

AN ELEMENTARY PROOF OF THE CHROMATIC SMITH FIXED POINT THEOREM

WILLIAM BALDERRAMA AND NICHOLAS J. KUHN

(communicated by J. P. C. Greenlees)

Abstract

A recent theorem by T. Barthel, M. Hausmann, N. Naumann, T. Nikolaus, J. Noel, and N. Stapleton says that if A is a finite abelian p -group of rank r , then any finite A -space X which is acyclic in the n th Morava K -theory with $n \geq r$ will have its subspace X^A of fixed points acyclic in the $(n - r)$ th Morava K -theory. This is a chromatic homotopy version of P. A. Smith's classical theorem that if X is acyclic in mod p homology, then so is X^A .

The main purpose of this paper is to give an elementary proof of this new theorem that uses minimal background, and follows, as much as possible, the reasoning in standard proofs of the classical theorem. We also give a new fixed point theorem for finite dimensional, but possibly infinite, A -CW complexes, which suggests some open problems.

1. Introduction

Fixing a prime p and finite group G , say that a G -space X is a finite G -space if its p -localization is a retract of the p -localization of a finite G -CW complex in the G -equivariant homotopy category. We let X^G denote its subspace of fixed points.

Let $K(n)_*$ denote the n th Morava K -theory at the prime p . In particular, $K(1)_*$ is a summand of complex K -theory with mod p coefficients, and $K(0)_*$ is rational homology.

A key result in [6A19] can be stated as follows.

Theorem 1.1. *Let A be a finite abelian p -group of rank r , and let X be a finite A -space. If $\tilde{K}(n)_*(X) = 0$ with $n \geq r$, then $\tilde{K}(n - r)_*(X^A) = 0$.*

This is a chromatic homotopy theory analogue of the following classical theorem of P. A. Smith [S41].

Theorem 1.2. *Let P be a finite p -group, and let X be a finite dimensional P -space. If $\tilde{H}_*(X; \mathbb{Z}/p) = 0$, then $\tilde{H}_*(X^P; \mathbb{Z}/p) = 0$.*

Received March 3, 2023, revised March 24, 2023; published on February 21, 2024.

2020 Mathematics Subject Classification: Primary 55M35; Secondary 55N20, 55P42, 55P91.

Key words and phrases: fixed point, chromatic homotopy, Morava K -theory.

Article available at <http://dx.doi.org/10.4310/HHA.2024.v26.n1.a8>

Copyright © 2024, International Press. Permission to copy for private use granted.

We note that Theorem 1.1 follows by iteration from the cases when $A = C_{p^k}$, the cyclic p -group of order p^k , and Theorem 1.2 follows from the case when $P = C_p$.

The main purpose of this note is to give an elementary proof of Theorem 1.1 that follows, as much as possible, the reasoning in standard proofs of the classical theorem. All needed background material about $K(n)_*$ and related theories is in papers published by 2000. We hope that our presentation will lend some clarity about the interesting remaining problems in this area.

The classical theorem includes the statement that $X^P \neq \emptyset$, while the version of Theorem 1.1 proved in [6A19] implicitly assumes that X has a point fixed by A , an assumption we do not need to make. Indeed, the first steps in our proof hold when X is just assumed to be finite dimensional, and not necessarily finite, and lead to the following new fixed point theorem.

Theorem 1.3. *Let A be a finite abelian p -group of rank r , and X a finite dimensional A -CW complex. If $\tilde{K}(n)_*(X) = 0$ with $n \geq r$, then $X^A \neq \emptyset$.*

We note that, if X is any space, then $\tilde{K}(n)_*(X) = 0 \Rightarrow \tilde{K}(r)_*(X) = 0$ for $n \geq r \geq 1$ [Bou99]. (This generalized Ravenel's result [Rav84, Thm. 2.11] about finite X .) Thus this fixed point theorem for all n is implied by the special case when $n = r$.

In §2, we recall some needed background material, and deduce some simple consequences. The theorems are then quickly proved in §3.

A final section has various remarks and speculations. In particular, we wonder if some weakening of the finiteness hypothesis of Theorem 1.1 might be possible, and we are curious about the existence of examples showing that Theorem 1.3 is as strong as possible.

Acknowledgments

The second author is a PI of RTG NSF grant DMS-1839968, which has supported the research of the first author. Some of the writing of this paper was done while the second author was visiting the Utrecht Geometry Center, with support from Simons Foundation Collaboration Grant 709802. We thank Neil Strickland for helpful email about literature references, and Tom Goodwillie and Ian Leary for their helpful answers to a Mathoverflow query (see §4.3).

2. Background material and a localization result

2.1. Morava E -theory and $E_n^*(BA)$

Recall the Brown–Peterson homology theory BP , with coefficient ring $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$. We work with complex oriented theories E that are p -local and with p -typical formal group laws. The coefficient ring E_* of such a theory is a BP_* -algebra, and is said to be Landweber exact and v_n -periodic if $v_0, v_1, \dots$ acts as a regular sequence on E_* with v_n acting as a unit on $E_*/(v_0, \dots, v_{n-1})$. (Throughout, we let $v_0 = p$, as is standard.)

Lemma 2.1 ([H95a, Cor. 1.12]). *Let E_* be Landweber exact and v_n -periodic. A spectrum is E_* -acyclic if and only if it is $K(m)_*$ -acyclic for $0 \leq m \leq n$. Thus, if a finite spectrum is E^* -acyclic then it is $K(n)_*$ -acyclic.*

Now let E_n be the n th Morava E -theory, as in [H95b], [HS99], or [HKR00]. There are variants of these, so for concreteness, we will say that

$$E_n^* = \mathbb{Z}_p[u^{\pm 1}][[v_1, \dots, v_{n-1}]],$$

where $u \in E_n^{-2}$, and with complex orientation $y \in E_n^2(BU(1))$ whose associated formal group law F has p -series of the form

$$[p](y) = py +_F v_1 y^p +_F \dots +_F v_{n-1} y^{p^{n-1}} +_F u^{p^n-1} y^{p^n}.$$

In particular, $v_n = u^{p^n-1}$. This is a Landweber exact and v_n -periodic theory, with the following additional property.

Lemma 2.2 ([H95b, Prop. 3.6]). *If $f: X \rightarrow Y$ is a map of spectra, then f is an E_n^* -isomorphism if and only if it is a $K(n)_*$ -isomorphism.*

Proposition 2.3. *Let G be a finite group and X a G -space. If $\tilde{K}(n)_*(X) = 0$, then the map $EG \times_G X \rightarrow BG$ induces an isomorphism*

$$E_n^*(BG) \simeq E_n^*(EG \times_G X).$$

Proof. There are implications

$$\begin{aligned} \tilde{K}(n)_*(X) = 0 &\Leftrightarrow K(n)_*(X) \simeq K(n)_*(pt) \\ &\Rightarrow K(n)_*(EG \times_G X) \simeq K(n)_*(BG) \\ &\Leftrightarrow E_n^*(BG) \simeq E_n^*(EG \times_G X). \end{aligned}$$

The previous lemma gives the third implication. The second implication is a special case of a general fact: if $f: X \rightarrow Y$ is a G -equivariant map between G -spaces that is an isomorphism in a generalized homology theory E_* , then it induces an isomorphism on the associated Borel theory $E_*(EG \times_G X) \xrightarrow{\sim} E_*(EG \times_G Y)$. $\square$

We recall some basic calculations as in [HKR00, §5]. First the complex orientation $y \in E_n^2(BU(1))$ defines an isomorphism $E_n^*(BU(1)) \simeq E_n^*[[y]]$, and the p^k -series satisfies $[p^k](y) \equiv u^{p^{nk}-1} y^{p^{nk}} \pmod{(v_0, \dots, v_{n-1}, y^{p^{nk}+1})}$.

The standard inclusion $i_k: C_{p^k} \hookrightarrow U(1)$ induces an isomorphism

$$E_n^*(BC_{p^k}) \simeq E_n^*[[y_k]]/([p^k](y_k)),$$

where $y_k = i_k^*(y)$, and it then follows from the Weierstrass preparation theorem that $E_n^*(BC_{p^k})$ is a free E_n^* -module with basis $1, y_k, y_k^2, \dots, y_k^{p^{nk}-1}$.

Similarly, if $l < k$ and $q: C_{p^k} \rightarrow C_{p^l}$ is the standard quotient map, then the map $q^*: E_n^*(BC_{p^l}) \rightarrow E_n^*(BC_{p^k})$ satisfies $q^*(y_l) = [p^{k-l}](y_k)$, and induces an isomorphism

$$E_n^*(BC_{p^k}) \simeq E_n^*(BC_{p^l})[[y_k]]/([p^{k-l}](y_k) - y_l).$$

This makes $E_n^*(BC_{p^k})$ into a free $E_n^*(BC_{p^l})$ module of rank $p^{n(k-l)}$.

From these calculations a couple of general facts follow.

Lemma 2.4.

(a) *If A is a finite abelian p -group, then $E_n^*(BA)$ is a free E_n^* -module of rank $|A|^n$, and there is a natural isomorphism*

$$E_n^*(BA) \otimes_{E_n^*} E_n^*(X) \simeq E_n^*(BA \times X)$$

for any space X .

- (b) If $q: A \rightarrow \bar{A}$ is an epimorphism between two finite abelian p -groups, then the induced map $q^*: E_n^*(B\bar{A}) \rightarrow E_n^*(BA)$ which makes $E_n^*(BA)$ into a free $E_n^*(B\bar{A})$ -module of rank $|\text{Ker } q|^n$.

2.2. Localization

Let A be a finite abelian p -group. Observe that the group $C_p^\times$ of automorphisms of C_p acts freely on the set of nontrivial homomorphisms $\alpha: A \rightarrow C_p$ via postcomposition.

If $\alpha: A \rightarrow C_p$ is a homomorphism, we let $e(\alpha) = \alpha^*(y_1) \in E_n^2(BA)$. This is the Euler class of the 1-dimensional complex representation defined by the composite $A \xrightarrow{\alpha} C_p \hookrightarrow U(1)$. With this notation, we define two Euler classes

$$e(A), \bar{e}(A) \in E_n^*(BA)$$

as follows.

Definitions 2.5. Let $e(A) = \prod e(\alpha)$, with the product over all nontrivial homomorphisms $\alpha: A \rightarrow C_p$. Let $\bar{e}(A) = \prod e(\alpha)$, with the product over one representative of each $C_p^\times$ -orbit. The definition of $\bar{e}(A)$ involves choices, but, for concreteness, we let $\bar{e}(C_p) = y_1$, i.e. we choose the identity $C_p \rightarrow C_p$.

Basic facts about these are summarized in the next lemma.

Lemma 2.6.

- (a) If $i: A' \hookrightarrow A$ is the inclusion of a proper subgroup, then $i^*(e(A)) = i^*(\bar{e}(A)) = 0$.
(b) $e(A)^{-1}E_n^*(BA) = \bar{e}(A)^{-1}E_n^*(BA)$.

Proof. $e(A)$ and $\bar{e}(A)$ are products of the various $e(\alpha)$. If A' is a proper subgroup of A , then at least one α restricts to the trivial representation of A' . As the Euler class of the trivial representation vanishes, statement (a) follows.

To prove statement (b), we first note that, since $\bar{e}(A)$ divides $e(A)$, inverting $e(A)$ also inverts $\bar{e}(A)$. To see the converse, we need to show that, given a nontrivial $\alpha: A \rightarrow C_p$, inverting $e(\alpha)$ also inverts $e(\beta)$ for all β in the $C_p^\times$ -orbit of α . Such β are given by the composites $A \xrightarrow{\alpha} C_p \xrightarrow{m} C_p$, with $m \in \{1, \dots, p-1\}$.

Since $e(m \circ \alpha) = [m](e(\alpha))$, it suffices to show that inverting $e(\alpha)$ also inverts $[m](e(\alpha))$ for any $m \in \{1, \dots, p-1\}$. One way to see this goes as follows. If one chooses s such that $sm \equiv 1 \pmod{p}$, then $y_1 = [s]([m](y_1)) \in E_n^*(BC_p)$. Applying α^* to this equation, we learn that $e(\alpha) = [s]([m](e(\alpha))) \in E_n^*(BA)$. Since s always divides $[s](x)$, we see that $[m](e(\alpha))$ divides $e(\alpha)$, and so inverting $e(\alpha)$ also inverts $[m](e(\alpha))$. $\square$

Proposition 2.7. Let A be a finite abelian p -group, and let e be either $e(A)$ or $\bar{e}(A)$. If X is a finite dimensional A -CW complex, then the inclusion $X^A \hookrightarrow X$ induces an isomorphism

$$e^{-1}E_n^*(EA \times_A X) \xrightarrow{\sim} e^{-1}E_n^*(BA \times X^A).$$

Proof. This is an application of classic localization theory, as in [tD87, Chapter III],

and a simple proof goes as follows. For notational simplicity, let $F = X^A$. The A -equivariant cofibration sequence $F \rightarrow X \rightarrow X/F$ induces a cofibration sequence

$$BA \times F \rightarrow EA \times_A X \rightarrow EA_+ \wedge_A X/F,$$

so we need to show that $e^{-1}\tilde{E}_n^*(EA_+ \wedge_A X/F) = 0$. Since X is finite dimensional, X/F has a finite filtration by its equivariant skeleta $(X/F)^j$, so it suffices to show $e^{-1}\tilde{E}_n^*(EA_+ \wedge_A (X/F)^j / (X/F)^{j-1}) = 0$ for each j . One has an equivariant equivalence of the form

$$(X/F)^j / (X/F)^{j-1} \simeq \bigvee_i \Sigma^j(A/A_i)_+$$

where each A_i is a *proper* subgroup of A . Since $EA_+ \wedge_A (A/A_i)_+ \simeq BA_{i+}$, we need to show that $e^{-1} \prod_i E_n^*(BA_i) = 0$. But this follows immediately from Lemma 2.6(a). $\square$

3. Proofs of the theorems

Proposition 2.3, Proposition 2.7, and Lemma 2.4 combine to prove the following theorem, which gets us much of the way towards the proofs of Theorem 1.1 and Theorem 1.3.

Theorem 3.1. *Let A be a finite abelian p -group, and let X be a finite dimensional A -CW complex. If $\tilde{K}(n)_*(X) = 0$, then, with e either $e(A)$ or $\bar{e}(A)$, the map $X^A \rightarrow pt$ induces an isomorphism*

$$e^{-1}E_n^*(BA) \xrightarrow{\sim} e^{-1}E_n^*(BA) \otimes_{E_n^*} E_n^*(X^A).$$

Now we need to know something about $e^{-1}E_n^*(BA)$. The following algebraic result of Mark Hovey and Hal Sadofsky will suffice to prove Theorem 1.1.

Proposition 3.2. *Let $e = \bar{e}(C_p) = y \in E_n^*(BC_p) \simeq E_n^*[[y]]/([p](y))$. The ring*

$$e^{-1}E_n^*(BC_p) \simeq E_n^*((y))/([p](y))$$

is Landweber exact and v_{n-1} -periodic.

Proof. This is given an elementary proof in [HS96, p. 3583]. For completeness, and to emphasize its elementary nature, we give a slightly different short proof.

As $E_n^*(BC_p)$ is a free E_n^* -module, and regular sequences are preserved by localization, the ring $e^{-1}E_n^*(BC_p)$ is Landweber exact. We must only show that it is v_{n-1} -periodic. Let $I = (v_0, v_1, \dots, v_{n-2})$. Then we must show that $(e^{-1}E_n^*(BC_p))/I$ is nonzero and that v_{n-1} is a unit in this ring.

Recall $v_n = u^{p^n-1}$. We have $[p](y) \equiv v_{n-1}y^{p^{n-1}} +_F v_n y^{p^n} \pmod{I}$, so that

$$v_{n-1}y^{p^{n-1}} \equiv [-1](v_n y^{p^n}) \pmod{(I, [p](y))}$$

where $[-1](y)$ is the (-1) -series associated to the formal group law F .

When $n \geq 2$, it follows that

$$\begin{aligned} (e^{-1}E_n^*(BC_p))/I &\simeq \mathbb{Z}/p[u^{\pm 1}][[v_{n-1}]]((y))/(v_{n-1}y^{p^{n-1}} +_F v_n y^{p^n}) \\ &\simeq \mathbb{Z}/p[u^{\pm 1}][[v_{n-1}]]((y))/(v_{n-1} - y^{-p^{n-1}} \cdot [-1](v_n y^{p^n})) \\ &\simeq \mathbb{Z}/p[u^{\pm 1}]((y)) \end{aligned}$$

since $y^{-p^{n-1}} \cdot [-1](v_n y^{p^n}) = -v_n y^{p^n - p^{n-1}} \epsilon(y)$, where $\epsilon(y)$ is a monic power series in y . A similar computation applies when $n = 1$, only one ends up with

$$e^{-1}E_1^*(BC_p) \simeq \mathbb{Z}_p[u^{\pm 1}]((y))/(p - y^{-1} \cdot [-1](v_1 y^p)),$$

which will be a free $\mathbb{Q}_p[u^{\pm 1}]$ -module with basis $1, y, \dots, y^{p-2}$. In either case, this ring is visibly nonzero and contains v_{n-1} as a unit as claimed. $\square$

A variant of Proposition 3.2 will suffice to prove Theorem 1.3.

Proposition 3.3. *Let $n \geq r$. The ring $e(C_p^r)^{-1}E_n^*(BC_p^r)$ is not zero.*

Proof. The special case when $r = n$ was analyzed in [HKR00, §6.2]: the localized ring $e(C_p^n)^{-1}E_n^*(BC_p^n)$ is the nonzero ring called $L_1(E_*)$ there, and is in fact easily shown to be finite and faithfully flat over $p^{-1}E_n^*$. For the general case, note that the projection $C_p^n \rightarrow C_p^r$ onto the first r coordinates induces an algebra map from $e(C_p^r)^{-1}E_n^*(BC_p^r)$ to the nonzero ring $e(C_p^n)^{-1}E_n^*(BC_p^n)$. $\square$

Both of the propositions imply similar results for more general abelian groups.

Corollary 3.4.

- (a) *The ring $e(C_{p^k})^{-1}E_n^*(BC_{p^k})$ is Landweber exact and v_{n-1} -periodic.*
- (b) *Let A be a finite abelian p -group of rank r . Let $n \geq r$ and let $e \in E_n^*(BA)$ be $e(A)$ or $\bar{e}(A)$. Then the ring $e^{-1}E_n^*(BA)$ is nonzero.*

Proof. If A has rank r , then any surjection $q: A \rightarrow C_p^r$ induces a bijection

$$q^*: \text{Hom}(C_p^r, C_p) \xrightarrow{\sim} \text{Hom}(A, C_p).$$

It follows that $q^*(e(C_p^r)) = e(A)$. Lemma 2.4(b) tells us that, via q^* , $E_n^*(BA)$ is a finitely generated free $E_n^*(BC_p^r)$ -algebra, so $e(A)^{-1}E_n^*(BA)$ will be a finitely generated free $e(C_p^r)^{-1}E_n^*(BC_p^r)$ -algebra. Thus Proposition 3.2 proves statement (a) and Proposition 3.3 implies statement (b). $\square$

3.1. Proof of Theorem 1.3

Let A have rank r . Theorem 3.1 tells us that if $\tilde{K}(n)_*(X) = 0$, then

$$e^{-1}E_n^*(BA) \xrightarrow{\sim} e^{-1}E_n^*(BA) \otimes_{E_n^*} E_n^*(X^A).$$

Corollary 3.4(b) tells us that, if $n \geq r$, then $e^{-1}E_n^*(BA) \neq 0$. (When $r = 1$, Corollary 3.4(a) already shows this.) Thus $E_n^*(X^A) \neq 0$, and so $X^A \neq \emptyset$.

3.2. Proof of Theorem 1.1

We first observe that if A is abelian and A' is a subgroup of A , then we have $X^A = (X^{A'})^{A/A'}$. Thus it suffices to prove Theorem 1.1 when A is cyclic, as the general case will follow by iteration.

So let $C = C_{p^k}$, and let X be a finite C -space such that $\tilde{K}(n)_*(X) = 0$ for some $n \geq 1$. As before, Theorem 3.1 tells us that $X^C \neq \emptyset$ and then that

$$e(C)^{-1}E_n^*(BC) \otimes_{E_n^*} \tilde{E}_n^*(X^C) = 0.$$

Given a finite CW complex Y , let $h^*(Y) = e(C)^{-1}E_n^*(BC) \otimes_{E_n^*} \tilde{E}_n^*(Y)$. By Corollary 3.4(a), this defines a reduced cohomology theory with coefficient ring that is Landweber exact and v_{n-1} -periodic. Since $h^*(X^C) = 0$, Lemma 2.1 tells us that $\tilde{K}(n-1)_*(X^C) = 0$.

4. Further remarks

4.1. Comparison with other proofs

The first thing to say is that every proof we know of Theorem 1.1 involves inverting Euler classes at some point in the argument.

In particular, the proof of this theorem in the special case when $A = C_p$ by Strickland [S10, Thm. 16.9] (unpublished) or Balmer and Sanders [BS17] uses results of [HS96], and Barthel et al. [6A19] invert Euler classes in the key section 3 of their paper. (In all these papers, Theorem 1.1 is stated in terms of the geometric fixed point functor Φ^A applied to a compact object in A -spectra, but this is easily seen to be equivalent to Theorem 1.1 as stated above, only with the added assumption that there exists a fixed point.)

Unlike those proofs, our proof here switches to E_n -cohomology, which leads to our Theorem 3.1, and also allows for our simple deduction of Theorem 1.1 for all A from the case when $A = C_p$ using Corollary 3.4, which itself has an easy proof.

Stronger than we need for our purposes is the statement that, if A is a finite abelian p -group of rank r , and $n \geq r$, then $e(A)^{-1}E_n^*(BA)$ is Landweber exact and v_{n-r} -periodic. In [T02], T. Torii shows this when A is elementary abelian. In [MNN19, Prop. 5.28], A. Mathew, N. Naumann, and J. Noel show this for general A , as an application of Greenlees and Strickland's analysis of the rings $E_n^*(BA)/(v_0, \dots, v_t)$ in [GS99].

Regarding our fixed point theorem, Theorem 1.3, we note that application of the main theorem in [KL20] leads to the deduction of a generalization of Theorem 1.3 for all finite p -groups P [KL20, Thm. 2.20], but specialized to the finite P -space case. Our argument here proving Theorem 1.3 for abelian groups is simpler, and, of course, applies to all finite dimensional complexes with appropriate A -actions.

In a different direction, the second author has noted [K21] that Theorem 1.1 in the case when $n = r$ can be immediately 'read off of' the generalized character theory of [HKR00], and that the theorem in the general case can be similarly deduced from Stapleton's more general transchromatic characters [S13]. (This argument yields the full unbased version of Theorem 1.1.) Constructing these characters also involves inverting appropriate classes in the Morava E -theory of abelian p -groups (and then assembling the localized rings into an appropriate universal ring). For the purposes

of proving Theorem 1.1, our proof here uses much less analysis of $E_n^*(BA)$ than is used in [HKR00] and [S13].

4.2. Generalizations to non-abelian groups

The paper [BS17] reduced the problem of understanding the topology of the Balmer spectrum of a stable equivariant homotopy category to a problem that can be posed as follows: given $n \geq 0$, and a subgroup Q of a finite p -group P , compute $r_n(P, Q)$, where $r_n(P, Q)$ is the minimal r such that if X is a finite P -space and X^Q is $K(n+r)_*$ -acyclic, then X^P is $K(n)_*$ -acyclic.

As discussed in [KL20], iterated use of Theorem 1.1 shows that $r_n(P, Q) \leq r(P, Q)$, where $r(P, Q)$ is the minimal r such that there exists a sequence of subgroups

$$Q = K_0 \triangleleft K_1 \triangleleft \cdots \triangleleft K_r = P$$

with each K_{i-1} normal in K_i and each K_i/K_{i-1} cyclic. (In particular, $r(P, Q)$ is the rank of P/Q when P is abelian.)

One might ask if this upper bound is best possible. To show this for a particular n and pair $Q < P$, one needs to find a finite P -space X such that

$$\tilde{K}(n+r)_*(X^Q) = 0 \text{ but } \tilde{K}(n)_*(X^P) \neq 0,$$

with $r = r(P, Q) - 1$. The authors of [6A19] find such examples when P is abelian.

The main theorem of [KL20] says that the statement “for all finite P -spaces X , if X^Q is $K(n+r)_*$ -acyclic then X^P is $K(n)_*$ -acyclic” implies the apparently stronger statement “for all finite P -spaces X , there is an inequality

$$\dim_{K(n+r)_*} K(n+r)_*(X^Q) \geq \dim_{K(n)_*} K(n)_*(X^P).”$$

(This sort of conclusion is analogous to a theorem of Floyd [F52] in the classical situation.) This leads both to interesting applications of Theorem 1.1 [KL21] and to more families of examples showing that $r_n(P, Q) = r(P, Q)$ [KL20].

Some steps in our proof of Theorem 1.1 are quite formal, and can be generalized to nonabelian groups. But the proof ultimately hinges on Corollary 3.4, which fails in the strongest possible way for nonabelian groups: one can define an Euler class $e \in E_n^*(BG)$ for any finite group G , but $e^{-1}E_n^*(BG) = 0$ whenever G is nonabelian. If there are pairs $Q < P$ for which $r_n(P, Q)$ is strictly less than $r(P, Q)$, then some clever new ideas will be needed to prove this. [KL20] describes a number of pairs $Q < P$ for which $r_n(P, Q)$ is not yet known. Here we will just advertise one of these: let C be a noncentral subgroup of order 2 in the dihedral group D of order 16. Is $r_n(D, C)$ equal to 2 or 3? (We note that $r(D, C) = 3$ and $r_n(D, \{e\}) = r(D, \{e\}) = 2$.)

4.3. Questions about finite dimensional complexes

An obvious question is whether Theorem 1.1 might hold for finite dimensional complexes that aren’t necessarily finite. Finiteness appears in our proof to ensure that $e^{-1}E_n^*(BC) \otimes_{E_n^*} E_n^*(X^C) = 0 \implies K(n-1)_*(X^C) = 0$. Still one can ask:

Question 4.1. Let A be a finite abelian p -group of rank r , and let X be a finite dimensional A -CW complex. If $\tilde{K}(n)_*(X) = 0$ with $n \geq r$, must $\tilde{K}(n-r)_*(X^A) = 0$?

The first thing to say is that the answer is no, if $n = r$. If $M(\mathbb{Q})$ is the mapping telescope of $S^1 \xrightarrow{2} S^1 \xrightarrow{3} S^1 \xrightarrow{4} \cdots$, then $\tilde{H}_*(M(\mathbb{Q}); \mathbb{Z}) = \mathbb{Q}$, concentrated in dimension 1.

Thus if we let an abelian p -group A act trivially on $M(\mathbb{Q})$, then $\tilde{K}(n)_*(M(\mathbb{Q})) = 0$ for all $n > 0$ (and all p), but $\tilde{K}(0)_*(M(\mathbb{Q})^A) \neq 0$. But this is really a non-equivariant example, and Bousfield's result in [Bou99], that if X is a space then $\tilde{K}(n)_*(X) = 0$ implies $\tilde{K}(n-r)_*(X) = 0$ for $n-r \geq 1$, makes it plausible that the question could have a positive answer for $n-r \geq 1$. We note that [KL20] shows that such a chromatic Smith theorem would imply that the analogous chromatic Floyd theorem would hold, and, furthermore, the fixed point theorem [KL20, Thm. 2.20] would generalize to finite dimensional complexes with an action of a finite p -group.

One might also wonder if our fixed point theorem Theorem 1.3 is best possible.

Question 4.2. For all $r \geq 1$, does there exist a finite dimensional C_p^r -space X that is $K(r-1)_*$ -acyclic, and has no fixed points?

The answer is yes, when $r = 1$. The second author asked on Mathoverflow if C_p could act on a rationally trivial finite dimensional complex without fixed points. As answers, Tom Goodwillie [G22] described a free action of C_2 on a rationally acyclic 2-dimensional complex, and Ian Leary pointed to a paper of his that constructs fixed point free actions of any finite group on a 3-dimensional rationally acyclic complex [L05, Thm. 13.1].

References

- [BS17] P. Balmer and B. Sanders, *The spectrum of the equivariant stable homotopy theory of a finite group*, Invent. Math. **208** (2017), 283–326.
- [6A19] T. Barthel, M. Hausmann, N. Naumann, T. Nikolaus, J. Noel, and N. Stapleton, *The Balmer spectrum of the equivariant homotopy category of a finite abelian group*, Invent. Math. **216** (2019), 215–240.
- [Bou99] A. K. Bousfield, *On $K(n)$ -equivalences of spaces*, Homotopy invariant algebraic structures (Baltimore), Amer. Math. Soc. Contemp. Math. Ser. **239**(1999), 85–89.
- [tD87] T. tom Dieck, *Transformation groups*, De Gruyter Studies in Mathematics **8**, de Gruyter, 1987.
- [F52] E. E. Floyd, *On periodic maps and the Euler characteristics of associated spaces*, Trans. Amer. Math. Soc. **72** (1952), 138–147.
- [GS99] J. P. C. Greenlees and N. P. Strickland, *Varieties and local cohomology for chromatic group rings*, Topology **38** (1999), 1093–1139.
- [G22] T. Goodwillie, answer to *Can a cyclic group of prime order act on a rationally acyclic finite dimensional complex and have no fixed points?* asked on Mathoverflow, <https://mathoverflow.net/q/438014>.
- [HKR00] M. J. Hopkins, N. J. Kuhn, and D. C. Ravenel, *Generalized group characters and complex oriented cohomology theories*, J. Amer. Math. Soc. **13** (2000), 553–594.
- [H95a] M. Hovey, *Bousfield localization functors and Hopkins' chromatic splitting conjecture*, J. Amer. Math. Soc. Contemp. Math. **181** (1995), 225–250.

- [H95b] M. Hovey, *Cohomological Bousfield classes*, J. Pure Appl. Algebra **103** (1995), 45–59.
- [HS96] M. Hovey and H. Sadofsky, *Tate cohomology lowers chromatic Bousfield classes*, Proc. Amer. Math. Soc. **124** (1996), 3579–3585.
- [HS99] M. Hovey and N. P. Strickland, *Morava K -theories and localizations*, Mem. Amer. Math. Soc. **666**, 1999.
- [KL20] N. J. Kuhn and C. J. R. Lloyd, *Chromatic Fixed Point Theory and the Balmer spectrum for extraspecial 2-groups*, Amer. J. Math., to appear. Preprint 2020.
- [KL21] N. J. Kuhn and C. J. R. Lloyd, *Computing the Morava K -theory of real Grassmanians using chromatic fixed point theory*, Algebr. Geom. Topol., to appear. Preprint 2021.
- [K21] N. J. Kuhn, *A short proof of the chromatic Smith fixed point theorem*, Preprint 2021. To be renamed *Generalized character theory implies the chromatic Smith fixed point theorem*.
- [L05] I. Leary, *On finite subgroups of groups of type VF* , Geom. Topol. **9** (2005), 1953–1976.
- [MNN19] A. Mathew, N. Naumann, and J. Noel, *Derived induction and restriction theory*, Geom. Topol. **23** (2019), 541–636.
- [Rav84] D. C. Ravenel, *Localization with respect to certain periodic homology theories*, Amer. J. Math. **106** (1984), 351–414.
- [S41] P. A. Smith, *Fixed point theorems for periodic transformations*, Amer. J. Math. **63**(1941), 1–8.
- [S13] N. Stapleton, *Transchromatic generalized character maps*, Algebr. Geom. Topol. **13** (2013), 171–203.
- [S10] N. P. Strickland, *Thick ideals of finite G -spectra*, unpublished notes from 2010.
- [T02] T. Torii, *The geometric fixed point spectrum of $(\mathbb{Z}/p)^r$ Borel cohomology for E_n and its completion*, Amer. Math. Soc. Contemp. Math. **293** (2002), 343–369.

William Balderrama eqr8nm@virginia.edu

Department of Mathematics, University of Virginia, Charlottesville, VA 22904, USA

Nicholas J. Kuhn [nj4x@virginia.edu](mailto:njk4x@virginia.edu)

Department of Mathematics, University of Virginia, Charlottesville, VA 22904, USA