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ORIGINAL PAPER

# Variation of holonomy for projective structures and an application to drilling hyperbolic 3-manifolds

**Martin Bridgeman**[1] · **Kenneth Bromberg**[1]

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**Abstract**
We bound the derivative of complex length of a geodesic under variation of the projective structure on a closed surface in terms of the norm of the Schwarzian in a neighborhood of the geodesic. One application is to cone-manifold deformations of acylindrical hyperbolic 3-manifolds.

**Keywords** Projective structure · Hyperbolic drilling

**Mathematics Subject Classification** 30F60 · 32G15

## 1 Introduction

Let $X$ be a Riemann surface or, equivalently, a hyperbolic surface and $\gamma$ a closed geodesic on $X$. A projective structure $\Sigma$ on $X$ determines a holonomy representation of $\pi_1(X)$. If the holonomy of $\gamma$ is not parabolic then $\mathscr{L}_\gamma(\Sigma)$ is the *complex length* of $\gamma$ which is a complex number whose imaginary part is well defined modulo $2\pi$. We let $P(X)$ be the space of projective structures on $X$ and $P_\gamma(X)$ the subspace where the holonomy of $\gamma$ is not parabolic or the identity. Then $\mathscr{L}_\gamma$ is a smooth function on $P_\gamma(X)$. The goal of this note is to gain quantitative control of the derivative of $\mathscr{L}_\gamma$.

We begin by describing a formula for the derivative. We first identify the universal cover $\tilde{X}$ with the upper half plane $\mathbb{U}$ normalized in such a way that the imaginary axis is a lift of $\gamma$. The projective structure $\Sigma$ determines a developing map $f\colon \mathbb{U} \to \widehat{\mathbb{C}}$ which we semi-normalize so that $f(e^\ell z) = e^{\mathscr{L}_\gamma(\Sigma)} f(z)$ where $\ell$ is the length of $\gamma$ in $X$. We define the holomorphic vector field $\mathbf{n} = z\frac{\partial}{\partial z}$ on $\widehat{\mathbb{C}}$. Then the pull back $f^*\mathbf{n}$ is a holomorphic vector field on $\mathbb{U}$ that descends to a vector field on the annular cover $X_\gamma \to X$ associated to $\gamma$. We also denote

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✉ Martin Bridgeman
bridgem@bc.edu

1 Department of Mathematics, Boston College, Chestnut Hill, MA, USA

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this vector field on $X_\gamma$ as $f^*\mathbf{n}$. Note that $f$ is only determined up to post-composition with an element of $\mathsf{PSL}_2(\mathbb{C})$ that fixes 0 and $\infty$. Since $\mathbf{n}$ is invariant under any such element, we have that $f^*\mathbf{n}$ is well defined.

The tangent space $T_\Sigma P(X)$ is canonically identified with $Q(X)$, the space of holomorphic quadratic differentials on $X$. The pairing of a holomorphic vector field with a holomorphic quadratic differential $\phi$ is a holomorphic 1-form, so $f^*\mathbf{n}\cdot\phi$ is a holomorphic (and hence closed) 1-form on $X_\gamma$. Our formula for the derivative of $\mathscr{L}_\gamma$ is

**Theorem 1.1** *Given $\Sigma \in P_\gamma(X)$ with normalized developing map $f$ and $\phi \in Q(X) \cong T_\Sigma(P_\gamma(X))$ we have*

$$d\mathscr{L}_\gamma(\phi) = -\int_\gamma f^*\mathbf{n}\cdot\phi.$$

In the special case when $\Sigma$ is Fuchsian then $\mathbf{n} = f^*\mathbf{n}$, and, as we will see below, the integral can be computed explicitly. In general, to estimate the the integral we need quantitive control over the difference between $\mathbf{n}$ and $f^*\mathbf{n}$. This is the content of our next result.

Before stating our estimates, we define some norms. If $\Phi$ is a quadratic differential on $X$ then $|\Phi|$ is an area form, so its ratio with the hyperbolic area form is a non-negative function. We let $\|\Phi(z)\|$ be this function. It is a natural pointwise norm of $\Phi$ and we let $\|\Phi\|_p$ be the corresponding $L^p$-norms with respect to the hyperbolic area form. We also let $\|\cdot\|$ be the hyperbolic length of vector fields. For all norms we write $\|\cdot\|_\gamma$ to represent the sup norm over the curve $\gamma$.

With these definitions, we can now state our estimate.

**Theorem 1.2** *Let $\Phi$ be a quadratic differential on $\mathbb{U}$ and assume that in an $r$-neighborhood (in hyperbolic metric) of the geodesic $\gamma$ given by the imaginary axis we have $\|\Phi(z)\| \leq K$ with $r < 1/2$ and $K/r < 1/4$. Then there exists a locally univalent map $f : \mathbb{U} \to \widehat{\mathbb{C}}$ such that*

- *$\Phi = Sf$, the Schwarzian derivative of $f$;*
- *$f(it) \to 0$ as $t \to 0$ and $f(it) \to \infty$ as $t \to \infty$;*
- *$\|f^*\mathbf{n} - \mathbf{n}\|_\gamma \leq \frac{9K}{r}$ and $\|f_*\mathbf{n} - \mathbf{n}\|_{f\circ\gamma} \leq \frac{9K}{r}$*

*where $\|\cdot\|_\gamma$ is the supremum of the hyperbolic length of the vector field along the curve $\gamma$ in $\mathbb{U}$ and $\|\cdot\|_{f\circ\gamma}$ is the supremum of the Euclidean length of the vector field along the curve $f\circ\gamma$ in $\mathbb{C}^*$.*

The proof of Theorem 1.2 is more complicated than one might expect. It involves the use of *Epstein surfaces* and estimates in 3-dimensional hyperbolic space.

An elementary consequence of Theorem 1.2 is the following approximation for $d\mathscr{L}_\gamma$.

**Theorem 1.3** *Let $\Sigma$ be a projective structure and $\gamma$ a closed geodesic of length $\ell$ such that $\|\Phi\| \leq K$ in the $r$-neighborhood of $\gamma$ with $r \leq 1/2$ and $K/r \leq 1/4$. Then*

$$\left| d\mathscr{L}_\gamma(\phi) + \int_\gamma \mathbf{n}\cdot\phi \right| \leq \frac{9K\ell}{r}\|\phi\|_\gamma.$$

***Proof*** We have by Theorems 1.1 and 1.2

$$\left| d\mathscr{L}_\gamma(\phi) + \int_\gamma \mathbf{n}\cdot\phi \right| \leq \int_\gamma |(f^*\mathbf{n} - \mathbf{n})\cdot\phi|$$

$$\leq \|f^*\mathbf{n} - \mathbf{n}\|_\gamma \cdot \|\phi\|_\gamma \int_\gamma \|dz\|_{\mathbb{H}^2}$$
$$= \frac{9K\ell}{r}\|\phi\|_\gamma$$

where $\|dz\|_{\mathbb{H}^2}$ is hyperbolic line element. □

### 1.1 Application to deformations of hyperbolic manifolds

Let $N$ be an acylindrical hyperbolizable 3-manifold with boundary $S = \partial N$. Then for any (noded) conformal structure $Y$ in the Weil-Petersson completion $\overline{\mathscr{T}(S)}$ of Teichmüller space there is a unique geometrically finite hyperbolic structure $M_Y$ on $N$ with conformal boundary $Y$. The hyperbolic structure $M_Y$ determines a projective structure on $Y$ with Schwarzian quadratic differential denoted $\Phi_Y$. One question that naturally arises is "if the $L^2$-norm of $\Phi_Y$ is small, does this imply that the $L^\infty$-norm is also small and therefore the manifold $M_Y$ has almost geodesic convex core boundary?". This is not the case as the $L^2$-norm may be small but there are short curves where the $L^\infty$-norm is large. In [2] we analysed this problem. We showed that for $Y \in \mathscr{T}(S)$ if the $L^2$-norm $\|\Phi_Y\|_2$ is sufficiently small then there is a *nearby* $\hat{Y} \in \overline{\mathscr{T}(S)}$ (in the Weil-Petersson metric) such that the $L^\infty$-norm $\|\Phi_{\hat{Y}}\|_\infty$ is *small*. This $\hat{Y}$ will generally be a noded surface pinched along curves where the $L^\infty$-norm in $Y$ was not small.

The bounds in [2] for *nearby* and *small* were linear in $\|\Phi_Y\|_2^{\frac{1}{2n(S)+3}}$ where $n(S)$ is the maximum number of disjoint geodesics, in particular if $S$ is closed connected of genus $g$, then $n(S) = 3g - 3$. Our application is to use our variation bound in Theorem 1.3 to significantly improve this estimate and replace the power $\frac{1}{2n(S)+3}$ with the constant power $\frac{1}{2}$. Thus the bounds become independent of the topology. Namely we prove:

**Theorem 1.4** *There exist $K, C > 0$ such that the following holds; Let $N$ be an acylindrical hyperbolizable manifold with boundary $S = \partial N$ and $Y \in \mathscr{T}(S)$ with $\|\Phi_Y\|_2 \leq K$. Then there exists a $\hat{Y} \in \overline{\mathscr{T}(S)}$ with*

1. $d_{\mathrm{WP}}(Y, \hat{Y}) \leq C\sqrt{\|\Phi_Y\|_2}$;
2. $\|\Phi_{\hat{Y}}\|_\infty \leq C\sqrt{\|\Phi_Y\|_2}$.

Our proof also holds for *relative acylindrical* 3-manifolds as in [2] but for simplicity we will restrict to the acylindrical case.

## 2 Variation of holonomy

### Preliminaries

We consider $X$ a Riemann surface structure on a surface $S$ and $P(X)$ the space of projective structures on $X$. Associated to $\Sigma \in P(X)$ is the holonomy representation $\rho \in \mathrm{Hom}(\pi_1(S), \mathsf{PSL}_2(\mathbb{C}))$ (unique up to conjugacy). The space $P(X)$ can be parametrized by $Q(X)$ the space of holomorphic quadratic differentials by taking the Schwarzian derivative of its developing map $f : \mathbb{U} \to \hat{\mathbb{C}}$. As $Q(X)$ is a vector space, it is its own tangent space, and a variation of $\Sigma$ is given by a quadratic differential $\phi \in Q(X)$. Throughout the paper we will use $\phi$ to denote deformations of projective structures and $\Phi$ (or $\Phi_t$) to denote the Schwarzian of projective structures.

The advantage of the Schwarzian $\Phi$ is that it is uniquely determined by $\Sigma$ while the holonomy representation $\rho$ and developing map $f$ are not. Given a smooth 1-parameter family of projective structures $\Sigma_t$ we get a smooth family $\Phi_t \in Q(X)$ of Schwarzians. While the developing maps $f_t : \mathbb{U} \to \widehat{\mathbb{C}}$ are not uniquely determined, we can choose them to vary smoothly and it will be convenient to do this after making some normalizations.

We will be interested in the *complex length* of an element $\gamma \in \pi_1(S)$ that represents a closed geodesic for the hyperbolic structure on $X$. After fixing $\gamma$, we identify $\mathbb{U}$ with $\tilde{X}$ so that the deck action of $\gamma$ on $\mathbb{U}$ is given by $\gamma(z) = e^{\ell_\gamma} z$ where $\ell_\gamma$ is the length of the geodesic representative of $\gamma$. Under the covering map $\mathbb{U} \to X$ the imaginary axis is taken to this geodesic. Our second normalization is to choose the developing map $f$ and corresponding holonomy representation $\rho$ such that $\rho(\gamma)(z) = e^{\mathscr{L}_\gamma} z$. Then $\mathscr{L}_\gamma$ is the complex length which we also denote by $\mathscr{L}_\gamma(\Sigma)$ to indicate the projective structure when needed. Note that this is only well defined modulo $2\pi i$.

Extending these normalizations to the 1-parameter family $\Sigma_t$, we get a smooth family of developing maps $f_t$ and holonomy representations with $\rho_t(\gamma)(z) = e^{\mathscr{L}_\gamma(t)} z$. Note that while $\mathscr{L}_\gamma$ is only well defined modulo $2\pi i$, after this choice is made there is a unique choice of $\mathscr{L}_\gamma(t)$ that makes the path continuous. In particular the time zero derivative $\dot{\mathscr{L}}_\gamma$ of $\mathscr{L}_\gamma(t)$ is well defined.

We also let $v$ be the vector field on $\mathbb{U}$ such that $f_* v(z)$ is the tangent vector of the path $f_t(z)$ at $t = 0$. The vector field $v$ is not equivariant under the deck action but a standard computation gives

$$v - \beta_* v = f^* \dot{\rho}(\beta)$$

for all $\beta \in \pi_1(S)$ where the vector field $\dot{\rho}(\beta)$ at $z$ is the tangent vector at time $t = 0$ of the path $(\rho_t(\beta)\rho_0(\beta)^{-1})(z)$. As $\rho_t(\gamma)(z) = e^{\mathscr{L}_\gamma(t)} z$ we have that $\dot{\rho}(\gamma) = \dot{\mathscr{L}}_\gamma \left(z \frac{\partial}{\partial z}\right)$ so

$$v - \gamma_* v = f^* \left( \dot{\mathscr{L}}_\gamma \, z \frac{\partial}{\partial z} \right). \tag{2.1}$$

If $U$ is an open neighborhood in $\mathbb{U}$ where $f_t$ is injective then $(U, f_t)$ is a projective chart for $\Sigma_t$. In the chart $(U, f)$ the vector field $v$ is represented as $g \frac{\partial}{\partial z}$ where $g$ is a holomorphic function. In this chart the *Schwarzian derivative* of $v$ is $g_{zzz} dz^2$. This is a quadratic differential and a computation gives that it is $\phi$ the time zero derivative of the path $\Phi_t$.

### 2.1 Model deformations

We are now ready to prove Theorem 1.1. Here's the outline:

- We first construct a model deformation $\phi_\lambda$ on the projective structure on the annulus $X_\gamma$.
- The model deformation will be very explicit so that we can directly calculate the line integral of $f^* \mathbf{n} \cdot \phi_\lambda$ over $\gamma$.
- We then find a specific $\lambda$ so that the 1-form $f^* \mathbf{n} \cdot \phi - f^* \mathbf{n} \cdot \phi_\lambda$ is exact. Then the line integrals over both 1-forms will be equal so our previous calculation will give the theorem.

A minor issue with this outline is that the model deformation will only be defined on a sub-annulus of $X_\gamma$. We begin with this difficulty.

The set $f^{-1}(\{0, \infty\})$ will be discrete in $\widehat{\mathbb{C}}$ and $\gamma$-invariant so it will descend to a discrete set in $X_\gamma$. Let $\gamma'$ be a smooth curve, homotopic to the core curve of annulus $X_\gamma$, that misses this set and let $A$ be an annular neighborhood of $\gamma'$ that is also disjoint from this set. Then $\tilde{A}$, the pre-image of $A$ in $\mathbb{U}$, will also be disjoint from $f^{-1}(\{0,\infty\})$. As $\tilde{A}$ is simply connected we can choose a well defined function $\log f$ on $\tilde{A}$ and for $\lambda \in \mathbb{C}$ define the vector field $v_\lambda$ on $\tilde{A}$ by $f_* v_\lambda(w) = \lambda f(w) \log f(w) \frac{\partial}{\partial z}$. We let $\phi_\lambda$ be the Schwarzian of $v_\lambda$ and differentiating $\lambda z \log z$ three times we see that $\phi_\lambda = f^*\left(-\frac{\lambda}{z^2} dz^2\right)$. Then $\phi_\lambda$ is our model deformation.

Note the choice of log defines $\mathscr{L}_\gamma$ as a complex number rather than just a number modulo $2\pi i$. We will use this in the rest of the proof. Next we compute the line integral of $f^*\mathbf{n} \cdot \phi_\lambda$ over $\gamma'$:

**Lemma 2.1**

$$\int_{\gamma'} f^*\mathbf{n} \cdot \phi_\lambda = -\lambda \mathscr{L}_\gamma$$

***Proof*** Let $\sigma : [0,1] \to \mathbb{U}$ be a smooth path that projects to $\gamma'$ in $X_\gamma$. In particular $\sigma(1) = \gamma(\sigma(0))$ and $f(\sigma(1)) = f(\gamma(\sigma(0)) = e^{\mathscr{L}_\gamma} f(\sigma(0))$. Then

$$\begin{aligned}
\int_{\gamma'} f^*\mathbf{n} \cdot \phi_\lambda &= \int_\sigma f^*\left(z\frac{\partial}{\partial z} \cdot -\frac{\lambda}{z^2} dz^2\right) \\
&= -\lambda \int_{f\circ\sigma} \frac{dz}{z} \\
&= -\lambda(\log f(\sigma(1)) - \log f(\sigma(0)) \\
&= -\lambda \mathscr{L}_\gamma.
\end{aligned}$$

□

Next we give a criteria for the form $f^*\mathbf{n} \cdot \phi$ to be exact.

**Lemma 2.2** *Let $v$ be a $\gamma$-equivariant holomorphic vector field on $\tilde{A}$ with Schwarzian $\phi$. Then $f^*\mathbf{n} \cdot \phi$ descends to an exact form on $A$.*

***Proof*** The vector field $v$ is a section of the holomorphic tangent bundle $T_\mathbb{C} X_\gamma$. We have the differential operator $\partial$ which takes a $(p,q)$-differential to a $(p+1,q)$-differential. Therefore, associated to vector field $v$ (i.e. a $(-1,0)$-differential), we define the function $v_z = \partial v$, the 1-form $v_{zz} = \partial^2 v$ and the quadratic differential $v_{zzz} = \partial^3 v$ which we also denote by $\phi$. Therefore $f^*\mathbf{n} \cdot v_{zz} - v_z$ is a holomorphic function on $X_\gamma$ and we'll show that it is a primitive of $f^*\mathbf{n} \cdot \phi$.

We can do this calculation in a chart. Namely choose an open neighborhood $U$ in $\mathbb{U}$ such $f$ is injective on $U$. Then as above $(U, f)$ is a chart for the projective structure and in this chart $v$ has the form $g\frac{\partial}{\partial z}$ where $g$ is a holomorphic function on $f(U)$ and $\phi$ is $g_{zzz} dz^2$. It follows that on this chart $v_z$ is $g_z$, $v_{zz}$ is $g_{zz} dz$ and $\phi = v_{zzz}$ is $g_{zzz} dz^2$. As the derivative of $zg_{zz} - g_z$ is $g_{zzz}$ this shows that $f^*\mathbf{n} \cdot v_{zz} - v_z$ is a primitive for $f^*\mathbf{n} \cdot \phi$ as claimed. □

***Proof of Theorem 1.1*** Since $f^*\mathbf{n} \cdot \phi$ is holomorphic it is closed and we have

$$\int_\gamma f^*\mathbf{n} \cdot \phi = \int_{\gamma'} f^*\mathbf{n} \cdot \phi.$$

Next we calculate to see that

$$v_\lambda - \gamma_* v_\lambda = f^*\left(\lambda \mathscr{L}_\gamma \, z\frac{\partial}{\partial z}\right)$$

so if $\lambda = \dot{\mathscr{L}}_\gamma/\mathscr{L}_\gamma$ then by (2.1) we have that $v - v_{\dot{\mathscr{L}}_\gamma/\mathscr{L}_\gamma}$ is $\gamma$-invariant. Therefore by Lemma 2.2 the 1-form $f^*\mathbf{n}\cdot(\phi - \phi_{\dot{\mathscr{L}}_\gamma/\mathscr{L}_\gamma})$ is exact and

$$\int_{\gamma'} f^*\mathbf{n}\cdot\phi = \int_{\gamma'} f^*\mathbf{n}\cdot\phi_{\dot{\mathscr{L}}_\gamma/\mathscr{L}_\gamma} = -\dot{\mathscr{L}}_\gamma$$

where the last equality comes from Lemma 2.1. Combining the first and last equality gives the theorem. □

We conclude this section with a simple formula for the line integral when the projective structure is Fuchsian. For this, rather than representing the annulus as a quotient of the upper half plane $\mathbb{U}$ for this calculation it is convenient to represent the annulus as an explicit subset of $\mathbb{C}$. Namely, let

$$A_\ell = \left\{ z \in \mathbb{C} \mid e^{\frac{-\pi^2}{\ell}} < |z| < e^{\frac{\pi^2}{\ell}} \right\}.$$

This annulus is conformally equivalent to $X_\gamma$ when $\ell = \ell_\gamma(X)$, and the circle $|z| = 1$ is the closed geodesic of length $\ell$ in $A_\ell$. For this representation of the annulus the vector field $\mathbf{n}$ is written as $\mathbf{n} = \frac{2\pi i}{\ell} z \frac{\partial}{\partial z}$ and we observe that $\mathbf{n}$ extends to a holomorphic vector field on all of $\widehat{\mathbb{C}}$ that is 0 at $z = 0$ and $z = \infty$.

To decompose the quadratic differential $\phi$ on $A_\ell$ we let $D^+$ and $D^-$ be the disks in $\widehat{\mathbb{C}}$ with $|z| > e^{\frac{-\pi^2}{\ell}}$ and $|z| < e^{\frac{\pi^2}{\ell}}$, respectively. We recall that any holomorphic function $\psi$ on $A_\ell$ can be written as $\psi_+ + \psi_0 + \psi_-$ where $\psi_+$ and $\psi_-$ extend to holomorphic functions on $D^+$ and $D^-$ that are zero at $z = \infty$ and $z = 0$, respectively, and $\psi_0$ is constant. Then $\phi$ can be written as $\phi = \frac{\psi}{z^2} dz^2$ so the decomposition of $\psi$ gives

$$\phi = \phi_+ + \phi_0 + \phi_- \tag{2.2}$$

where $\phi_+$ and $\phi_-$ extend to holomorphic quadratic differentials on $D^+$ and $D^-$ with simple poles at $z = \infty$ and $z = 0$, respectively, and $\phi_0$ is a constant multiple of $\frac{dz^2}{z^2}$.

**Lemma 2.3** *Let $\phi$ be a holomorphic quadratic differential on $X_\gamma$. Then*

$$\int_\gamma \mathbf{n}\cdot\phi = \int_\gamma \mathbf{n}\cdot\phi_0$$

*and*

$$\left| \int_\gamma \mathbf{n}\cdot\phi \right| = \ell \|\phi_0\|_\infty$$

***Proof*** In the annulus $A_\gamma$ the line integral along $\gamma$ is the line integral over the circle $|z| = 1$.

Note that $\mathbf{n}\cdot\phi_\pm$ extends to a 1-form on $D^\pm$ so

$$\int_{|z|=1} \mathbf{n}\cdot\phi_\pm = 0$$

since by Cauchy's Theorem for any line integral of a closed curve over a holomorphic 1-form on a simply connected region is zero. Therefore

$$\int_{|z|=1} \mathbf{n}\cdot\phi = \int_{|z|=1} \mathbf{n}\cdot\phi_+ + \mathbf{n}\cdot\phi_0 + \mathbf{n}\cdot\phi_- = \int_{|z|=1} \mathbf{n}\cdot\phi_0.$$

We have the covering map $\mathbb{U} \rightarrow A_\ell$ given by $z \rightarrow (-iz)^{\frac{2\pi i}{\ell}}$. Thus pushing forward the hyperbolic metric on $\mathbb{U}$ we have that the hyperbolic metric $g_{A_\ell}$ on $A_\ell$ is

$$g_{A_\ell} = \left(\frac{\ell}{2\pi|z|\cos\left(\frac{\ell}{2\pi}\log|z|\right)}\right)^2 |dz|^2.$$

As $\phi_0 = \frac{\psi_0}{z^2}dz^2$ we therefore have

$$\|\phi_0\|_\infty = \sup_z \frac{|\phi_0(z)|}{g_{A_\ell}(z)} = \frac{4\pi^2|\psi_0|}{\ell^2}.$$

Thus

$$\int_{|z|=1} \mathbf{n}\cdot\phi_0 = \frac{2\pi i\psi_0}{\ell}\int_{|z|=1}\frac{dz}{z} = \frac{-4\pi^2\psi_0}{\ell}$$

and

$$\left|\int_{|z|=1} \mathbf{n}\cdot\phi_0\right| = \ell\|\phi_0\|_\infty.$$

□

## 3 Derivative bounds on univalent maps

Next we prove Theorem 1.2. We begin by reducing it to a more explicit statement.

**Theorem 3.1** *Let $\Phi$ be a quadratic differential on $\mathbb{U}$ and assume that on an $r$-neighborhood of the imaginary axis (in the hyperbolic metric) we have $\|\Phi(z)\| \leq K$ with $r < 1/2$ and $K/r < 1/4$. Then there exists a locally univalent map $f : \mathbb{U} \rightarrow \widehat{\mathbb{C}}$ such that*

1. *$Sf = \Phi$;*
2. *$f(it) \rightarrow 0$ as $t \rightarrow 0$ and $f(it) \rightarrow \infty$ as $t \rightarrow \infty$;*
3. *$f(i) = i$, $|f'(i) - 1| \leq \frac{9K}{r}$ and $|\frac{1}{f'(i)} - 1| \leq \frac{9K}{r}$.*

We now see how this implies Theorem 1.3:

***Proof of Theorem 1.2 assuming Theorem 3.1*** Given $ie^t \in \mathbb{U}$ with $t \in \mathbb{R}$ we let $\Phi^t$ be the pull back of $\Phi$ by the isometry $\gamma_t(z) = e^t z$. As $\gamma_t$ preserves the imaginary axis we still have that $\|\Phi^t(z)\| \leq K$ for $z$ in an $r$-neighborhood of the axis. Therefore by Theorem 3.1 we have a locally univalent map $f^t : \mathbb{U} \rightarrow \widehat{\mathbb{C}}$ satisfying the 3 bullets.

We now let $f = f^0$. Since $Sf^t = \Phi^t$ we have that $S(f^t \circ \gamma_{-t}) = \Phi$. Since $f$ and $f^t \circ \gamma_{-t}$ have the same Schwarzian, they differ by post-composition with an element of $\mathsf{PSL}_2(\mathbb{C})$. Since these two maps have the same behavior as $it \rightarrow 0$ and $it \rightarrow \infty$ this element of $\mathsf{PSL}_2(\mathbb{C})$ must be of the form $z \mapsto e^\lambda z$. We note that $\mathbf{n}$ is invariant under these maps (which includes the maps $\gamma_t$). In particular this implies that

$$\left|\frac{1}{(f^t)'(i)}i - i\right| = \|(f^t)^*(\mathbf{n}(i)) - \mathbf{n}(i)\|_{g_{\mathbb{H}^2}} = \|f^*(\mathbf{n}(f(it))) - \mathbf{n}(it)\|_{g_{\mathbb{H}^2}}$$

so the inequality 3 above gives

$$\|f^*(\mathbf{n}(f(it)) - \mathbf{n}(it)\|_{g_{\mathbb{H}^2}} \leq \frac{9K}{r}.$$

Similarly if $g_{euc}$ is the Euclidean metric on $\mathbb{C}^*$ then

$$|(f^t)'(i)i - i| = |(f^t)_*(\mathbf{n}(i)) - \mathbf{n}(i)| = \|f_*(\mathbf{n}(it)) - \mathbf{n}(f(it))\|_{g_{euc}}$$

□

Here is a brief outline of the proof of Theorem 3.1:

- Given the quadratic differential $\Phi$ on $\mathbb{U}$ there is an immersion $f_0 : \mathbb{U} \to \mathbb{H}^3$ such that composition of $f_0$ with the *hyperbolic Gauss map* is a map $f : \mathbb{U} \to \widehat{\mathbb{C}}$ with $Sf = \Phi$.
- The surface $f_0 : \mathbb{U} \to \mathbb{H}^3$ is the *Epstein surface* for $\Phi$. In [9], Epstein gives formulas for the metric and shape operator of this surface in terms of the hyperbolic metric on $\mathbb{U}$ and $\Phi$.
- We will use Epstein's formulas to show that the curve $t \mapsto f_0(ie^t)$ is nearly unit speed and has small curvature. This will imply that $f_0(it)$ limits to distinct points as $t \to 0$ and $t \to \infty$. We then normalize so that these limiting points are 0 and $\infty$.
- The proof is then completed by a calculation of the hyperbolic Gauss map using the Minkowski model for hyperbolic space.

Before starting the proof of Theorem 3.1, we review the necessary facts about *Epstein surfaces*. These surfaces are defined for any conformal metric on $\mathbb{U}$ and a holomorphic quadratic differential $\Phi$. Here, we will restrict to the hyperbolic metric.

The *projective second fundamental form* for the hyperbolic metric $g_{\mathbb{H}^2}$ is

$$\mathrm{II} = \Phi + \bar{\Phi} + g_{\mathbb{H}^2}.$$

The *projective shape operator* $\hat{B}$ is given by the formula

$$g_{\mathbb{H}^2}(\hat{B}v, w) = \mathrm{II}(v, w).$$

We then define a *dual pair* by

$$g = \frac{1}{4}\left(\mathrm{id} + \hat{B}\right)^* g_{\mathbb{H}^2} \qquad B = \left(\mathrm{id} + \hat{B}\right)^{-1}\left(\mathrm{id} - \hat{B}\right). \tag{3.3}$$

By inverting it follows also that

$$g_{\mathbb{H}^2} = (\mathrm{id} + B)^* g \quad \hat{B} = (\mathrm{id} + B)^{-1}(\mathrm{id} - B). \tag{3.4}$$

We will also use two maps on the unit tangent bundle $T^1\mathbb{H}^3$. First we have the projection $\pi : T^1\mathbb{H}^3 \to \mathbb{H}^3$. We also have the hyperbolic Gauss map $\mathfrak{g} : T^1\mathbb{H}^3 \to \widehat{\mathbb{C}}$ which takes each unit tangent vector to the limit of the geodesic ray tangent to the vector.

**Theorem 3.2** (Epstein [9]) *Given any holomorphic quadratic differential $\Phi$ on $\mathbb{U}$ there exists a smooth map $\hat{f} : \mathbb{U} \to T^1\mathbb{H}^3$ with*

1. *Where $B$ is non-singular, the map $f_0 = \pi \circ \hat{f}$ is smooth, $g = f_0^* g_{\mathbb{H}^3}$ and $B$ is the shape operator for the immersed surface given by $f_0$.*
2. *The map $f = \mathfrak{g} \circ \hat{f}$ is locally univalent with $Sf = \Phi$.*
3. *The eigenvalues of $\hat{B}$ are $1 \pm 2\|\Phi(z)\|$.*

### 3.1 Geodesic curvature

The path $\gamma(t) = ie^t$ is a unit speed parameterization of the imaginary axis. We will now begin the computation of the curvature of $\alpha = f_0 \circ \gamma$.

We begin with a few preliminaries. Let $\hat{B}_0$ be the traceless part of $\hat{B}$. Then $\mathrm{II}_0(X, Y) = g_{\mathbb{H}^2}(\hat{B}_0 X, Y)$ is the traceless part of II. We then have $\hat{B} = \mathrm{id} + \hat{B}_0$ and $\mathrm{II}_0 = \Phi + \bar{\Phi}$. We also need to relate the Riemannian connection $\hat{\nabla}$ for $g_{\mathbb{H}^2}$ and the Riemannian connection $\nabla$ for $g$. A simple calculation (see [11, Lemma 5.2]) gives

$$(\mathrm{id} + \hat{B})\nabla_X Y = \hat{\nabla}_X((\mathrm{id} + \hat{B})Y).$$

**Lemma 3.3** *Define the holomorphic function h by*

$$h = \Phi(\mathbf{n}, \mathbf{n}).$$

*Then*

$$\left(\mathrm{id} + \hat{B}\right)\nabla_{\dot\gamma}\dot\gamma = 4\,\mathrm{Re}\,dh(\dot\gamma)\bar{\mathbf{n}} \quad \textit{and} \quad \|\nabla_{\dot\gamma}\dot\gamma\|_g = |dh(\dot\gamma)|.$$

***Proof*** We work in the complexified tangent bundle of $\mathbb{U}$. We want to compute $\nabla_{\dot\gamma}\dot\gamma$ where $\dot\gamma$ is the tangent vector to the path $\gamma$. As $\left(\mathrm{id} + \hat{B}\right)\nabla = \hat{\nabla}\left(\mathrm{id} + \hat{B}\right)$ then

$$\begin{aligned}\left(\mathrm{id} + \hat{B}\right)\nabla_{\dot\gamma}\dot\gamma &= \hat{\nabla}_{\dot\gamma}\left(\mathrm{id} + \hat{B}\right)\dot\gamma \\ &= 2\hat{\nabla}_{\dot\gamma}\dot\gamma + \hat{\nabla}_{\dot\gamma}\hat{B}_0\dot\gamma \\ &= \hat{\nabla}_{\dot\gamma}\hat{B}_0\dot\gamma\end{aligned}$$

since $\hat{\nabla}_{\dot\gamma}\dot\gamma = 0$ as $\gamma$ is a geodesic in $g_{\mathbb{H}^2}$.

To compute this term we note $\dot\gamma = \mathbf{n} + \bar{\mathbf{n}}$. Therefore, since $\mathrm{II}_0$ is symmetric we have

$$\begin{aligned}g_{\mathbb{H}^2}\left(\hat{B}_0\dot\gamma, \mathbf{n}\right) &= \mathrm{II}_0(\mathbf{n} + \bar{\mathbf{n}}, \mathbf{n}) \\ &= \Phi(\mathbf{n}, \mathbf{n}) = h.\end{aligned}$$

Using the compatibility of the metric with the Riemannian connection and that $\mathbf{n}$ is parallel along $\gamma$ we have

$$dh(\dot\gamma) = g_{\mathbb{H}^2}\left(\hat{\nabla}_{\dot\gamma}\left(\hat{B}_0\dot\gamma\right), \mathbf{n}\right).$$

A similar calculation gives

$$d\bar{h}(\dot\gamma) = g_{\mathbb{H}^2}\left(\hat{\nabla}_{\dot\gamma}\left(\hat{B}_0\dot\gamma\right), \bar{\mathbf{n}}\right).$$

Let

$$v = 2dh(\dot\gamma)\bar{\mathbf{n}} + 2d\bar{h}(\dot\gamma)\mathbf{n}.$$

Note that we are extending $g_{\mathbb{H}^2}$ to be $\mathbb{C}$-linear on $T\mathbb{U} \otimes \mathbb{C}$ and therefore

$$g_{\mathbb{H}^2}(\mathbf{n}, \mathbf{n}) = g_{\mathbb{H}^2}(\bar{\mathbf{n}}, \bar{\mathbf{n}}) = 0 \quad \text{and} \quad g_{\mathbb{H}^2}(\mathbf{n}, \bar{\mathbf{n}}) = 1/2.$$

It follows that

$$g_{\mathbb{H}^2}(v, \mathbf{n}) = dh(\dot\gamma) \quad \text{and} \quad g_{\mathbb{H}^2}(v, \bar{\mathbf{n}}) = d\bar{h}(\dot\gamma).$$

As $\mathbf{n}$ and $\bar{\mathbf{n}}$ span the tangent space this implies

$$\left(\mathrm{id} + \hat{B}\right) \nabla_{\dot{\gamma}} \dot{\gamma} = \hat{\nabla}_{\dot{\gamma}} \hat{B}_0(\dot{\gamma}) = v.$$

Since $\|v\|_{g_{\mathbb{H}^2}} = 2|dh|$ this gives

$$\|\nabla_{\dot{\gamma}} \dot{\gamma}\|_g = \frac{1}{2}\| \left(\mathrm{id} + \hat{B}\right) \nabla_{\dot{\gamma}} \dot{\gamma}\|_{g_{\mathbb{H}^2}} = \frac{1}{2}\|\hat{\nabla}_{\dot{\gamma}} \hat{B}_0 \dot{\gamma}\|_{g_{\mathbb{H}^2}} = |dh(\dot{\gamma})|.$$

□

Next we use the Cauchy integral formula to bound $dh$.

**Lemma 3.4** *Assume that $r \leq 1/2$ and $K < 1$. If $\|\Phi(z)\| \leq K$ for $z$ in the $r$-neighborhood of $\gamma$ then the geodesic curvature $\kappa_\gamma$ of $\gamma$ in the metric $g$ satisfies*

$$\kappa_\gamma \leq \frac{5K}{4r(1-K)^2}.$$

***Proof*** We have

$$\kappa_\gamma = \frac{\|(\nabla_{\dot{\gamma}} \dot{\gamma})^\perp\|_g}{\|\dot{\gamma}\|_g^2} \leq \frac{\|\nabla_{\dot{\gamma}} \dot{\gamma}\|_g}{\|\dot{\gamma}\|_g^2}$$

where $(\nabla_{\dot{\gamma}} \dot{\gamma})^\perp$ is the component of $\nabla_{\dot{\gamma}} \dot{\gamma}$ perpendicular to $\dot{\gamma}$.

To bound $\|\nabla_{\dot{\gamma}} \dot{\gamma}\|_g$ we will work in the disk model for $\mathbb{H}^2$ with $\gamma$ the geodesic following the real axis and we will bound the curvature at the origin. With this normalization we have $\mathbf{n} = \frac{1-z^2}{2} \frac{\partial}{\partial z}$ and therefore

$$h(z) = \frac{\Phi(z)(1-z^2)^2}{4}.$$

(Here we are not distinguishing between $\Phi$ as quadratic differential and $\Phi$ as holomorphic function.) At zero, $\dot{\gamma} = \frac{1}{2}\partial_x$ so by Lemma 3.3 we have

$$\|\nabla_{\dot{\gamma}} \dot{\gamma}\|_g = |dh(\dot{\gamma})| = |h_x(0)|/2 = |h_z(0)|/2$$

where the last equality uses that $h$ is holomorphic.

We now bound $|h_z(0)|$ using the Cauchy integral formula. Given that $\|\Phi(z)\| \leq K$ in the $r$-neighborhood of the origin (in the hyperbolic metric) we have

$$|h(z)| \leq \frac{1}{4}|1-z^2|^2|\Phi(z)| \leq \frac{|1-z^2|^2}{(1-|z|^2)^2} K$$

when $|z| \leq R = \tanh(r/2)$. Therefore

$$\begin{aligned}
|h_z(0)| &= \frac{1}{2\pi}\left|\int_{|z|=R} \frac{h(z)}{z^2} dz\right| \\
&\leq \frac{KR}{2\pi R^2(1-R^2)^2} \int_0^{2\pi} \left|1 - R^2 e^{2i\theta}\right|^2 d\theta \\
&= \frac{K}{2\pi R(1-R^2)^2} \int_0^{2\pi} \left(1 - 2R^2\cos(2\theta) + R^4\right) d\theta \\
&= \frac{K(1+R^4)}{R(1-R^2)^2}.
\end{aligned}$$

If $r \leq 1/2$ then, since $\tanh(r/2) \leq r/2$, we have $R \leq 1/4$. As the derivative of $\tanh^{-1} R$ is $1/(1-R^2)$ when $R \leq 1/4$ we have $\tanh^{-1} R \leq 16R/15$ and $\tanh(r/2) \geq 15r/32$. Combining this with our estimate on $|h_z(0)|$ we have

$$|h_z(0)| \leq \frac{32}{15r} \cdot \frac{\left(1+(1/4)^4\right)K}{\left(1-(1/4)^2\right)^2} \leq \frac{5K}{2r}$$

and

$$\|\nabla_{\dot{\gamma}}\dot{\gamma}\|_g \leq \frac{5K}{4r}.$$

We now obtain a lower bound on $\|\dot{\gamma}\|_g$. For this we recall that by Theorem 3.2 the eigenvalues of $\hat{B}$ at $z \in \mathbb{H}^2$ are $1 \pm 2\|\Phi(z)\|$. If $z \in \gamma$ then $\|\Phi(z)\| \leq K$ giving

$$\|\dot{\gamma}\|_g = \frac{1}{2}\|\left(\mathrm{id}+\hat{B}\right)\dot{\gamma}\|_{g_{\mathbb{H}^2}} \geq 1-K.$$

Therefore

$$\kappa_\gamma \leq \frac{\|\nabla_{\dot{\gamma}}\dot{\gamma}\|_g}{\|\dot{\gamma}\|_g^2} < \frac{5K}{4r(1-K)^2}.$$

□

We can now bound the curvature of $\alpha$ in $\mathbb{H}^3$.

**Lemma 3.5** *Assume that $r \leq 1/2$ and $K < 1$. Then if $\|Sf(z)\| \leq K$ for all $z$ in the $r$-neighborhood of $\gamma$ then the geodesic curvature $\kappa_\alpha$ of $\alpha = f_0 \circ \gamma$ in $\mathbb{H}^3$ satisfies*

$$\kappa_\alpha \leq \frac{3K}{2r(1-K)^2}.$$

***Proof*** To bound the geodesic curvature of the curve $\alpha = f_0 \circ \gamma$ we again need to bound the covariant derivative $\bar{\nabla}_{\dot{\alpha}}\dot{\alpha}$. This will have a component tangent to the immersed surface which is $(f_0)_*\nabla_{\dot{\gamma}}\dot{\gamma}$ plus an orthogonal component whose length is $\|B(\dot{\gamma})\|_g$. This last norm is bounded by the product of the maximal eigenvalue of $B$ and $\|\dot{\gamma}\|_g$. As $B = (\mathrm{id}+\hat{B})^{-1}(\mathrm{id}-\hat{B})$ then eigenvalues of $B$ at a point $f(z)$ in the immersed surface are $-\frac{\|\Phi(z)\|}{\|\Phi(z)\|\pm 1}$. If $z$ is in $\gamma$ we have $\|\Phi(z)\| \leq K$ so the maximum eigenvalue is bounded above by $K/(1-K)$. It follows that

$$\kappa_\alpha \leq \sqrt{\kappa_\gamma^2 + \left(\frac{K}{1-K}\right)^2} \leq \frac{K}{r(1-K)^2}\sqrt{(5/4)^2+(1/2)^2} \leq \frac{3K}{2r(1-K)^2}.$$

□

### 3.2 Curves with geodesic curvature $\kappa_g < 1$

It is well known that curves in $\mathbb{H}^n$ with geodesic curvature $\leq \kappa < 1$ are quasi-geodesics. In particular, a bi-infinite path with curvature $\leq \kappa$ will limit to distinct endpoints. We need a quantatative version of this statement.

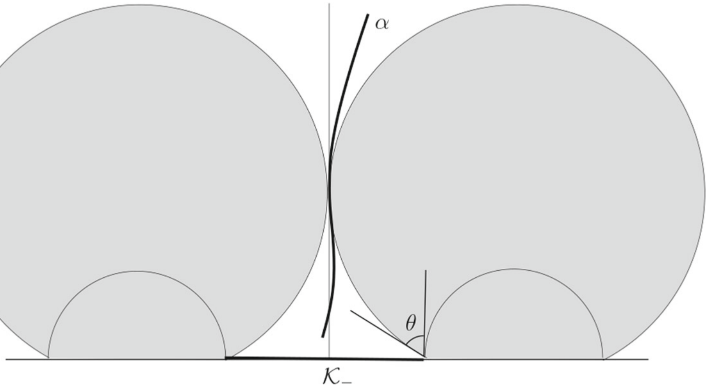


**Fig. 1** In this 2-dimensional figure the union of the $H_\delta$ is the shaded region. In $\mathbb{H}^3$ one rotates this region about the vertical axis. The relationship between the angle $\theta$ and a $\kappa$ is given by $\kappa = \sin(\theta)$. One then computes that $\mathscr{K}_-$ is a disk of radius $\frac{1}{\kappa}\left(1-\sqrt{1-\kappa^2}\right)$

**Lemma 3.6** *Let* $\alpha\colon \mathbb{R} \to \mathbb{H}^3$ *be a smooth, bi-infinite curve with curvature at most* $\kappa < 1$. *Normalize* $\alpha$ *in the upper half space model of* $\mathbb{H}^3$ *so that* $\alpha(0) = (0, 0, 1)$ *and* $\alpha'(0) = (0, 0, \lambda)$ *with* $\lambda > 0$. *Then there are distinct points* $z_-, z_+ \in \widehat{\mathbb{C}}$ *with* $\lim_{t\to\pm\infty} \alpha(t) = z_\pm$ *with*

$$|z_\pm|^{\mp 1} \leq \frac{1}{\kappa}\left(1-\sqrt{1-\kappa^2}\right) \leq \kappa.$$

*Let* $P_t$ *be the hyperbolic plane orthogonal to* $\alpha$ *at* $\alpha(t)$. *Then* $\partial P_t \to z_\pm$ *at* $t \to \pm\infty$.

***Proof*** Let $H_\delta$ be a convex region in $\mathbb{H}^3$ bounded by a plane of constant curvature $\delta < 1$. If $\alpha'(0)$ is tangent to $\partial H_\delta$ and $\delta > \kappa$ then in a neighborhood of zero the only intersection of $\alpha$ with $H_\delta$ will be at $\alpha(0)$. Now suppose there is some $t_0$ with, say, $t_0 > 0$ such that $\alpha(t_0) \in H_\delta$. Then by compactness of the interval $[0, t_0]$ there is a minimum $\epsilon > 0$ such that $\alpha$ restricted $[0, t_0]$ is contained in the $\epsilon$-neighborhood of $H_\delta$. This will be a convex region $H_{\delta'}$ bounded by a plane of constant curvature $\delta' > \delta$ and $\alpha$ will be tangent to the boundary of this region. However, all of $\alpha$ restricted to $[0, t_0]$ will be contained in $H_{\delta'}$ contradicting that at the point of tangency the intersection of $\alpha$ and $H_{\delta'}$ will be an isolated point. This implies that $\alpha(0)$ is the unique point where $\alpha$ intersects $H_\delta$ and $\alpha$ is disjoint from the interior of $H_\delta$.

Now take the union of the interior of all regions $H_\delta$ with $\delta > \kappa$ that are tangent to $\alpha'(0)$ and let $\mathscr{K}$ be its complement. Then the image of $\alpha$ will be contained in $\mathscr{K}$.

The intersection of the closure $\mathscr{K}$ with $\widehat{\mathbb{C}} = \partial\mathbb{H}^3$ will be two regions $\mathscr{K}_-$ and $\mathscr{K}_+$ with the accumulation set of $\alpha(t)$ as $t \to \pm\infty$ contained in $\mathscr{K}_\pm$. Then $\mathscr{K}_-$ is the disk $|z| \leq \frac{1}{\kappa}\left(1-\sqrt{1-\kappa^2}\right)$ (see Fig. 1) and by symmetry $\mathscr{K}_+$ is the region with $1/|z| \leq \frac{1}{\kappa}\left(1-\sqrt{1-\kappa^2}\right)$ (including $\infty \in \widehat{\mathbb{C}}$).

Let $z_-$ be a point in the accumulation set of $\alpha(t)$ as $t \to -\infty$ and let $b_-$ be a Buseman function based at $z_-$. We then observe that the angle between $-\alpha'(t)$ and the gradient $\nabla b_-$ is $\leq \theta$. It is enough to do this calculation when $t = 0$. We then let $\mathfrak{h}$ be the horosphere based at $z_-$ that goes through $\alpha(0)$. Then (along $\mathfrak{h}$) the gradient $\nabla b_-$ is the inward pointing normal vector field to $\mathfrak{h}$. The angle will be greatest when $z_-$ is in $\partial\mathscr{K}_-$ and a direct computation gives that the angle in this case is $\theta$. The bound on the angle implies that $b_-(\alpha(t)) \to \infty$ as $t \in -\infty$ and therefore $\gamma(t) \to z_-$ as $t \to -\infty$.

A similar argument shows that $\alpha(t) \to z_+$ as $t \to \infty$ for some $z_+ \in \mathscr{K}_+$. Let $\beta$ be the geodesic in $\mathbb{H}^3$ with endpoints $z_-$ and $z_+$.

Let $z$ be a point in $\partial P_t$ and let $\mathfrak{h}_z$ be the horosphere based at $z$ that goes through $\alpha(t)$ (and is therefore tangent to $\alpha'(t)$). Let $\mathfrak{h}_{z,\beta}$ be the horosphere based at $z$ that is tangent to $\beta$. We claim that the hyperbolic distance between these two horospsheres is $\leq \tanh^{-1}(\kappa)$. For this we can assume $t = 0$ and let $\mathscr{H}$ be the convex hull of the regions $\mathscr{K}_\pm$. Then $P_0 \cap \mathscr{H}$ is a disk $D$ of radius $\tanh^{-1}(\kappa)$ and $\beta$ will be contained in $\mathscr{H}$ and intersect $D$. Now let $\mathfrak{h}_\pm$ be the two horospheres based at $z$ that are tangent to $\mathscr{H}$. The distance between $\mathfrak{h}_z$ and each of the $\mathfrak{h}_\pm$ will be $\tanh^{-1}(\kappa)$ and $\mathfrak{h}_{z,\beta}$ will lie between the $\mathfrak{h}_\pm$. This implies the distance bound.

We now adjust the picture and so that $z_- = 0$ and $z_+ = \infty$ in the upper half space model for $\mathbb{H}^3$. Assume that $t_i \to -\infty$ and let $z_i$ be points in $\partial P_{t_i}$. We'll show that $z_i \to 0$. Assume not. Then, after passing to a subsequence we can assume that $|z_i|$ is bounded below away from zero. Then for each $i$ we apply the isometry $z \mapsto z/|z_i|$ to the points $\alpha(t_i)$ and the horospheres $\mathfrak{h}_{z_i}$ and $\mathfrak{h}_{z_i,\beta}$ to get new objects $\bar{\alpha}(t_i)$, $\bar{\mathfrak{h}}_{z_i}$ and $\bar{\mathfrak{h}}_{z_i,\beta}$. This isometry fixes $\beta$ and as $|z_i|$ is bounded below we still have that $\bar{\alpha}(t_i) \to 0 \in \widehat{\mathbb{C}}$. Then horospheres $\bar{\mathfrak{h}}_{z_i,\beta}$ will have Euclidean radius 1 while the Euclidean radius of the $\bar{\mathfrak{h}}_{z_i}$ will go to infinity, contradicting that the distance between these horospheres is bounded by $\tanh^{-1}(\kappa)$. Therefore $|z_i| \to 0$, proving the last claim. □

### 3.3 Gauss map

Let $S$ be an immersed surface in $\mathbb{H}^3$ and

$$n \colon S \to T^1\mathbb{H}^n$$

be the lift to the unit tangent bundle. The Gauss map for $S$ is

$$\mathfrak{g}_S \colon S \to \widehat{\mathbb{C}}$$

with $\mathfrak{g}_S = \mathfrak{g} \circ n$. The following lemma gives the derivative of $\mathfrak{g}_S$.

**Lemma 3.7** *Let $S$ be an oriented, immersed surface in $\mathbb{H}^3$ with $\hat{S}$ its lift to $T^1\mathbb{H}^3$. Normalize it so that $p = (0, 1) \in S$ and the lift of this point to $\hat{S}$ is the vector $\partial_y$. Let $G \colon T_pS \to T_i\mathbb{C}$ be the linear map with $G(\partial_x) = \partial_x$ and $G(\partial_t) = \partial_y$ and let $\mathfrak{g}_S \colon S \to \widehat{\mathbb{C}}$ be the Gauss map. Then*

$$(\mathfrak{g}_S)_*(p) = G \circ (\mathrm{id} + B)$$

*where $B \colon T_pS \to T_pS$ is the shape operator.*

The proof is a straightforward calculation that is simplest to do in the Minkowski model for $\mathbb{H}^n$.

Let $\langle , \rangle$ be the inner product on $\mathbb{R}^{n,1}$. Then

$$\mathbb{H}^n = \{x \in \mathbb{R}^{n,1} \mid \langle x, x\rangle = -1\}$$

with tangent space

$$T_x\mathbb{H}^n = \{v \in \mathbb{R}^{n,1} \mid \langle x, v\rangle = 0\}.$$

Then the restriction of $\langle , \rangle$ to each $T_x\mathbb{H}^n$ is positive definite and gives a metric of constant sectional curvature equal to $-1$. This is a model for hyperbolic space.

In this model the sphere at infinity for $\mathbb{H}^n$ is the projectivized light cone which we identify with the unit sphere in the plane $x_{n+1} = 1$. In this model there is a very simple formula for the Gauss map.

**Lemma 3.8** (Bryant [7]) *For $x \in \mathbb{H}^n$ and $v \in T^1_x\mathbb{H}^n$ the hyperbolic Gauss map is given by*

$$\mathfrak{g}(x, v) = \frac{x + v}{\langle x + v, (0, \ldots, 0, 1)\rangle}.$$

***Proof*** Note that this formula is clear when $x = (0, \ldots, 0, 1)$. Then general case the follows from equivariance. □

The formula for the Riemannian connection on $\mathbb{H}^n$ is also very simple. The Minkowski connection $\hat{\nabla}$ on $\mathbb{R}^{3,1}$ is flat with $\hat{\nabla}_X Y = X(Y)$. Thus if $\nabla$ is the Riemannian connection on $\mathbb{H}^3$ then by compatibility we have

$$X(Y) = \nabla_X Y + \langle X(Y), N\rangle N.$$

where $N$ is the normal to $\mathbb{H}^3$ in $\mathbb{R}^{3,1}$. We note that $N : \mathbb{H}^3 \to \mathbb{R}^{3,1}$ is given by $N(x) = x$.

From this formula for $\nabla$ we can also calculate the shape operator $B$. We have

$$BX = \nabla_X(n) = X(n) - \langle X(n), N\rangle N.$$

As $N(p) = p$ then $X(N) = X$ giving

$$\langle X(n), N\rangle = -\langle n, X(N)\rangle = -\langle n, X\rangle = 0$$

so

$$BX = X(n).$$

Note that tangent spaces in $\mathbb{R}^{3,1} \cong \mathbb{R}^4$ are canonically identified and if $p = (0, 0, 0, 1)$ then the linear map $G\colon T_pS \to T_{\mathfrak{g}_S(p)}\partial\mathbb{H}^3$ (from Lemma 3.7) is the identity map. Therefore, we need to show that

$$(\mathfrak{g}_S)_*(p) = \mathrm{id} + B.$$

With these preliminaries done we can now prove the lemma.

***Proof of Lemma 3.7*** By Lemma 3.8 we have

$$\mathfrak{g}_S(x) = \frac{x + n(x)}{\langle x + n(x), (0, 0, 0, 1)\rangle}.$$

Note that for $v \in T_pS$ the derivative of $x \mapsto x + n(x)$ in the direction of $v$ is $v + Bv$ by the calculation above. As $\langle v + Bv, (0, 0, 0, 1)\rangle = 0$ this implies that at $p$ the derivative of the denominator above is zero which gives that

$$(\mathfrak{g}_S)_*(p)v = v + Bv = (\mathrm{id} + B)v$$

as claimed. □

Using the above, we prove the following bound on the derivative of $f$.

**Lemma 3.9** *Let $f : \mathbb{U} \to \hat{\mathbb{C}}$ be a locally univalent map with Epstein maps $f_0\colon \mathbb{U} \to \mathbb{H}^3$ and $\hat{f}_0\colon \mathbb{U} \to T^1\mathbb{H}^3$ normalized as follows:*

1. $f_0(i) = (0, 1)$;

2. $\hat{f}_0(i) = \partial_y$;
3. $(f_0)_*(i)\partial_y = \lambda\partial_t$ *with* $\lambda = \|\partial_y\|_g$.

*Then*

$$|f'(i)| = 1 \quad \text{and} \quad |f'(i) - 1| \leq 2\|\Phi(i)\|$$

*where* $\Phi = Sf$ *is the Schwarzian.*

***Proof*** Let $P \subset T_{(0,1)}\mathbb{H}^3$ be the image of $T_i\mathbb{U}$ under the map $(f_0)_*(i)$. We then have the following sequence of isometries:

$$(T_i\mathbb{U}, g_{\mathbb{H}^2}) \xrightarrow{\text{id}+B} (T_i\mathbb{U}, g) \xrightarrow{(f_0)_*(i)} (P, g_{\mathbb{H}^3}) \xrightarrow{G} (T_i\mathbb{U}, g_{\mathbb{H}^2})$$

and by Lemma 3.7 the composition of these maps is the map $f_*(i)$. In particular $f_*(i)$ is an isometry of $T_i(U)$ to itself so $|f'(i)| = 1$.

As all the above maps are isometries, we can do the computation in any of the metrics. For convenience we will do it in the $g$-metric. Note that $|f'(i) - 1|$ is the distance between the vectors $f'(i)\partial_x$ and $\partial_x$ in $T_i\mathbb{U}$ or, since $f$ is conformal, the distance between $f'(i)\partial_y$ and $\partial_y$ with the hyperbolic metric. In the $g$-metric this is the distance between $(\text{id} + B)\partial_y$ and $\frac{\partial_y}{\lambda}$ (by our normalization (3)). That is

$$\begin{aligned} |f'(i) - 1| &= \left\| (\text{id} + B)\partial_y - \frac{\partial_y}{\lambda} \right\|_g = \left\| (\lambda - 1)\frac{\partial_y}{\lambda} + B\partial_y \right\|_g \\ &\leq |\lambda - 1| \cdot \left\| \frac{\partial_y}{\lambda} \right\|_g + \|B\partial_y\|_g = |\lambda - 1| + \|B\partial_y\|_g. \end{aligned}$$

As $(\text{id} + B)^{-1} = \frac{1}{2}(\text{id} + \hat{B})$ and by Theorem 3.2 the eigenvalues of $\hat{B}$ are $1 \pm 2\|Sf(i)\|$, we have

$$\lambda = \|\partial_y\|_g = \frac{1}{2}\|(\text{id} + \hat{B})\partial_y\|_{g_{\mathbb{H}^2}} \leq 1 + \|Sf(i)\|.$$

Therefore

$$|\lambda - 1| \leq \|Sf(i)\|.$$

Also $(\text{id} + \hat{B})B = (\text{id} - \hat{B})$, giving

$$\|B\partial_y\|_g = \frac{1}{2}\|(\text{id} + \hat{B})B\partial_y\|_{g_{\mathbb{H}^2}} = \frac{1}{2}\|(\text{id} - \hat{B})\partial_y\|_{g_{\mathbb{H}^2}} = \|Sf(i)\|.$$

Combining these bounds, we obtain the result. □

### 3.4 Proof of Theorem 3.1

Let $f^1 \colon \mathbb{U} \to \widehat{\mathbb{C}}$ be the map given by Lemma 3.9 and $f_0^1 \colon U \to \widehat{\mathbb{C}}$ the associated Epstein map. By our assumptions, $r < 1/2$ and $K/r < 1/4$, so $K < 1/8$. Then Lemma 3.5 gives that

$$\kappa_\alpha < \frac{2K}{r} < 1$$

where $\kappa_\alpha$ is the curvature of $\alpha = f_0^1 \circ \gamma$ in $\mathbb{H}^3$. The normalization (3) in Lemma 3.9 and the bound on $\kappa_\alpha$ allow us to apply Lemma 3.6. Therefore we have $z_\pm \in \widehat{\mathbb{C}}$ such that

$\lim_{t\to\pm\infty} f_0^1(it) = z_\pm$ and if the $P_t$ are the planes perpendicular to $\alpha$ at $\alpha(t)$, then $\lim_{t\to\pm\infty} P_t = z_\pm$. Thus as $f^1(it) \in \partial P_t \cap \widehat{\mathbb{C}}$ then $\lim_{t\to\pm\infty} f^1(it) = z_\pm$.

We let $f = m \circ f^1$ where

$$m(z) = i \cdot \frac{i - z^+}{i - z^-} \cdot \frac{z - z^-}{z - z^+}.$$

Then $m(z^-) = 0, m(z^+) = \infty$ and $m(i) = i$. Therefore $f$ has the desired normalization, and we are left to bound the derivative at $i$. This will follow from Lemma 3.9 if we can bound the derivative of $m$ at $i$. Computing, we have

$$|m'(i) - 1| = \left|\frac{z^-}{i - z^-} - \frac{i/z^+}{i/z^+ - 1}\right| \leq \frac{2\kappa_\alpha}{1 - \kappa_\alpha}.$$

For the reciprocal, we have

$$\left|\frac{1}{m'(i)} - 1\right| = \left|\frac{(i - z_-)^2}{i/z_+(z_-/z_+ - 1)} - \frac{z_-}{i}\right| \leq \frac{\kappa_\alpha(1 + \kappa_\alpha)^2}{1 - \kappa_\alpha^2} + \kappa_\alpha = \frac{2\kappa_\alpha}{1 - \kappa_\alpha}.$$

By assumption $K/r < 1/4$, and by Lemma 3.5, $\kappa_\alpha < 2K/r$. Therefore we have $\kappa_\alpha < 1/2$. Combining these, we get that

$$\left|m'(i)^{\pm 1} - 1\right| < \frac{2\kappa_\alpha}{1 - \kappa_\alpha} < \frac{8K}{r}.$$

By Lemma 3.9 we have $|(f^1)'(i)| = 1$ and $|(f^1)'(i) - 1| < 2K$, giving

$$\begin{aligned}\left|f'(i)^{\pm 1} - 1\right| &= \left|((f^1)'(i)m'(i))^{\pm 1} - 1\right| \\ &< \left|(f^1)'(i)\right|^{\pm 1} \cdot \left|m'(i)^{\pm 1} - 1\right| + \left|(f^1)'(i)^{\mp 1} - 1\right| \\ &< \frac{8K}{r} + 2K < \frac{9K}{r}.\end{aligned}$$

□

## 4 Application to hyperbolic three-manifolds

We now describe an application of the above to hyperbolic three-manifolds. For simplicity, we consider $N$ an acylindrical hyperbolizable 3-manifold with boundary $S$. The space of convex-cocompact structures $CC(N)$ on $N$ is parametrized by the conformal structures on the boundary $S$, i.e. the Teichmuller space $\mathscr{T}(S)$. Explicitly, for $Y \in \mathscr{T}(S)$ we denote by $M_Y$ the convex cocompact structure on $N$ with conformal boundary $Y$. Associated to $M_Y$ is a projective structure $\Sigma_Y$ on $S$ in the conformal class of $Y$. By taking the Schwarzian derivative, this gives a quadratic differential $\Phi_Y$ on $Y$. One natural question that arose in prior work of the authors (see [2]) was whether having small $\|\Phi_Y\|_2$ implied that $\|\Phi_Y\|_\infty$ was small which would further imply that $M_Y$ has convex core boundary close to being totally geodesic. This does not hold, as there may be short curves in $Y$ with $\Phi_Y$ having large norm in their collar neighborhood, but which does not contribute much to the $\|\Phi\|_2$. Nevertheless, in [2] we were able to quantify this behavior and show that if $\|\Phi_Y\|_2$ is small, then there is a nearby noded surface $\hat{Y}$ (in the Weil–Petersson metric), with $\|\Phi_{\hat{Y}}\|_\infty$ small. Our approach used drilling via cone-deformations to control the geometry of a hyperbolic manifold, specifically its projective structure, while drilling out the short curves that had $L^\infty$

norm large. By applying the analysis in this paper, we are able to remove the exponential dependence on genus to obtain new bounds that are universal and near optimal (linear in $\sqrt{\|\Phi_Y\|_2}$).

In [1] we used these bounds to obtain bounds on the volumes of convex cores of (relatively) acylindrical hyperbolic 3-manifolds. There is one ingredient in this proof that is not effective—McMullen's contraction constant for the skinning map. However, if one could obtain effective bounds of this constant (for example show that it was universal) than the improved by bounds here would give significantly better bounds on the volume growth than were obtained in [1].

**Theorem 4.1** *There exist $K, C > 0$ such that the following holds; Let $N$ be an acylindrical hyperbolizable manifold with boundary $S = \partial N$ and $Y \in \mathscr{T}(S)$ with $\|\Phi_Y\|_2 \leq K$. Then there exists a $\hat{Y} \in \overline{\mathscr{T}(S)}$ with*

1. $d_{\mathrm{WP}}(Y, \hat{Y}) \leq C\sqrt{\|\Phi_Y\|_2}$;
2. $\|\Phi_{\hat{Y}}\|_\infty \leq C\sqrt{\|\Phi_Y\|_2}$.

The above result actually holds for *relative acylindrical 3-manifolds* as in [2]. For simplicity we are restricting to the acylindrical setting here. At most points the proof is the same as in [2]. However, at one key point we will introduce a new argument which improves the estimate, removing genus dependence. We will restrict to proving that improved estimate and allow the reader to refer to Bridgeman et al. [2] for the remainder of the proof.

Let $\mathscr{C}$ be a collection of disjoint essential simple closed curves on $S$ and $\hat{N}$ the 3-manifold obtained from removing the curves in $\mathscr{C}$ from the level surface $S \times \{1/2\}$ in a collar neighborhood $S \times [0, 1]$ of $N$. Given a $Y \in \mathscr{T}(S)$ there is a unique hyperbolic structure $\hat{M}_Y$ on $\hat{N}$ with conformal boundary $Y$. Again there is a projective structure on $Y$ with Schwarzian $\hat{\Phi}_Y$. Key to our bounds is choosing $\mathscr{C}$ such that in the complement of the standard collars of $\mathscr{C}$ we have bounds on the pointwise norm of $\hat{\Phi}_Y$. It is at this stage that we improve our estimate.

For a complete hyperbolic surface $X$ let $X^{<\epsilon}$ be the set of points of injectivity radius $< \epsilon$. Similarly we let $X^{>\epsilon}$ be the set of points of injectivity radius $> \epsilon$. Recall that by the collar lemma [8] there is an $\epsilon_2 > 0$ such that $X^{<\epsilon_2}$ is a collection of standard collars about simple closed geodesics of length $< 2\epsilon_2$ and cusps. Explicitly we choose $\epsilon_2 = \sinh^{-1}(1)$, the Margulis constant. If $\gamma$ is a closed geodesic of length $< 2\epsilon_2$, we let $U_\gamma$ be this collarm and for a collection of curves $\mathscr{C}$ we let $U_{\mathscr{C}}$ be the union of these standard collars. We can also choose $\overline{\epsilon}_2 > 0$ so that for any point in $X^{<\overline{\epsilon}_2}$ the disk of radius $\mathrm{inj}_X(z)$ is contained in $X^{<\epsilon_2}$. Again, by elementary calculation, we can choose $\overline{\epsilon}_2 = \sinh^{-1}(1/\sqrt{3})$. We denote by $\hat{U}_\gamma$ and $\hat{U}_{\mathscr{C}}$ the corresponding sub-annuli of $U_\gamma$ and $U_{\mathscr{C}}$. We note that if $\ell_\gamma(X) < 2\overline{\epsilon}_2$ then $d(\hat{U}_\gamma, X - U_\gamma) \geq \overline{\epsilon}_2$.

We can now state our improved estimate:

**Theorem 4.2** *There exist constants $C_0, C_1$ such that if $\|\Phi_Y\|_2 \leq C_0$ then there exists a collection of curves $\mathscr{C}$ such that $\ell_{\mathscr{C}}(Y) \leq 2\|\Phi_Y\|_2$ and*

$$\left\|\hat{\Phi}_Y(z)\right\| \leq C_1\sqrt{\|\Phi_Y\|}$$

*for $z \in Y - \hat{U}_{\mathscr{C}}$.*

The proof of Theorem 1.4 follows by combining Theorem 4.2 and Bridgeman et al. [2, Theorem 3.5]. We now briefly outline this and refer the reader to Bridgeman et al. [2] for further details.

***Proof of Theorem 1.4 using Theorem 4.2*** In [2, Theorem 3.5] we prove that we can find curves $\mathscr{C}$ such that

1. $d_{\mathrm{WP}}(Y, \hat{Y}) \leq C\sqrt{\ell_{\mathscr{C}}(Y)}$;
2. $\|\Phi_{\hat{Y}}\|_\infty \leq C\sqrt{\ell_{\mathscr{C}}(Y)}$.

In [2] we were only able to find $\mathscr{C}$ such that $\ell_c(Y) \leq \|\Phi_Y\|_2^{\frac{2}{2n(S)+3}}$ for all $c \in \mathscr{C}$ which gave the genus dependent bound $\ell_{\mathscr{C}}(Y) \leq n(S)\|\Phi_Y\|_2^{\frac{2}{2n(S)+3}}$ on the total length of $\mathscr{C}$. Substituting this bound gave us our original result in [2].

To obtain the new bounds in Theorem 1.4 we apply Bridgeman et al. [2, Theorem 3.5] with the improved bound $\ell_{\mathscr{C}}(Y) \leq 2\|\Phi_Y\|_2$ from Theorem 4.2. □

### 4.1 $L^2$ and $L^\infty$-norms for quadratic differentials

In order to prove our estimate, we will first need some results relating the $L^2$ and $L^\infty$-norms for holomorphic quadratic differentials.

We have the following bound for the pointwise norm in terms of the $L_2$-norm of $\phi$.

**Lemma 4.3** *Given $\Phi \in Q(X)$ we have:*

- *(Teo [12]) If $z \in X^{\geq\epsilon}$ then*
$$\|\Phi(z)\| \leq C(\epsilon)\|\Phi\|_2$$
*where $C(x) = \frac{4\pi}{3}(1 - \operatorname{sech}^6(x/2))^{-1/2}$.*
- *(Bridgeman–Wu [4]) If $z \in \hat{U}_\gamma$ where $\gamma$ is a closed geodesic of length $< 2\overline{\epsilon}_2$ then*
$$\|\Phi(z)\| \leq \frac{\|\Phi|_{U_\gamma}\|_2}{\sqrt{\mathrm{inj}_X(z)}}.$$

***Proof*** The first statement is the main result of Teo [12]. The second statement doesn't appear explicitly in [4] but is a simple consequence of it. By Bridgeman and Wu [4, Proposition 3.3, part 4] we have for $z \in \hat{U}_\gamma$
$$\|\Phi(z)\| \leq G(\mathrm{inj}_X(z))\ \|\Phi|_{U_\gamma}\|_2$$
for some explicit function $G$. Then by direct computation in [4, Eq. (3.13)] we prove that $G(t) \leq 1/\sqrt{t}$ for $t \leq \epsilon_2$. The result then follows. □

For a sufficiently short closed geodesic $\gamma$ on $X$ we can lift $\Phi \in Q(X)$ to the annular cover $X_\gamma$. The covering map is injective on $U_\gamma$ in $X_\gamma$ so on $U_\gamma$ we have our annular decomposition given in (2.2)
$$\Phi = \Phi_-^\gamma + \Phi_0^\gamma + \Phi_+^\gamma.$$

We will need the following $L^\infty$-bound in the thin part. In [13, Lemma 11]) Wolpert proved a bound which is qualitatively the same but we will need the following quantitative version which we derive independently.

**Lemma 4.4** *Let $z \in \hat{U}_\gamma$. Then*
$$\|\Phi(z)\| \leq \|\Phi_0^\gamma\|_\infty + 2C(\overline{\epsilon}_2)\ \|\Phi|_{U_\gamma}\|_2$$

***Proof*** This is again essentially also contained in [4]. We have on $\hat{U}_\gamma$ the splitting of $\Phi$ into

$$\Phi = \Phi_0^\gamma + \Phi_+^\gamma + \Phi_-^\gamma.$$

Therefore

$$\|\Phi(z)\| \leq \|\Phi_0^\gamma(z)\| + \|\Phi_+^\gamma(z)\| + \|\Phi_-^\gamma(z)\|.$$

We need to bound the norms $\|\Phi_0^\gamma(z)\|$, $\|\Phi_+^\gamma(z)\|$, and $\|\Phi_-^\gamma(z)\|$. By definition $\|\Phi_0^\gamma(z)\| \leq \|\Phi_0^\gamma\|_\infty$. By Bridgeman and Wu [4, Proposition 3.3, part 3]

$$\|\Phi_\pm^\gamma(z)\| \leq C(\overline{\epsilon}_2)\ \|\Phi|_{U_\gamma}\|_2.$$

Thus it follows that

$$\|\Phi(z)\| \leq \|\Phi_0^\gamma\|_\infty + 2C(\overline{\epsilon}_2)\ \|\Phi|_{U_\gamma}\|_2.$$

□

Our next estimate is a statement about holomorphic quadratic differentials that we will use to choose the curves $\mathscr{C}$.

**Lemma 4.5** *Let $\Phi \in Q(X)$ with $\|\Phi\|_2 < 1/2$. Let $\mathscr{C}$ be the collection of simple closed curves $\gamma$ such that $\ell_\gamma(X) \leq 2\overline{\epsilon}_2$ and $\|\Phi|_{\hat{U}_c}\|_\infty \geq \sqrt{\|\Phi\|_2}$. Then*

$$\ell_{\mathscr{C}}(X) \leq 2\|\Phi\|_2.$$

*Furthermore if $z \in X - \hat{U}_{\mathscr{C}}$ then $\|\Phi(z)\| \leq \sqrt{\|\Phi\|_2}$.*

***Proof*** For each $\gamma \in \mathscr{C}$ choose $z_\gamma \in \hat{U}_\gamma$ with $\|\Phi(z_\gamma)\| \geq \sqrt{\|\Phi\|_2}$. Squaring the second bound from Lemma 4.3 gives the bound

$$\|\Phi\|_2 \leq \|\Phi(z_\gamma)\|^2 \leq \frac{\left\|\ \Phi|_{U_\gamma}\right\|_2^2}{\mathrm{inj}_X(z_\gamma)}.$$

Using that $\mathrm{inj}_X(z_\gamma) \geq \ell_\gamma(X)/2$ and summing gives

$$\frac{\|\Phi\|_2}{2}\sum_{\gamma\in\mathscr{C}}\ell_\gamma(X) \leq \sum_{\gamma\in\mathscr{C}}\|\Phi|_{U_\gamma}\|_2^2 \leq \|\Phi\|_2^2$$

which rearranges to give

$$\ell_{\mathscr{C}}(X) = \sum_{\gamma\in\mathscr{C}}\ell_\gamma(X) \leq 2\|\Phi\|_2.$$

By the first bound in Lemma 4.3 for $z \in X^{\geq\overline{\epsilon}_2}$ we have as $C(\overline{\epsilon}_2) \leq 1.1$

$$\|\Phi(z)\| \leq C(\overline{\epsilon}_2)\|\Phi\|_2 \leq \sqrt{\|\Phi\|_2}$$

where the second inequality uses the inequality $\|\Phi\|_2 \leq 1/2$. If $z \in X^{<\overline{\epsilon}_2}$ but $z \notin \hat{U}_\gamma$ for some $\gamma \in \mathscr{C}$ we have

$$\|\Phi(z)\| \leq \sqrt{\|\Phi\|_2}$$

by the definition of $\mathscr{C}$. These two inequalities give the second bound. □

### 4.2 Drilling

We recall our setting. We consider $N$ an acylindrical hyperbolizable 3-manifold with boundary $S = \partial N$ and for any (noded) conformal structure $Y$ in the Weil–Petersson completion $\overline{\mathscr{T}(S)}$ of Teichmüller space there is a unique geometrically finite hyperbolic structure $M_Y$ on $N$ with conformal boundary $Y$.

The proof of Theorem 4.2 uses the deformation theory of hyperbolic cone-manifolds developed by Hodgson and Kerkhoff [10] and generalized by the second author in [5, 6]. In particular if $\mathscr{C}$ is sufficiently short in $M_Y$ (or in $Y$) then there will be a one-parameter family of hyperbolic cone-manifolds $M_t$ where the complement of the singular locus is homeomorphic $\hat{N}$, the conformal boundary of $M_t$ is $Y$ and the cone angle is $t$. The $M_t$ will also determine projective structures on $Y$ with Schwarzian quadratic differentials $\Phi_t$ and derivative $\phi_t = \dot{\Phi}_t$. Then when $t = 2\pi$ we have $M_Y = M_{2\pi}$, $\Phi_Y = \Phi_{2\pi}$ and when $t = 0$ we have $\hat{M}_Y = M_0$.

Motivated by Lemma 4.5 above, we will fix $\mathscr{C}$ to be the collection of closed geodesics $\gamma$ on $Y$ with $\ell_\gamma(Y) \leq 2\overline{\epsilon}_2$ and $\| \Phi_Y|_{\hat{U}_\gamma} \|_\infty \geq \sqrt{\|\Phi_Y\|_2}$. Then by the above $\ell_{\mathscr{C}}(Y) \leq 2\|\Phi_Y\|_2$. We first will need to apply the following theorem.

**Theorem 4.6** (Bridgeman–Bromberg [3]) *There exists an $L_0 > 0$ and $c_{\mathrm{drill}} > 0$ such that the following holds. Let $M$ be a convex cocompact hyperbolic 3-manifold with incompressible boundary and $\mathscr{C}$ a collection of simple closed geodesics in $M$ each of whose length is less than or equal $L_0$. Let $M_t$ be the unique hyperbolic cone-manifold with cone angle $t \in (0, 2\pi]$ about $\mathscr{C}$ and $\Sigma_t$ the projective structure on the boundary. If $\Phi_t$ is the Schwarzian of the uniformization map for $\Sigma_t$ and $\phi_t = \dot{\Phi}_t$ then*

$$\|\phi_t\|_2 \leq c_{\mathrm{drill}}\sqrt{L_{\mathscr{C}}(M)}.$$

By the Bers inequality, $L_{\mathscr{C}}(M) \leq 2\ell_{\mathscr{C}}(Y) \leq 4\|\Phi_Y\|_2$. Thus we obtain

**Theorem 4.7** *If $N$ is an acylindrical hyperbolic manifold with $\|\Phi_Y\|_2 < L_0/4$ then there exists a family of curves $\mathscr{C}$ such that $\ell_{\mathscr{C}}(Y) \leq 2\|\Phi_Y\|_2$ and*

$$\|\phi_t\|_2 \leq 2c_{drill}\sqrt{\|\Phi_Y\|_2}$$

*for $t \in (0, 2\pi]$.*

We now promote the $L^2$-bound to a pointwise bound in the complement of the collars of $\mathscr{C}$. We first need the following lemma.

**Lemma 4.8** *There exist a universal constants $D > 0$ such that if $\|\Phi_Y\|_2 < L_0/4$ and $\|\Phi_t(z)\| \leq 1/36$ for all $z \in Y - \hat{U}_{\mathscr{C}}$ then for $z \in Y - \hat{U}_{\mathscr{C}}$ we have*

$$\|\phi_t(z)\| \leq D\sqrt{\|\Phi_Y\|_2}.$$

***Proof*** We have the $L^2$-bound $\|\phi_t\|_2 \leq 2c_{drill}\sqrt{\|\Phi_Y\|_2}$. We need to convert this to a pointwise bound.

For $z \in Y^{\geq \overline{\epsilon}_2}$ we have

$$\|\phi_t(z)\| \leq C(\overline{\epsilon}_2)\|\phi_t\|_2 \leq 2c_{drill}C(\overline{\epsilon}_2)\sqrt{\|\Phi_Y\|_2}.$$

Now assume $\ell_\gamma(Y) \leq 2\overline{\epsilon}_2$ but $\gamma \notin \mathscr{C}$. Then $\phi_t$ has annular decomposition on $U_\gamma$ given by

$$\phi_t = (\phi_t)^\gamma_0 + (\phi_t)^\gamma_+ + (\phi_t)^\gamma_-.$$

For $z \in \hat{U}_\gamma$ by Lemma 4.4 we have

$$\|\phi_t(z)\| \leq \left\|(\phi_t)_0^\gamma\right\|_\infty + 2C(\overline{\epsilon}_2)\|\phi_t\|_2 \leq \left\|(\phi_t)_0^\gamma\right\|_\infty + 4c_{drill}C(\overline{\epsilon}_2)\sqrt{\|\Phi_Y\|_2}.$$

We let $D_0 = 4c_{drill}C(\overline{\epsilon}_2)$. Since the above holds for all $z \in \hat{U}_\gamma$ we also have

$$\|\phi_t\|_\gamma \leq \left\|(\phi_t)_0^\gamma\right\|_\infty + D_0\sqrt{\|\Phi_Y\|_2}. \tag{4.5}$$

To finish the proof we need to bound $\left\|(\phi_t)_0^\gamma\right\|_\infty$ by a constant multiple of $\sqrt{\|\Phi_Y\|_2}$. By definition, for any closed geodesic $\beta$ with $\ell_\beta(Y) \leq 2\overline{\epsilon}_2$ then $d(\hat{U}_\beta, Y - U_\beta) \geq \overline{\epsilon}_2$. Therefore $d(\gamma, \hat{U}_{\mathscr{C}}) \geq 2\overline{\epsilon}_2 > 1/2$ and the $1/2$ neighborhood of $\gamma$ is in $Y - \hat{U}_{\mathscr{C}}$. By our assumptions $\|\Phi_t(z)\| \leq 1/36$ for $z \in Y - \hat{U}_{\mathscr{C}}$ and therefore by Theorem 1.3 we have

$$\left|\frac{\dot{\mathscr{L}}_\gamma(t)}{\ell_\gamma(Y)} + \frac{1}{\ell_\gamma(Y)}\int_\gamma \mathbf{n}\cdot\phi_t\right| \leq \frac{9\cdot(1/36)}{1/2}\|\phi_t\|_\gamma = \frac{1}{2}\|\phi_t\|_\gamma.$$

Applying Lemma 2.3 to the integral and rearranging and then applying (4.5) gives

$$\left|\frac{\dot{\mathscr{L}}_\gamma(t)}{\ell_\gamma(Y)}\right| \geq \left\|(\phi_t)_0^\gamma\right\|_\infty - \frac{1}{2}\|\phi_t\|_\gamma \geq \frac{1}{2}\left\|(\phi_t)_0^\gamma\right\|_\infty - \frac{D_0}{2}\sqrt{\|\Phi_Y\|_2}. \tag{4.6}$$

Next we bound the ratio on the left. In [5], the second author analysed the change in length of geodesics under cone deformations. We now apply a number of results from [5]. By Bromberg [5, Eq. (4.6)] we have

$$|\dot{\mathscr{L}}_\gamma(t)| \leq 4L_{\mathscr{C}}(t)L_\gamma(t). \tag{4.7}$$

We need to bound the two terms on the right. By Bromberg [5, Proposition 4.1] we have that $L_{\mathscr{C}}(t) \leq L_{\mathscr{C}}(2\pi)$ for $t \in (0, 2\pi]$. When $t = 2\pi$, the manifold is non-singular so we can apply the Bers inequality to see that $L_{\mathscr{C}}(2\pi) \leq 2\ell_{\mathscr{C}}(Y)$. Since $|\dot{L}_\gamma(t)| \leq |\dot{\mathscr{L}}_\gamma(t)|$ we can combine (4.7) with the bound on $L_{\mathscr{C}}(t)$ to get

$$\left|\frac{\dot{L}_\gamma(t)}{L_\gamma(t)}\right| \leq \left|\frac{\dot{\mathscr{L}}_\gamma(t)}{L_\gamma(t)}\right| \leq 8\ell_{\mathscr{C}}(Y)$$

which integrates to give

$$L_\gamma(t) \leq L_\gamma(2\pi)e^{8\pi\ell_{\mathscr{C}}(Y)} \leq 2\ell_\gamma(Y)e^{16\pi\|\Phi_Y\|_2}.$$

Here the second inequality comes from applying the Bers inequality to $\gamma$ in $M = M_{2\pi}$ and the bound $\ell_{\mathscr{C}}(Y) \leq 2\|\Phi_Y\|_2$. As we have assumed a universal bound $\|\Phi_Y\|_2 \leq L_0/4$ these two inequalities give

$$\left|\frac{\dot{\mathscr{L}}_\gamma(t)}{\ell_\gamma(Y)}\right| \leq 16e^{16\pi\|\Phi_Y\|_2}\ell_{\mathscr{C}}(Y) \leq D_1\|\phi_Y\|_2. \tag{4.8}$$

where $D_1 = 16e^{4\pi L_0}$ is a universal constant. Combining (4.6) and (4.8) gives

$$\left\|(\phi_t)_0^\gamma\right\|_\infty \leq (D_0 + 2D_1)\sqrt{\|\Phi_Y\|_2}.$$

Finally by Eq. (4.5) we have for $z \in \hat{U}_\gamma$

$$\|\phi_t(z)\| \leq D\sqrt{\|\Phi_Y\|_2}$$

where $D = 2D_0 + 2D_1 = 32e^{4\pi L_0} + 8c_{drill}C(\overline{\epsilon_2})$. □

Finally we can now prove Theorem 4.2 which we restate.

**Theorem 4.9** *There exist constants $C_0, C_1$ such that if $\|\Phi_Y\|_2 \leq C_0$ then there exists collection of curves $\mathscr{C}$ such that $\ell_{\mathscr{C}}(Y) \leq 2\|\Phi_Y\|_2$ and*

$$\left\|\hat{\Phi}_Y(z)\right\| \leq C_1\sqrt{\|\Phi_Y\|}$$

*for $z \in Y - \hat{U}_{\mathscr{C}}$.*

***Proof*** We let $C_0 = \min\{(1/64)^2, 1/(128\pi D)^2, L_0/4\}$ and $C_1 = 2\pi D + 1$. If $\|\Phi_Y\|_2 < C_0$ then by Lemma 4.5 for $z \in Y - \hat{U}_{\mathscr{C}}$ we have

$$\|\Phi_Y(z)\| \leq \sqrt{\|\Phi_Y\|_2} \leq \frac{1}{64}.$$

We now show that $\|\Phi_t(z)\| \leq 1/36$ for $z \in Y - \hat{U}_{\mathscr{C}}$ and $t \in (0, 2\pi]$. If not, then by continuity there is a $t_0 > 0$ such that $\|\Phi_t(z)\| = 1/36$ and $\|\Phi_t(z)\| < 1/36$ for $t \in (t_0, 2\pi]$. For $t \geq t_0$ as $\|\Phi_Y\|_2 < L_0/4$ by Lemma 4.8 above, we have if $z \in Y - \hat{U}_{\mathscr{C}}$ then $\|\phi_t(z)\| \leq D\sqrt{\|\Phi_Y\|_2}$. Therefore integrating we have

$$\frac{1}{64} \leq \|\Phi_{t_0}(z) - \Phi_Y(z)\| \leq \int_{t_0}^{2\pi} \|\phi_t(z)\| dt < 2\pi D\sqrt{\|\Phi_Y\|_2} \leq \frac{2\pi D}{128\pi D} \leq \frac{1}{64}.$$

This gives our contradiction. Thus we can apply Lemma 4.8 and integrate to get

$$\|\hat{\Phi_Y}(z) - \Phi_Y(z)\| \leq \int_0^{2\pi} \|\phi_t(z)\| dt \leq 2\pi D\sqrt{\|\Phi_Y\|_2}.$$

Thus

$$\|\hat{\Phi_Y}(z)\| \leq \|\Phi_Y(z)\| + 2\pi D\sqrt{\|\Phi_Y\|_2} \leq (2\pi D + 1)\sqrt{\|\Phi_Y\|_2}.$$

□

**Author Contributions** Both authors contributed equally to this work.

## Declarations

**Conflict of interest** The authors declare no competing interests.

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