PIZZA AND 2-STRUCTURES

RICHARD EHRENBORG, SOPHIE MOREL AND MARGARET READDY

ABSTRACT. Let $\mathcal{H}$ be a Coxeter hyperplane arrangement in n-dimensional Euclidean space. Assume that the negative of the identity map belongs to the associated Coxeter group W. Furthermore assume that the arrangement is not of type A_1^n. Let K be a measurable subset of the Euclidean space with finite volume which is stable by the Coxeter group W and let a be a point such that K contains the convex hull of the orbit of the point a under the group W. In a previous article the authors proved the generalized pizza theorem: that the alternating sum over the chambers T of $\mathcal{H}$ of the volumes of the intersections $T \cap (K + a)$ is zero. In this paper we give a dissection proof of this result. In fact, we lift the identity to an abstract dissection group to obtain a similar identity that replaces the volume by any valuation that is invariant under affine isometries. This includes the cases of all intrinsic volumes. Apart from basic geometry, the main ingredient is a theorem of the authors where we relate the alternating sum of the values of certain valuations over the chambers of a Coxeter arrangement to similar alternating sums for simpler subarrangements called 2-structures introduced by Herb to study discrete series characters of real reduced groups.

1. INTRODUCTION

The 2-dimensional pizza theorem is the following result: Given a disc in the plane, choose a point on this disc and cut the disc by $2k$ equally spaced lines passing through the point, where $k \geq 2$. The alternating sum of the areas of the resulting slices is then equal to zero. This was first proved by Goldberg [Gol68]. Frederickson gave a dissection proof [Fre12] based on dissection proofs of Carter-Wagon in the case $k = 2$ (see [CW94]) and of Allen Schwenk (unpublished) in the cases $k = 3, 4$. Frederickson deduced dissection proofs of a similar sharing result for the pizza crust and of the so-called calzone theorem, which is the analogue of the pizza theorem for a ball in $\mathbb{R}^3$ that is cut by one horizontal plane and by $2k$ equally-spaced vertical planes all meeting at one point in the ball.

To generalize the pizza problem, consider a finite central hyperplane arrangement $\mathcal{H}$ in $\mathbb{R}^n$ and fix a base chamber of this arrangement. Each chamber T has a sign $(-1)^T$ determined by the parity of the number of hyperplanes separating it from the base chamber. If K is a measurable subset of $\mathbb{R}^n$ of finite volume, what can we say about the pizza quantity $\sum_T (-1)^T \operatorname{Vol}(T \cap K)$, where the sum runs over all the chambers T of $\mathcal{H}$? The original pizza theorem is the case where $n = 2$, $\mathcal{H}$ has the type of the dihedral arrangement $I_2(2k)$ and K is a disc containing the origin. The calzone theorem is the case where $n = 3$, $\mathcal{H}$ has the type of the product arrangement $I_2(2k) \times A_1$ and K is a ball containing the origin.

The following generalization of the pizza and calzone theorems was proved in [EMR22b, Theorem 1.2] by analytic means. We recently learned that Brailov had proved independently this result in the case a of ball for the type B_n arrangement using similar methods [Bra22].

Theorem 1.1 (Ehrenborg–Morel–Readdy)**.** *Let $\mathcal{H}$ be a Coxeter arrangement with Coxeter group W that contains the negative of the identity map, denoted by $-\mathrm{id}$. Assume that $\mathcal{H}$ is not of type A_1^n.*

Date: October 17, 2023.

2020 *Mathematics Subject Classification.* Primary 52B45, 20F55, 51F15; Secondary 51M20, 51M25.

Key words and phrases. Coxeter arrangements, 2-structures, dissections, Pizza Theorem, reflection groups, intrinsic volumes, pseudo-root systems, Bolyai-Gerwien Theorem.

Let K be a set of finite measure that is stable by the group W. Then for every point $a \in \mathbb{R}^n$ such that K contains the convex hull of $\{w(a) : w \in W\}$, we have

$$\sum_T (-1)^T \operatorname{Vol}(T \cap (K + a)) = 0.$$

The proof of this result uses an expression for $\sum_T (-1)^T \operatorname{Vol}(T \cap (K + a))$ as an alternating sum of pizza quantities over subarrangements of $\mathcal{H}$ of the form $\{e_1^\perp, \dots, e_n^\perp\}$ with $(e_1, \dots, e_n)$ an orthonormal basis of $\mathbb{R}^n$, in other words, subarrangements of type A_1^n.

In the paper [EMR22a], we study a different sum $\sum_T (-1)^T \nu(\overline{T})$, where ν is a valuation defined on closed convex polyhedral cones of $\mathbb{R}^n$ that takes integer values. Under the same condition that $\mathcal{H}$ is a Coxeter arrangement, we rewrite this quantity as an alternating sum of similar quantities for certain subarrangements of $\mathcal{H}$ that are products of rank 1 and rank 2 arrangements [EMR22a, Theorem 3.2.1], and then deduce an expression for it. These subarrangements, called 2-*structures*, were introduced by Herb [Her00] to study characters of discrete series of real reductive groups. In fact, the identity of [EMR22a, Theorem 3.2.1] is valid for any valuation and its proof uses only basic properties of Coxeter systems and closed convex polyhedral cones.

In this paper we use the setting of 2-structures and [EMR22a, Corollary 3.2.4] (recalled in Theorem 2.5) to obtain a dissection proof of the higher-dimensional pizza theorem of [EMR22b, Theorem 1.2] that is independent of the results and methods of [EMR22b]:

Theorem 1.2 (Abstract pizza theorem; see Theorem 3.5.)**.** *With the notation and hypotheses of Theorem 1.1, we have*

$$\sum_T (-1)^T [\overline{T} \cap (K + a)] = \sum_T (-1)^T [T \cap (K + a)] = 0,$$

where the brackets denote classes in the abstract dissection group of Definition 3.1.

As we take into account lower-dimensional sets when defining our abstract dissection group, this result implies generalizations of the higher-dimensional pizza theorem to all the intrinsic volumes when K is convex.

The idea of the proof of Theorem 1.2 is the following: by expanding the expression using 2-structures, we can reduce to a sum where each term is a similar expression for an arrangement that is a product of arrangements of types A_1 and $I_2(2^k)$. We then adapt the dissection proof of Frederickson to an arrangement of type $I_2(2m) \times \mathcal{H}'$. We also explain how to keep track of lower-dimensional regions of the dissection. If our product arrangement contains at least one dihedral factor, then its contribution is zero, and we immediately get a dissection proof of the result. However, if all the product arrangements that appear are of type A_1^n, then their individual contributions are not zero. We need one extra step in the proof to show that the contributions cancel. This uses a slight refinement of the Bolyai-Gerwien Theorem explained in Section 4.

An interesting point to note is that the shape of the pizza plays absolutely no role in this proof, as long as it has the same symmetries as the arrangement and contains the convex hull of $\{w(a) : w \in W\}$. In particular, we no longer need to assume that it is measurable and of finite volume.

The plan of the paper is as follows. Section 2 contains a review of 2-structures and of the results from [EMR22a] that we will need. Section 3 contains the statement and proof of the abstract pizza theorem (Theorem 3.5), and Section 4, as we already mentioned, contains a Bolyai-Gerwien type result that is needed in the proof of the abstract pizza theorem.

Let us mention some interesting questions that remain open:

(1) The paper [EMR22b] proves the pizza theorem for more general arrangements (the condition is that the arrangement $\mathcal{H}$ is a Coxeter arrangement and that the number of hyperplanes is greater than the dimension n and has the same parity as that dimension), but only in the case of the ball; see [EMR22b, Theorem 1.1]. Is it possible to give a dissection proof of this result?
(2) Mabry and Deiermann [MD09] show that the two-dimensional pizza theorem does not hold for a dihedral arrangement having an odd number of lines. More precisely, they determine the sign of the quantity $\sum_T (-1)^T \operatorname{Vol}(T \cap K)$, where K is a disc containing the origin, and show that it vanishes if and only if the center of K lies on one of the lines. Their methods are analytic. As far as we know, there exists no dissection proof of this result either. The higher-dimensional case where $\mathcal{H}$ is a Coxeter arrangement and the number of its hyperplanes does not have the same parity as n also remains wide open.

2. Review of 2-structures and of the basic identity

Let V be a finite-dimensional real vector space with an inner product $(\cdot,\cdot)$. For every $\alpha \in V$, we denote by H_α the hyperplane $\alpha^\perp$ and by s_α the orthogonal reflection in the hyperplane H_α.

We say that a subset Φ of V is a *normalized pseudo-root system* if:

(a) Φ is a finite set of unit vectors;
(b) for all $\alpha, \beta \in \Phi$, we have $s_\beta(\alpha) \in \Phi$ (in particular, taking $\alpha = \beta$, we get that $-\alpha \in \Phi$).

Elements of Φ are called *pseudo-roots*. The *rank* of Φ is the dimension of its span.

We call such objects *pseudo-root systems* to distinguish them from the (crystallographic) root systems that appear in representation theory. If Φ' is a root system then the set $\Phi = \{\alpha/\|\alpha\| : \alpha \in \Phi'\}$ is a normalized pseudo-root system. Not every normalized pseudo-root system arises in this manner; see for instance the pseudo-root systems of type H_3 and H_4.

We say that a normalized pseudo-root system Φ is *irreducible* if, whenever $\Phi = \Phi_1 \sqcup \Phi_2$ with Φ_1 and Φ_2 orthogonal, we have either $\Phi_1 = \varnothing$ or $\Phi_2 = \varnothing$. Every normalized pseudo-root system can be written uniquely as a disjoint union of pairwise orthogonal irreducible normalized pseudo-root systems. Irreducible normalized pseudo-root systems are classified: they are in one of the infinite families A_n ($n \geq 1$), B_n/C_n ($n \geq 2$),[1] D_n ($n \geq 4$), $I_2(m)$ ($m \geq 3$) or one of the exceptional types E_6, E_7, E_8, F_4, H_3 or H_4, with types $I_2(3)$ and A_2 isomorphic, as well as types $I_2(4)$ and B_2. (See [GB85, Chapter 5] or Table 1 in [BB05, Appendix A].)

We say that a subset $\Phi^+ \subset \Phi$ is a *positive system* if there exists a total ordering $<$ on the $\mathbb{R}$-vector space V such that $\Phi^+ = \{\alpha \in \Phi : \alpha > 0\}$ (see [Hum90, Section 1.3]). The *Coxeter group* of Φ is the group of isometries W of V generated by the reflections s_α for $\alpha \in \Phi$. This group preserves Φ by definition of a normalized pseudo-root system, and it acts simply transitively on the set of positive systems by [Hum90, Section 1.4]. In particular, the Coxeter group W is finite.

Let E be a finite set of unit vectors of V such that $E \cap (-E) = \varnothing$. The corresponding *hyperplane arrangement* is the set of hyperplanes $\mathcal{H} = \{H_e : e \in E\}$. A *chamber* of $\mathcal{H}$ is a connected component of $V - \bigcup_{e \in E} H_e$; we denote by $\mathscr{T}(\mathcal{H})$ the set of chambers of $\mathcal{H}$. Fix a chamber T_0 to be the *base chamber*. For a chamber $T \in \mathscr{T}(\mathcal{H})$ we denote by $S(T, T_0)$ the set of $e \in E$ such that the two chambers T and T_0 are on different sides of the hyperplane H_e, and define the *sign* of T to be $(-1)^T = (-1)^{|S(T,T_0)|}$.

We say that $\mathcal{H}$ is a *Coxeter arrangement* if it is stable by the orthogonal reflections in each of its hyperplanes. In that case, the set $\Phi = E \cup (-E)$ is a normalized pseudo-root system. We call its Coxeter group the Coxeter group of the arrangement. The map sending a positive system $\Phi^+ \subset \Phi$ to the set $\{v \in V : \forall \alpha \in \Phi^+ (v, \alpha) > 0\}$ is a bijection from the positive systems in Φ to the chambers of $\mathcal{H}$. See, for example, [Bou68, Chapitre V § 4 № 8 Proposition 9 p. 99] and the discussion following

[1] The pseudo-root systems of types B_n and C_n are identical after normalizing the lengths of the roots.

it. Conversely, if $\Phi \subset V$ is a normalized pseudo-root system with Coxeter group W and $\Phi^+ \subset \Phi$ is a positive system, then $\mathcal{H} = \{H_\alpha : \alpha \in \Phi^+\}$ is a Coxeter hyperplane arrangement, and in that case we always take the base chamber T_0 to be the chamber corresponding to Φ^+.

We now define product arrangements. Let V_1 and V_2 be two finite-dimensional real vector spaces equipped with inner products, and suppose that we are given hyperplane arrangements $\mathcal{H}_1$ and $\mathcal{H}_2$ on V_1 and V_2 respectively. We consider the product space $V_1 \times V_2$, where the factors are orthogonal. The *product arrangement* $\mathcal{H}_1 \times \mathcal{H}_2$ is then the arrangement on $V_1 \times V_2$ with hyperplanes $H \times V_2$ for $H \in \mathcal{H}_1$ and $V_1 \times H'$ for $H' \in \mathcal{H}_2$. If $\mathcal{H}_1$ is the empty arrangement, then we write $V_1 \times \mathcal{H}_2$ instead of the confusing $\varnothing \times \mathcal{H}_2$. Similarly, if $\mathcal{H}_2$ is the empty arrangement, we write $\mathcal{H}_1 \times V_2$. If the arrangements $\mathcal{H}_1$ and $\mathcal{H}_2$ arise from normalized pseudo-root systems $\Phi_1 \subset V_1$ and $\Phi_2 \subset V_2$, then their product $\mathcal{H}_1 \times \mathcal{H}_2$ arises from the normalized pseudo-root system $\Phi_1 \times \{0\} \cup \{0\} \times \Phi_2 \subset V_1 \times V_2$. We also denote this pseudo-root system by $\Phi_1 \times \Phi_2$.

The notion of 2-structures was introduced by Herb for root systems to study the characters of discrete series representations; see, for example, the review article [Her00]. The definition we give here is Definition B.2.1 of [EMR22a]. It has been slightly adapted to work for pseudo-root systems.

Definition 2.1. Let Φ be a normalized pseudo-root system with Coxeter group W. A 2-*structure* for Φ is a subset φ of Φ satisfying the following properties:

(a) The subset φ is a disjoint union $\varphi = \varphi_1 \sqcup \varphi_2 \sqcup \cdots \sqcup \varphi_r$, where the φ_i are pairwise orthogonal subsets of φ and each of them is an irreducible pseudo-root system of type A_1, B_2 or $I_2(2^k)$ for $k \geq 3$.
(b) Let $\varphi^+ = \varphi \cap \Phi^+$. If $w \in W$ is such that $w(\varphi^+) = \varphi^+$ then $\det(w) = 1$.

We denote by $\mathcal{T}(\Phi)$ the set of 2-structures for Φ.

Proposition 2.2. *Let Φ be a normalized pseudo-root system with Coxeter group W.*

(i) *The group W acts transitively on the set of 2-structures $\mathcal{T}(\Phi)$.*
(ii) *The pseudo-root system Φ and its 2-structures have the same rank if and only if there exists $w \in W$ whose restriction to $\mathrm{Span}(\Phi)$ is equal to $-\mathrm{id}_{\mathrm{Span}(\Phi)}$.*

Proof. (i) See the start of Section 4 of [Her00] and Proposition B.2.4 of [EMR22a].
(ii) For Φ arising from a root system Φ', these two conditions are equivalent to the fact that Φ' is spanned by strongly orthogonal roots; see, for example, the top of page 2559 of [Her01]. For general pseudo-root systems, see the classification of 2-structures in Section B.4 of [EMR22a]. ⊛

To each 2-structure $\varphi \subset \Phi$, we can associate a sign $\epsilon(\varphi) = \epsilon(\varphi, \Phi^+)$ (see the start of Section 5 and Lemma 5.1 of [Her01] and Definition B.2.8 of [EMR22a]).

We next introduce the abstract pizza quantity. Let $\mathcal{H}$ be a central hyperplane arrangement on V. Let $\mathcal{C}_\mathcal{H}(V)$ be the set of closed convex polyhedral cones in V that are intersections of closed half-spaces bounded by hyperplanes H where $H \in \mathcal{H}$, and let $K_\mathcal{H}(V)$ be the quotient of the free abelian group $\bigoplus_{K \in \mathcal{C}_\mathcal{H}(V)} \mathbb{Z}[K]$ on $\mathcal{C}_\mathcal{H}(V)$ by the relations $[K] + [K'] = [K \cup K'] + [K \cap K']$ for all $K, K' \in \mathcal{C}_\mathcal{H}(V)$ such that $K \cup K' \in \mathcal{C}_\mathcal{H}(V)$. For $K \in \mathcal{C}_\mathcal{H}(V)$, we still denote the image of K in $K_\mathcal{H}(V)$ by $[K]$.

Definition 2.3. Suppose that we have fixed a base chamber of $\mathcal{H}$. The *abstract pizza quantity* of $\mathcal{H}$ is

$$P(\mathcal{H}) = \sum_{T \in \mathscr{T}(\mathcal{H})} (-1)^T [\overline{T}] \in K_\mathcal{H}(V).$$

Remark 2.4. By Lemma 3.2.3 of [EMR22a], we have

$$P(\mathcal{H}) = \sum_{T \in \mathscr{T}(\mathcal{H})} (-1)^T [T].$$

We use this alternative definition of $P(\mathcal{H})$ in our proofs.

The following result is Corollary 3.2.4 of [EMR22a]. It shows how to evaluate the pizza quantity for a Coxeter arrangement in terms of the associated 2-structures.

Theorem 2.5. *Let $\Phi \subset V$ be a normalized pseudo-root system. Choose a positive system $\Phi^+ \subset \Phi$ and let $\mathcal{H}$ be the hyperplane arrangement $(H_\alpha)_{\alpha \in \Phi^+}$ on V with base chamber corresponding to Φ^+. For every 2-structure $\varphi \in \mathcal{T}(\Phi)$, we write $\varphi^+ = \varphi \cap \Phi^+$ and we denote by $\mathcal{H}_\varphi$ the hyperplane arrangement $(H_\alpha)_{\alpha \in \varphi^+}$ with base chamber corresponding to φ^+. Then we have*

$$P(\mathcal{H}) = \sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(\varphi) P(\mathcal{H}_\varphi).$$

If $\varphi \in \mathcal{T}(\Phi)$ then the closures of the chambers of $\mathcal{H}_\varphi$ are elements of $\mathcal{C}_\mathcal{H}(V)$, so $P(\mathcal{H}_\varphi)$ makes sense as an element of $K_\mathcal{H}(V)$.

3. A dissection proof of the higher-dimensional pizza theorem

Definition 3.1. Let $\mathcal{C}(V)$ be a nonempty family of subsets of V that is stable by finite intersections and affine isometries and such that, if $C \in \mathcal{C}(V)$ and D is a closed affine half-space of V, then $C \cap D \in \mathcal{C}(V)$. Furthermore, we assume that $\mathcal{C}(V)$ is closed with respect to Cartesian products, that is, if $C_i \in \mathcal{C}(V_i)$ for $i = 0, 1$ then $C_0 \times C_1 \in \mathcal{C}(V_0 \times V_1)$. For example, we could take $\mathcal{C}(V)$ to be the family of all convex subsets of V, or of all closed (or compact) convex subsets, or of all convex polyhedra.

We denote by $K(V)$ the quotient of the free abelian group $\bigoplus_{C \in \mathcal{C}(V)} \mathbb{Z}[C]$ on $\mathcal{C}(V)$ by the relations:

- $[\varnothing] = 0$;
- $[C \cup C'] + [C \cap C'] = [C] + [C']$ for all $C, C' \in \mathcal{C}(V)$ such that $C \cup C' \in \mathcal{C}(V)$;
- $[g(C)] = [C]$, for every $C \in \mathcal{C}(V)$ and every affine isometry g of V.

For $C \in \mathcal{C}(V)$, we still denote the image of C in $K(V)$ by $[C]$.

Definition 3.2. A *valuation* on $\mathcal{C}(V)$ with values in an abelian group A is a function $\mathcal{C}(V) \longrightarrow A$ that can be extended to a morphism of groups $K(V) \longrightarrow A$.

Remark 3.3. Define $\mathcal{B}(V)$ to be the relative Boolean algebra generated by $\mathcal{C}(V)$, that is, the smallest collection of subsets of V that contains $\mathcal{C}(V)$ and is closed under finite unions, finite intersections and set differences. Groemer's Integral Theorem states that a valuation on $\mathcal{C}(V)$ can be extended to a valuation on the Boolean algebra $B(V)$; see [Gro78] and also [KR97, Chapter 2]. Applying this to the valuation $C \longmapsto [C]$ with values in $K(V)$, we see that we can make sense of $[C]$ for any $C \in \mathcal{B}(V)$. For instance, we have $[C_1 \cup C_2] = [C_1] + [C_2] - [C_1 \cap C_2]$ and $[C_1 - C_2] = [C_1] - [C_1 \cap C_2]$. Moreover, if $\mathcal{C}(V)$ is the set of all convex polyhedra in V, then $\mathcal{B}(V)$ contains all polyhedra (convex or not), and also half-open polyhedra.

Next we have the following straightforward lemma, whose proof we omit, which states that the class symbol is well-behaved with respect to Cartesian products.

Lemma 3.4. *The two class identities $[C_0] = [D_0]$ and $[C_1] = [D_1]$ in $K(V_0)$ and $K(V_1)$, respectively, imply that $[C_0 \times C_1] = [D_0 \times D_1]$ in $K(V_0) \times K(V_1)$.*

Let $\mathcal{H}$ be a central hyperplane arrangement on V with fixed base chamber. If $K \in \mathcal{C}(V)$, we have a morphism of groups $e_K : K_{\mathcal{H}}(V) \longrightarrow K(V)$ induced by the map $\mathcal{C}_{\mathcal{H}}(V) \longrightarrow \mathcal{C}(V)$, $C \longmapsto C \cap K$.

We denote by $P(\mathcal{H}, K)$ the image of $P(\mathcal{H})$ by this morphism e_K; in other words, we have

$$P(\mathcal{H}, K) = \sum_{T \in \mathscr{T}(\mathcal{H})} (-1)^T [\overline{T} \cap K].$$

By Remark 2.4, we also have

$$P(\mathcal{H}, K) = \sum_{T \in \mathscr{T}(\mathcal{H})} (-1)^T [T \cap K].$$

We state the main theorem of this paper. First for $u, v \in V$ define the half-open line segment $(u, v]$ by $\{(1-\lambda)u + \lambda v : 0 < \lambda \leq 1\}$. Our main result is the following:

Theorem 3.5 (The Abstract Pizza Theorem)**.** *Let $\mathcal{H}$ be a Coxeter hyperplane arrangement with Coxeter group W in an n-dimensional space V such that $-\mathrm{id}_V \in W$. Let $K \in \mathcal{C}(V)$ and $a \in V$. Suppose that K is stable by the group W and contains the convex hull of the set $\{w(a) : w \in W\}$.*

(i) If $\mathcal{H}$ is not of type A_1^n, we have $P(\mathcal{H}, K + a) = 0$ in $K(V)$.

(ii) If $\mathcal{H}$ has type A_1^n, Φ is the normalized pseudo-root system corresponding to $\mathcal{H}$ and $\Phi^+ = \{e_1, \ldots, e_n\}$ where $\Phi^+ \subset \Phi$ is the positive system corresponding to the base chamber of $\mathcal{H}$, then the following identity holds:

$$P(\mathcal{H}, K + a) = \left[\prod_{i=1}^{n} (0, 2(a, e_i)e_i] \right]. \tag{3.1}$$

Here we are using Remark 3.3 to make sense of the right-hand side of equation (3.1).

The conditions on K are satisfied if for example K is convex, contained in $\mathcal{C}(V)$, stable by W and $0 \in K + a$. Indeed, the last condition implies that $-a \in K$; as $-\mathrm{id}_V \in W$ by assumption, this in turns implies that $a \in K$, hence that K contains the convex hull of the set $\{w(a) : w \in W\}$.

We will give the proof of Theorem 3.5 at the end of the section. This proof does not use Theorem 1.2 of [EMR22b], so we obtain a new proof of that result.

Let $V_0, \ldots, V_n$ denote the intrinsic volumes on V (see [Sch14, Section 4.2]).

Lemma 3.6. *Let $(v_1, \ldots, v_k)$ be an orthogonal family of vectors in V. Then*

$$V_i((0, v_1] \times \ldots \times (0, v_k]) = 0$$

for $0 \leq i \leq k - 1$.

Proof. By Lemma 14.2.1 of [SW08] or Proposition 4.2.3 of [KR97], it suffices to prove that, if $a < b$ are real numbers, the 0th intrinsic volume of the half-open segment $(a, b] \subset \mathbb{R}$ is 0. As the 0th intrinsic volume is the Euler-Poincaré characteristic with compact support, this is clear. ⊛

Corollary 3.7. *We keep the notation and hypotheses of Theorem 3.5. If $\mathcal{H}$ is not of type A_1^n, we have*

$$\sum_{T \in \mathscr{T}(\mathcal{H})} (-1)^T V_i(T \cap (K + a)) = 0 \tag{3.2}$$

for every $0 \leq i \leq n$, where K is assumed to be convex if $i \neq n$. If $\mathcal{H}$ has type A_1^n and K is convex then equation (3.2) holds for $0 \leq i \leq n - 1$.

Proof. If $\mathcal{H}$ is not of type A_1^n, then equation (3.2) actually holds for any valuation on $\mathcal{C}(V)$ that is invariant under the group of affine isometries; this includes the intrinsic volumes.

Suppose that $\mathcal{H}$ is of type A_1^n. Then we know that equation (3.1) holds. The result then follows from Lemma 3.6. ⊛

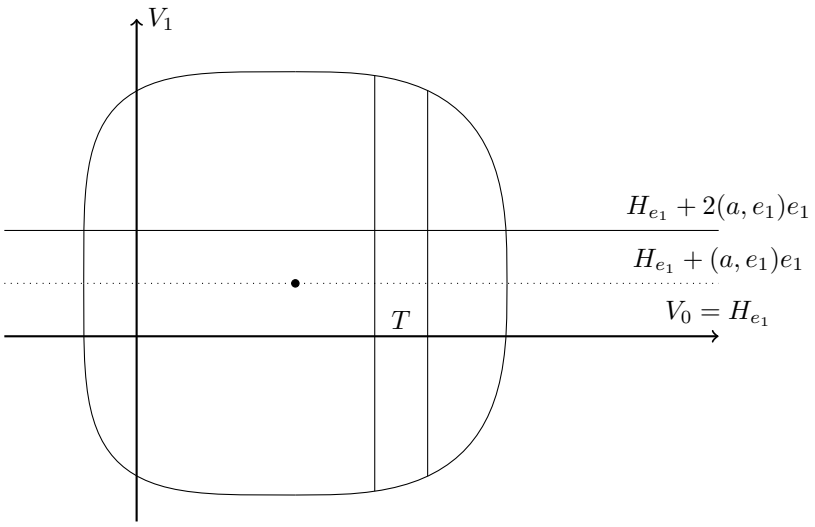

FIGURE 1. A schematic sketch of $V_0 \times V_1$ for the proof of Lemma 3.9.

Remark 3.8. Theorem 3.5 immediately implies generalizations to our higher-dimensional case of the "thin crust" and "thick crust" results of Confection 3 and Leftovers 1 of [MD09] for an even number of cuts.

We obtain the "thin crust" result by evaluating the $(n-1)$st intrinsic volume on $P(\mathcal{H}, K + a)$. Note that this result holds for a pizza of any (convex) shape and even in the case where we only make n cuts, where n is the dimension.

To generalize the "thick crust" result, consider two sets $K \subset L$ stable by W and in $\mathcal{C}(V)$. If $a \in V$ is such that K contains the convex hull of the set $\{w(a) : w \in W\}$, then

$$P(\mathcal{H}, (L - K) + a) = P(\mathcal{H}, L + a) - P(\mathcal{H}, K + a) = 0,$$

so in particular

$$\sum_{T \in \mathscr{T}(\mathcal{H})} (-1)^T \operatorname{Vol}(T \cap ((L - K) + a)) = 0.$$

The case where K and L are balls with the same center is the "thick crust" result.

We now state and prove some lemmas that will be used in the proof of Theorem 3.5.

Lemma 3.9. *Let $\mathcal{H}_i$ be a hyperplane arrangement on V_i for $i = 0, 1$. Assume furthermore that $\mathcal{H}_1 = \{H_e\}_{e \in E_1}$ has type A_1^r and $\dim(V_1) = r$. Let $E_1 = \{e_1, \ldots, e_r\}$ be the index set of $\mathcal{H}_1$. Let $\mathcal{H}$ and V be the Cartesian products $\mathcal{H}_0 \times \mathcal{H}_1$ and $V_0 \times V_1$, respectively. Then for every $K \in \mathcal{C}(V)$ that is stable under the orthogonal reflections in the hyperplanes $V_0 \times H_{e_1}, \ldots, V_0 \times H_{e_r}$ and for every $a \in V_1$, if $L = K + a$, we have the identity*

$$P(\mathcal{H}, L) = P\left(\mathcal{H}_0 \times V_1, L \cap (V_0 \times (0, 2(a, e_1)e_1] \times \cdots \times (0, 2(a, e_r)e_r])\right),$$

where $\mathcal{H}_0 \times V_1$ is the product of $\mathcal{H}_0$ and the empty hyperplane arrangement on V_1.

Proof. By a straightforward induction, we may assume that $r = \dim V_1 = 1$. Also, after changing the sign of e_1, we may assume that $(a, e_1) \geq 0$. See Figure 1 for a sketch of the situation. Let T be a chamber of the arrangement $\mathcal{H}_0$. The classes of the two regions $(T \times \mathbb{R}_{>0}e_1) \cap L$ and $(T \times \mathbb{R}_{<0}e_1) \cap L$ of $\mathcal{H}$ occur with opposite signs in the pizza quantity $P(\mathcal{H}, L)$. Note that the region $(T \times \mathbb{R}_{>2(a,e_1)}e_1) \cap L$ is the orthogonal reflection of the region $(T \times \mathbb{R}_{<0}e_1) \cap L$ in the affine hyperplane $H_{e_1} + (a, e_1)e_1 = H_{e_1} + a$. Hence these regions have the same class in $K(V)$ which cancels in the pizza quantity

$P(\mathcal{H}, L)$, and the class of the region $(T \times (0, 2(a, e_1)e_1]) \cap L = (T \times V_1) \cap L \cap (V_0 \times (0, 2(a, e_1)e_1])$ remains. As the map $T \longmapsto T \times V_1$ is a sign-preserving bijection from $\mathscr{T}(\mathcal{H}_0)$ to $\mathscr{T}(\mathcal{H}_0 \times V_1)$, this completes the proof. ⊛

We now consider the case of a hyperplane arrangement that is the product of a 2-dimensional dihedral arrangement and another arrangement. Suppose that $V = V_0 \times V_1$, where the factors are orthogonal, and that $\mathcal{H} = \mathcal{H}_0 \times \mathcal{H}_1$, where $\mathcal{H}_i$ is a hyperplane arrangement in V_i. Suppose also that $\dim V_0 = 2$ and that $\mathcal{H}_0$ is an arrangement of type $I_2(2m)$ with Coxeter group W_0 where $m \geq 2$. We view W_0 as a group of isometries of V by making $w \in W_0$ act on $V = V_0 \times V_1$ by $w \times \mathrm{id}_{V_1}$. We also choose a family $\mathcal{C}(V)$ as in Definition 3.1.

Let $a \in V_0$. We will describe a dissection of V_0. The case where $m = 4$ is shown in Figure 2. We call $L_0, \ldots, L_{2m-1}$ the lines of $\mathcal{H}_0$ (numbered so that the angle between L_0 and L_i is an increasing function of i) and we assume that the point a is in a chamber between L_{m-1} and L_m. Choose a closed half-space D bounded by L_0 and containing a (this choice is unique if $a \notin L_0$). Then, for $0 \leq i \leq 2m-1$, we denote by T_i the unique chamber of $\mathcal{H}$ contained in D and with boundary contained in $L_i \cup L_{i+1}$. We assume that $(-1)^{T_0} = 1$ for concreteness. The point a is in the closure of the chamber T_{m-1}.

We write $\mathscr{T}_+ = \{T \in \mathscr{T}(\mathcal{H}_0) : (-1)^T = 1\}$ and $\mathscr{T}_- = \{T \in \mathscr{T}(\mathcal{H}_0) : (-1)^T = -1\}$. Let W_a be the group of affine isometries generated by the orthogonal reflections in the lines $L + a$, for $L \in \mathcal{H}_0$. We take $R_0(a)$ to be the convex hull of the points $w(0)$ for $w \in W_a$. This is the shaded polygon on Figure 2, where the darker slices are the intersections with the closures of chambers in $\mathscr{T}_+$. We have the inclusion $R_0(a) \subset \bigcup_{i=0}^{2m-2} \overline{T}_i$. Finally we set

$$R_{0,\pm}(a) = R_0(a) \cap \bigcup_{T \in \mathscr{T}_\pm} T.$$

Lemma 3.10. *The following three identities hold in $K(V_0 \times V_1)$:*

(i) Let $K \in \mathcal{C}(V)$ such that K is stable by W_0 and let $L = K + a$. Then

$$\sum_{T \in \mathscr{T}} (-1)^T [L \cap ((T - R_0(a)) \times V_1)] = 0.$$

(ii) Let $K \in \mathcal{C}(V)$ such that K is stable by W_0 and let $L = K + a$ then

$$P(\mathcal{H}, L) = P(V_0 \times \mathcal{H}_1, L \cap (R_{0,+}(a) \times V_1)) - P(V_0 \times \mathcal{H}_1, L \cap (R_{0,-}(a) \times V_1)).$$

(iii) If $K_1 \subset V_1$ is such that $R_0(a) \times K_1 \in \mathcal{C}(V)$, then

$$[R_{0,+}(a) \times K_1] = [R_{0,-}(a) \times K_1].$$

Proof. We begin by proving (i). For $0 \leq i \leq 2m-1$, we denote by $R_{i,\pm}$ the unique chamber of $\mathcal{H}$ not contained in D and with boundary contained in $L_i \cup L_{i+1}$, that is, the image of T_i by the symmetry with center 0; we write $R_{i,+}$ if this chamber has sign $+1$, or equivalently if i is even, and $R_{i,-}$ if this chamber has sign -1, or equivalently if i is odd. For every $0 \leq j \leq m-1$, we denote by $R_{2j,-}$ and $R_{2j+1,+}$ the orthogonal reflection in the line $L_{2j+1}^{\perp} + a$ of $R_{2j,+}$ and $R_{2j+1,-}$, respectively. Note that $R_{2j,-} \subset T_{2j+1}$ and $R_{2j+1,+} \subset T_{2j}$. For $0 \leq j \leq m-1$ again, we denote by $S_{j,+}$ the interior of $T_{2j} - (R_0(a) \cup R_{2j+1,+})$ and by $S_{j,-}$ the interior of $T_{2j+1} - (R_0(a) \cup R_{2j,-})$. Then $T_{2j} - R_0(a)$ is the disjoint union of $R_{2j+1,+}$, $S_{j,+}$ and an open ray D_{2j} starting at an extremal point of $R_0(a)$ (the image of 0 by the orthogonal reflection in the line $L_{2j+1}^{\perp} + a$) and parallel to L_{2j}. Similarly $T_{2j+1} - R_0(a)$ is the disjoint union of $R_{2j,-}$, $S_{j,-}$ and an open ray D_{2j+1} starting at the same extremal point of $R_0(a)$ and parallel to L_{2j+2}. See Figure 2 for the case $m = 4$, where the rays D_{2j} and D_{2j+1} are dashed.

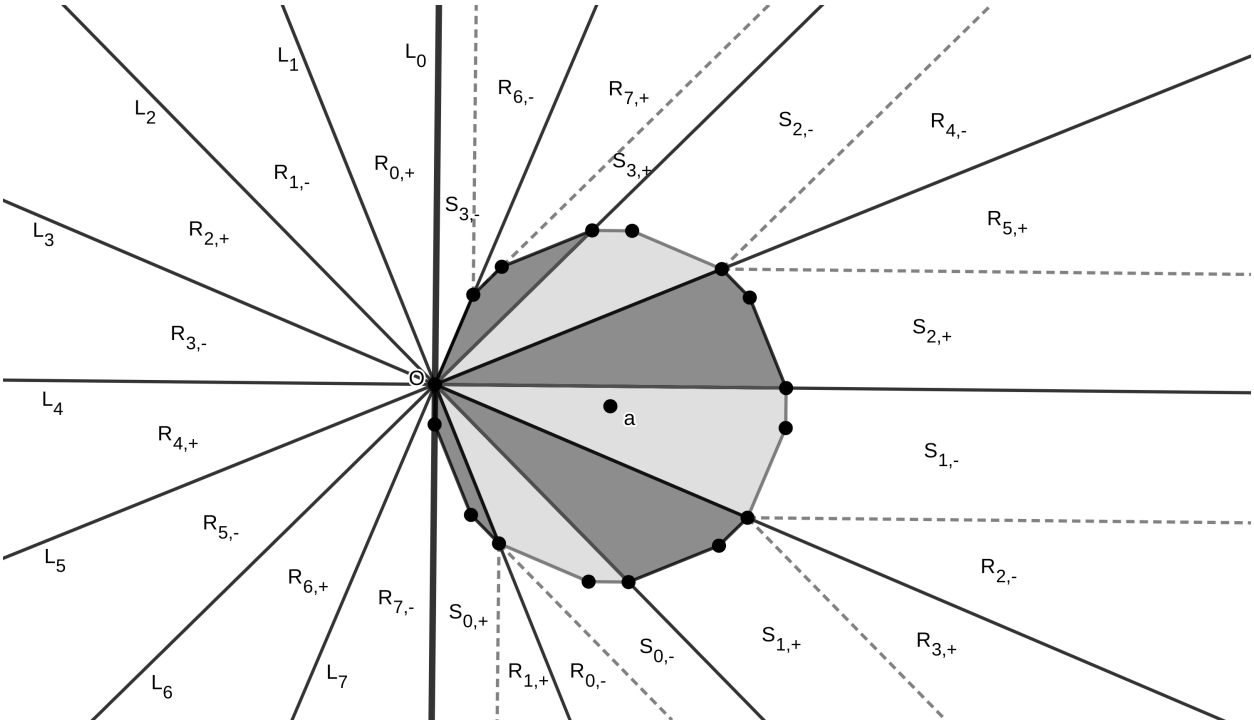

FIGURE 2. A picture of the regions in the proof of Lemma 3.10(i) for the case of $I_2(8)$.

The union $\bigcup_{T\in\mathscr{T}_+} T$ is equal to the disjoint union of the set $R_{0,+}(a)$, the regions $R_{i,+}$ for $0 \le i \le 2m-1$, the regions $S_{j,+}$ for $0 \le j \le m-1$ and the rays D_{2j} for $0 \le j \le m-1$. On the other hand, the union $\bigcup_{T\in\mathscr{T}_-} T$ is equal to the disjoint union of the set $R_{0,-}(a)$, the regions $R_{i,-}$ for $0 \le i \le 2m-1$, the regions $S_{j,-}$ for $0 \le j \le m-1$ and the rays D_{2j+1} for $0 \le j \le m-1$. Consider the following four observations:

- For $0 \le i \le 2m-1$ the region $R_{i,-}$ is the image of $R_{i,+}$ by the orthogonal reflection in the affine line $L^{\perp}_{2\lfloor i/2\rfloor+1} + a$.
- For $0 \le j \le m-1$ the region $S_{j,-}$ is the image of $S_{j,+}$ by the rotation with center a and angle π/m.
- For $0 \le j \le m-2$ the ray D_{2j+3} is the image of the ray D_{2j} by the rotation with center a and angle $2\pi/m$.
- The ray D_{2m-2} is the image of D_1 by the orthogonal reflection in the affine line $L_m + a$.

Each of them is of the form: the set X is the image of the set Y under an affine isometry g belonging to the group W_a. Since the set $L = K + a$ is invariant under g, we obtain that the set $L \cap (X \times V_1)$ is the image of $L \cap (Y \times V_1)$, and hence that $[L \cap (X \times V_1)] = [L \cap (Y \times V_1)]$. Statement (i) follows by summing over all pairs of sets X and Y.

Next we prove (ii). There is a bijection $\mathscr{T}(\mathcal{H}_0) \times \mathscr{T}(\mathcal{H}_1) \xrightarrow{\sim} \mathscr{T}(\mathcal{H})$ where $(T, T') \longmapsto T \times T'$ and $(-1)^{T\times T'} = (-1)^T(-1)^{T'}$ for all $T \in \mathscr{T}(\mathcal{H}_0)$ and $T' \in \mathscr{T}(\mathcal{H}_1)$. Hence

$$P(\mathcal{H}, L) = \sum_{T'\in\mathscr{T}(\mathcal{H}_1)} \sum_{T\in\mathscr{T}(\mathcal{H}_0)} (-1)^T(-1)^{T'}[L \cap (T \times T')]. \tag{3.3}$$

Fix $T' \in \mathscr{T}(\mathcal{H}_1)$ for a moment. The fact that K is stable by W_0 implies that $K \cap (V_0 \times T')$ is also stable by W_0. Hence applying statement (i) to the set $(K \cap (V_0 \times T')) + a = L \cap (V_0 \times T')$ yields

$$\sum_{T \in \mathscr{T}(\mathcal{H}_0)} (-1)^T [L \cap ((T - R_0(a)) \times T')] = 0. \tag{3.4}$$

Multiplying equation (3.4) with the sign $(-1)^{T'}$, summing over all $T' \in \mathscr{T}(\mathcal{H}_1)$, and subtracting the result from equation (3.3) yields

$$\begin{aligned} P(\mathcal{H}, L) &= \sum_{T' \in \mathscr{T}(\mathcal{H}_1)} \sum_{T \in \mathscr{T}(\mathcal{H}_0)} (-1)^T (-1)^{T'} [L \cap ((T \cap R_0(a)) \times T')] \\ &= \sum_{T' \in \mathscr{T}(\mathcal{H}_1)} (-1)^{T'} [L \cap (R_{0,+}(a) \times T')] - \sum_{T' \in \mathscr{T}(\mathcal{H}_1)} (-1)^{T'} [L \cap (R_{0,-}(a) \times T')] \\ &= P(V_0 \times \mathcal{H}_1, L \cap (R_{0,+}(a) \times V_1)) - P(V_0 \times \mathcal{H}_1, L \cap (R_{0,-}(a) \times V_1)). \end{aligned}$$

Finally we consider (iii). By Lemma 3.4, it suffices to show that, if we take $\mathcal{C}(V_0)$ to be the set of convex polygons in V_0, then $[R_{0,+}(a)] = [R_{0,-}(a)]$ in $K(V_0)$. This follows from Corollary 4.2 and from the fact that the intrinsic volumes of $R_{0,+}(a)$ and $R_{0,-}(a)$ are equal (which is an easy calculation), but we also give a direct proof. We consider the following dissection of the polygon $R_0(a)$; see Figure 3 for the case $m = 4$. For $0 \leq i \leq 2m-1$ let P_i be the image of 0 by the orthogonal reflection in the line $L_i^{\perp} + a$; note that P_i is a boundary point of $R_0(a)$, and that it is on L_i. We describe the pieces of the dissection of $R_0(a)$:

- For $1 \leq i \leq m-2$ consider the pair of isosceles triangles $B_{i,+}$ and $B_{i,-}$ that have one side equal to the segment $[0, P_{2i}]$, angles equal to $\pi/2m$ at the vertices 0 and P_i, and such that $B_{i,\pm}$ is in a chamber with sign ± 1; in other words, the triangle $B_{i,-}$ is in the chamber T_{2i-1}, and $B_{i,+}$ is in the chamber T_{2i}.
- Let $B_{0,+}$ be the isosceles triangle contained in T_0 with one side equal to the segment $[0, P_0]$ and angles equal to $\pi/2m$ at the vertices 0 and P_0.
- Consider the isosceles triangle contained in T_{2m-3} with one side equal to the segment $[0, P_{2m-2}]$ and angles equal to $\pi/2m$ at the vertices 0 and P_{2m-2}; this splits into an isosceles triangle $B_{0,-}$ congruent to $B_{0,+}$ and an isosceles trapezoid $B_{m-1,-}$ having one edge equal to $[0, P_{2m-2}]$.
- Let $B_{m-1,+}$ be the image of $B_{m-1,-}$ by the orthogonal reflection in the line L_{2m-2}; then $B_{m-1,+}$ is contained in the chamber T_{2m-2}.

To finish the dissection of $R_0(a)$, we note that, for $0 \leq i \leq 2m-3$, we still have a quadrilateral piece Q_i left over in $T_i \cap R_0(a)$. Then for $1 \leq i \leq 2m-3$ the quadrilateral Q_i is the image of Q_{i-1} by the rotation with center a and angle π/m. Indeed, this is true for the intersections of these quadrilaterals with the boundary of $R_0(a)$ (which consist of two edges with endpoints P_{i-1} and P_{i+1}), and it is easy to calculate the angles at the vertices of this intersection and to see that they correspond: If $1 \leq i \leq 2m-2$ and i is even, respectively odd, the angle of Q_{i-1} at P_i is $(i-1)\pi/2m$, respectively $i\pi/2m$. If $0 \leq i \leq 2m-3$ and i is even, respectively odd, the angle of Q_i at P_i is $(2m-2-i)\pi/2m$, respectively $(2m-1-i)\pi/2m$.

This is Frederickson's dissection; see pages 28–31 of [Fre12]. That paper is only interested in giving a dissection proof that the alternating sum of the areas is equal to zero. Hence it can safely ignore line segments of area zero, whereas we are proving an identity in $K(V_0)$ and have to be careful with all regions, including lower dimensional ones.

The set $R_{0,+}(a)$ is the disjoint union of the following subsets:

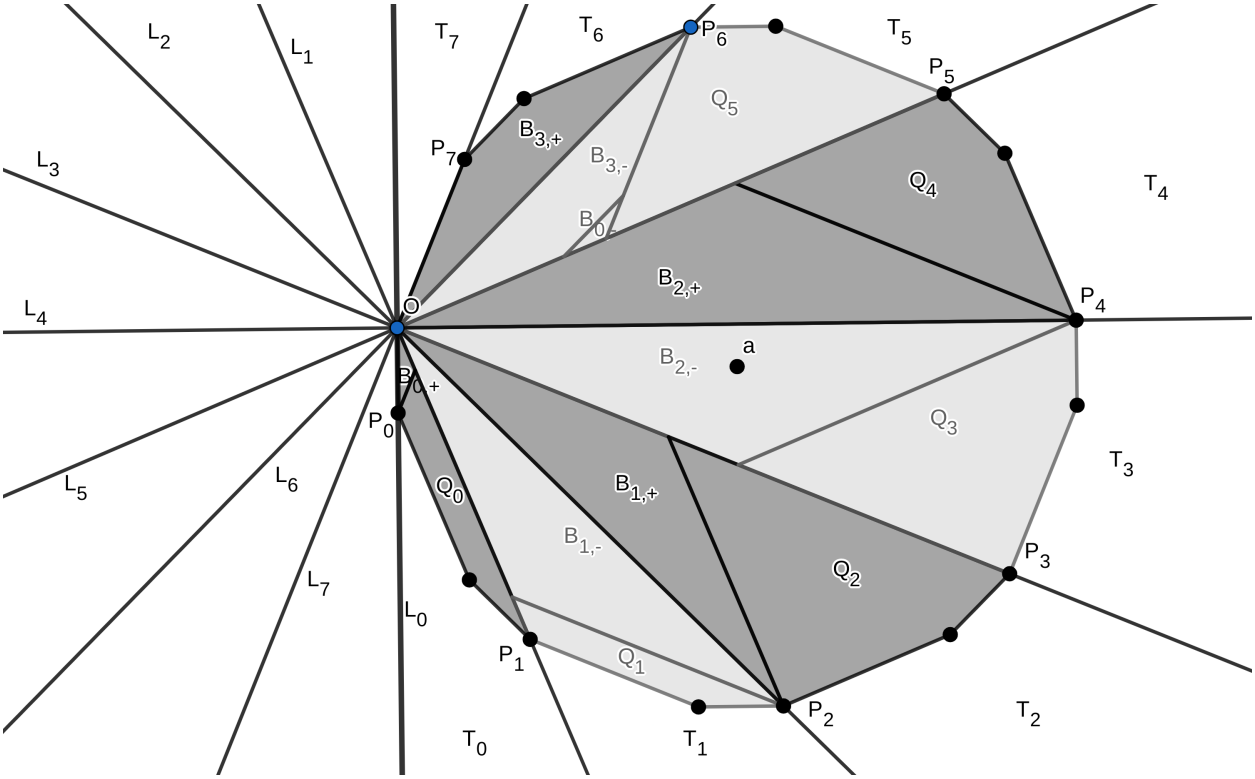

FIGURE 3. The dissection of the polygon $R_0(a)$ in the case of $I_2(8)$. The darker shaded regions are the intersections with the chambers which have a positive sign.

- For $0 \leq i \leq m-2$ let $B'_{i,+}$ be the intersection of the triangle $B_{i,+}$ with the interior of the chamber containing $B_{i,+}$. In other words, $B'_{i,+}$ is the union of the interior of $B_{i,+}$ and the relative interior of one of its two equal sides.
- Let $B'_{m-1,+}$ be the intersection of the trapezoid $B_{m-1,+}$ and T_{2m-2}. That is, $B'_{m-1,+}$ is the union of the interior of $B_{m-1,+}$ and the intersection of its boundary with the boundary of $R_0(a)$, minus the two extremal points of this intersection.
- For $0 \leq j \leq m-2$ let Q'_{2j} be the intersection of the quadrilateral Q_{2j} and T_{2j}. That is, Q'_{2j} is the union of the interior of Q_{2j+1} and the intersection of its boundary with the boundary of $R_0(a)$, minus the two extremal points of this intersection.

As for the set $R_{0,-}(a)$, it is the disjoint union of the following subsets:

- Let $B'_{0,-}$ be the union of the interior of the triangle $B_{0,-}$ and the relative interior of the side that it shares with Q_{2m-3}.
- For $1 \leq i \leq m-2$ let $B'_{i,-}$ be the intersection of the triangle $B_{i,-}$ with the interior of the chamber containing $B'_{i,-}$. That is, $B'_{i,-}$ is the union of the interior of $B_{i,-}$ and the relative interior of one of its two equal sides.
- Let $B'_{m-1,-}$ be the intersection of the trapezoid $B_{m-1,-}$ and T_{2m-3}.
- For $0 \leq j \leq m-2$ let Q'_{2j+1} be the intersection of the quadrilateral Q_{2j+1} and T_{2j+1}. That is, Q'_{2j+1} is the union of the interior of Q_{2j+1} and the intersection of its boundary with the boundary of $R_0(a)$, minus the two extremal points of this intersection.

We obtain that $[R_{0,+}(a)] = [R_{0,-}(a)]$ since the regions $B'_{i,+}$ and $B'_{i,-}$ are isometric for every $0 \leq i \leq m-1$, as are the regions Q_{2j} and Q_{2j+1} for $0 \leq j \leq m-2$. ⊛

Lemma 3.11. *Suppose that we have $V = V_1^{(1)} \times \cdots \times V_1^{(r)} \times V_2^{(1)} \times \cdots \times V_2^{(s)}$, where the factors of the product are pairwise orthogonal, and that $\mathcal{H}$ is a product $\mathcal{H}_1^{(1)} \times \cdots \times \mathcal{H}_1^{(r)} \times \mathcal{H}_2^{(1)} \times \cdots \times \mathcal{H}_2^{(s)}$, where each $\mathcal{H}_i^{(j)}$ is a hyperplane arrangement on $V_i^{(j)}$. Suppose further that:*

(a) *If $1 \leq j \leq r$ then $V_1^{(j)}$ is 1-dimensional, and we have a unit vector $e^{(j)}$ in $V_1^{(j)}$ yielding the hyperplane arrangement $\mathcal{H}_1^{(j)} = \{0\}$.*

(b) *If $1 \leq j \leq s$ then $V_2^{(j)}$ is 2-dimensional, and the arrangement $\mathcal{H}_2^{(j)}$ is of type $I_2(2m^{(j)})$ for some $m^{(j)} \geq 2$.*

Let $a \in V$ and $K \in \mathcal{C}(V)$. Suppose that K is stable by the Coxeter group W and contains the convex hull of the set $\{w(a) : w \in W\}$. Then the following two statements hold:

(i) *If $s \geq 1$ we have $P(\mathcal{H}, K + a) = 0$ in $K(V)$.*

(ii) *If $s = 0$ we have in $K(V)$ the identity*

$$P(\mathcal{H}, K + a) = \left[(0, 2(a, e^{(1)})e^{(1)}] \times \cdots \times (0, 2(a, e^{(r)})e^{(r)}]\right].$$

Proof. Let $L = K + a$. Since $-\mathrm{id}_V \in W$ we have $-a \in K$ and so $0 \in L + a$; also, the sets $\{w(a) : w \in W\}$ and $\{w(-a) : w \in W\}$ are equal. Let W_a be the group of affine isometries of V generated by the orthogonal reflections in the hyperplanes $a + H$, for $H \in \mathcal{H}$. The conditions on K imply that L is stable by W_a and contains the convex hull of the set $\{u(0) : u \in W_a\} = \{w(-a) + a : w \in W\}$.

Write $a = (a_1^{(1)}, \ldots, a_1^{(r)}, a_2^{(1)}, \ldots, a_2^{(s)})$, with $a_i^{(j)} \in V_i^{(j)}$. For $1 \leq i \leq r$ let $S^{(i)}$ denote the half-open line-segment $(0, 2(a, e^{(i)})e^{(i)}]$. For $1 \leq j \leq s$ we consider subsets $R_{+1}^{(j)} = R_{0,+}(a_2^{(j)})$ and $R_{-1}^{(j)} = R_{0,-}(a_2^{(j)})$ of $V_2^{(j)}$ as in Lemma 3.10. By Lemmas 3.9 and 3.10(ii), we have that

$$P(\mathcal{H}, L) = \sum_{(\epsilon_1, \ldots, \epsilon_s) \in \{\pm 1\}^s} \epsilon_1 \cdots \epsilon_s \left[L \cap \left(S^{(1)} \times \cdots \times S^{(r)} \times R_{\epsilon_1}^{(1)} \times \cdots \times R_{\epsilon_s}^{(s)}\right)\right].$$

Consider the polyhedron

$$P = [0, 2(a, e^{(1)})e^{(1)}] \times \cdots \times [0, 2(a, e^{(r)})e^{(r)}] \times R_0(a_2^{(1)}) \times \cdots \times R_0(a_2^{(s)}).$$

Then P is the convex hull of the set $\{u(0) : u \in W_a\}$ by definition of the polygons $R_0(a_2^{(j)})$, hence it is contained in L and so is its subset $S^{(1)} \times \cdots \times S^{(r)} \times R_{\epsilon_1}^{(1)} \times \cdots \times R_{\epsilon_s}^{(s)}$ for every $(\epsilon_1, \ldots, \epsilon_s) \in \{\pm 1\}^s$. So we obtain

$$P(\mathcal{H}, L) = \sum_{(\epsilon_1, \ldots, \epsilon_s) \in \{\pm 1\}^s} \epsilon_1 \cdots \epsilon_s \left[S^{(1)} \times \cdots \times S^{(r)} \times R_{\epsilon_1}^{(1)} \times \cdots \times R_{\epsilon_s}^{(s)}\right].$$

If $s = 0$ this implies statement (ii). Suppose that $s \geq 1$. By point (iii) of Lemma 3.10, we know that

$$\left[S^{(1)} \times \cdots \times S^{(r)} \times R_{\epsilon_1}^{(1)} \times \cdots \times R_{\epsilon_s}^{(s)}\right] = \left[S^{(1)} \times \cdots \times S^{(r)} \times R_{+1}^{(1)} \times \cdots \times R_{+1}^{(s)}\right]$$

for every $(\epsilon_1, \ldots, \epsilon_s) \in \{\pm 1\}^s$. As $\sum_{(\epsilon_1, \ldots, \epsilon_s) \in \{\pm 1\}^s} \epsilon_1 \cdots \epsilon_s = 0$, this finishes the proof of (i). ⊛

Proof of Theorem 3.5. Statement (ii) is exactly Lemma 3.11(ii). We now prove statement (i), so we assume that $\mathcal{H}$ is not of type A_1^n. By Theorem 2.5, we have

$$P(\mathcal{H}, K + a) = \sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(\varphi) P(\mathcal{H}_\varphi, K + a).$$

By definition, any 2-structure for Φ is of type $A_1^r \times \prod_{k \geq 2} I_2(2^k)^{s_k}$ with $\sum_{k \geq 2} s_k$ finite and, as W acts transitively on the set of 2-structures (Proposition 2.2(i)), the integers r and s_k, for $k \geq 2$, do not depend on the 2-structure but only on Φ. Also, by Proposition 2.2(ii), we have $\dim V =$

$r + \sum_{k\geq 2} 2s_k$, so we are in the situation of Lemma 3.11. Suppose that $\sum_{k\geq 2} s_k \geq 1$. Then by Lemma 3.11(i) we have $P(\mathcal{H}_\varphi, K+a) = 0$ for every $\varphi \in \mathcal{T}(\Phi)$ and hence $P(\mathcal{H}, K+a) = 0$. Assume now that $\sum_{k\geq 2} s_k = 0$, that is, $s_k = 0$ for every k. Statement (ii) of the same lemma implies that

$$P(\mathcal{H}, K+a) = \sum_{\varphi\in\mathcal{T}(\Phi)} \epsilon(\varphi) \left[\prod_{e\in\varphi\cap\Phi^+} (0, 2(a,e)e] \right]. \tag{3.5}$$

This is an alternating sum of classes of half-open rectangular parallelotopes in V. So we can apply Theorem 4.1 to prove that $P(\mathcal{H}, K+a) = 0$ in $K(V)$. We know that $V_i(P(\mathcal{H}, K+a)) = 0$ if $0 \leq i \leq n-1$ by Lemma 3.6, so it remains to prove that $V_n(P(\mathcal{H}, K+a)) = 0$, that is, that the alternating sum of the volumes of the parallelotopes $\prod_{e\in\varphi\cap\Phi^+}(0, 2(a,e)e]$ is equal to zero. This follows from Theorem 1.2 of the paper [EMR22b]. However, we now give a direct proof (that does not use analysis) using the method of that corollary. Let $f : V \longrightarrow \mathbb{R}$ be the function defined by

$$f(a) = \sum_{\varphi\in\mathcal{T}(\Phi)} \epsilon(\varphi) \prod_{e\in\varphi\cap\Phi^+} 2(a,e).$$

Note that f is a polynomial homogeneous of degree n on V. Furthermore equation (3.5) implies that

$$\mathrm{Vol}(P(\mathcal{H}, K+a)) = f(a)$$

for every convex subset K of V of finite volume that is stable by W and every $a \in V$ such that $0 \in K + a$. The polynomial f satisfies $f(w(a)) = \det(w) f(a)$ for every $w \in W$ and every $a \in V$ (this is easy to see; see, for example, Corollary 2.3 of [EMR22b]), so it vanishes on every hyperplane of $\mathcal{H}$. But if $f \neq 0$, then the vanishing set of f must be of degree at most n, which contradicts the fact that, as $\mathcal{H}$ is not of type A_1^n, we have $|\mathcal{H}| > n$. Hence we must have $f = 0$, and this gives the desired result. ⊛

Remark 3.12. In the paper [Hir+99], J., M. D., J. K., A. D. and P. M. Hirschhorn proved that if a circular pizza is cut into $4m$ slices by $2m$ cuts at equal angles to each other and if m people share the pizza by each taking every mth slice then they receive equal shares. If $m = 4$, Frederickson gives a dissection-based proof of this fact on page 32 of [Fre12], and Proposition 9.1 of [EMR22b] generalizes the result to pizzas of more general shapes. We cannot lift this result to the group $K(V)$, because it does not hold in that group. For example, if we consider the pizza of Figure 3, then it is not true in general that the sums of the perimeters of the pizza pieces in all the shares will be equal.

However, we can lift the generalization of the Hirschhorns's result to the quotient $K_0(V)$ of the group $K(V)$ by the subgroup generated by all the elements $[C]$ with $C \in \mathcal{C}(V)$ contained in a line of V. More precisely, consider a Coxeter arrangement of type $I_2(2m)$ in $\mathbb{R}^2$ with m even and let W be the Coxeter group of this arrangement. Let $T_0, \ldots, T_{4m-1}$ be the chambers of $\mathcal{H}$, indexed so that that T_i and T_{i+1} share a wall. Let $K \in \mathcal{C}(V)$ and $a \in V$. Suppose that K is stable by W and contains the convex hull of the set $\{w(a) : w \in W\}$. Then the quantity

$$\sum_{i=0}^{3} [T_{r+mi} \cap (K+a)] \in K_0(V)$$

is independent of $0 \leq r \leq m-1$. (This implies the Hirschhorns's result even in the case where k is odd: Just apply the previous statement with $m = 2k$, and share the pizza between k people by giving the pth person the eight slices contained in the chambers T_{2p+mi} and $T_{2p+1+mi}$, for $0 \leq i \leq 3$.)

Let us prove this result. Let $0 \leq r \leq m-2$. We want to show that

$$\sum_{i=0}^{3}[T_{r+im} \cap (K+a)] = \sum_{i=0}^{3}[T_{r+1+im} \cap (K+a)]$$

in $K_0(V)$. Suppose that we know that

$$\sum_{i=0}^{3}[T_{r+im} \cap R_+(a) \cap (K+a)] = \sum_{i=0}^{3}[T_{r+1+im} \cap R_-(a) \cap (K+a)] \tag{3.6}$$

in $K_0(V)$, where $R_+(a)$ and $R_-(a)$ are given by

$$R_\pm(a) = (V - R_0(a)) \cap \bigcup_{\substack{T \in \mathscr{T}(\mathcal{H}) \\ (-1)^T = \pm 1}} T.$$

Then it remains to see that

$$\sum_{i=0}^{3}[T_{r+im} \cap R_0(a) \cap (K+a)] = \sum_{i=0}^{3}[T_{r+1+im} \cap R_0(a) \cap (K+a)]$$

in that same quotient. But now all the regions appearing in the sums are polygons, so the equality of the sums of their classes in $K_0(V)$ is equivalent to the equality of the sums of their areas, by the Bolyai-Gerwien Theorem; see [Bol78, Section 5]. This last equality follows either from Proposition 9.1 of [EMR22b], or from the Hirschhorns's result. (The Hirschhorns only consider the case of a circular pizza, but, by equation (3.6), their result for a circular pizza implies the result for the polygonal pizza $R_0(a)$.)

We first suppose that r is even. To prove equation (3.6), we suppose that $T_0, \dots, T_{2m-1}$ denote the same chambers as in the proof of Lemma 3.10, and we use the notation of that proof. In particular, for $0 \leq i \leq 2m-1$, the chamber T_{2m+i} is equal to the region $R_{i,\epsilon}$, where ϵ is the sign $(-1)^i$. We have $T_r \cap R_+(a) = S_{r/2,+} \cup R_{r+1,+}$, $T_{r+m} \cap R_+(a) = S_{(r+m)/2,+} \cup R_{r+m+1,+}$, $T_{r+2m} \cap R_+(a) = T_{r+2m} = R_{r,+}$ and $T_{r+3m} \cap R_+(a) = T_{r+3m} = R_{r+m,+}$. On the other hand, we have $T_{r+1} \cap R_-(a) = S_{r/2,-} \cup R_{r,-}$, $T_{r+1+m} \cap R_-(a) = S_{(r+m)/2,-} \cup R_{r+m,-}$, $T_{r+1+2m} \cap R_-(a) = T_{r+1+2m} = R_{r+1,-}$ and $T_{r+1+3m} \cap R_-(a) = T_{r+1+3m} = R_{r+m+1,-}$. This implies equation (3.6).

We now consider the case where r is odd. We again suppose that $T_0, \dots, T_{2m-1}$ denote the same chambers as in the proof of Lemma 3.10 and the notation of that lemma, but we use a different dissection, that is illustrated in Figure 4 in the case $m = 4$. For $0 \leq i \leq 2m-2$, we consider the same region $R_{i,\pm}$ as in the proof of Lemma 3.10, but we denote by $R'_{2m-1,-}$ the chamber T_{2m-1}. For every $1 \leq j \leq m-1$, we denote by $R'_{2j-1,+}$ (respectively, $R'_{2j,-}$) the orthogonal reflection of $R_{2j,+}$ (respectively, $R_{2j+1,-}$) in the line $L_{2j}^\perp + a$; note that $R'_{2j-1,+} \subset T_{2j}$ and $R'_{2j,-} \subset T_{2j-1}$. We also denote by $R'_{0,-}$ (respectively, $R'_{2m-1,+}$) the orthogonal reflection of $R_{0,+}$ (respectively, $R'_{2m-1,-}$) in the line $L_m^\perp + a$. For $1 \leq j \leq m-1$ again, we denote by $S'_{j,-}$ the interior of $T_{2j-1} - (R_0(a) \cup R'_{2j,-})$ and by $S'_{j,+}$ the interior of $T_{2j} - (R_0(a) \cup R'_{2j-1,+})$. Finally, we denote by $S'_{0,-}$ the interior of $T_{4m-1} - (R_0(a) \cup R'_{0,-})$ and $S'_{0,+}$ the interior of $T_0 - (R_0(a) \cup R'_{2m-1,+})$.

For $0 \leq i \leq 2m-2$, the chamber T_{2m+i} is then equal to the region $R_{i,\epsilon}$, where ϵ is the sign $(-1)^i$; also, the chamber T_{2m-1} is equal to $R'_{2m-1,-}$. We have $T_r \cap R_-(a) = S_{(r+1)/2,-} \cup R_{r+1,-}$, $T_{r+m} \cap R_-(a) = S_{(r+m+1)/2,-} \cup R_{r+m+1,-}$, $T_{r+2m} \cap R_-(a) = T_{r+2m} = R_{r,-}$ and $T_{r+3m} \cap R_-(a) = T_{r+3m} = R_{r+m,-}$. On the other hand, we have $T_{r+1} \cap R_+(a) = S_{(r+1)/2,+} \cup R_{r,+}$, $T_{r+1+m} \cap R_+(a) = S_{(r+1+m)/2,+} \cup R_{r+m,+}$, $T_{r+1+2m} \cap R_+(a) = T_{r+1+2m} = R_{r+1,+}$ and $T_{r+1+3m} \cap R_+(a) = T_{r+1+3m} = R_{r+m+1,+}$. This implies equation (3.6).

As in Frederickson's article, there should also be a dissection-based proof of the equality of areas that we use to finish the proof, but we were not courageous enough to look for it.

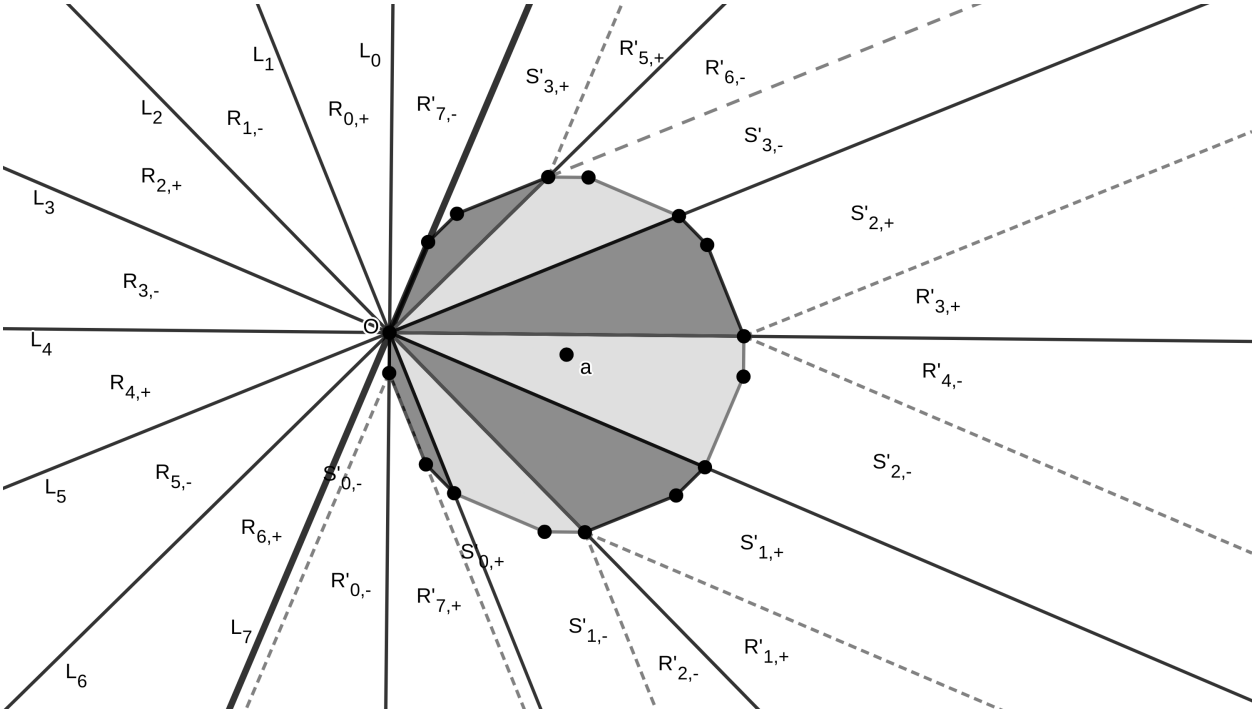

FIGURE 4. The dissection that we use for odd r in the case of $I_2(8)$.

4. The Bolyai-Gerwien Theorem

The classical Bolyai-Gerwien Theorem states that two polygons are scissors congruent if and only if they have the same area. There is also a well-known generalization in higher dimensions that applies to parallelotopes; it follows from the characterization of translational scissors congruences in arbitrary dimensions, and was proved independently by Jessen-Thorup and Sah; see the beginning of Section 7 of [JT78] or Theorem 1.1 in Chapter 4 of [Sah79]. In this section, we state a slight refinement of this generalization, Theorem 4.1, that keeps track of lower-dimensional faces; in other words, we do not want to ignore the boundaries.

As in the previous sections, let V be an n-dimensional real vector space with an inner product $(\cdot,\cdot)$. If $(v_1,\dots,v_r)$ is a linearly independent list of elements of V, we define the parallelotope

$$P(v_1,\dots,v_r)=\left\{\sum_{i=1}^{r} a_i v_i : 0\le a_i\le 1\right\}.$$

We denote by $\mathcal{P}(V)$ the set of all convex polytopes in V (including lower-dimensional ones) and by $\mathcal{Z}(V)$ the subfamily of polytopes that are translates of parallelotopes of the form $P(v_1,\dots,v_r)$. The set $\mathcal{P}(V)$ satisfies the conditions of Definition 3.1, so we can define an abelian group $K_{\mathcal{P}}(V)$ as in that definition. We denote by $K_{\mathcal{Z}}(V)$ the subgroup of $K_{\mathcal{P}}(V)$ generated by the classes $[P]$ for $P\in\mathcal{Z}(V)$. Remark 3.3 implies that, if $\mathcal{P}_{\mathrm{ext}}(V)$ is the relative Boolean algebra generated by $\mathcal{P}(V)$, then we can define the class $[P]$ in $K_{\mathcal{P}}(V)$ of any element P in $\mathcal{P}_{\mathrm{ext}}(V)$. We denote by $\mathcal{Z}_{\mathrm{ext}}(V)$ the set of elements P of $\mathcal{P}_{\mathrm{ext}}(V)$ such that $[P]\in K_{\mathcal{P}}(V)$ is in the subgroup $K_{\mathcal{Z}}(V)$. For example, the set $\mathcal{Z}_{\mathrm{ext}}(V)$ contains $\mathcal{Z}(V)$, and it also contains all half-open parallelotopes.

Recall $V_0,\dots,V_n$ denotes the intrinsic volumes on V; see [Sch14, Section 4.2]. These are valuations on the set of all compact convex subsets of V, and in particular on $\mathcal{P}(V)$, so they induce

morphisms of groups from $K_{\mathcal{P}}(V)$ to $\mathbb{R}$, which we still denote by $V_0, \dots, V_n$. Note that V_0 is the Euler-Poincaré characteristic with compact support, so the image of $K_{\mathcal{P}}(V)$ is $\mathbb{Z}$.

The main result of this section is the following isomorphism.

Theorem 4.1. *The morphism $(V_0, V_1, \dots, V_n) : K_{\mathcal{Z}}(V) \longrightarrow \mathbb{Z} \times \mathbb{R}^n$ is an isomorphism. In particular, if $P, P' \in \mathcal{Z}_{\mathrm{ext}}(V)$ are such that $V_i(P) = V_i(P')$ for every $0 \leq i \leq n$, then $[P] = [P']$ in $K_{\mathcal{Z}}(V)$.*

We will give the proof of the theorem at the end of this section.

Corollary 4.2. *Suppose that $\dim(V) = 2$. Then the triple (V_0, V_1, V_2) induces an isomorphism from $K_{\mathcal{P}}(V)$ to $\mathbb{Z} \times \mathbb{R}^2$. In particular, if P, P' are two elements of $\mathcal{P}_{\mathrm{ext}}(V)$, then their classes in $K_{\mathcal{P}}(V)$ are equal if and only if $V_i(P) = V_i(P')$ for all $0 \leq i \leq 2$.*

Proof. By Theorem 4.1 it suffices to prove that $K_{\mathcal{P}}(V) = K_{\mathcal{Z}}(V)$. As points and segments are parallelotopes, it suffices to prove that every polygon P is scissors congruent to a parallelogram, which follows from the Bolyai-Gerwien Theorem (see [Bol78, Section 5]). ⊛

Remark 4.3.
(1) If we take the quotient $K_{\mathcal{P},0}(V)$ of $K_{\mathcal{P}}(V)$ by the subgroup generated by the classes of lower-dimensional polytopes, then two polytopes have the same class in $K_{\mathcal{P},0}(V)$ if and only if they are scissors congruent, and the Bolyai-Gerwien Theorem (see [Bol78, Section 5]), says that, if $\dim(V) = 2$, the area V_2 induces an isomorphism from $K_{\mathcal{P},0}(V)$ to $\mathbb{R}$.
(2) If $\dim(V) \geq 3$, then $K_{\mathcal{Z}}(V) \neq K_{\mathcal{P}}(V)$. Otherwise, every element of $\mathcal{P}(V)$ of positive volume would be scissors congruent to an element of $\mathcal{Z}(V)$, hence to a cube, and this is not true by the negative solution to Hilbert's third problem (see for example [Bol78]).

Let $\mathcal{Z}'(V)$ be the set of translates of parallelotopes of the form $P(v_1, \dots, v_r)$, for $(v_1, \dots, v_r)$ a linearly independent family of elements of V such that $r \leq n-1$, and let $K'_{\mathcal{Z}}(V)$ be the subgroup of $K_{\mathcal{Z}}(V)$ generated by the classes of elements of $\mathcal{Z}'(V)$. We also write $\mathcal{Z}'_{\mathrm{ext}}(V)$ for the set of $P \in \mathcal{Z}_{\mathrm{ext}}(V)$ such that $[P] \in K'_{\mathcal{Z}}(V)$.

If $P, P' \in \mathcal{Z}(V)$, we write $P \sim P'$ if there exist $P_1, \dots, P_r, Q_1, \dots, Q_s \in \mathcal{P}_{\mathrm{ext}}(V)$, $a_1, \dots, a_r$, $b_1, \dots, b_s \in V$ and $R, R' \in \mathcal{Z}'_{\mathrm{ext}}(V)$ such that

$$P \sqcup Q_1 \sqcup \cdots \sqcup Q_s = R \sqcup P_1 \sqcup \cdots \sqcup P_r \text{ and } P' \sqcup (Q_1 + b_1) \sqcup \cdots \sqcup (Q_s + b_s) = R' \sqcup (P_1 + a_1) \sqcup \cdots \sqcup (P_r + a_r).$$

It is not hard to see that this is an equivalence relation, and that equivalent parallelotopes have the same volume.

Lemma 4.4. *Let W, W' be subspaces of V such that $V = W \times W'$. We do not assume that W and W' are orthogonal. Let $P, P' \in \mathcal{Z}(W)$ and $S, S' \in \mathcal{Z}(W')$ such that $P \sim P'$ and $S \sim S'$. Then the relation $P \times S \sim P' \times S'$ holds.*

Proof. As $\sim$ is transitive and as W and W' play symmetric roles, it suffices to treat the case where $S = S'$. We choose $P_1, \dots, P_r, Q_1, \dots, Q_s \in \mathcal{P}_{\mathrm{ext}}(W)$, $a_1, \dots, a_r, b_1, \dots, b_s \in W$ and $R, R' \in \mathcal{Z}'_{\mathrm{ext}}(W)$ such that

$$P \sqcup Q_1 \sqcup \cdots \sqcup Q_s = R \sqcup P_1 \sqcup \cdots \sqcup P_r \text{ and } P' \sqcup (Q_1 + b_1) \sqcup \cdots \sqcup (Q_s + b_s) = R' \sqcup (P_1 + a_1) \sqcup \cdots \sqcup (P_r + a_r).$$

Then

$$(P \times S) \sqcup (Q_1 \times S) \sqcup \cdots \sqcup (Q_s \times S) = (R \times S) \sqcup (P_1 \times S) \sqcup \cdots \sqcup (P_r \times S)$$

and

$$(P' \times S) \sqcup ((Q_1 \times S) + b_1) \sqcup \cdots \sqcup ((Q_s \times S) + b_s) = (R' \times S) \sqcup ((P_1 \times S) + a_1) \sqcup \cdots \sqcup ((P_r \times S) + a_r).$$

As $R \times S$ and $R' \times S$ are in $\mathcal{Z}'_{\mathrm{ext}}(V)$, this implies that $P \times S \sim P' \times S$. ⊛

FIGURE 5. Proof of Lemma 4.5.

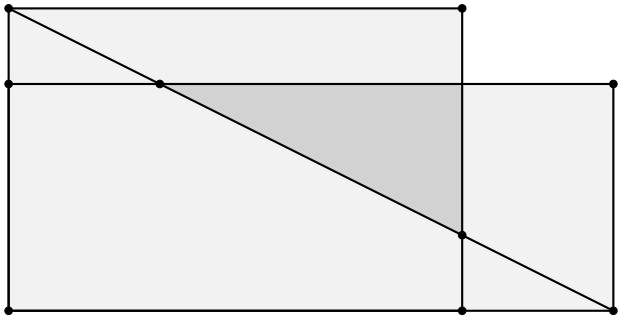

FIGURE 6. Proof of Lemma 4.5.

For two real numbers $a, b \in \mathbb{R}$ recall that the half open interval is given by $(a, b] = \{t \in \mathbb{R} : a < t \leq b\}$ and the closed interval by $[a, b] = \{t \in \mathbb{R} : a \leq t \leq b\}$.

Lemma 4.5. *Let $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$ be bases of V, and let $P = P(v_1, \ldots, v_n)$ and $P' = P(w_1, \ldots, w_n)$. Then $V_n(P) = V_n(P')$ if and only if there exists an isometry g of V such that $P \sim g(P')$.*

In particular, if $V = \mathbb{R}^n$, then, for every $P \in \mathcal{Z}(V)$, the classes of P and of $(0, 1]^{n-1} \times (0, V_n(P)]$ in $K_{\mathcal{Z}}(V)$ are equal modulo $K'_{\mathcal{Z}}(V)$.

Proof. We already know that $V_n(P) = V_n(P')$ if $P \sim g(P')$ with g an isometry of V. We prove the converse by induction on n. It suffices to show that, for every basis $(v_1, \ldots, v_n)$ of V, there exists an orthonormal basis $(e_1, \ldots, e_n)$ of V and $a \in \mathbb{R}_{\geq 0}$ such that $P(v_1, \ldots, v_n) \sim P(e_1, \ldots, e_{n-1}, a \cdot e_n)$; we then must have $a = V_n(P(v_1, \ldots, v_n))$. There is nothing to prove if $n = 0$, and the claim is clear if $n = 1$. Suppose that $n = 2$. The classical proofs that two parallelograms that have the same basis and the same height are scissors congruent and that rectangles that have parallel sides and the same area are scissors congruent use only translations to move the pieces of the decompositions (see for example [EucBC, Proposition 35] and Figure 30 on page 52 of [Bol78]; we reproduce the relevant decompositions in Figures 5 and 6). As the boundaries of the polygons that we ignore when we are talking about scissors congruence are in $\mathcal{Z}'_{\text{ext}}(V)$ when $\dim(V) = 2$, this gives the claim.

Suppose that $n \geq 3$. Let $(v_1, \ldots, v_n)$ be a basis of V, and let $P = P(v_1, \ldots, v_n)$. By the claim for $n = 2$ and Lemma 4.4, there exists an orthonormal basis (e_1, e_2) of $\text{Span}(v_1, v_2)$ and $a \in \mathbb{R}_{\geq 0}$ such that $P \sim P(e_1, a \cdot e_2, v_3, \ldots, v_n)$. Applying the $n = 2$ case in $\text{Span}(e_1, v_3)$ and Lemma 4.4, we can find $v'_3 \in \text{Span}(e_1, v_3)$ orthogonal to e_1 such that $P \sim P(e_1, a \cdot e_2, v'_3, v_4, \ldots, v_n)$. Now applying the $n = 2$ case in $\text{Span}(e_2, v'_3)$, noting that this space is orthogonal to e_1, and using Lemma 4.4, we can find a unit vector $e_3 \in \text{Span}(e_2, v'_3)$ that is orthogonal to e_1 and e_2 and $b \in \mathbb{R}_{\geq 0}$ such that $P \sim P(e_1, e_2, b \cdot e_3, v_4, \ldots, v_n)$. Continuing in this way, we finally obtain the claim.

We prove the last sentence of the lemma. If P is a translate of $P(v_1,\dots,v_k)$ with $k \leq n-1$, then the classes of P and of $(0,1]^{n-1}\times(0,V_n(P)]$ are both in $K'_{\mathcal{Z}}(V)$. Suppose that P is a translate of $P(v_1,\dots,v_n)$, with $(v_1,\dots,v_n)$ a basis of V. By the first assertion, there exists an isometry g of V such that $g\cdot P \sim [0,1]^{n-1}\times[0,V_n(P)]$, so the classes of P and of $[0,1]^{n-1}\times[0,V_n(P)]$ in $K_{\mathcal{Z}}(V)$ are equal modulo $K'_{\mathcal{Z}}(V)$. As the difference between the classes of $[0,1]^{n-1}\times[0,V_n(P)]$ and of $(0,1]^{n-1}\times(0,V_n(P)]$ is in $K'_{\mathcal{Z}}(V)$, the result follows. ⊛

Lemma 4.6. *(i) Let H be a hyperplane of V. The inclusion $\mathcal{Z}(H)\subset\mathcal{Z}(V)$ induces a morphism $K_{\mathcal{Z}}(H)\longrightarrow K_{\mathcal{Z}}(V)$ whose image is $K'_{\mathcal{Z}}(V)$.* [2]
(ii) The subgroup $K'_{\mathcal{Z}}(V)$ is the kernel of the morphism $V_n : K_{\mathcal{Z}}(V)\longrightarrow\mathbb{R}$.

Proof. (i) The existence of the morphism $K_{\mathcal{Z}}(H)\longrightarrow K_{\mathcal{Z}}(V)$ is clear, as well as the fact that its image is contained in $K'_{\mathcal{Z}}(V)$. Conversely, any translate of a $P(v_1,\dots,v_k)$ with $k\leq n-1$ can be moved by an affine isometry to lie in H, so its class is in the image of $K_{\mathcal{Z}}(H)$.

(ii) We may assume that $V=\mathbb{R}^n$. Any polytope in $\mathcal{Z}'(V)$ has volume zero, so $K'_{\mathcal{Z}}(V)$ is included in the kernel of V_n. We prove the reverse inclusion. Let x be an element of $\operatorname{Ker} V_n$, and write $x=\sum_{i=1}^r \alpha_i[P_i]$, with $\alpha_i\in\{\pm1\}$ and $P_i\in\mathcal{Z}(V)$. We want to show that $x\in K'_{\mathcal{Z}}(V)$. By Lemma 4.5, for every $1\leq i\leq r$, the class of P_i is equal to the class of $(0,1]^{n-1}\times(0,V_n(P_i)]$ modulo $K'_{\mathcal{Z}}(V)$. So x is equal modulo $K'_{\mathcal{Z}}(V)$ to the sum

$$\sum_{i=1}^r \alpha_i[(0,1]^{n-1}\times(0,V_n(P_i)]] = [(0,1)^{n-1}\times(0,V_+]] - [(0,1]^{n-1}\times(0,V_-]],$$

where $V_\pm=\sum_{1\leq i\leq r,\ \alpha_i=\pm1}\operatorname{Vol}(P_i)$. As $V_+-V_-=V_n(x)=0$ by assumption, we conclude that $x\in K'_{\mathcal{Z}}(V)$. ⊛

Proof of Theorem 4.1. Let $V_*=(V_0,V_1,\dots,V_n):K_{\mathcal{Z}}(V)\longrightarrow\mathbb{Z}\times\mathbb{R}^n$. Then the morphism V_* sends the class of a point to $(1,0,\dots,0)$, so its image contains the factor $\mathbb{Z}$. Denote by $(e_1,\dots,e_n)$ the canonical basis of $\mathbb{R}^n$. If $i\in\{1,\dots,n\}$ and $a\in\mathbb{R}_{\geq0}$, then by Lemma 3.6 V_* sends the class of the half-open rectangular parallelotope $(0,1]^{i-1}\times(0,a]\times\{0\}^{n-i}$ to $(0,a\cdot e_i)\in\mathbb{Z}\times\mathbb{R}^n$, so the image of V_* contains $\mathbb{R}\cdot e_i$. This shows that V_* is surjective.

We now prove the injectivity of V_* by induction on $\dim(V)$. If $\dim(V)=0$, the result is clear. Suppose that $\dim(V)>0$ and that we know the result for spaces of smaller dimension. Let $x\in K_{\mathcal{Z}}(V)$ such that $V_i(x)=0$ for $0\leq i\leq n$. In particular, we have $V_n(x)=0$, so $x\in K'_{\mathcal{Z}}(V)$ by Lemma 4.6(ii). Let H be a hyperplane of V. Then x is in the image of the morphism $K_{\mathcal{Z}}(H)\longrightarrow K_{\mathcal{Z}}(V)$ by Lemma 4.6(ii); choose a preimage $y\in K_{\mathcal{Z}}(H)$ of x. If $0\leq i\leq n-1$, then we have $V_i(y)=V_i(x)$ because intrinsic volumes do not depend on the dimension of the ambient space (see the top of page 214 of [Sch14]), so $V_i(y)=0$. It follows from the induction hypothesis that $y=0$, and we conclude that $x=0$. ⊛

Acknowledgements

The authors thank Dominik Schmid for introducing them to the Pizza Theorem, and Ramon van Handel for dispelling some of their misconceptions and lending them a copy of [Sch14]. They made extensive use of Geogebra to understand the 2-dimensional situation and to produce some of the figures. This work was partially supported by the LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon, within the program "Investissements d'Avenir" (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR), and by Princeton University. This work was also partially supported by grants from the Simons Foundation (#429370 and #854548 to

[2] In fact, it follows from Theorem 4.1 that $K_{\mathcal{Z}}(H)\longrightarrow K_{\mathcal{Z}}(V)$ is injective, so we get an isomorphism from $K_{\mathcal{Z}}(H)$ to $K'_{\mathcal{Z}}(V)$.

Richard Ehrenborg and #422467 to Margaret Readdy). The third author was also supported by NSF grant DMS-2247382.

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Department of Mathematics, University of Kentucky, Lexington, KY 40506-0027, USA.
http://www.math.uky.edu/~jrge/, richard.ehrenborg@uky.edu.

ENS de Lyon, Unité De Mathématiques Pures Et Appliquées, 69342 Lyon Cedex 07, France.
http://perso.ens-lyon.fr/sophie.morel/, sophie.morel@ens-lyon.fr.

Department of Mathematics, University of Kentucky, Lexington, KY 40506-0027, USA.
http://www.math.uky.edu/~readdy/, margaret.readdy@uky.edu.