

DECOUPLING INEQUALITIES FOR QUADRATIC FORMS

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Abstract

We prove sharp $\ell^q L^p$ decoupling inequalities for $p, q \in [2, \infty)$ and arbitrary tuples of quadratic forms. Connections to prior results on decoupling inequalities for quadratic forms are also explained. We also include some applications of our results to exponential sum estimates and to Fourier restriction estimates. The proof of our main result is based on scale-dependent Brascamp–Lieb inequalities.

Contents

1. Introduction	387
2. Some applications	398
3. Examples: Old and new	400
4. Transversality	408
5. Induction on scales	418
6. Lower bounds in Theorem 1.1	428
7. Sharp $\ell^q L^p$ decoupling inequalities with $q > p$	433
8. Proofs of Corollaries 1.3–1.5	434
9. Fourier restriction: Proof of Corollary 2.3	436
References	441

1. Introduction

Let $n, d \geq 1$. We denote by $\mathbf{Q}(\xi) = (Q_1(\xi), \dots, Q_n(\xi))$ an n -tuple of real quadratic forms in d variables. The graph of such a tuple, $S_{\mathbf{Q}} = \{(\xi, \mathbf{Q}(\xi)) \in [0, 1]^d \times \mathbb{R}^n\}$, is a d -dimensional submanifold of $\mathbb{R}^{d+n}$. We often write a spatial vector in $\mathbb{R}^{d+n}$ as (x, y) with $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$. Similarly, we often write a frequency vector in $\mathbb{R}^{d+n}$ as (ξ, η) with $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$ and $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{R}^n$.

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Let $\square \subset [0, 1]^d$. Define the Fourier extension operator

$$E_{\square}^{\mathbf{Q}} g(x, y) := \int_{\square} g(\xi) e^{i(x \cdot \xi + y \cdot \mathbf{Q}(\xi))} d\xi, \quad (1.1)$$

with $x \in \mathbb{R}^d$, $y \in \mathbb{R}^n$. For $q, p \geq 2$ and dyadic $\delta \in (0, 1)$, let $D_{q,p}(\mathbf{Q}, \delta)$ be the smallest constant D such that

$$\|E_{[0,1]^d}^{\mathbf{Q}} g\|_{L^p(w_B)} \leq D \left(\sum_{\substack{\square \subset [0,1]^d \\ l(\square) = \delta}} \|E_{\square}^{\mathbf{Q}} g\|_{L^p(w_B)}^q \right)^{1/q} \quad (1.2)$$

holds for every measurable function g and every ball $B \subset \mathbb{R}^{d+n}$ of radius δ^{-2} , where w_B is a smooth version of the indicator function of B (see (1.31) in the subsection of notation), and the sum on the right-hand side runs through all dyadic cubes of side length δ . In the current paper, we determine an optimal asymptotic behavior of $D_{q,p}(\mathbf{Q}, \delta)$ as δ tends to zero, for every choice of $q, p \geq 2$ and $\mathbf{Q}$.

Before stating our main theorem, let us first review related results in the literature. Decoupling theory originated from the work of Wolff [58] and was further developed by Łaba and Wolff [42], Łaba and Pramanik [41], Garrigós and Seeger [25], [26], and Bourgain [8]. A breakthrough came with the resolution of the ℓ^2 -decoupling conjecture for paraboloids by Bourgain and Demeter [11]. Subsequently, Bourgain, Demeter, and Guth [17] resolved the main conjecture in Vinogradov's mean value theorem using decoupling theory. We also refer to Bourgain and Demeter [13], Bourgain, Demeter and Guo [16], Guo and Zhang [30] and Guo and Zorin-Kranich [32] for extensions of [17] to higher dimensions.

In the current paper, we study sharp decoupling inequalities for quadratic d -surfaces in $\mathbb{R}^{d+n}$ with $d, n \geq 1$. The cases $n = 1, d \geq 1$, that is, quadratic hyper-surfaces, were the objects studied by Bourgain and Demeter [11], [15]. Since these works, there have been a number of other works studying sharp decoupling inequalities for quadratic d -surfaces in $\mathbb{R}^{d+n}$ with $n \geq 2$, that is, manifolds of codimension greater than 1. Bourgain's improvement on the Lindelöf hypothesis [9] relies on a decoupling inequality in the case $d = n = 2$, which was later generalized and extended to a more general family of manifolds with dimension and codimension 2 in [12]. Further sharp decoupling inequalities for (nondegenerate) quadratic d -surfaces of codimension 2 were proved, for $2 \leq d \leq 4$, in [22] and [31]. More recently, in [34], the classification of sharp decoupling inequalities for quadratic 3-surfaces in $\mathbb{R}^5$ was completed, and sharp decoupling inequalities were proved in the “degenerate” cases, which were not covered by the previously mentioned works. The approach to the “degenerate” cases in [34] stands out from the previously mentioned works, in that it relies on small cap decouplings for the parabola and the 2-surface $(\xi_1, \xi_2, \xi_1^2, \xi_1 \xi_2)$ (we refer also to [23] for further discussion of small cap decouplings). For manifolds

of codimension $n > 2$, the only result for quadratic forms that are not monomials prior to the current article was due to Oh [52], who proved sharp decoupling inequalities for nondegenerate 3-surfaces in $\mathbb{R}^6$. In the current paper, we provide a unified approach that takes care of all the above examples, and indeed all quadratic d -surfaces in $\mathbb{R}^{d+n}$ for arbitrary combinations of d and n . In Section 3 we will explain in more detail how the above-mentioned results fit into our main theorem.

Before we finish reviewing the related decoupling literature, we would also like to mention that Li [44], [45], building partially on [38], [53], quantified the ϵ loss implicit in (1.8) in the case of the parabola.

Beyond decoupling theory, problems associated with quadratic d -surfaces ($d \geq 2$) of codimension bigger than 1 have also attracted much attention, in particular in Fourier restriction theory and related areas. We refer to Christ [19], [20], Mockenhaupt [47], Bak and Lee [1], Bak, Lee, and Lee [2], Lee and Lee [43], and Guo and Oh [33] for the restriction problems associated with manifolds of codimension 2 and higher, Bourgain [10], Rogers [56], and Oberlin [51] for the planar variant of the Kakeya problem, and Oberlin [49] for sharp $L^p \rightarrow L^q$ improving estimates for a quadratic 3-surface in $\mathbb{R}^5$. Recently, Gressman [27]–[29] has made significant progress in proving sharp L^p -improving estimates for Radon transforms of intermediate dimensions. Perhaps more interestingly, he connected this problem with Brascamp–Lieb inequalities and geometric invariant theory.

One major difficulty in the development of the above-mentioned problems in the setting of codimension bigger than 1 is the lack of a good notion of “curvature.” This is in strong contrast with the case of codimension 1, where we have Gaussian curvatures and the notion of rotational curvatures, introduced by Phong and Stein in [54] and [55].

Next let us turn to the main theorem of the current paper. We will formulate a slightly more general (and essentially equivalent) version of (1.2). This version uses functions with Fourier supports in small neighborhoods of $S_{\mathbf{Q}}$, instead of Fourier extension operators, and lends itself more readily to induction on dimension d .

It is convenient to define the Fourier supports in terms of symmetries of $S_{\mathbf{Q}}$. The group $\mathbf{A}$ generated by translations and scalings of $\mathbb{R}^d$ consists of affine maps of the form $A(\xi) = \delta\xi + a$ with $a \in \mathbb{R}^d$ and $\delta \in (0, \infty)$ (in particular, $\mathbf{A} \cong \mathbb{R}^d \rtimes (0, \infty)$). This group acts on $\mathbb{R}^{d+n}$ by affine transformations

$$\mathcal{A}(A)(\xi, \eta) = (\delta\xi + a, \delta^2\eta + \delta\nabla\mathbf{Q}(a) \cdot \xi + \mathbf{Q}(a)). \quad (1.3)$$

This $\mathbf{A}$ -action leaves $S_{\mathbf{Q}}$ invariant. For a cube $\square \subset \mathbb{R}^d$, let $A_{\square} \in \mathbf{A}$ be the map such that $A_{\square}([0, 1]^d) = \square$, and denote the corresponding affine transformation on $\mathbb{R}^{d+n}$ by $\mathcal{A}_{\square} := \mathcal{A}(A_{\square})$. We define the associated uncertainty region by

$$\mathcal{U}_{\square} = \mathcal{U}_{\square}(\mathbf{Q}) := \mathcal{A}_{\square} \left([-2, 2]^d \times \prod_{j=1}^n 4d (\|\text{Hess } \mathcal{Q}_j\| + 1) [-1, 1] \right). \quad (1.4)$$

The main feature of the definition (1.4) is that the uncertainty region $\mathcal{U}_{\square}$ contains the convex hull of the graph of $\mathbf{Q}$ on $\square$ and is not much larger than this convex hull. Another convenient property is that

$$2\square \subseteq 2\square' \implies \mathcal{U}_{\square} \subseteq \mathcal{U}_{\square'}. \quad (1.5)$$

We will denote by $f_{\square}$ an arbitrary function with $\text{supp } \widehat{f_{\square}} \subseteq \mathcal{U}_{\square}$.

Let $q, p \geq 2$. Let $\delta < 1$ be a dyadic number. Let $\mathcal{P}(\delta)$ be the partition of $[0, 1]^d$ into dyadic cubes of side length δ . Let $\mathcal{D}_{q,p}(\mathbf{Q}, \delta)$ be the smallest constant D such that

$$\left\| \sum_{\square \in \mathcal{P}(\delta)} f_{\square} \right\|_{L^p(\mathbb{R}^{d+n})} \leq D \left(\sum_{\square \in \mathcal{P}(\delta)} \|f_{\square}\|_{L^p(\mathbb{R}^{d+n})}^q \right)^{1/q} \quad (1.6)$$

holds for every $f_{\square}$ with $\text{supp } \widehat{f_{\square}} \subseteq \mathcal{U}_{\square}$. If $p = q$, we often write

$$\mathcal{D}_p(\mathbf{Q}, \delta) := \mathcal{D}_{q,p}(\mathbf{Q}, \delta). \quad (1.7)$$

Let $\Gamma_{q,p}(\mathbf{Q})$ be the smallest constant Γ such that, for every $\epsilon > 0$, we have

$$\mathcal{D}_{q,p}(\mathbf{Q}, \delta) \leq C_{p,q,\mathbf{Q},\epsilon} \delta^{-\Gamma-\epsilon}, \quad \text{for every dyadic } \delta < 1, \quad (1.8)$$

where $C_{p,q,\mathbf{Q},\epsilon}$ is a constant that is allowed to depend on $p, q, \mathbf{Q}$ and ϵ . If $p = q$, we often write

$$\Gamma_p(\mathbf{Q}) := \Gamma_{p,p}(\mathbf{Q}). \quad (1.9)$$

For a tuple $\widetilde{\mathbf{Q}} = (\widetilde{\mathcal{Q}}_1(\xi), \dots, \widetilde{\mathcal{Q}}_{\widetilde{n}}(\xi))$ of quadratic forms with $\xi \in \mathbb{R}^d$, denote

$$\text{NV}(\widetilde{\mathbf{Q}}) := |\{1 \leq d' \leq d : \partial_{\xi_{d'}} \widetilde{\mathcal{Q}}_{\widetilde{n}'} \not\equiv 0 \text{ for some } 1 \leq \widetilde{n}' \leq \widetilde{n}\}|. \quad (1.10)$$

Here for a function F , we use $F \not\equiv 0$ to mean that it does not vanish constantly, and $\text{NV}(\widetilde{\mathbf{Q}})$ refers to “the number of variables that $\widetilde{\mathbf{Q}}$ depends on.” For instance, for $\widetilde{\mathbf{Q}} = ((\xi_1 + \xi_3)^2, (\xi_1 + \xi_3 + \xi_4)^2)$, we have that $\text{NV}(\widetilde{\mathbf{Q}}) = 3$.

For $0 \leq n' \leq n$ and $0 \leq d' \leq d$, define

$$\mathfrak{d}_{d',n'}(\mathbf{Q}) := \inf_{\substack{M \in \mathbb{R}^{d \times d} \\ \text{rank}(M) = d'}} \inf_{\substack{M' \in \mathbb{R}^{n' \times n} \\ \text{rank}(M') = n'}} \text{NV}(M' \cdot (\mathbf{Q} \circ M)), \quad (1.11)$$

where $\mathbf{Q} \circ M$ is the composition of $\mathbf{Q}$ with M . We abbreviate $\mathfrak{d}_{n'}(\mathbf{Q}) := \mathfrak{d}_{d,n'}(\mathbf{Q})$.

As these quantities will be crucial throughout the entire paper, it may make sense to look at them from a few angles. Let $H \subset \mathbb{R}^d$ be a linear subspace of codimension m . Let Rot_H be a rotation¹ on $\mathbb{R}^d$ that maps $\{\xi \in \mathbb{R}^d : \xi_{d-m+1} = \dots = \xi_d = 0\}$ to H . We define

$$Q|_H(\xi') := Q((\xi', \mathbf{0}) \cdot (\text{Rot}_H)^T),$$

with $\xi' \in \mathbb{R}^{d-m}$ and $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^m$. (1.12)

Here $(\text{Rot}_H)^T$ refers to the transpose of Rot_H . Similarly, for $\mathbf{Q} = (Q_1, \dots, Q_n)$, we denote

$$\mathbf{Q}|_H := (Q_1|_H, \dots, Q_n|_H). \quad (1.13)$$

From the Bourgain–Guth argument used in Proposition 5.6 below and from Lemma 6.1 below, one can see clearly why restricting to subspaces is natural. With the above notation, we can also write

$$\mathfrak{d}_{d',n'}(\mathbf{Q}) = \inf_H \inf_{M \in \text{GL}_{d'}} \inf_{\substack{M' \in \mathbb{R}^{n' \times n} \\ \text{rank}(M') = n'}} \text{NV}(M' \cdot (\mathbf{Q}|_H \circ M)), \quad (1.14)$$

where H runs through all linear subspaces in $\mathbb{R}^d$ of dimension d' . In other words, this is the minimal number of variables that n' many of the forms in $\mathbf{Q}$ depend on, after restricting them to subspaces of dimension d' , up to linear changes of variables in their definition domain $\mathbb{R}^{d'}$ and their value domain $\mathbb{R}^n$. From (1.14), one can also see that

$$\mathfrak{d}_{d',n'}(\mathbf{Q}) = \inf_{H \text{ of dim } d'} \mathfrak{d}_{d',n'}(\mathbf{Q}|_H). \quad (1.15)$$

For example, if we take $d = 3, n = 3$ and $\mathbf{Q} = ((\xi_1 + \xi_2 + \xi_3)^2, (\xi_1 + \xi_2)^2, (\xi_1 + \xi_2)^2)$, then $\mathfrak{d}_0(\mathbf{Q}) = \mathfrak{d}_1(\mathbf{Q}) = 0$, $\mathfrak{d}_2(\mathbf{Q}) = 1$ and $\mathfrak{d}_3(\mathbf{Q}) = 2$.

THEOREM 1.1

Let $d \geq 1$ and $n \geq 1$. Let $\mathbf{Q} = (Q_1, \dots, Q_n)$ be a collection of quadratic forms in d variables. Let $2 \leq q \leq p < \infty$. Then the $\ell^q L^p$ decoupling exponent for the d -surface $S_{\mathbf{Q}} = \{(\xi, \mathbf{Q}(\xi)) : \xi \in [0, 1]^d\}$ equals

$$\begin{aligned} \Gamma_{q,p}(\mathbf{Q}) = & \max_{0 \leq d' \leq d} \max_{0 \leq n' \leq n} \left(d' \left(1 - \frac{1}{p} - \frac{1}{q} \right) \right. \\ & \left. - \mathfrak{d}_{d',n'}(\mathbf{Q}) \left(\frac{1}{2} - \frac{1}{p} \right) - \frac{2(n-n')}{p} \right). \end{aligned} \quad (1.16)$$

¹There are infinitely many such rotations: we pick an arbitrary one.

Moreover, for $2 \leq p < q \leq \infty$, we have

$$\Gamma_{q,p}(\mathbf{Q}) = \Gamma_{p,p}(\mathbf{Q}) + d(1/p - 1/q). \quad (1.17)$$

The expression (1.16) simplifies in the case $q = p$ in the following way.

COROLLARY 1.2

In the situation of Theorem 1.1, we have

$$\Gamma_p(\mathbf{Q}) = \max_{d/2 < d' \leq d} \max_{0 \leq n' \leq n} \left((2d' - \mathfrak{d}_{d',n'}(\mathbf{Q})) \left(\frac{1}{2} - \frac{1}{p} \right) - \frac{2(n-n')}{p} \right) \quad (1.18)$$

for every $p \geq 2$.

Proof

For every $d' \leq d/2$ and $0 \leq n' \leq n$, we have $2d' - \mathfrak{d}_{d',n'}(\mathbf{Q}) \leq d \leq 2d - \mathfrak{d}_{d,n'}(\mathbf{Q})$. Hence, the (d', n') term in (1.16) is not larger than the (d, n') term. $\square$

Taking $d' = d$ and $n' \in \{0, n\}$ in (1.18), we see that, for every tuple $\mathbf{Q} = (Q_1, \dots, Q_n)$ of quadratic forms depending on d variables, it always holds that

$$\Gamma_p(\mathbf{Q}) \geq \max \left(d \left(\frac{1}{2} - \frac{1}{p} \right), 2d \left(\frac{1}{2} - \frac{1}{p} \right) - \frac{2n}{p} \right) \quad \text{for every } p \geq 2. \quad (1.19)$$

Similarly,

$$\Gamma_{2,p}(\mathbf{Q}) \geq \max \left(0, d \left(\frac{1}{2} - \frac{1}{p} \right) - \frac{2n}{p} \right) \quad \text{for every } p \geq 2. \quad (1.20)$$

We say that $\mathbf{Q} = (Q_1, \dots, Q_n)$ is *strongly nondegenerate* if

$$\mathfrak{d}_{d-m,n'}(\mathbf{Q}) \geq n'd/n - m \quad (1.21)$$

for every n' and every m with $0 \leq m \leq d$.

COROLLARY 1.3 (Best possible $\ell^2 L^p$ decoupling)

We have

$$\Gamma_{2,p}(\mathbf{Q}) = \max \left(0, d \left(\frac{1}{2} - \frac{1}{p} \right) - \frac{2n}{p} \right) \quad \text{for every } 2 \leq p < \infty \quad (1.22)$$

if and only if $\mathbf{Q}$ is *strongly nondegenerate*.

We say that $\mathbf{Q} = (Q_1, \dots, Q_n)$ is *nondegenerate* if

$$\mathfrak{d}_{d-m,n'}(\mathbf{Q}) \geq n'd/n - 2m, \quad (1.23)$$

for every n' and every m with $0 \leq m < d/2$.

COROLLARY 1.4 (Best possible $\ell^p L^p$ decoupling)

We have

$$\Gamma_{p,p}(\mathbf{Q}) = \max \left(d \left(\frac{1}{2} - \frac{1}{p} \right), 2d \left(\frac{1}{2} - \frac{1}{p} \right) - \frac{2n}{p} \right) \quad \text{for every } 2 \leq p < \infty \quad (1.24)$$

if and only if $\mathbf{Q}$ is nondegenerate.

In view of (1.20) and (1.19), Corollaries 1.3 and 1.4 characterize tuples of quadratic forms that possess “best possible” $\ell^2 L^p$ decoupling constants and $\ell^p L^p$ decoupling constants, respectively.

We say that $\mathbf{Q} = (Q_1, \dots, Q_n)$ is *weakly nondegenerate* if

$$\mathfrak{d}_{d-m,n}(\mathbf{Q}) \geq d - 2m, \quad (1.25)$$

for every $0 \leq m < d/2$.

COROLLARY 1.5

A tuple $\mathbf{Q} = (Q_1, \dots, Q_n)$ of quadratic forms is weakly nondegenerate if and only if there exists some $p_c > 2$ such that

$$\Gamma_p(\mathbf{Q}) = d \left(\frac{1}{2} - \frac{1}{p} \right), \quad 2 \leq p \leq p_c. \quad (1.26)$$

If $\mathbf{Q}$ is weakly nondegenerate, then the largest possible p_c is given by

$$2 + \min \left(\frac{4(n - n')}{d - (\mathfrak{d}_{d-m,n'}(\mathbf{Q}) + 2m)} \right), \quad (1.27)$$

where the minimum on the right-hand side is taken over all n' and m satisfying $n' \leq n - 1$, $m < d/2$ and $d > \mathfrak{d}_{d-m,n'}(\mathbf{Q}) + 2m$.

One reason that we are interested in the exponent p_c in Corollary 1.5 is that, when applying our main results to exponential sum estimates (see Corollary 2.1 below), the exponent p_c is the largest for which we can still expect square root cancellation; see immediately below Corollary 2.1 for what we mean by “square root cancellation.”

The connections of these three notions of nondegeneracies will be discussed in forthcoming examples (see, e.g., Corollary 3.4. We leave the proof of Corollaries 1.3–1.5 to Section 8.

Table 1. Minimal number of variables on which lower-dimensional restrictions of $\mathbf{Q}_1$ and $\mathbf{Q}_2$ depend.

	$\mathfrak{d}_{d',n'}(\mathbf{Q}_1)$					$\mathfrak{d}_{d',n'}(\mathbf{Q}_2)$		
$n' =$	0	1	2	3	4	0	1	2
$d' = 3$	0	1	2	3	3	0	1	3
$d' = 2$	0	0	0	0	2	0	0	1

At the end of the introduction, we would like to make a few remarks on Theorem 1.1 and how the double max on the right-hand side of (1.16) is connected to different regimes in scale-dependent Brascamp–Lieb inequalities (see Theorem 4.3 below). To enable comparison with previous approaches, we specialize to $p = q$, so that (1.16) simplifies to (1.18), and consider the tuples of quadratic forms

$$\mathbf{Q}_1(\xi) = (\xi_1\xi_2, \xi_1\xi_3, \xi_2\xi_3, \xi_3^2), \quad \mathbf{Q}_2(\xi) = (\xi_1^2, \xi_2^2 + \xi_1\xi_3), \quad (1.28)$$

for which sharp $\ell^p L^p$ decoupling inequalities were previously proved in [32] and [34], respectively. The reasons for picking these two examples will soon become clear. The numbers of variables (1.11) that appear in (1.18) in these cases are summarized in Table 1.

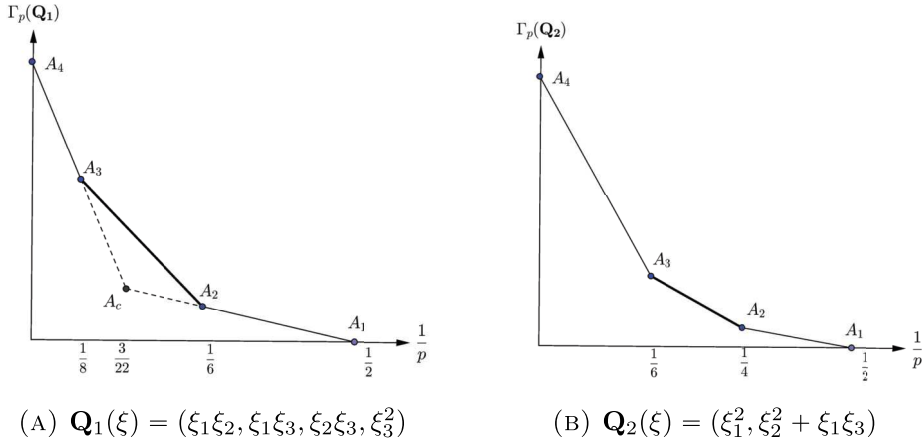
To illustrate the kind of arguments used to obtain the entries in Table 1, let us consider $\mathfrak{d}_{3,3}(\mathbf{Q}_1)$. If $\mathfrak{d}_{3,3}(\mathbf{Q}_1) \leq 2$, then there is a 3-dimensional subspace $\mathcal{Q}$ of the linear space of quadratic forms spanned by $\mathbf{Q}_1$, such that all forms in the subspace $\mathcal{Q}$ depend on at most two variables. The space of quadratic forms depending on any two variables has dimension 3, so $\mathcal{Q}$ consists of all quadratic forms depending on these two variables. In particular, $\mathcal{Q}$ contains two linearly independent quadratic forms that are complete squares. Therefore, $\text{lin } \mathbf{Q}_1$ contains a quadratic form of the form $(a\xi_1 + b\xi_2 + c\xi_3)^2$ with $(a, b) \neq (0, 0)$. But such a quadratic form includes one of the monomials ξ_1^2, ξ_2^2 with a nonzero coefficient, contradicting the fact that it lies in $\text{lin } \mathbf{Q}_1$. Hence, $\mathfrak{d}_{3,3}(\mathbf{Q}_1) > 2$, and since $\mathfrak{d}_{3,3}(\mathbf{Q}_1) \leq 3$, we obtain $\mathfrak{d}_{3,3}(\mathbf{Q}_1) = 3$. Upper bounds for $\mathfrak{d}_{d',n'}$ are usually easier to obtain. For example, with the notation from (1.14), the upper bounds for $\mathfrak{d}_{2,n'}(\mathbf{Q}_1)$ are obtained with $H = \{\xi_3 = 0\}$ and the upper bounds for $\mathfrak{d}_{2,n'}(\mathbf{Q}_2)$ with $H = \{\xi_1 = 0\}$.

Substituting the numbers in Table 1 into (1.18), we see that the decoupling exponents are given by

$$\Gamma_p(\mathbf{Q}_1) = \max\left(\frac{3}{2} - \frac{3}{p}, 2 - \frac{6}{p}, 3 - \frac{14}{p}\right) \quad \text{and} \quad (1.29)$$

$$\Gamma_p(\mathbf{Q}_2) = \max\left(\frac{3}{2} - \frac{3}{p}, \frac{5}{2} - \frac{7}{p}, 3 - \frac{10}{p}\right) \quad (1.30)$$

for every $p \geq 2$. These decoupling exponents are sketched as functions of $1/p$ in Figure 1 (not to scale). The kinks A_2, A_3 in the graph 1a and B_2, B_3 in the graph


 Figure 1. Decoupling exponents for $\mathbf{Q}_1, \mathbf{Q}_2$.

1b are called *critical points*: After proving sharp decoupling inequalities at these two points, we can simply interpolate them with trivial estimates at L^2 and L^∞ and obtain sharp $\ell^p L^p$ decoupling inequalities for every $p \geq 2$. One reason for picking these two examples: on one hand, graph (A) and graph (B) in Figure 1 look similar, in that they both have two critical points; on the other hand, the nature of the critical points in the respective graphs is entirely different, and this can sometimes be misleading when it comes to trying to come up with a unified proof strategy for both examples. In the following several paragraphs, let us try to explain the different nature of the critical points, together with one main ingredient of the paper—scale-dependent Brascamp–Lieb inequalities.

The tuple $\mathbf{Q}_1$, together with the linear forms ξ_1, ξ_2, ξ_3 , is the Arkhipov–Chubarikov–Karatsuba (ACK) system generated by the monomial $\xi_1\xi_2\xi_3^2$, restricted to degree 2. To be more precise, we take all possible partial derivatives of $\xi_1\xi_2\xi_3^2$, collect the resulting terms, and throw away the terms that are of degree higher than 2. Decoupling exponents for all ACK systems were found in [32]. The main argument to handle $\mathbf{Q}_1$ there was purely guided by the “fake” kink A_c , and the critical points A_2 and A_3 were deliberately neglected (see [30, Section 7] for a detailed discussion on the critical points and the “fake” kink in Parsell–Vinogradov systems). Parsell–Vinogradov systems are special cases of ACK systems and that discussion applies equally well to ACK systems. The example (1.28) is perhaps the simplest system of quadratic monomials that admits more than one critical point and is therefore discussed here in detail. The way A_c appears in [32] can be summarized as follows: For all $p \geq 2$, the article [32] uses only the global/scale-invariant Brascamp–Lieb

inequality due to Bennett et al. [5]; the difference between $p > p_c := 22/3$ and $p < p_c$ is in the scheme of induction on scales there.

Let us describe the current argument of handling $\mathbf{Q}_1$. In the current paper, we still use the argument of induction on scales (see Section 5), and a phase transition also happens at the “fake” kink A_c . However, for $p \leq p_c$, we will use local Brascamp–Lieb inequalities as in [6, Theorem 2.2], and for $p \geq p_c$, we will use discrete Brascamp–Lieb inequalities as in [6, Theorem 2.5]. For $p = p_c$, local and discrete Brascamp–Lieb inequalities coincide and become scale-invariant, as in [5, Theorem 1.15]. The significance of local and discrete Brascamp–Lieb inequalities may not be fully reflected from this example, as they coincide at some point. The next example explains what we could do in a case where they do not coincide at any point.

Let us turn to the tuple $\mathbf{Q}_2$. As mentioned above, sharp decoupling inequalities for $\mathbf{Q}_2$ were proved in [34]. The tuple $\mathbf{Q}_2$ is an interesting example as it is perhaps the simplest nontrivial example that is degenerate, in the sense that the scale-invariant Brascamp–Lieb inequality cannot be used for any $p \in [2, \infty)$. To handle this degeneracy, several specialized tools were introduced in [34], including a partial small ball decoupling inequality for the tuple $(\xi_1 \xi_2, \xi_2^2)$. In [34], sharp decoupling inequalities were first proved at the critical points B_2 and B_3 , and interpolation with trivial estimates at B_1 and B_4 was used to cover the whole range $p \geq 2$.

Let us describe our current approach. First of all, the counterpart of A_c in Graph 1b is not significant anymore. Secondly, it turns out that for $2 \leq p \leq 4$ we can still use the local Brascamp–Lieb inequality and for $p \geq 6$ the discrete Brascamp–Lieb inequality (it is a general principle that the local Brascamp–Lieb inequality works for small p and the discrete one for large p , although the ranges of p ’s in which they work may be empty). There is, however, a new regime for $p \in (4, 6)$. Although this segment can be filled in by interpolation between $p = 4$ and $p = 6$, our proof works directly for all p , which becomes necessary when the number of kinks increases further. To this end, we use the family of scale-dependent Brascamp–Lieb inequalities due to Maldague [46], which unifies scale-invariant, discrete, and local Brascamp–Lieb inequalities due to Bennett et al. [5], [6].

Organization of the paper

In Section 2 we state a few applications of our main theorem. In Section 3, we compute the decoupling exponent provided by the main theorem more explicitly for several examples of tuples of quadratic forms $\mathbf{Q}$, including some of those tuples $\mathbf{Q}$ for which sharp decoupling inequalities have been previously established in the literature, and a few tuples $\mathbf{Q}$ (in particular, arbitrary pairs of forms and tuples of simultaneously diagonalizable forms) for which our results are new.

The upper bounds of $\Gamma_{q,p}(\mathbf{Q})$ in Theorem 1.1 with $q \leq p$ are proved in Sections 4 and 5. The lower bounds of $\Gamma_{q,p}(\mathbf{Q})$ in Theorem 1.1 with $q \leq p$ are proved in Section 6. In Section 7, we show that the optimal decoupling inequalities for $q > p$ follow from the case $q = p$ of Theorem 1.1.

In Section 8, we provide the proofs of Corollary 1.3, Corollary 1.4 and Corollary 1.5. In Section 9, we prove the Fourier restriction estimate in Corollary 2.3.

Notation

For two positive constants A_1, A_2 and a set $\mathcal{I}$ of parameters, we use $A_1 \lesssim_{\mathcal{I}} A_2$ to mean that there exists $C > 0$ depending on the parameters in $\mathcal{I}$ such that $A_1 \leq CA_2$. Typically, $\mathcal{I}$ will be taken to be $\{\mathbf{Q}, d, n, p, q, \epsilon\}$, where $\epsilon > 0$ is a small number. Similarly, we define $A_1 \gtrsim A_2$. Moreover, $A_1 \sim A_2$ means $A_1 \lesssim A_2$ and $A_1 \gtrsim A_2$.

Let $\delta \in (0, 1)$ be a dyadic number. We denote by $\mathcal{P}(Q, \delta)$ the dyadic cubes of side length δ in Q for every dyadic cube $Q \subset [0, 1]^d$. Let $\mathcal{P}(\delta)$ be the partition of $[0, 1]^d$ into dyadic cubes of side length δ . Let $\square$ be a cube with side length $l(\square)$. We use $c \cdot \square$ to denote the cube of side length $c \cdot l(\square)$ and of the same center as $\square$.

For two linear spaces V, V' , we use $V' \leq V$ to mean that V' is a linear subspace of V . For a sequence of real numbers $\{A_j\}_{j=1}^M$, we abbreviate $\prod A_j := (\prod_{j=1}^M |A_j|)^{1/M}$. For $E > 0$ and a ball $B = B(c_B, r_B) \subset \mathbb{R}^{d+n}$ with center c_B and radius r_B , define an associated weight

$$w_{B,E}(\cdot) := \left(1 + \frac{|\cdot - c_B|}{r_B}\right)^{-E}. \quad (1.31)$$

The power E is large number depending on d, n (e.g., $E = 10(d + n)$) and will be omitted from the notation $w_{B,E}$. All implicit constants in the paper are allowed to depend on E . Also, we define the following averaged integrals:

$$\|f\|_{\mathcal{L}^p(B)} := \left(\frac{1}{|B|} \int_B |f|^p\right)^{1/p} \quad \text{and} \quad \|f\|_{\mathcal{L}^p(w_B)} := \left(\frac{1}{|B|} \int |f|^p w_B\right)^{1/p}.$$

For a dyadic box $\square \subset [0, 1]^d$, a function $f_{\square}$ is always implicitly assumed to satisfy $\text{supp } \widehat{f_{\square}} \subset \mathcal{U}_{\square}$, unless otherwise stated.

We would like to make the convention that all vectors are column vectors, unless they are variables of functions or otherwise stated. Following are a few more conventions we make on notation: We will use dyadic cubes of side lengths δ, δ^b with $b < 1$ and $1/K_j$ with $1 \leq j \leq d$. One can always keep in mind that $\log_{K_j} \log_{K_j}(1/\delta) \geq K_j$. We will always use $\square$ to denote a dyadic cube of the smallest scale δ , J to denote a dyadic cube of an intermediate scale δ^b , and W or W_j to denote a dyadic cube of a large scale $1/K_j$. We will introduce certain multilinear estimates during the proof, and the degree of the multilinearity will always be called M .

2. Some applications

2.1. Exponential sum estimates

Let $\mathbf{Q} = (Q_1, \dots, Q_n)$ be a collection of quadratic forms of integral coefficients defined on $\mathbb{R}^d$. Let $\mathbf{w} = (w_1, \dots, w_d) \in \mathbb{N}^d$.

COROLLARY 2.1

For every $d, n \geq 1$, every $p \geq 2, \epsilon > 0$, there exists $C_{\mathbf{Q}, \epsilon, p}$ such that

$$\left\| \sum_{1 \leq d' \leq d} \sum_{0 \leq w_{d'} \leq W} e^{2\pi i(\mathbf{w} \cdot \mathbf{x} + \mathbf{Q}(\mathbf{w}) \cdot \mathbf{y})} \right\|_{L^p([0,1]^d \times [0,1]^n)} \leq C_{\mathbf{Q}, \epsilon, p} W^{\Gamma_p(\mathbf{Q}) + \frac{d}{p} + \epsilon} \quad (2.1)$$

for every integer W .

If $\Gamma_p(\mathbf{Q}) = d(1/2 - 1/p)$, then the above corollary says that

$$\begin{aligned} & \left\| \sum_{1 \leq d' \leq d} \sum_{0 \leq w_{d'} \leq W} e^{2\pi i(\mathbf{w} \cdot \mathbf{x} + \mathbf{Q}(\mathbf{w}) \cdot \mathbf{y})} \right\|_{L^p([0,1]^d \times [0,1]^n)} \\ & \leq C_{\mathbf{Q}, \epsilon, p} W^\epsilon \left\| \sum_{1 \leq d' \leq d} \sum_{0 \leq w_{d'} \leq W} e^{2\pi i(\mathbf{w} \cdot \mathbf{x} + \mathbf{Q}(\mathbf{w}) \cdot \mathbf{y})} \right\|_{L^2([0,1]^d \times [0,1]^n)}, \end{aligned} \quad (2.2)$$

by which we mean square root cancellation holds for the exponential sum at such p .

The derivation of exponential sum estimates of the form in the above corollary from decoupling inequalities has been standard (see, e.g., [11, Section 2] and [17, Section 4]). We will not repeat the argument here.

Let $s \geq 1$ be an integer. Consider the system of Diophantine equations

$$\begin{aligned} \mathbf{w}_1 + \dots + \mathbf{w}_s &= \mathbf{w}_{s+1} + \dots + \mathbf{w}_{2s}, \\ \mathbf{Q}(\mathbf{w}_1) + \dots + \mathbf{Q}(\mathbf{w}_s) &= \mathbf{Q}(\mathbf{w}_{s+1}) + \dots + \mathbf{Q}(\mathbf{w}_{2s}). \end{aligned} \quad (2.3)$$

For a large integer W , let $\mathcal{J}_{\mathbf{Q}}(W)$ be the number of solutions to (2.3), where $0 \leq w_{d'} \leq W$ for every d' . As an immediate corollary of (2.1), we obtain the following.

COROLLARY 2.2

For every $d, n \geq 1$, integer $s \geq 1$, and every $\epsilon > 0$, there exists $C_{\mathbf{Q}, \epsilon, s}$ such that

$$\mathcal{J}_{\mathbf{Q}}(W) \leq C_{\mathbf{Q}, \epsilon, s} W^{2s\Gamma_{2s}(\mathbf{Q}) + d + \epsilon}, \quad (2.4)$$

for every W .

2.2. Fourier restriction estimates

Let $\mathbf{Q} = (Q_1, \dots, Q_n)$ be a collection of quadratic forms defined on $\mathbb{R}^d$. We say that $\mathbf{Q}$ is *linearly independent* if $Q_1, \dots, Q_n$ are linearly independent. We are interested in the Fourier restriction problems: Find an optimal range of p such that

$$\|E_{[0,1]^d}^{\mathbf{Q}} g\|_{L^p(\mathbb{R}^{d+n})} \lesssim_{d,n,p,\mathbf{Q}} \|g\|_p \quad (2.5)$$

holds true for every function g . By a simple change of variables, one can see that the restriction estimate (2.5) cannot hold true for any $p < \infty$ if $\mathbf{Q}$ is linearly dependent. As an application of Corollary 1.2, we prove some restriction estimates for every linearly independent $\mathbf{Q}$ for some range of p .

COROLLARY 2.3

Let $\mathbf{Q} = (Q_1, \dots, Q_n)$ be a collection of linearly independent quadratic forms defined on $\mathbb{R}^d$. Then

$$\|E_{[0,1]^d}^{\mathbf{Q}} g\|_{L^p(\mathbb{R}^{d+n})} \lesssim_{d,n,p,\mathbf{Q}} \|g\|_p \quad (2.6)$$

for every

$$p > p_{\mathbf{Q}} := 2 + \max_{m \geq 1} \max_{n' \leq n} \frac{4n'}{2m + \mathfrak{d}_{d-m,n'}(\mathbf{Q})}. \quad (2.7)$$

The proof of this corollary will be presented in Section 9. One significance of this corollary is that the range (2.7) is sharp for Parsell–Vinogradov systems. Let us be more precise. Let $d \geq 2$. Denote $\xi^\alpha := \xi_1^{\alpha_1} \dots \xi_d^{\alpha_d}$ for $\xi = (\xi_1, \dots, \xi_d)$ and a multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$. For $\mathbf{Q} := (\xi^\alpha)_{|\alpha|=2}$, we have $n = d(d+1)/2$, and it has been shown by Christ [20] and Mockenhaupt [47] that (2.6) holds if and only if

$$p > 2 + \frac{4n}{d+1} = 2d + 2. \quad (2.8)$$

Let us also mention that, for this tuple $\mathbf{Q}$, the full range of $L^q \rightarrow L^p$ estimates generalizing (2.6) has been obtained in [1] and [50]. The next claim shows that the range (2.7) coincides with (2.8).

CLAIM 2.4

Let $\mathbf{Q} := (\xi^\alpha)_{|\alpha|=2}$. Then

$$\max_{m \geq 1} \max_{n' \leq n} \frac{4n'}{2m + \mathfrak{d}_{d-m,n'}(\mathbf{Q})} = \frac{4n}{2 + \mathfrak{d}_{d-1,n}(\mathbf{Q})} = \frac{4n}{d+1} = 2d. \quad (2.9)$$

In other words, the max is attained at $m = 1, n' = n$.

Proof of Claim 2.4

By definition, $\mathfrak{d}_{d-1,n}(\mathbf{Q}) = d - 1$. Hence it suffices to show that the leftmost expression in (2.9) is equal to $2d$.

Fix $m \geq 1$ and $n' \leq n$. Denote $l := \mathfrak{d}_{d-m,n'}(\mathbf{Q})$. Notice that by the definition of $\mathfrak{d}_{d-m,n'}(\mathbf{Q})$, we see that $l + m \leq d$. Our goal is to show that $\frac{4n'}{2m+l} \leq 2d$. We claim that

$$n' \leq \binom{l+1}{2} + \binom{m+1}{2} + m(d-m). \quad (2.10)$$

Indeed, by definition (1.14), there exist a linear subspace $H \subset \mathbb{R}^d$ of dimension $d - m$ and a linear subspace $\mathcal{Q}$ of the span of $\mathbf{Q}$ of dimension n' such that the restrictions of the forms from $\mathcal{Q}$ to H depend only on l variables. Since the system $\mathbf{Q} = (\xi^\alpha)_{|\alpha|=2}$ is a basis for the space of all quadratic forms in d variables, the above statement does not depend on H and the l variables inside H , so we may assume $H = \{\xi : \xi_{d-m+1} = \dots = \xi_d = 0\}$ and the l variables are $\xi_1, \dots, \xi_l$. In this case, $\mathcal{Q}$ is contained in the space of all quadratic forms that depend either only on $\xi_1, \dots, \xi_l$, or on at least one of the variables $\xi_{d-m+1}, \dots, \xi_d$. The right-hand side of (2.10) is precisely the dimension of the latter space, which concludes the proof of (2.10).

Given (2.10), it remains to show

$$4\left(\frac{l(l+1)}{2} + \frac{m(m+1)}{2} + m(d-m)\right) \leq 2d(2m+l), \quad (2.11)$$

which is equivalent to

$$2(l+m)(l-m+1) \leq 2dl. \quad (2.12)$$

This holds because $l + m \leq d$ and $l - m + 1 \leq l$. $\square$

3. Examples: Old and new

3.1. Example: Hypersurfaces with nonvanishing Gaussian curvatures

We take $n = 1$. Let Q be a quadratic form depending on d variables. Without loss of generality, we assume that $\mathfrak{d}_1(Q) = d$. Via a change of coordinate, we can write $Q(\xi)$ as $\xi_1^2 \pm \xi_2^2 \pm \dots \pm \xi_d^2$. This is the (hyperbolic) paraboloid case. It is easy to see $\mathfrak{d}_0(Q) = 0, \mathfrak{d}_1(Q) = d$.

LEMMA 3.1

Let $\tilde{Q} : \mathbb{R}^d \rightarrow \mathbb{R}$ be a quadratic form. Let $M \in M_{d \times d}$ with rank d' . Then

$$\mathfrak{d}_1(\tilde{Q}(\cdot M)) \geq \mathfrak{d}_1(\tilde{Q}(\cdot)) - 2(d - d'). \quad (3.1)$$

Proof of Lemma 3.1

A lemma of this form was already proved and used by Bourgain and Demeter (see [15, Lemma 2.6]). We use $\text{Hess}(\tilde{Q})$ to denote the Hessian of the quadratic form $\tilde{Q}$. What we need to prove is, for every $M \in M_{d \times d}$ with rank d' , it holds that

$$\text{rank}(M \text{Hess}(\tilde{Q}) M^T) \geq \text{rank}(\text{Hess}(\tilde{Q})) - 2(d - d'). \quad (3.2)$$

This follows immediately from Sylvester's rank inequality,

$$\text{rank}(AB) \geq \text{rank}(A) + \text{rank}(B) - n, \quad (3.3)$$

for two arbitrary matrices $A, B \in M_{n \times n}$. $\square$

By Lemma 3.1, we know that $\mathfrak{d}_1(Q|_H) \geq d - 2m$ for every linear subspace of codimension m , which means Q is nondegenerate. Therefore, we can apply Corollary 1.4 and obtain

$$\Gamma_p(Q) = \max\left(d - \frac{2d+2}{p}, d\left(\frac{1}{2} - \frac{1}{p}\right)\right). \quad (3.4)$$

This recovers the $\ell^p L^p$ decoupling results of Bourgain and Demeter in [11] and [15]. Moreover, if we take $Q(\xi) = \xi_1^2 + \dots + \xi_d^2$, then it is elementary to see that $\mathfrak{d}_1(Q|_H) \geq d - m$ for every linear subspace of codimension m , which means Q is strongly nondegenerate. Therefore, we can apply Corollary 1.3 and obtain

$$\Gamma_{2,p}(Q) = \max\left(0, \frac{d}{2} - \frac{d+2}{p}\right). \quad (3.5)$$

This recovers the $\ell^2 L^p$ decoupling results of Bourgain and Demeter in [11].

3.2. Example: Codimension 2 manifolds in $\mathbb{R}^4$

Take $d = n = 2$. Let $Q_1(\xi) = A_1 \xi_1^2 + 2A_2 \xi_1 \xi_2 + A_3 \xi_2^2$ and $Q_2(\xi) = B_1 \xi_1^2 + 2B_2 \xi_1 \xi_2 + B_3 \xi_2^2$. Under the assumption that

$$\text{rank} \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{bmatrix} = 2, \quad (3.6)$$

Bourgain and Demeter [12] proved that

$$\Gamma_p(\mathbf{Q}) = \max\left(2\left(\frac{1}{2} - \frac{1}{p}\right), 2\left(1 - \frac{4}{p}\right)\right), \quad (3.7)$$

with $\mathbf{Q} = (Q_1, Q_2)$. This decoupling inequality is particularly interesting as it is one key ingredient in Bourgain's improvement on the Lindelöf hypothesis in [9].

Let us see how Theorem 1.1 recovers this result. We take $d = n = 2$ and notice that $\mathfrak{d}_2(\mathbf{Q}) = 2$ (indeed, if $\mathfrak{d}_2(\mathbf{Q}) \leq 1$, then Q_1, Q_2 would be linearly dependent, since the space of quadratic forms in one variable is 1-dimensional, contradicting (3.6)). Moreover, it is straightforward to see that $\mathfrak{d}_1(\mathbf{Q}) > 0$, as the assumption (3.6) says that Q_1 and Q_2 are linearly independent. Therefore, $\mathbf{Q}$ is nondegenerate in the sense of (1.23). We can apply Corollary 1.4 and recover the result of Bourgain and Demeter [12].

3.3. Example: Degenerate 3-dimensional submanifolds of $\mathbb{R}^5$

Take $d = 3$, $n = 2$ and $\mathbf{Q} = (\xi_1^2, \xi_2^2 + \xi_1\xi_3)$. Note that $\mathfrak{d}_0(\mathbf{Q}) = 0$, $\mathfrak{d}_1(\mathbf{Q}) = 1$, $\mathfrak{d}_2(\mathbf{Q}) = 3$, and therefore $\mathbf{Q}$ fails to satisfy the nondegeneracy condition (1.23). On the other hand, one can also compute, for instance via (1.14), that $\mathfrak{d}_{2,2}(\mathbf{Q}) = 1$, $\mathfrak{d}_{2,1}(\mathbf{Q}) = 0$ and $\mathfrak{d}_{d',n'}(\mathbf{Q}) = 0$ whenever $d' \leq 1$. We apply Theorem 1.1 and obtain that

$$\Gamma_p(\mathbf{Q}) = \max\left(3\left(\frac{1}{2} - \frac{1}{p}\right), \frac{5}{2} - \frac{7}{p}, 3 - \frac{10}{p}\right), \quad (3.8)$$

after some elementary computation. This recovers the main result of Guo et al. [34], via an entirely different approach: The proof in [34] relies on bilinear Fourier restriction estimates, small cap decoupling inequalities for the parabola and the manifold $(\xi_1, \xi_2, \xi_1^2, \xi_1\xi_2)$ and a more sophisticated induction argument; while the proof in the current paper relies on more sophisticated Brascamp–Lieb inequalities and multilinear Fourier restriction estimates.

3.4. Simultaneously diagonalizable forms

COROLLARY 3.2

Let $\mathbf{Q} = (Q_1, \dots, Q_n)$ be a collection of quadratic forms without mixed terms. Then

$$\Gamma_p(\mathbf{Q}) = \max_{0 \leq n' \leq n} \left(d \left(\frac{1}{2} - \frac{1}{p} \right) + \left(\frac{1}{2} - \frac{1}{p} \right) (d - \mathfrak{d}_{n'}(\mathbf{Q})) - \frac{2(n - n')}{p} \right), \quad (3.9)$$

for every $p \geq 2$.

Proof of Corollary 3.2

We first apply Corollary 1.2 and obtain

$$\Gamma_p(\mathbf{Q}) = \max_{d/2 \leq d' \leq d} \max_{0 \leq n' \leq n} \left((2d' - \mathfrak{d}_{d',n'}(\mathbf{Q})) \left(\frac{1}{2} - \frac{1}{p} \right) - \frac{2(n - n')}{p} \right). \quad (3.10)$$

In order to obtain (3.9), it suffices to prove that

$$\max_{d/2 \leq d' \leq d} (2d' - \mathfrak{d}_{d',n'}(\mathbf{Q})) = 2d - \mathfrak{d}_{n'}(\mathbf{Q}), \quad (3.11)$$

for every n' . By the equivalent definition of $\mathfrak{d}_{d',n'}(\mathbf{Q})$ as in (1.14), this is equivalent to proving

$$\min_{0 \leq m \leq d/2} \inf_{\substack{H \text{ of} \\ \text{Setcodimm}}} (\mathfrak{d}_{n'}(\mathbf{Q}|_H) + 2m) = \mathfrak{d}_{n'}(\mathbf{Q}), \quad (3.12)$$

for every n' , which is the same as saying

$$\mathfrak{d}_{n'}(\mathbf{Q}) - 2m \leq \mathfrak{d}_{n'}(\mathbf{Q}|_H) \quad (3.13)$$

for every $1 \leq n' \leq n$ and every plane H of codimension m with $1 \leq m \leq d/2$.

We argue by contradiction and assume that

$$\mathfrak{d}_{n'}(\mathbf{Q}|_H) \leq \mathfrak{d}_{n'}(\mathbf{Q}) - 2m - 1, \quad (3.14)$$

for some n' and some linear subspace H of codimension m . By the definition (1.11), we can find $M_{d-m} \in GL_{d-m}(\mathbb{R})$ and $M' \in M_{n \times n'}$ of rank n' such that

$$NV(\bar{\mathbf{P}}) = \mathfrak{d}_{n'}(\mathbf{Q}|_H), \quad (3.15)$$

where for $\xi' \in \mathbb{R}^{d-m}$ we define

$$\begin{aligned} \bar{\mathbf{P}}(\xi') &:= (Q_1|_H(\xi' \cdot M_{d-m}), \dots, Q_n|_H(\xi' \cdot M_{d-m})) \cdot M' \\ &= (Q_1((\xi' \cdot M_{d-m}, \mathbf{0}) \cdot \text{Rot}_H), \dots, Q_n((\xi' \cdot M_{d-m}, \mathbf{0}) \cdot \text{Rot}_H)) \cdot M'. \end{aligned} \quad (3.16)$$

Here $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^m$ and Rot_H is a rotation matrix acting on $\mathbb{R}^d$. Let $M_d \in GL_d(\mathbb{R})$ be a matrix such that

$$(\xi' \cdot M_{d-m}, \mathbf{0}) = (\xi', \mathbf{0}) \cdot M_d, \quad \text{for every } \xi' \in \mathbb{R}^{d-m}. \quad (3.17)$$

With this notation, we can write

$$\begin{aligned} \bar{\mathbf{P}}(\xi') &= (Q_1((\xi', \mathbf{0}) \cdot M_d \cdot \text{Rot}_H), \dots, Q_n((\xi', \mathbf{0}) \cdot M_d \cdot \text{Rot}_H)) \cdot M' \\ &=: (\bar{P}_1(\xi'), \dots, \bar{P}_{n'}(\xi')). \end{aligned} \quad (3.18)$$

Recall (3.15). It implies that

$$NV(\lambda_1 \bar{P}_1 + \dots + \lambda_{n'} \bar{P}_{n'}) \leq \mathfrak{d}_{n'}(\mathbf{Q}|_H) \leq \mathfrak{d}_{n'}(\mathbf{Q}) - 2m - 1, \quad (3.19)$$

for all choices of $\lambda_1, \dots, \lambda_{n'} \in \mathbb{R}$. Now if we denote

$$\bar{\mathbf{Q}}(\xi) := (\bar{Q}_1(\xi), \dots, \bar{Q}_{n'}(\xi)) := (Q_1(\xi), \dots, Q_n(\xi)) \cdot M', \quad (3.20)$$

then from the definition of $\mathfrak{d}_{n'}(\mathbf{Q})$ and the fact that $Q_1, \dots, Q_n$ are diagonal quadratic forms, we can find some $\lambda_1, \dots, \lambda_{n'}$ such that

$$\mathfrak{d}_1(\lambda_1 \bar{Q}_1 + \cdots + \lambda_{n'} \bar{Q}_{n'}) = \text{NV}(\lambda_1 \bar{Q}_1 + \cdots + \lambda_{n'} \bar{Q}_{n'}) \geq \mathfrak{d}_{n'}(\mathbf{Q}). \quad (3.21)$$

Recall the definition of $\bar{\mathbf{P}}$ in (3.16) and the relation in (3.17) and (3.18). Lemma 3.1 then says that

$$\text{NV}(\lambda_1 \bar{P}_1 + \cdots + \lambda_{n'} \bar{P}_{n'}) \geq \mathfrak{d}_{n'}(\mathbf{Q}) - 2m, \quad (3.22)$$

which is a contradiction to (3.19). $\square$

COROLLARY 3.3

For $1 \leq n' \leq n$, define

$$Q_{n'}(\xi) := \sum_{1 \leq d' \leq d} a_{n',d'} \xi_{d'}^2. \quad (3.23)$$

Then for every $p \geq 2$, with $\mathbf{Q} = (Q_1, \dots, Q_n)$,

$$\Gamma_p(\mathbf{Q}) = \max \left(d \left(\frac{1}{2} - \frac{1}{p} \right), 2d \left(\frac{1}{2} - \frac{1}{p} \right) - \frac{2n}{p} \right) \quad (3.24)$$

if and only if, for every $1 \leq n' \leq n$, every $n \times (\lfloor d - \frac{n'd}{n} \rfloor + 1)$ submatrix of

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,d} \\ \cdots & & & \\ a_{n,1} & a_{n,2} & \cdots & a_{n,d} \end{bmatrix} \quad (3.25)$$

has rank at least $n - n' + 1$. Here for $A \in \mathbb{R}$, $\lfloor A \rfloor$ refers to the largest integer $\leq A$.²

When $n = 2$, a condition of the form (3.25) already appeared in Heath-Brown and Pierce [39]. Let $\mathbf{Q} = (Q_1, Q_2)$ be a pair of quadratic forms with integer coefficients. Heath-Brown and Pierce [39] studied the problem of representing a pair of integers (n_1, n_2) by the pair of (Q_1, Q_2) for general Q_1 and Q_2 . If Q_1 and Q_2 are assumed to be simultaneously diagonalizable, say of the form (3.23), then the condition in [39] becomes that every 2×2 minor of

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,d} \\ a_{n,1} & a_{n,2} & \cdots & a_{n,d} \end{bmatrix} \quad (3.26)$$

has rank 2 (see [39, Condition 3]).

Proof of Corollary 3.3

Let us show the “only if” part by contradiction. Suppose that, for some $1 \leq n' \leq n$, some $n \times (\lfloor d - \frac{n'd}{n} \rfloor + 1)$ submatrix of (3.25) has rank $n - n'$ or less. Then

²This notation is used only in Corollary 3.3 and its proof.

$$\mathfrak{d}_{n'}(\mathbf{Q}) \leq d - \left(\left\lfloor d - \frac{n'd}{n} \right\rfloor + 1 \right) < \frac{n'd}{n}. \quad (3.27)$$

Therefore, $\mathbf{Q}$ is not nondegenerate in the sense of (1.23), and (3.24) cannot hold true by Corollary 1.4.

Let us show the other direction of the equivalence. First of all, notice that the two terms on the right-hand side of (3.24) match at $p = p_{n,d} := 2 + 4n/d$. By Corollary 3.2, it suffices to show that

$$d\left(\frac{1}{2} - \frac{1}{p}\right) + \left(\frac{1}{2} - \frac{1}{p}\right)(d - \mathfrak{d}_{n'}(\mathbf{Q})) - \frac{2(n - n')}{p} \leq 2d\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{2n}{p} \quad (3.28)$$

for every $1 \leq n' \leq n$ and every $p \geq p_{n,d}$. By rearranging the terms, what we need to show becomes

$$\mathfrak{d}_{n'}(\mathbf{Q}) \geq n'd/n \quad (3.29)$$

for every $1 \leq n' \leq n$. We argue by contradiction and assume that

$$\mathfrak{d}_{n'}(\mathbf{Q}) < n'd/n \quad (3.30)$$

for some $1 \leq n' \leq n$. By definition, there exist $M \in \mathbb{R}^{d \times d}$ of rank d and $M' \in \mathbb{R}^{n \times n}$ of rank n' such that

$$\mathfrak{d}_{n'}(\mathbf{Q}) = \text{NV}(M' \cdot (\mathbf{Q} \circ M)). \quad (3.31)$$

Since the assumption (3.25) is invariant under the row operations, we may assume that M' is a diagonal matrix with diagonal entries $1, \dots, 1, 0, \dots, 0$. By the inequality (3.30), we have

$$\dim \bigcap_{i=1}^{n'} \bigcap_{\xi \in \mathbb{R}^d} \ker \nabla Q_i(\xi) > d - \frac{n'd}{n}. \quad (3.32)$$

It remains to observe that

$$\ker \nabla Q_i(\xi) = \left\{ \eta \in \mathbb{R}^d \sum_{j=1}^d \xi_j \eta_j a_{i,j} = 0 \right\} \quad \text{and}$$

$$\bigcap_{\xi \in \mathbb{R}^d} \ker \nabla Q_i(\xi) = \{ \eta \in \mathbb{R}^d \eta_j a_{i,j} = 0 \text{ for all } j = 1, \dots, d \},$$

so that (3.32) implies that an $n' \times (\lfloor d - \frac{n'd}{n} \rfloor + 1)$ submatrix of (3.25) vanishes. $\square$

3.5. Decoupling theory for two quadratic forms

COROLLARY 3.4

Let $\mathbf{Q} = (Q_1, Q_2)$ be two linearly independent quadratic forms defined on $\mathbb{R}^d$ satisfying $\mathfrak{d}_2(\mathbf{Q}) = d$.

- (1) Let $1 \leq k < d/2$. Then $\mathbf{Q}$ satisfies $\mathfrak{d}_1(\mathbf{Q}) = k$ and the weakly nondegenerate condition if and only

$$\Gamma_p(\mathbf{Q}) = \max\left(d\left(\frac{1}{2} - \frac{1}{p}\right), (2d - k)\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{2}{p}, 2d\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{4}{p}\right), \quad (3.33)$$

for every $p \geq 2$.

- (2) $\mathbf{Q}$ is nondegenerate if and only if it is weakly nondegenerate and satisfies $\mathfrak{d}_1(\mathbf{Q}) \geq d/2$.

Proof of Corollary 3.4

Let us start with proving the first part of the corollary. We denote the right-hand side of (3.33) by $\Gamma'_p(\mathbf{Q})$. By Corollary 1.2, $\Gamma_p(\mathbf{Q})$ is given by

$$\max_{d/2 \leq d' \leq d} \max\left((2d' - \mathfrak{d}_{d',2}(\mathbf{Q}))\left(\frac{1}{2} - \frac{1}{p}\right), (2d' - \mathfrak{d}_{d',1}(\mathbf{Q}))\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{2}{p}, 2d\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{4}{p}\right). \quad (3.34)$$

Let us first show that (3.33) holds; that is, $\Gamma_p(\mathbf{Q}) = \Gamma'_p(\mathbf{Q})$ for every $p \geq 2$, if and only if

$$\begin{aligned} \max_{d'}(2d' - \mathfrak{d}_{d',1}(\mathbf{Q})) &= 2d - k, \\ \max_{d'}(2d' - \mathfrak{d}_{d',2}(\mathbf{Q})) &= d. \end{aligned} \quad (3.35)$$

To show that (3.35) implies (3.33), we apply (3.34), move the $\max_{d/2 \leq d' \leq d}$ inside the second max, and obtain (3.33). To show the other direction of the equivalence, the constraint $k < d/2$ will come into play. Notice that under this assumption,

$$\Gamma'_p(\mathbf{Q}) = \begin{cases} d\left(\frac{1}{2} - \frac{1}{p}\right) & \text{if } p \leq 2 + \frac{4}{d-k}, \\ (2d - k)\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{2}{p} & \text{if } 2 + \frac{4}{d-k} \leq p \leq 2 + \frac{4}{k}, \\ 2d\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{4}{p} & \text{if } p \geq 2 + \frac{4}{k}. \end{cases} \quad (3.36)$$

Note that we are now under the assumption that $\Gamma_p(\mathbf{Q}) = \Gamma'_p(\mathbf{Q})$ for every $p \geq 2$. When p is slightly larger than 2, we have

$$\Gamma_p(\mathbf{Q}) = \max_{d/2 \leq d' \leq d} (2d' - \mathfrak{d}_{d',2}(\mathbf{Q})) \left(\frac{1}{2} - \frac{1}{p} \right), \quad (3.37)$$

as the contributions from the other two terms in (3.34) are already negative. This implies

$$\max_{d/2 \leq d' \leq d} (2d' - \mathfrak{d}_{d',2}(\mathbf{Q})) = d. \quad (3.38)$$

We use (3.38) to further simplify $\Gamma_p(\mathbf{Q})$ to

$$\begin{aligned} & \max \left(d \left(\frac{1}{2} - \frac{1}{p} \right), \max_{d/2 \leq d' \leq d} (2d' - \mathfrak{d}_{d',1}(\mathbf{Q})) \left(\frac{1}{2} - \frac{1}{p} \right) - \frac{2}{p}, \right. \\ & \left. 2d \left(\frac{1}{2} - \frac{1}{p} \right) - \frac{4}{p} \right). \end{aligned} \quad (3.39)$$

By comparing $\Gamma_p(\mathbf{Q})$ with $\Gamma'_p(\mathbf{Q})$ for $2 + \frac{4}{d-k} \leq p \leq 2 + \frac{4}{k}$, we see that

$$\max_{d/2 \leq d' \leq d} (2d' - \mathfrak{d}_{d',1}(\mathbf{Q})) = 2d - k. \quad (3.40)$$

This finishes the proof that (3.33) is equivalent to (3.35).

It remains to show that (3.35) is equivalent to that $\mathbf{Q}$ is weakly nondegenerate and satisfies $\mathfrak{d}_1(\mathbf{Q}) = k$. Since the second equation in (3.35) is already equivalent to the weakly nondegenerate condition, what we need to prove becomes $\mathfrak{d}_{d,1}(\mathbf{Q}) = \mathfrak{d}_1(\mathbf{Q}) = k$ if and only if

$$\max_{d'} (2d' - \mathfrak{d}_{d',1}(\mathbf{Q})) = 2d - k, \quad (3.41)$$

which follows immediately from

$$\max_{d'} (2d' - \mathfrak{d}_{d',1}(\mathbf{Q})) = 2d - \mathfrak{d}_{d,1}(\mathbf{Q}). \quad (3.42)$$

To prove (3.42), it suffices to prove the following.

CLAIM 3.5

We have

$$\mathfrak{d}_{d,1}(\mathbf{Q}) - \mathfrak{d}_{d',1}(\mathbf{Q}) \leq 2(d - d') \quad (3.43)$$

for every $d/2 \leq d' \leq d$.

The proof of Claim 3.5 will be presented in the end of this subsection. So far, we have finished the proof of the first part of the corollary.

Let us turn to the second part and show that $\mathbf{Q}$ is nondegenerate if and only if it is weakly nondegenerate and satisfies $\mathfrak{d}_1(\mathbf{Q}) \geq d/2$. By definition, we need to show that $\mathfrak{d}_1(\mathbf{Q}) \geq d/2$ if and only if

$$\mathfrak{d}_{d-m,1}(\mathbf{Q}) \geq d/2 - 2m \quad (3.44)$$

for every $0 \leq m \leq d/2$. By taking $m = 0$, we see that (3.44) implies $\mathfrak{d}_1(\mathbf{Q}) \geq d/2$. The other direction immediately follows from Claim 3.5. This finishes the second part of the corollary. $\square$

Proof of Claim 3.5

We take $M_0 \in \mathbb{R}^{d \times d}$ of rank d' and $M'_0 \in \mathbb{R}^{2 \times 2}$ of rank 1 such that

$$\mathfrak{d}_{d',1}(\mathbf{Q}) = \inf_{\substack{M \in \mathbb{R}^{d \times d} \\ \text{rank}(M) = d'}} \inf_{\substack{M' \in \mathbb{R}^{n \times n} \\ \text{rank}(M') = 1}} \text{NV}(M' \cdot (\mathbf{Q} \circ M)) = \text{NV}(M'_0 \cdot (\mathbf{Q} \circ M_0)). \quad (3.45)$$

Therefore, there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$\mathfrak{d}_{d',1}(\mathbf{Q}) = \text{NV}(\tilde{Q}(\cdot M_0)) = \mathfrak{d}_{d,1}(\tilde{Q}(\cdot M_0)), \quad (3.46)$$

where $\tilde{Q} = \lambda_1 Q_1 + \lambda_2 Q_2$ and $\mathbf{Q} = (Q_1, Q_2)$. We now apply Lemma 3.1 and obtain

$$\begin{aligned} \mathfrak{d}_{d',1}(\mathbf{Q}) - \mathfrak{d}_{d,1}(\mathbf{Q}) &= \mathfrak{d}_{d,1}(\tilde{Q}(\cdot M_0)) - \mathfrak{d}_{d,1}(\mathbf{Q}) \\ &\geq \mathfrak{d}_{d,1}(\tilde{Q}(\cdot M_0)) - \mathfrak{d}_{d,1}(\tilde{Q}) \\ &\geq -2(d - d'). \end{aligned} \quad (3.47)$$

This completes the proof of Claim 3.5. $\square$

4. Transversality

4.1. Brascamp–Lieb inequalities

Central tools in most existing proofs of decoupling inequalities are the Brascamp–Lieb inequalities for products of functions in $\mathbb{R}^m$ which are constant along some linear subspaces. Scale-invariant inequalities of this kind have been characterized in [5]. A novelty of our approach is that we for the first time take full advantage of scale-dependent versions of Brascamp–Lieb inequalities. First inequalities of this kind were proved in [5] and [6], and a unified description taking into account both minimal and maximal scales was obtained in [46]. We will only use the results of [46] in a special symmetric case when all functions f_j below play similar roles. This special case is captured in the following definition.

Definition 4.1

Let $m, m' \in \mathbb{N}$. Let $(V_j)_{j=1}^M$ be a tuple of linear subspaces $V_j \subseteq \mathbb{R}^m$ of dimension m' . For a linear subspace $V \subseteq \mathbb{R}^m$, let $\pi_V : \mathbb{R}^m \rightarrow V$ denote the orthogonal projection onto V . For $0 \leq \alpha \leq M$ and $R \geq 1$, we denote by $\text{BL}((V_j)_{j=1}^M, \alpha, R, \mathbb{R}^m)$ (for *Brascamp–Lieb constant*) the smallest constant such that the inequality

$$\begin{aligned} & \int_{[-R, R]^m} \prod_{j=1}^M f_j(\pi_{V_j}(x))^\alpha \, dx \\ & \leq \text{BL}((V_j)_{j=1}^M, \alpha, R, \mathbb{R}^m) \prod_{j=1}^M \left(\int_{V_j} f_j(x_j) \, dx_j \right)^\alpha \end{aligned} \quad (4.1)$$

holds for any functions $f_j : V_j \rightarrow [0, \infty)$ that are constant at scale 1, in the sense that V_j can be partitioned into cubes with unit side length on each of which f_j is constant. If the dimension m of the total space $\mathbb{R}^m$ is clear from the context, $\text{BL}((V_j)_{j=1}^M, \alpha, R, \mathbb{R}^m)$ is often abbreviated to $\text{BL}((V_j)_{j=1}^M, \alpha, R)$.

We also need a *Kekeya* variant of *Brascamp–Lieb* inequalities, in which each function $f_j \circ \pi_{V_j}$ is replaced by a sum of functions of the form $f_{j,l} \circ \pi_{V_{j,l}}$, where $V_{j,l}$ are different subspaces. The first almost optimal inequality of this kind was the multilinear *Kekeya* inequality, proved in [7], which generalizes the *Loomis–Whitney* inequality. A simplified induction on scales proof was later given by Guth [36]. An endpoint version of the multilinear *Kekeya* inequality was proved by Guth [35] using the polynomial method. Endpoint *Kekeya*-type extensions of *Brascamp–Lieb* inequalities were further developed in [21], [59], and [60]. It will be convenient to use the following formulation, although a non-endpoint result such as [46, Theorem 2] would also suffice for the purpose of proving decoupling inequalities with the optimal range of exponents.

THEOREM 4.2 (*Kekeya–Brascamp–Lieb* [60])

Fix integers $m' \leq m$. Let $\mathcal{V}_j$, $1 \leq j \leq M$, be families of linear subspaces of $\mathbb{R}^m$ of dimension m' . Let $1 \leq \alpha \leq M$ and $R \geq 1$. Assume that

$$A := \sup_{V_1 \in \mathcal{V}_1, \dots, V_M \in \mathcal{V}_M} \text{BL}((V_j)_{j=1}^M, \alpha, R) < \infty. \quad (4.2)$$

Then, for any nonnegative integrable functions $f_{j,V_j} : V_j \rightarrow \mathbb{R}$ constant at scale 1, we have

$$\int_{B(0, R)} \prod_{j=1}^M \left(\sum_{V_j \in \mathcal{V}_j} f_{j,V_j}(\pi_{V_j}(x)) \right)^\alpha \, dx$$

$$\leq C^\alpha A \prod_{j=1}^M \left(\sum_{V_j \in \mathcal{V}_j} \int_{V_j} f_{j,V_j}(x) dx \right)^\alpha, \quad (4.3)$$

where the constant C depends only on the dimension m .

The uniform bound (4.2) is clearly necessary for (4.3) to hold. In the scale-invariant case, such uniform bounds for Brascamp–Lieb constants were obtained in [4] and [3]. We need the following corresponding result in the scale-dependent case.

THEOREM 4.3 ([46, Theorem 3])

In the situation of Definition 4.1, fix a tuple $(V_j)_{j=1}^M$ and an exponent $1 \leq \alpha \leq M$. Let

$$\kappa := \sup_{V \leq \mathbb{R}^m} \left(\dim V - \frac{\alpha}{M} \sum_{j=1}^M \dim \pi_{V_j} V \right), \quad (4.4)$$

where the supremum is taken over all linear subspaces of $\mathbb{R}^m$.

Then there exists a constant $C_0 < \infty$ and a neighborhood of the tuple $(V_j)_{j=1}^M$ in the M th power of the Grassmanian manifold of all linear subspaces of dimension m' of $\mathbb{R}^m$ such that, for any tuple $(\tilde{V}_j)_{j=1}^M$ in this neighborhood and any $R \geq 1$, we have

$$\text{BL}((\tilde{V}_j)_{j=1}^M, \alpha, R) \leq C_0 R^\kappa. \quad (4.5)$$

4.2. Transversality for quadratic forms

Let $\mathbf{Q} = (Q_1, \dots, Q_n)$ be a sequence of quadratic forms defined on $\mathbb{R}^d$. The subspaces in the subsequent application of Takeya–Brascamp–Lieb inequalities will be the tangent spaces to the manifold $S_{\mathbf{Q}}$:

$$V_\xi = V_\xi(\mathbf{Q}) := \text{lin}\{(e_j, \partial_j \mathbf{Q}(\xi)), j = 1, \dots, d\}, \quad \xi \in \mathbb{R}^d. \quad (4.6)$$

Here e_j is the j th coordinate vector, and lin refers to linear span. Transversality of pieces of this manifold will be measured by the exponent κ defined in (4.4) evaluated at tangent spaces somewhere at the respective pieces: the smaller the exponent, the more transverse are the pieces. It is an observation going back to [13] (for scale-invariant Brascamp–Lieb inequalities) that the most transverse situations arise when the pieces are not concentrated near a low-degree subvariety in the following sense.

Definition 4.4

A subset $\mathcal{W} \subseteq \mathcal{P}(1/K)$ will be called θ -uniform if, for every nonzero polynomial P in d variables with real coefficients of degree $\leq d$, we have

$$|\{W \in \mathcal{W} : 2W \cap Z_P \neq \emptyset\}| \leq \theta |\mathcal{W}|.$$

Here Z_P refers to the zero set of P . When using the notation $\mathcal{W} = \{W_1, \dots, W_M\} = \{W_j\}_{j=1}^M$ for θ -uniform sets, we always mean that the W_j 's are pairwise distinct.

LEMMA 4.5

Let $\theta \in [0, 1]$, $\alpha \geq 1$, and $K \in 2^{\mathbb{N}}$. Then there exists $C_{\theta, K, \alpha} < \infty$ such that, for every θ -uniform set $\mathcal{W} = \{W_1, \dots, W_M\} \subseteq \mathcal{P}(1/K)$ with $\alpha \leq M$ and every $R \geq 1$, we have

$$\sup_{\xi_j \in W_j} \text{BL}((V_{\xi_j})_{j=1}^M, \alpha, R, \mathbb{R}^{d+n}) \leq C_{\theta, K, \alpha} R^{\kappa_Q(\alpha \cdot (1-\theta))},$$

where

$$\kappa_Q(\alpha) := \sup_{V \leq \mathbb{R}^{d+n}} (\dim V - \alpha \sup_{\xi \in \mathbb{R}^d} \dim \pi_{V_\xi} V). \quad (4.7)$$

In the remaining part, if $\mathbf{Q}$ is clear from the context, we often abbreviate $\kappa_Q(\alpha)$ to $\kappa(\alpha)$.

Proof of Lemma 4.5

Since there are only finitely many θ -uniform sets $\mathcal{W}$, and, for any fixed θ -uniform set $\mathcal{W} = \{W_1, \dots, W_M\}$, the set $\prod_{j=1}^M \overline{W_j}$ is compact, by Theorem 4.3, it suffices to show that, for any $\xi_j \in \overline{W_j}$, and every subspace $V \leq \mathbb{R}^{d+n}$, we have

$$\dim V - \frac{\alpha}{M} \sum_{j=1}^M \dim \pi_{V_{\xi_j}} V \leq \dim V - \alpha(1-\theta) \sup_{\xi \in \mathbb{R}^d} \dim \pi_{V_\xi} V.$$

This is equivalent to

$$\frac{1}{M} \sum_{j=1}^M \dim \pi_{V_{\xi_j}} V \geq (1-\theta) \sup_{\xi \in \mathbb{R}^d} \dim \pi_{V_\xi} V.$$

If $v_1, \dots, v_m$ is a basis of V , then

$$\dim \pi_{V_\xi} V = \text{rank} \begin{pmatrix} e_1 & \partial_1 \mathbf{Q}(\xi) \\ \vdots & \vdots \\ e_d & \partial_d \mathbf{Q}(\xi) \end{pmatrix} \cdot (v_1 \quad \dots \quad v_m), \quad (4.8)$$

where on the right-hand side we have the product of two matrices. Each minor determinant of this matrix is a polynomial of degree at most d . Consider the largest minor (of size $d' \times d'$, say) whose determinant is a nonvanishing polynomial; call this polynomial P (if $d' = 0$, then $P = 1$). Then

$$d' = \sup_{\xi \in \mathbb{R}^d} \dim \pi_{V_\xi} V.$$

By Definition 4.4, we have $P(\xi_j) \neq 0$ for at least $(1 - \theta)M$ many j 's. Therefore,

$$\frac{1}{M} \sum_{j=1}^M \dim \pi_{V_{\xi_j}} V \geq \frac{1}{M} \sum_{j: P(\xi_j) \neq 0} \dim \pi_{V_{\xi_j}} V \geq \frac{1}{M} \sum_{j: P(\xi_j) \neq 0} d' \geq (1 - \theta)d'. \quad \square$$

This finishes the proof of the lemma.

From the proof of Lemma 4.5, we see that the sup in $\sup_{\xi \in \mathbb{R}^d} \dim \pi_{V_\xi} V$ is attained at almost every point, with respect to the d -dimensional Lebesgue measure. Therefore, we introduce the following notation:

$$\dim \pi V := \sup_{\xi \in \mathbb{R}^d} \dim \pi_{V_\xi} V. \quad (4.9)$$

Next, we will find a more explicit description of the exponent (4.7) in terms of the quadratic forms $\mathbf{Q}$. The following result relates the terms in (4.7) to the quantities introduced in (1.11).

LEMMA 4.6

Let $\mathbf{Q}$ be an n -tuple of quadratic forms in d variables. For a linear subspace $V \subseteq \mathbb{R}^{d+n}$, let

$$d' := \dim \pi V, \quad n' := \dim V - \dim \pi V.$$

Then

$$n' \leq n \quad \text{and} \quad \mathfrak{d}_{n'}(\mathbf{Q}) \leq d'.$$

Lemma 4.6 relies on the following algebraic result.

LEMMA 4.7

Let $\mathbb{F} = \mathbb{R}(\xi_1, \dots, \xi_d)$ be the field of rational functions in d variables. Let $A = (\sum_k a_{i,j,k} \xi_k)_{i,j}$ be a $(N_1 \times N_2)$ -matrix whose entries are linear maps with real coefficients. Suppose that $\text{rank}_{\mathbb{F}} A = r$. Then there exist real invertible matrices B, B' such that

$$BAB' = \begin{pmatrix} * & * \\ * & 0 \end{pmatrix}, \quad (4.10)$$

where the zero block has size $(N_1 - r) \times (N_2 - r)$.

Standard linear algebra shows that there exist invertible matrices B, B' with entries in $\mathbb{F}$ such that (4.7) holds. The point of Lemma 4.7 is that we can find B, B'

with real entries. Note that Lemma 4.7 may fail if entries of A are not assumed to be linear forms. We will include a proof of Lemma 4.7 below, and we also note that after finishing the first version of the paper, Zipei Nie [48] pointed out to us that Lemma 4.7 is in fact known in the literature and follows from [24, Lemma 1]. We thank him for this comment.

Proof of Lemma 4.6 assuming Lemma 4.7

The claim $n' \leq n$ follows from the fact that the tangent spaces V_ξ have codimension n .

After linear changes of variables in $\mathbb{R}^d$ and $\mathbb{R}^n$, we may assume that V is spanned by linearly independent vectors of the form

$$(e_1, v_1), \dots, (e_s, v_s), (0, \tilde{e}_1), \dots, (0, \tilde{e}_j),$$

where e_i are unit coordinate vectors in $\mathbb{R}^d$, $\tilde{e}_i$ are unit coordinate vectors in $\mathbb{R}^n$, v_i are vectors in $\mathbb{R}^n$, and $s \leq \min(d, \dim V)$. Note also that $\dim V = s + j$ and that $j \leq n$. As in (4.8), we have

$$\dim \pi_{V_\xi} V = \text{rank}_{\mathbb{R}} \begin{pmatrix} e_1 & \partial_1 \mathbf{Q}(\xi) \\ \vdots & \vdots \\ e_d & \partial_d \mathbf{Q}(\xi) \end{pmatrix} \cdot \begin{pmatrix} e_1^T & \dots & e_s^T & 0 & \dots & 0 \\ v_1^T & \dots & v_s^T & \tilde{e}_1^T & \dots & \tilde{e}_j^T \end{pmatrix}.$$

Since all entries of the product matrix on the right-hand side are polynomials in ξ , we have

$$d' = \sup_{\xi} \dim \pi_{V_\xi} V = \text{rank}_{\mathbb{F}} \begin{pmatrix} e_1 & \partial_1 \mathbf{Q}(\xi) \\ \vdots & \vdots \\ e_d & \partial_d \mathbf{Q}(\xi) \end{pmatrix} \cdot \begin{pmatrix} e_1^T & \dots & e_s^T & 0 & \dots & 0 \\ v_1^T & \dots & v_s^T & \tilde{e}_1^T & \dots & \tilde{e}_j^T \end{pmatrix},$$

where $\mathbb{F}$ is the field of rational functions in d variables. This is because the rank equals the size of the largest minor with nonvanishing determinant, and the determinant of any minor, viewed as an element of $\mathbb{F}$, vanishes if and only if its value vanishes for every ξ . The latter matrix can be written in the block form

$$\begin{pmatrix} I + L_1 & L_3 \\ L_2 & B \end{pmatrix}, \quad (4.11)$$

where I is the $s \times s$ identity matrix, L_1, L_2, L_3 are matrices whose entries are linear combinations of monomials of degree 1, and

$$B = \begin{pmatrix} \partial_{s+1} Q_1(\xi) & \dots & \partial_{s+1} Q_j(\xi) \\ \vdots & & \vdots \\ \partial_d Q_1(\xi) & \dots & \partial_d Q_j(\xi) \end{pmatrix}.$$

Let $r := \text{rank}_{\mathbb{F}} B$. Any $r \times r$ -minor determinant P of the matrix B is a homogeneous polynomial of degree r , and P coincides with the lowest-degree homogeneous part of the corresponding $(r+s) \times (r+s)$ -minor determinant of (4.11), obtained by adjoining the first s rows and columns. Therefore,

$$d' \geq s + r.$$

Let us continue to prove $\mathfrak{d}_{n'}(\mathbf{Q}) \leq d'$. Recall that $n' = s + j - d' \leq j - r$. By the definition in (1.11), it suffices to find linear changes of variables in $\mathbb{R}^d$ and $\mathbb{R}^n$, after which $Q_{r+1}, \dots, Q_j$ no longer depend on variables $\xi_{s+r+1}, \dots, \xi_d$. Notice that row and column operations on B with coefficients in $\mathbb{R}$ correspond to linear changes of variables in $\mathbb{R}^d$ and $\mathbb{R}^n$, respectively. By Lemma 4.7, by row and column operations with coefficients in $\mathbb{R}$, B can be brought in a form in which it has a $(j-r) \times (d-s-r)$ -block of zeros. This means that, after a change of variables, $Q_{r+1}, \dots, Q_j$ do not depend on variables $\xi_{s+r+1}, \dots, \xi_d$. $\square$

Proof of Lemma 4.7

Let k_1 be the largest index such that ξ_{k_1} appears in A . Swapping rows and columns, we may assume $a_{1,1,k_1} \neq 0$. Using elementary row and column operations, we may further assume that $a_{1,1,k_1} = 1$, $a_{1,j,k_1} = 0$, and $a_{i,1,k_1} = 0$ for all $j \neq 0$ and $i \neq 0$. Thus, we may assume

$$A = \begin{pmatrix} \xi_{k_1} + * & * \\ * & A' \end{pmatrix},$$

where ξ_{k_1} does not appear in entries $*$ and A' is an $(N_1 - 1) \times (N_2 - 1)$ -matrix. If $A' \neq 0$, we repeat the same procedure in A' , and so on. If this process stops after at most r iterations, then we are done. Otherwise, we have brought the upper-left corner of A into the form

$$\begin{pmatrix} \xi_{k_1} + * & * & \dots & * \\ * & \xi_{k_2} + * & \dots & * \\ \vdots & & \ddots & \vdots \\ * & \dots & * & \xi_{k_{r+1}} + * \end{pmatrix}, \quad (4.12)$$

where $a_{i,j,k} = 0$ if $i \neq j$ and $k \geq k_{\min(i,j)}$. The determinant of this matrix is a polynomial whose leading term in the lexicographic ordering is $\xi_{k_1} \cdots \xi_{k_{r+1}}$:

$$\det(4.12) = \xi_{k_1} \cdots \xi_{k_{r+1}} + \text{lower-order terms}.$$

This can be seen by induction on the size of this matrix. Indeed, if $k_1 = \dots = k_l > k_{l+1}$, then ξ_{k_1} appears in this matrix only in the first l diagonal entries, so

$$\det(4.12) = \xi_{k_1}^l \cdot \det \begin{pmatrix} \xi_{k_{l+1}} + * & * & \dots & * \\ * & \xi_{k_{l+2}} + * & \dots & * \\ \vdots & & \ddots & \vdots \\ * & \dots & * & \xi_{k_{r+1}} + * \end{pmatrix} \\ + \text{lower-order terms.}$$

In particular, the matrix (4.12) is invertible (over $\mathbb{F}$), so that $\text{rank}_{\mathbb{F}} A \geq r + 1$, a contradiction. $\square$

COROLLARY 4.8

For any $\alpha \geq 1$, the exponent defined in (4.7) satisfies

$$\kappa(\alpha) \leq \sup_{0 \leq n' \leq n} (n' + (1 - \alpha)\mathfrak{d}_{n'}(\mathbf{Q})). \quad (4.13)$$

Proof of Corollary 4.8

Let $V \subseteq \mathbb{R}^{d+n}$ be a linear subspace. With the notation from Lemma 4.6, we obtain

$$\dim V - \alpha \dim \pi V = (d' + n') - \alpha d' = n' + (1 - \alpha)d' \leq n' + (1 - \alpha)\mathfrak{d}_{n'}(\mathbf{Q}).$$

The conclusion follows after taking the supremum over all subspaces V . $\square$

4.3. Ball inflation

A so-called ball inflation inequality, based on scale-invariant Keakeya–Brascamp–Lieb inequalities, was first introduced in [17, Theorem 6.6]. Here, we formulate a version of this inequality based on scale-dependent Keakeya–Brascamp–Lieb inequalities. Recall that $\mathcal{U}_J$ was defined in (1.4).

PROPOSITION 4.9 (Ball inflation)

Let $K \in 2^{\mathbb{N}}$ be a dyadic integer and $0 < \rho \leq 1/K$. Let $\{W_j\}_{j=1}^M \subset \mathcal{P}(1/K)$ be a θ -uniform set of cubes. Then, for any $1 \leq t \leq p < \infty$, any functions f_J with $\text{supp } \widehat{f_J} \subset \mathcal{U}_J$ and any $x_0 \in \mathbb{R}^{d+n}$, we have

$$\left\| \prod_{j=1}^M \left(\sum_{J \in \mathcal{P}(W_j, \rho)} \|f_J\|_{\mathbb{E}^t(w_{B(x_0, 1/\rho)})}^t \right)^{1/t} \right\|_{\mathbb{E}^p_{x \in B(x_0, \rho^{-2})}} \\ \leq C_{\theta, K, p, t} \rho^{-\left(\frac{d}{t} - \frac{d+n}{p} + \frac{\kappa((1-\theta)p/t)}{p}\right)} \\ \times \prod_{j=1}^M \left(\sum_{J \in \mathcal{P}(W_j, \rho)} \|f_J\|_{\mathbb{E}^t(w_{B(x_0, \rho^{-2})})}^t \right)^{1/t}. \quad (4.14)$$

Proof of Proposition 4.9

Let $R := \rho^{-1}$. Without loss of generality, we set $x_0 = 0$. Let $\Omega := B(0, R^2)$. Let $\alpha := p/t$. The p th power of the left-hand side of (4.14) equals

$$\oint_{x \in \Omega} \prod_{j=1}^M \left(\sum_{J \in \mathcal{P}(W_j, \rho)} \|f_J\|_{L^t(w_{B(x, 1/\rho)})}^t \right)^\alpha, \quad (4.15)$$

where $\oint_\Omega := |\Omega|^{-1} \int_\Omega$ denotes the average integral. For each cube $J \in \mathcal{P}(W_j, \rho)$ with center ξ_J , we cover Ω with a family $\mathcal{T}_J$ of disjoint tiles T_J , which are rectangular boxes with n long sides of length $2\rho^{-2}$ centered at 0 pointing in the directions $V_{\xi_J}^\perp$ and d short sides of length ρ^{-1} pointing in complementary directions (the length of the long sides equals the diameter of $B(x_0, \rho^{-2})$, so that we only need one layer of tiles in the directions $V_{\xi_J}^\perp$). We can choose these tiles so that they are contained in $C_0\Omega$ with $C_0 \lesssim 1$. We let $T_J(x)$ be the tile containing x , and for $x \in \bigcup_{T_J \in \mathcal{T}_J} T_J$ we define

$$F_J(x) := \sup_{y \in T_J(x)} \|f_J\|_{L^t(w_{B(y, 1/\rho)})}.$$

Then

$$\oint_{x \in \Omega} \prod_{j=1}^M \left(\sum_{J \in \mathcal{P}(W_j, \rho)} \|f_J\|_{L^t(w_{B(x, 1/\rho)})}^t \right)^\alpha \leq \oint_\Omega \prod_{j=1}^M \left(\sum_{J \in \mathcal{P}(W_j, \rho)} |F_J|^t \right)^\alpha.$$

Since the function F_J is constant on each tile $T_J \in \mathcal{T}_J$, we can write its restriction to Ω in the form $\tilde{F}_J \circ \pi_J$, where π_J is the orthogonal projection onto V_{ξ_J} . To apply Theorem 4.2, we apply the change of variables $x \rightarrow Rx$ such that the resulting functions $F_J(Rx)$ are constant at the unit scale:

$$\oint_\Omega \prod_{j=1}^M \left(\sum_{J \in \mathcal{P}(W_j, \rho)} |F_J|^t \right)^\alpha = R^{-(d+n)} \int_{B(0, R)} \prod_{j=1}^M \left(\sum_{J \in \mathcal{P}(W_j, \rho)} F_J(R \cdot)^t \right)^\alpha. \quad (4.16)$$

By Theorem 4.2 and Lemma 4.5 with $R = \rho^{-1}$, we bound the last expression by

$$\begin{aligned} & R^{-(d+n)} C_{\theta, K, \alpha} R^{\kappa(\alpha(1-\theta))} \prod_{j=1}^M \left(\sum_{J \in \mathcal{P}(W_j, \rho)} \int_{B(0, C_0 R) \subset \mathbb{R}^d} (\tilde{F}_J(R \cdot))^t \right)^\alpha \\ & \lesssim R^{-(d+n)} C_{\theta, K, \alpha} R^{\kappa(\alpha(1-\theta))} R^{\alpha d} \\ & \quad \times \prod_{j=1}^M \left(\sum_{J \in \mathcal{P}(W_j, \rho)} \oint_{B(0, C_0 R^2) \subset \mathbb{R}^{d+n}} |F_J|^t \right)^\alpha. \end{aligned} \quad (4.17)$$

The conclusion will now follow from the bound

$$\|F_J\|_{\mathcal{L}^t(C_0\Omega)} \lesssim \|f_J\|_{\mathcal{L}^t(w_\Omega)}, \quad (4.18)$$

which is a standard application of the uncertainty principle (see, e.g., [32, (3.13)]). $\square$

The ball inflation inequality in Proposition 4.9 is sufficient for proving $\ell^p L^p$ decoupling. For the proof of $\ell^q L^p$ decoupling with $q < p$ we need a slightly more general statement.

COROLLARY 4.10 (Ball inflation, $\ell^q L^t$ version)

In the situation of Proposition 4.9, for any $1 \leq \tilde{q} \leq t$, we have

$$\begin{aligned} & \left\| \prod_{j=1}^M \left(\sum_{J \in \mathcal{P}(W_j, \rho)} \|f_J\|_{\mathcal{L}^t(w_{B(x, 1/\rho)})}^{\tilde{q}} \right)^{1/\tilde{q}} \right\|_{\mathcal{L}^p_{x \in B(x_0, \rho^{-2})}} \\ & \leq C_{\theta, K, p, t, \tilde{q}} (|\log \rho| + 2)^{K^d} \rho^{-\left(\frac{d}{t} - \frac{d+n}{p} + \frac{\kappa((1-\theta)p/t)}{p}\right)} \\ & \quad \times \prod_{j=1}^M \left(\sum_{J \in \mathcal{P}(W_j, \rho)} \|f_J\|_{\mathcal{L}^t(w_{B(x_0, \rho^{-2})})}^{\tilde{q}} \right)^{1/\tilde{q}}. \end{aligned} \quad (4.19)$$

Proof of Corollary 4.10

This follows from Proposition 4.9 by a dyadic pigeonholing argument in the proof of [17, Theorem 6.6]. For the sake of completeness, we still include the proof here. We follow the presentation in [32, Appendix A].

For each $1 \leq j \leq M$, partition

$$\mathcal{P}(W_j, \rho) = \mathcal{J}_{j, \infty} \cup \bigcup_{\iota=0}^{\lfloor d \log_2(1/\rho) \rfloor} \mathcal{J}_{j, \iota}, \quad (4.20)$$

where for $0 \leq \iota \leq \log(1/\rho)$

$$\begin{aligned} \mathcal{J}_{j, \iota} &:= \left\{ J \in \mathcal{P}(W_j, \rho) : 2^{-\iota-1} < \frac{\|f_J\|_{\mathcal{L}^t(w_B)}}{\max_{J' \in \mathcal{P}(W_j, \rho)} \|f_{J'}\|_{\mathcal{L}^t(w_B)}} \leq 2^{-\iota} \right\}, \\ \mathcal{J}_{j, \infty} &:= \left\{ J \in \mathcal{P}(W_j, \rho) : \right. \\ & \quad \left. \|f_J\|_{\mathcal{L}^t(w_B)} \leq 2^{-\lfloor d \log_2(1/\rho) \rfloor} \max_{J' \in \mathcal{P}(W_j, \rho)} \|f_{J'}\|_{\mathcal{L}^t(w_B)} \right\}. \end{aligned} \quad (4.21)$$

Since $M \leq K^d$, the claim (4.19) follows by the triangle inequality from

$$\begin{aligned}
& \left\| \prod_{j=1}^M \left(\sum_{J \in \mathcal{J}_{j,\iota_j}} \|f_J\|_{\mathbf{L}^t(w_{B(x,1/\rho)})}^{\tilde{q}} \right)^{1/\tilde{q}} \right\|_{\mathbf{L}_{x \in \Omega}^p} \\
& \leq C_{\theta,K,p,t,\tilde{q}} \rho^{-(\frac{d}{t} - \frac{d+n}{p} + \frac{\kappa((1-\theta)p/t)}{p})} \prod_{j=1}^M \left(\sum_{J \in \mathcal{P}(W_j,\rho)} \|f_J\|_{\mathbf{L}^t(w_\Omega)}^{\tilde{q}} \right)^{1/\tilde{q}}, \quad (4.22)
\end{aligned}$$

which we will show for every choice of $\iota_1, \dots, \iota_M \in \{0, \dots, \lfloor d \log_2(1/\rho) \rfloor\} \cup \{\infty\}$.

Since $\tilde{q} \leq t$, by Hölder's inequality, the left-hand side of (4.22) is bounded by

$$\left(\prod_{j=1}^M |\mathcal{J}_{j,\iota_j}|^{\frac{1}{\tilde{q}} - \frac{1}{t}} \right) \left\| \prod_{j=1}^M \left(\sum_{J \in \mathcal{J}_{j,\iota_j}} \|f_J\|_{\mathbf{L}^t(w_{B(x,1/\rho)})}^t \right)^{1/t} \right\|_{\mathbf{L}_{x \in B}^p}. \quad (4.23)$$

By Proposition 4.9, the last display is bounded by

$$\begin{aligned}
& C_{\theta,K,p,t,\tilde{q}} \rho^{-(\frac{d}{t} - \frac{d+n}{p} + \frac{\kappa((1-\theta)p/t)}{p})} \left(\prod_{j=1}^M |\mathcal{J}_{j,\iota_j}|^{\frac{1}{\tilde{q}} - \frac{1}{t}} \right) \\
& \times \prod_{j=1}^M \left(\sum_{J \in \mathcal{J}_{j,\iota_j}} \|f_J\|_{\mathbf{L}^t(w_B)}^t \right)^{1/t}. \quad (4.24)
\end{aligned}$$

It remains to observe that, for every ι , we have

$$|\mathcal{J}_{j,\iota}|^{\frac{1}{\tilde{q}} - \frac{1}{t}} \left(\sum_{J \in \mathcal{J}_{j,\iota}} \|f_J\|_{\mathbf{L}^t(w_B)}^t \right)^{1/t} \lesssim \left(\sum_{J \in \mathcal{P}(W_j,\rho)} \|f_J\|_{\mathbf{L}^t(w_B)}^{\tilde{q}} \right)^{1/\tilde{q}}. \quad (4.25)$$

If $\iota \neq \infty$, this follows, as the summands on the left-hand side are comparable. For $\iota = \infty$, we have

$$\begin{aligned}
& |\mathcal{J}_{j,\infty}|^{\frac{1}{\tilde{q}} - \frac{1}{t}} \left(\sum_{J \in \mathcal{J}_{j,\infty}} \|f_J\|_{\mathbf{L}^t(w_B)}^t \right)^{1/t} \\
& \leq |\mathcal{J}_{j,\infty}|^{\frac{1}{\tilde{q}}} \max_{J \in \mathcal{J}_{j,\infty}} \|f_J\|_{\mathbf{L}^t(w_B)} \\
& \leq \rho^{-d} 2^{-\lfloor d \log_2(1/\rho) \rfloor} \max_{J' \in \mathcal{P}(W_j,\rho)} \|f_{J'}\|_{\mathbf{L}^t(w_B)} \\
& \lesssim \left(\sum_{J \in \mathcal{P}(W_j,\rho)} \|f_J\|_{\mathbf{L}^t(w_B)}^{\tilde{q}} \right)^{1/\tilde{q}}. \quad \square
\end{aligned}$$

5. Induction on scales

The upper bounds of $\Gamma_{q,p}(\mathbf{Q})$ in Theorem 1.1 will be proved by induction on dimension d . The main inductive step is contained in the following result, whose proof will

occupy the whole Section 5. One can apply Theorem 5.1 repeatedly and then obtain the upper bounds in part (1.16) of Theorem 1.1.

THEOREM 5.1

Let $d \geq 1$ and $n \geq 1$. Let $\mathbf{Q} = (Q_1, \dots, Q_n)$ be a collection of quadratic forms in d variables. Let $2 \leq q \leq p < \infty$ and

$$\Lambda := \sup_H \Gamma_{q,p}(\mathbf{Q}|_H), \quad (5.1)$$

where the sup is taken over all hyperplanes $H \subset \mathbb{R}^d$ that pass through the origin. Then

$$\Gamma_{q,p}(\mathbf{Q}) \leq \max\left(\Lambda, \max_{0 \leq n' \leq n} \left[d \left(1 - \frac{1}{p} - \frac{1}{q}\right) - \mathfrak{d}_{n'}(\mathbf{Q}) \left(\frac{1}{2} - \frac{1}{p}\right) - \frac{2(n - n')}{p} \right] \right). \quad (5.2)$$

In the proof of Theorem 5.1, we may assume that

$$\Gamma := \Gamma_{q,p}(\mathbf{Q}) > \Lambda, \quad (5.3)$$

since otherwise (5.2) already holds. The assumption (5.3) is convenient, because it means that the multilinear terms in Proposition 5.6 below dominate the lower-dimensional terms. On a technical level, it allows us to define the quantities (5.32) that are central to the bootstrapping argument.

5.1. Stability of decoupling constants and lower-dimensional contributions

In order to make use of the quantity (5.1), we need to show that we have a bound for the decoupling constants $\mathcal{D}_{q,p}(\mathbf{Q}|_H, \delta)$ that is uniform in the hyperplanes H . More generally, it turns out that decoupling constants can be bounded locally uniformly in the coefficients of the quadratic forms $\mathbf{Q}$. Although it is possible to obtain such uniform bounds by keeping track of the dependence on $\mathbf{Q}$ in all our proofs, we use this opportunity to record a compactness argument for decoupling constants whose validity is not restricted to quadratic forms.

THEOREM 5.2

For every $2 \leq q \leq p < \infty$, $\epsilon > 0$, and real quadratic forms $Q_1, \dots, Q_n$ in d variables, there exist $C_{\mathbf{Q}, \epsilon, q, p} < \infty$ and a neighborhood $\mathcal{Q}$ of $(Q_1, \dots, Q_n)$ such that, for every $(\tilde{Q}_1, \dots, \tilde{Q}_n) \in \mathcal{Q}$ and every $\delta \in (0, 1)$, we have

$$\mathcal{D}_{q,p}((\tilde{Q}_1, \dots, \tilde{Q}_n), \delta) \leq C_{\mathbf{Q}, \epsilon, q, p} \delta^{-\Gamma_{q,p}(\mathbf{Q}) - \epsilon},$$

where $\Gamma_{q,p}(\mathbf{Q})$ is given by (1.8).

LEMMA 5.3 (Affine rescaling)

Let $2 \leq q \leq p < \infty$. For any dyadic numbers $0 < \delta \leq \sigma \leq 1$ and every $J \in \mathcal{P}(\sigma)$,

$$\left\| \sum_{\square \in \mathcal{P}(J, \delta)} f_{\square} \right\|_{L^p} \leq \mathcal{D}_{q,p}(\mathbf{Q}, \delta/\sigma) \left(\sum_{\square \in \mathcal{P}(J, \delta)} \|f_{\square}\|_{L^p}^q \right)^{1/q}. \quad (5.4)$$

Such a lemma has also been standard in the decoupling literature (see, e.g., [11, Section 4] or [31, Lemma 1.23]).

Proof of Theorem 5.2

Let $\sigma = \sigma(\mathbf{Q}, \epsilon, q, p)$ be a small number, which will be determined later. We may assume that $\delta \leq \sigma/4$. We consider a tuple $(\tilde{Q}_1, \dots, \tilde{Q}_n)$ such that

$$\sup_i \|\text{Hess}(\tilde{Q}_i - Q_i)\| < \sigma^2/(10d + 10). \quad (5.5)$$

Then, for every $J \in \mathcal{P}(\sigma)$ and $\square \in \mathcal{P}(J, \delta)$, we have

$$\mathcal{U}_{\square}(\tilde{\mathbf{Q}}) \subseteq \mathcal{U}_J(\mathbf{Q}). \quad (5.6)$$

Take a collection of functions $f_{\square}$ with $\text{supp } \hat{f}_{\square} \subset \mathcal{U}_{\square}(\tilde{\mathbf{Q}})$ for each $\square \in \mathcal{P}(\delta)$. Using (5.6) and the definition of $\Gamma_{q,p}(\mathbf{Q})$, we obtain

$$\begin{aligned} \left\| \sum_{\square \in \mathcal{P}(\delta)} f_{\square} \right\|_{L^p} &\leq C_{\mathbf{Q}, \epsilon} \sigma^{-\Gamma_{q,p}(\mathbf{Q}) - \epsilon/2} \left(\sum_{J \in \mathcal{P}(\sigma)} \|f_J\|_{L^p}^q \right)^{1/q} \\ &\leq C_{\mathbf{Q}, \epsilon} \sigma^{-\Gamma_{q,p}(\mathbf{Q}) - \epsilon/2} \mathcal{D}_{q,p}(\tilde{\mathbf{Q}}, \delta/\sigma) \left(\sum_{\square \in \mathcal{P}(\delta)} \|f_{\square}\|_{L^p}^q \right)^{1/q}. \end{aligned} \quad (5.7)$$

The last inequality follows from the affine rescaling (Lemma 5.3). Hence, we obtain

$$\mathcal{D}_{q,p}(\tilde{\mathbf{Q}}, \delta) \leq C_{\mathbf{Q}, \epsilon} \sigma^{-\Gamma_{q,p}(\mathbf{Q}) - \epsilon/2} \mathcal{D}_{q,p}(\tilde{\mathbf{Q}}, \delta/\sigma). \quad (5.8)$$

We iterate this inequality $\log_{\sigma^{-1}}(\delta^{-1})$ -times and obtain

$$\mathcal{D}_{q,p}(\tilde{\mathbf{Q}}, \delta) \leq C_{\mathbf{Q}, \epsilon} (\delta^{-1})^{\log_{\sigma^{-1}} C_{\mathbf{Q}, \epsilon}} \delta^{-\Gamma_{q,p}(\mathbf{Q}) - \epsilon/2}. \quad (5.9)$$

It suffices to take σ small enough so that $\log_{\sigma^{-1}} C_{\mathbf{Q}, \epsilon} \leq \epsilon/2$. $\square$

COROLLARY 5.4

Let $2 \leq q \leq p < \infty$. For each $\epsilon > 0$, there exists $C_{\mathbf{Q}, \epsilon, q, p} < \infty$ such that, for every linear subspace $H \subset \mathbb{R}^d$ of codimension 1, we have

$$\mathcal{D}_{q,p}(\mathbf{Q}|_H, \delta) \leq C_{\mathbf{Q}, \epsilon, q, p} \delta^{-\Lambda - \epsilon}, \quad (5.10)$$

where Λ was defined in (5.1).

Proof

Recall from (1.12) that the $\mathbf{Q}|_H$ are parameterized by the orthogonal group $O(d)$. Since $\mathbf{Q}|_H$ depends continuously on the rotation used to define it, the group $O(d)$ is compact, and by Theorem 5.2, we obtain the claim. $\square$

To prepare for the broad-narrow analysis of Bourgain and Guth [18] in the following section, we need the following lemma that takes care of the case when frequency cubes are clustered near subvarieties of low degrees.

LEMMA 5.5 ([31, Corollary 2.18])

For every $d \geq 1$, $D > 1$ and $\epsilon > 0$, there exists $c = c(D, \epsilon) > 0$ such that the following holds. For every sufficiently large K , there exist

$$K^c \leq K_1 \leq K_2 \leq \cdots \leq K_D \leq \sqrt{K} \quad (5.11)$$

such that for every nonzero polynomial P in d variables of degree at most D , there exist collections of pairwise disjoint cubes $\mathcal{W}_j \subset \mathcal{P}(1/K_j)$, $j = 1, 2, \dots, D$, such that

$$\mathcal{N}_{1/K}(Z_P) \cap [0, 1]^d \subset \bigcup_{j=1}^D \bigcup_{W \in \mathcal{W}_j} W \quad (5.12)$$

and

$$\left\| \sum_{W \in \mathcal{W}_j} f_W \right\| \lesssim_{D, \mathbf{Q}, \epsilon, q, p} K_j^{\Lambda + \epsilon} \left(\sum_{W \in \mathcal{W}_j} \|f_W\|_p^q \right)^{1/q}. \quad (5.13)$$

Here $\mathcal{N}_{1/K}(Z_P)$ denotes the $1/K$ neighborhood of the zero set of P .

This lemma was stated in [31] only for $p = q$ and with Fourier support condition that is slightly different from (1.4). The same proof works also for $q \leq p$ and with Fourier support condition (1.4) without any change, and we will therefore not repeat it here. The main hypothesis of [31, Corollary 2.18] is [31, Hypothesis 2.4], which is exactly what we verified in Corollary 5.4.

5.2. Multilinear decoupling

For a positive integer K , a transversality parameter $\theta > 0$, and $0 < \delta < K^{-1}$, the *multilinear decoupling constant*

$$\text{MulDec}(\delta, \theta, K) = \text{MulDec}(\mathbf{Q}, \delta, \theta, K) \quad (5.14)$$

is the smallest constant such that the inequality

$$\begin{aligned}
& \left(\int_{\mathbb{R}^{d+n}} \left(\prod_{j=1}^M \|f_{W_j}\|_{L^p(B(x,K))} \right)^p dx \right)^{1/p} \\
& \leq \text{MulDec}_{q,p}(\mathbf{Q}, \delta, \theta, K) \prod_{j=1}^M \left(\sum_{\square \in \mathcal{P}(W_j, \delta)} \|f_{\square}\|_{L^p(\mathbb{R}^{d+n})}^q \right)^{1/q} \quad (5.15)
\end{aligned}$$

holds for every choice of functions $f_{\square}$ and every θ -uniform set $\{W_1, \dots, W_M\} \subseteq \mathcal{P}(K^{-1})$ with $1 \leq M \leq K^d$.

We use a version of the Bourgain–Guth reduction of linear to multilinear estimates (see [18]). Estimates of a similar form already appeared in works of Bourgain and Demeter (see, e.g., [15], [13]). The version below is a minor variant of [31, Proposition 2.33]. This is the place where the uniform bound in Theorem 5.2 is used.

PROPOSITION 5.6

Let $2 \leq q \leq p < \infty$. Let Λ be given by (5.1). Then, for each $\epsilon > 0$ and $\theta > 0$, there exists K such that

$$\begin{aligned}
\mathcal{D}_{q,p}(\mathbf{Q}, \delta) & \lesssim_{\epsilon, \theta} \delta^{-\Lambda - \epsilon} \\
& + \delta^{-\epsilon} \max_{\delta \leq \delta' \leq 1; \delta' \text{ dyadic}} \left[\left(\frac{\delta'}{\delta} \right)^{\Lambda} \text{MulDec}_{q,p}(\mathbf{Q}, \delta', \theta, K) \right]. \quad (5.16)
\end{aligned}$$

Proof of Proposition 5.6

Let $\{f_{\square}\}_{\square \in \mathcal{P}(\delta)}$ be a collection of functions with $\text{supp } \hat{f}_{\square} \subset \mathcal{U}_{\square}$. In the proof, for each dyadic cube J with $l(J) \geq \delta$, we denote

$$f_J := \sum_{\square \in \mathcal{P}(J, \delta)} f_{\square}. \quad (5.17)$$

Let K be a large constant that is to be determined. For each ball $B' \subset \mathbb{R}^{d+n}$ of radius K , we initialize

$$\mathcal{S}_0(B') := \{W \in \mathcal{P}(1/K) \mid \|f_W\|_{L^p(B')} \geq K^{-d} \max_{W' \in \mathcal{P}(1/K)} \|f_{W'}\|_{L^p(B')}\}. \quad (5.18)$$

We repeat the following algorithm. Let $\iota \geq 0$. If $\mathcal{S}_{\iota}(B') = \emptyset$ or $\mathcal{S}_{\iota}(B')$ is θ -uniform, then we set

$$\mathcal{T}(B') := \mathcal{S}_{\iota}(B') \quad (5.19)$$

and terminate. Otherwise, there exists a subvariety Z of degree at most d such that

$$|\{W \in \mathcal{S}_{\iota}(B') \mid 2W \cap Z \neq \emptyset\}| \geq \theta |\mathcal{S}_{\iota}(B')|. \quad (5.20)$$

Fix any such variety Z . Note that $2W \cap Z \neq \emptyset \implies W \subseteq \mathcal{N}_{2^d/K}(Z)$. For $j \in \{1, \dots, d\}$, let $\mathcal{W}_{t,j}(B') := \mathcal{W}_j$ be as in Lemma 5.5 with K replaced by $K/2^d$. Repeat this algorithm with

$$\mathcal{S}_{t+1}(B') := \mathcal{S}_t(B') \setminus \bigcup_{j=1}^d \bigcup_{W \in \mathcal{W}_{t,j}(B')} \mathcal{P}(W, 1/K). \quad (5.21)$$

This algorithm terminates after $O(\log K)$ steps, with an implicit constant depending on θ , as in each step we remove at least the set on the left-hand side of (5.20), which constitutes a fixed proportion θ of $\mathcal{S}_t(B')$.

To process the cubes in $\mathcal{W}_{t,j}$ and to avoid multiple counting, we define

$$\widetilde{\mathcal{W}}_{t,j} := \left(\mathcal{W}_{t,j} \setminus \bigcup_{0 \leq t' < t} \mathcal{W}_{t',j} \right) \setminus \bigcup_{1 \leq j' < j} \bigcup_{t'} \bigcup_{W \in \mathcal{W}_{t',j'}} \mathcal{P}(W, 1/K_j). \quad (5.22)$$

So far, we see that every cube in (5.18) can be covered by exactly one cube in

$$\left(\bigcup_t \bigcup_j \widetilde{\mathcal{W}}_{t,j} \right) \cup \mathcal{T}(B'). \quad (5.23)$$

Therefore, by the triangle inequality we obtain

$$\begin{aligned} \left\| \sum_{\square \subset [0,1]^d} f_{\square} \right\|_{L^p(B')} &\leq \left(\sum_{W \in \mathcal{P}(1/K)} \|f_W\|_{L^p(B')}^q \right)^{1/q} \\ &\quad + \sum_{t \lesssim \log K} \sum_{j=1}^d \left\| \sum_{W \in \widetilde{\mathcal{W}}_{t,j}} f_W \right\|_{L^p(B')} \\ &\quad + \sum_{W \in \mathcal{T}(B')} \|f_W\|_{L^p(B')}. \end{aligned} \quad (5.24)$$

On the right-hand side of (5.24), the first term is used to take care of the cubes that are not counted in (5.18). Next, we will see how to handle all these three terms. The second term on the right-hand side will be processed via a standard localization argument (see, e.g., [31, Remark 1.24]) and Lemma 5.5. It is bounded by

$$C_{Q,\epsilon,p,q} \log K \sum_{j=1}^d K_j^{\Lambda+\epsilon} \left(\sum_{W \in \mathcal{P}(1/K_j)} \|f_W\|_{L^p(w_{B'})}^q \right)^{1/q}. \quad (5.25)$$

Recall in Lemma 5.5 that $K^c \leq K_j \leq \sqrt{K}$ for some $c = c(d, \epsilon)$ and every j . This allows us to absorb $\log K$ by K_j^{ϵ} , which is the only place where the lower bound K^c in (5.11) is used. To bound the last term, we use the definition of $\mathcal{T}(B')$ and obtain

$$K^d \max_{W \in \mathcal{T}(B')} \|f_W\|_{L^p(B')} \leq K^{2d} \left(\sum_{\substack{\{W_1, \dots, W_M\} \subseteq \mathcal{P}(1/K) \\ \theta\text{-uniform}}} \prod_{j=1}^M \|f_{W_j}\|_{L^p(B')}^p \right)^{1/p}. \quad (5.26)$$

The above estimate seems rather crude, but we can allow any K -dependent constant in the estimate for this term. We plug (5.25) and (5.26) in (5.24), integrate over the centers of balls B' , and obtain

$$\begin{aligned} & \left\| \sum_{\square \in \mathcal{P}(\delta)} f_{\square} \right\|_{L^p(\mathbb{R}^{d+n})} \\ & \leq C_{\mathbf{Q}, \epsilon, q, p} \sum_{j=0}^d K_j^{\Lambda+2\epsilon} \left(\sum_{W \in \mathcal{P}(1/K_j)} \|f_W\|_{L^p(\mathbb{R}^{d+n})}^q \right)^{1/q} \\ & \quad + K^{2d} \sum_{\substack{\{W_1, \dots, W_M\} \subseteq \mathcal{P}(1/K) \\ \theta\text{-uniform}}} \left(\sum_{B' \subset \mathbb{R}^{d+n}} \prod_{j=1}^M \|f_{W_j}\|_{L^p(B')}^p \right)^{1/p}. \end{aligned} \quad (5.27)$$

Here we let $K_0 := K$. The terms under the sum in the former term have the same form as that on the left-hand side and therefore are ready for an iteration argument. In other words, we will apply (rescaled versions of) (5.27) to each term $\|f_W\|_{L^p(\mathbb{R}^{d+n})}$ under the sum in the former term. By the definition of the multilinear decoupling constant, the latter term can be controlled by

$$K^{2d} 2^{K^d} \text{MulDec}_{q,p}(\mathbf{Q}, \delta, \theta, K) \left(\sum_{\square \in \mathcal{P}(\delta)} \|f_{\square}\|_{L^p(\mathbb{R}^{d+n})}^q \right)^{1/q}, \quad (5.28)$$

where we used that there are only 2^{K^d} subsets of $\mathcal{P}(1/K)$, and hence at most that many θ -uniform subsets. We plug (5.28) in (5.27). Now it is a standard argument to iterate (5.27) and obtain the desired estimate in the proposition. We leave out the details and refer to [13, Section 8] or [13, Proposition 8.4]. $\square$

Recall that we have assumed (5.3). For most of Section 5, we fix some $0 < \epsilon < \Gamma - \Lambda$, a transversality parameter $\theta > 0$, and a corresponding K as in Proposition 5.6.

The multilinear decoupling constant will be estimated by the same procedure as in [11]. For a detailed exposition of this argument, we refer to [14, Theorem 10.16] or [31, Section 2.6]. We use a compressed version of this argument, in which each step is expressed as an inequality between the quantities (5.32) below. This version of the Bourgain–Demeter argument was originally motivated by decoupling for higher-degree polynomials (see [32]).

For a θ -uniform set $\{W_j\}_{j=1}^M \subset \mathcal{P}(1/K)$ and a choice of functions $f_{\square}$, $\square \in \mathcal{P}(\delta)$, we write

$$\begin{aligned}\tilde{A}_2(b) &:= \left\| \prod_{j=1}^M \left(\sum_{J \in \mathcal{P}(W_j, \delta^b)} \|f_J\|_{L^2(w_{B(x, \delta^{-2b})})}^2 \right)^{1/2} \right\|_{L^p_{x \in \mathbb{R}^{d+n}}}, \\ \tilde{A}_t(b) &:= \left\| \prod_{j=1}^M \left(\sum_{J \in \mathcal{P}(W_j, \delta^b)} \|f_J\|_{L^t(w_{B(x, \delta^{-b})})}^{\tilde{q}} \right)^{1/\tilde{q}} \right\|_{L^p_{x \in \mathbb{R}^{d+n}}}, \\ \tilde{A}_p(b) &:= \left\| \prod_{j=1}^M \left(\sum_{J \in \mathcal{P}(W_j, \delta^b)} \|f_J\|_{L^p(w_{B(x, \delta^{-2b})})}^q \right)^{1/q} \right\|_{L^p_{x \in \mathbb{R}^{d+n}}},\end{aligned}$$

where $0 < b \leq 1$ and

$$\frac{1}{t} = \frac{1/2}{p} + \frac{1/2}{2}, \quad \frac{1}{\tilde{q}} = \frac{1/2}{q} + \frac{1/2}{2}. \quad (5.29)$$

Note that $2 \leq \tilde{q} \leq t \leq p$. For

$$0 < b < 1 \quad \text{and} \quad * = 2, t, p, \quad (5.30)$$

let $a_*(b)$ be the infimum over all exponents a such that, for every θ -uniform set $\{W_j\}_{j=1}^M \subseteq \mathcal{P}(1/K)$, every $\delta < 1/K$, and every choice of functions $f_\square$, $\square \in \mathcal{P}(\delta)$, we have

$$\tilde{A}_*(b) \lesssim_{a, \theta, K} \delta^{-a} \prod_{j=1}^M \left(\sum_{\square \in \mathcal{P}(W_j, \delta)} \|f_\square\|_{L^p(\mathbb{R}^{d+n})}^q \right)^{1/q}, \quad (5.31)$$

with the implicit constant independent of the choice of the tuples (W_j) and $(f_\square)$, and in particular independent of b , as we will send $b \rightarrow 0$. It follows from Hölder's inequality that this $a_*(b) < \infty$. Recall that $\Gamma := \Gamma_{q,p}(\mathbf{Q})$. As in [32, Section 3.6], we define

$$a_* := \liminf_{b \rightarrow 0} \frac{\Gamma - a_*(b)}{b}, \quad * \in \{2, t, p\}. \quad (5.32)$$

The next lemma says that a_* is nontrivial.

LEMMA 5.7

Under the above notation, it holds that

$$a_* < \infty, \quad (5.33)$$

for $* = 2, t, p$.

Proof of Lemma 5.7

By Hölder's inequality and Bernstein's inequality, the left-hand side of (5.15) is bounded by

$$\delta^{-Cb} \tilde{A}_*(b) \quad (5.34)$$

for any $*$ $\in \{2, t, p\}$ and any $0 < b < 1$ with some constant C depending on $*$. Therefore, we obtain that

$$\text{MulDec}_{q,p}(\mathbf{Q}, \delta, \theta, K) \lesssim_{\epsilon, \theta, K} \delta^{-(Cb + a_*(b) + \epsilon)}, \quad (5.35)$$

for every $\epsilon > 0$ and $1 > b > 0$. This, together with Proposition 5.6 and the assumption (5.3), implies that

$$\Gamma \leq Cb + a_*(b). \quad (5.36)$$

This finishes the proof of the lemma. $\square$

5.3. Using linear decoupling

By Hölder's inequality, we obtain

$$\tilde{A}_p(b) \leq \prod_{j=1}^M \left\| \left(\sum_{J \in \mathcal{P}(W_j, \delta^b)} \|f_J\|_{L^p(w_{B(x, \delta^{-2b})})}^q \right)^{1/q} \right\|_{L^p_{x \in \mathbb{R}^d + n}}. \quad (5.37)$$

By Minkowski's inequality, this is further bounded by

$$\begin{aligned} & \prod_{j=1}^M \left(\sum_{J \in \mathcal{P}(W_j, \delta^b)} \|f_J\|_{L^p(w_{B(x, \delta^{-2b})})}^q \right)^{1/q}_{L^p_{x \in \mathbb{R}^d + n}} \\ & \lesssim \prod_{j=1}^M \left(\sum_{J \in \mathcal{P}(W_j, \delta^b)} \|f_J\|_{L^p(\mathbb{R}^d + n)}^q \right)^{1/q}. \end{aligned} \quad (5.38)$$

By the definition of the decoupling exponent and affine scaling (Lemma 5.3), this is

$$\lesssim_{\epsilon} \delta^{-(\Gamma + \epsilon)(1-b)} \prod_{j=1}^M \left(\sum_{\square \in \mathcal{P}(W_j, \delta)} \|f_{\square}\|_{L^p}^q \right)^{1/q}. \quad (5.39)$$

Hence

$$a_p(b) \leq (\Gamma + \epsilon)(1 - b),$$

for every $\epsilon > 0$, which means $a_p(b) \leq \Gamma(1 - b)$. It follows that

$$a_p \geq \Gamma. \quad (5.40)$$

5.4. Using L^2 orthogonality

By L^2 orthogonality (see, e.g., [32, Appendix B] for details), we have

$$\begin{aligned} \tilde{A}_2(b) &= \left\| \prod_{j=1}^M \left(\sum_{J \in \mathcal{P}(W_j, \delta^b)} \|f_J\|_{\mathbf{L}^2(w_{B(x, \delta^{-2b})})}^2 \right)^{1/2} \right\|_{L^p_{x \in \mathbb{R}^{d+n}}} \\ &\lesssim \left\| \prod_{j=1}^M \left(\sum_{J \in \mathcal{P}(W_j, \delta^{2b})} \|f_J\|_{\mathbf{L}^2(w_{B(x, \delta^{-2b})})}^2 \right)^{1/2} \right\|_{L^p_{x \in \mathbb{R}^{d+n}}}. \end{aligned} \quad (5.41)$$

We further apply Hölder's inequality and obtain

$$\lesssim \delta^{-d \cdot 2b(1/2 - 1/\tilde{q})} \left\| \prod_{j=1}^M \left(\sum_{J \in \mathcal{P}(W_j, \delta^{2b})} \|f_J\|_{\mathbf{L}^{\tilde{q}}(w_{B(x, \delta^{-2b})})}^{\tilde{q}} \right)^{1/\tilde{q}} \right\|_{L^p_{x \in \mathbb{R}^{d+n}}} \quad (5.42)$$

Note that the last expression is exactly $\delta^{-db(1-2/\tilde{q})} \tilde{A}_t(2b)$. Hence

$$a_2(b) \leq db(1 - 2/\tilde{q}) + a_t(2b).$$

It follows that

$$a_2 \geq -d(1 - 2/\tilde{q}) + 2a_t. \quad (5.43)$$

5.5. Ball inflation

Using Corollary 4.10 with $\rho = \delta^b$ and taking L^p -norms in x_0 on both sides of (4.19), we obtain

$$\begin{aligned} \tilde{A}_t(b) &= \left\| \prod_{j=1}^M \left(\sum_{J \in \mathcal{P}(W_j, \delta^b)} \|f_J\|_{\mathbf{L}^{\tilde{q}}(w_{B(x, \delta^{-b})})}^{\tilde{q}} \right)^{1/\tilde{q}} \right\|_{L^p_{x \in \mathbb{R}^{d+n}}} \\ &\lesssim_{\epsilon} \delta^{-b(\gamma + \epsilon)} \left\| \prod_{j=1}^M \left(\sum_{J \in \mathcal{P}(W_j, \delta^{2b})} \|f_J\|_{\mathbf{L}^{\tilde{q}}(w_{B(x, \delta^{-2b})})}^{\tilde{q}} \right)^{1/\tilde{q}} \right\|_{L^p_{x \in \mathbb{R}^{d+n}}}, \end{aligned} \quad (5.44)$$

for every $\epsilon > 0$, where

$$\begin{aligned} \gamma &:= \frac{d}{t} - \frac{d+n}{p} + \frac{\kappa((1-\theta)p/t)}{p} \\ &\leq \frac{d}{t} - \frac{d+n}{p} + \frac{1}{p} \sup_{0 \leq n' \leq n} \left(n' + \left(1 - \frac{p}{t}(1-\theta)\right) \mathfrak{d}_{n'}(\mathbf{Q}) \right), \end{aligned} \quad (5.45)$$

and the log factors in Corollary 4.10 have been absorbed by $\delta^{-b\epsilon}$. In the last step we used Corollary 4.8. In the end, we apply Hölder's inequality to the last term in (5.44) and obtain

$$\tilde{A}_t(b) \lesssim \delta^{-b(\gamma+\epsilon)} \tilde{A}_2(b)^{1/2} \tilde{A}_p(b)^{1/2}. \quad (5.46)$$

It follows that

$$a_t(b) \leq b\gamma + a_p(b)/2 + a_2(b)/2.$$

Substituting this inequality into the definition (5.32), we obtain

$$a_t \geq -\gamma + a_p/2 + a_2/2. \quad (5.47)$$

5.6. Proof of Theorem 5.1

Inequalities (5.29), (5.40), (5.43), and (5.47) imply

$$\Gamma \leq a_p \leq 2\gamma - a_2 + 2a_t \leq 2\gamma + d(1 - 2/\tilde{q}) = 2\gamma + d(1/2 - 1/q).$$

Inserting the definitions of the respective terms into this inequality, we obtain

$$\Gamma_{q,p}(\mathbf{Q}) \leq 2\left(\frac{d}{t} - \frac{d+n}{p} + \frac{1}{p} \sup_{0 \leq n' \leq n} \left(n' + \left(1 - \frac{p}{t}(1-\theta)\right) \mathfrak{d}_{n'}(\mathbf{Q})\right)\right) + d\left(\frac{1}{2} - \frac{1}{q}\right).$$

Both sides of this inequality depend continuously on θ , and we consider its limit when $\theta \rightarrow 0$. This gives

$$\Gamma_{q,p}(\mathbf{Q}) \leq 2\left(\frac{d}{t} - \frac{d+n}{p} + \sup_{0 \leq n' \leq n} \left(\frac{n'}{p} + \left(\frac{1}{p} - \frac{1}{t}\right) \mathfrak{d}_{n'}(\mathbf{Q})\right)\right) + d\left(\frac{1}{2} - \frac{1}{q}\right).$$

Substituting the ansatz (5.29) for t , we obtain

$$\begin{aligned} \Gamma_{q,p}(\mathbf{Q}) &\leq d\left(\frac{1}{p} + \frac{1}{2}\right) - 2\frac{d+n}{p} + \sup_{0 \leq n' \leq n} \left(\frac{2n'}{p} + \left(\frac{2}{p} - \frac{1}{p} - \frac{1}{2}\right) \mathfrak{d}_{n'}(\mathbf{Q})\right) \\ &\quad + d\left(\frac{1}{2} - \frac{1}{q}\right) \\ &= \sup_{0 \leq n' \leq n} \left(\left(\frac{1}{2} - \frac{1}{p}\right)(d - \mathfrak{d}_{n'}) - \frac{2(n-n')}{p}\right) + d\left(\frac{1}{2} - \frac{1}{q}\right). \end{aligned}$$

This finishes the proof of Theorem 5.1.

6. Lower bounds in Theorem 1.1

In this section, we show the lower bounds for $\ell^q L^p$ decoupling constants in Theorem 1.1 for $q \leq p$. We will prove that

$$\begin{aligned} \Gamma_{q,p}(\mathbf{Q}) &\geq \max\left(\sup_H \Gamma_{q,p}(\mathbf{Q}|_H), \right. \\ &\quad \left. \max_{0 \leq n' \leq n} \left(d\left(1 - \frac{1}{p} - \frac{1}{q}\right) - \mathfrak{d}_{n'}(\mathbf{Q})\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{2(n-n')}{p}\right)\right), \quad (6.1) \end{aligned}$$

where H is a hyperplane passing through the origin, for every $p \geq 2, q \geq 2$. Note that here we do not necessarily require $q \leq p$. One can apply the above inequality repeatedly and then obtain the lower bounds in Theorem 1.1 for $q \leq p$.

First of all, we relate the decoupling exponent with the decoupling exponents on subspaces. Here, the distinction between the cases $q \leq p$ and $q > p$ becomes apparent.

LEMMA 6.1

Let $\mathbf{Q}$ be an n -tuple of quadratic forms in d variables, and let $H \subseteq \mathbb{R}^d$ be a linear subspace of dimension d' . Then, for any $2 \leq q \leq p < \infty$, we have

$$\Gamma_{q,p}(\mathbf{Q}) \geq \Gamma_{q,p}(\mathbf{Q}|_H), \quad (6.2)$$

and, for any $2 \leq p < q \leq \infty$, we have

$$\Gamma_{q,p}(\mathbf{Q}) \geq \Gamma_{q,p}(\mathbf{Q}|_H) + (d - d') \left(\frac{1}{p} - \frac{1}{q} \right). \quad (6.3)$$

Proof of Lemma 6.1

For notational convenience, assume that $\mathbb{R}^d = H \times \mathbb{R}^{d''}$ with $d'' = d - d'$. The bound (6.2) will follow from

$$\mathcal{D}_{q,p}(\mathbf{Q}, C\delta) \gtrsim \mathcal{D}_{q,p}(\mathbf{Q}|_H, \delta), \quad (6.4)$$

for some absolute constant C . To see this, let $\{\tilde{f}_{\square' \square''} \in \mathcal{P}([0, 1]^{d'}, \delta)\}$ be a tuple of functions on $\mathbb{R}^{d'+n}$ that nearly extremizes the inequality (1.6) for $\mathbf{Q}|_H$. Fix a bump function ϕ such that $\text{supp } \hat{\phi} \subseteq B(0, \delta^2) \subset \mathbb{R}^{d''}$ and, for

$$\square = \square' \times \square'' \in \mathcal{P}([0, 1]^d, \delta) = \mathcal{P}([0, 1]^{d'}, \delta) \times \mathcal{P}([0, 1]^{d''}, \delta),$$

consider the functions

$$f_{\square} = f_{\square' \times \square''} = \begin{cases} \tilde{f}_{\square'} \otimes \phi & \square'' = \square''_0 := [0, \delta]^{d''}, \\ 0 & \square'' \neq \square''_0. \end{cases}$$

Then $\text{supp } \widehat{f_{\square}} \subseteq C \mathcal{U}_{\square}$ and

$$\left\| \sum_{\square} f_{\square} \right\|_p = \|\phi\|_p \left\| \sum_{\square'} f_{\square'} \right\|_p, \quad \|f_{\square' \times \square''}\|_p = \mathbf{1}_{\square'' = \square''_0} \|\phi\|_p \|f_{\square'}\|_p,$$

which implies (6.4). Here $\mathbf{1}$ denotes an indicator function, which takes the value 1 if the statement in the subscript is true, and 0 otherwise.

To see (6.3), we define $f_{\square' \times \square''_0}$, as above. For other $\square'' \in \mathcal{P}([0, 1]^{d''}, \delta)$, let $a'' \in \mathbb{R}^{d''}$ be the center of $\square''$ and define

$$f_{\square' \times \square''} := A_{\square''} f_{\square' \times \square''_0}(\cdot + c_{\square''}),$$

where $c_{\square''} \in \mathbb{R}^{d+n}$ are very large vectors and the linear operators $A_{\square''}$ are given by affine transformations in the Fourier space:

$$\widehat{A_{\square''} f}(\xi, \eta) := \widehat{f}(\xi - (0, a''), \eta + Q(0, a'') - \nabla Q(0, a'') \cdot \xi).$$

If $c_{\square''}$ are sufficiently far apart, then functions $f_{\square' \times \square''}$ and $f_{\tilde{\square}' \times \tilde{\square}''}$ are almost disjointly supported for $\square'' \neq \tilde{\square}''$, so that

$$\begin{aligned} \left\| \sum_{\square} f_{\square} \right\|_p &\sim \left(\sum_{\square''} \left\| \sum_{\square'} f_{\square' \times \square''} \right\|_p^p \right)^{1/p} \\ &= \|\phi\|_p \left(\sum_{\square''} \left\| \sum_{\square'} \tilde{f}_{\square'} \right\|_p^p \right)^{1/p} = \|\phi\|_p \delta^{-d''/p} \left\| \sum_{\square'} \tilde{f}_{\square'} \right\|_p \end{aligned}$$

and

$$\begin{aligned} \left(\sum_{\square} \|f_{\square}\|_p^q \right)^{1/q} &= \left(\sum_{\square''} \sum_{\square'} \|f_{\square' \times \square''}\|_p^q \right)^{1/q} \\ &= \|\phi\|_p \left(\sum_{\square''} \sum_{\square'} \|\tilde{f}_{\square'}\|_p^q \right)^{1/q} = \|\phi\|_p \delta^{-d''/q} \left(\sum_{\square'} \|\tilde{f}_{\square'}\|_p^q \right)^{1/q}. \end{aligned}$$

This implies

$$\mathcal{D}_{q,p}(\mathbf{Q}, C\delta) \gtrsim \delta^{-d''(1/p-1/q)} \mathcal{D}_{q,p}(\mathbf{Q}|_H, \delta),$$

and therefore (6.3). $\square$

To show the lower bound in (6.1), it remains to prove the following.

PROPOSITION 6.2

Let $\mathbf{Q}$ be an n -tuple of quadratic forms in d variables. For $0 \leq n' \leq n$ and $2 \leq q, p \leq \infty$, we have

$$\Gamma_{q,p}(\mathbf{Q}) \geq d \left(1 - \frac{1}{p} - \frac{1}{q} \right) - \mathfrak{d}_{n'}(\mathbf{Q}) \left(\frac{1}{2} - \frac{1}{p} \right) - \frac{2(n-n')}{p}. \quad (6.5)$$

Proof of Proposition 6.2

Let $d' = \mathfrak{d}_{n'}(\mathbf{Q})$. After linear changes of variables, we may assume that $Q_1, \dots, Q_{n'}$ depend only on $\xi_1, \dots, \xi_{d'}$. Write frequency points in $\mathbb{R}^{d+n}$ as

$$(\xi', \xi'', \eta', \eta'') \in \mathbb{R}^{d'+d''+n'+n''}, \quad (6.6)$$

with

$$\begin{aligned}\xi' &= (\xi_1, \dots, \xi_{d'}), \xi'' = (\xi_{d'+1}, \dots, \xi_d), \\ \eta' &= (\eta_1, \dots, \eta_{n'}), \eta'' = (\eta_{n'+1}, \dots, \eta_n),\end{aligned}\tag{6.7}$$

and $d' + d'' = d, n' + n'' = n$. Similarly, we write spatial points in $\mathbb{R}^{d+n}$ as

$$(x', x'', y', y'') \in \mathbb{R}^{d'+d''+n'+n''}.\tag{6.8}$$

For a dyadic cube $\square \in \mathcal{P}(\delta)$, write $\square = \square' \times \square''$ with $\square' \subset \mathbb{R}^{d'}$ and $\square'' \subset \mathbb{R}^{d''}$. Choose functions $f_\square$ of the form

$$f_\square(x', x'', y', y'') = g_{\square'}(x', y') h_\square(x'', y'')\tag{6.9}$$

with the following properties:³

- (1) $\widehat{g_{\square'}}$ and $\widehat{h_\square}$ are positive smooth functions satisfying

$$\int \widehat{g_{\square'}} = \int \widehat{h_\square} = 1.\tag{6.10}$$

- (2) $\widehat{g_{\square'}}$ is supported on a ball of radius $\approx \delta^2$ contained in

$$\{(\xi', \eta') : \xi' \in \delta \cdot \square', |\eta_1 - Q_1(\xi')| \leq \delta^2, \dots, |\eta_{n'} - Q_{n'}(\xi')| \leq \delta^2\},\tag{6.11}$$

where $\delta \cdot \square'$ is the box of the same center as $\square'$ and side length δ times that of $\square'$.

- (3) $\widehat{h_\square}$ is supported on a rectangular box of dimensions comparable to

$$\underbrace{\delta^1 \times \dots \times \delta^1}_{d'' \text{ times}} \times \underbrace{\delta^2 \times \dots \times \delta^2}_{n'' \text{ times}}\tag{6.12}$$

contained in

$$\bigcup_{\xi' \in \delta \cdot \square'} \{(\xi'', \eta'') : \xi'' \in \square'', |\eta_{\tilde{n}'} - Q_{\tilde{n}'}(\xi', \xi'')| \leq \delta^2, n' < \tilde{n}' \leq n\}.\tag{6.13}$$

On one hand, by the uncertainty principle,

$$\|f_\square\|_p \sim \delta^{-(2d'+d''+2n)/p},\tag{6.14}$$

and by definition we have

$$\begin{aligned}\left\| \sum_{\square \in \mathcal{P}(\delta)} f_\square \right\|_p &\leq \mathcal{D}_{q,p}(\mathbf{Q}, \delta) \left(\sum_{\square \in \mathcal{P}(\delta)} \|f_\square\|^q \right)^{1/q} \\ &\sim \mathcal{D}_{q,p}(\mathbf{Q}, \delta) \delta^{-d/q} \delta^{-(2d'+d''+2n)/p}.\end{aligned}\tag{6.15}$$

³Here and below we use $\square$ instead of $\square''$ in $h_\square$, as $Q_{n'+1}, \dots, Q_n$ still depend on ξ' .

On the other hand, with $U = \{(x'', y'') \in \mathbb{R}^{d''} \times \mathbb{R}^{n''} \mid |x''|, |y''| \leq 10^{-d-n}/(\sup_j \|\text{Hess } Q_j\| + 1)\}$, we have

$$\begin{aligned} \left\| \sum_{\square \in \mathcal{P}(\delta)} f_{\square} \right\|_p &\gtrsim \inf_{(x'', y'') \in U} \left\| \sum_{\square \in \mathcal{P}(\delta)} f_{\square} \right\|_{L^p(\mathbb{R}^{d'} \times \{x''\} \times \mathbb{R}^{n'} \times \{y''\})} \\ &= \inf_{(x'', y'') \in U} \left\| \sum_{\square'} c_{\square', x'', y''} g_{\square'} \right\|_{L^p(\mathbb{R}^{d'} \times \mathbb{R}^{n'})}, \end{aligned} \quad (6.16)$$

where

$$c_{\square', x'', y''} := \sum_{\square''} h_{\square' \times \square''}(x'', y'') = \sum_{\square''} h_{\square}(x'', y'') \quad (6.17)$$

satisfies

$$|c_{\square', x'', y''}| \sim \delta^{-d''} \quad (6.18)$$

uniformly in $\square'$ and $(x'', y'') \in U$. This is because $h_{\square}(0, 0) = 1$ and

$$\begin{aligned} |h_{\square}(x'', y'') - h_{\square}(0, 0)| &\leq \int |e(x'' \cdot \xi'' + y'' \cdot \eta'') - 1| |\widehat{h_{\square}}(\xi'', \eta'')| d\xi'' d\eta'' \\ &\leq \frac{1}{2} \int |\widehat{h_{\square}}(\xi'', \eta'')| d\xi'' d\eta'' = \frac{1}{2}, \end{aligned}$$

so that all summands in (6.17) are close to 1.

Let $\phi_{\delta}(\cdot) = \phi(\delta^2 \cdot)$, where ϕ is a fixed positive Schwartz function on $\mathbb{R}^{d'} \times \mathbb{R}^{n'}$ with $\text{supp } \widehat{\phi} \subset B(0, 1/10)$. Then, by Hölder's inequality,

$$\begin{aligned} &\left\| \sum_{\square'} c_{\square', x'', y''} g_{\square'} \right\|_{L^p(\mathbb{R}^{d'} \times \mathbb{R}^{n'})} \\ &\geq \|\phi_{\delta}\|_{1/(1/2-1/p)}^{-1} \left\| \phi_{\delta} \sum_{\square'} c_{\square', x'', y''} g_{\square'} \right\|_{L^2(\mathbb{R}^{d'} \times \mathbb{R}^{n'})} \\ &\sim \delta^{2 \cdot (d' + n')(1/2-1/p)} \left\| \sum_{\square'} c_{\square', x'', y''} \phi_{\delta} \cdot g_{\square'} \right\|_{L^2(\mathbb{R}^{d'} \times \mathbb{R}^{n'})}. \end{aligned} \quad (6.19)$$

Since the Fourier supports of $\phi_{\delta} \cdot g_{\square'}$ are disjoint for different $(\square')$'s for sufficiently small δ , we obtain

$$\begin{aligned} &\left\| \sum_{\square'} c_{\square', x'', y''} \phi_{\delta} \cdot g_{\square'} \right\|_{L^2(\mathbb{R}^{d'} \times \mathbb{R}^{n'})} \\ &= \left(\sum_{\square'} |c_{\square', x'', y''}|^2 \|\phi_{\delta} \cdot g_{\square'}\|_{L^2(\mathbb{R}^{d'} \times \mathbb{R}^{n'})}^2 \right)^{1/2} \\ &\sim \delta^{-d'/2} \cdot \delta^{-d''} \cdot \delta^{-2 \cdot (d' + n')/2}, \end{aligned} \quad (6.20)$$

uniformly in $(x'', y'') \in U$. Combining the above estimates, we obtain

$$\begin{aligned} \mathcal{D}_{q,p}(\mathbf{Q}, \delta) &\delta^{-d/q} \delta^{-(2d'+d''+2n)/p} \\ &\gtrsim \delta^{2 \cdot (d'+n')(1/2-1/p)} \cdot \delta^{-d'/2} \cdot \delta^{-d''} \cdot \delta^{-2 \cdot (d'+n')/2}. \end{aligned} \quad (6.21)$$

This implies

$$\Gamma_{q,p}(\mathbf{Q}) \geq d(1 - 1/q - 1/p) - d'(1/2 - 1/p) - 2(n - n')/p,$$

as desired. $\square$

7. Sharp $\ell^q L^p$ decoupling inequalities with $q > p$

Proof of Theorem 1.1 with $q > p$

The upper bound $\leq$ follows from the Hölder inequality between ℓ^p and ℓ^q sums in the definitions of $\Gamma_{q,p}$ and $\Gamma_{p,p}$.

Let us prove the lower bound. Recall from Corollary 1.2 that

$$\Gamma_{p,p}(\mathbf{Q}) = \max_{d/2 \leq d' \leq d} \max_{0 \leq n' \leq n} \left((2d' - \mathfrak{d}_{d',n'}(\mathbf{Q})) \left(\frac{1}{2} - \frac{1}{p} \right) - \frac{2(n - n')}{p} \right). \quad (7.1)$$

We will show that

$$\begin{aligned} \Gamma_{q,p}(\mathbf{Q}) &\geq \max_{d' \leq d} \max_{0 \leq n' \leq n} \left((2d' - \mathfrak{d}_{d',n'}(\mathbf{Q})) \left(\frac{1}{2} - \frac{1}{p} \right) - \frac{2(n - n')}{p} \right) \\ &\quad + d(1/p - 1/q) \end{aligned} \quad (7.2)$$

via an induction on d . The base case $d = 1$ is easy, as quadratic forms depending on one variable ξ_1 are all multiples of ξ_1^2 . Let us assume we have proved (7.2) for $d = d_0$, that is, we have established (7.2) for all $\mathbf{Q}$ depending on d_0 variables. We aim to prove it for $d = d_0 + 1$, that is, for $\mathbf{Q}$ depending on $d_0 + 1$ variables. First of all, we apply Proposition 6.2 and obtain

$$\begin{aligned} \Gamma_{q,p}(\mathbf{Q}) &\geq \max_{0 \leq n' \leq n} \left((2(d_0 + 1) - \mathfrak{d}_{d_0+1,n'}(\mathbf{Q})) \left(\frac{1}{2} - \frac{1}{p} \right) - \frac{2(n - n')}{p} \right) \\ &\quad + (d_0 + 1) \left(\frac{1}{p} - \frac{1}{q} \right), \end{aligned} \quad (7.3)$$

which is the right-hand side of (7.2) with $d' = d_0 + 1$. It remains to prove that

$$\begin{aligned} \Gamma_{q,p}(\mathbf{Q}) &\geq \max_{d' \leq d_0} \max_{0 \leq n' \leq n} \left((2d' - \mathfrak{d}_{d',n'}(\mathbf{Q})) \left(\frac{1}{2} - \frac{1}{p} \right) - \frac{2(n - n')}{p} \right) \\ &\quad + (d_0 + 1) \left(\frac{1}{p} - \frac{1}{q} \right). \end{aligned} \quad (7.4)$$

Let $H \subset \mathbb{R}^{d_0+1}$ be a linear subspace of dimension d_0 . By Lemma 6.1, we obtain

$$\Gamma_{q,p}(\mathbf{Q}) \geq \Gamma_{q,p}(\mathbf{Q}|_H) + \frac{1}{p} - \frac{1}{q}. \quad (7.5)$$

Now we apply our induction hypothesis to $\Gamma_{q,p}(\mathbf{Q}|_H)$, as $\mathbf{Q}|_H$ depend on d_0 variables, and obtain

$$\begin{aligned} \Gamma_{q,p}(\mathbf{Q}) &\geq \max_{d' \leq d_0} \max_{n' \leq n} \left((2d' - \mathfrak{d}_{d',n'}(\mathbf{Q}|_H)) \left(\frac{1}{2} - \frac{1}{p} \right) - \frac{2(n-n')}{p} \right) \\ &\quad + (d_0 + 1) \left(\frac{1}{p} - \frac{1}{q} \right). \end{aligned} \quad (7.6)$$

In order to prove (7.4), we first take the sup over H in (7.6) and realize that it suffices to prove

$$\inf_H \mathfrak{d}_{d',n'}(\mathbf{Q}|_H) \leq \mathfrak{d}_{d',n'}(\mathbf{Q}), \quad (7.7)$$

for every H of dimension d_0 and every $d' \leq d_0$. This follows from the definition of $\mathfrak{d}_{d',n'}$. $\square$

The following example shows that Proposition 6.2 does not by itself always give the correct lower bound for $\Gamma_{q,p}$ when $q > p$. Let us take the extreme case $q = \infty$.

Example 7.1

Let $d = 4$, $n = 2$, and

$$\mathbf{Q} = (\xi_1^2 + \xi_2 \xi_4, \xi_3 \xi_4).$$

We have

$$\mathfrak{d}_{4,2} = 4, \quad \mathfrak{d}_{4,1} = 2, \quad \mathfrak{d}_{3,2} = 1,$$

and all other $\mathfrak{d}_{d',n'}$ are 0. Let $p = 2 + 4n/d = 4$. Then direct computation shows that $\Gamma_{\infty,p} = 9/4$. However, Proposition 6.2 only shows $\Gamma_{\infty,p} \geq 2$.

8. Proofs of Corollaries 1.3–1.5

8.1. Proof of Corollary 1.3

We apply Theorem 1.1 with $q = 2$ to the tuple of quadratic forms $\mathbf{Q}$, and by (1.20), we know that (1.22) holds true if and only if

$$\begin{aligned} &\max_{0 \leq d' \leq d} \max_{0 \leq n' \leq n} \left((d' - \mathfrak{d}_{d',n'}(\mathbf{Q})) \left(\frac{1}{2} - \frac{1}{p} \right) - \frac{2(n-n')}{p} \right) \\ &\leq \max \left(0, d \left(\frac{1}{2} - \frac{1}{p} \right) - \frac{2n}{p} \right) \end{aligned} \quad (8.1)$$

for every $p \geq 2$. Both sides of (8.1) are finite maxima of affine linear functions in $1/p$. The two arguments of the max on the right-hand side coincide at $p_0 := 2 + 4n/d$. Hence, (8.1) holds for every $p \in [2, \infty]$ if and only if it holds for all $p \in \{2, p_0, \infty\}$.

For $p = 2$, we have $\text{LHS}(8.1) = 0 = \text{RHS}(8.1)$. For $p = \infty$, we have

$$\text{LHS}(8.1) = \max_{0 \leq d' \leq d} \max_{0 \leq n' \leq n} (d' - \mathfrak{d}_{d',n'}(\mathbf{Q}))/2 = d/2,$$

where the maximum is attained at $d' = d$ and $n' = 0$, and therefore (8.1) holds with equality at $p = \infty$. For $p = p_0$, (8.1) is equivalent to

$$\max_{0 \leq d' \leq d} \max_{0 \leq n' \leq n} \left((d' - \mathfrak{d}_{d',n'}(\mathbf{Q})) \frac{2n}{dp_0} - \frac{2(n-n')}{p_0} \right) \leq 0. \quad (8.2)$$

A direct calculation shows that (8.2) is equivalent to the strong nondegeneracy condition (1.21).

8.2. Proof of Corollary 1.4

The proof is basically the same as that for Corollary 1.3. We apply Corollary 1.2 to the tuple of quadratic forms $\mathbf{Q}$, and by (1.19), we know that (1.24) holds true if and only if

$$\begin{aligned} & \max_{d/2 < d' \leq d} \max_{0 \leq n' \leq n} \left((2d' - \mathfrak{d}_{d',n'}(\mathbf{Q})) \left(\frac{1}{2} - \frac{1}{p} \right) - \frac{2(n-n')}{p} \right) \\ & \leq \max \left(d \left(\frac{1}{2} - \frac{1}{p} \right), 2d \left(\frac{1}{2} - \frac{1}{p} \right) - \frac{2n}{p} \right) \end{aligned} \quad (8.3)$$

for every $p \geq 2$. The two numbers on the right-hand side coincide at $p_0 = 2 + 4n/d$. As in the proof of Corollary 1.3, (8.3) holds for every $p \in [2, \infty]$ if and only if it holds for all $p \in \{2, p_0, \infty\}$. For $p \in \{2, \infty\}$, the condition (8.3) again always holds with equality. Hence, (8.3) holds for every $p \in [2, \infty]$ if and only if it holds at $p = p_0$, which is further equivalent to

$$\max_{d/2 < d' \leq d} \max_{0 \leq n' \leq n} \left((2d' - \mathfrak{d}_{d',n'}(\mathbf{Q})) \frac{2n}{dp_0} - \frac{2(n-n')}{p_0} \right) \leq \frac{2n}{p_0}. \quad (8.4)$$

A direct calculation shows (8.4) is equivalent to the nondegeneracy condition (1.23).

8.3. Proof of Corollary 1.5

Recall from Corollary 1.2 that

$$\begin{aligned} \Gamma_p(\mathbf{Q}) &= \max_{d/2 < d' \leq d} \max_{0 \leq n' \leq n} \gamma_{d',n'}(1/p), \\ \gamma_{d',n'}(1/p) &= (2d' - \mathfrak{d}_{d',n'}(\mathbf{Q})) \left(\frac{1}{2} - \frac{1}{p} \right) - \frac{2(n-n')}{p}. \end{aligned}$$

The functions $\gamma_{d',n'}$ are affine. For $n' < n$ and arbitrary d' , we have $\gamma_{d',n'}(1/2) < 0$. Moreover, for arbitrary d' , we have $\gamma_{d',n}(1/2) = 0$. For every $p \in (2, \infty)$, the condition (1.25) is equivalent to

$$\forall d' \in (d/2, d], \quad \gamma_{d',n}(1/p) \leq d(1/2 - 1/p).$$

In particular, if (1.25) fails, then (1.26) fails for any $p_c > 2$.

Suppose now that the condition (1.25) is satisfied. Then, in particular, $\mathfrak{d}_{d,n}(\mathbf{Q}) = d$, and it follows that Then, Corollary 1.2 implies

$$\Gamma_p(\mathbf{Q}) = \max\left(d\left(\frac{1}{2} - \frac{1}{p}\right), \max_{d/2 < d' \leq d} \max_{0 \leq n' < n} \gamma_{d',n'}(1/p)\right).$$

Since the latter double maximum is a piecewise affine function of $1/p$ and is strictly negative for $p = 2$, we see that there exists $p_c > 2$ satisfying (1.26). The largest possible p_c is the minimum of solutions $p \in (2, \infty)$ of the equations

$$d\left(\frac{1}{2} - \frac{1}{p}\right) = \gamma_{d',n'}(1/p) \quad (8.5)$$

for $d/2 < d' = d - m \leq d$ and $0 \leq n' \leq n - 1$. These solutions are given by the formula

$$p(d', n') = 2 + \frac{4n - 4n'}{2d' - d - \mathfrak{d}_{d',n'}(\mathbf{Q})}. \quad (8.6)$$

This shows (1.27), since the minimum in (1.27) is restricted in such a way as to be taken over numbers in $(2, \infty)$.

We note also that, for $n' = 0$ and $m = 0$, we have $\mathfrak{d}_{d,0}(\mathbf{Q}) = 0$, which shows that the minimum in (1.27) is taken over a nonempty set, and is at most $2 + 4n/d$.

9. Fourier restriction: Proof of Corollary 2.3

In this section we prove Corollary 2.3. The proof is standard, and it relies on an epsilon removal lemma of Tao [57], the broad-narrow analysis of Bourgain and Guth [18], and the decoupling inequalities established in the current paper. The use of decoupling inequalities in this context is also standard (see, e.g., Guth [37]). As $\mathbf{Q}$ will be fixed throughout the proof, we will leave out the dependence of the extension function $E^{\mathbf{Q}}g$ on $\mathbf{Q}$ and simply write Eg .

Let us begin with the epsilon removal lemma. In order to prove (2.6), it suffices to prove that for every $\epsilon > 0$, there exists $C_{d,n,p,\mathbf{Q},\epsilon} = C_\epsilon$ such that

$$\|E_{[0,1]^d} g\|_{L^p(B)} \leq C_\epsilon \delta^{-\epsilon} \|g\|_p, \quad (9.1)$$

for every $\delta \leq 1, \epsilon > 0, p > p_{\mathbf{Q}}$ and every ball $B \subset \mathbb{R}^{d+n}$ of radius δ^{-2} . Here and below, we will leave out the dependence of our implicit constants on d, n, p and $\mathbf{Q}$.

Such a reduction first appeared in [57] (see also [18], [40]). For a version of epsilon removal lemmas for manifolds of codimension bigger than 1, we refer to [33, Section 4].

In order to prove (9.1), we will apply the broad-narrow analysis and the decoupling inequalities in the current paper, together with an induction argument on δ . Let us assume that we have proved (9.1) with δ' in place of δ for every $1 \geq \delta' \geq 2\delta$. Under this induction hypothesis, we will prove (9.1). Let us begin with one corollary of Proposition 4.9.

COROLLARY 9.1 (Multilinear restriction estimate)

Let $K \in 2^{\mathbb{N}}$ be a dyadic integer and $0 < \delta \leq 1/K$. Let $\theta > 0$ and $\{W_j\}_{j=1}^M \subseteq \mathcal{P}(1/K)$ be a θ -uniform set of cubes. Let $B \subset \mathbb{R}^{d+n}$ be a ball of radius δ^{-2} . Then, for each $2 \leq p < \infty$ and $\epsilon' > 0$, we have

$$\left\| \prod_{j=1}^M |E_{W_j} g| \right\|_{L^p(B)} \leq C_{\theta, K, \epsilon'} \delta^{-\gamma(p, \theta, \mathbf{Q}) - \epsilon'} \prod_{j=1}^M \|g\|_{L^2(W_j)}, \quad (9.2)$$

where

$$\gamma(p, \theta, \mathbf{Q}) := \sup_{0 \leq n' \leq n} \left(\frac{2n'}{p} + \left(\frac{2}{p} - (1 - \theta) \right) \mathfrak{d}_{n'}(\mathbf{Q}) \right). \quad (9.3)$$

Proof of Corollary 9.1

The proof is essentially via the argument of passing from multilinear Kakeya estimates to multilinear restriction estimates as in Bennett, Carbery, and Tao [7]. Let us first show that

$$\begin{aligned} \left\| \prod_{j=1}^M |E_{W_j} g| \right\|_{L^p(B)} &\leq C_{\theta, K, \epsilon'} \delta^{-2(\frac{d}{2} - \frac{d+n}{p} + \frac{\kappa((1-\theta)p/2)}{p}) - \epsilon'} \\ &\quad \times \prod_{j=1}^M \left(\sum_{\square \in \mathcal{P}(W_j, \delta)} \|E_{\square} g\|_{L^2(w_B)}^2 \right)^{1/2}, \end{aligned} \quad (9.4)$$

for every $\epsilon' > 0$. We take a Schwartz function ψ such that ψ is positive on the ball of radius 1 centered at the origin, and Fourier transform of ψ has a compact support. Let us define the function $\psi_B(x) := \psi(\delta^2 x)$. By Hölder's inequality and L^2 -orthogonality (see, e.g., [32, Appendix B]), we see that

$$\begin{aligned}
& \left\| \prod_{j=1}^M |E_{W_j} g| \right\|_{\mathbf{L}^p(B)} \\
& \lesssim \left\| \prod_{j=1}^M |\psi_B E_{W_j} g| \right\|_{\mathbf{L}^p(B)} \\
& \lesssim \delta^{-\epsilon'(d+n)/2} \left\| \prod_{j=1}^M \|\psi_B E_{W_j} g\|_{\mathbf{L}^2(B(x, \delta^{-\epsilon'}))} \right\|_{\mathbf{L}^p_{x \in B}} \\
& \lesssim \delta^{-\epsilon'(d+n)/2} \left\| \prod_{j=1}^M \left(\sum_{J \in \mathcal{P}(W_j, \delta^{\epsilon'})} \|\psi_B E_J g\|_{\mathbf{L}^2(w_{B(x, \delta^{-\epsilon'})})}^2 \right)^{1/2} \right\|_{\mathbf{L}^p_{x \in B}}. \quad (9.5)
\end{aligned}$$

We apply (4.14) and L^2 -orthogonality, and bound the above term by

$$\begin{aligned}
& \delta^{-\epsilon'(d+n)/2} \delta^{-\epsilon'(\frac{d}{2} - \frac{d+n}{p} + \frac{\kappa((1-\theta)p/2)}{p})} \\
& \times \left\| \prod_{j=1}^M \left(\sum_{J \in \mathcal{P}(W_j, \delta^{\epsilon'})} \|\psi_B E_J g\|_{\mathbf{L}^2(w_{B(x, \delta^{-2\epsilon'})})}^2 \right)^{1/2} \right\|_{\mathbf{L}^p_{x \in B}} \\
& \lesssim \delta^{-\epsilon'(d+n)/2} \delta^{-\epsilon'(\frac{d}{2} - \frac{d+n}{p} + \frac{\kappa((1-\theta)p/2)}{p})} \\
& \times \left\| \prod_{j=1}^M \left(\sum_{J \in \mathcal{P}(W_j, \delta^{2\epsilon'})} \|\psi_B E_J g\|_{\mathbf{L}^2(w_{B(x, \delta^{-2\epsilon'})})}^2 \right)^{1/2} \right\|_{\mathbf{L}^p_{x \in B}}. \quad (9.6)
\end{aligned}$$

We repeat this process and obtain

$$\begin{aligned}
& \delta^{-\epsilon'(d+n)/2} \delta^{-2(\frac{d}{2} - \frac{d+n}{p} + \frac{\kappa((1-\theta)p/2)}{p})} \\
& \times \left\| \prod_{j=1}^M \left(\sum_{J \in \mathcal{P}(W_j, \delta)} \|\psi_B E_J g\|_{\mathbf{L}^2(w_{B(x, \delta^{-2})})}^2 \right)^{1/2} \right\|_{\mathbf{L}^p_{x \in B}}. \quad (9.7)
\end{aligned}$$

We rename $\epsilon'(d+n)/2$ by ϵ' , and the above term is bounded by the right-hand side of (9.4).

By the Plancherel theorem, we see that

$$\|E_{\square} g\|_{\mathbf{L}^2(w_B)} \lesssim \delta^d \|g\|_{L^2(\square)}. \quad (9.8)$$

Therefore, by (9.4), we obtain that

$$\left\| \prod_{j=1}^M |E_{W_j} g| \right\|_{L^p(B)}$$

$$\leq C_{\theta, K, \epsilon'} \delta^{-2\kappa((1-\theta)p/2)/p-\epsilon'} \prod_{j=1}^M \|g\|_{L^2(W_j)}. \quad (9.9)$$

It suffices to apply Corollary 4.8 to bound κ . $\square$

We let θ be a small number, which will be determined later. Its choice depends only on how close p is to p_Q . Therefore the dependence of the forthcoming constants on θ will also be compressed. Readers can take $\theta = 0$ for convenience. We define p_c to be the smallest number such that $\gamma(p_c, \theta, \mathbf{Q}) = 0$. More explicitly,

$$p_c = \max_{1 \leq n' \leq n} \left(2 + \frac{2n' + 2\theta \mathfrak{d}_{n'}(\mathbf{Q})}{(1-\theta)\mathfrak{d}_{n'}(\mathbf{Q})} \right). \quad (9.10)$$

To prove (9.1), we run the broad-narrow analysis of Bourgain and Guth [18] in a way that is almost the same as in the proof of Proposition 5.6. We repeat the proof there until before step (5.27), with $q = p$ and $f_\square$ replaced by $\psi_B E_\square g$ for every dyadic box $\square$ of side length δ . Next, instead of summing over all balls B' of radius K in $\mathbb{R}^{d+n}$, we sum over $B' \subset B$, a ball of radius δ^{-2} , and obtain

$$\begin{aligned} & \left\| \sum_{\square \subset [0,1]^d} \psi_B E_\square g \right\|_{L^p(B)} \\ & \leq C_{\epsilon'} \sum_{j=0}^d K_j^{\Lambda+2\epsilon'} \left(\sum_{W \in \mathcal{P}(1/K_j)} \|\psi_B E_W g\|_{L^p(w_B)}^p \right)^{1/p} \\ & \quad + K^d \sum_{1 \leq M \leq K^d} \sum_{\substack{W_1, \dots, W_M \in \mathcal{P}(1/K) \\ \theta\text{-uniform}}} \left(\sum_{B' \subset B} \prod_{j=1}^M \|\psi_B E_{W_j} g\|_{L^p(B')}^p \right)^{1/p}, \quad (9.11) \end{aligned}$$

where

$$\Lambda := \sup_H \Gamma_p(\mathbf{Q}|_H), \quad (9.12)$$

and the sup is taken over all hyperplanes $H \subset \mathbb{R}^d$ that pass through the origin. Regarding the second term on the right-hand side of (9.11), we notice that each term $|E_{W_j} g|$ is essentially constant on B' , and therefore we can apply Corollary 9.1 and bound it by $C_{\epsilon', K} \delta^{-\epsilon'} \|g\|_2$, whenever $p > p_c$. So far we have obtained

$$\begin{aligned} \|\psi_B E_{[0,1]^d} g\|_{L^p(B)} & \leq C_{\epsilon'} \sum_{j=0}^d K_j^{\Lambda+2\epsilon'} \left(\sum_{W \in \mathcal{P}(1/K_j)} \|\psi_B E_W g\|_{L^p(w_B)}^p \right)^{1/p} \\ & \quad + C_{\epsilon', K} \delta^{-\epsilon'} \|g\|_2, \quad (9.13) \end{aligned}$$

for every $\epsilon' > 0$ and $p > p_c$. After arriving at this form, we are ready to apply an inductive argument, as the terms on the right-hand side of (9.13) are of the same form as that on the left-hand side, with just different scales. To be precise, we will apply our induction hypothesis to each $\|\psi_B E_W g\|_{L^p(w_B)}$. All these terms can be handled in exactly the same way. Without loss of generality, we take $W = [0, 1/K_j]^d$. Recall that

$$E_W g(x, y) = \int_W g(\xi) e(\xi \cdot x + \mathbf{Q}(\xi) \cdot y) d\xi, \quad (9.14)$$

where $x \in \mathbb{R}^d$, $y \in \mathbb{R}^n$. We apply the change of variables $\xi \mapsto \xi/K_j$ and the induction hypothesis and obtain

$$\|\psi_B E_W g\|_{L^p(w_B)} \leq C C_\epsilon \delta^{-\epsilon} K_j^{-d} K_j^{\frac{d+2n}{p}} K_j^{\frac{d}{p}} \|g\|_{L^p(W)}, \quad (9.15)$$

where C is some new large constant that is allowed to depend on d, n, p and $\mathbf{Q}$. This, together with (9.13), implies that

$$\begin{aligned} & \|E_{[0,1]^d} g\|_{L^p(B)} \\ & \leq C C_{\epsilon'} C_\epsilon \delta^{-\epsilon} \sum_{j=0}^d K_j^{\Lambda-d+\frac{2d+2n}{p}+2\epsilon'} \|g\|_p + C_{\epsilon',K} \delta^{-\epsilon'} \|g\|_p, \end{aligned} \quad (9.16)$$

for every $\epsilon' > 0$. Recall from (5.11) that there exists a small number $c = c_{\epsilon'}$ such that

$$K^c \leq K_1 \leq K_2 \leq \dots \leq K_d \leq \sqrt{K}. \quad (9.17)$$

From (9.16) we see that if p is such that

$$\Lambda - d + (2d + 2n)/p < 0, \quad (9.18)$$

then we can pick ϵ' small enough and K sufficiently large, depending on ϵ' , such that

$$C C_{\epsilon'} K^{\Lambda-d+(2d+2n)/p+2\epsilon'} \leq 1/(2(d+1)). \quad (9.19)$$

After fixing ϵ' and K , we see that in order to control the second term in (9.16), we just need to set the constant C_ϵ from (9.1) to be $2C_{\epsilon',K}$ and then we can close the induction step.

Notice that there were two constraints on p , including $p > p_c$ and (9.18). Recall the definition of Λ in (9.12). One can apply Theorem 1.1 and see that

$$\Lambda = \max_{d' \leq d-1} \max_{0 \leq n' \leq n} \left[(2d' - \mathfrak{d}_{d',n'}(\mathbf{Q})) \left(\frac{1}{2} - \frac{1}{p} \right) - \frac{2(n-n')}{p} \right]. \quad (9.20)$$

Elementary computation shows that

$$p > \max\left(p_c, 2 + \max_{m \geq 1} \max_{n' \leq n} \frac{4n'}{2m + \mathfrak{d}_{d-m,n'}(\mathbf{Q})}\right) = \max(p_c, p_{\mathbf{Q}}). \quad (9.21)$$

As p_c is a continuous function depending on θ , to see that we have the range $p > p_{\mathbf{Q}}$, it suffices to show that

$$\max_{1 \leq n' \leq n} \left(2 + \frac{2n'}{\mathfrak{d}_{n'}(\mathbf{Q})}\right) \leq 2 + \max_{m \geq 1} \max_{n' \leq n} \frac{4n'}{2m + \mathfrak{d}_{d-m,n'}(\mathbf{Q})}. \quad (9.22)$$

This inequality follows from

$$2\mathfrak{d}_{d,n'}(\mathbf{Q}) \geq 2 + \mathfrak{d}_{d-1,n'}(\mathbf{Q}), \quad (9.23)$$

for every $n' \geq 1$, which holds true because $\mathfrak{d}_{d,n'}(\mathbf{Q}) > \mathfrak{d}_{d-1,n'}(\mathbf{Q})$ as long as $\mathfrak{d}_{d,n'}(\mathbf{Q}) > 0$. Recall that we assumed $\mathbf{Q}$ is linearly independent, and therefore we indeed have that $\mathfrak{d}_{d,n'}(\mathbf{Q}) > 0$ for every $n' \geq 1$. This verifies the range $p > p_{\mathbf{Q}}$ and thus finishes the proof of the corollary.

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