$

Finally, using (6.38) again, as well as the upper bound for $\tilde{u}$ in (6.4), yields

$$\mathcal{E}_R(t) \leq C\int_0^\infty \frac{x+\log(t+1)}{t+1} e^{-x-\mu(t)-\frac{(x+\log(t+1))^2}{C(t+1)}}\, dx \leq \frac{C\log(t+1)e^{-\mu(t)}}{t+1}.$$

By the already proved Proposition 6.1.(i), we know that $-\mu(t) \leq \overline{L}$, and, thus,

$$\mathcal{E}_R(t) \leq \frac{C\log(t+1)}{t+1}. \tag{6.51}$$

Putting together (6.49), (6.50), and (6.51), we obtain (6.46), which completes the proof of Lemma 6.4. □

6.3 The second step: the behavior of $\tilde{u}$ at $x = t^\gamma$

We now identify the constant order term in the expansion of $m(t)$. This is done in the manner of [46]. The main step is the following proposition.

Proposition 6.5. *Let u be the solution to either (1.1) or (1.24) under the assumptions of Theorem 1.3 or Theorem 1.4 with $\tilde{u}$ defined by (6.1). Then there exists $\alpha_\infty > 0$ such that, for any $\gamma \in (0, 1/2)$, we have*

$$\lim_{t\to\infty} e^{t^\gamma} \tilde{u}(t, t^\gamma) = \alpha_\infty.$$

As the proof of this is similar to that of [2, Corollary 6.3], which is, in turn, based on the proof of [46, Lemma 4.2], we merely provide an outline of the main points, as well as a description of the changes needed to be made in the present setting.

As the proof is significantly simpler for solutions to the reaction-diffusion equation (1.1) than for solutions to the reactive conservation law (1.24), we begin with the proof in former case.

6.3.1 The proof of Proposition 6.5 for the reaction-diffusion equation (1.1)

In this subsection we only use $u_{\rm rd}$, so we drop the "rd" subscript. We begin with the weighted Hopf-Cole transform

$$v(t,x) = e^{\Gamma}\tilde u(t,x) = \exp\Big\{x + \int_x^\infty \alpha(\tilde u(t,y))dy\Big\}\tilde u(t,x),$$

which satisfies

$$v_t + \frac{1}{2(t+1)}(v_x - v) - v_{xx} = G, \tag{6.52}$$

with

$$G = e^{\Gamma}\Big(f(\tilde u) - \tilde u - \alpha(\tilde u)^2\tilde u + 2\alpha(\tilde u)\tilde u_x\Big) + u\Big(- \int_x^\infty \alpha''(\tilde u)\tilde u_y^2 dy + \int_x^\infty \alpha'(u) f(\tilde u)\,dy\Big).$$

See the details of Proposition 4.1 in order to see how (6.52) is computed. Due to Proposition 4.1, we have

$$G \le 0. \tag{6.53}$$

Furthermore, for all $\gamma > 0$, $t \ge 1$, and $x \ge 0$, we have

$$G(t, x + t^\gamma) \ge -C\exp(-x - t^\gamma). \tag{6.54}$$

Here, we are using that, for u small, $f(u) - u = O(u^2)$, and, for all $t \ge 1$ and $x \ge 0$, we have

$$\tilde u(t,x) \le Ce^{-x} \quad \text{and} \quad -\tilde u_x \le Ce^{-x}.$$

The former is due to the upper bounds in Proposition 6.1 and (6.4), while the latter follows from the former by parabolic regularity theory.

Additionally, the argument of [2, Lemma 6.1] applies nearly verbatim, so we assert its conclusion without proof: for all $t \ge 1$,

$$v(t,x) \le C \quad \text{for all } x \ge 0, \quad \text{and} \quad v(t,x) \ge \frac{1}{C} \quad \text{for all } x \le \frac{\sqrt t}{C}. \tag{6.55}$$

Finally, we claim that, for all $t \ge 1$, we have

$$v_x(t, t^\gamma) \le 0 \quad \text{and} \quad |v_x(t, -t^\gamma)| \le Ce^{-\alpha'(1)t^\gamma}. \tag{6.56}$$

We see this as follows: using (6.3), we write

$$v_x(t,x) = (\tilde u_x(t,x) + \eta_*(\tilde u(t,x)))e^{\Gamma} = -we^{\Gamma} \le 0, \tag{6.57}$$

giving the first inequality in (6.56).

We now justify the second inequality in (6.56). Using the computation (6.57), there are three terms to bound: $\tilde u_x$, $\eta_*(\tilde u)$, and $\exp(\Gamma)$. First, observe that, by Proposition 6.1 and (6.4), we have

$$0 \le 1 - \tilde u(t,x) \le Ce^{\alpha'(1)x}. \tag{6.58}$$

Thus, by parabolic regularity theory, we have

$$|\tilde u_x(t, -t^\gamma)| \le Ce^{-\alpha'(1)t^\gamma}. \tag{6.59}$$

Next, from (6.58) and a Taylor approximation (recall that $\alpha(1)=1$), we find

$$0 \le 1-\alpha(\tilde{u}(t,x)) \le Ce^{\alpha'(1)x}.$$

This yields

$$\eta_*(\tilde{u}(t,-t^\gamma)) = \tilde{u}(t,-t^\gamma)(1-\alpha(\tilde{u}(t,-t^\gamma))) \le (1-\alpha(\tilde{u}(t,-t^\gamma))) \le Ce^{-\alpha'(1)t^\gamma}. \tag{6.60}$$

In addition, using (6.4) and a Taylor approximation again, we find, for $x \le 0$,

$$\begin{aligned}\exp\Big|x+\int_x^\infty \alpha(\tilde{u}(t,y))\,dy\Big| &= \exp\Big|\int_x^0 (\alpha(\tilde{u}(t,y))-1)dy + \int_0^\infty \alpha(\tilde{u}(t,y))dy\Big| \\ &\le \exp\Big|\int_x^0 Ce^{\alpha'(1)y}dy\Big| \cdot \exp\Big|\int_0^\infty Ce^{-y}dy\Big| \le C.\end{aligned}$$

The combination of (6.57), (6.59) and (6.60) yields (6.56).

As discussed in detail in [2, Section 6], these ingredients, that is, (6.53), (6.54), (6.55), and (6.56), are all that is needed to prove Proposition 6.5. This concludes the proof of Proposition 6.5 in the case of the reaction-diffusion equation (1.1).

6.3.2 The proof of Proposition 6.5 for the reactive conservation law (1.24)

We now drop the "rcl" subscript from $u_{\rm rcl}$ and denote by u the solution to (1.24). We begin by reviewing the key ingredients in Section 6.3.1 for the reaction-diffusion case. Clearly it is possible to establish (6.55) and (6.56) verbatim in the reactive conservation law case. In order to check (6.53) and (6.54), we need to consider the weighted Hopf-Cole transform

$$v(t,x) = e^\Gamma \tilde{u}(t,x) = e^{x+\int_x^\infty \alpha(\tilde{u}(t,y))dy}\tilde{u}(t,x).$$

Changing to the moving frame from (1.24), we have

$$\tilde{u}_t - \Big(2-\frac{1}{2(t+1)}\Big)\tilde{u}_x + (\tilde{u}\alpha(\tilde{u}))_x = \tilde{u}_{xx} + \tilde{u}(1-\alpha(\tilde{u})). \tag{6.61}$$

This allows us to compute

$$\begin{aligned}v_t + \frac{1}{2(t+1)}(v_x - v) - v_{xx} &= \Gamma_t v + e^\Gamma \tilde{u}_t + \frac{1}{2(t+1)}(e^\Gamma\tilde{u}_x - \alpha v) - ((1-\alpha)v + e^\Gamma\tilde{u}_x)_x \\ &= \Gamma_t v + e^\Gamma\Big(\Big(2-\frac{1}{2(t+1)}\Big)\tilde{u}_x - (\tilde{u}\alpha(\tilde{u}))_x + \tilde{u}_{xx} + \tilde{u}(1-\alpha(\tilde{u}))\Big) + \frac{1}{2(t+1)}(e^\Gamma\tilde{u}_x - \alpha v) \\ &\quad - \big(-\alpha'(\tilde{u})\tilde{u}_x v + (1-\alpha)^2 v + 2(1-\alpha)e^\Gamma\tilde{u}_x + e^\Gamma\tilde{u}_{xx}\big) \\ &= \Gamma_t v - \frac{\alpha}{2(t+1)}v - \alpha(\tilde{u})e^\Gamma\tilde{w}.\end{aligned}$$

To compute Γ_t, we use again (6.61), as well as (6.3) and integration by parts, to find

$$\begin{aligned}\Gamma_t &= \int_x^\infty \alpha'(\tilde{u})\Big(\Big(2-\frac{1}{2(t+1)}\Big)\tilde{u}_x - (\tilde{u}\alpha(\tilde{u}))_x + \tilde{u}_{xx} + \tilde{u}(1-\alpha(\tilde{u}))\Big)\,dy \\ &= \frac{\alpha}{2(t+1)} + \tilde{u}\alpha\alpha' - \tilde{u}_x\alpha'(\tilde{u}) + \int_x^\infty \big(2\alpha'\tilde{u}_x + \alpha''(\tilde{u})\tilde{u}\alpha(\tilde{u})\tilde{u}_x - \alpha''(\tilde{u})\tilde{u}_x^2 + \alpha'(\tilde{u})\eta_*(\tilde{u})\big)\,dy \\ &= \frac{\alpha}{2(t+1)} + \alpha'\tilde{w} + \int_x^\infty \big(-\tilde{u}\alpha''\tilde{u}_x + \alpha'\tilde{u}_x + \alpha''(\tilde{u})\tilde{u}\alpha(\tilde{u})\tilde{u}_x - \alpha''(\tilde{u})\tilde{u}_x^2 + \alpha'(\tilde{u})\eta_*(\tilde{u})\big)\,dy \\ &= \frac{\alpha}{2(t+1)} + \alpha'\tilde{w} - \int_x^\infty (\alpha' - \alpha''\tilde{u}_x)\tilde{w}\,dy.\end{aligned}$$

Combining both computations above yields

$$v_t + \frac{1}{2(t+1)}(v_x - v) - v_{xx} = e^{\Gamma}\Big((\tilde{u}\alpha' - \alpha)\tilde{w} - \tilde{u}\int_x^{\infty}(\alpha' - \alpha''\tilde{u}_x)\tilde{w}\,dy\Big). \tag{6.62}$$

Unfortunately, due to the convexity of $\alpha(u)$, the first term in the right side of (6.62) is positive. Amazingly, it is zero for exactly one choice, $\alpha(u) = u$, which was by coincidence considered in [2]. As a result, the analogue of (6.53) does not hold in the reactive conservation law case.

Let us now outline how to bypass this difficulty. Arguing exactly as in Section 6.3.1, it is easy to check that the analogue of (6.54) holds; that is,

$$v_t + \frac{1}{2(t+1)}(v_x - v) - v_{xx} \geq -Ce^{-x} \qquad \text{for } x \geq t^{\gamma}.$$

In addition, as we have observed, $v(t,x)$ still satisfies the first inequality in (6.56): $v_x(t,t^{\gamma}) \leq 0$. Therefore, a subsolution for $v(t,x)$ can be found as the solution to

$$\underline{v}_t + \frac{1}{2(t+1)}(\underline{v}_x - \underline{v}) - \underline{v}_{xx} = -Ce^{-x} \quad \text{for } x \geq t^{\gamma},$$

with the Neumann boundary condition $\underline{v}_x(t,t^{\gamma}) = 0$. That is, if $v(T,x) = \underline{v}(T,x)$ for all $x \geq T^{\gamma}$ at some time T, then

$$v(t,x) \geq \underline{v}(t,x), \quad \text{for all } t \geq T \text{ and } x \geq t^{\gamma}.$$

This part of the proof is unaffected and proceeds exactly as in [2].

In order to construct a supersolution for $v(t,x)$, we note that

$$e^{\Gamma}\Big((\tilde{u}\alpha' - \alpha)\tilde{w} - \tilde{u}\int_x^{\infty}(\alpha' - \alpha''\tilde{u}_x)\tilde{w}\,dy\Big) \leq Cv\tilde{w}.$$

Therefore, a supersolution for $v(t,x)$ is given by the solution to

$$\overline{v}_t + \frac{1}{2(t+1)}(\overline{v}_x - \overline{v}) - \overline{v}_{xx} = C\overline{v}\tilde{w} \quad \text{for } x \geq -t^{\gamma},$$

with the boundary condition

$$\overline{v}_x(t,-t^{\gamma}) = -Ce^{-\alpha'(1)t^{\gamma}}. \tag{6.63}$$

Here, we took into account the second inequality in (6.56).

Changing to self-similar variables

$$\overline{V}(\tau,y) = \overline{v}(e^{\tau}, e^{\tau/2}y),$$

yields

$$\overline{V}_{\tau} + \mathcal{L}\overline{V} + \frac{1}{2}e^{-\tau/2}\overline{V}_y = C\overline{V}e^{\tau}\tilde{w}(e^{\tau}, e^{\tau/2}y) \quad \text{for } y \geq -e^{-\tau(1/2-\gamma)}. \tag{6.64}$$

Here, $\mathcal{L}$ is the linear operator associated to the heat equation in self-similar variables:

$$\mathcal{L} := -\partial_y^2 - \frac{y}{2}\partial_y - \frac{1}{2}.$$

A key feature required to make the proof strategy of [2, Corollary 6.3] work is that, the right hand side of (6.64) must be 'suitably' bounded and tend to zero in a 'suitable' fashion as $\tau \to +\infty$. In particular, one needs to show that

$$\int_0^{\infty} \|e^{\tau}\tilde{w}(e^{\tau}, e^{\tau/2}\cdot)\|_{L^2_{\mu_G}}\,d\tau < \infty. \quad \text{and} \quad \sup_{\tau\geq 0}\|e^{\tau}\tilde{w}(e^{\tau},\cdot)\|_{L^{\infty}} < \infty \tag{6.65}$$

with the Gaussian-weighted measure

$$d\mu_G = e^{y^2/4}\, dy.$$

Roughly, this allows to treat the right side of (6.64) as an error term when pursuing $L^2_{\mu_G}$ estimates.

The bounds in (6.65) are a consequence of the following lemma, that we state here and prove in Section 6.5.

Lemma 6.6. *For all $t \geq 1$, we have*

$$\tilde{w}(t,x) \leq \begin{cases} Ct^{-1} & \text{if } x < 1, \\ Ct^{-1} x e^{-x-\frac{x^2}{5t}} & \text{if } x \geq 1. \end{cases} \tag{6.66}$$

We now discuss how the above subsolution and supersolution can be used to prove the the convergence of $V(\tau, y) = v(e^\tau, e^{\tau/2} y)$ in a slightly more detail. First, for the sake of a simple discussion, we assume that both $\overline{V}$ and $\underline{V}$ (defined analogously using $\underline{v}$) satisfy Neumann boundary conditions – it is easy to account for the the errors coming from the 'approximate' Neumann boundary condition (6.63) as they are exponentially small in t^γ.

Fix $T \gg 1$ and impose initial conditions

$$\underline{V}(T,\cdot) = \overline{V}(T,\cdot) = V(T,\cdot).$$

Notice that, changing to self-similar variables, the equation for $\underline{V}$ becomes

$$\underline{V}_\tau + \mathcal{L}\underline{V} + \frac{1}{2} e^{-\tau/2} \underline{V}_y = -Ce^{-ye^{\tau/2}} \qquad \text{for } y \geq e^{-\tau(1/2-\gamma)}. \tag{6.67}$$

Recall the equation for $\overline{V}$ is given by (6.64). Let us take for granted that $\underline{V}(t,\cdot)$, and $\overline{V}(t,\cdot)$ are bounded in $L^2_{\mu_G} \cap L^\infty$ uniformly in t and T (this is easily established for V in exactly the manner of [2, 34, 46] and it is then inherited by $\underline{V}$ and $\overline{V}$).

We consider the projection onto the principal eigenfunction

$$\psi(y) = \frac{1}{Z} e^{-y^2/4}.$$

Here, Z is chosen so that $\|\psi\|_{L^2_{\mu_G}} = 1$. For $\overline{V}$, we find

$$\frac{d}{dt}\langle \overline{V}, \psi\rangle_{L^2_{\mu_G}} = -\frac{1}{2} e^{-\tau/2} \overline{V}(\tau, 0) + Ce^\tau \|\tilde{w}(e^\tau, e^{\tau/2}\cdot)\|_{L^2_{\mu_G}},$$

and, for $\underline{V}$, we find

$$\frac{d}{dt}\langle \underline{V}, \psi\rangle_{L^2_{\mu_G}} = -\frac{1}{2} e^{-\tau/2} \underline{V}(\tau, 0) - Ce^{-\tau/2 - e^{\gamma\tau}}.$$

Integrating both identities above and using the initial data, it follows that, for any $\tau > T$, we have

$$\begin{aligned} -Ce^{-T/2} &\leq \langle \underline{V}(\tau,\cdot), \psi\rangle_{L^2_{\mu_G}} - \langle V(T,\cdot),\psi\rangle_{L^2_{\mu_G}} \leq \langle V(\tau,\cdot), \psi\rangle_{L^2_{\mu_G}} - \langle V(T,\cdot),\psi\rangle_{L^2_{\mu_G}} \\ &\leq \langle \overline{V}(\tau,\cdot), \psi\rangle_{L^2_{\mu_G}} - \langle V(T,\cdot),\psi\rangle_{L^2_{\mu_G}} \leq C\int_T^\tau e^\tau \|\tilde{w}(\tau',\cdot)\|_{L^2_{\mu_G}}\, d\tau'. \end{aligned}$$

As the left and right sides tend to zero as $T \to \infty$, due to (6.65), we deduce that $\langle V(\tau,\cdot),\psi\rangle_{L^2_{\mu_G}}$ is a Cauchy sequence in τ. It follows that there is α_∞ so that

$$\langle V, \psi\rangle_{L^2_{\mu_G}} \to \alpha_\infty \qquad \text{as } \tau \to \infty.$$

By (6.55), we also know that $\alpha_\infty > 0$.

Similar arguments using the spectral gap of $\mathcal{L}$ show that

$$\|V - \langle V, \psi\rangle_{L^2_{\mu_G}} \psi\|_{L^2_{\mu_G}} \to 0 \quad \text{as } \tau \to \infty,$$

which implies that

$$\|V - \alpha_\infty \psi\|_{L^2_{\mu_G}} \to 0 \quad \text{as } \tau \to \infty.$$

Then, using parabolic regularity theory and the boundedness of the right hand sides of (6.67) and (6.64) (here we are using the L^∞ bound of $e^\tau \tilde{w}$ (6.65)), this can be upgraded to

$$\|V - \alpha_\infty \psi\|_{L^\infty_{\text{loc}}} \to 0 \quad \text{as } \tau \to \infty.$$

After undoing the change of variables, this is precisely Proposition 6.5. As the technical details are exactly those in [2, 46], except for the small changes indicated above, we omit them.

6.4 The third step: convergence to the wave

We now conclude the proof of Theorems 1.3.(ii) and 1.4. We note that the proof of the final step here is different from that in [2, 46, 47]. It is no extra effort to consider both cases at the same time, so we use $\tilde{u}$ for both $\tilde{u_{\text{rd}}}$ and $\tilde{u_{\text{rcl}}}$.

Fix $\gamma \in (0, 1/10)$. We let $\alpha_\infty > 0$ be the constant from Proposition 6.5, and choose x_∞ be such that

$$\lim_{t\to\infty} e^{t^\gamma} U_\infty(x_\infty + t^\gamma) = \alpha_\infty. \tag{6.68}$$

Consider the difference

$$s(t,x) = e^x(\tilde{u}(t,x) - U_*(x_\infty + \varepsilon + x)).$$

We point out that, due to (6.68) and Proposition 6.5, we have

$$0 < s(t, t^\gamma) < 2\varepsilon, \tag{6.69}$$

for t sufficiently large. Using the identities

$$\tilde{w} = -\tilde{u}_x - \tilde{u} + A(\tilde{u}) \quad \text{and} \quad 0 = -U_*' - U_* + A(U_*),$$

we derive a first order ODE for s:

$$s_x = e^x(\tilde{u}_x + \tilde{u} - U_*' - U_*) = e^x(-\tilde{w} + A(\tilde{u}) - A(U_*)) = -e^x\tilde{w} + \xi s \tag{6.70}$$

where

$$\xi = \frac{A(\tilde{u}) - A(\tilde{U}_*)}{\tilde{u} - U_*} \geq 0.$$

As $\tilde{w} \geq 0$ and $s(t, t^\gamma) > 0$, (6.70) implies that

$$s(t,x) > 0 \qquad \text{for all } x < t^\gamma. \tag{6.71}$$

Next, by (6.38), in the reaction diffusion case, and Lemma 6.6, in the reactive conservation law case, we have

$$\tilde{w}(t,x) \leq \frac{C(x + \log t + 1)}{t} e^{-x - \frac{(x+\log t)^2}{Ct}} \qquad \text{for all } x > -\log t,$$

so that

$$s_x > -\frac{C(x + \log t + 1)}{t} e^{-\frac{(x+\log t)^2}{Ct}}, \quad \text{for all } x \in (-\log t, t^\gamma).$$

Integrating this, we find, for any $x > -\log(t)$, we have

$$s(t,x) - s(t,t^\gamma) < C \int_x^{t^\gamma} \frac{(y + \log t + 1)}{t} e^{-\frac{(y+\log t)^2}{Ct}}\, dy \leq \frac{C}{\sqrt{t}}.$$

In view of (6.69), we deduce that

$$s(t,x) < \frac{C}{\sqrt{t}} + 2\varepsilon. \tag{6.72}$$

Putting together (6.71) and (6.72) and undoing the change of variables, we obtain

$$0 < \tilde{u}(t,x) - U_*(x_\infty + \varepsilon + x) < e^{-x}\Big(\frac{C}{\sqrt{t}} + 2\varepsilon\Big) \qquad \text{for all } x < t^\gamma.$$

Notice also that

$$\|U_*(x_\infty + \varepsilon + \cdot) - U_*(x_\infty + \cdot)\|_{L^\infty} \leq \|U_*'\|_{L^\infty}\varepsilon \leq C\varepsilon.$$

As a consequence of the previous two inequalities, we obtain

$$\sup_{x\in[-\log\frac{1}{2\varepsilon}, t^\gamma]} |\tilde{u}(t,x) - U_*(x_\infty + x)| \leq \frac{C}{\sqrt{t}} + C\sqrt{\varepsilon}. \tag{6.73}$$

On the other hand, clearly we have

$$\begin{aligned} \sup_{x<-\log\frac{1}{2\varepsilon}} |\tilde{u}(t,x) - U_*(x_\infty + x)| &\leq \sup_{x<-\log\frac{1}{2\varepsilon}} |\tilde{u}(t,x) - 1| + \sup_{x<-\log\frac{1}{2\varepsilon}} |1 - U_*(x_\infty + x)| \\ &\leq Ce^{-\alpha'(1)\log\frac{1}{2\varepsilon}} = C\varepsilon^{\alpha'(1)/2} \end{aligned} \tag{6.74}$$

and

$$\sup_{x>t^\gamma} |\tilde{u}(t,x) - U_*(x + x_\infty)| \leq \sup_{x>t^\gamma} \tilde{u}(t,x) + \sup_{x>t^\gamma} U_*(x + x_\infty) \leq Ce^{-t^\gamma}. \tag{6.75}$$

These inequalities can be seen by (6.4), Proposition 6.1 and (6.6).

The proof is now finished by combining (6.73), (6.74), and (6.75). □

6.5 The proof of Lemma 6.6

For notational ease, we drop the 'rcl' subscript for the remainder of the proof as we only work with the reactive conservation law in this section.

As the construction of the supersolution is somewhat complicated and opaque, let us first describe, roughly, why the claim holds. First, fix $T \gg 1$ and recall that $\tilde{w}$ satisfies

$$\tilde{w}_t - \Big(2 - \frac{1}{2(t+T)}\Big)\tilde{w}_x + A'(\tilde{u})\tilde{w}_x \;=\; \tilde{w}_{xx} + \tilde{w}(1 - A'(\tilde{u})).$$

Ignoring the A' terms, momentarily, the work of [34, equation (20)], suggests that, for $x \geq 1$, we should have

$$\tilde{w}(t,x) \leq \frac{Cx}{t} e^{-x - \frac{x^2}{(4+o(1))t}}.$$

This is exactly the desire bound on $x \geq 1$. On the other hand, in order to obtain a bound on $x < 1$, we must understand the reaction coefficient $1 - A'$. An important observation is that $A'(1) > 1$, which follows from the assumptions: A is convex, $A'(0) = 0$, and $A(1) = 1$. Hence, since $A'(\tilde{u}) \approx A'(1) > 1$

for $x < 1$ due to Proposition 6.1 (up to a constant shift), the comparison principle immediately yields

$$\sup_{x<1} \tilde{w}(t,x) \le \max\{e^{-t/C}, \tilde{w}(t,1)\} \le \frac{C}{t}.$$

The above two inequalities give (6.66).

We use the above heuristics to construct a suitable supersolution for $\tilde{w}$. Additionally, slightly abusing the notation, we let

$$\tilde{w}(t,x) = w(t, x + 2t - \tfrac{1}{2}\log(t+T) - C) \quad \text{and} \quad \tilde{u}(t,x) = u(t, x + 2t - \tfrac{1}{2}\log(t+T) - C).$$

Here, we choose C so that, in view of Proposition 6.1,

$$\inf_{t\ge 0, x\le 10} \tilde{u}(t,x) \ge u_*, \tag{6.76}$$

with $u_* \in (0,1)$ chosen so that $A'(u_*) > 1$. Note that it suffices to prove (6.66) for this shift of $\tilde{w}$.

The function $\tilde{w}$ satisfies

$$\tilde{w}_t - \Big(2 - \frac{1}{2(t+T)}\Big)\tilde{w}_x + A'(\tilde{u})\tilde{w}_x = \tilde{w}_{xx} + \tilde{w}(1 - A'(\tilde{u})). \tag{6.77}$$

Let us define two auxiliary functions:

$$\theta(t) = \frac{T}{t+T}$$

and, for $\kappa \in (0, 1/4)$ to be chosen,

$$\begin{aligned}\omega(t,x) &= x\frac{T}{t+T}\exp\Big\{4 - 2\sqrt{\frac{T}{t+T}} - \frac{x^2}{4(t+T)}\Big(1 - \kappa\sqrt{\frac{T}{t+T}}\Big)\Big\}\\ &= x\theta(t)\exp\Big\{4 - 2\sqrt{\theta(t)} - \frac{x^2}{4(t+T)}(1 - \kappa\sqrt{\theta(t)})\Big\}.\end{aligned}$$

We also set

$$\bar{w}(t,x) = \begin{cases} \frac{T}{t+T} & \text{for all } x \le 1,\\ \min\big\{\frac{T}{t+T}, e^{-x}\omega(t,x)\big\} & \text{for all } x > 1.\end{cases}$$

Clearly, it suffices to show that there is $A > 0$ such that

$$w(t,x) \le A\overline{w}(t,x) \qquad \text{for all } t \ge 1, x \in \mathbb{R}.$$

To this end, by the comparison principle, it is enough to show the following:

(i) $w(1,\cdot) \le A\overline{w}(1,\cdot)$,

(ii) $\bar{w} = e^{-x}\omega$ on $[10,\infty)$,

(iii) $e^{-1}\omega(t,1) > \theta(t)$ for all $t \ge 0$,

(iv) θ is a supersolution to (6.77) on $(-\infty, 10]$, and

(v) $e^{-x}\omega$ is a supersolution to (6.77) on $(0,\infty)$.

Before proceeding, let us discuss the purpose of (ii) and (iiii). The former allows us to only check (iv) on the domain $(-\infty, 10]$. This is crucial because θ is *not* a supersolution to (6.77) on $\mathbb{R}$ but only when $\tilde{u} \approx 1$. On the other hand, (iii) guarantees the continuity of $\bar{w}$. As such, we see that $\bar{w}$ is a minimum of two supersolutions on $[1, 10]$ and, thus, is itself a supersolution.

We now check conditions (i)-(v). As w satisfies a parabolic equation with L^1 initial data:

$$\int |w(0,x)|\,dx = \int w(0,x)\,dx = \int (-\partial_x u_0 - \eta_*(u_0))\,dx \le \int (-\partial_x u_0)\,dx = 1,$$

it can easily be bounded by

$$w(1,x) \le Ce^{-x^2/C}.$$

Up to increasing T and A, it is clear that

$$Ce^{-x^2/C} \le A\overline{w},$$

whence (i) is established.

Both (ii) and (iii) are elementary, by increasing T if necessary. Additionally, after increasing T further and recalling (6.76), we find

$$\theta_t - \Big(2 - \frac{1}{2(t+T)}\Big)\theta_x + A'(\tilde u)\theta_x - \theta_{xx} - \theta(1 - A'(\tilde u)) = -\frac{\theta}{t+T} + \theta(A'(u_*) - 1) > 0,$$

for $x < 10$, which implies (iv).

Finally, we check (v). This computation is made easier by noting that it suffices to check that

$$\omega_t + \frac{1}{2(t+T)}(\omega_x - \omega) + A'(\tilde u)\omega_x - \omega_{xx} \ge 0. \tag{6.78}$$

A direct computation yields

$$\begin{aligned}
\omega_t+&\frac{1}{2(t+T)}(\omega_x - \omega) + A'(\tilde u)\omega_x - \omega_{xx} \\
&= \omega\Bigg[\frac{\dot\theta}{\theta} - \frac{\dot\theta}{\sqrt{\theta}} + \frac{x^2}{4(t+T)^2}(1-\kappa\sqrt{\theta}) + \frac{x^2}{4(t+T)}\kappa\frac{\dot\theta}{2\sqrt{\theta}} \\
&\qquad + \Big(\frac{1}{2(t+T)} + A'(\tilde u)\Big)\Big(\frac{1}{x} - \frac{x}{2(t+T)}(1-\kappa\sqrt{\theta})\Big) - \frac{1}{2(t+T)} \\
&\qquad - \Big(\Big(\frac{1}{x} - \frac{x}{2(t+T)}(1-\kappa\sqrt{\theta})\Big)^2 + \frac{1}{x^2} + \frac{1}{2(t+T)}\Big(1-\kappa\sqrt{\theta}\Big)\Big)\Bigg] \\
&= \omega\Bigg[-\frac{\dot\theta}{\sqrt{\theta}} + \frac{x^2}{4(t+T)^2}\kappa\sqrt{\theta}\Big(1-\kappa\sqrt{\theta} + \frac{\dot\theta}{2\theta}\Big) \\
&\qquad + \left(\frac{1}{2(t+T)} + A'(\tilde u)\right)\left(\frac{1}{x} - \frac{x}{2(t+T)}(1-\kappa\sqrt{\theta})\right) - \frac{3}{2(t+T)}\kappa\sqrt{\theta}\Bigg].
\end{aligned} \tag{6.79}$$

Up to increasing T, we have

$$-\frac{\dot\theta}{\sqrt{\theta}} = \frac{\sqrt{\theta}}{t+T} > \frac{3\kappa\sqrt{\theta}}{t+T} \qquad \text{and} \qquad 1 - \kappa\sqrt{\theta} + \frac{\dot\theta}{2\theta} \ge \frac{1}{2}$$

(recall that $\kappa \in (0, 1/4)$ and $\theta \le 1$). Using these inequalities and that $A' \ge 0$, (6.79) becomes

$$\begin{aligned}
\omega_t+&\frac{1}{2(t+T)}(\omega_x - \omega) + A'(\tilde u)\omega_x - \omega_{xx} \\
&\ge \omega\left[\frac{\sqrt{\theta}}{2(t+T)} + \frac{x^2}{4(t+T)^2}\frac{\kappa\sqrt{\theta}}{2} + \left(\frac{1}{2(t+T)} + A'(\tilde u)\right)\left(\frac{1}{x} - \frac{x}{2(t+T)}\right)\right].
\end{aligned} \tag{6.80}$$

The first and second terms in the right side of (6.80) are positive. Next, we show that they dominate the potentially negative last term.

When $x \leq \sqrt{2(t+T)}$, the final term in (6.80) is positive as well. We obtain

$$\omega_t + \frac{1}{2(t+T)}(\omega_x - \omega) + A'(\tilde{u})\omega_x - \omega_{xx} \geq 0 \qquad \text{when } x \leq \sqrt{2(t+T)}, \tag{6.81}$$

as desired. Hence, we need only consider the case $x > \sqrt{2(t+T)}$.

In this case, we first show that $A'(\tilde{u})$ is small when $x > \sqrt{2(t+T)}$. Indeed, using Proposition 6.1 and (6.4), we see that, up to increasing T, we have

$$\tilde{u}(t,x) < \tilde{u}(t, 2\sqrt{t+T}) \leq C\exp(-\sqrt{t+T}).$$

Due to the assumptions on A, we then deduce

$$A'(\tilde{u}(t,x)) \leq C\tilde{u}(t,x) \leq C\exp(-\sqrt{t+T}) \leq \frac{1}{2(t+T)}. \tag{6.82}$$

Applying (6.82) and Young's inequality, we have

$$\begin{aligned}\Big(\frac{1}{2(t+T)} + A'(\tilde{u})\Big)\Big(\frac{1}{x} - \frac{x}{2(t+T)}\Big) &\geq -\frac{x}{4(t+T)^2} \\ &\geq -\frac{x^2}{8(t+T)^{5/2}} - \frac{1}{8(t+T)^{3/2}} = -\frac{1}{\sqrt{T}}\left(\frac{x^2}{8(t+T)^2}\sqrt{\theta} + \frac{\sqrt{\theta}}{2(t+T)}\right).\end{aligned}$$

Clearly, up to increasing T, this, along with (6.80) implies that

$$\omega_t + \frac{1}{2(t+T)}(\omega_x - \omega) + A'(\tilde{u})\omega_x - \omega_{xx} \geq 0 \qquad \text{when } x \geq \sqrt{2(t+T)}. \tag{6.83}$$

The combination of (6.81) and (6.83) yields (6.78). Due to the equivalence of (6.78), this yields (v), which concludes the proof. □

7 Semi-FKPP fronts: proof of Theorem 1.3.(i)

In this section, we show that there is x_0, depending on the initial condition, such that solutions u to (1.17) with $\chi \in [0,1)$, satisfy

$$u(t, x + 2t - \tfrac{3}{2}\log t + x_0) \to U_*(x), \qquad \text{as } t \to \infty.$$

The general structure of this proof is similar to the previous section. First, we obtain upper and lower bounds of u that show that the front $2t - (3/2)\log t + O(1)$. Second, we examine the behavior at $2t + t^\gamma$ to identify a candidate for the precise constant shift term. Finally, we use the 'pulled' nature of the front to show that closeness of the u and the traveling wave $O(t^\gamma)$ ahead of the front yields convergence everywhere.

The first two steps proceed as in the analogous case ($\beta < 2$) in the Burgers-FKPP setting of [2], with the use of the weighted Hopf-Cole transform in Proposition 4.1. This is recalled briefly in Section 7.1, where Proposition 7.1 is proved.

On the other hand, the final step, convergence to the traveling wave, is significantly more difficult than in [2]. This is mostly due to the fact that the equation for the weighted Hopf-Cole transform of u involves terms that did not appear in [2] where $\alpha(u) = u$. This part of the proof is presented in Section 7.2. The key result here is the differential inequality in Lemma 7.3 that follows from yet another surprising and intricate set of algebraic cancellations.

In this section, we use the tilde for a change to the moving frame $x \mapsto x + 2t - (3/2)\log(t+1)$, such as

$$\tilde{u}(t,x) := u(t, x + 2t - \frac{3}{2}\log(t+1)).$$

7.1 The precise behavior at $2t + t^\gamma$

The goal of this section is to establish the following proposition, the analogue of [2, Lemma 4.1].

Proposition 7.1. *Under the assumptions of Theorem 1.3.(i), there exists $\omega_\infty > 0$, depending on the initial condition, such that, for any $\gamma \in (0, 1/2)$, we have*

$$\lim_{t\to\infty} \frac{e^{t^\gamma}}{t^\gamma} \tilde{u}(t, t^\gamma) = \omega_\infty.$$

We discuss below the essential ingredients of [2, Lemma 4.1] and show that they are present in our setting. We begin with a preliminary upper bound that is required to show that the integral term in the weighted Hopf-Cole transform is not 'too big.' Notice that, by the comparison principle,

$$u \le u_{\mathrm{pp}},$$

where u_{pp} is the pushmi-pullyu front associated to $\chi = 1$, with the same initial condition. It follows immediately from Theorem 1.3.(ii) and (6.4) that

$$\tilde{u}(t,x) \le \min(1, C\exp\{-x + \log(t+1)\}) = \min(1, C(t+1)e^{-x}). \tag{7.1}$$

We deduce that

$$\int_x^\infty \tilde{u}(t,y)\,dy \le \begin{cases} C + \log(t+1) - x & \text{if } x \le \log(t+1), \\ C(t+1)e^{-x} & \text{if } x \ge \log(t+1). \end{cases} \tag{7.2}$$

Next, we work with the weighted Hopf-Cole transform of $\tilde{u}$:

$$v(t,x) = \exp\Big(x + \sqrt{\chi}\int_x^\infty \alpha(\tilde{u}(t,y))\,dy\Big)\tilde{u}(t,x).$$

Following the computations in the proof of Proposition 4.1, we see that

$$v_t - v_{xx} + \frac{3}{2(t+1)}(v_x - v) - \frac{g[\tilde{u}]}{\tilde{u}}v = 0, \tag{7.3}$$

where

$$\begin{aligned} g[\tilde{u}] :=& f(\tilde{u}) - \tilde{u} - \chi\alpha(\tilde{u})^2\tilde{u} + 2\sqrt{\chi}\alpha(\tilde{u})\tilde{u}_x \\ &+ \tilde{u}\Big(-\sqrt{\chi}\int_x^\infty \alpha''(\tilde{u})\tilde{u}_y^2\,dy + \sqrt{\chi}\int_x^\infty \alpha'(\tilde{u})f(\tilde{u})\,dy\Big) \le 0. \end{aligned}$$

Using (7.2), we make two crucial observations:

$$\begin{aligned} &v(t, -t^\gamma) \le C\exp\big(-\tfrac{1}{2}(1-\sqrt{\chi})t^\gamma\big) \qquad \text{and} \\ &\frac{g[\tilde{u}](t,t^\gamma)}{\tilde{u}(t,t^\gamma)} \ge -C\exp\big(-\tfrac{1}{2}t^\gamma\big), \end{aligned} \tag{7.4}$$

for any $\gamma > 0$.

The key ingredients in the proof of [2, Lemma 4.1] are

(i) (subsolution) v satisfies

$$v_t - v_{xx} + \frac{3}{2(t+1)}(v_x - v) \le 0;$$

(ii) (not too far from supersolution) for any $\gamma \in (0, 1/2)$ and $x > t^\gamma$,

$$v_t - v_{xx} + \frac{3}{2(t+1)}(v_x - v) \ge -ce^{-t^\gamma/C};$$

(iii) (approximate Dirichlet boundary condition) for any $\gamma > 0$,

$$v(t, -t^\gamma) \le Ce^{-t^\gamma/C}. \tag{7.5}$$

It is clear that (i) follows from (7.3)-(7.4) while (ii) and (iii) follow from (7.4).

From this point, the proof of Proposition 7.1 is verbatim the same as [2, Lemma 4.1] and is omitted.

7.2 Convergence to the wave

7.2.1 The setup

We now use Proposition 7.1 to finish the proof of Theorem 1.3.(i). While we follow the approach of [2], there is a technical complication due to extra terms that appear in the general setting, such as the nontiviality of α''.

To begin, we make two observations. Firstly, the combinations of Proposition 7.1 and (7.1) yields

$$\lim_{t\to\infty} \frac{v(t, t^\gamma)}{t^\gamma} = \omega_\infty. \tag{7.6}$$

In fact, this is the bound we use, not Proposition 7.1. Secondly, since

$$\tilde{u} v_x = v\tilde{u}_x + (1 - \sqrt{\chi}\alpha(\tilde{u}))\tilde{u}v,$$

we can rewrite the equation for v as

$$v_t - v_{xx} + \frac{3}{2t}(v_x - v) - 2\sqrt{\chi}\alpha(\tilde{u})v_x - G[\tilde{u}]v = 0. \tag{7.7}$$

Here, for a suitably smooth and decaying function ρ, we define

$$\begin{aligned} G[\rho] &:= \frac{g[\rho]}{\rho} - \frac{2\sqrt{\chi}\alpha(\rho)\rho_x}{\rho} - 2\sqrt{\chi}\alpha(\rho)(1 - \sqrt{\chi}\alpha(\rho)) \\ &= \frac{f(\rho)}{\rho} - 1 - \chi\alpha(\rho)^2 - 2\sqrt{\chi}\alpha(\rho)(1 - \sqrt{\chi}\alpha(\rho)) \\ &\quad - \sqrt{\chi}\int_x^\infty \alpha''(\rho)\rho_y^2\, dy + \sqrt{\chi}\int_x^\infty \alpha'(\rho)f(\rho)\, dy. \end{aligned}$$

With some arithmetic, this can be simplified to the form we use below:

$$\begin{aligned} G[\rho] = &- (1+\chi)\alpha(\rho) - 2\sqrt{\chi}(1 - \sqrt{\chi})\alpha(\rho) + \chi\rho\alpha'(\rho)(1 - \alpha(\rho)) \\ &- \sqrt{\chi}\int_x^\infty \alpha''(\rho)\rho_y^2\, dy + \sqrt{\chi}\int_x^\infty \alpha'(\rho)f(\rho)\, dy \end{aligned}$$

The re-writing of (7.7) in terms of G, at the expense of introducing an extra drift term in the left side of (7.7) compared to (7.3), is crucial in the proof of the main Lemma 7.2, below. This algebraic ingredient is a major difference with the proofs of [2, 46, 47].

We now take the weighted Hopf-Cole transform of a suitable shift of the traveling wave. To this end, for $\omega \in (\omega_\infty/2, 2\omega_\infty)$, we let

$$\varphi_\omega(t,x) = U_*(x + \zeta_\omega(t)), \tag{7.8}$$

with ζ_ω to be chosen. This satisfies

$$\partial_t \varphi_\omega - \partial_x^2 \varphi_\omega - \Big(2 - \frac{3}{2(t+1)}\Big)\partial_x \varphi_\omega - f(\varphi_\omega) = \Big(\frac{3}{2(t+1)} + \dot{\zeta}_\omega\Big)\partial_x \varphi_\omega. \tag{7.9}$$

Next, we define its Hopf-Cole transform:

$$\psi_\omega(t,x) = \exp\Big(x + \sqrt{\chi}\int_x^\infty \alpha(\varphi_\omega(t,y))\,dy\Big)\varphi_\omega(t,x),$$

and fix the shift $\zeta_\omega(t)$ by the normalization

$$\psi_\omega(t,t^\gamma) = \omega t^\gamma. \tag{7.10}$$

Recall that there are constants $D>0$ and $B \in \mathbb{R}$ so that the traveling wave has the asymptotics

$$U_*(x) = Dxe^{-x} + Be^{-x} + o(e^{-x}) \qquad \text{for } x \gg 1.$$

Thus, we have

$$\zeta_\omega(t) = -\log\big(\frac{\omega}{D}\big) + \frac{1}{t^\gamma}\Big(\frac{B}{D} - \log\big(\frac{\omega}{D}\big)\Big) + o(t^{-\gamma}), \tag{7.11}$$

and

$$|\dot{\zeta}_\omega(t)| \le \frac{C}{t^{1+\gamma}}. \tag{7.12}$$

The constant C is independent of ω due to the restriction $\omega \in (\omega_\infty/2, 2\omega_\infty)$. From (7.9) and (7.12), we find

$$\Big|\partial_t \psi_\omega - \partial_x^2 \psi_\omega + \frac{3}{2t}(\partial_x \psi_\omega - \psi_\omega) - 2\sqrt{\chi}\alpha(\varphi_\omega)\partial_x \psi_\omega - G[\varphi_\omega]\psi_\omega\Big| \le \frac{C}{t^{1-\gamma}} \qquad \text{for } |x| < t^\gamma. \tag{7.13}$$

7.2.2 The main lemma and heuristic description of the proof of Theorem 1.3.(i)

The main step in the proof of Theorem 1.3.(i) is the following analogue of [2, Lemma 4.3] bounding the difference between the weighted Hopf-Cole transforms.

Lemma 7.2. *Under the assumptions of Theorem 1.3.(i), if $\varepsilon \in (0, \omega_\infty/2)$ and if $\gamma < 1/3$, then there is $\lambda > 0$ so that*

$$v(t,x) - \psi_{\omega_\infty+\varepsilon}(t,x) \le \frac{C}{t^\lambda} \qquad \textit{for all } t > T_\varepsilon \textit{ and } |x| < t^\gamma.$$

The proof of Lemma 7.2 is substantially more intricate than [2, Lemma 4.3]. We postpone it to Section 7.2.3.

A simple consequence of Lemma 7.2 useful in the proof of Theorem 1.3.(i) is that, for any $L > 0$, we have

$$\sup_{|x| \le L} v(t,x) \le C. \tag{7.14}$$

Proof of Theorem 1.3.(i)

While the proof of Theorem 1.3.(i) is very similar to that of [2, Theorem 1.1 for $\beta < 2$], we give the complete details due to the subtle but important difference between Lemma 7.2 and [2, Lemma 4.3]; that is, that we use v directly, here, whereas [2] uses an auxiliary function.

The two key inequalities that we prove are: for any $L, \varepsilon > 0$,

$$\inf_{x<t^\gamma} (\tilde{u} - \varphi_{\omega_\infty - \varepsilon}) \geq 0 \quad \text{and} \quad \lim_{t\to\infty} \sup_{x\in[-L,L]} (\tilde{u} - \varphi_{\omega_\infty + \varepsilon}) \leq C\varepsilon. \tag{7.15}$$

Before establishing these, we show how to use them to conclude the proof.

We begin with an observation:

$$\Big\| \frac{e^{x_+}}{x_+ + 1} (\varphi_{\omega_\infty} - \varphi_{\omega_\infty - \varepsilon}) \Big\|_{L^\infty} \leq C\varepsilon, \tag{7.16}$$

due to the regularity of the traveling wave U_*, the definition of φ_ω (7.8), and the expansion of ζ_ω (7.11).

We immediately see from (7.15) and (7.16) that

$$\lim_{t\to\infty} \|\tilde{u} - \varphi_{\omega_\infty}\|_{L^\infty([-L,L])} = 0. \tag{7.17}$$

To handle the domain $(-\infty, -L)$, it suffices to use both inequalities in (7.16) to find: for all $x \in (-\infty, -L)$ and t sufficiently large,

$$\begin{aligned} |\varphi_{\omega_\infty}(t,x) - \tilde{u}(t,x)| &\leq \|\varphi_{\omega_\infty} - \varphi_{\omega_\infty-\varepsilon}\|_{L^\infty} + |\varphi_{\omega_\infty-\varepsilon}(t,x) - \tilde{u}(t,x)| \\ &\leq C\varepsilon + |\varphi_{\omega_\infty-\varepsilon}(t,x) - \tilde{u}(t,x)| = C\varepsilon + \tilde{u}(t,x) - \varphi_{\omega_\infty-\varepsilon}(t,x) \\ &\leq C\varepsilon + 1 - \varphi_{\omega_\infty-\varepsilon}(t,x) \leq C\varepsilon + Ce^{-\alpha'(1)L}. \end{aligned} \tag{7.18}$$

To handle the domain (L, ∞), we argue similarly to find: for all $x \in (L, +\infty)$ and t sufficiently large,

$$|\varphi_{\omega_\infty}(t,x) - \tilde{u}(t,x)| \leq C\varepsilon + Ce^{-L}. \tag{7.19}$$

The claim is established after putting together (7.17), (7.19), and (7.18) and then sequentially taking t to infinity, ε to zero, and L to infinity. Hence, our goal is now to prove (7.15).

We begin with the lower bound in (7.15). For t sufficiently large, it follows from (7.10) and the smallness of the integrals in the respective definitions of the weighted Hopf-Cole transforms that

$$\tilde{u}(t, t^\gamma) > \varphi_{\omega_\infty-\varepsilon}(t, t^\gamma).$$

Since $\tilde{u}$ is steeper than U_* and, thus, than $\varphi_{\omega_\infty-\varepsilon}$, we deduce that

$$\tilde{u}(t,x) > \varphi_{\omega_\infty-\varepsilon}(t,x) \qquad \text{for all } x < t^\gamma, \tag{7.20}$$

which concludes the proof of the first inequality in (7.15).

Next, we consider the second inequality in (7.15). We decompose the difference as

$$\begin{aligned} &\tilde{u}(t,x) - \varphi_{\omega_\infty+\varepsilon}(t,x) \\ &\quad = \exp\Big\{ - x - \sqrt{\chi} \int_x^\infty \alpha(\varphi_{\omega_\infty+\varepsilon}(t,y))\, dy \Big\} \big[v(t,x) - \psi_{\omega_\infty+\varepsilon}(t,x) \big] \\ &\qquad + e^{-x} v(t,x) \Big[\exp\Big\{ - \sqrt{\chi} \int_x^\infty \alpha(\tilde{u}(t,y))\, dy \Big\} - \exp\Big\{ - \sqrt{\chi} \int_x^\infty \alpha(\varphi_{\omega_\infty+\varepsilon}(t,y))\, dy \Big\} \Big]. \end{aligned} \tag{7.21}$$

The first term is easy to estimate using Lemma 7.2:

$$\exp\Big\{-x-\sqrt{\chi}\int_x^\infty \alpha(\varphi_{\omega_\infty+\varepsilon}(t,y))\,dy\Big\}\big[v(t,x)-\psi_{\omega_\infty+\varepsilon}(t,x)\big]\le e^L\frac{C}{t^\lambda}, \tag{7.22}$$

as desired.

Next, we consider the second term in (7.21). Using (7.1), it is clear that, up to an error term, we can restrict the limits of integration slightly:

$$\begin{aligned}
&e^{-x}v(t,x)\Big[\exp\Big\{-\sqrt{\chi}\int_x^\infty \alpha(\tilde u(t,y))\,dy\Big\}-\exp\Big\{-\sqrt{\chi}\int_x^\infty \alpha(\varphi_{\omega_\infty+\varepsilon}(t,y))\,dy\Big\}\Big]\\
&\qquad\le Ce^{-\frac12 t^\gamma}+e^{-x}v(t,x)\Big[\exp\Big\{-\sqrt{\chi}\int_x^{t^\gamma}\alpha(\tilde u(t,y))\,dy\Big\}\\
&\qquad\qquad-\exp\Big\{-\sqrt{\chi}\int_x^{t^\gamma}\alpha(\varphi_{\omega_\infty+\varepsilon}(t,y))\,dy\Big\}\Big].
\end{aligned}$$

We further decompose the second term above

$$\begin{aligned}
&\exp\Big\{-\sqrt{\chi}\int_x^{t^\gamma}\alpha(\tilde u(t,y))\,dy\Big\}-\exp\Big\{-\sqrt{\chi}\int_x^{t^\gamma}\alpha(\varphi_{\omega_\infty+\varepsilon}(t,y))\,dy\Big\}\\
&=\Big[\exp\Big\{-\sqrt{\chi}\int_x^{t^\gamma}\alpha(\tilde u(t,y))\,dy\Big\}-\exp\Big\{-\sqrt{\chi}\int_x^{t^\gamma}\alpha(\varphi_{\omega_\infty-\varepsilon}(t,y))\,dy\Big\}\Big]\\
&\quad+\Big[\exp\Big\{-\sqrt{\chi}\int_x^{t^\gamma}\alpha(\varphi_{\omega_\infty-\varepsilon}(t,y))\,dy\Big\}-\exp\Big\{-\sqrt{\chi}\int_x^{t^\gamma}\alpha(\varphi_{\omega_\infty+\varepsilon}(t,y))\,dy\Big\}\Big].
\end{aligned}$$

The first term in the right side is nonpositive because α is increasing and $\tilde u\ge\varphi_{\omega_\infty-\varepsilon}$ (recall (7.20)). For the second term, we use a Taylor expansion and (7.16) to conclude that

$$\begin{aligned}
&\exp\Big\{-\sqrt{\chi}\int_x^{t^\gamma}\alpha(\varphi_{\omega_\infty-\varepsilon}(t,y))\,dy\Big\}-\exp\Big\{-\sqrt{\chi}\int_x^{t^\gamma}\alpha(\varphi_{\omega_\infty+\varepsilon}(t,y))\,dy\Big\}\\
&=e^{-\sqrt{\chi}\int_x^{t^\gamma}\alpha(\varphi_{\omega_\infty-\varepsilon}(t,y))\,dy}\Big[1-\exp\Big\{-\sqrt{\chi}\int_x^{t^\gamma}(\alpha(\varphi_{\omega_\infty+\varepsilon}(t,y))-\alpha(\varphi_{\omega_\infty-\varepsilon}(t,y)))\,dy\Big\}\Big]\\
&\le 1-\exp\Big\{-\sqrt{\chi}\int_x^{t^\gamma}(\alpha(\varphi_{\omega_\infty+\varepsilon}(t,y))-\alpha(\varphi_{\omega_\infty-\varepsilon}(t,y)))\,dy\Big\}\\
&\le\sqrt{\chi}\int_x^{t^\gamma}\big(\alpha(\varphi_{\omega_\infty+\varepsilon}(t,y))-\alpha(\varphi_{\omega_\infty-\varepsilon}(t,y))\big)\,dy\le C\varepsilon.
\end{aligned}$$

Using the boundedness of $e^{-x}v$ (7.14) as well as the work above, we find

$$\begin{aligned}
&e^{-x}v(t,x)\Big[\exp\Big\{-\sqrt{\chi}\int_x^\infty \alpha(\tilde u(t,y))\,dy\Big\}-\exp\Big\{-\sqrt{\chi}\int_x^\infty \alpha(\varphi_{\omega_\infty+\varepsilon}(t,y))\,dy\Big\}\Big]\\
&\qquad\le Ce^{-\frac12 t^\gamma}+\frac{C}{t^\lambda}+C\varepsilon.
\end{aligned}\tag{7.23}$$

By applying (7.22) and (7.23) in (7.21), we obtain

$$\tilde u(t,x)-\varphi_{\omega_\infty+\varepsilon}\le Ce^{-\frac12 t^\gamma}+\frac{C}{t^\lambda}+C\varepsilon.$$

This yields exactly the second inequality in (7.15), which completes the proof. □

7.2.3 Proof of Lemma 7.2

A key step in proving Lemma 7.2 is the derivation of a differential inequality for

$$s(t,x) = v(t,x) - \psi_{\omega_\infty+\varepsilon}(t,x).$$

As the computation is somewhat complicated, we state it here as a further lemma and prove it after we show how it is used to prove Lemma 7.2.

Lemma 7.3. *In the setting of Lemma 7.2, whenever $s \geq 0$, we have*

$$s_t - s_{xx} + \frac{3}{2t}(s_x - s) - 2\sqrt{\chi}\alpha(\tilde{u})s_x \leq \frac{C}{t^{1-\gamma}}, \qquad \textit{for } |x| \leq t^\gamma. \tag{7.24}$$

Proof of Lemma 7.2

We prove this via the construction of a supersolution for (7.24) on $[-t^\gamma, t^\gamma]$. First, we check the boundary conditions for s. Using (7.5), we find

$$s(t, -t^\gamma) \leq C \exp\big(-\tfrac{1}{C}t^\gamma\big). \tag{7.25}$$

Additionally, using (7.6) and (7.10) and the choice of $\gamma = \omega_\infty + \varepsilon$ with $\varepsilon > 0$, we find

$$s(t, t^\gamma) \leq 0. \tag{7.26}$$

We now define a supersolution to (7.24). Fix any $\lambda \in (0, 1-4\gamma)$ and $Q \geq 1$, and let

$$\overline{s}(t,x) = \frac{Q}{t^\lambda}\cos\Big(\frac{x+t^\gamma}{2t^\gamma}\Big) = \frac{Q}{t^\lambda}\cos\Big(\frac{x}{2t^\gamma} + \frac{1}{2}\Big).$$

We first notice that $\overline{s}_x \leq 0$ on $[-t^\gamma, t^\gamma]$. This allows us to drop a positive term in the first step below, after which we directly compute:

$$\begin{aligned}\overline{s}_t - \overline{s}_{xx} &+ \frac{3}{2t}(\overline{s}_x - \overline{s}) - 2\sqrt{\chi}\alpha(\tilde{u})\overline{s}_x \geq \overline{s}_t - \overline{s}_{xx} + \frac{3}{2t}(\overline{s}_x - \overline{s}) \\ &= -\frac{\lambda}{t}\overline{s} + \frac{\gamma Q x}{2t^{1+\lambda+\gamma}}\sin\Big(\frac{x}{2t^\gamma} + \frac{1}{2}\Big) + \frac{1}{4t^{2\gamma}}\overline{s} + \frac{3}{2t}\Big(-\frac{xQ}{2t^{\gamma+\lambda}}\sin\Big(\frac{x}{2t^\gamma} + \frac{1}{2}\Big) - \overline{s}\Big).\end{aligned}$$

Taking t sufficiently large and then using that, for $|x| \leq t^\gamma$, we have $\overline{s}(t,x) \geq Q\cos(1)/t^\lambda$, we see

$$\overline{s}_t - \overline{s}_{xx} + \frac{3}{2t}(\overline{s}_x - \overline{s}) - 2\sqrt{\chi}\alpha(\tilde{u})\overline{s}_x \geq \frac{1}{8t^{2\gamma}}\overline{s} - \frac{CQ}{t^{1+\lambda}} \geq \frac{Q\cos(1)}{8t^{2\gamma+\lambda}} - \frac{CQ}{t^{1+\lambda}} \geq \frac{Q\cos(1)}{16t^{2\gamma+\lambda}}.$$

The last inequality follows because $2\gamma + \lambda < 1 + \lambda$ (recall that $\gamma < 1/3$). Recalling that $Q \geq 1$ and decreasing λ so that $2\gamma + \lambda < 1 - \gamma$ yields

$$\overline{s}_t - \overline{s}_{xx} + \frac{3}{2t}(\overline{s}_x - \overline{s}) - 2\sqrt{\chi}\alpha(\tilde{u})\overline{s}_x \geq \frac{C}{t^{1-\gamma}} \tag{7.27}$$

for t sufficiently large. In other words, $\overline{s}$ is a supersolution to (7.24) for t sufficiently large (independent of Q) and $|x| < t^\gamma$. We let T_0 be such that (7.27) holds for $t \geq T_0$.

By Lemma 7.3 and the comparison principle, we have

$$s(t,x) \leq \overline{s}(t,x) \qquad \text{on } |x| \leq t^\gamma \tag{7.28}$$

for all $t \geq T_0$ sufficiently large as long as we can verify that $s \leq \overline{s}$ on the parabolic boundary.

We first check the portion of the parabolic boundary $t = T_0$. There, it is easy to see that we can choose Q such that $s(T_0, x) \leq \overline{s}(T_0, x)$ for all $|x| < T_0^\gamma$; indeed, $s(T_0, \cdot)$ is bounded above and $\overline{s}/Q$ is bounded below on $[-t^\gamma, t^\gamma]$.

Next we check the portion of the parabolic boundary $|x| = t^\gamma$. Notice that

$$\overline{s}(t, \pm t^\gamma) \geq \frac{Q\cos(1)}{t^\lambda}.$$

From (7.25) and (7.26), we see that

$$s(t, \pm t^\gamma) \leq Ce^{-t^\gamma/C}.$$

Clearly $s(t, \pm t^\gamma) \leq \overline{s}(t, \pm t^\gamma)$ for $t \geq T_1$ for some T_1. Hence, up to increasing Q to handle the range $t \in [T_0, \max\{T_0, T_1\}]$, we have

$$s(t, \pm t^\gamma) \leq \overline{s}(t, \pm t^\gamma) \qquad \text{for all } t \geq T_0.$$

The previous two paragraphs show that $s \leq \overline{s}$ on the parabolic boundary. The inequality (7.28) follows, from which the conclusion of Lemma 7.2 follows. □

Proof of Lemma 7.3

We now establish the differential inequality (7.24). As ω remains fixed throughout the proof below, we drop it notationally, for ease. We start by obtaining a few preliminary results that help us establish (7.24). First, note that

$$\tilde{u} > \varphi \quad \text{whenever} \quad s > 0. \tag{7.29}$$

Indeed, take x_t to be the 'farthest right' point where $s(x_t) = 0$ (it is clearly negative at $+\infty$, so this is well defined). Then, we have

$$\tilde{u}(t, x_t) \exp\Big(x_t + \sqrt{\chi}\int_{x_t}^\infty \alpha(\tilde{u}(t, y))\, dy\Big) = \varphi(t, x_t) \exp\Big(x_t + \sqrt{\chi}\int_{x_t}^\infty \alpha(\varphi(t, y))\, dy\Big).$$

Suppose that $\tilde{u}(t, x_t) < \varphi(t, x_t)$. In this case, as $\tilde{u}$ is steeper than φ, it follows that $\tilde{u} < \varphi$ on (x_t, ∞), so we find, using monotonicity of $\alpha(u)$,

$$\tilde{u}(t, x_t) < \varphi(t, x_t) = \tilde{u}(t, x_t) \exp\Big(- \sqrt{\chi}\int_{x_t}^\infty (\alpha(\varphi(t, y)) - \alpha(\tilde{u}(t, y)))\, dy\Big) < \tilde{u}(t, x_t),$$

which is clearly a contradiction. It follows that $\tilde{u}(t, x_t) \geq \varphi(t, x_t)$, and another use of the steepness comparison together with the choice of x_t implies that

$$\tilde{u}(t, x) > \varphi(t, x) \quad \text{for all } x < x_t. \tag{7.30}$$

Now, by construction of x_t, if $s(t, x) > 0$ it must be that $x < x_t$. Thus, by (7.30), $\tilde{u}(t, x) > \varphi(t, x)$, as desired. This completes the proof of the claim (7.29).

Next, since $u(t, \cdot)$ and U_* are strictly decreasing, we can find the function $\eta(t, u)$ such that

$$- u_x(t, x) = \eta(t, u(t, x)).$$

One can see immediately that $\eta(t, u(t, x)) = w(t, x) + \eta_*(u(t, x))$, and the positivity of w implies that, for any $z \in (0, 1)$,

$$\eta(t, z) > \eta_*(z). \tag{7.31}$$

In addition, Proposition 3.2 and Lemma A.4 yield

$$f(z) = \eta_*(z)(2 - \eta_*'(z)) \quad \text{and} \quad \sqrt{\chi}(z - A(z)) \le \eta_*(z) \le z - A(z). \tag{7.32}$$

We are now ready to prove (7.24). By combining (7.7) and (7.13), we see that

$$\begin{aligned} &s_t - s_{xx} + \frac{3}{2t}(s_x - s) - 2\sqrt{\chi}\alpha(\tilde u)s_x \\ &\qquad \le \frac{C}{t^{1-\gamma}} + 2\sqrt{\chi}(\alpha(\tilde u) - \alpha(\varphi))\psi_x + G[\tilde u]v - G[\varphi]\psi \\ &\qquad \le \frac{C}{t^{1-\gamma}} + 2\sqrt{\chi}(\alpha(\tilde u) - \alpha(\varphi))\psi_x + G[\tilde u]s + \Big(G[\tilde u] - G[\varphi]\Big)\psi. \end{aligned}$$

Thus, it suffices to prove

$$2\sqrt{\chi}(\alpha(\tilde u) - \alpha(\varphi))\psi_x + G[\tilde u]s + \Big(G[\tilde u] - G[\varphi]\Big)\psi \le 0. \tag{7.33}$$

This is the reason for the change from using g to G, cf. (7.3) and (7.7). As we see below, the key observation in the proof of (7.33), and, as a consequence, a key step in the proof of Theorem 1.3.(i), is that

$$G[\tilde u] - G[\varphi] \le 0. \tag{7.34}$$

The analogous term had we not made the 'swap' is $g[\tilde u] - g[\varphi]$; however, we are unable to prove this is nonpositive. On the other hand, the cost of making the 'swap' is the $-2\chi\alpha(\tilde u)s_x$ term in the left hand side of (7.24). As we have seen in the proof of Lemma 7.2, this term, while it may not have a definite sign, did not pose an issue with constructing a supersolution.

The easiest term in (7.33) is ψ_x. Here, we simply use its relationship to φ and (7.32) to find:

$$\psi_x = \frac{\psi}{\varphi}(\varphi_x + (1 - \sqrt{\chi}\alpha(\varphi))\varphi) \le \frac{\psi}{\varphi}(-\sqrt{\chi}\varphi(1 - \alpha(\varphi)) + (1 - \sqrt{\chi}\alpha(\varphi))\varphi) = (1 - \sqrt{\chi})\psi.$$

Now, we prove (7.34). We estimate integral terms in the definition of G using (7.31):

$$\begin{aligned} &-\sqrt{\chi}\int_x^\infty \big(\alpha''(\tilde u)\tilde u_y^2 - \alpha'(\tilde u)f(\tilde u)\big)\,dy + \sqrt{\chi}\int_x^\infty \big(\alpha''(\varphi)\partial_y\varphi^2 - \alpha'(\varphi)f(\varphi)\big)\,dy \\ &\qquad = -\sqrt{\chi}\int_0^{\tilde u}\Big(\alpha''(z)\eta(t,z) - \frac{\alpha'(z)f(z)}{\eta(t,z)}\Big)\,dz + \sqrt{\chi}\int_0^{\varphi}\Big(\alpha''(z)\eta_*(z) - \frac{\alpha'(z)f(z)}{\eta_*(z)}\Big)\,dz \\ &\qquad \le -\sqrt{\chi}\int_0^{\tilde u}\Big(\alpha''(z)\eta_*(z) - \frac{\alpha'(z)f(z)}{\eta_*(z)}\Big)\,dz + \sqrt{\chi}\int_0^{\varphi}\Big(\alpha''(z)\eta_*(z) - \frac{\alpha'(z)f(z)}{\eta_*(z)}\Big)\,dz \\ &\qquad = -\sqrt{\chi}\int_\varphi^{\tilde u}\Big(\alpha''(z)\eta_*(z) - \frac{\alpha'(z)f(z)}{\eta_*(z)}\Big)\,dz. \end{aligned}$$

On the other hand, we can convert some of the non-integral terms in G to integral terms:

$$\begin{aligned} &\chi\tilde u\alpha'(\tilde u)(1 - \alpha(\tilde u)) - \chi\varphi\alpha'(\varphi)(1 - \alpha(\varphi)) \\ &\qquad = \chi\int_\varphi^{\tilde u}(z\alpha'(z)(1 - \alpha(z))'\,dz = \chi\int_\varphi^{\tilde u}(\alpha'(z)(z - A(z))'\,dz \\ &\qquad = \chi\int_\varphi^{\tilde u}\alpha''(z)(z - A(z))\,dz + \chi\int_\varphi^{\tilde u}\alpha'(z)(1 - A'(z))\,dz \\ &\qquad = \chi\int_\varphi^{\tilde u}\alpha''(z)(z - A(z))\,dz + \chi\int_\varphi^{\tilde u}\alpha'(z)\Big(1 - \frac{1}{\chi}\Big(\frac{f(z)}{z - A(z)} - 1\Big)\Big)\,dz. \end{aligned}$$

Using then (7.32), we find

$$\begin{aligned}&\chi\tilde{u}\alpha'(\tilde{u})(1-\alpha(\tilde{u}))-\chi\varphi\alpha'(\varphi)(1-\alpha(\varphi))\\&\quad\leq\sqrt{\chi}\int_{\varphi}^{\tilde{u}}\Big(\alpha''(z)\eta_*(z)+\chi\int_{\varphi}^{\tilde{u}}\alpha'(z)\Big(1-\frac{1}{\chi}\Big(\sqrt{\chi}\frac{f(z)}{\eta_*(z)}-1\Big)\Big)\,dz.\end{aligned}$$

Hence, we have

$$\begin{aligned}&-\sqrt{\chi}\int_x^{\infty}\big(\alpha''(\tilde{u})\tilde{u}_y^2-\alpha'(\tilde{u})f(\tilde{u})\big)\,dy+\sqrt{\chi}\int_x^{\infty}\big(\alpha''(\varphi)\partial_y\varphi^2-\alpha'(\varphi)f(\varphi)\big)\,dy\\&+\chi\tilde{u}\alpha'(\tilde{u})(1-\alpha(\tilde{u}))-\chi\varphi\alpha'(\varphi)(1-\alpha(\varphi))\\&\quad\leq\chi\int_{\varphi}^{\tilde{u}}\alpha'(z)\Big(1+\frac{1}{\chi}\Big)\,dz=(\chi+1)(\alpha(\tilde{u})-\alpha(\varphi)).\end{aligned}$$

To summarize, we have so far arrived at:

$$\begin{aligned}G[\tilde{u}]-G[\varphi]\leq&(\chi+1)(\alpha(\tilde{u})-\alpha(\varphi))-(1+\chi)(\alpha(\tilde{u})-\alpha(\varphi))\\&-2\sqrt{\chi}(\alpha(\tilde{u})-\alpha(\varphi))(1-\sqrt{\chi})=-2\sqrt{\chi}(1-\sqrt{\chi})(\alpha(\tilde{u})-\alpha(\varphi)).\end{aligned}$$

A similar computation yields

$$G[\tilde{u}]\leq-2\sqrt{\chi}(1-\sqrt{\chi})\alpha(\tilde{u}).$$

Combining all of the above, we have

$$\begin{aligned}&2\sqrt{\chi}(\alpha(\tilde{u})-\alpha(\varphi))\psi_x+G[\tilde{u}]s+(G[\tilde{u}]-G[\varphi])\psi\\&\quad\leq2\sqrt{\chi}(\alpha(\tilde{u})-\alpha(\varphi))(\varphi_x+(1-\sqrt{\chi}\alpha(\varphi))\varphi)\frac{\psi}{\varphi}\\&\qquad-2\sqrt{\chi}(1-\sqrt{\chi})\alpha(\tilde{u})s-2\sqrt{\chi}(1-\sqrt{\chi})(\alpha(\tilde{u})-\alpha(\varphi))\psi.\end{aligned}$$

Using again (7.32) to estimate φ_x, in addition to the fact that $\tilde{u}>\varphi$ (recall (7.29)) and the nonnegativity of ψ, we find

$$\begin{aligned}&2\sqrt{\chi}(\alpha(\tilde{u})-\alpha(\varphi))\psi_x+G[\tilde{u}]s+(G[\tilde{u}]-G[\varphi])\psi\\&\quad\leq2\sqrt{\chi}(\alpha(\tilde{u})-\alpha(\varphi))(-\sqrt{\chi}\varphi(1-\alpha(\varphi))+(1-\sqrt{\chi}\alpha(\varphi))\varphi)\frac{\psi}{\varphi}\\&\qquad-2\sqrt{\chi}(1-\sqrt{\chi})\alpha(\tilde{u})s-2\sqrt{\chi}(1-\sqrt{\chi})(\alpha(\tilde{u})-\alpha(\varphi))\psi\\&\quad=-2\sqrt{\chi}(1-\sqrt{\chi})\alpha(\tilde{u})s\leq0,\end{aligned}$$

where the last inequality follows due to the assumed nonnegativity of s.

Thus, (7.33) is established, completing the proof. □

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A Auxiliary results on nonlinearities and traveling waves

In this appendix, we discuss some auxiliary elementary facts related to the models introduced in this paper.

A.1 Non-triviality of the semi-FKPP range

We first show that the semi-FKPP range is nontrivial.

Lemma A.1. *Suppose that $A \in C^2([0,1])$ satisfies (2.10) and χ_{FKPP} is defined by (3.5). Then, we have $\chi_{FKPP} \le 1/2$.*

We note that a sharper bound than $1/2$ could be obtained by taking into account the order of vanishing where A "lifts away" from 0. If the order is $n-1$, one obtains $1/n$. We opt for a simpler proof since sharpness is not our goal here.

Proof. If $A''(0) > 0$, one obtains this from L'Hopital's rule:

$$\begin{aligned}
\chi_{FKPP} &\le \lim_{u\to 0} \frac{A(u)}{A'(u)(u - A(u))} = \lim_{u\to 0} \frac{A'(u)}{A'(u)(1 - A'(u)) + A''(u)(u - A(u))} \\
&= \lim_{u\to 0} \frac{A''(u)}{A''(u)(1 - A'(u)) - A'(u)A''(u) + A'''(u)(u - A(u)) + A''(u)(1 - A'(u))} \\
&= \frac{A''(0)}{A''(0) + A''(0)} = \frac{1}{2}.
\end{aligned}$$

This concludes the proof in this case.

If $A''(0) = 0$, let u_ε be the smallest $u > 0$ so that $A''(u) = \varepsilon$:

$$u_\varepsilon = \inf\{u : A''(u) = \varepsilon\}.$$

The existence of such a u_ε follows from (2.10) for ε sufficiently small. Clearly, u_ε is increasing in ε. Hence, there is $u_0 \in [0,1]$ such that $u_\varepsilon \to u_0$ as $\varepsilon \to 0$. By construction, we have

$$A(u_0) = A'(u_0) = A''(u_0) = 0.$$

The definition of u_ε and (2.10) imply that

$$A'(u_\varepsilon) = \int_0^{u_\varepsilon} A''(u)du \le \varepsilon u_\varepsilon,$$

and, more generally, we have

$$A'(u) = \int_0^u A''(v)dv \le \varepsilon u, \quad \text{for all } 0 \le u \le u_\varepsilon.$$

This gives

$$A(u_\varepsilon) = \int_0^{u_\varepsilon} A'(u)du \le \varepsilon u_\varepsilon^2/2.$$

Arguing again by L'Hopital's rule, we find

$$\begin{aligned}
\chi_{FKPP} &\le \lim_{u\to u_0} \frac{A(u)}{A'(u)(u - A(u))} = \lim_{\varepsilon\to 0} \frac{A(u_\varepsilon)}{A'(u_\varepsilon)(u_\varepsilon - A(u_\varepsilon)} \\
&= \lim_{\varepsilon\to 0} \frac{A'(u_\varepsilon)}{A'(u_\varepsilon)(1 - A'(u_\varepsilon)) + A''(u_\varepsilon)(u_\varepsilon - A(u_\varepsilon))} = \lim_{\varepsilon\searrow 0} \frac{1}{1 - A'(u_\varepsilon) + \frac{A''(u_\varepsilon)}{A'(u_\varepsilon)}(u_\varepsilon - A(u_\varepsilon))} \\
&\le \lim_{\varepsilon\to 0} \frac{1}{1 - \varepsilon u_\varepsilon + \frac{\varepsilon}{\varepsilon u_\varepsilon}(u_\varepsilon - \varepsilon u_\varepsilon^2/2)} = \lim_{\varepsilon\to 0} \frac{1}{2 - 3\varepsilon u_\varepsilon/2} = \frac{1}{2}.
\end{aligned}$$

Thus, the proof is complete. □

A.2 The pushed case

We have mentioned in the introduction that waves are pushed when $\chi > 1$ in (1.17). Although the pushed regime is not a focus of this paper, we provide a short proof of this fact for the sake of completeness.

Proposition A.2. *Suppose that $\chi > 1$, the function $A(u)$ satisfies (1.18), and*

$$f(u) = f_\chi(u) := f'(0)(u - A(u))(1 + \chi A'(u)). \tag{A.1}$$

Then, the minimal speed c_* is given by

$$c_* = \sqrt{f'(0)}\Big(\frac{1}{\sqrt{\chi}} + \sqrt{\chi}\Big) > 2\sqrt{f'(0)},$$

and the minimal speed traveling wave profile function is

$$\eta_*(u) = \sqrt{\chi}(u - A(u)). \tag{A.2}$$

As a consequence, traveling wave solutions of (1.1) have a purely exponential decay, as in (1.10).

Proof. We may assume that $f'(0) = 1$ without loss of generality. To obtain an upper bound on c_*, first observe that

$$f(u) = p_f(u)(c_\chi - p_f'(u)) \qquad \text{where } \ c_\chi = \sqrt{\chi} + \frac{1}{\sqrt{\chi}} \ \text{ and } \ p_f(u) = \sqrt{\chi}(u - A(u)).$$

Then applying (1.4), we obtain the upper bound

$$c_* \le \sup_{u\in[0,1]} \Big(p_f'(u) + \frac{f(u)}{p_f(u)}\Big) = c_\chi.$$

To prove the lower bound

$$c_* \ge c_\chi,$$

we argue by contradiction, assuming that $c_* < c_\chi$. Let η_* be the traveling wave profile function. It follows from (3.11), (3.13), and the remark in between that, since $c_* < c_\chi$, we have

$$\eta_*'(0) = \frac{c_* + \sqrt{c_*^2 - 4}}{2} < \frac{c_\chi + \sqrt{c_\chi^2 - 4}}{2} = \sqrt{\chi} = p_f'(0). \tag{A.3}$$

As we see below, the contradiction comes from the fact that (A.3) says that the traveling wave moving with the speed $c_\chi > c_*$ decays faster that the minimal speed wave U_*.

Proposition 3.1 implies that

$$p_f(u)(c_\chi - p_f'(u)) = f(u) = \eta_*(u)(c_* - \eta_*'(u)). \tag{A.4}$$

Integrating the identity (A.4) from 0 to u, letting P and N be the anti-derivatives of p_f and η_*, respectively, and rearranging terms yields

$$c_\chi P(u) - c_* N(u) = \frac{p_f^2(u)}{2} - \frac{\eta_*^2(u)}{2}. \tag{A.5}$$

Observe that, due to (A.3), we know that $p_f(u) > \eta_*(u)$ for all u that are positive and sufficiently small. Hence, we have $c_\chi P(u) > c_* N(u)$ for small u as well. Let u_0 be the smallest $u > 0$ so that $p_f(u_0) = \eta_*(u_0)$. This must exist because $p_f(1) = \eta_*(1)$.

From the construction of u_0, we have that

$$p_f(u) > \eta_*(u) \qquad \text{for all } u \in (0, u_0).$$

Recalling that $c_* < c_\chi$, by assumption, it follows that

$$c_\chi P(u_0) > c_* P(u_0) = c_* \int_0^{u_0} p_f(u)du > c_* \int_0^{u_0} \eta_*(u)du = c_* N(u_0). \tag{A.6}$$

On the other hand, by (A.5) and the fact that $p_f(u_0) = \eta_*(u_0)$, we find

$$c_\chi P(u_0) = c_* N(u_0) + \frac{p_f(u_0)^2}{2} - \frac{\eta_*(u_0)^2}{2} = c_* N(u_0).$$

This contradicts (A.6). We conclude that $c_* \geq c_\chi$, finishing the proof. □

As a consequence of Proposition A.2 and the comparison principle, we have the following.

Corollary A.3. *Suppose that $\chi > 1$, the function $A(u)$ satisfies (1.18), $f(u)$ satisfies (1.2), and*

$$f(u) \geq f'(0)(u - A(u))(1 + \chi A'(u)).$$

Then, the minimal speed c_* obeys a lower bound

$$c_* \geq \sqrt{f'(0)}\Big(\frac{1}{\sqrt{\chi}} + \sqrt{\chi}\Big) > 2\sqrt{f'(0)}.$$

As a consequence, traveling wave solutions of (1.1) are pushed.

A.3 The generality of the class of nonlinearities

We discuss here how restrictive the assumptions (1.17)-(1.18) on the nonlinearity $f(u)$are. The main point is that, while there is always at least one pair (χ, A) such that the form (1.17) holds for a given $f(u)$ satisfying assumptions (1.2), it is not clear when A satisfies the conditions in (1.18). For simplicity, we assume that $f'(0) = 1$.

We first show the existence of a pair (χ, A). Indeed, letting χ be the larger solution of

$$c_* = \frac{1}{\sqrt{\chi}} + \sqrt{\chi},$$

and using the traveling wave profile function η_*, we can define, as in (A.2),

$$A(u) = u - \frac{1}{\sqrt{\chi}}\eta_*(u). \tag{A.7}$$

Then, from Proposition 3.3, we have:

$$\begin{aligned} f(u) &= \eta_*(u)(c_* - \eta_*'(u)) = \sqrt{\chi}(u - A(u))(c_* - \sqrt{\chi}(1 - A'(u))) \\ &= \sqrt{\chi}(u - A(u))\big((c_* - \sqrt{\chi}) + \sqrt{\chi}A'(u)\big) \\ &= \sqrt{\chi}(u - A(u))\Big(\frac{1}{\sqrt{\chi}} + \sqrt{\chi}A'(u)\Big) = (u - A(u))(1 + \chi A'(u)). \end{aligned}$$

Hence $f(u)$ is in the form of (1.17). However, the function $A(u)$ defined in (A.7) need not satisfy the conditions in (1.18). Actually, when the minimal speed traveling wave has an extra linear factor in the exponential asymptotics (1.11), we do know from Proposition 3.2 that $\eta_*(u)$ is not $C^2[0,1]$. Thus, in that case this construction does not give a function $A(u)$ satisfying the regularity assumptions in (1.18).

Neither does $A(u)$ given by (A.7) have to be convex. Indeed, take any non-convex, smooth, positive $\eta(u)$ satisfying $\eta'(0) = 1$ and $\eta(0) = 0 = \eta(1)$, for example,

$$\eta(u) = \Big(\frac{1}{2}u + \frac{1}{2}\frac{\sin(10u)}{10}\Big)(1-u).$$

Then, defining $A(u)$ by (A.7) with $\chi = 1$, and setting

$$f(u) = (u - A(u))(1 + A'(u))$$

we obtain a nonlinearity of the form (1.17) and satisfying (1.1)-(1.2), but for which the choice of $A(u)$ in (A.7) is not convex.

Next we discuss the uniqueness of χ and A. Due to Proposition A.2, uniqueness of $\chi > 1$ and $A(u)$ satisfying (1.18) holds for nonlinearities of the form (A.1). On the other hand, uniqueness is not necessarily true in the pulled case. Indeed, consider the following simple example: for any $n \geq 3$, take the nonlinearity

$$f(u) = u - u^n.$$

Immediately, we see that, with $\chi = 0$ and $A(u) = u^n$, this has the form of (1.17):

$$f(u) = (u - u^n)(1 + 0 \cdot (nu^{n-1})) = (u - A(u))(1 + 0 \cdot A'(u)).$$

There is, however, another decomposition of the form (1.17): choosing $m = (n+1)/2$, $\tilde{\chi} = 1/(m-1)$, and $\tilde{A}(u) = u^m$, we can write

$$\begin{aligned} f(u) &= u - u^m + u^m - u^n = u - u^m + u^{m-1}(u - u^m) = (u - u^m)(1 + u^{m-1}) \\ &= (u - \tilde{A}(u))(1 + \tilde{\chi}\tilde{A}'(u)). \end{aligned}$$

Hence, the choice of χ and $A(u)$ is not necessarily unique in the pulled case.

Let us comment that we saw above an example of χ and $A(u)$ for which (u) has the form (1.17) but for which $A(u)$ is not convex. However, this example was pulled, so we do not necessarily have uniqueness of this decomposition. It is, thus, possible that $f(u)$ has another decomposition of the form (1.17) with a new $A(u)$ that is convex. We leave it as an open question to determine conditions on $f(u)$ that guarantee the existence of a *suitable* χ and $A(u)$ such that (1.17)-(1.18) hold.

A.4 Bounds for the traveling wave profile

Lemma A.4. *Let $f(u)$ be of the form (1.17) with $\chi \in [0,1]$ and $A(u)$ that satisfies* (1.18). *Then, the minimal speed traveling wave profile satisfies*

$$\sqrt{\chi}(z - A(z)) \leq \eta_*(z) \leq z - A(z). \tag{A.8}$$

Proof. The minimal speed traveling wave $U(x) = U_*(x)$ satisfies

$$-c_*U' = U'' + f'(0)(U - A(U))(1 + \chi A'(U)). \tag{A.9}$$

Introducing $V = -U' > 0$, we write (A.9) as

$$\frac{dU}{dx} = -V, \quad \frac{dV}{dx} = -c_* V + f'(0)(U - A(U))(1 + \chi A'(U)).$$

This leads to

$$\frac{dV}{dU} = c_* - \frac{f'(0)(U - A(U))(1 + \chi A'(U))}{V}.$$

We claim that when $0 \le \chi < 1$, all trajectories are trapped in the region D_1 bounded by the curves

$$\ell_1 = \{V = \sqrt{\chi f'(0)}(U - A(U))\},$$

and $\ell_2 = \{V = \sqrt{f'(0)}(U - A(U))\}$. The reason is that along ℓ_1, we have

$$\frac{dV}{dU} = c_* - \frac{\sqrt{f'(0)}}{\sqrt{\chi}}(1 + \chi A'(U)),$$

and the slope of the curve ℓ_1 itself is $\sqrt{\chi f'(0)}(1 - A'(U))$. Since U is decreasing along the trajectory, the trajectories point into the region D_1 along ℓ_1 if

$$c_* - \frac{\sqrt{f'(0)}}{\sqrt{\chi}}(1 + \chi A'(U)) \le \sqrt{\chi f'(0)}(1 - A'(U)), \quad \text{for all } U \in [0, 1].$$

This condition is satisfied, as

$$c_* = 2\sqrt{f'(0)} \le \frac{\sqrt{f'(0)}}{\sqrt{\chi}} + \sqrt{\chi f'(0)}.$$

On the other hand, the trajectories point into the region D_1 along ℓ_2 if

$$c_* - \sqrt{f'(0)}(1 + \chi A'(U)) \ge \sqrt{f'(0)}(1 - A'(U)), \quad \text{for all } U \in [0, 1].$$

This condition is also satisfied, as

$$c_* = 2\sqrt{f'(0)} \ge 2\sqrt{f'(0)} + \sqrt{f'(0)}(\chi - 1)A'(U),$$

for all $U \in [0, 1]$ and $0 \le \chi \le 1$, because $A(U)$ is increasing. Therefore, the minimal speed traveling wave trajectory lies in the region D_1. Thus, for any $0 < z < 1$ such that $z = U_*(x)$, we have

$$\sqrt{f'(0)}\sqrt{\chi}(z - A(z)) \le \eta_*(z) \le \sqrt{f'(0)}(z - A(z)),$$

which is (A.8). □

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