

Electromagnetic Casimir–Polder Interaction for a Conducting Cone

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Abstract: Using the formulation of the electromagnetic Green’s function of a perfectly conducting cone in terms of analytically continued angular momentum, we compute the Casimir–Polder interaction energy of a cone with a polarizable particle. We introduce this formalism by first reviewing the analogous approach for a perfectly conducting wedge, and then demonstrate the calculation through numerical evaluation of the resulting integrals.

Keywords: Casimir–Polder interaction; Kontorovich–Lebedev transformation; electromagnetic Green’s function

1. Introduction

The Casimir–Polder interaction between an uncharged conducting object on a polarizable particle [1–3] provides one of the simplest examples of a mesoscopic fluctuation-based force. Since the particle can be treated as a delta-function potential, its effects can be evaluated in any basis. As a result, in the scattering formalism, the interaction energy between the particle and conducting object can be determined directly from the full electromagnetic Green’s function in the presence of the object. In contrast, for the Casimir force between two objects, one needs the scattering T -matrices for each object connected by the free Green’s function expressed in each object’s scattering basis to propagate fluctuations between objects [4–9].

Along with the standard plane, cylinder, and sphere geometries, for which there exist analytic expressions in terms of scattering modes for the Green’s function in the presence of a perfect conductor, the conducting wedge [10,11], which also models a cosmic string [12], is a case where the Green’s function can be obtained analytically as a mode sum, by imposing the wedge boundary conditions through a discrete, fractional, angular momentum index. However, one can also use analytic continuation to a continuous, complex angular momentum to express this Green’s function in terms of the T -matrix for scattering in the angular (rather than radial) variable [13–16], an approach that then extends to the case of the cone [16] and puts the Green’s function into a form that is more directly analogous to the sphere, cylinder, and plane results. Mathematically, this approach is based on the Mehler–Fock and Kontorovich–Lebedev transforms [17]. The complex angular momentum approach requires that one consider only imaginary frequencies, however, so though it is well-suited to equilibrium problems at both zero and nonzero temperature, it cannot be applied to heat transfer [18], which must be computed on the real axis.

All of these calculations allow for investigation of the Casimir–Polder interaction near a sharp edge or tip, where the derivative expansion approach [19–23] is not applicable, yielding semi-analytic results in terms of a small number of integrals and sums. However, this approach is limited to perfect conductors and, as a result, complements calculations based on surface current methods [24–27], which are more complex numerically, but applicable to more general geometries and materials. Recent work using the multiple scattering surface method, in which one combines expansions in scattering between and within objects [28], provides a particularly relevant comparison by demonstrating the Casimir force between a dielectric wedge and plane.



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Here we use the analytically continued scattering formalism to calculate the Casimir–Polder force of a conducting cone on a polarizable atom, as might arise, for example, in the case of a particle beam passing by an atomic force microscope. We begin by reviewing the wedge calculation in the discrete angular momentum approach, and show how to obtain the same result using the analytic continuation approach. We then extend this calculation to the case of the cone, obtaining a result in terms of a sum and integral over angular momentum variables. For a special case where the particle lies on the cone axis, the calculation simplifies to a single integral. This calculation can be straightforwardly extended to frequency-dependent polarizability and nonzero temperature, although, in those cases, an additional sum or integral over frequency must be done numerically.

2. Review of Casimir–Polder Wedge

Let us begin by reviewing the Casimir–Polder interaction energy for a conducting wedge, which was computed in Refs. [10,29] and considered in the context of repulsive forces in Refs. [30,31]. Let the wedge run parallel to the z -axis and have a half-opening angle $0 < \theta_0 < \pi$ around $\theta = 0$ with the wedge vertex located at $x = y = 0$, and consider imaginary wavenumber $k = i\kappa$ with $\kappa > 0$. Note that by allowing $\theta_0 > \frac{\pi}{2}$, one is able to consider a case where the particle is inside the wedge. For a particle located at angle $\theta \in [0, 2\pi]$ obeying $\theta_0 < \theta < 2\pi - \theta_0$, one can write the full Green’s function for the wedge in terms of ordinary cylindrical wavefunctions of fractional order [10,11],

$$\mathfrak{G}(\mathbf{r}_1, \mathbf{r}_2, \kappa) = -\frac{p}{\pi} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \sum_{\ell=-\infty}^{\infty} \left(M_{\ell k_z \kappa}^{\text{outgoing}} \otimes M_{\ell k_z \kappa}^{\text{regular}*} - N_{\ell k_z \kappa}^{\text{outgoing}} \otimes N_{\ell k_z \kappa}^{\text{regular}*} \right), \quad (1)$$

in terms of the magnetic (transverse electric) and electric (transverse magnetic) modes, respectively,

$$\begin{aligned} M_{\ell k_z \kappa}(\mathbf{r}) &= \frac{1}{\sqrt{\kappa^2 + k_z^2}} \nabla \times \left[\hat{\mathbf{z}} f_{|\ell p|} \left(\sqrt{\kappa^2 + k_z^2} r \right) e^{ik_z z} \cos(\ell p(\theta - \theta_0)) \right] \\ N_{\ell k_z \kappa}(\mathbf{r}) &= \frac{1}{\kappa \sqrt{\kappa^2 + k_z^2}} \nabla \times \nabla \times \left[\hat{\mathbf{z}} f_{|\ell p|} \left(\sqrt{\kappa^2 + k_z^2} r \right) e^{ik_z z} \sin(\ell p(\theta - \theta_0)) \right] \end{aligned} \quad (2)$$

with ℓ the quantum number, the hat denoting unit vector, and $p = \frac{\pi}{2(\pi - \theta_0)}$, where the regular (outgoing) function is evaluated at the point $\mathbf{r}_1$ or $\mathbf{r}_2$ with the smaller (larger) value of the cylindrical radius r and the radial functions given in terms of Bessel functions for regular and outgoing modes as:

$$f_{|\ell p|}^{\text{regular}}(\sqrt{\kappa^2 + k_z^2} r) = I_{|\ell p|}(\sqrt{\kappa^2 + k_z^2} r) \quad \text{and} \quad f_{|\ell p|}^{\text{outgoing}}(\sqrt{\kappa^2 + k_z^2} r) = K_{|\ell p|}(\sqrt{\kappa^2 + k_z^2} r). \quad (3)$$

This Green’s function then obeys:

$$(\nabla \times \nabla \times + \kappa^2) \mathfrak{G}(\mathbf{r}_1, \mathbf{r}_2, \kappa) = \delta^{(3)}(\mathbf{r}_1 - \mathbf{r}_2) \quad (4)$$

in the presence of the conducting wedge, whereas the free Green’s function $\mathfrak{G}_0(\mathbf{r}_1, \mathbf{r}_2, \kappa)$, given by setting $p = 1$ and replacing the trigonometric functions $\sin(\ell p(\theta - \theta_0))$ and $\cos(\ell p(\theta - \theta_0))$ in Equation (2) with $e^{i\ell\theta}/\sqrt{2}$, obeys the same equation in empty space.

One can then use the “TGTG” (T -matrix/free Green’s function) [4–9] formulation of the Casimir energy, considering only the lowest-order interaction with the potential for a particle with polarizability α at position $\mathbf{r}$,

$$V(\mathbf{r}') = -4\pi\alpha\kappa^2\delta^{(3)}(\mathbf{r} - \mathbf{r}'), \quad (5)$$

which can be expressed in any basis since it is a delta function, $\delta^{(3)}(\mathbf{r} - \mathbf{r}')$.

The result for the interaction energy of a particle with isotropic polarizability α becomes [10,29]:

$$\begin{aligned} U(\mathbf{r}) &= -\frac{\hbar c}{2\pi} \int_0^\infty \text{Tr}[V(\mathbf{r})(\mathfrak{G}(\mathbf{r}, \mathbf{r}', \kappa) - \mathfrak{G}_0(\mathbf{r}, \mathbf{r}', \kappa))] d\kappa \\ &= 2\alpha\hbar c \int_0^\infty \kappa^2 \text{tr}[\mathfrak{G}(\mathbf{r}, \mathbf{r}, \kappa) - \mathfrak{G}_0(\mathbf{r}, \mathbf{r}, \kappa)] d\kappa \\ &= -\frac{3\alpha\hbar c}{8\pi r^4 \sin^4(p(\theta - \theta_0))} \left[p^4 - \frac{2}{3} p^2 (p^2 - 1) \sin^2(p(\theta - \theta_0)) - \frac{1}{135} (p^2 - 1)(p^2 + 11) \sin^4(p(\theta - \theta_0)) \right], \quad (6) \end{aligned}$$

where $\hbar$ is the reduced Planck's constant, c denotes the speed of light, 'Tr' denotes the trace, which includes the trace over the spatial coordinate, and 'tr' denotes the trace only over polarizations. In this approach, there is not a straightforward way to subtract the free contribution mode-by-mode, so one instead uses a point-splitting argument to subtract the entire contribution from the free Green's function at once.

For the case of the cone, there does not exist an analog of this full Green's function written in terms of a rescaled order. As a result, we next recompute the result for the wedge using a different form of the Green's function, which generalizes more readily to the case of the cone. In this approach, the angular momentum sum is replaced via analytic continuation by an integral, yielding for the free Green's function [13,16]:

$$\begin{aligned} \mathfrak{G}_0(\mathbf{r}_1, \mathbf{r}_2, \kappa) &= -\frac{1}{\pi^2} \int_{-\infty}^\infty \frac{dk_z}{2\pi} \int_0^\infty d\lambda \left(M_{\lambda k_z \kappa}^{\text{outgoing},+} \otimes M_{\lambda k_z \kappa}^{\text{regular},+*} + M_{\lambda k_z \kappa}^{\text{outgoing},-} \otimes M_{\lambda k_z \kappa}^{\text{regular},-*} \right. \\ &\quad \left. - N_{\lambda k_z \kappa}^{\text{outgoing},+} \otimes N_{\lambda k_z \kappa}^{\text{regular},+*} - N_{\lambda k_z \kappa}^{\text{outgoing},-} \otimes N_{\lambda k_z \kappa}^{\text{regular},-*} \right), \quad (7) \end{aligned}$$

where the transverse modes are:

$$M_{\lambda k_z \kappa}(\mathbf{r}) = \frac{1}{\sqrt{\kappa^2 + k_z^2}} \nabla \times \left[\hat{\mathbf{z}} K_{i\lambda} \left(\sqrt{\kappa^2 + k_z^2} r \right) e^{ik_z z} f_\lambda(\theta) \right] \quad \text{and} \quad N_{\lambda k_z \kappa}(\mathbf{r}) = \frac{1}{\kappa} \nabla \times M_{\lambda k_z \kappa}(\mathbf{r}) \quad (8)$$

and $\theta \in [-\pi, \pi]$. One has both even and odd modes, with regular modes given by:

$$f_\lambda^{\text{regular},+}(\theta) = \cosh(\lambda\theta) \quad \text{and} \quad f_\lambda^{\text{regular},-}(\theta) = \sinh(\lambda\theta) \quad (9)$$

and outgoing modes given by

$$f_\lambda^{\text{outgoing},+}(\theta) = \cosh(\lambda(\pi - |\theta|)) \quad \text{and} \quad f_\lambda^{\text{outgoing},-}(\theta) = \sinh(\lambda(\pi - |\theta|)) \text{sgn } \theta, \quad (10)$$

where λ denotes the analytically continued angular quantum number, and the regular (outgoing) functions are evaluated at the point $\mathbf{r}_1$ or $\mathbf{r}_2$ with the smaller (larger) value of $|\theta|$. Note that the star indicates the conjugation of the complex exponential part of the function only.

Although not needed for the computation, the corresponding longitudinal mode is:

$$L_{\lambda k_z \kappa}(\mathbf{r}) = \frac{1}{\kappa} \nabla \left[K_{i\lambda} \left(\sqrt{\kappa^2 + k_z^2} r \right) e^{ik_z z} f_\lambda(\theta) \right]. \quad (11)$$

If its contribution is added to the free Green's function, the result is equal to the scalar Green's function, $\frac{1}{2\pi} \int_{-\infty}^\infty \frac{dk_z}{2\pi} K_0 \left(\sqrt{\kappa^2 + k_z^2} \left| r_1 e^{i\theta_1} - r_2 e^{i\theta_2} \right| \right) e^{ik_z(z_> - z_<)}$, times the identity matrix; without this contribution, one obtains the same scalar times the projection matrix on to the transverse components. Here, $z_>$ ($z_<$) is the z coordinate associated with the point with the larger (smaller) value of $|\theta|$.

In this approach, we take the wedge to be located at $\theta = \pm\theta_0$ and the particle's location will always have $|\theta| > \theta_0$. One then obtains the full Green's function by replacing the

regular solution with a combination of regular and outgoing solutions given in terms of the T -matrix:

$$\mathcal{T}_\lambda^{M,+} = \frac{\sinh(\lambda\theta_0)}{\sinh(\lambda(\pi - \theta_0))} = -\mathcal{T}_\lambda^{N,-} \quad \text{and} \quad \mathcal{T}_\lambda^{M,-} = \frac{\cosh(\lambda\theta_0)}{\cosh(\lambda(\pi - \theta_0))} = -\mathcal{T}_\lambda^{N,+}, \quad (12)$$

so that it now obeys the conducting boundary conditions on the wedge, yielding:

$$\begin{aligned} \mathfrak{G}(\mathbf{r}_1, \mathbf{r}_2, \kappa) = & -\frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \int_0^{\infty} d\lambda \left[M_{\lambda k_z \kappa}^{\text{outgoing},+} \otimes \left(M_{\lambda k_z \kappa}^{\text{regular},+*} + \mathcal{T}_\lambda^{M,+} M_{\lambda k_z \kappa}^{\text{outgoing},+*} \right) \right. \\ & + M_{\lambda k_z \kappa}^{\text{outgoing},-} \otimes \left(M_{\lambda k_z \kappa}^{\text{regular},-*} + \mathcal{T}_\lambda^{M,-} M_{\lambda k_z \kappa}^{\text{outgoing},-*} \right) \\ & - N_{\lambda k_z \kappa}^{\text{outgoing},+} \otimes \left(N_{\lambda k_z \kappa}^{\text{regular},+*} + \mathcal{T}_\lambda^{N,+} N_{\lambda k_z \kappa}^{\text{outgoing},+*} \right) \\ & \left. - N_{\lambda k_z \kappa}^{\text{outgoing},-} \otimes \left(N_{\lambda k_z \kappa}^{\text{regular},-*} + \mathcal{T}_\lambda^{N,-} N_{\lambda k_z \kappa}^{\text{outgoing},-*} \right) \right]. \end{aligned} \quad (13)$$

In this form, one can subtract the free Green's function mode-by-mode, leaving only the terms with outgoing waves multiplied by the T -matrix. One obtains for the energy:

$$\begin{aligned} U(\mathbf{r}) = & -\frac{\alpha \hbar c}{\pi^3 r^2} \int_0^{\infty} d\kappa \int_{-\infty}^{\infty} dk_z \int_0^{\infty} d\lambda \frac{1}{\sinh(2(\pi - \theta_0)\lambda)} \\ & \left\{ K_{i\lambda}(\sqrt{\kappa^2 + k_z^2} r) \left[\left(r^2(\kappa^2 + k_z^2) + \lambda^2 \right) \cosh(2(\pi - \theta)\lambda) \sinh \pi \lambda \right. \right. \\ & \left. \left. + \left(r^2(\kappa^2 + k_z^2) + \frac{\kappa^2 - k_z^2}{\kappa^2 + k_z^2} \lambda^2 \right) \sinh((\pi - 2\theta_0)\lambda) \right] \right. \\ & \left. + r^2 \left(\frac{\partial}{\partial r} K_{i\lambda}(\sqrt{\kappa^2 + k_z^2} r) \right)^2 \left(\cosh(2(\pi - \theta)\lambda) \sinh \pi \lambda + \frac{k_z^2 - \kappa^2}{k_z^2 + \kappa^2} \sinh((\pi - 2\theta_0)\lambda) \right) \right\} \\ = & -\frac{\alpha \hbar c}{\pi r^4} \int_0^{\infty} d\lambda \left[\frac{\lambda + \lambda^3}{3} \coth \pi \lambda - \frac{1}{3} \coth(2(\pi - \theta_0)\lambda) + \frac{\cosh(2(\pi - \theta)\lambda)}{\sinh(2(\pi - \theta_0)\lambda)} \right], \end{aligned} \quad (14)$$

where the integrals over κ and k_z are done using polar coordinates. After carrying out the λ integral, one obtains agreement with Equation (6).

3. Electromagnetic Cone Green's Function

Let us now construct the Green's function for the perfectly conducting cone with half-opening angle $0 < \theta_0 < \pi$, centered on the z -axis with the cone vertex at $z = 0$, as shown in Figure 1. We again consider imaginary wavenumber $k = i\kappa$ with $\kappa > 0$. Note that by allowing $\theta_0 > \frac{\pi}{2}$, one can consider a case where the particle is inside the cone. From Ref. [16], one has magnetic (transverse electric) and electric (transverse magnetic) transverse modes:

$$\mathbf{M}_{\lambda m \kappa}(\mathbf{r}) = \nabla \times \left[\mathbf{r} k_{i\lambda - \frac{1}{2}}(\kappa r) e^{im\phi} f_{\lambda m}(\mathbf{r}) \right] \quad \text{and} \quad \mathbf{N}_{\lambda m \kappa}(\mathbf{r}) = \frac{1}{\kappa} \nabla \times \mathbf{M}_{\lambda m \kappa}(\mathbf{r}) \quad (15)$$

with

$$f_{\lambda m}^{\text{regular}}(\mathbf{r}) = P_{i\lambda - \frac{1}{2}}^{-m}(\cos \theta) \quad \text{and} \quad f_{\lambda m}^{\text{outgoing}}(\mathbf{r}) = P_{i\lambda - \frac{1}{2}}^m(-\cos \theta), \quad (16)$$

where $P_{i\lambda - \frac{1}{2}}^m(\cos \theta)$ is the Legendre function of the first kind and $k_{i\lambda - \frac{1}{2}}^m(\kappa r) = \sqrt{\frac{2}{\pi \kappa r}} K_{i\lambda}(\kappa r)$ is the modified spherical Bessel function of the third kind, both with complex degree/order $\ell = i\lambda - \frac{1}{2}$. The “ghost” mode [16] is:

$$\mathbf{R}_{\lambda m \kappa} = \mathbf{r} \times \nabla \left[k_{i\lambda - \frac{1}{2}}(\kappa r) P_{i\lambda - \frac{1}{2}}^{-|m|}(\pm \cos \theta) e^{im\phi} \right], \quad (17)$$

where the $\pm$ sign is for regular ('+') and outgoing ('-') modes. Its contribution arises from the contour integral used to turn the sum over the angular momentum quantum number ℓ into the integral over its analytic continuation λ , in which it cancels the contribution from the $\ell = 0$ mode, since that mode does not exist in electromagnetism. As a result, it is only ever evaluated at $\lambda = \frac{1}{2i}$, corresponding to $\ell = 0$.

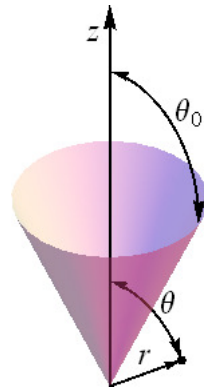


Figure 1. Geometry of cone with half-opening angle θ_0 and particle at radius r and angle $\theta > \theta_0$.

In this basis, the free Green's function is [16]:

$$\begin{aligned} \mathfrak{G}_0(r_1, r_2, \kappa) = & -\frac{\kappa}{4\pi} \left[\sum_{m=-\infty}^{\infty} \int_0^{\infty} \frac{\lambda \tanh \pi \lambda}{\lambda^2 + \frac{1}{4}} d\lambda \left(M_{\lambda m \kappa}^{\text{outgoing}} \otimes M_{\lambda m \kappa}^{\text{regular}*} - N_{\lambda m \kappa}^{\text{outgoing}} \otimes N_{\lambda m \kappa}^{\text{regular}*} \right) \right. \\ & \left. + \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \Gamma(|m|) \Gamma(|m| + 1) R_{\lambda m \kappa}^{\text{outgoing}} \otimes R_{\lambda m \kappa}^{\text{regular}*} \right]_{\lambda=\frac{1}{2i}}, \end{aligned} \quad (18)$$

where the regular (outgoing) function is evaluated at the point r_1 or r_2 with the smaller (larger) value of $|\theta|$, and $\Gamma(x)$ is the gamma function. Note that, as above, the star indicates the conjugation of the complex exponential part of the function only. Here, the integral over $\lambda = \frac{1}{i} \left(\ell + \frac{1}{2} \right)$ represents the analytic continuation of the sum over ℓ .

For completeness, we also give the longitudinal mode:

$$L_{\lambda m \kappa} = -\frac{\sqrt{\lambda^2 + \frac{1}{4}}}{\kappa} \nabla \left[k_{i\lambda-\frac{1}{2}}(\kappa r) e^{im\phi} f_{\lambda m}(r) \right] \quad (19)$$

for this geometry. If its contribution is added to the free Green's function, the result is equal to the scalar Green's function $\frac{\kappa}{4\pi} k_0(\kappa|r_1 - r_2|)$ times the identity matrix; without this contribution, one obtains the same scalar times the projection matrix on to the transverse components.

The full Green's function in the presence of the conducting boundary $\mathfrak{G}$ is then given by the same expression with the replacement, $\chi^{\text{regular}*} \rightarrow \chi^{\text{regular}*} + \mathcal{T}_{\lambda m}^{\chi} \chi^{\text{outgoing}*}$, where $\chi = M, N, R$; again, the star indicates the conjugation of the complex exponential part of the function only, and $\mathcal{T}_{\lambda m}^{\chi}$ is the corresponding T -matrix element [16],

$$\mathcal{T}_{\lambda m}^N = -\frac{P_{i\lambda-\frac{1}{2}}^{-m}(\cos \theta_0)}{P_{i\lambda-\frac{1}{2}}^m(-\cos \theta_0)}, \quad \mathcal{T}_{\lambda m}^M = -\frac{\frac{\partial}{\partial \theta_0} P_{i\lambda-\frac{1}{2}}^{-m}(\cos \theta_0)}{\frac{\partial}{\partial \theta_0} P_{i\lambda-\frac{1}{2}}^m(-\cos \theta_0)}, \quad \text{and} \quad \mathcal{T}_{\lambda m}^R = \frac{P_{i\lambda-\frac{1}{2}}^{-|m|}(\cos \theta_0)}{P_{i\lambda-\frac{1}{2}}^{-|m|}(-\cos \theta_0)}, \quad (20)$$

in terms of θ_0 , the half-opening angle of the cone. Subtracting the contribution from the free Green's function then cancels the term with the regular solution, leaving only the product of outgoing solutions in the interaction energy.

4. Electromagnetic Cone Casimir–Polder Energy

After some algebra and simplification, one obtains the Casimir–Polder interaction energy for an atom with isotropic and frequency-independent polarizability α at distance r from the cone vertex and angle θ from the cone axis, with $|\theta| > \theta_0$, as:

$$\begin{aligned}
 U(r) &= 2\alpha\hbar c \int_0^\infty \kappa^2 \text{tr}[\mathfrak{G}(\mathbf{r}, \mathbf{r}, \kappa) - \mathfrak{G}_0(\mathbf{r}, \mathbf{r}, \kappa)] d\kappa \\
 &= -\frac{\alpha\hbar c}{2\pi} \int_0^\infty \kappa^3 \text{tr} \left[\sum_{m=-\infty}^\infty \int_0^\infty \frac{\lambda \tanh \pi\lambda}{\lambda^2 + \frac{1}{4}} d\lambda \left(\mathcal{T}_{\lambda m}^M \mathbf{M}_{\lambda m \kappa}^{\text{outgoing}} \otimes \mathbf{M}_{\lambda m \kappa}^{\text{outgoing}*} - \mathcal{T}_{\lambda m}^N \mathbf{N}_{\lambda m \kappa}^{\text{outgoing}} \otimes \mathbf{N}_{\lambda m \kappa}^{\text{outgoing}*} \right) \right. \\
 &\quad \left. + \sum_{\substack{m=-\infty \\ m \neq 0}}^\infty \Gamma(|m|) \Gamma(|m|+1) \mathcal{T}_{\lambda m}^R \mathbf{R}_{\lambda m \kappa}^{\text{outgoing}} \otimes \mathbf{R}_{\lambda m \kappa}^{\text{outgoing}*} \right]_{\lambda=\frac{1}{2r}} d\kappa \\
 &= \frac{\alpha\hbar c}{2\pi r^2} \sum_{m=-\infty}^\infty \int_0^\infty \kappa d\kappa \left\{ \int_0^\infty d\lambda \lambda \tanh \pi\lambda \left[\left(\frac{\partial}{\partial \theta} P_{i\lambda-\frac{1}{2}}^m(-\cos \theta) \right)^2 + \frac{m^2}{\sin^2 \theta} P_{i\lambda-\frac{1}{2}}^m(-\cos \theta)^2 \right] \right. \\
 &\quad \times \left(\mathcal{T}_{\lambda m}^N \left(\frac{\partial}{\partial r} (rk_{i\lambda-\frac{1}{2}}(\kappa r)) \right)^2 - \mathcal{T}_{\lambda m}^M \kappa^2 r^2 k_{i\lambda-\frac{1}{2}}(\kappa r)^2 \right) \frac{1}{\left(\lambda^2 + \frac{1}{4} \right)} \\
 &\quad \left. + \mathcal{T}_{\lambda m}^N \left(\lambda^2 + \frac{1}{4} \right) P_{i\lambda-\frac{1}{2}}^m(-\cos \theta)^2 k_{i\lambda-\frac{1}{2}}(\kappa r)^2 \right] - \left(\frac{1+\cos \theta}{1-\cos \theta} \right)^{|m|} \left(\frac{1-\cos \theta_0}{1+\cos \theta_0} \right)^{|m|} \frac{2|m|}{\sin^2 \theta} e^{-2\kappa r} \left. \right\} \\
 &= \frac{\alpha\hbar c}{8r^4} \left\{ \sum_{m=-\infty}^\infty \int_0^\infty d\lambda \lambda \text{sech } \pi\lambda \tanh \pi\lambda \left[2\mathcal{T}_{\lambda m}^N \left(\lambda^2 + \frac{1}{4} \right) P_{i\lambda-\frac{1}{2}}^m(-\cos \theta)^2 \right. \right. \\
 &\quad \left. \left. + \left(\mathcal{T}_{\lambda m}^N - \mathcal{T}_{\lambda m}^M \right) \left(\left(\frac{\partial}{\partial \theta} P_{i\lambda-\frac{1}{2}}^m(-\cos \theta) \right)^2 + \frac{m^2}{\sin^2 \theta} P_{i\lambda-\frac{1}{2}}^m(-\cos \theta)^2 \right) \right] - \frac{1}{\pi} \frac{\sin^2 \theta_0}{(\cos \theta - \cos \theta_0)^2} \right\}, \quad (21)
 \end{aligned}$$

where the last term arises from the “ghost” mode contribution. Here,

$$\frac{\lambda^2 + \frac{1}{4}}{2r^2} \int_0^\infty \kappa k_{i\lambda-\frac{1}{2}}(r\kappa)^2 d\kappa = \int_0^\infty \kappa^3 k_{i\lambda-\frac{1}{2}}(r\kappa)^2 d\kappa = \int_0^\infty \frac{\kappa}{r^2} \left[\frac{\partial}{\partial r} (rk_{i\lambda-\frac{1}{2}}(r\kappa)) \right]^2 d\kappa = \frac{\pi}{4r^4} \left(\lambda^2 + \frac{1}{4} \right) \text{sech } \pi\lambda \quad (22)$$

is used [32] to carry out the integral over κ , with integrals involving derivatives with respect to r obtained by differentiating under the integral sign. The ghost term can be computed using the derivative of a geometric series,

$$\sum_{m=1}^\infty m \left(\frac{1+\cos \theta}{1-\cos \theta} \right)^m \left(\frac{1-\cos \theta_0}{1+\cos \theta_0} \right)^m = \frac{\sin^2 \theta \sin^2 \theta_0}{4(\cos \theta - \cos \theta_0)^2}, \quad (23)$$

along with an elementary κ integral.

One can check the result (21) numerically in the case of $\theta_0 = \pi/2$, when this result becomes the Casimir–Polder energy of a particle at a distance $d = r|\cos \theta|$ from a conducting plane, $U(r) = -\frac{3\alpha\hbar c}{8\pi d^4}$. Let us also note that the difference of T -matrices simplifies to

$$\mathcal{T}_{\lambda m}^N - \mathcal{T}_{\lambda m}^M = \frac{4 \cosh \pi\lambda}{\pi \sin \theta_0} \frac{1}{\frac{\partial}{\partial \theta_0} [P_{i\lambda-\frac{1}{2}}^m(-\cos \theta_0)^2]} \quad (24)$$

by using the Wronskian relation between $P_\ell^m(z)$ and $P_\ell^m(-z)$.

Carefully taking the limit $\theta \rightarrow \pi$, one obtains a special case where the particle lies on the cone axis. Here, the only contributions arise from $m = -1, 0, +1$, leading to a result that simplifies to:

$$U(r) = \frac{\alpha\hbar c}{2\pi r^2} \int_0^\infty \kappa d\kappa \left\{ \int_0^\infty d\lambda \lambda \left(\lambda^2 + \frac{1}{4} \right) \tanh \pi\lambda \left[\left(\mathcal{T}_{\lambda 0}^N - \kappa^2 r^2 \mathcal{T}_{\lambda 1}^M \right) k_{i\lambda-\frac{1}{2}}(\kappa r)^2 \right. \right.$$

$$\begin{aligned}
 & + \mathcal{T}_{\lambda 1}^N \left(\frac{\partial}{\partial r} \left(r k_{i\lambda-\frac{1}{2}}(\kappa r) \right) \right)^2 \Big] - e^{-2\kappa r} \tan^2 \frac{\theta_0}{2} \Big\} \\
 = & \frac{\alpha \hbar c}{8r^4} \left\{ \int_0^\infty d\lambda \lambda \left(\lambda^2 + \frac{1}{4} \right) \operatorname{sech} \pi \lambda \tanh \pi \lambda \left[2\mathcal{T}_{\lambda 0}^N + \left(\lambda^2 + \frac{1}{4} \right) (\mathcal{T}_{\lambda 1}^N - \mathcal{T}_{\lambda 1}^M) \right] - \frac{1}{\pi} \tan^2 \frac{\theta_0}{2} \right\} \quad (25)
 \end{aligned}$$

for the cone–particle interaction energy when $\theta = \pi$. Here, it is helpful to obtain the result in the first line, before integration over κ , because that result can straightforwardly be extended to nonzero temperature and frequency-dependent polarization, as will be described in more detail below.

Within this special case, it is illustrative to consider $\theta_0 = \frac{\pi}{2}$, where the cone becomes a plane, for which $\mathcal{T}_{\lambda 0}^N = -1$ and $\mathcal{T}_{\lambda 1}^N = -\mathcal{T}_{\lambda 1}^M = -\frac{1}{\left(\lambda^2 + \frac{1}{4}\right)}$. One then uses the κ integrals above along with the integrals [32,33] (the second integral does not appear to have been obtained previously):

$$\begin{aligned}
 \int_0^\infty k_{i\lambda-\frac{1}{2}}(\kappa r)^2 \lambda \tanh \pi \lambda d\lambda &= \frac{1}{2\kappa r} e^{-2\kappa r} \\
 \int_0^\infty k_{i\lambda-\frac{1}{2}}(\kappa r)^2 \lambda^3 \tanh \pi \lambda d\lambda &= \frac{1}{2\kappa r} \left(\kappa r + \frac{1}{4} \right) e^{-2\kappa r} \\
 \int_0^\infty \left(\frac{\partial}{\partial r} \left(r k_{i\lambda-\frac{1}{2}}(\kappa r) \right) \right)^2 \lambda \tanh \pi \lambda d\lambda &= \frac{1}{2\kappa r} \left(\kappa^2 r^2 - \kappa r + \frac{1}{2} \right) e^{-2\kappa r} \\
 \int_0^\infty d\lambda \lambda \left(\lambda^2 + \frac{1}{4} \right) \operatorname{sech} \pi \lambda \tanh \pi \lambda &= \frac{1}{2\pi}, \quad (26)
 \end{aligned}$$

where, again, the integrals involving derivatives with respect to r are obtained by differentiating under the integral sign, to do both the κ and λ integrals explicitly and in either order and obtain the standard results for the plane,

$$\begin{aligned}
 U(r) &= -\frac{\alpha \hbar c}{2\pi r^2} \int_0^\infty \kappa d\kappa \left\{ \int_0^\infty d\lambda \lambda \tanh \pi \lambda \left[\left(\lambda^2 + \frac{1}{4} + \kappa^2 r^2 \right) k_{i\lambda-\frac{1}{2}}(\kappa r)^2 + \left(\frac{\partial}{\partial r} \left(r k_{i\lambda-\frac{1}{2}}(\kappa r) \right) \right)^2 \right] + e^{-2\kappa r} \right\} \\
 &= -\frac{\alpha \hbar c}{4\pi r^3} \int_0^\infty d\kappa \left(2\kappa^2 r^2 + 2\kappa r + 1 \right) e^{-2\kappa r} \\
 &= -\frac{\alpha \hbar c}{2r^4} \left[\int_0^\infty d\lambda \lambda \left(\lambda^2 + \frac{1}{4} \right) \operatorname{sech} \pi \lambda \tanh \pi \lambda + \frac{1}{4\pi} \right] = -\frac{3\alpha \hbar c}{8\pi r^4}, \quad (27)
 \end{aligned}$$

where, in the second line, we have done the λ integral first and, in the third line, we have done the κ integral first. The former expression shows that if the ghost contribution is grouped with the electric modes, the contributions from the electric and magnetic modes match the planar calculation individually as functions of κ , and, as a result, reproduce the 5:1 ratio of the total contributions of the electric and magnetic modes [34].

5. Anisotropic Polarizability

By repeating the above calculation in the case where α is a matrix, one can extend these results to the case of an anisotropic particle. We write the polarizability in the general form:

$$\alpha = \begin{pmatrix} \alpha_\perp \cos^2 \beta & \alpha_{xy} - i\gamma_z & \alpha_{xz} + i\gamma_y \\ \alpha_{xy} + i\gamma_z & \alpha_\perp \sin^2 \beta & \alpha_{yz} - i\gamma_x \\ \alpha_{xz} - i\gamma_y & \alpha_{yz} + i\gamma_x & \alpha_{zz} \end{pmatrix}, \quad (28)$$

which includes both the symmetric and antisymmetric (nonreciprocal) off-diagonal components. Without loss of generality, we take the particle to be at $\phi = 0$, so that it lies in the xz -plane. In terms of these parameters, one obtains for the energy:

$$\begin{aligned}
 U(r) = & \frac{\hbar c}{8r^4} \left\{ \sum_{m=-\infty}^{\infty} \int_0^{\infty} d\lambda \lambda \operatorname{sech} \pi \lambda \tanh \pi \lambda \right. \\
 & \times \left[\left(\frac{\partial}{\partial \theta} P_{i\lambda-\frac{1}{2}}^m(-\cos \theta) \right)^2 \left((\alpha_{\perp} \cos^2 \beta \cot^2 \theta - 2\alpha_{xz} \cot \theta + \alpha_{zz}) \mathcal{T}_{\lambda m}^N - \alpha_{\perp} \sin^2 \beta \mathcal{T}_{\lambda m}^M \right) \right. \\
 & + 2m \frac{\gamma_x + \gamma_z \cot \theta}{\sin \theta} \left(\frac{\partial}{\partial \theta} P_{i\lambda-\frac{1}{2}}^m(-\cos \theta) \right) P_{i\lambda-\frac{1}{2}}^m(-\cos \theta) (\mathcal{T}_{\lambda m}^N - \mathcal{T}_{\lambda m}^M) \\
 & + P_{i\lambda-\frac{1}{2}}^m(-\cos \theta)^2 \left(m^2 (2\alpha_{xz} \cot \theta - \alpha_{\perp} \cos^2 \beta \cot^2 \theta - \alpha_{zz}) \mathcal{T}_{\lambda m}^M \right. \\
 & + 2 \left(\frac{1}{4} + \lambda^2 \right) \left(\alpha_{zz} \cos^2 \theta + \frac{2m^2 \alpha_{\perp}}{(1+4\lambda^2)} \csc^2 \theta \sin^2 \beta + 2\alpha_{xz} \cos \theta \sin \theta + \alpha_{\perp} \cos^2 \beta \sin^2 \theta \right) \mathcal{T}_{\lambda m}^N \left. \right] \\
 & \left. + \frac{(2\alpha_{xz} \cos \theta \sin \theta - \alpha_{\perp} + (\alpha_{\perp} \cos^2 \beta - \alpha_{zz}) \sin^2 \theta) \sin^2 \theta_0}{2\pi (\cos \theta - \cos \theta_0)^2} \right\}, \quad (29)
 \end{aligned}$$

which, on the axis $\theta = \pi$, simplifies to

$$U(r) = \frac{\hbar c}{4r^4} \left\{ \int_0^{\infty} d\lambda \lambda \left(\frac{1}{4} + \lambda^2 \right) \operatorname{sech} \pi \lambda \tanh \pi \lambda \left[\alpha_{zz} \mathcal{T}_{\lambda 0}^N + \left(\frac{\alpha_{\perp}}{4} + \frac{\gamma_z}{2} \right) \left(\frac{1}{4} + \lambda^2 \right) (\mathcal{T}_{\lambda 1}^N - \mathcal{T}_{\lambda 1}^M) \right] - \frac{\alpha_{\perp} \tan^2(\theta_0/2)}{4\pi} \right\}. \quad (30)$$

Of particular interest is the γ_z term, which generates a nonreciprocal torque around the z -axis. Comparing the α_{zz} and $\alpha_{\perp}$ contributions also enables us to compare whether a particle with a single polarization axis prefers to be aligned with or perpendicular to the axis of the cone.

6. Results and Discussion

To visualize these results numerically, in Figure 2, the Casimir–Polder interaction energy of an isotropic particle is plotted scaled by the fourth power of $r \sin(\theta - \theta_0)$, which gives the perpendicular distance from the particle to the plane in the case where $\theta - \theta_0 < \frac{\pi}{2}$. For $\theta_0 = \pi/2$, the result in units of $\alpha \hbar c$ is $-\frac{3}{8\pi} \approx -0.1194$, and past this inflection point, as the cone envelops the particle, its interaction becomes much stronger.

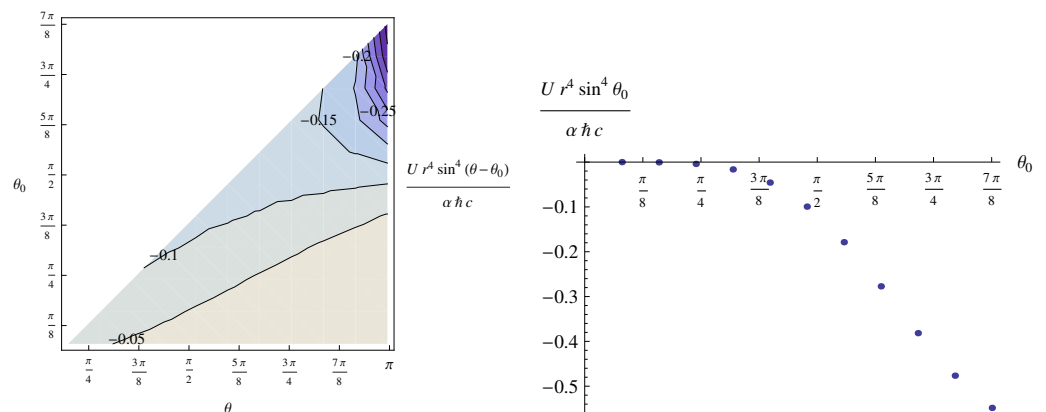


Figure 2. Scaled Casimir–Polder interaction energy, $\frac{U r^4 \sin^4(\theta - \theta_0)}{\alpha \hbar c}$, for an isotropic particle as a function of θ and θ_0 (left) and as a function of θ_0 for $\theta = \pi$ (right). See text for details.

All of the above calculations can be extended to nonzero equilibrium temperature T , in which case the integral over κ from 0 to ∞ is replaced by $\frac{2\pi k_B T}{\hbar c}$ times the sum

over Matsubara frequencies, $\kappa_n = \frac{2\pi n k_B T}{\hbar c}$, for all $n = 0, 1, 2, 3, \dots$, where the $n = 0$ contribution is counted with a weight of $\frac{1}{2}$. This term must be considered carefully, since the Bessel function has a logarithmic singularity as $\kappa \rightarrow 0$ for fixed λ . For the special case of $\theta_0 = \pi/2$, one can see explicitly from the above that this singularity disappears when the integral over λ is done first, which should remain the case in general. In all of these calculations, one can also straightforwardly move α inside the κ integral or sum to model a frequency-dependent polarizability. However, introducing either or both nonzero temperature and frequency-dependent polarizability then requires the κ sum or integral to be carried out numerically.

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References

1. Casimir, H.B.G.; Polder, D. The influence of retardation on the London-van der Waals forces. *Phys. Rev.* **1948**, *73*, 360–372. [\[CrossRef\]](#)
2. Sukenik, C.I.; Boshier, M.G.; Cho, D.; Sandoghdar, V.; Hinds, E.A. Measurement of the Casimir–Polder force. *Phys. Rev. Lett.* **1993**, *70*, 560–563. [\[CrossRef\]](#) [\[PubMed\]](#)
3. Harber, D.M.; Obrecht, J.M.; McGuirk, J.M.; Cornell, E.A. Measurement of the Casimir–Polder force through center-of-mass oscillations of a Bose–Einstein condensate. *Phys. Rev. A* **2005**, *72*, 033610. [\[CrossRef\]](#)
4. Langbein, D. *Theory of Van der Waals Attraction*; Springer: Berlin/Heidelberg, Germany, 1974. [\[CrossRef\]](#)
5. Lambrecht, A.; Maia Neto, P.A.; Reynaud, S. The Casimir effect within scattering theory. *New J. Phys.* **2006**, *8*, 243. [\[CrossRef\]](#)
6. Emig, T.; Jaffe, R.L.; Kardar, M.; Scardicchio, A. Casimir interaction between a plate and a cylinder. *Phys. Rev. Lett.* **2006**, *96*, 080403. [\[CrossRef\]](#) [\[PubMed\]](#)
7. Kenneth, O.; Klich, I. Opposites attract: A theorem about the Casimir force. *Phys. Rev. Lett.* **2006**, *97*, 160401. [\[CrossRef\]](#) [\[PubMed\]](#)
8. Emig, T.; Graham, N.; Jaffe, R.; Kardar, M. Casimir forces between arbitrary compact objects. *Phys. Rev. Lett.* **2007**, *99*, 170403. [\[CrossRef\]](#) [\[PubMed\]](#)
9. Rahi, S.J.; Emig, T.; Graham, N.; Jaffe, R.L.; Kardar, M. Scattering theory approach to electrodynamic Casimir forces. *Phys. Rev. D* **2009**, *80*, 085021. [\[CrossRef\]](#)
10. Deutsch, D.; Candelas, P. Boundary effects in quantum field theory. *Phys. Rev. D* **1979**, *20*, 3063–3080. [\[CrossRef\]](#)
11. Brevik, I.; Lygren, M. Casimir effect for a perfectly conducting wedge. *Ann. Phys.* **1996**, *251*, 157–179. [\[CrossRef\]](#)
12. Helliwell, T.M.; Konkowski, D.A. Vacuum fluctuations outside cosmic strings. *Phys. Rev. D* **1986**, *34*, 1918–1920. [\[CrossRef\]](#) [\[PubMed\]](#)
13. Oberhettinger, F. Diffraction of waves by a wedge. *Commun. Pure Appl. Math.* **1954**, *7*, 551–563. [\[CrossRef\]](#)
14. Carslaw, H.S. The scattering of sound waves by a cone. *Math. Ann.* **1914**, *75*, 133–147. [\[CrossRef\]](#)
15. Felsen, L. Plane-wave scattering by small-angle cones. *IEEE Trans. Anten. Propag.* **1957**, *5*, 121–129. [\[CrossRef\]](#)
16. Maghrebi, M.F.; Rahi, S.J.; Emig, T.; Graham, N.; Jaffe, R.L.; Kardar, M. Analytical results on Casimir forces for conductors with edges and tips. *Proc. Natl. Acad. Sci. USA* **2011**, *108*, 6867–6871. [\[CrossRef\]](#)
17. Yakubovich, S.B. *Index Transforms*; World Scientific: Singapore, 1996. [\[CrossRef\]](#)
18. Krüger, M.; Bimonte, G.; Emig, T.; Kardar, M. Trace formulas for nonequilibrium Casimir interactions, heat radiation, and heat transfer for arbitrary objects. *Phys. Rev. B* **2012**, *86*, 115423. [\[CrossRef\]](#)
19. Fosco, C.D.; Lombardo, F.C.; Mazzitelli, F.D. Proximity force approximation for the Casimir energy as a derivative expansion. *Phys. Rev. D* **2011**, *84*, 105031. [\[CrossRef\]](#)
20. Bimonte, G.; Emig, T.; Kardar, M. Casimir–Polder interaction for gently curved surfaces. *Phys. Rev. D* **2014**, *90*, 081702. [\[CrossRef\]](#)
21. Bimonte, G.; Emig, T.; Jaffe, R.L.; Kardar, M. Casimir forces beyond the proximity approximation. *Europhys. Lett. (EPL)* **2012**, *97*, 50001. [\[CrossRef\]](#)
22. Bimonte, G.; Emig, T.; Jaffe, R.L.; Kardar, M. Spectroscopic probe of the van der Waals interaction between polar molecules and a curved surface. *Phys. Rev. A* **2016**, *94*, 022509. [\[CrossRef\]](#)
23. Bimonte, G.; Emig, T.; Kardar, M. Material dependence of Casimir forces: Gradient expansion beyond proximity. *Appl. Phys. Lett.* **2012**, *100*, 074110. [\[CrossRef\]](#)

24. Rodriguez, A.; Ibanescu, M.; Iannuzzi, D.; Joannopoulos, J.D.; Johnson, S.G. Virtual photons in imaginary time: Computing exact Casimir forces via standard numerical electromagnetism techniques. *Phys. Rev. A* **2007**, *76*, 032106. [[CrossRef](#)]
25. Homer Reid, M.T.; Efficient computation of Casimir interactions between arbitrary 3D objects. *Phys. Rev. Lett.* **2009**, *103*, 040401. [[CrossRef](#)] [[PubMed](#)]
26. Homer Reid, M.T.; White, J.; Johnson, S.G. Fluctuating surface currents: An algorithm for efficient prediction of Casimir interactions among arbitrary materials in arbitrary geometries. *Phys. Rev. A* **2013**, *88*, 022514. [[CrossRef](#)]
27. Bimonte, G.; Emig, T. Unifying theory for Casimir forces: Bulk and surface formulations. *Universe* **2021**, *7*, 225. [[CrossRef](#)]
28. Emig, T.; Bimonte, G. Multiple scattering expansion for dielectric media: Casimir effect. *Phys. Rev. Lett.* **2023**, *130*, 200401. [[CrossRef](#)] [[PubMed](#)]
29. Brevik, I.; Lygren, M.; Marachevsky, V. Casimir–Polder effect for a perfectly conducting wedge. *Ann. Phys.* **1998**, *267*, 134–142. [[CrossRef](#)]
30. Milton, K.A.; Abalo, E.K.; Parashar, P.; Pourtolami, N.; Brevik, I.; Ellingsen, S.Å. Casimir–Polder repulsion near edges: Wedge apex and a screen with an aperture. *Phys. Rev. A* **2011**, *83*, 062507. [[CrossRef](#)]
31. Milton, K.A.; Abalo, E.K.; Parashar, P.; Pourtolami, N.; Brevik, I.; Ellingsen, S.Å. Repulsive Casimir and Casimir–Polder forces. *J. Phys. Math. Theor.* **2012**, *45*, 374006. [[CrossRef](#)]
32. Gradshteyn, I.S.; Ryzhik, I.M. *Table of Integrals, Series, and Products*; Academic Press: San Diego, CA, USA, 2007. Available online: <https://www.sciencedirect.com/book/9780123736376/table-of-integrals-series-and-products> (accessed on 15 September 2023).
33. Oberhettinger, F.; Higgins, T.P. *Tables of Lebedev, Mehler and Generalized Mehler Transforms*; Boeing Scientific Research Laboratories: Washington, DC, USA, 1961. Available online: <https://apps.dtic.mil/sti/tr/pdf/AD0267210.pdf> (accessed on 15 September 2023).
34. Milton, K.A. Casimir–Polder forces in inhomogeneous backgrounds. *J. Opt. Soc. Am. B* **2019**, *36*, C41–C45. [[CrossRef](#)]

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