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The error term in the truncated Perron formula for the logarithm of an L-function

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Abstract. We improve upon the traditional error term in the truncated Perron formula for the logarithm of an L-function. All our constants are explicit.

1 Introduction

The truncated Perron formula relates the summatory function of an arithmetic function to a contour integral that may be estimated using techniques from complex analysis. Let $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ be absolutely convergent on $\operatorname{Re} s > c_F$; examples include the Riemann zeta function, Dirichlet L-functions, the Dedekind zeta function associated with a number field, and Artin L-functions. The truncated Perron formula tells us that if $x > 0$ is not an integer, $T \geq 1$, and $c > c_F$, then

$$\sum_{n\leq x} f(n) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} F(s)\frac{x^s}{s}\,ds + O^*\left(\sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^c |f(n)| \min\left\{1, \frac{1}{T|\log\frac{x}{n}|}\right\}\right), \tag{1.1}$$

in which $O^*(g(x)) = h(x)$ means $|h(x)| \leq g(x)$ (see [8, Chapter 7], [10, Section 5.1], [11, Example 4.4.15], and [15, Section II.2]). We let T depend on x, and let $c = c_F + 1/\log x$, so that $x^c = ex^{c_F}$. A variation of (1.1) improves the order of the error term by truncating the integral at $\pm T^*$ for an unknown $T^* \in [T, O(T)]$ [3], although this is inconvenient if one must avoid T^* that correspond to the ordinates of nontrivial zeros of $F(s)$. The authors of [3] have also informed us in a personal communication that their paper inherited an unfortunate typo from another paper, so the error term in their variation of the truncated Perron formula could be worse by a factor of $\log x$; this means that our main result (Theorem 1.1) will be comparable in strength *and* more straightforward to apply when compared against the outcome of their result.

For $\operatorname{Re} s > 1$, the logarithm of the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is $\log\zeta(s) = \sum_{n=1}^{\infty}\Lambda(n)(\log n)^{-1}n^{-s}$, in which $\Lambda(n)$ is the von Mangoldt function. The logarithm of a typical L-function is of the form $\sum_{n=1}^{\infty}\Lambda(n)a_n(\log n)^{-1}n^{-s}$, in which the a_n are easily controlled. For example, $|a_n| \leq 1$ for Dirichlet L-functions and

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$|a_n| \le d$ for Artin L-functions of degree d (see [7, Chapter 5]. In these cases, the error term in (1.1) with $c = 1/\log x$ is on the order of

$$\sum_{n=2}^{\infty}\left(\frac{x}{n}\right)^{\frac{1}{\log x}}\frac{\Lambda(n)}{n\log n}\min\left\{1, \frac{1}{T|\log\frac{x}{n}|}\right\} = O\left(\frac{\log x}{T}\right). \tag{1.2}$$

Granville and Soundararajan used (1.2) with Dirichlet L-functions to study large character sums [6, equation (8.1)]. Cho and Kim applied it to Artin L-functions to obtain asymptotic bounds on Dedekind zeta residues [1, Proposition 3.1]. A bilinear relative of (1.2) appears in Selberg's work on primes in short intervals [14, Lemma 4]. Analogous sums arise with the logarithmic derivative of an L-function in [4, p. 106] and [12, p. 44].

We improve upon (1.2) asymptotically and explicitly in the following result.

Theorem 1.1 *If $x \ge 3.5$ is a half integer and $T \ge (\log\frac{3}{2})^{-1} > 2.46$, then*

$$\sum_{n=2}^{\infty}\left(\frac{x}{n}\right)^{\frac{1}{\log x}}\frac{\Lambda(n)}{n\log n}\min\left\{1, \frac{1}{T|\log\frac{x}{n}|}\right\} \le \frac{R(x)}{T}, \tag{1.3}$$

in which

$$R(x) = 40.23\log\log x + 58.12 + \frac{3.87}{\log x} + \frac{5.22\log x}{\sqrt{x}} - \frac{1.84}{\sqrt{x}}. \tag{1.4}$$

Our result has a wide and explicit range of applicability. For example, the following corollary employs (1.1) with $T = x$ and $c = 1/\log x$. Since one can use analytic techniques to see the integral below is asymptotic to $\log L(1,\chi)$, one can relate $\log L(1,\chi)$ to a short sum. We hope to do so explicitly in the future.

Corollary 1.2 *Let $L(s,\chi)$ be an entire Artin L-function of degree d such that*

$$L(s,\chi) = \prod_p \prod_{i=1}^{d}\left(1 - \frac{\alpha_i(p)}{p^s}\right)^{-1} \quad \textit{for } \operatorname{Re} s > 1$$

with $a(p^k) = \alpha_1(p)^k + \cdots + \alpha_d(p)^k$ for prime p. Then, with $R(x)$ as in (1.4), we have

$$\sum_{1<n<x}\frac{\Lambda(n)a(n)}{n\log n} = \frac{1}{2\pi i}\int_{\frac{1}{\log x}-ix}^{\frac{1}{\log x}+ix}\frac{x^s}{s}\log L(1+s,\chi)\,ds + O^*\left(\frac{d\,R(x)}{x}\right).$$

2 Preliminaries

Here, we establish several lemmas needed for the proof of Theorem 1.1.

Lemma 2.1 *If $\sigma > 0$, then $\log\zeta(1+\sigma) \le -\log\sigma + \gamma\sigma$.*

Proof For $s > 1$, we have $\zeta(s) \le e^{\gamma(s-1)}/(s-1)$ [13, Lemma 5.4]. Let $s = 1+\sigma$ and take logarithms to obtain the desired result. ∎

For real z, w, the equation $z = we^w$ can be solved for w if and only if $z \ge -e^{-1}$. There are two branches for $-e^{-1} \le z < 0$. The lower branch defines the *Lambert* $W_{-1}(z)$

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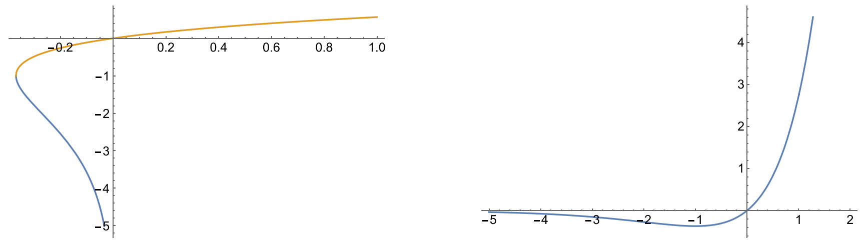
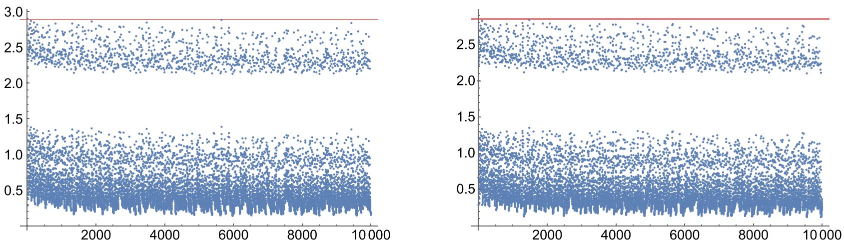


(a) The two branches of $z = we^w$. The lower branch (blue) is the Lambert function $W_{-1}(z)$, the upper branch (gold) is $W_0(z)$.

(b) $f(w) = we^w$ decreases on $(-\infty, -1]$.

Figure 1: Graphs relevant to the construction of the sequence y_n.



(a) $F_1(x)$ is bounded above by C (red).

(b) $F_2(x)$ is bounded above by C (red).

Figure 2: The functions $F_1(x)$ and $F_2(x)$ behave erratically.

function [2], which decreases to $-\infty$ as $z \to 0^-$ (see Figure 1a). For $n \geq 6 > 2e$, we define the strictly increasing sequence

$$y_n = \frac{-n}{2} W_{-1}\left(\frac{-2}{n}\right) \qquad \text{for } n \geq 6. \tag{2.1}$$

Lemma 2.2 *For $n \geq 8$, we have $\frac{2y_n}{\log y_n} = n$ and $y_n \geq \frac{n}{2} \log n$.*

Proof For $n \geq 6$, the definition of W_{-1} and (2.1) confirm that $\frac{2y_n}{\log y_n} = n$. Thus, the desired inequality is equivalent to $W_{-1}(\frac{-2}{n}) \leq -\log n$. Since $f(w) = we^w$ decreases on $(-\infty, -1]$ (Figure 1b) and $-\frac{1}{e} < -\frac{2}{n} < 0$, the desired inequality is equivalent to

$$-\frac{2}{n} \geq f(-\log n) = (-\log n)e^{-\log n} = -\frac{\log n}{n},$$

which holds whenever $\log n \geq 2$. This occurs for $n \geq e^2 \approx 7.38906$. ∎

Remark 2.3 For all $-e^{-1} \leq x < 0$, the bound $W_{-1}(x) \leq \log(-x) - \log(-\log(-x))$ is valid (see [9, equations (8) and (39)]). It follows from this observation and (2.1) that

$$y_n \geq \frac{n}{2}\left(\log\left(\frac{1}{2}\log\frac{n}{2}\right) + \log n\right),$$

which also implies Lemma 2.2 for $n \geq 15$.

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The next lemma is needed later to handle a few exceptional primes.

Lemma 2.4 *Let $x > 1$ be a half integer, and let $C = \frac{1284699552}{444215525} = 2.89206\ldots$.*

1. *Let $p_{-8} < p_{-7} < \cdots < p_{-1} < x$ denote the largest eight primes (if they exist) in the interval $(\frac{x}{2}, x)$. We have the sharp bound*

$$F_1(x) = \sum_{1\le n\le 8} \frac{1}{x - p_{-n}} \le C \tag{2.2}$$

(see Figure 2a). *The corresponding summand in (2.2) is zero if p_{-n} does not exist.*

2. *Let $x < p_1 < p_2 < \cdots < p_8$ denote the smallest eight primes (if they exist) in the interval $(x, \frac{3x}{2})$. We have the sharp bound*

$$F_2(x) = \sum_{1\le n\le 8} \frac{1}{p_n - x} \le C \tag{2.3}$$

(see Figure 2a). *The corresponding summand in (2.3) is zero if p_n does not exist.*

Proof (a) If $x \ge 10.5$, then $2, 3, 5 \notin (\frac{x}{2}, x)$. Computation confirms that

$$F_1(x) \le F_1(3.5) = \frac{8}{3} = 2.66\ldots$$

for $x \le 9.5$. If $x \ge 10.5$, then any prime in $(\frac{x}{2}, x)$ is congruent to one of $1, 7, 11, 13, 17, 19, 23, 29 \pmod{30}$. There are finitely many patterns modulo 30 that the $p_{-8}, p_{-7}, \ldots, p_{-1}$ may assume. Among these, computation confirms that $F_1(x)$ is maximized if

$$\begin{aligned}
p_{-1} &= \lfloor x \rfloor \equiv 19 \pmod{30}, & p_{-5} &= \lfloor x \rfloor - 12 \equiv 7 \pmod{30},\\
p_{-2} &= \lfloor x \rfloor - 2 \equiv 17 \pmod{30}, & p_{-6} &= \lfloor x \rfloor - 18 \equiv 1 \pmod{30},\\
p_{-3} &= \lfloor x \rfloor - 6 \equiv 13 \pmod{30}, & p_{-7} &= \lfloor x \rfloor - 20 \equiv 29 \pmod{30},\\
p_{-4} &= \lfloor x \rfloor - 8 \equiv 11 \pmod{30}, & p_{-8} &= \lfloor x \rfloor - 26 \equiv 23 \pmod{30},
\end{aligned}$$

which yields the desired upper bound C. This prime pattern first occurs for $x = 88{,}819.5$ (see https://oeis.org/A022013).

(b) If $x \ge 5.5$, then $2, 3, 5 \notin (x, \frac{3x}{2})$. Observe that $F_2(x) \le 2$ for $x \le 4.5$ (attained at $x = 1.5, 2.5, 4.5$). If $x \ge 5.5$, then (as in (a)), any prime in $(x, \frac{3x}{2})$ is congruent to one of $1, 7, 11, 13, 17, 19, 23, 29 \pmod{30}$. It follows that $F_2(x)$ is maximized if

$$\begin{aligned}
p_1 &= \lceil x \rceil \equiv 11 \pmod{30}, & p_5 &= \lceil x \rceil + 12 \equiv 23 \pmod{30},\\
p_2 &= \lceil x \rceil + 2 \equiv 13 \pmod{30}, & p_6 &= \lceil x \rceil + 18 \equiv 29 \pmod{30},\\
p_3 &= \lceil x \rceil + 6 \equiv 17 \pmod{30}, & p_7 &= \lceil x \rceil + 20 \equiv 1 \pmod{30},\\
p_4 &= \lceil x \rceil + 8 \equiv 19 \pmod{30}, & p_8 &= \lceil x \rceil + 26 \equiv 7 \pmod{30},
\end{aligned}$$

which yields the desired upper bound C. This prime pattern occurs for $x = 10.5$, but not all eight primes lie in $(x, \frac{3}{2}x)$. Therefore, the first admissible value is $x = 15{,}760{,}090.5$ (see https://oeis.org/A022011). ■

We also need an elementary estimate on kth powers in intervals.

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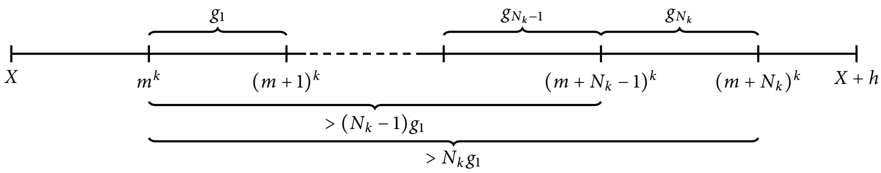

Figure 3: Proof of Lemma 2.5.

Lemma 2.5 *Let $X > 1$ be a noninteger, $h > 1$, and $k \geq 2$.*

(1) *There are at most $N_k + 1$ perfect kth powers in $[X, X+h)$, in which $N_k \leq \frac{h}{k\sqrt{X}}$.*

(2) *The shortest gap between kth powers in $[X, X+h)$ (if they exist) is $G_k \geq k\sqrt{X}$.*

Proof We may assume that X is so large that $N_k \geq 1$. Let $m = \lceil X^{\frac{1}{k}} \rceil$ so that m^k is the first kth power larger than X. Consider the gaps $g_1, g_2, \ldots, g_{N_k}$ between the N_k consecutive kth powers in $[X, X+h)$ (see Figure 3). Then

$$G_k = \min\{g_1, g_2, \ldots, g_{N_k}\} = g_1 = (m+1)^k - m^k \geq km^{k-1} \geq kX^{\frac{k-1}{k}} \geq k\sqrt{X}.$$

The desired inequality follows since $N_k G_k \leq g_1 + g_2 + \cdots + g_{N_k} \leq h$. ∎

Finally, we need an estimate on the nth *harmonic number* $H_n = \sum_{j=1}^n \frac{1}{j}$:

$$\frac{1}{2n + \frac{2}{5}} < H_n - \log n - \gamma < \frac{1}{2n + \frac{1}{3}} \leq \frac{3}{7} \quad \text{for} \quad n \geq 1, \tag{2.4}$$

in which γ is the Euler–Mascheroni constant [16]. We require the upper bound

$$\begin{aligned}
\sum_{\ell=1}^{n} \frac{1}{2\ell - 1} &= \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n}\right) - \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) = H_{2n} - \frac{1}{2}H_n \\
&\leq \left(\log 2n + \gamma + \frac{1}{4n + \frac{1}{3}}\right) - \frac{1}{2}\left(\log n + \gamma + \frac{1}{2n + \frac{2}{5}}\right) \\
&\leq \frac{1}{2}\log n + \frac{1}{2}\gamma + \log 2 + \frac{1}{4n + \frac{1}{3}} - \frac{1}{4n + \frac{4}{5}} \\
&= \frac{1}{2}\log n + \frac{1}{2}\gamma + \log 2 + \frac{7}{240n^2 + 68n + 4} \\
&\leq \frac{1}{2}\log n + \frac{1}{2}\gamma + \log 2 + \frac{7}{312} \quad \text{for} \quad n \geq 1. \qquad (2.5)
\end{aligned}$$

3 Proof of Theorem 1.1

In what follows, $x \in \mathbb{N} + \frac{1}{2} = \{\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots\}$ and $c = \frac{1}{\log x}$. Minor improvements below are possible; these were eschewed in favor of a final estimate of simple shape.

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3.1 When n is very far from x

Suppose that $n \le \frac{x}{2}$ or $\frac{3x}{2} \ge n$. Then $\log \frac{x}{n} \le -\log \frac{3}{2}$ or $\log \frac{3}{2} < \log 2 \le \log \frac{x}{n}$, so $|\log \frac{x}{n}| \ge \log \frac{3}{2}$. If $T \ge (\log \frac{3}{2})^{-1} > 2.46$, then

$$\min\left\{1, \frac{1}{T|\log \frac{x}{n}|}\right\} \le \frac{1}{T|\log \frac{x}{n}|} \le \frac{1}{T\log \frac{3}{2}}.$$

For such T, the previous inequality and Lemma 2.1 imply (recall that $c = \frac{1}{\log x}$)

$$\begin{aligned}\sum_{\substack{n\le \frac{x}{2} \text{ or}\\ n\ge \frac{3x}{2}}} \left(\frac{x}{n}\right)^c \frac{\Lambda(n)}{n\log n} \min\left\{1, \frac{1}{T|\log \frac{x}{n}|}\right\} &\le x^c \sum_{\substack{n\le \frac{x}{2} \text{ or}\\ n\ge \frac{3x}{2}}} \frac{1}{n^{1+c}}\left(\frac{\Lambda(n)}{\log n}\right)\left(\frac{1}{T\log \frac{3}{2}}\right)\\ &\le \frac{x^c}{T\log \frac{3}{2}} \log \zeta(1+c)\\ &\le \frac{x^c}{T\log \frac{3}{2}}(-\log c + \gamma c)\\ &= \frac{e}{T\log \frac{3}{2}}\left(\log\log x + \frac{\gamma}{\log x}\right). \qquad (3.1)\end{aligned}$$

3.2 Reduction to a sum over prime powers

Suppose that $\frac{x}{2} < n < \frac{3x}{2}$. Let $z = 1 - \frac{n}{x}$ and observe that $|z| < \frac{1}{2}$. Then

$$\log \frac{x}{n} = -\log(1-z) = z\left(-\frac{\log(1-z)}{z}\right),$$

in which the function in parentheses is positive and achieves its minimum value $2\log \frac{3}{2} = 0.81093\ldots$ on $|z| < \frac{1}{2}$ at its left endpoint $-\frac{1}{2}$ (see Figure 4a). Then

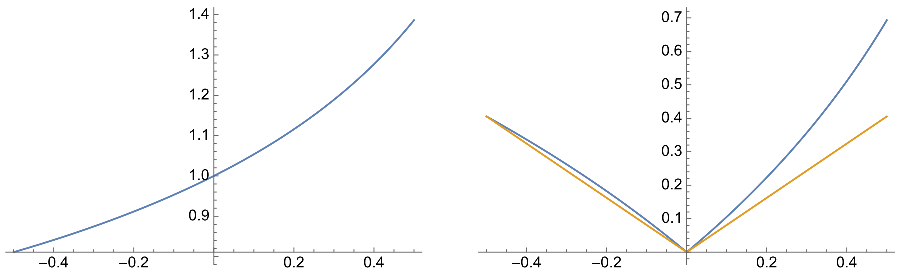
$$|\log(1-z)| > \left(2\log \frac{3}{2}\right)|z| \quad \text{for} \quad |z| < \frac{1}{2}, \qquad (3.2)$$

whose validity is illustrated in Figure 4b. Therefore,

$$\left|\log \frac{x}{n}\right| > \left(2\log \frac{3}{2}\right)\left|1 - \frac{n}{x}\right| \quad \text{for} \quad \frac{x}{2} < n < \frac{3x}{2}, \qquad (3.3)$$

and hence

$$\begin{aligned}&\sum_{\frac{x}{2}<n<\frac{3x}{2}} \left(\frac{x}{n}\right)^c \frac{\Lambda(n)}{n\log n} \min\left\{1, \frac{1}{T|\log \frac{x}{n}|}\right\}\\ &\le \frac{x^c}{T} \sum_{\frac{x}{2}<n<\frac{3x}{2}} \left(\frac{\Lambda(n)}{n^{1+c}\log n}\right)\left(\frac{1}{|\log \frac{x}{n}|}\right)\end{aligned}$$



(a) $-\frac{\log(1-z)}{z}$ on $|z| \leq \frac{1}{2}$.

(b) $|\log(1-z)|$ (blue) and $(2\log\frac{3}{2})|z|$ (gold) on $|z| \leq \frac{1}{2}$.

Figure 4: Graphs relevant to the derivation of (3.2).

$$
\begin{aligned}
&\leq \frac{x^c}{T} \sum_{\frac{x}{2}<n<\frac{3x}{2}} \left(\frac{\Lambda(n)}{n^{1+c}\log n}\right) \frac{1}{(2\log\frac{3}{2})|1-\frac{n}{x}|} && \text{(by (3.3))}\\
&\leq \frac{x^c}{T(2\log\frac{3}{2})} \sum_{\frac{x}{2}<n<\frac{3x}{2}} \frac{2}{x}\left(\frac{\Lambda(n)}{n^{c}\log n}\right) \frac{1}{|1-\frac{n}{x}|} && \text{(since } \tfrac{x}{2}<n)\\
&\leq \frac{x^c}{T\log\frac{3}{2}} \sum_{\frac{x}{2}<n<\frac{3x}{2}} \left(\frac{\Lambda(n)}{n^{c}\log n}\right) \frac{1}{|x-n|}\\
&\leq \frac{x^c}{T\log\frac{3}{2}} \sum_{\frac{x}{2}<p^k<\frac{3x}{2}} \left(\frac{\log p}{(p^k)^{c}k\log p}\right) \frac{1}{|x-p^k|} && \text{(def. of } \Lambda)\\
&\leq \frac{e}{T\log\frac{3}{2}} \sum_{\frac{x}{2}<p^k<\frac{3x}{2}} \frac{1}{k|x-p^k|} && \text{(since } c=\tfrac{1}{\log x}), \qquad (3.4)
\end{aligned}
$$

in which the final two sums run over all prime powers p^k in the stated interval.

The remainder of the proof uses ideas from [5, Lemma 2] to estimate

$$
\sum_{\frac{x}{2}<p^k<\frac{3x}{2}} \frac{1}{k|x-p^k|} = \underbrace{\sum_{\frac{x}{2}<p<\frac{3x}{2}} \frac{1}{|x-p|}}_{S_{\text{prime}}(x)} + \underbrace{\sum_{\substack{\frac{x}{2}<p^k<\frac{3x}{2}\\ k\geq 2}} \frac{1}{k|x-p^k|}}_{S_{\text{power}}(x)}. \tag{3.5}
$$

3.3 The sum over primes

First observe that

$$
S_{\text{prime}}(x) \leq \underbrace{\sum_{\frac{x}{2}<p<x} \frac{1}{x-p}}_{S^-_{\text{prime}}(x)} + \underbrace{\sum_{x<p<\frac{3x}{2}} \frac{1}{p-x}}_{S^+_{\text{prime}}(x)}.
$$

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We require the Brun–Titchmarsh theorem (see [10, Corollary 2]):

$$\pi(X+Y)-\pi(X)\leq \frac{2Y}{\log Y}, \quad \text{where } \pi(x)=\sum_{p\leq x}1,\ X>0, \text{ and } Y>1. \tag{3.6}$$

3.3.1 The lower sum over primes

Let $p_{-k}<p_{-(k-1)}<\cdots<p_{-2}<p_{-1}$ be the primes in $(\frac{x}{2},x)$; note that $k\leq\frac{x}{2}$. Apply (3.6) with $X=x-y_n$ and $Y=y_n$ to get

$$0\leq \pi(x)-\pi(x-y_n)\leq \frac{2y_n}{\log y_n}=n \qquad \text{for } 6\leq n\leq k$$

by Lemma 2.2, so $(x-y_n,x]$ contains at most n primes. Thus, $p_{-(n+1)}\leq x-y_n$ and

$$\frac{1}{x-p_{-(n+1)}}\leq\frac{1}{y_n} \qquad \text{for } 6\leq n\leq k-1. \tag{3.7}$$

Then Lemma 2.2, which requires $k\geq 8$, and the integral test provide

$$\begin{aligned}
\sum_{\frac{x}{2}<p<x}\frac{1}{x-p} &= \sum_{1\leq n\leq 8}\frac{1}{x-p_{-n}}+\sum_{9\leq n\leq k}\frac{1}{x-p_{-n}}\\
&\leq F_1(x)+\sum_{8\leq n\leq k-1}\frac{1}{y_n} && \text{(by (2.2) and (3.7))}\\
&\leq C+2\sum_{7<n\leq\frac{x}{2}}\frac{1}{n\log n} && \text{(by Lemma 2.2)}\\
&\leq C+2\log\log x,
\end{aligned}$$

which is valid for $k\leq 7$ since Lemma 2.4a shows that the sum is majorized by C.

3.3.2 The upper sum over primes

Let $p_1<p_2<\cdots<p_k$ denote the primes in $(x,\frac{3x}{2})$ and note that $k\leq\frac{x}{2}$. Then (3.6) with $X=x$ and $Y=y_n$ ensures that

$$0\leq\pi(x+y_n)-\pi(x)\leq\frac{2y_n}{\log y_n}=n \qquad \text{for} \qquad 6\leq n\leq k$$

by Lemma 2.2, so $(x,x+y_n]$ contains at most n primes. Thus, $p_{n+1}\geq x+y_n$ and

$$\frac{1}{p_{n+1}-x}\leq\frac{1}{y_n} \qquad \text{for} \qquad 6\leq n\leq k. \tag{3.8}$$

An argument similar to that above reveals that

$$\sum_{x<p<\frac{3x}{2}}\frac{1}{p-x}\leq\sum_{1\leq n\leq 8}\frac{1}{p_n-x}+\sum_{9\leq n\leq k}\frac{1}{p_n-x}\leq C+2\log\log x.$$

3.3.3 Final bound over primes

For $x \in \mathbb{N} + \frac{1}{2}$, the previous inequalities yield

$$S_{\text{prime}}(x) = \sum_{\frac{x}{2} < p < \frac{3x}{2}} \frac{1}{|x-p|} \le 2C + 4 \log\log x. \tag{3.9}$$

3.4 The sum over prime powers

We now majorize

$$S_{\text{power}}(x) = \sum_{\substack{\frac{x}{2} < p^k < \frac{3x}{2} \\ k \ge 2}} \frac{1}{k|x-p^k|}.$$

3.4.1 Initial reduction

To bound $S_{\text{power}}(x)$ it suffices to majorize

$$S_{\text{sqf}}(x) = \sum_{\substack{\frac{x}{2} < n^k < \frac{3x}{2} \\ k \ge 2 \\ n \ge 2 \text{sq. free}}} \frac{1}{k|x-n^k|}, \tag{3.10}$$

in which the prime powers p^k are replaced with the powers n^k of square free $n \ge 2$. The square-free restriction ensures that powers such as $2^6 = (2^2)^3 = (2^3)^2$ are not counted multiple times in (3.10). If $\frac{x}{2} < n^k < \frac{3x}{2}$ and $k \ge 2$, then (since $n \ge 2$)

$$k \le \frac{\log \frac{3x}{2}}{\log 2} \le \lfloor 2.4 \log x \rfloor \qquad \text{for} \qquad x \ge 3.5. \tag{3.11}$$

3.4.2 Nearest-power sets

The largest contributions to $S_{\text{sqf}}(x)$ come from the powers closest to x. We handle those summands separately and split the sum (3.10) accordingly. For each $k \ge 2$, the inequalities $\lfloor x^{\frac{1}{k}} \rfloor^k < x < \lceil x^{\frac{1}{k}} \rceil^k$ exhibit the two kth powers nearest to x. Define

$$\mathcal{N}_k \subseteq \left\{ \lfloor x^{\frac{1}{k}} \rfloor^k, \lceil x^{\frac{1}{k}} \rceil^k \right\} \tag{3.12}$$

according to the following rules:

- $\mathcal{N}_k$ contains $\lfloor x^{\frac{1}{k}} \rfloor^k$ if it is square free and belongs to $(\frac{x}{2}, \frac{3x}{2})$.
- $\mathcal{N}_k$ contains $\lceil x^{\frac{1}{k}} \rceil^k$ if it is square free and belongs to $(\frac{x}{2}, \frac{3x}{2})$.

Consequently, $\mathcal{N}_k$, if nonempty, contains only powers that satisfy the restrictions in (3.10). The square-free condition ensures that $\mathcal{N}_j \cap \mathcal{N}_k = \varnothing$ for $j \ne k$.

Write $S_{\text{sqf}}(x) = S_{\text{near}}(x) + S_{\text{far}}(x)$, in which

$$S_{\text{near}}(x) = \sum_{\substack{\frac{x}{2} < n^k < \frac{3x}{2} \\ k \ge 2 \\ n \ge 2 \text{ sq. free} \\ n^k \in \mathcal{N}_k}} \frac{1}{k|x-n^k|} \quad \text{and} \quad S_{\text{far}}(x) = \sum_{\substack{\frac{x}{2} < n^k < \frac{3x}{2} \\ k \ge 2 \\ n \ge 2 \text{ sq. free} \\ n^k \notin \mathcal{N}_k}} \frac{1}{k|x-n^k|}. \tag{3.13}$$

3.4.3 Near Sum

For $x \geq 3.5$, a nearest-neighbor overestimate provides

$$\begin{aligned}
S_{\text{near}}(x) &= \sum_{\substack{\frac{x}{2}<n^k<\frac{3x}{2}\\ k\geq 2\\ n\geq 2 \text{ sq. free}\\ n^k\in\mathcal{N}_k}} \frac{1}{k|x-n^k|} && \text{(by 3.13)}\\
&= \sum_{k=2}^{\lfloor 2.4\log x\rfloor} \sum_{m\in\mathcal{N}_k} \frac{1}{k|x-m|} && \text{(by 3.11)}\\
&\leq \frac{1}{2}\sum_{k=2}^{\lfloor 2.4\log x\rfloor} \sum_{m\in\mathcal{N}_k} \frac{1}{|x-m|} && \text{(since } k\geq 2)\\
&\leq \frac{1}{2}\sum_{j=0}^{\lfloor 2.4\log x\rfloor -2} \left(\frac{1}{x-(\lfloor x\rfloor - j)} + \frac{1}{(\lceil x\rceil + j)-x}\right) && \text{(see below)}\\
&\leq \frac{1}{2}\sum_{\ell=1}^{\lfloor 2.4\log x\rfloor -1} \frac{2}{\ell-\frac{1}{2}} \quad < \quad 2\sum_{\ell=1}^{\lfloor 2.4\log x\rfloor} \frac{1}{2\ell-1}\\
&\leq \log(\lfloor 2.4\log x\rfloor) + \gamma + 2\log 2 + \frac{14}{312} && \text{(by 2.5)}\\
&< \log\log x + \gamma + 2\log 2 + \log 2.4 + \frac{7}{156}. && (3.14)
\end{aligned}$$

Let us elaborate on a crucial step above. Consider the at most $\lfloor 2\log x\rfloor - 1$ pairs of values $|x-m|$ that arise as m ranges over each $\mathcal{N}_k$ with $2\leq k\leq \lfloor 2\log x\rfloor$ (since $\mathcal{N}_j(x)\cap\mathcal{N}_k = \varnothing$ for $j\neq k$, no m appears more than once). Replace these values with the absolute deviations of x from its $2\times(\lfloor 2\log x\rfloor - 1)$ nearest neighbors $\lfloor x\rfloor - j$ (to the left) and $\lceil x\rceil + j$ (to the right), in which $0\leq j\leq \lfloor 2\log x\rfloor - 2$. Since $x\in\mathbb{N}+\frac{1}{2}$, these deviations are of the form $\ell - \frac{1}{2}$ for $1\leq \ell\leq \lfloor 2\log x\rfloor - 1$.

3.4.4 Splitting the second sum

From (3.13), the second sum in question is

$$S_{\text{far}}(x) = \sum_{\substack{\frac{x}{2}<n^k<\frac{3x}{2}\\ k\geq 2\\ n\geq 2 \text{ sq. free}\\ n^k\notin\mathcal{N}_k}} \frac{1}{k|x-n^k|} \leq \sum_{\substack{\frac{x}{2}<n^k<\frac{3x}{2}\\ n,k\geq 2\\ n^k\notin\mathcal{N}_k}} \frac{1}{k|x-n^k|} = S_{\text{far}}^-(x) + S_{\text{far}}^+(x),$$

in which

$$S_{\text{far}}^-(x) = \sum_{k\geq 2}\sum_{\substack{\frac{x}{2}<n^k<x\\ n\geq 2, n^k\notin\mathcal{N}_k}} \frac{1}{k|x-n^k|} \quad\text{and}\quad S_{\text{far}}^+(x) = \sum_{k\geq 2}\sum_{\substack{x<n^k<\frac{3x}{2}\\ n\geq 2, n^k\notin\mathcal{N}_k}} \frac{1}{k|x-n^k|}. \tag{3.15}$$

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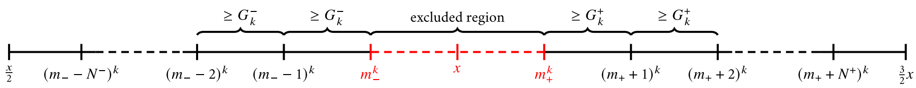

Figure 5: Analysis of kth powers in $[\frac{x}{2}, \frac{3x}{2}]$, in which $m_- = \lfloor x^{1/k} \rfloor$ and $m_+ = \lceil x^{1/k} \rceil$ are excluded from consideration. There are at most N_k^- admissible kth powers in $[\frac{x}{2}, x)$, with minimal gap size G_k^-, and at most N_k^+ admissible kth powers in $[x, \frac{3x}{2})$, with minimal gap size G_k^+.

For $k \geq 2$, Lemma 2.5 with $X = h = \frac{x}{2}$, then with $X = x$ and $h = \frac{x}{2}$, implies that

$$G_k^- \geq \frac{k\sqrt{x}}{\sqrt{2}}, \quad N_k^- \leq \frac{\sqrt{x}}{2\sqrt{2}}, \quad \text{and} \quad G_k^+ \geq k\sqrt{x}, \quad N_k^+ \leq \frac{\sqrt{x}}{4} \tag{3.16}$$

are admissible in Figure 5. For $1 \leq j \leq N_k^-$ and $1 \leq j \leq N_k^+$, respectively,

$$|x - (m_- - j)^k| \geq \frac{jk\sqrt{x}}{\sqrt{2}} \qquad \text{and} \qquad |x - (m_+ + j)^k| \geq jk\sqrt{x}.$$

Let $N_k^{\pm} \geq 1$, since otherwise the corresponding sum estimated below is zero. Then

$$\sum_{\substack{\frac{x}{2} < n^k < x \\ n \geq 2, n^k \notin \mathcal{N}_k}} \frac{1}{k|x - n^k|} = \sum_{j=1}^{N_k^-} \frac{1}{k|x - (m_- - j)^k|} \leq \frac{\sqrt{2} H_{N_k^-}}{k^2 \sqrt{x}} \tag{3.17}$$

and

$$\sum_{\substack{x < n^k < \frac{3x}{2} \\ n \geq 2, n^k \notin \mathcal{N}_k}} \frac{1}{k|x - n^k|} = \sum_{j=1}^{N_k^+} \frac{1}{k|x - (m_+ + j)^k|} \leq \frac{H_{N_k^+}}{k^2 \sqrt{x}}. \tag{3.18}$$

Therefore,

$$\begin{aligned} S_{\text{far}}^-(x) &= \sum_{k \geq 2} \sum_{\substack{\frac{x}{2} < n^k < x \\ n \geq 2, n^k \notin \mathcal{N}_k}} \frac{1}{k|x - n^k|} \leq \frac{\sqrt{2}}{\sqrt{x}} \sum_{k \geq 2} \frac{H_{N_k^-}}{k^2} && \text{(by (3.15) and (3.17))} \\ &\leq \frac{\sqrt{2}}{\sqrt{x}} \left(\frac{1}{2} \log x - \frac{3}{2} \log 2 + \gamma + \frac{3}{7} \right) \sum_{k \geq 2} \frac{1}{k^2} && \text{(by (2.4) and (3.16))} \\ &= \frac{\pi^2 - 6}{3\sqrt{2x}} \left(\frac{1}{2} \log x - \frac{3}{2} \log 2 + \gamma + \frac{3}{7} \right) && \left(\text{since } \zeta(2) - 1 = \tfrac{\pi^2 - 6}{6}\right) \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} S_{\text{far}}^+(x) &= \sum_{k \geq 2} \sum_{\substack{x < n^k < \frac{3x}{2} \\ n \geq 2, n^k \notin \mathcal{N}_k}} \frac{1}{k|x - n^k|} \leq \frac{1}{\sqrt{x}} \sum_{k \geq 2} \frac{H_{N_k^+}}{k^2} && \text{(by (3.15) and (3.18))} \\ &\leq \frac{1}{\sqrt{x}} \left(\frac{1}{2} \log x - 2 \log 2 + \gamma + \frac{3}{7} \right) \sum_{k \geq 2} \frac{1}{k^2} && \text{(by (2.4) and (3.16))} \end{aligned}$$

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$$= \frac{\pi^2-6}{6\sqrt{x}}\left(\frac{1}{2}\log x - 2\log 2 + \gamma + \frac{3}{7}\right) \qquad \left(\text{since } \zeta(2)-1 = \tfrac{\pi^2-6}{6}\right). \tag{3.20}$$

3.4.5 Final prime-power estimate

Using (3.14), (3.19), and (3.20), we can bound

$$S_{\text{power}}(x) \le S_{\text{sqf}}(x) = S_{\text{near}}(x) + S^-_{\text{far}}(x) + S^+_{\text{far}}(x).$$

We postpone doing this explicitly until the finale below.

4 Conclusion

For $x \ge 3.5$, with $T \ge (\log \frac{3}{2})^{-1}$, the sum (1.3) is bounded by

$$\underbrace{\frac{e}{T\log\frac{3}{2}}\left(\log\log x + \frac{\gamma}{\log x}\right)}_{\text{by (3.1)}} + \underbrace{\frac{e}{T\log\frac{3}{2}}\sum_{\frac{x}{2}<p^k<\frac{3x}{2}}\frac{1}{k|x-p^k|}}_{\text{by (3.4)}}$$

$$\le \frac{e}{T\log\frac{3}{2}}\left(\log\log x + \frac{\gamma}{\log x}\right) + \frac{e}{T\log\frac{3}{2}}\underbrace{\left(S_{\text{prime}}(x) + S_{\text{sqf}}(x)\right)}_{\text{by (3.5) and (3.10)}}$$

$$\le \frac{e}{T\log\frac{3}{2}}\left(\log\log x + \frac{\gamma}{\log x}\right) + \frac{e}{T\log\frac{3}{2}}\Bigg[\underbrace{2C + 4\log\log x}_{S_{\text{prime}}(x)\text{ bounded by (3.9)}}$$

$$+ \underbrace{\left(\log\log x + \gamma + 2\log 2 + \log 2.4 + \frac{7}{156}\right)}_{S_{\text{near}}(x)\text{ bounded by (3.14)}} + \underbrace{\frac{\pi^2-6}{3\sqrt{2x}}\left(\frac{1}{2}\log x - \frac{3}{2}\log 2 + \gamma + \frac{3}{7}\right)}_{S^-_{\text{far}}\text{ bounded by (3.19)}}$$

$$+ \underbrace{\frac{\pi^2-6}{6\sqrt{x}}\left(\frac{1}{2}\log x - 2\log 2 + \gamma + \frac{3}{7}\right)}_{S^+_{\text{far}}(x)\text{ bounded by (3.20)}}\Bigg]$$

$$< \frac{1}{T}\left(40.22465\log\log x + 58.11106 + \frac{3.86972}{\log x} + \frac{5.21918\log x}{\sqrt{x}} - \frac{1.85268}{\sqrt{x}}\right).$$

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