



Letter

One-loop quantum stress-energy tensor for the kink and sine-Gordon solitons

N. Graham^{a,*,*}, H. Weigel^{b,*,*}

^a Department of Physics, Middlebury College, Middlebury, VT 05753, USA

^b Institute for Theoretical Physics, Physics Department, Stellenbosch University, Matieland 7602, South Africa

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ABSTRACT

We compute the renormalized one-loop quantum corrections to the energy density $T_{00}(x)$ and pressure $T_{11}(x)$ for solitons in the $1+1$ dimensional scalar sine-Gordon and kink models. We show how precise implementation of counterterms in dimensional regularization resolves previously identified discrepancies between the integral of $T_{00}(x)$ and the known correction to the total energy.

1. Introduction

In hadron physics, the form factors of the energy-momentum tensor have recently attracted broader attention because they can be related to generalized parton distributions [1] that are experimentally accessible [2]. The empirical status of these form factors has been summarized recently in a mini-review [3]. Theoretical input is available from lattice simulations [4] as well as from calculations in the bag [5], Skyrme, [6] and quark soliton models [7]. Even though soliton models for baryons in three space dimensions are not fully renormalizable, estimates [8] with renormalization prescriptions guided by chiral perturbation theory [9] indicate that there are substantial quantum contributions to the baryon mass. Similar results should thus also hold for the densities contained in the energy-momentum tensor. It is therefore important to gain a full understanding of the quantum contributions to the energy-momentum tensor in soliton models. As a first step we therefore explore this problem in low-dimensional, renormalizable models.

Topological solitons in $1+1$ -dimensional scalar field theory, such as the ϕ^4 kink and the sine-Gordon soliton, provide an ideal theoretical laboratory in which to study the effects of quantum fluctuations [10]. Because the resulting potentials for small-amplitude quantum fluctuations are of the exactly solvable Pöschl-Teller form, one can carry out analytic calculations to obtain exact results at one loop, such as the quantum correction to the soliton's energy [11]. These calculations often highlight subtleties of renormalization. For example, in the extension to a supersymmetric model, one might expect the quantum correction to the soliton's energy to be zero, since the spectra of Bose and Fermi modes are identical except for zero modes, and enter with opposite signs. However, by Levinson's theorem, the mismatch between zero modes also implies a difference in the phase shifts for the quantum fluctuations about the soliton. Since these phase shifts parameterize the change of the density of states generated by the soliton, there is a nonzero correction to the energy. Because this correction is negative, it appears to violate the BPS bound, but the central charge receives an equal correction by the same mechanism [12]. This result reflects the presence of an anomaly in the supersymmetric theory [13], an example of the more general relationship between anomalies and Levinson's Theorem [14].

Recent work [15,16] studying local densities has highlighted another such subtlety of these quantum corrections, for the case of local densities in a scalar theory. Here again the subtleties arise from the zero mode. Since it corresponds to translation of the soliton, it must be quantized as a collective coordinate rather than as a small-amplitude quantum fluctuation. But when one carries out this calculation for the one-loop quantum correction to the energy density, it appears that the integral over x of the result does not agree with the known correction to the total energy. To address this discrepancy, Ref. [15] introduces an additional counterterm for the stress-energy tensor, while Ref. [16] obtains a spatially constant additional correction. However, because the former is an *ad hoc* modification of the original theory and the latter represents a redefinition of the renormalization condition for the cosmological constant, both of these modifications are expected to change the total energy as well. In the

* Corresponding authors.

E-mail addresses: ngraham@middlebury.edu (N. Graham), weigel@sun.ac.za (H. Weigel).

following, we argue that this puzzle is instead resolved by a subtle aspect of the renormalization process. When the counterterm is specified precisely in dimensional regularization, one obtains a result that agrees with the known total energy with no need for additional modifications.

We begin by reviewing the scalar field theory models and their soliton solutions in Sec. 2. Then in Sec. 3 we introduce the quantum corrections, obtained using spectral methods in dimensional regularization, leading to analytic results for both models in Sec. 4, and we include additional supporting calculations in two Appendices.

2. Model

We consider a scalar model defined by the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \phi'^2 - U(\phi) \quad (1)$$

in $D = 1 + 1$ spacetime dimensions, with dot and prime denoting time and space derivatives of ϕ , respectively. The equation of motion is

$$\ddot{\phi}^2 = \phi'' - U'(\phi), \quad (2)$$

where $U'(\phi)$ refers to the derivative of U with respect to its argument. Then a static solution $\phi_0(x)$ obeys

$$\frac{d\phi_0(x)}{dx} = \pm \sqrt{2U(\phi_0(x))} \quad (3)$$

for the soliton and antisoliton. We will consider both the sine-Gordon soliton, for which

$$U^{(\text{SG})}(\phi) = m^2(1 - \cos \phi) \quad \text{and} \quad \phi_0^{(\text{SG})}(x) = 4 \arctan e^{\pm mx}, \quad (4)$$

and the ϕ^4 kink, for which

$$U^{(\text{kink})}(\phi) = \frac{m^2}{8} (\phi^2 - 1)^2 \quad \text{and} \quad \phi_0^{(\text{kink})}(x) = \pm \tanh \frac{mx}{2}, \quad (5)$$

where in both cases m is the mass of perturbative fluctuations.

In a fuller analysis of the theory, the Lagrangian would include an overall factor of $\frac{m^2}{2\lambda}$, which does not affect the equations of motion. It tracks the order of the loop expansion in terms of the four-point coupling constant λ , with the classical contribution entering at order λ^{-1} and the one-loop contribution that we focus on here entering at order λ^0 . In that analysis, it is convenient to introduce the unscaled boson field $\varphi = \frac{m}{\sqrt{2\lambda}}\phi$. We will use this scaling in Appendix A, where we carry out a perturbative expansion in the coupling.

Expanding to quadratic order, we obtain the equation for small oscillations $\eta(x, t) = \eta_k(x)e^{-i\omega t}$ around the soliton,

$$-\frac{d^2\eta_k(x)}{dx^2} + V(x)\eta_k(x) = k^2\eta_k(x), \quad (6)$$

where $\omega = \sqrt{k^2 + m^2}$ is the mode frequency and the small-oscillation potential is given by

$$V(x) = U''(\phi_0(x)) - m^2 = -\frac{\ell + 1}{\ell} m^2 \text{sech}^2 \frac{mx}{\ell} \quad (7)$$

with $\ell = 1$ for the sine-Gordon soliton and $\ell = 2$ for the kink.

Both potentials are reflectionless and are exactly solvable. The continuum modes are

$$\eta_k^{(\text{SG})}(x) = \frac{i}{\sqrt{k^2 + m^2}} (k + im \tanh mx) e^{ikx} \quad (8)$$

for the sine-Gordon soliton and

$$\eta_k^{(\text{kink})}(x) = \frac{1}{\sqrt{k^2 + m^2} \sqrt{k^2 + \frac{m^2}{4}}} \left(\frac{m^2}{4} + k^2 + \frac{3}{2} imk \tanh \frac{mx}{2} - \frac{3}{4} m^2 \tanh^2 \frac{mx}{2} \right) e^{ikx} \quad (9)$$

for the kink. These continuum modes are normalized such that $\lim_{x \rightarrow \pm\infty} |\eta_k(x)| = 1$. Both models have zero modes with $\omega = 0$,

$$\eta_0^{(\text{SG})}(x) = \sqrt{\frac{m}{2}} \text{sech} mx = \frac{1}{\sqrt{8m}} \frac{d}{dx} \phi_0^{(\text{SG})}(x) \quad (10)$$

and

$$\eta_0^{(\text{kink})}(x) = \sqrt{\frac{3m}{8}} \text{sech}^2 \frac{mx}{2} = \sqrt{\frac{3}{2m}} \frac{d}{dx} \phi_0^{(\text{kink})}(x). \quad (11)$$

The kink also has a “shape” mode

$$\eta_1^{(\text{kink})}(x) = \sqrt{\frac{3m}{4}} \tanh \frac{mx}{2} \text{sech} \frac{mx}{2} \quad (12)$$

with frequency $\omega = \sqrt{\frac{3}{4}}m$. We use standard normalization for the bound state wave-functions, $\int_{-\infty}^{\infty} dx |\eta_i(x)|^2 = 1$.

We define the Green's function with outgoing wave boundary conditions,

$$G(x_1, x_2, k) = \frac{i}{2k} \eta_{-k}(x_{<}) \eta_k(x_{>}), \quad (13)$$

where $x_{<}$ ($x_{>}$) is the smaller (larger) of x_1 and x_2 , along with the corresponding free Green's function

$$G_0(x_1, x_2, k) = \frac{i}{2k} e^{-ikx_{<}} e^{ikx_{>}} = \frac{i}{2k} e^{ik|x_1 - x_2|}. \quad (14)$$

3. Spectral method

We wish to compute one-loop quantum corrections to the stress-energy tensor. By symmetry, its off-diagonal components vanish, so we only need to calculate T_{00} and T_{11} .

Following the approach of Ref. [16], we express the calculation of the stress-energy tensor

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \mathcal{L} \quad (15)$$

in terms of four components $T_1(x)$, $T_2(x)$, $T_3(x)$, and $T_4(x)$. In this decomposition, $T_1(x)$ and $T_2(x)$ are the contributions from the quadratic field fluctuations, while $T_3(x)$ and $T_4(x)$ are the contributions from linear fluctuations that are of the same order in λ as $T_1(x)$ and $T_2(x)$, cf. Appendix A. Furthermore $T_1(x)$ and $T_3(x)$ are the derivative terms, which enter with the same sign in both T_{00} and T_{11} , while $T_2(x)$ and $T_4(x)$ are the potential terms, which enter with opposite signs.

We begin by considering the quadratic contribution to T_{00} ,

$$T_1(x) + T_2(x) = \frac{1}{2} \langle \dot{\eta}(x, t)^2 \rangle + \frac{1}{2} \langle \eta'(x, t)^2 \rangle + V(x) \langle \eta(x, t)^2 \rangle = \langle \dot{\eta}(x, t)^2 \rangle + \frac{1}{4} \left\langle \frac{d^2}{dx^2} (\eta(x, t)^2) \right\rangle, \quad (16)$$

where angle brackets refer to renormalized expectation values with the zero mode contribution omitted, and we have used the equations of motion to write the same quantity in two different forms.

We will first apply the spectral method [17,18] to the second line of Eq. (16). In order to implement the renormalization counterterms precisely, we use dimensional regularization and introduce n transverse dimensions [19], so that the soliton becomes a domain wall in $n+1$ space dimensions. We obtain

$$\begin{aligned} T_1(x) + T_2(x) &= -\frac{\Gamma\left(-\frac{n+1}{2}\right)}{(4\pi)^{\frac{n+1}{2}}} \left\{ \sum_{\text{bound states } j} \frac{1}{2} \left(\omega_j^{n+1} - m^{n+1} \right) \left(\eta_j(x)^2 + \frac{n+1}{4\omega_j^2} \frac{d^2}{dx^2} (\eta_j(x)^2) \right) \right. \\ &\quad \left. + \int_0^\infty \frac{dk}{2\pi} \left((k^2 + m^2)^{\frac{n+1}{2}} - m^{n+1} \right) \frac{2k}{i} \left[G(x, x, k) - G_0(x, x, k) \left(1 + \frac{V(x)}{2k^2} \right) \right. \right. \\ &\quad \left. \left. + \frac{n+1}{4(k^2 + m^2)} \frac{d^2}{dx^2} G(x, x, k) + \frac{im(n+1)}{4k(k^2 + m^2)^2} \frac{d^2}{dx^2} (\eta_0(x)^2) \right] \right\} \\ &= -\frac{1}{2(4\pi)^{\frac{n+1}{2}} \Gamma\left(\frac{n+3}{2}\right)} \int_m^\infty d\kappa (\kappa^2 - m^2)^{\frac{n+1}{2}} 2\kappa \left[G(x, x, i\kappa) - G_0(x, x, i\kappa) \left(1 - \frac{V(x)}{2\kappa^2} \right) \right. \\ &\quad \left. + \frac{n+1}{4(m^2 - \kappa^2)} \frac{d^2}{dx^2} G(x, x, i\kappa) + \frac{m(n+1)}{4\kappa(m^2 - \kappa^2)^2} \frac{d^2}{dx^2} (\eta_0(x)^2) \right]. \end{aligned} \quad (17)$$

There are two terms involving the free Green's function. The first one, just $G_0(x, x, k)$, subtracts the vacuum contribution and defines the zero of energy. The second one, proportional to $G_0(x, x, k)V(x)$, is the counterterm contribution, cf. Eq. (A.3) in Appendix A,

$$\Delta T_{\mu\nu} = -g_{\mu\nu} \Delta \mathcal{L}. \quad (18)$$

These counterterms are defined precisely and unambiguously via dimensional regularization [20]. For the models under consideration, they are [21]

$$\Delta \mathcal{L}^{(\text{SG})} = C^{(\text{SG})} (1 - \cos \phi) \quad \text{and} \quad \Delta \mathcal{L}^{(\text{kink})} = C^{(\text{kink})} (\phi^2 - 1). \quad (19)$$

The coefficients are chosen such that the tadpole diagram is exactly canceled. As a result, we obtain a fully continuum formulation without a need for any discretization or additional counterterms. For real k , we have used the completeness relation

$$\sum_{\text{bound states } j} \eta_j(x)^2 + \int_0^\infty \frac{dk}{\pi} \frac{2k}{i} [G(x, x, k) - G_0(x, x, k)] = 0, \quad (20)$$

which is the local equivalent of Levinson's theorem, to replace ω by $\omega - m$ in both the continuum and bound state contributions, as is necessary to obtain convergence of the integral at small k . We stress that the above sums over bound states include the zero mode. The last term in Eq. (17) removes the contribution from the zero mode, since it should be excluded in this calculation. Then we have used contour integration to write the integral on the imaginary axis $k = i\kappa$. The poles enclosed by this contour exactly cancel the explicit contributions from the bound states.

Next we use the equations of motion to recast this result into the form of the first line of Eq. (16). We obtain

$$T_1(x) + T_2(x) = -\frac{1}{2(4\pi)^{\frac{n+1}{2}} \Gamma\left(\frac{n+3}{2}\right)} \int_m^\infty d\kappa (\kappa^2 - m^2)^{\frac{n+1}{2}} 2\kappa \frac{(n+1)}{2(m^2 - \kappa^2)} \left[\frac{1-n}{1+n} (m^2 - \kappa^2) (G(x, x, i\kappa) - G_0(x, x, i\kappa)) \right. \\ \left. + \frac{d}{dx} \frac{d}{dy} (G(x, y, i\kappa) - G_0(x, y, i\kappa)) \Big|_{y=x} + \frac{m}{\kappa(m^2 - \kappa^2)} \left(\frac{d\eta_0(x)}{dx} \right)^2 \right. \\ \left. + V(x) \left(G(x, x, i\kappa) - G_0(x, x, i\kappa) \frac{\kappa^2 - m^2}{\kappa^2(n+1)} \right) + m^2 (G(x, x, i\kappa) - G_0(x, x, i\kappa)) + \frac{m}{\kappa(m^2 - \kappa^2)} (V(x) + m^2) \eta_0(x)^2 \right], \quad (21)$$

which we emphasize is simply an algebraic reorganization of Eq. (17) using the small amplitude fluctuation equation (6).

The key expression to focus on here is the factor of $\frac{\kappa^2 - m^2}{\kappa^2(n+1)}$ multiplying the free Green's function in the potential term. The need for this factor would not be apparent from the original expression, since one would just expect to subtract the free Green's function; this factor approaches unity at large κ and $n = 0$, but its difference from unity makes a finite contribution to the energy density, yielding an expression with the correct integral over space, the well-established vacuum polarization energy (VPE), cf. Eq. (34) below. Because the spectral method provides the exact counterterm in dimensional regularization, this factor is unambiguously required in order for the resulting expression to be consistent with Eq. (17).

To compute T_{11} , we will also need the difference $T_1(x) - T_2(x)$ between the derivative and potential terms. The only subtlety here is how to divide the counterterm contribution described above between the two terms. We make this determination by requiring that the contribution linear in $V(x)$ vanishes. With this condition, we obtain

$$T_1(x) - T_2(x) = -\frac{1}{2(4\pi)^{\frac{n+1}{2}} \Gamma\left(\frac{n+3}{2}\right)} \int_m^\infty d\kappa (\kappa^2 - m^2)^{\frac{n+1}{2}} 2\kappa \frac{(n+1)}{2(m^2 - \kappa^2)} \left[\frac{1-n}{1+n} (m^2 - \kappa^2) (G(x, x, i\kappa) - G_0(x, x, i\kappa)) \right. \\ \left. + \frac{d}{dx} \frac{d}{dy} (G(x, y, i\kappa) - G_0(x, y, i\kappa)) \Big|_{y=x} + \frac{m}{\kappa(m^2 - \kappa^2)} \left(\frac{d\eta_0(x)}{dx} \right)^2 \right. \\ \left. - V(x) \left(G(x, x, i\kappa) - G_0(x, x, i\kappa) \frac{\kappa^2 - 2m^2}{\kappa^2(n+1)} \right) - m^2 (G(x, x, i\kappa) - G_0(x, x, i\kappa)) - \frac{m}{\kappa(m^2 - \kappa^2)} (V(x) + m^2) \eta_0(x)^2 \right]. \quad (22)$$

In this expression, we note the factor of $\frac{\kappa^2 - 2m^2}{\kappa^2(n+1)}$ in the counterterm contribution needed to implement the renormalization condition, where one might have expected the same factor as in Eq. (21), without the factor of 2. This difference arises as a result of contributions to the integrand that formally vanish at $n = 0$, but which give a finite, nonzero contribution to the integral in dimensional regularization, and can be viewed as a consequence of the trace anomaly [22]. With this choice, we obtain the correct contribution to the trace $T_\mu{}^\mu$ due to the soliton,

$$2T_2(x) = \frac{1}{4\pi} V(x) + \langle \eta(x)^2 \rangle (V(x) + m^2), \quad (23)$$

where the first term on the right-hand side reflects the contribution from the anomaly, and we have computed the renormalized expectation value $\langle \eta(x)^2 \rangle$ in Appendix B. The soliton's translational variance yields $\partial_\mu T^{\mu 1} \neq 0$, so the composite operator $T^{\mu 1}$ is not protected against further renormalization.

4. Results

We can now carry out the integrals using Mathematica and take the limit $n \rightarrow 0$. We obtain

$$T_1^{(\text{SG})}(x) + T_2^{(\text{SG})}(x) = -\frac{m^2 \text{sech}^2 mx}{2\pi}, \quad T_1^{(\text{SG})}(x) - T_2^{(\text{SG})}(x) = 0 \quad (24)$$

for the sine-Gordon model and

$$T_1^{(\text{kink})}(x) + T_2^{(\text{kink})}(x) = m^2 \left(\frac{1}{4\sqrt{3}} - \frac{3}{8\pi} \right) \text{sech}^2 \frac{mx}{2} - \frac{17m^2}{32\sqrt{3}} \text{sech}^4 \frac{mx}{2} + \frac{5m^2}{16\sqrt{3}} \text{sech}^6 \frac{mx}{2}, \\ T_1^{(\text{kink})}(x) - T_2^{(\text{kink})}(x) = \frac{\sqrt{3}m^2}{32} \text{sech}^4 \frac{mx}{2} - \frac{m^2}{16\sqrt{3}} \text{sech}^6 \frac{mx}{2} \quad (25)$$

for the kink.

In contrast to the total energy, the densities have contributions linear in the quantum fluctuations

$$T_{\mu\mu}^{(\text{linear})} = 2\partial_\mu \eta \partial_\mu \phi_0 - g_{\mu\mu} [\partial_\nu \eta \partial^\nu \phi_0 - U'(\phi_0) \eta], \quad (26)$$

where the index μ is not summed. Following Ref. [16] and as outlined in Appendix A, we compute the expectation value

$$\langle \eta(x) \rangle = -\frac{1}{2} \int_{-\infty}^{\infty} dy U'''(\phi_0(y)) \int_0^{\infty} dq \frac{1}{2\pi \sqrt{q^2 + m^2}} \frac{2q}{i} (G(y, y, q) - G_0(y, y, q)), \int_0^{\infty} dk \frac{1}{\pi(k^2 + m^2)} \frac{2k}{i} G(x, y, k), \quad (27)$$

which in turn gives the contributions from $T_{\mu\mu}^{(\text{linear})}$ to the densities

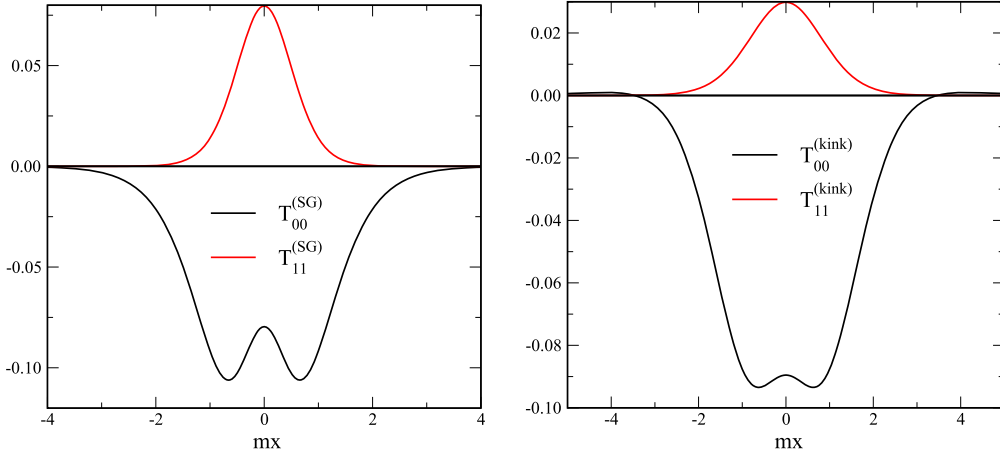


Fig. 1. Quantum corrections to the energy and pressure densities $T_{00}(x)$ and $T_{11}(x)$, respectively, for the sine-Gordon (left panel) and kink (right panel) solitons, shown in units of the perturbative mass squared, m^2 .

$$T_3(x) = \phi'_0(x) \frac{d}{dx} \langle \eta(x) \rangle \quad \text{and} \quad T_4(x) = U'(\phi_0(x)) \langle \eta(x) \rangle. \quad (28)$$

Explicit calculations yield

$$T_3^{(\text{SG})}(x) = -\frac{m^2}{4\pi} \text{sech}^2 mx + \frac{m^2}{2\pi} \text{sech}^4 mx, \quad T_4^{(\text{SG})}(x) = -\frac{m^2}{4\pi} \text{sech}^2 mx + \frac{m^2}{4\pi} \text{sech}^4 mx \quad (29)$$

for the sine-Gordon model and [16]

$$\begin{aligned} T_3^{(\text{kink})}(x) &= m^2 \left(\frac{\sqrt{3}}{32} mx \tanh \frac{mx}{2} - \frac{7}{32\sqrt{3}} - \frac{3}{16\pi} \right) \text{sech}^4 \frac{mx}{2} + m^2 \left(\frac{\sqrt{3}}{16} + \frac{9}{32\pi} \right) \text{sech}^6 \frac{mx}{2}, \\ T_4^{(\text{kink})}(x) &= m^2 \left(\frac{\sqrt{3}}{32} mx \tanh \frac{mx}{2} - \frac{1}{8\sqrt{3}} - \frac{3}{16\pi} \right) \text{sech}^4 \frac{mx}{2} + m^2 \left(\frac{1}{8\sqrt{3}} + \frac{3}{16\pi} \right) \text{sech}^6 \frac{mx}{2} \end{aligned} \quad (30)$$

for the kink. By construction of the soliton, Eq. (3), we have $\int_{-\infty}^{\infty} dx [T_3(x) + T_4(x)] = 0$ as a consistency check.

Putting these results together using

$$T_{00}(x) = T_1(x) + T_2(x) + T_3(x) + T_4(x), \quad T_{11}(x) = T_1(x) - T_2(x) + T_3(x) - T_4(x) \quad (31)$$

we obtain

$$T_{00}^{(\text{SG})}(x) = -\frac{m^2}{\pi} \text{sech}^2 mx + \frac{3m^2}{4\pi} \text{sech}^4 mx, \quad T_{11}^{(\text{SG})}(x) = \frac{m^2}{4\pi} \text{sech}^4 mx \quad (32)$$

for the sine-Gordon model and

$$\begin{aligned} T_{00}^{(\text{kink})}(x) &= m^2 \left(\frac{1}{4\sqrt{3}} - \frac{3}{8\pi} \right) \text{sech}^2 \frac{mx}{2} + m^2 \left(\frac{5}{8\sqrt{3}} + \frac{15}{32\pi} \right) \text{sech}^6 \frac{mx}{2} + m^2 \left(\frac{\sqrt{3}}{16} mx \tanh \frac{mx}{2} - \frac{7}{8\sqrt{3}} - \frac{3}{8\pi} \right) \text{sech}^4 \frac{mx}{2}, \\ T_{11}^{(\text{kink})}(x) &= \frac{3m^2}{32\pi} \text{sech}^6 \frac{mx}{2} \end{aligned} \quad (33)$$

for the kink. These functions are shown in Fig. 1. Like Ref. [15] but unlike Ref. [16], we find that $|T_{00}|$ has a local minimum at the center of the soliton. However, this minimum is not zero for the sine-Gordon model.

From the above densities we obtain the total energies

$$\int_{-\infty}^{\infty} dx T_{00}^{(\text{SG})}(x) = -\frac{m}{\pi} \quad \text{and} \quad \int_{-\infty}^{\infty} dx T_{00}^{(\text{kink})}(x) = m \left(\frac{1}{4\sqrt{3}} - \frac{3}{2\pi} \right) \quad (34)$$

in the no-tadpole renormalization scheme, which are the well-established results [11,23,24]. Similarly, the integrated pressures are

$$\int_{-\infty}^{\infty} dx T_{11}^{(\text{SG})}(x) = \frac{m}{3\pi} \quad \text{and} \quad \int_{-\infty}^{\infty} dx T_{11}^{(\text{kink})}(x) = \frac{m}{5\pi}. \quad (35)$$

5. Conclusions

We have explained calculations of the renormalized corrections to $T_{00}(x)$ and $T_{11}(x)$ for solitons in the sine-Gordon and ϕ^4 kink models, with counterterms precisely specified through dimensional regularization. The theory is defined in the continuum, without the need for artificial boundary conditions, and all counterterms are fully specified by the no-tadpole renormalization condition. When written in terms of quadratic expectation

values, the tadpole graph counterterm enters with a kinematic factor that would otherwise be unexpected. Incorporating this term gives a result for $T_{00}(x)$ whose integral gives the correct total energy, resolving a discrepancy observed in previous studies. A similar analysis of $T_{11}(x)$ shows that this calculation also captures the effects of the trace anomaly. Overall, this result provides a valuable example of the subtleties of renormalization in the context of exactly solvable models.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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Appendix A. Renormalization of $\langle\eta(x)\rangle$ in the kink model

In this appendix we describe the renormalization of $\langle\eta(x)\rangle$ in kink model. At the end we will write it in a general form that also applies to the sine-Gordon model.

Our starting point is the original Lagrangian in terms of the unscaled field $\varphi = \frac{m}{\sqrt{2\lambda}}\phi$,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - U(\varphi) \quad \text{with} \quad U(\varphi) = \frac{\lambda}{4}\left[\varphi^2 - \frac{m^2}{2\lambda}\right]^2, \quad (\text{A.1})$$

with the λ coupling constant explicitly included to allow for a perturbation expansion. The Lagrangian in Eq. (5) is obtained by taking $\lambda = \frac{m^2}{2}$. The counterterm Lagrangian at one-loop is

$$\mathcal{L}_{\text{ct}} = c\lambda\left[\varphi^2 - \frac{m^2}{2\lambda}\right] = \frac{c}{3}\left[U''(\varphi) - m^2\right] \xrightarrow{\varphi \rightarrow \varphi_0} \frac{c}{3}V(x), \quad (\text{A.2})$$

where the coupling constant is factored out to simplify subsequent expressions.

In the no-tadpole scheme, we have to remove the term linear in V from the effective action

$$\int d^2x \left[\mathcal{L} + \mathcal{L}_{\text{ct}}\right] + \frac{i}{2}\text{Trlog}[\partial_x^2 + m^2 + V(x)] = \dots + \int d^2x V(x) \left[\frac{c}{3} - \frac{i}{2} \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 - m^2}\right] \Rightarrow c = \frac{3i}{2} \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 - m^2} = \frac{3i}{2}\Delta_F^{(0)}(0). \quad (\text{A.3})$$

Here we implicitly take all loop-integrals to be dimensionally regularized. The free Feynman propagator is defined so that $(\partial_x^2 + m^2)\Delta_F^{(0)}(x) = -\delta^2(x)$. Below we will also encounter the Feynman propagator of the full theory, which solves $(\partial_x^2 + m^2 + V(x))\Delta_F(x, y) = -\delta^2(x - y)$.

As in the case of the VPE, we expect the densities $T^{\mu\nu}$ to be $\mathcal{O}(\lambda^0)$. Since $\varphi_0 = \mathcal{O}(1/\sqrt{\lambda})$ we must therefore collect all terms that contribute to $\langle\eta(x)\rangle$ at $\mathcal{O}(\sqrt{\lambda})$. To this end we need to compute the path integral

$$\begin{aligned} \langle\eta(x)\rangle &= \frac{1}{Z} \int D\eta \eta(x) e^{\frac{i}{2} \int \eta \Delta_F^{-1} \eta + i\lambda \int d^2z [2c\varphi_0\eta - \varphi_0\eta^3]} = \frac{i\lambda}{Z} \int D\eta \eta(x) \int d^2z \varphi_0(z) [2c\eta(z) - \eta(z)^3] e^{\frac{i}{2} \int \eta \Delta_F^{-1} \eta} + \dots \\ &= 3i\lambda \int d^2z \varphi_0(z) \Delta_F(x, z) \left[\Delta_F(z, z) - \Delta_F^{(0)}(0)\right], \end{aligned} \quad (\text{A.4})$$

where we have used $\langle\eta(x)\eta(z)\rangle = i\Delta_F(x, z)$, Wick's theorem for $\langle\eta(x)\eta(z)^3\rangle$, and Eq. (A.3) for the counterterm coefficient. Note that the cubic term arises from expanding $U(\varphi_0 + \eta)$, so that the coefficient is $U'''(\varphi_0)/6$. The linear term originates from the first-order expansion of the counterterm, which by itself is proportional to $U'''(\varphi_0)$, up to an additive constant. Hence the general form of the expectation value is

$$\langle\eta(x)\rangle = \frac{i}{2} \int d^2z U'''(\varphi_0) \Delta_F(x, z) \left[\Delta_F(z, z) - \Delta_F^{(0)}(0)\right]. \quad (\text{A.5})$$

For static φ_0 , the integrals over the frequencies in the propagators and the time coordinate z_0 are straightforward, resulting in Eq. (27).

Appendix B. Expectation value of $\langle\eta(x)^2\rangle$

The expectation value of $\langle\eta(x)^2\rangle$ can be computed by the same methods as we have used above for $T_{\mu\nu}$, providing a simpler example of the subtleties of renormalization. The relevant expression is similar to Eq. (17), but without the derivative terms and with two fewer powers of $\omega = \sqrt{m^2 - \kappa^2}$,

$$\langle\eta(x)^2\rangle = \frac{1}{(4\pi)^{\frac{n+1}{2}} \Gamma\left(\frac{n+3}{2}\right)} \int_m^\infty d\kappa \left(\kappa^2 - m^2\right)^{\frac{n-1}{2}} \kappa \left[G(x, x, i\kappa) - G_0(x, x, i\kappa) + \frac{m}{\kappa(m^2 - \kappa^2)} \eta_0(x)^2\right], \quad (\text{B.1})$$

which becomes

$$\langle \eta^{(\text{SG})}(x)^2 \rangle = 0 \quad \text{and} \quad \langle \eta^{(\text{kink})}(x)^2 \rangle = \frac{1}{4\sqrt{3}} \operatorname{sech}^2 \frac{mx}{2} \tanh^2 \frac{mx}{2} \quad (\text{B.2})$$

for the two models. Direct calculation shows that

$$-\frac{1}{2} \frac{dV(x)}{dx} \langle \eta(x)^2 \rangle = \frac{d}{dx} (T_1(x) - T_2(x)) \quad (\text{B.3})$$

in both models. Letting $\eta = \eta(x, t) = \eta_k(x) e^{-i\omega t}$, from Eq. (15) we have

$$T_1 - T_2 = \frac{1}{2} \left\langle \left(\frac{\partial \eta}{\partial t} \right)^2 + \left(\frac{\partial \eta}{\partial x} \right)^2 - U''(\phi_0) \eta^2 \right\rangle \quad (\text{B.4})$$

$$\Rightarrow \quad \frac{d}{dx} (T_1 - T_2) = - \left\langle \frac{\partial \eta}{\partial x} \left(\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial^2 \eta}{\partial x^2} + U''(\phi_0) \eta \right) \right\rangle + \left\langle \frac{\partial}{\partial t} \left(\frac{\partial \eta}{\partial t} \frac{\partial \eta}{\partial x} \right) \right\rangle - \frac{1}{2} \frac{dU''(\phi_0)}{dx} \langle \eta^2 \rangle. \quad (\text{B.5})$$

The first term on the right-hand side vanishes by the field equation for η . The second term is the time derivative of T_{01} , which by itself is zero. Hence Eq. (B.3) properly reflects the continuity equation for T_{v1} , and thus corroborates the finite part of the renormalization in Eq. (22).

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