On the determination of a coefficient in a space-fractional equation with operators of Abel type

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Abstract

We consider the inverse problem of recovering an unknown, spatially-dependent coefficient $q(x)$ from the fractional order equation $\mathbb{L}_\alpha u = f$ defined in a region of $\mathbb{R}^2$ from boundary information. Here $\mathbb{L}_\alpha = D_x^{\alpha_x} + D_y^{\alpha_y} + q(x)$ where the operators $D_x^{\alpha_x}$, $D_y^{\alpha_y}$ denote fractional derivative operators based on the Abel fractional integral. In the classical case this reduces to $-\triangle u + q(x)u = f$ and this has been a well-studied problem. We develop both uniqueness and reconstruction results and show how the ill-conditioning of this inverse problem depends on the geometry of the region and the fractional powers α_x and α_y.

Keywords: Inverse problems, fractional PDE, coefficient identification, Newton's method, eigenvalue problem

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1 Introduction

The use of fractional derivatives in physical applications is now commonplace as deficiencies in certain models using only integer order derivatives have been widely explored. These include diffusion models where Brownian motion is not the underlying modality and the mean square path length of a particle is not proportional to t itself but to a fractional power t^α. In the case of $0 < \alpha < 1$ such a process is labelled as *subdiffusive* and is characterised by waiting times with a non-finite mean. In classical damping for the wave equation a term of the form $-b\,\triangle u_t$ is included and leads to exponential decay of all frequencies. On the other hand a fractional operator here, $-b\triangle D_t^\alpha u$ gives a very different situation: decay is now a power law that does have a frequency dependency. This was in fact probably one of the first applications of fractional derivatives in a partial differential equation and dates from the 1960's, see Caputo [?]. Perhaps 1.
the most important distinction is that fractional derivatives are nonlocal operators leading not only to new physics but requiring different mathematical tools.

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The use of fractional derivatives in inverse problems for partial differential equations is more recent but has seen an exponential increase in the number of papers over the last decade. A survey of the early part of this work can be found in [**?**] and more recently in the book [**?**].

While the majority of this body of research involves fractional derivatives in time in models such as $D_t^\alpha u - \mathbb{L}u = f(x,t,u)$ where D^α is a derivative based on the Abel fractional integral and $\mathbb{L}$ is an elliptic operator, there is also the possibility to consider fractional derivatives in space. From a particle-diffusive standpoint this leads to *superdiffusion* where now the variance of the mean square distance is non-finite leading to arbitrary long particle jump lengths, see [**?**]. In this case there are many possible definitions. One is to look at a fractional power of the elliptic operator $-\mathbb{L}$ and again there are several versions here, see, e.g. [**?**, **?**]. Another is to again use combinations of fractional derivatives of the Abel type and this is the approach taken in this paper.

Our specific situation is to view our differential operator to be in potential form and we seek to recover the unknown, spatially-varying coefficient as the potential $q(x)$ in such a fractional elliptic operator from boundary measurements. We assume the setting to be in two spatial dimensions, although as it will become apparent, the problem extends to higher spatial dimensions. To recover such an unknown coefficient with domain $\Omega \subset \mathbb{R}^d$ from boundary measurements requires the full Dirichlet-to-Neumann map: for each member of a complete family of Dirichlet conditions $\{u_n\}$ on $\partial\Omega$, we must provide the corresponding Neumann measurements $\left\{\frac{\partial u_n}{\partial \nu}\right\}$. We then have $(d-1)+(d-1)$ dimensional information from which to recover a d-dimensional unknown. In the case $d \geq 3$ the problem is overposed while for $d=2$ the dimension count is exact but is drastically under-posed if $d=1$. In the $\mathbb{R}^d$, $d \geq 2$ setting there is an extensive literature on this problem not only in the classical case but also when the leading term Laplacian operator is replaced by the so-called fractional Laplacian $(-\triangle)^\beta$ with $1<\beta<2$. See, for example, [**?**, **?**].

We do not envision such an extensive measurement set and will in fact take only a single Dirichlet boundary condition and measure the corresponding Neumann values on a subset of the boundary of Ω. Clearly, such a restricted problem must constrain the coefficient q and we shall assume that it is a function of a single variable $x \in \mathbb{R}$ (with an exception being made in Remark 4.1). This in turn will constrain the geometrical situation, but we will actually focus on the harder (and at the same time more realistic) case of measuring in a direction orthogonal to the direction of variability of q. The resulting inverse problem therefore must be expected to be severely ill-posed. Moreover, rather than looking at versions of the fractional Laplacian, we are interested in fractional derivatives based on the Abel integral operator, cf. (7), (8), (9) in order to model anisotropic behaviour in the sense that memory due to nonlocality acts in a certain direction.

Our operators thus consist of one-sided fractional derivatives of Djrbashian-Caputo type together with left and right averages of these. In the latter situation we look at the Riesz derivative which is a symmetric combination of left and right fractional derivatives and uses the Riemann-Liouville formulation to allow a larger class of solutions. The connection here to the random walk model is that the Riesz derivative with Dirichlet boundary conditions is the generator of a stopped, α-stable Lévy motion. See, for example, [**?**].

1.1 Problem Configuration

Let Ω be the rectangle $(0,1)\times(0,L)$ (or more generally $\Omega=(0,1)\times D$ for some domain $D\subseteq\mathbb{R}^{d-1}$) and $u(x,y)$ be defined in Ω by

$$\mathbb{L}_\alpha u = f \qquad \text{in } \Omega \tag{1}$$

with

$$u(x,0)=u(x,L)=0,\quad u(0,y)=\phi_0(y)\quad u(1,y)=\phi_1(y). \tag{2}$$

Here $\mathbb{L}_\alpha$ is given by either

$$\mathbb{L}_\alpha u = -D_x^\alpha u - u_{yy} + q(x)u, \tag{3}$$

or

$$\mathbb{L}_\alpha u = -u_{xx} - D_y^\alpha u + q(x)u, \tag{4}$$

where $\alpha\in(1,2]$. Models such as these fractional advection-dispersion flow equations occur frequently in the literature; see for example, [**?**] and references within. In (2) f, ϕ_0, ϕ_1 are given information and we must recover $q(x)$ from the overposed value

$$g^\delta(y)\approx g(y)=u_x(0,y),\quad y\in(0,L), \tag{5}$$

where g^δ is the actually available (noisy) data.
In the higher dimensional case of replacing $(0,L)$ by $D\subseteq\mathbb{R}^{d-1}$, these data would be given by

$$g^\delta(y)\approx g(y)=u_x(0,y),\quad y\in\Gamma\subseteq\overline{D}, \tag{6}$$

where pure dimension count would still admit Γ to be just a one-dimensional set (e.g., part of the boundary of a two-dimensional domain D) as the quantity q to be reconstructed depends on the single variable x only. Since variation of the data occurs in a direction orthogonal to x, we expect the reconstruction problem to be severely ill-posed. Indeed the influence of the order of differentiation in the PDE is only minor, as well will see in Section 3.
The operator D_x^α will either be a one-sided Djrbashian-Caputo derivative (and there are two sub-cases here depending on the starting position being either $x=0$ or $x=1$ in (3)), or the Riemann-Liouville version of the Riesz derivative taken as the symmetric combination of the one-sided derivatives, which we will consider in both settings (3), (4).
With the one-sided Abel integral operators

$$\begin{aligned} {}_aI_x^\gamma v(x) &:= \frac{1}{\Gamma(\gamma)}\int_a^x \frac{v(t)}{(x-t)^{1-\gamma}}dt, \quad \text{for } x>a,\\ {}_xI_b^\gamma v(x) &:= \frac{1}{\Gamma(\gamma)}\int_x^b \frac{v(t)}{(t-x)^{1-\gamma}}dt, \quad \text{for } x<b. \end{aligned} \tag{7}$$

in the case $\alpha\in(1,2)$ relevant here, the one-sided Djrbashian-Caputo derivative on $(0,1)$ is defined by

$${}_0D_x^\alpha v(x) = {}_0I_x^{2-\alpha} v_{xx}(x) \tag{8}$$

and the two-sided Riemann-Liouville Riesz derivative on the interval $(-1,1)$ by

$$ {}^R D_x^\alpha v(x) = \frac{1}{2\cos((2-\alpha)\pi/2)} \frac{d^2}{dx^2} \left({}_{-1}I_x^{2-\alpha} + {}_xI_1^{2-\alpha} \right) v(x), \tag{9} $$

(and likewise for ${}^R D_y^\alpha$), which can be transformed to the interval $(0,1)$ or $(0,L)$ by the map $x \mapsto (x+1)/2$ or $y \mapsto L(y+1)/2$.

An important property of the Abel integral operator that we make use of throughout this paper is its coercivity

$$ \int_0^1 (I^\gamma v)(x) v(x)\, dx \geq \cos(\gamma\pi/2) \|v\|_{H_*^{-\gamma/2}(0,1)}, \quad \gamma \in [0,1), \quad v \in H_*^{-\gamma/2}(0,1) \tag{10} $$

with $\|v\|_{H_*^s(0,1)}^2 := \int_{\mathbb{R}} (1+\omega^2)^s \left| \int_0^1 e^{-i\omega t} v(t)\, dt \right|^2 d\omega$ for $s \in \mathbb{R}$ and the corresponding function space is defined as $H_*^s(0,1) = \{v \in L^2(0,1) : \|v\|_{H_*^s(0,1)} < \infty\}$ for $s > 0$ while $H_*^s(0,1)$ is the completion of $L^2(0,1)$ in the above defined $H^s(0,1)$ norm for $s < 0$, cf. [**?**, **?**]. It is readily 2., 3.
checked that the Riesz operator with homogeneous Dirichlet boundary conditions is selfadjoint

$$ \begin{aligned} \langle -{}^R D_x^\alpha v_1, v_2 \rangle &= \frac{1}{2\cos((2-\alpha)\pi/2)} \int_0^1 v_1'(x) \left({}_0I_x^{2-\alpha} v_2' + {}_xI_1^{2-\alpha} v_2' \right)(x)\, dx \\ &= \langle v_1, -{}^R D_x^\alpha v_2 \rangle, \qquad v_1, v_2 \in H_0^1(0,1), \end{aligned} \tag{11} $$

thus by (10)

$$ \langle -{}^R D_y^\alpha v, v \rangle = \|v'\|_{H_*^{\alpha/2-1}(0,1)}. \tag{12} $$

While case (4) is naturally restricted to a 1-d setting in the y direction, we can replace $-u_{yy}$ with homogeneous Dirichlet boundary conditions at $y \in \{0, L\}$ in (3) by a more general symmetric elliptic differential operator in higher space dimensions with homogeneous boundary conditions of more general type, leading to a selfadjoint positive definite operator $\mathscr{A} : \mathscr{D}(\mathscr{A}) \to \mathscr{H}$.

An example is the negative Laplacian $\mathscr{A} = -\triangle_y$ equipped with homogeneous Dirichlet, Neumann, or impedance boundary conditions and $\mathscr{H} = L^2(D)$; more generally, we can set

$$ \mathscr{A} = -\rho \nabla_y \cdot (a \nabla_y), \quad \mathscr{H} = L^2_{1/\rho}(D), \quad \mathscr{D}(\mathscr{A}) \subseteq H^2(\Omega), \quad D \subseteq \mathbb{R}^d \text{ a Lipschitz domain} \tag{13} $$

with the weighted L^2 inner product $\langle v, w \rangle = \int_D \frac{1}{\rho} v w\, dx$ and coefficients

$$ a \in W^{1,\infty}(D), \quad \rho \in L^\infty(D), \quad 0 < \underline{a} \leq a(y),\, 0 < \underline{\rho} \leq \rho(y),\, y \in D, \tag{14} $$

that are positive bounded away from zero and depend on y only.

Another option according to (12) is to use a Riesz fractional derivative also in the y direction (in addition to the fractional DC or Riesz one in x direction in (3)), that is

$$ \mathscr{A} = -{}^R D_y^\beta, \quad \mathscr{H} = L^2(0,L), \quad \mathscr{D}(\mathscr{A}) \subseteq H_*^\beta(\Omega), \quad D = (0,L). \tag{15} $$

These various combinations will turn out to give quite different answers to our ability to recover the unknown $q(x)$ from the overposed boundary data and it is a primary purpose of this paper to highlight this aspect.

A generic application is to a layered medium that varies through $q(x)$ only in one direction (the x direction). As a more specific example of this we mention wave propagation in the frequency domain with $q(x) = -\frac{\omega^2}{c^2(x)}$, where ω is the frequency and c the speed of sound.

The remainder of this paper is organised as follows. We first of all derive a reconstruction method based on separation of variables and application of Newton's method in Section 2. The performance of this method is illustrated by numerical tests for all of the described cases in Section 3. In Section 4 we provide a uniqueness proof in the y-direction fractional case (4) based on inverse Sturm-Liouville theory and discuss to what extent its ideas carry over to the x-direction Riesz fractional case (3). Besides the main problem of reconstructing q we will here also provide a result on unique recovery of the fractional order α in the x fractional Riesz case. Finally, in Section 5, we provide some foundation for the Newton-type methods devised and used in Sections 2 and 3.

2 Reconstruction by Newton's method

In this section we derive a reconstruction scheme for recovering q in (1), (2), (3), (4), from boundary data (5). This also comprises considerations on the evaluation of the forward map F and its derivative F'.

2.1 The x-fractional case (3)

Slightly more abstractly than (3), we consider the problem with fractional derivative in x direction for $u \in H^2(0,1;\mathscr{H}) \cap L^2(0,1;\mathscr{D}(\mathscr{A}))$, $f \in L^2(0,1;\mathscr{H})$, $q \in L^2(0,1)$, with a selfadjoint operator $\mathscr{A} \in L(\mathscr{D}(\mathscr{A}),\mathscr{H})$ with compact inverse

$$-{}_0D^\alpha u(x) + \mathscr{A} u(x) + q(x)u(x) = f(x), \quad x \in (0,1),$$

with boundary conditions

$$u(0) = \phi_0, \qquad u(1) = \phi_1,$$

that covers (3) with the special setting $\mathscr{H} = L^2(D)$, $\mathscr{D}(\mathscr{A}) = H^2(D) \cap H_0^1(D)$, $\mathscr{A} = -\triangle_y$ when equipped with homogeneous Dirichlet boundary conditions.

Separation of variables and the expansion $u(x) = \sum_{j=1}^\infty u_j(x)\varphi_j$ in terms of the eigenfunctions φ_j of $\mathscr{A}$ (based on the spectral theorem applied to the compact selfadjoint operator $\mathscr{A}^{-1}$) leads to problems of the form

$$-{}_0D^\alpha u_j(x) + \lambda_j u_j(x) + q(x)u_j(x) = f_j(x), \quad x \in (0,1), \tag{16}$$

with boundary conditions

$$u_j(0) = \phi_{0,j}, \qquad u_j(1) = \phi_{1,j}, \tag{17}$$

and the overposed data

$$u_j'(0) = b_j, \tag{18}$$

where $f_j(x) = \langle f(x), \varphi_j \rangle_{\mathscr{H}}$, $\phi_{i,j} = \langle \phi_i, \varphi_j \rangle_{\mathscr{H}}$, $i \in \{0,1\}$, $b_j = \langle g, \varphi_j \rangle_{\mathscr{H}}$, and λ_j is the eigenvalue of $\mathscr{A}$ corresponding to φ_j. 4.

The inverse problem now can be written as

$$F(q) = \underline{b} \tag{19}$$

with $F(q) = (F_j(q))_{j\in\mathbb{N}}$, $F : L^2(0,1) \to \ell^2$, $F_j(q) = u_j'(0)$ where u_j solves the boundary value problem (16), (17), and $\underline{b} = (b_j)_{j\in\mathbb{N}}$ with b_j as in (18).

To evaluate the forward operator F, for each $j \in \mathbb{N}$ we have to solve (16), (17). For this purpose we use a solver $w = S(f, w_0, w_1, \lambda)$ of the boundary value problem for $q = 0$

$$-{}_0D^\alpha w(x) + \lambda w(x) = f(x), \quad x \in (0,1), \quad w(0) = w_0, \quad w(1) = w_1 \tag{20}$$

and proceed by a fixed point iteration. Thus we start with $u_j^0(x) = S(f_j, \phi_{0,j}, \phi_{1,j}, \lambda_j)$ and the solution u_j to (16), (17) is constructed by successive approximations

$$u_j^{n+1}(x) = S(f_j - q(s)u_j^n, \phi_{0,j}, \phi_{1,j}, \lambda_j). \tag{21}$$

In the Djrbashian-Caputo case we obtain $w = S(f, w_0, w_1, \lambda)$ via a Green's function as follows. From e.g., [**?**, Proposition 4.5], [**?**, Thm 5.4] with $b = w_x(0)$, $G(t) = t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)$ we have

$$w(x) = -\int_0^x G(x-t)f(t)\,dt + w_0 E_{\alpha,1}(\lambda x^\alpha) + bxE_{\alpha,2}(\lambda x^\alpha).$$

Now we eliminate b by using the right hand boundary value

$$w_1 = -\int_0^1 G(1-t)f(t)\,dt + w_0 E_{\alpha,1}(\lambda) + bE_{\alpha,2}(\lambda), \tag{22}$$

and this implies that

$$b = \frac{1}{E_{\alpha,2}(\lambda)}\Big(w_1 + \int_0^1 G(1-t)f(t)\,dt - w_0 E_{\alpha,1}(\lambda)\Big).$$

Finally we obtain

$$w(x) = \int_0^1 K(x,t)f(t)\,dt + w_0\, e_0(x) + w_1\, e_1(x), \tag{23}$$

where

$$\begin{aligned} K(x,t) &= \begin{cases} e_1(x)G(1-t) - G(x-t) & \text{if } t \le x \\ e_1(x)G(1-t) & \text{if } t > x \end{cases} \\ e_0(x) &= E_{\alpha,1}(\lambda x^\alpha) - e_1(x)E_{\alpha,1}(\lambda), \quad e_1(x) = x\frac{E_{\alpha,2}(\lambda x^\alpha)}{E_{\alpha,2}(\lambda)} \end{aligned} \tag{24}$$

In the Riesz case, to construct $w = S(f, w_0, w_1, \lambda)$ we use a solver based on a Galerkin discretisation with Jacobi polynomials as described in [?] on the symmetric interval $(-1,1)$ and extend it to inhomogeneous Dirichlet boundary conditions and an additional term λw in the equation. For real α, $\beta > -1$ let $P_n^{\alpha,\beta}$ be the classical Jacobi polynomials with respect the weight function $\omega^{\alpha,\beta}(x) = (1-x)^{\frac{\alpha}{2}}(1+x)^{\frac{\beta}{2}}$ over $[-1,1]$. These are such that

$$\begin{aligned} &\int_{-1}^{1} P_n^{\alpha,\beta}(x) P_m^{\alpha,\beta}(x)\omega^{\alpha,\beta}(x)\,dx = \gamma_n^{\alpha,\beta}\delta_{mn} \quad \text{where} \\ &\gamma_n^{\alpha,\beta} = \|P_m^{\alpha,\beta}\|^2_{L_{\omega^{\alpha,\beta}}} = \frac{2^{\frac{\alpha}{2}+\frac{\beta}{2}+1}\Gamma(n+\frac{\alpha}{2}+1)\Gamma(n+\frac{\beta}{2}+1)}{(2n+\frac{\alpha}{2}+\frac{\beta}{2}+1)\,n!\,\Gamma(n+\frac{\alpha}{2}+\frac{\beta}{2}+1)}. \end{aligned} \tag{25}$$

and satisfy a three term recursion scheme. For our purpose we only require the case where $\alpha = \beta$ and thus let $P_n^{\alpha}(x) = P_n^{\alpha,\alpha}(x)$. Again following standard notation we define

$$J_n^{-\alpha}(x) := (1-x^2)^{\frac{\alpha}{2}} P_n^{\alpha}(x), \quad \alpha > -2.$$

Then it can be verified that $J_n^{-\alpha}(-x) = (-1)^n J_n^{-\alpha}(x)$ and $\frac{d^k}{dx^k} J_n^{-\alpha}(\pm 1) = 0$ for $k = 0, 1, \ldots, \lceil \alpha \rceil - 1$ and it satisfies a three term recursion scheme analogous to the one for P_n^{α}, as well as the orthogonality condition

$$\int_{-1}^{1} J_n^{-\alpha} J_m^{-\alpha} \omega^{-\alpha,-\alpha}(x)\,dx = \gamma_n^{\alpha,\alpha}\delta_{mn}. \tag{26}$$

The key relation between these polynomials for our purpose is

$${}^{R}D^{\alpha} J_n^{-\alpha} = \frac{\Gamma(\ell+1+\alpha)}{\ell!} P_n^{\alpha},$$

see [?, Corollary 1].

To extend the Galerkin method described in [?] to inhomogeneous Dirichlet boundary conditions and an additional term λw in the equation, we first of all consider

$$-{}^{R}D^{\alpha}\tilde{w}(\tilde{x}) + \lambda\tilde{w}(\tilde{x}) = \tilde{f}(\tilde{x}), \quad \tilde{x} \in (-1,1), \quad \tilde{w}(-1) = w_0, \quad \tilde{w}(1) = w_1$$

which with $w(x) = \tilde{w}(\tilde{x})$, $f(x) = \tilde{f}(\tilde{x})$, $\tilde{x} = 2x - 1$ we transform back to $(0,1)$. With $\bar{\phi}(\tilde{x}) = \frac{1}{2}(w_1 + w_0 + (w_1 - w_0)x)$, the function $\hat{w} = \tilde{w} - \bar{\phi}$ satisfies homogeneous boundary conditions 5. and we can use the Ansatz

$$\tilde{f}(\tilde{x}) \approx \sum_{k=0}^{N} \tilde{f}_k P_k^{\alpha}(\tilde{x}), \quad \bar{\phi}(\tilde{x}) \approx \sum_{k=0}^{N} \bar{\phi}_k P_k^{\alpha}(\tilde{x}), \quad \hat{w}(\tilde{x}) \approx \sum_{k=0}^{N} \hat{w}_k J_k^{-\alpha}(\tilde{x}).$$

Taking the weighted inner product with P_ℓ^{α} and using orthogonality, we obtain the linear system

$$\frac{\Gamma(\ell+1+\alpha)}{\ell!}\hat{w}_\ell + \lambda\sum_{k=0}^{N} G_{k,\ell}\hat{w}_k = \tilde{f}_\ell - \lambda\bar{\phi}_\ell \quad \ell = 0, 1, \ldots, N, \tag{27}$$

where $G_{j,\ell} = \langle J_j^{-\alpha}, P_\ell^\alpha \rangle_{L^2_{\omega^{\alpha,\alpha}}}$.

The Jacobian for applying Newton's method to (19) is computed by choosing a basis $(p_i)_{i\in\mathbb{N}}$ of $L^2(0,1)$ and setting $F_j'(q)p_i = v_j'(0)$ where v_j solves the boundary value problem

$$\begin{aligned} &-{}_0D^\alpha v_j(x) + \lambda_j v_j(x) + q(x)v_j(x) = -p_k(x)u_j(x)\,, \quad x \in (0,1)\,,\\ &v_j(0) = 0\,, \quad v_j(1) = 0, \end{aligned} \tag{28}$$

that is, $v_j = S(-p_i u_j, 0, 0, \lambda_j)$. The Newton step at an iterate q^k is then computed as

$$\delta q = \sum_{i=1}^\infty c_i p_i, \text{ where } c = (c_i)_{i\in\mathbb{N}} \text{ solves } A^k c = r \text{ with } A^k_{ji} = F_j'(q^k)p_i,\ r_j = b_j^\delta - F_j(q^k).$$

Here $b_j^\delta = \langle g^\delta, \varphi_j \rangle$ with g^δ being the actually given noisy data. Regularisation can be achieved by truncating the expansion for δq and /or the number of components $F_j(q^k)$ taken into account. Alternatively or additionally to that, Tikhonov regularisation can be applied by using regularised approximations $c = (A^{k*}A^k + \gamma_k I)^{-1}A^{k*}r$ to the solution of $A^k c = r$, cf. Section 5.2.

Remark 2.1. *Alternatively one might think of considering the initial value problem for* (16) *prescribing*

$$u_j(0) = \phi_{0,j}\,, \qquad u_j'(0) = b_j\,,$$

*and define $\tilde{F}_j(q) = u_j(1)$, so the inverse problems reads as $\tilde{F}(q) = \tilde{\phi}_1 = (\phi_{1,j})_{j\in\mathbb{N}}$. The direct solver is now actually simpler, based on the fixed point equation (relying on the representation from, e.g., [**?**, Proposition 4.5], [**?**, Thm 5.4])*

$$\begin{aligned} u_j(x) = &E_{\alpha,1}(\lambda_j x^\alpha)\,\phi_{0,j} + xE_{\alpha,2}(\lambda_j x^\alpha)\,b_j \\ &+ \int_0^x (x-t)^{\alpha-1} E_{\alpha,\alpha}(\lambda_j (x-t)^\alpha) q(t) u_j(t)\,dt. \end{aligned} \tag{29}$$

However, due to $\lambda_j \to \infty$, the operator $\tilde{F}$ is unbounded.

2.2 The y-fractional case (4) with a Riesz derivative

Restricting again to one space dimension in the y-direction $D = (0,L)$, we can apply an analogous procedure by applying separation of variables with respect to eigenfunctions of $-{}^R D_y^\alpha$. While there are many open questions on the eigenvalue problem for the Djrbashian-Caputo derivative (8), see, e.g., [**?**], much more can be said about the Riesz derivative (9). The Riesz derivative operator $-{}^R D_y^\alpha : H_*^{\alpha/2}(0,L) \to (H_*^{\alpha/2}(0,1))^*$ when equipped with homogeneous Dirichlet boundary conditions is selfadjoint and positive definite, due to (11), (12), and thus boundedly invertible by the Lax-Milgram Lemma. Its inverse K, considered as an operator from $L^2(0,L)$ into itself is compact, due to the fact that $H_*^{\alpha/2}(0,L)$ compactly embeds into $L^2(0,L)$. Moreover K is also selfadjoint. Therefore by the spectral theorem for compact selfadjoint operators, K, whose nullspace is trivial, has a complete set of orthogonal eigenfunctions and the eigenvalues are all real, positive and tend to zero. The eigenfunctions of $-D_y^\alpha = K^{-1}$ are the same as those for K and therefore still complete in $L^2(0,L)$, and its eigenvalues are real, positive and tend to infinity.

With an eigensystem $(\lambda_j^\alpha, \varphi_j^\alpha)_{j\in\mathbb{N}}$ of $-{}^R D_y^\alpha$ with homogeneous Dirichlet boundary conditions, we can write $u(x,y) = \sum_{j=1}^\infty u_j(x)\varphi_j^\alpha(y)$, where u_j solves

$$-u_j''(x) + \lambda_j^\alpha u_j(x) + q(x) = f_j(x) \quad x \in (0,1), \quad u_j(0) = \phi_{0,j}(0),\ u_j(1) = \phi_{1,j}(1) \tag{30}$$

cf. (16), (17) and the overposed data

$$u_j'(0) = b_j,$$

where $f_j(x) = \langle f(x), \varphi_j^\alpha\rangle_{L^2}$, $\phi_{i,j} = \langle \phi_i, \varphi_j^\alpha\rangle_{L^2}$, $i \in \{0,1\}$, $b_j = \langle g, \varphi_j^\alpha\rangle_{L^2}$.

Again we can use successive approximations (21) to evaluate the forward operator and its Jacobian. The solution operator S defined by $w = S(f, w_0, w_1, \lambda)$ such that

$$-w''(x) + \lambda w(x) = f(x), \quad x \in (0,1), \quad w(0) = w_0, \quad w(1) = w_1 \tag{31}$$

can be obtained by simply taking the case $\alpha = 2$ in (23), (24) and $\mathscr{A}$ according to (15).

3 Reconstructions

The purpose of this section is to look at quantitative differences in the ability to reconstruct q from the various combinations of operators and their dependence on the associated fractional powers. We will indicate this by giving both reconstructions of $q(x)$ and also computing the singular values of the Jacobian matrix needed for its recovery using Newton's method. The latter gives a strong indication of the degree of ill-conditioning of the inverse problem.

As a baseline for the reconstructions we take a Lipschitz continuous function $q(x)$ (in fact a piecewise linear function). We seek recovery of this with a set of pure sine basis functions $\{\sin n\pi x\}_1^N$. The best fit to the actual q measured in $L^2(0,1)$ within this basis set for $N = 11$ is $\|q_{\text{act}} - q_{\text{recon}}\| = 0.04$. As we will see, the regularisation needed to stabilise the Newton iterations even under very low levels of noise in the data measurements would preclude effective use of a larger number of basis functions. This becomes evident when looking at the graphics containing the singular values of the Jacobian. So we also used $N = 11$ basis functions in the reconstructions.

We consider (3), (4), that is,

$$\mathbb{L}_\alpha u = D_x^{\alpha_x} u - D_y^{\alpha_y} u + q(x)u \tag{32}$$

where D^{α_x} is either the one-sided Djrbashian-Caputo derivative (8) or the symmetric Riesz derivative (9) with homogeneous Dirichlet boundary conditions and D^{α_y} is the symmetric Riesz derivative on an interval $(-L,L)$, with Dirichlet boundary conditions at $y = -L$ and $y = L$. When considering (3) the value $\alpha_y = 2$ will be a default, likewise for (4) with $\alpha_x = 2$, but we also use the case when the fractional exponent is the same in both directions, $\alpha_y = \alpha_x$.

The sub-cases we will consider are the following,

- $D_x^{\alpha_x}$ is a single fractional derivative of Djrbashian-Caputo type (8) with starting value the left endpoint of the interval and $D_y^{\alpha_y} = \frac{\partial^2}{\partial y^2}$. In this case we make the sub-cases of overposed data prescribed on the left and on the right as the inversion of the nonlocal operator $D_x^{\alpha_x}$ is very different in the cases of the overposed data being posed at the starting value of $D_x^{\alpha_x}$ as opposed to the end value. This phenomenon has been seen frequently in the case of a one-sided fractional derivative of Abel type; see [**?**][Sec 3.2] and [**?**][Chapter 10.2].

- $D_x^{\alpha_x}$ is the derivative of a symmetric combination of left and a right Abel fractional integrals, that is a one-dimensional Riesz operator (9) and again $D_y^{\alpha_y} = \frac{\partial^2}{\partial y^2}$.

- Both $D_x^{\alpha_x}$ and $D_y^{\alpha_y}$ are Riesz derivatives as in equation (9). There are further sub-cases here: $\alpha_x = 2$ or $\alpha_x = \alpha_y$. The second assumes the same fractional operator and exponent in both spatial directions, while the first assumes that the rectangular medium is "classical Brownian" in one direction and "anomalous" in the other.

- We allow the length of the rectangle to vary in the y-direction. The larger the value of L the smaller the eigenvalues in the $y-$direction and leading to a correspondingly less ill-conditioned overall inverse problem. This is also intuitive in view of the fact that the measurement interval is longer relative to the interval on which q is to be reconstructed.

For numerical computations we used the methods described in Section 2, that is, a solution representation by means of Green's functions for the one-sided Djrbashian-Caputo fractional derivatives and a spectral Galerkin methods using Jacobi polynomials for the Riesz cases. The Jacobian matrix was formed by solving the linearised forward problem (28) for each q basis function p_k and taking the endpoint values $v'(0)$ or $v'(1)$. Regularisation was of Tikhonov type with a diagonal regularisation matrix R weighted by powers of the frequencies.

The Djrbashian-Caputo case in the x direction

In figure 1 we show the first ten singular values of the Jacobian matrix computed about $q = 0$ for the exponents $\alpha = \{\frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2\}$. Here we took $L = 2$ and compared data measured on the left and with it measured on the right. Note that the derivative here is ${}_0^{DC}D_x^{\alpha}$ with starting value at the left endpoint $x = 0$ in the x-direction and $\frac{d^2}{dy^2}$ in the y direction. The exponential decay of the singular values is evident from this graphic as is the monotonic behaviour with respect to α and shows that the problem becomes more ill-conditioned with increasing α. However, the influence of α on this ill-posedness is very weak in the sense that the slope of the log-singular value decay hardly changes with α. The condition numbers of the Jacobians, that is, the ratios between their 1st and 10th singular values are as follows: $\alpha = 1.25$: 12.95, $\alpha = 1.5$: 14.05, $\alpha = 1.75$: 17.90, $\alpha = 2$: 20.43. As regards left-right measurements the distinction is quite 6.
clear for the $\alpha < 2$ cases but appears to be identical when $\alpha = 2$. This is exactly what should be expected for when $\alpha = 2$ we have the regular second derivative operator which gives symmetric results at both endpoints.

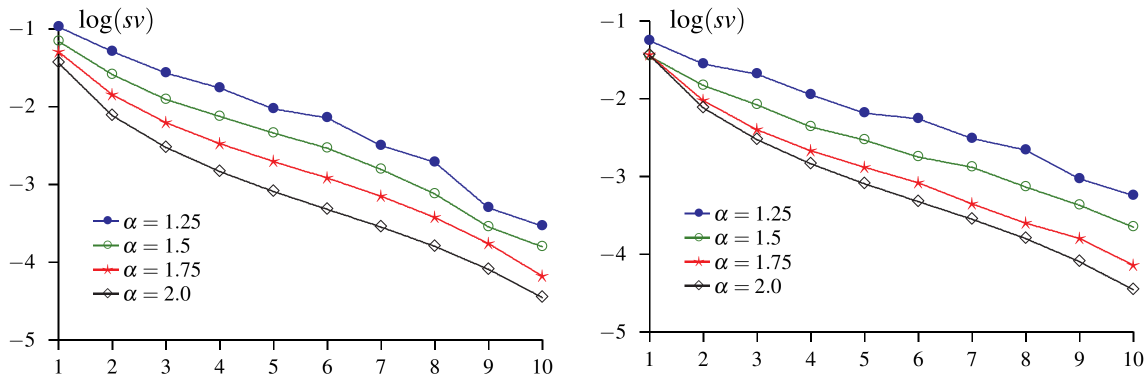

Figure 1: **Singular values of the Jacobian for** $\alpha = (1.25,\ 1.5,\ 1.75, 2.0)$, $L = 2$.
Leftmost figure with data on left, rightmost with data on right.

In figure 2 we show the reconstructions obtained by the Newton scheme for $\alpha = 1.5$ and taking the rectangle to be lengthened with $L = 6$. As to be expected, a large L value will lead to less ill-conditioning and hence superior reconstructions. We will see a direct comparison of this effect in the next subsection. The upper pair of figures is with no added noise, and the final relative norm differences in q are 0.08 in both cases. The lower pair has 0.1% added noise, giving norm differences in q of 0.113 and 0.120.

One might expect that the differences here would be almost totally insignificant, but it is a mark of the extreme degree of ill-conditioning that such small changes in noise level can make a recognisable change. Even the no added noise case required some regularisation and this explains in part why the reconstruction doesn't correspond exactly to the actual $q(x)$. The other contributing factor here is that we are using a restricted number N of basis elements in our reconstructed q which is in itself a form of regularization.

The Riesz case in the x direction

Here we have taken $D_x^{\alpha_x}$ to be the Riesz derivative in the x direction and $\frac{d^2}{dy^2}$ in the y direction. In this symmetric case we expect the one-sided derivative effect to vanish and this is indeed the case. However, nonlocality still plays a role and this may be expected to show up in reconstructions when the unknown q has a significant feature near one of the endpoints: prescribing data at this endpoint will give superior reconstructions than providing it at the further endpoint. This is indeed the situation as we see in figure 3.

We also provide the singular values of the Jacobian matrix for both the cases $L = 2$ (the region is a square) and $L = 6$. This clearly shows the decreased ill-conditioning when L is larger which corresponds to smaller eigenvalues for a given index number in this case. In turn this decreases the ill-conditioning in the x direction. This effect is in place for all α values including $\alpha = 2$.

Figure 3 shows a reconstruction of the standard $q(x)$ from the Riesz derivative in the $x-$ spatial direction. The left and right figures correspond to data measurements on the respective sides. Notice that the right-hand reconstruction is superior. In this case this has nothing to do with directionality of ${}^R D_x^\alpha$ as this is a symmetric operator on $[0,1]$. Rather, the right hand data is better able to handle the large spike in $q(x)$ near $x = 1$ than the left hand boundary measurement

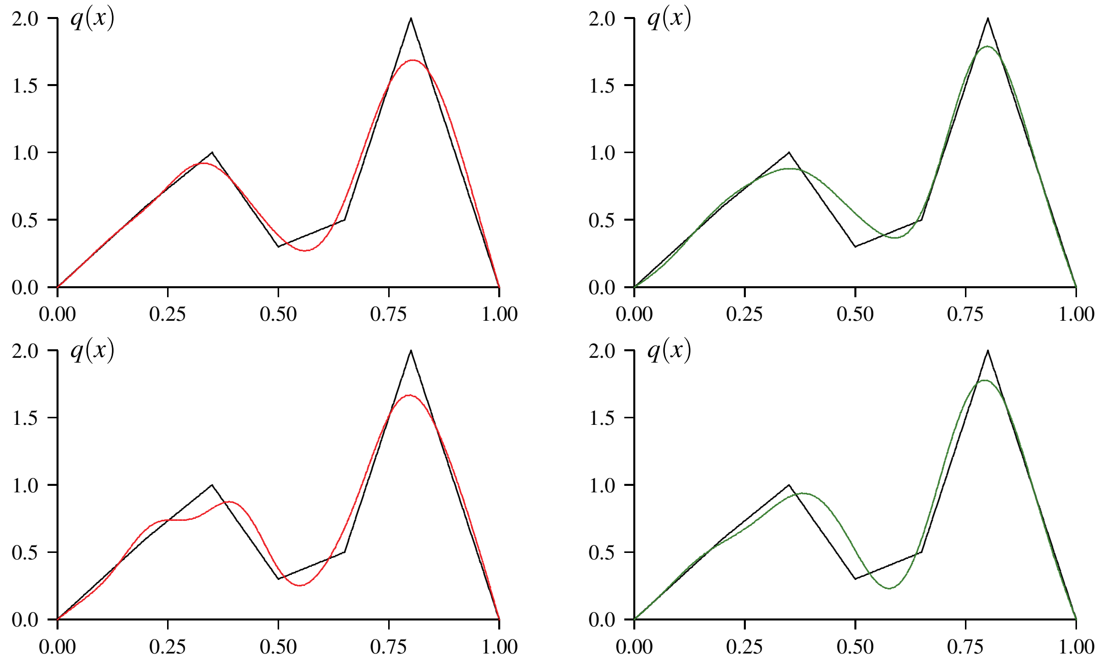

Figure 2: **Reconstruction of q in the DC case with $\alpha = 1.5$, $L = 6$:**
Leftmost figure with data on left, rightmost with data on right.
Upper row: 0 per cent noise, lower row: 0.1 per cent noise.

due to simple proximity. If $q(x)$ were in fact symmetric then these reconstructions would be identical. The value of $\alpha = 1.5$ is used in figure 3. In fact, the reconstruction accuracy depended on α to only a small degree; the L^2 relative error difference in $q(x)$ between the lowest and the highest alpha value was only a factor of about 1.5.

The Riesz case in the y direction

Here we isolate the fractional operator effect to the orthogonal direction in which the unknown q is defined; the differential operator in the x-direction is just $\frac{d^2}{dx^2}$. In the y-direction we take the operator to be of Riesz type. Figure 5 shows the resulting singular values of the Jacobian with the usual four α_y values. Notice the rapid decay indicating severe ill-conditioning due to the effect of inverting the classical operator, while there is a distinction between the α_y values. Figure 6 indicates that left and right placement of the data measurements are insignificant (as would be expected) and as in Figure 3, only the asymmetry of the target q leads to a slight difference in quality between left and right.

The Riesz case in the x and y direction

We have so far taken the assumption that the the underlying material has different properties in the x and y directions and giving rise to differing α_x and α_y but it is of course possible that there is a directional invariance and the same operator acts in both directions. The analysis is of course a special case of the above but we do show reconstructions in Figure 8 for the case with $\alpha = \alpha_x = \alpha_y = 1.5$. Note that according to the singular value decay shown in Figure 7, this leads

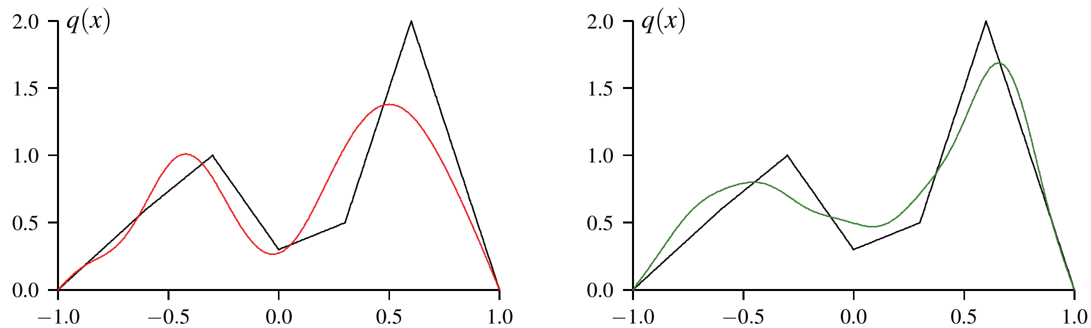

Figure 3: **Reconstruction of q in the Riesz-x case with $\alpha = 1.5$, $L = 6$, and 0.1 per cent noise: Leftmost figure with data on left, rightmost with data on right.**

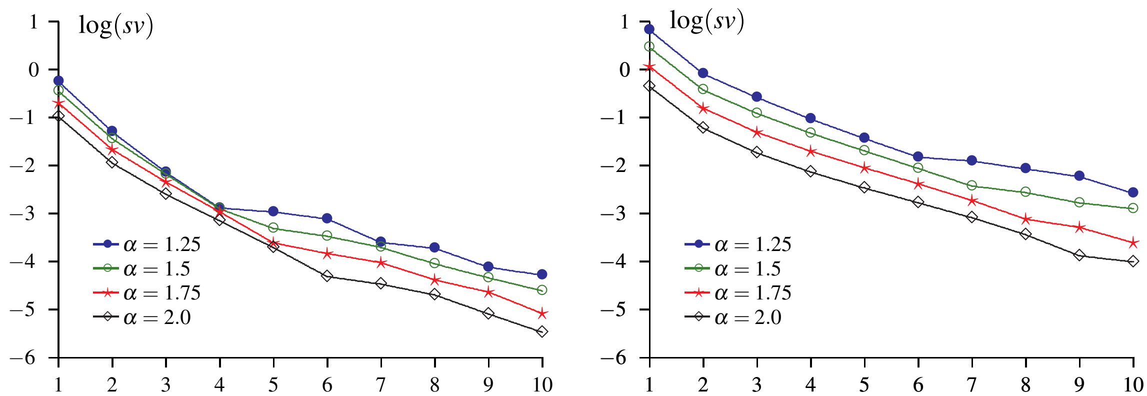

Figure 4: **Singular values of the Jacobian for $\alpha = (1.25,\ 1.5,\ 1.75, 2.0)$ for the Riesz-x case. Leftmost figure with $L = 2$, rightmost with $L = 6$.**

to a slight improvement of the ill-posedness as compared to the purely y fractional case, as to be expected from comparison with the x fractional cases above.

4 Uniqueness

In this section we will provide a uniqueness result for (3) based on inverse Sturm-Liouville theory. Section 4.2 discusses to what extent this approach can be transferred to the x-direction Riesz fractional case (4), but ends with the conclusion that this is prevented by some still-missing gaps in Riesz fractional inverse Sturm-Liouville theory.

4.1 Uniqueness in the y-direction Riesz fractional case

Consider

$$-u_{xx}(x,y) - {}_0D_y^{\alpha} u(x,y) + q(x)u(x,y) = 0, \text{ in } (0,1)\times(0,L) \tag{33}$$

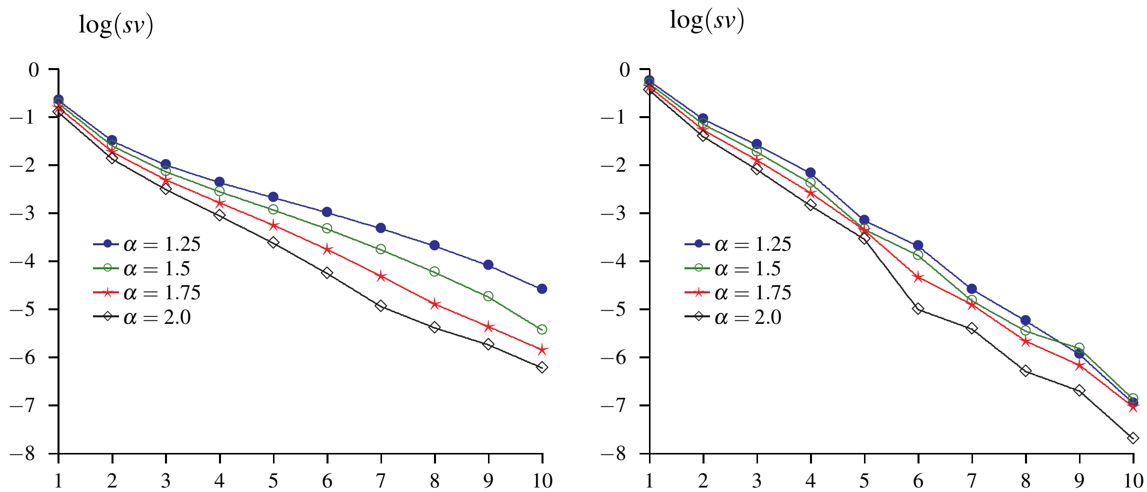

Figure 5: **Singular values of the Jacobian for $\alpha = (1.25,\ 1.5,\ 1.75, 2.0)$ for the Riesz-y case.** Leftmost figure with $L = 2$, rightmost with $L = 6$.

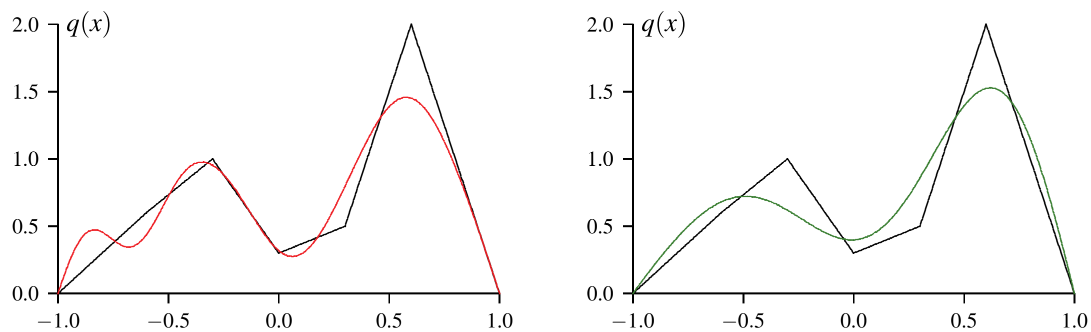

Figure 6: **Reconstruction of q in the Riesz-y case with $\alpha = 1.5$, $L = 6$ and 0.1 per cent noise: Leftmost figure with data on left, rightmost with data on right.**

where ${}_0D_y^\alpha$ is either the Djrbashian-Caputo or the Riesz derivative, with boundary conditions

$$u(x,0) = 0,\ u(x,L) = a(x), \quad u_x(0,y) - hu(0,y) = 0 \quad u_x(1,y) + Hu(1,y) = 0. \tag{34}$$

(including the Dirichlet or Neumann case with $h = H = \infty$ or $h = H = 0$), and corresponding overposed data

$$u_x(0,y) = g(y) \quad y \in (0,L). \tag{35}$$

With an eigensystem $(\mu_j, \psi_j)_{j\in\mathbb{N}}$ of $-\partial_{xx} + q$ with impedance boundary conditions $\psi_j'(0) - h\psi_j(0) = 0$, $\psi_j'(1) + H\psi_j(1) = 0$ and L^2 normalisation $\langle \psi_j, \psi_k \rangle = \delta_{jk}$ we can write $u(x,y) = \sum_{j=1}^\infty a_j w_j(y)\psi_j(x)$, where $a_j := \langle a, \psi_j \rangle$ and w_j solves

$$-{}_0D_y^\alpha w_j(y) + \mu_j w_j(y) = 0 \text{ in } (0,1), \quad w_j(0) = 0,\ w_j(L) = 1, \tag{36}$$

and thus

$$g(y) = \sum_{j=1}^\infty a_j w_j(y) \psi_j'(0) \text{ and } {}_0D_y^\alpha g(y) = -\sum_{j=1}^\infty a_j \mu_j w_j(y) \psi_j'(0). \tag{37}$$

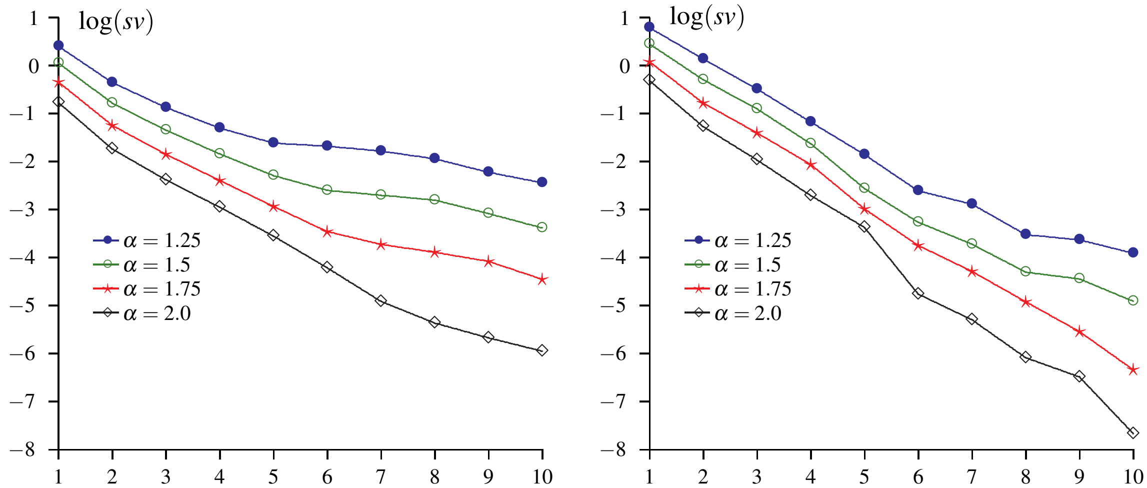

Figure 7: **Singular values of the Jacobian for** $\alpha = (1.25,\ 1.5,\ 1.75, 2.0)$ **for the Riesz-xy case**. Leftmost figure with $L = 2$, rightmost with $L = 6$.

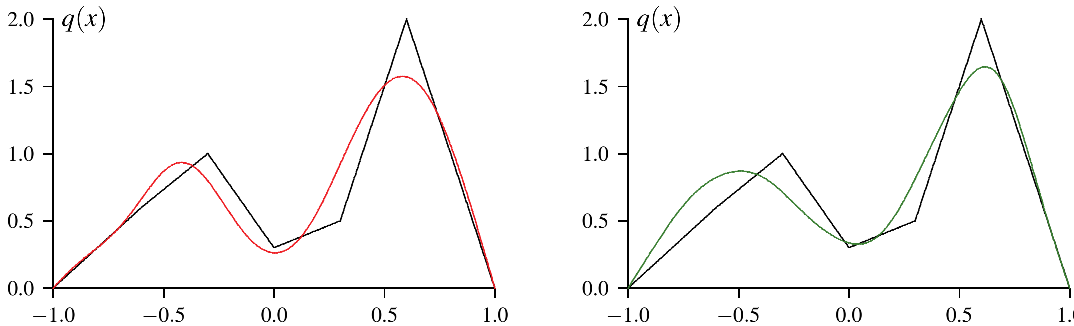

Figure 8: **Reconstruction of** $\mathbf{q}$ **in the Riesz-xy case with** $\alpha = 1.5$ **and** 0.1 **per cent noise: Leftmost figure with data on left, rightmost with data on right.**

If

$$a_j = \langle a, \psi_j \rangle \neq 0 \text{ for all } j \in \mathbb{N}, \tag{38}$$

this allows us to extract both $(\mu_j)_{j\in\mathbb{N}}$ and $(\psi_j'(0))_{j\in\mathbb{N}}$ as follows.

To this end, we first of all show that the functions $(w_j)_{j\in\mathbb{N}}$ are linearly independent. (Note that they are not eigenfunctions, because of the inhomogeneous boundary value at $y = L$ – if they were, linear independence would be immediate.) Assume that $0 = \sum_{j=1}^{\infty} c_j w_j =: \bar{w}$. Then for any $n \in \mathbb{N}$, also $0 = (-{}_0D_y^{\alpha})^n \bar{w}(y) = \sum_{j=1}^{\infty} c_j (\mu_j)^n w_j(y)$ for all $y \in (0, L)$ and thus, with $y \to L$ and $w_j(L) = 1$ we obtain $0 = \sum_{j=1}^{\infty} c_j (\mu_j)^n$. Since the Vandermonde determinant is nonzero for the distinct values $\mu_1 < \mu_2 < \cdots$, we conclude $c_j = 0$, $j \in \mathbb{N}$, that is, $(w_j)_{j\in\mathbb{N}}$ are linearly independent.

Thus, for each $j \in \mathbb{N}$, we have $w_j \notin \text{span}(w_i)_{i\in\mathbb{N}\setminus\{j\}}$ and a corollary of the Hahn-Banach Theorem yields existence of an element $w_j^* \in L^2(0, L)^* = L^2(0, L)$ such that $\langle w_j^*, w_i \rangle = \delta_{ij}$. Applying

w_j^* to (37) yields

$$\langle w_j^*, g\rangle = a_j \psi_j'(0)\,, \quad \langle w_j^*, {}_0D_y^\alpha g\rangle = -a_j\mu_j\psi_j'(0) = -\mu_j\langle w_j^*, g\rangle\,,$$

and so we obtain both μ_j and $\psi_j'(0)$, provided $a_j \neq 0$.
Sturm-Liouville theory (e.g., [**?**, Theorem 3.8.2] using the fact that endpoint data directly translates into norming constant data) therefore yields uniqueness of q.

Theorem 4.1. *The boundary data* (35) *uniquely determines the coefficient $q(x)$ in the boundary value problem* (33), (34)*, provided* (38) *holds.*

Condition (38) can be guaranteed by setting, for example, $a = \delta_0$, since then $a_j = \psi_j(0)$ and if this would vanish, the additional impedance boundary condition would imply $\psi_j = 0$.

Remark 4.1. *This result can be extended to a spatially higher dimensional version with respect to x by means of [**?**, Corollary 1.4]. Consider the identification of $q \in L^2(B)$ in the boundary value problem*

$$\begin{aligned} &-\triangle u(x,y) - {}_0D_y^\alpha u(x,y) + q(x)u(x,y) = 0 \quad \text{in } B\times(0,L) \\ &u(x,0) = 0,\ u(x,L) = a(x) \quad x\in B, \qquad u(x_0,y) = 0 \quad x_0 \in \partial B,\ y\in(0,L) \end{aligned} \tag{39}$$

from observations

$$\partial_\nu u(x_0,y) = g(y) \quad x_0 \in \partial B,\ y \in (0,L) \tag{40}$$

on the boundary of the smooth domain B.
Then, arguing as above we can uniquely recover both μ_j and $\partial_\nu\psi_j(x_0)$ for all $x_0 \in \partial B$ and $j \in \mathbb{N}$, provided (38) *holds. Applying [**?**, Corollary 1.4] yields uniqueness of q.*

4.2 On uniqueness in the x-direction Riesz fractional case

Considering the alternative setting

$$-D_x^\alpha u(x,y) - u_{yy}(x,y) + q(x)u(x,y) = 0, \text{ in } (0,1)\times(0,L), \tag{41}$$

where $D_x^\alpha = -{}^RD_x^\alpha$ is the Riesz derivative, with boundary conditions

$$u(x,0) = 0,\ u(x,L) = a(x), \quad u(0,y) = 0 \quad u(1,y) = 0, \tag{42}$$

and corresponding overposed data 7.

$$u_x(0,y) = g(y), \tag{43}$$

we can still perform separation of variables as well as reconstruction of eigenvalues and endpoint 8.
data as follows. For nonnegative $q \in L^\infty(0,1)$, arguing as in Section 2.2, using selfadjointness of the operator $-{}^RD_x^\alpha + q : H_*^{\alpha/2}(0,1) \to (H_*^{\alpha/2}(0,1))^*$ equipped with homogeneous Dirichlet boundary conditions and compactness of its inverse, we can conclude existence of its spectral decomposition and completeness of its eigenfunctions. 9.

Using the resulting eigensystem $(\mu_j,\psi_j)_{j\in\mathbb{N}}$ of $-{}^RD_x^\alpha+q$ with homogeneous Dirichlet boundary conditions, we can write $u(x,y)=\sum_{j=1}^\infty u_j(y)\psi_j(x)$, where u_j solves

$$-u_j''(y)+\mu_j u_j(y)=0 \text{ in } (0,L), \quad u_j(0)=0,\ u_j(L)=a_j:=\langle a,\psi_j\rangle,$$

thus

$$g(y)=\sum_{j=1}^\infty u_j(y)\psi_j'(0)=\sum_{j=1}^\infty a_j\frac{\sinh(\sqrt{\mu_j}y)}{\sinh(\sqrt{\mu_j})}\psi_j'(0) \tag{44}$$

If $a_j\neq 0$ for all $j\in\mathbb{N}$, this allows us to extract both $(\mu_j)_{j\in\mathbb{N}}$ and $(\psi_j'(0))_{j\in\mathbb{N}}$: Since the functions $\ell_j(y)=\frac{\sinh(\sqrt{\mu_j}y)}{\sinh(\sqrt{\mu_j})}\in L^2(0,L)$ are linearly independent, for each $j\in\mathbb{N}$, $\ell_j\notin\text{span}(\ell_i)_{i\in\mathbb{N}\setminus\{j\}}$ a corollary of the Hahn-Banach Theorem yields existence of an element $\ell_j^*\in L^2(0,L)^*=L^2(0,L)$ such that $\langle\ell_j^*,\ell_i\rangle=\delta_{ij}$. Applying ℓ_j^* to (44) and its second derivative yields

$$\langle\ell_j^*,g\rangle=a_j\psi_j'(0)\,,\quad \langle\ell_j^*,g''\rangle=-a_j\mu_j\psi_j'(0)=-\mu_j\langle\ell_j^*,g\rangle\,, \tag{45}$$

and so we obtain both μ_j and $\psi_j'(0)$, provided $a_j\neq 0$ ($\psi_j'(0)\neq 0$ follows as usual in the homogeneous Dirichlet case).

Remark 4.2. *If an inverse Sturm-Liouville theory were available in the fractional case $\alpha\in(1,2)$ as is the situation in the integer case $\alpha=2$, then the analogue of [**?**, Theorem 3.8.2] would give us uniqueness of q.*

We can still benefit from knowledge of the eigenvalues in other ways. As an example, consider recovery of the differentiation order α in case it is unknown. Then this can be obtained from the eigenvalue asymptotics according to the following result.

Lemma 4.1. *Let $q\in L^\infty(-1,1)$. Then for any $\alpha\in(1,2)$ there exists a constant $C_\alpha>0$ such that the eigenvalues $(\mu_j)_{j\in\mathbb{N}}$ of $-{}^RD_x^\alpha+q$ satisfy*

$$|\mu_j-(j\pi/2)^\alpha|\le\|q\|_{L^\infty(-1,1)}+\frac{C_\alpha}{j},\qquad j\in\mathbb{N}.$$

Proof. We rely on results from [**?**] for the eigenvalues of the fractional Laplacian on the domain $(-1,1)$ as well as on all of $\mathbb{R}$. To do so, we denote these operators by $(-_{(-1,1)}\triangle)^{\alpha/2}$ and $(-_{\mathbb{R}}\triangle)^{\alpha/2}$, respectively. Likewise we notationally distinguish between the two Riesz fractional versions $-{}_{(-1,1)}^{\ \ R}D_x^\alpha$ and $-{}_{\mathbb{R}}^{R}D_x^\alpha$ where the latter is obtained by replacing -1 and 1 by $-\infty$ and ∞ in the definition (9). It is known (see, e.g., [**?**]) that these operators coincide when defined on all of $\mathbb{R}$, that is,

$$-{}_{\mathbb{R}}^{R}D_x^\alpha=(-_{\mathbb{R}}\triangle)^{\alpha/2}\ \text{ on } H^\alpha(\mathbb{R}).$$

Moreover, we have

$$-{}_{\mathbb{R}}^{R}D_x^\alpha u=-{}_{(-1,1)}^{\ \ R}D_x^\alpha u\quad u\in\widetilde{H}^\alpha(-1,1):=\{u\in H^\alpha(\mathbb{R}):\text{supess}(u)\subseteq(-1,1)\}.$$

The results in [**?**] provide us with an approximate eigensystem $(\tilde{\mu}_j, \tilde{\psi}_j)_{j\in\mathbb{N}}$ of $(-_{\mathbb{R}}\triangle)^{\alpha/2}$ (and actually also of $(-_{(-1,1)}\triangle)^{\alpha/2}$) as follows. For

$$\tilde{\mu}_j := \Big(\frac{n\pi}{2} - \frac{(2-\alpha)\pi}{8}\Big)^{\alpha}, \quad \tilde{\psi}_j(x) = \eta(-x)F_j(1+x) + \eta(x)F_j(1-x)$$

with the smoothed Heaviside function

$$\eta(x) = \mathbf{1}_{(-1/3,0)}(x)\,9/2(x+1/3)^2 + \mathbf{1}_{[0,1/3)}(x)\,(1-9/2(x-1/3)^2) + \mathbf{1}_{[1/3,\infty)}(x)$$

and the eigenfunctions F_j of the fractional Laplacian on the half line $(-_{(0,\infty)}\triangle)^{\alpha/2}$, we have, [**?**, Lemmas 1, 2],

$$|e_j(x)| \le \frac{C(2-\alpha)}{\sqrt{\alpha}\,n} \quad x\in(-1,1) \quad \text{for } e_j := (-_{\mathbb{R}}\triangle)^{\alpha/2}\tilde{\psi}_j - \tilde{\mu}_j\tilde{\psi}_j\,,$$
$$\tilde{\psi}_j \in \widetilde{H}^{\alpha}(-1,1) \quad \text{and } |\|\tilde{\psi}_j\|_{L^2(-1,1)} - 1| \le \frac{C(2-\alpha)}{\sqrt{\alpha}\,j}$$

for some constant $C>0$ independent of α and j. Taking the inner products of the approximate eigenvalue equation with the eigenfunctions ψ_i^0 of the selfadjoint operator $-{}^{R}_{(-1,1)}D_x^{\alpha}$ (with corresponding eigenvalues μ_i^0) and summing up the squares of these generalised Fourier coefficients over $i\in\mathbb{N}$ we obtain

$$\sum_{i=1}^{\infty}(\mu_i^0-\tilde{\mu}_j)^2\langle\psi_i^0,\tilde{\psi}_j\rangle_{L^2(-1,1)}^2 = \sum_{i=1}^{\infty}\Big(\langle -{}^{R}_{\mathbb{R}}D_x^{\alpha}\psi_i^0,\tilde{\psi}_j\rangle_{L^2(-1,1)} - \tilde{\mu}_j\langle\psi_i^0,\tilde{\psi}_j\rangle_{L^2(-1,1)}\Big)^2$$
$$= \sum_{i=1}^{\infty}\Big(\langle\psi_i^0, -{}^{R}_{\mathbb{R}}D_x^{\alpha}\tilde{\psi}_j\rangle_{L^2(-1,1)} - \tilde{\mu}_j\langle\psi_i^0,\tilde{\psi}_j\rangle_{L^2(-1,1)}\Big)^2 = \sum_{i=1}^{\infty}\langle\psi_i^0,e_j\rangle_{L^2(-1,1)}^2.$$

The left hand side can be estimated from below by $\min_{i\in\mathbb{N}}(\mu_i^0-\tilde{\mu}_j)^2\|\tilde{\psi}_j\|_{L^2(-1,1)}^2$, since $(\psi_i^0)_{i\in\mathbb{N}}$ is a complete orthonormal system in $L^2(-1,1)$. Likewise, the right hand side equals $\|e_j\|_{L^2(-1,1)}^2$. Thus, upon renumbering the eigenvalues such that $(\mu_j^0-\tilde{\mu}_j)^2 = \min_{i\in\mathbb{N}}(\mu_i^0-\tilde{\mu}_j)^2$, altogether we obtain

$$|\mu_j^0-\tilde{\mu}_j| \le \sqrt{2}\Big(1-\frac{C(2-\alpha)}{\sqrt{\alpha}\,j}\Big)^{-1}\frac{C(2-\alpha)}{\sqrt{\alpha}\,j}\,. \tag{46}$$

To estimate the difference between μ_j^0 and μ_j, we proceed analogously, testing the identity $-{}^{R}_{\mathbb{R}}D_x^{\alpha}\psi_j^0 + q\psi_j^0 - \mu_j^0\psi_j^0 = q\psi_j^0$ with ψ_i and summing over i to obtain

$$\min_{i\in\mathbb{N}}(\mu_i-\mu_j^0)^2 \le \sum_{i=1}^{\infty}(\mu_i-\mu_j^0)^2\langle\psi_i,\psi_j^0\rangle_{L^2(-1,1)}^2$$
$$= \sum_{i=1}^{\infty}\Big(\langle\psi_i, -{}^{R}_{\mathbb{R}}D_x^{\alpha}\psi_j^0 + q\psi_j^0\rangle_{L^2(-1,1)} - \mu_j^0\langle\psi_i,\psi_j^0\rangle_{L^2(-1,1)}\Big)^2$$
$$= \sum_{i=1}^{\infty}\langle\psi_i,q\psi_j^0\rangle_{L^2(-1,1)}^2 = \|q\psi_j^0\|_{L^2(-1,1)}^2 \le \|q\|_{L^\infty(-1,1)}^2.$$

□

Lemma 4.1 provides us with an asymptotic formula for α in terms of the eigenvalues that we have previously obtained from (44), (45)

$$\alpha = \lim_{j\to\infty} \frac{\ln(\mu_j)}{\ln(j\pi/2)}, \tag{47}$$

where convergence actually takes place with a rate of at least $|\alpha - \frac{\ln(\mu_j)}{\ln(j\pi/2)}| = o(\frac{1}{\ln(j\pi/2)})$.
Thus we have proven the following result:

Theorem 4.2. *For the problem* (41), (42) *the value of the fractional exponent* α *can be obtained from the overposed conditions* (43) *using the formula* (47).

5 The forward problem and convergence of Newton's method

In this section we will consider the forward problem in the more general setting of q depending on x and y to first of all show well-posedness of the underlying initial boundary value problems in Section 5.1. Then in Section 5.2 this more general setting will allow us to verify a range invariance condition on the linearised forward operator and thus prove convergence of Newton's method.

5.1 Well-posedness of the forward problem

We provide an analysis of two types of x-fractional boundary value problems (3) involving the Djrbashian-Caputo derivative ${}_0D_x^\alpha$, namely

$$-({}_0D_x^{\alpha-1}u)_x + \mathscr{A}u + qu = f, \quad x \in (0,1), \quad u(0) = \phi_0, \quad u(1) = \phi_1 \tag{48}$$

$$-{}_0D_x^\alpha u + \mathscr{A}u + qu = f, \quad x \in (0,1), \quad u_x(0) = b_0, \quad u(1) = \phi_1 \tag{49}$$

and the Riesz derivative

$$-{}^RD_x^\alpha u + \mathscr{A}u + qu = f, \quad x \in (0,1), \quad u(0) = \phi_0, \quad u(1) = \phi_1 \tag{50}$$

and for the y-fractional case (4) with Riesz derivative

$$-u_{xx} - {}^RD_y^\alpha u + qu = f, \quad x \in (0,1), \quad y \in (0,L), \quad u(0) = \phi_0, \quad u(1) = \phi_1 \tag{51}$$

the latter in the spatially 1-d case. Here we assume $\alpha \in (1,2)$.
The anisotropy in space is taken into account by doing the analysis in Bochner spaces, with the x-direction as the distinguished direction.

The x-fractional cases (48)**,** (49)**,** (50)

Since with $I^{2-\alpha}v = k^{2-\alpha} * v$, we have

$$ {}_0D_x^\alpha v = {}_0I_x^{2-\alpha} v_{xx} = ({}_0I_x^{2-\alpha} v_x)_x - k^{2-\alpha}(x) v_x(0) = ({}_0D_x^{\alpha-1} v)_x - k^{2-\alpha}(x) v_x(0), $$

using $\bar\phi(x) = \phi_0 + x(\phi_1 - \phi_0)$ in case of (48) and $\bar\phi(x) = b_0 x + \phi_1 - b_0$ in case of (49) we can rewrite both cases as $u = \bar\phi + \tilde u$ with

$$ \begin{aligned} & -({}_0I_x^{2-\alpha}\tilde u_x)_x + \mathscr{A}\tilde u + q\tilde u = \tilde f := f - \mathscr{A}\bar\phi - q\bar\phi\,, \quad x \in (0,1), \\ & \begin{cases} \tilde u(0) = 0 & \text{for (48)} \\ \tilde u_x(0) = 0 & \text{for (49)} \end{cases} \quad \tilde u(1) = 0 \end{aligned} \tag{52} $$

where we have used ${}_0D_x^\alpha \bar\phi \equiv 0$ for $\alpha \geq 1$.

Note that the symmetric positive definite operator $\mathscr{A} : \mathscr{D}(\mathscr{A})(\subseteq \mathscr{H}) \to \mathscr{H}$ only acts in the directions y perpendicular to x, see, e.g., (13), (14). We set $\dot H^\sigma(D) := \mathscr{D}(\mathscr{A}^{\sigma/2})$ for any $\sigma \geq 0$; in case of $\mathscr{A} = -\triangle_y$ with homogeneous Dirichlet boundary conditions, we have, for example the following correspondences to classical Sobolev spaces: $\dot H^1(D) = H_0^1(D)$, $\dot H^2(D) = H_0^1(D) \cap H^2(D)$. In case of $\mathscr{A} = -{}_0^R D_y^\beta$ on $D = (0,L)$ with homogeneous Dirichlet boundary conditions, we have $\dot H^1(D) = H_*^{\sigma\beta}(0,L)$ (and due to the definition of this space via extension by zero outside $(0,L)$, no case distinction with respect to σ is needed).

The smoothness index σ will have to be chosen sufficiently large in order to be able to achieve certain embedding results; in particular, we will require $\dot H^\sigma(D) \subseteq L^\infty(D)$ with

$$ \|v\|_{L^\infty(D)} \leq C^D_{\dot H^\sigma \to L^\infty} \|\mathscr{A}^{\sigma/2} v\|_{\mathscr{H}}, \quad v \in \dot H^\sigma(D). \tag{53} $$

As previously noted, in this section we allow q to depend on x and y but assume that it is sufficiently close to an only x dependent nonnegative function $\bar q$

$$ \exists\, \bar q \in L^\infty(0,1), \quad \bar q \geq 0 \text{ a.e.}\,, \quad \|\bar q - q\|_{L^p(0,1;\dot H^\sigma(D))} \leq \bar c, \tag{54} $$

where under the $L^p(0,1;\dot H^\sigma(D))$ norm we formally interpret $\bar q$ as a function of x and y by setting $(\bar q(x))(y) \equiv \bar q(x)$.

To obtain an energy estimate for the solution u, we multiply (52) with $\mathscr{A}^\sigma \tilde u$ and integrate over $(0,1) \times D$ using integration by parts

$$ \begin{aligned} & \int_0^1 \Big(\langle {}_0I_x^{2-\alpha} \mathscr{A}^{\sigma/2} \tilde u_x(x), \mathscr{A}^{\sigma/2} \tilde u_x(x) \rangle_{\mathscr{H}} + \|\mathscr{A}^{(1+\sigma)/2} \tilde u(x)\|_{\mathscr{H}}^2 + \bar q(x) \|\mathscr{A}^{\sigma/2} \tilde u(x)\|_{\mathscr{H}}^2 \Big)\, dx \\ & = \int_0^1 \langle \mathscr{A}^{\sigma/2} \big((\bar q(x) - q(x)) \tilde u + \tilde f \big), \mathscr{A}^{\sigma/2} \tilde u \rangle_{\mathscr{H}}\, dx =: \text{rhs} \end{aligned} \tag{55} $$

Using coercivity of the Abel integral operator (10) and nonnegativity of $\bar q$, we can estimate the left hand side from below. The right hand side can be estimated from above by assuming that

$$ \|\mathscr{A}^{\sigma/2}[vw]\|_{\mathscr{H}} \leq C \Big(\|v\|_{L^\infty(D)} \|\mathscr{A}^{\sigma/2} w\|_{\mathscr{H}} + \|w\|_{L^\infty(D)} \|\mathscr{A}^{\sigma/2} v\|_{\mathscr{H}} \Big), \quad v, w \in L^\infty(D) \cap \dot H^\sigma(D). \tag{56} $$

The latter in case (13) with $a\equiv 1$, $c\equiv 1$ follows from the Kato-Ponce inequality

$$\|vw\|_{W^{\rho,r}(D)} \leq C\Big(\|v\|_{W^{\rho,p_1}(0,T)}\|w\|_{L^{q_1}(0,T)} + \|v\|_{L^{p_2}(0,T)}\|w\|_{W^{\rho,q_2}(0,T)}\Big) \tag{57}$$

for $\max\{r/d-d,0\}\leq\rho<1$ or $\rho\in 2\mathbb{N}$, $1<r<\infty$, p_1, p_2, q_1, $q_2\in(1,\infty]$, with $\frac{1}{r}=\frac{1}{p_i}+\frac{1}{q_i}$, $i=1,2$; see, e.g., [**?**], applied with $\rho=\sigma$, $r=p_1=q_2=2$, $q_1=p_2=\infty$. As a consequence of (53) and (56), $\dot{H}^\sigma$ is a Banach algebra, that is,

$$\|\mathscr{A}^{\sigma/2}[vw]\|_{\mathscr{H}} \leq 2CC^D_{\dot{H}^\sigma\to L^\infty}\|\mathscr{A}^{\sigma/2}v\|_{\mathscr{H}}\|\mathscr{A}^{\sigma/2}w\|_{\mathscr{H}}, \quad v,w\in\dot{H}^\sigma(D),$$

which clearly would not extend to $L^2(0,1;\dot{H}^\sigma(D))$.

We make the assumption $\alpha/2-1/2\geq-(p-1)/(2p)$, that is, $p\alpha\geq 1$ so that $H^{\alpha/2}_*(0,1)$ continuously embeds into $L^{2p/(p-1)}(0,1)$ and vice versa $L^{2p/(p+1)}(0,1)$ continuously embeds into $H^{-\alpha/2}_*(0,1)$ This together with Hölder's and Young's inequality yields

$$\begin{aligned}
\text{rhs} =& \int_0^1 \langle \mathscr{A}^{\sigma/2}\big((\bar{q}(x)-q(x))\tilde{u}+f-\mathscr{A}\bar{\phi}-q\bar{\phi}\big), \mathscr{A}^{\sigma/2}\tilde{u}\rangle_{\mathscr{H}}\,dx\\
\leq& 2CC^D_{\dot{H}^\sigma\to L^\infty}\bar{c}\|\mathscr{A}^{\sigma/2}\tilde{u}\|^2_{L^{2p/(p-1)}(0,1;\mathscr{H})}\\
&+\Big(2CC^D_{\dot{H}^\sigma\to L^\infty}\Big\|\,\|\mathscr{A}^{\sigma/2}q\|_{\mathscr{H}}\|\mathscr{A}^{\sigma/2}\bar{\phi}\|_{\mathscr{H}}\Big\|_{H^{-\alpha/2}_*(0,1)}\\
&\quad+\|\mathscr{A}^{\sigma/2}f\|_{H^{-\alpha/2}_*(0,1;\mathscr{H})}+\|\mathscr{A}^{1+\sigma/2}\bar{\phi}\|_{H^{-\alpha/2}_*(0,1;\mathscr{H})}\Big)\|\mathscr{A}^{\sigma/2}\tilde{u}\|_{H^{\alpha/2}_*(0,1;\mathscr{H})}\\
\leq& 2CC^D_{\dot{H}^\sigma\to L^\infty}\bar{c}\|\mathscr{A}^{\sigma/2}\tilde{u}\|^2_{L^{2p/(p-1)}(0,1;\mathscr{H})}+\frac{1}{2\underline{c}}C(q,\bar{\phi},f)^2+\frac{\underline{c}}{2}\|\mathscr{A}^{\sigma/2}\tilde{u}\|_{H^{\alpha/2}_*(0,1;\mathscr{H})}
\end{aligned}$$

where

$$\begin{aligned}
C(q,\bar{\phi},f) =& 2CC^D_{\dot{H}^\sigma\to L^\infty}\|\mathscr{A}^{\sigma/2}q\|_{L^p(0,1;\mathscr{H})}\|\mathscr{A}^{\sigma/2}\bar{\phi}\|_{L^{(p+1)/(p-1)}(0,1;\mathscr{H})}\\
&+\|\mathscr{A}^{\sigma/2}f\|_{H^{-\alpha/2}_*(0,1;\mathscr{H})}+\|\mathscr{A}^{1+\sigma/2}\bar{\phi}\|_{H^{-\alpha/2}_*(0,1;\mathscr{H})}.
\end{aligned} \tag{58}$$

Here we have assumed that $\bar{c}$ in (54) is small enough so that

$$\underline{c} := \cos((1-\alpha/2)\pi) - 2CC^D_{\dot{H}^\sigma\to L^\infty}\bar{c} > 0. \tag{59}$$

Applying these estimates to (55), we arrive at the energy estimate

$$\frac{\underline{c}}{2}\|\mathscr{A}^{\sigma/2}\tilde{u}\|_{H^{\alpha/2}_*(0,1;\mathscr{H})}+\|\mathscr{A}^{(1+\sigma)/2}\tilde{u}\|^2_{L^2(0,1;\mathscr{H})} \leq \frac{1}{2\underline{c}}C(q,\bar{\phi},f)^2. \tag{60}$$

Due to coercivity of the Riesz operator (12), these estimates remain valid for equation (50).

The y-fractional case (51)

This is covered by the above proof, with the replacements $d\to 1$, ${}_0D^\alpha_x\to\frac{d^2}{dx^2}$, $\mathscr{A}\to -{}^RD^\alpha_y$, based on (11), (12); the embedding estimate (53) can thus be achieved by assuming $\sigma\alpha>\frac{1}{2}$.

Theorem 5.1. *Let* (53), (56) *and* $p\alpha \geq 1$ *hold. Then for any* $f \in H_*^{-\alpha/2}(0,1;\dot{H}^\sigma(D))$, ϕ_0, ϕ_1, $b_0 \in \dot{H}^{2+\sigma}(D)$, $q \in L^p(0,1;\dot{H}^\sigma(D))$ *satisfying* (54), (59)*, each of the boundary value problems* (48), (49), (50), (51) *has a unique solution*

$$u \in U = H_*^{\alpha/2}(0,1;\dot{H}^\sigma(D)) \cap L^2(0,1;\dot{H}^{2+\sigma}(D)) \tag{61}$$

(with $D = (0,L)$ *in case of* (51)*). Moreover,* $\tilde{u} = u - \bar{\phi}$ *satisfies the bound* (60).

Proof. The full proof is based on a Faedo-Galerkin discretization using eigenfunctions φ_j of $\mathscr{A}$, that is, an ansatz $\tilde{u}(x,y) \approx \tilde{u}_N(x,y) := \sum_{j=1}^N \tilde{u}_j(x)\varphi_j(y)$ and uniform energy estimates on the Galerkin solutions $\tilde{u}_N$ derived as above. Uniform boundedness in Hilbert spaces yields a weakly convergent subsequence of the Galerkin solutions whose weak limit can be (easily, because of linearity) shown to yield a solution. Uniqueness results from the same energy estimates. □

Remark 5.1. *Assuming higher regularity of* q *with respect to* x *and multiplying, e.g., with* $-\mathscr{A}^\sigma \tilde{u}_{xx}$ *yields higher order energy estimates.*

5.2 Range invariance of the linearisation and convergence of frozen Newton

In a slightly different approach from Section 2, where we used separation of variables, we here define the forward operator $\mathbb{F} : \mathscr{D}(\mathbb{F})(\subseteq X) \to Y$, $\mathscr{D}(\mathbb{F}) = \{q \in X \,:\, q \text{ satisfies (54)}\}$ by $\mathbb{F}(q)(y) = u_x(x_0,y;q) = \mathbb{S}(q)_x(x_0,y)$, $y \in D$, where $x_0 \in \{0,1\}$ and $u = \mathbb{S}(q)$ solves (48).
(We will here exemplarily dicuss the case (48). The other settings (49), (50), (51) can be analysed analogously, based on Theorem 5.1).
Both X and Y are Hilbert spaces, in particular, in view of the analysis above,

$$X = L^2(0,1;\dot{H}^\sigma(D)), \quad Y = L^2(0,1;\mathscr{H}) \tag{62}$$

with σ such that (53), (56) is satisfied, that is, $\sigma = 1$ if $d = 2$, that is $D \subseteq \mathbb{R}^1$ and $\sigma = 2$ if $d = 3$, that is $D \subseteq \mathbb{R}^2$ (the latter would also cover the rather nonphysical case $d = 4$). As a regularisation term, in view of the fact that we aim for a potential that only depends on x, we may use an equivalent weighted norm in X

$$\|q\|_X^2 := \|q\|^2_{L^2(0,1;\dot{H}^\sigma(D))} + \rho \|\nabla_y q\|^2_{L^2(0,1;\mathscr{H})}$$

with a large penalty parameter ρ.
The derivative of $\mathbb{F}$ is defined by $\mathbb{F}'(q)\underline{dq} = (\mathbb{S}'(q)\underline{dq})_x(x_0,\cdot)$ where $v = \mathbb{S}'(q)\underline{dq}$ solves

$$-({}_0D_x^{\alpha-1}v)_x + \mathscr{A}v + qv = -\underline{dq}\,\mathbb{S}(q), \quad x \in (0,1), \quad v(0) = 0, \quad v(1) = 0, \tag{63}$$

or

$$-{}_0D_x^{\alpha}v + \mathscr{A}v + qv = -\underline{dq}\,\mathbb{S}(q), \quad x \in (0,1), \quad v_x(0) = 0, \quad v(1) = 0. \tag{64}$$

Thus, provided $u=\mathbb{S}(q)$ is bounded away from zero, $\mathbb{F}'$ satisfies the range invariance condition

$$\mathbb{F}'(\tilde{q})=\mathbb{F}'(q)R_q^{\tilde{q}} \text{ with } \|R_q^{\tilde{q}}-I\|_{X\to X}\leq C_R\|\tilde{q}-q\|_X \tag{65}$$

for

$$R_q^{\tilde{q}}\underline{dq}=\tfrac{1}{\mathbb{S}(q)}\Big((\tilde{q}-q)\,\mathbb{S}'(\tilde{q})\underline{dq}+\underline{dq}\,\mathbb{S}(\tilde{q})\Big) \tag{66}$$

due to the estimate

$$\begin{aligned}\|R_q^{\tilde{q}}\underline{dq}-\underline{dq}\|_X &\leq C_\rho\|\tfrac{1}{\mathbb{S}(q)}\Big((\tilde{q}-q)\,\mathbb{S}'(\tilde{q})\underline{dq}+\underline{dq}\,(\mathbb{S}(\tilde{q})-\mathbb{S}(q))\Big)\|_{L^2(0,1;\dot{H}^\sigma(D))}\\ &\leq C_\rho(2CC^D_{\dot{H}^\sigma\to L^\infty})^3\|\tfrac{1}{\mathbb{S}(q)}\|_{X\to X}\Big(\|\tilde{q}-q\|_{L^2(0,1;\dot{H}^\sigma(D))}\|\mathbb{S}'(\tilde{q})\underline{dq}\|_{L^\infty(0,1;\dot{H}^\sigma(D))}\\ &\qquad+\|\underline{dq}\|_{L^2(0,1;\dot{H}^\sigma(D))}\|\mathbb{S}(\tilde{q})-\mathbb{S}(q)\|_{L^\infty(0,1;\dot{H}^\sigma(D))}\Big).\end{aligned}$$

Note that this range invariance would not hold on a space of functions depending on x only, since multiplication with, e.g., $\mathbb{S}(\tilde{q})$ in (66) would add variability in y.

Here it suffices to work with the Gâteaux derivative $\mathbb{F}'(q)\underline{dq}=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(\mathbb{F}(q+\varepsilon\underline{dq})-\mathbb{F}(q))$ of $\mathbb{F}$, since by the Fundamental Theorem of Calculus, for any element $y^*\in Y^*$ in the dual of Y, this satisfies

$$\langle y^*,\mathbb{F}(q+\underline{dq})-\mathbb{F}(q)\rangle_{Y^*,Y}=\int_0^1\langle y^*,\mathbb{F}'(q+\theta\underline{dq})\,d\theta\underline{dq}\rangle_{Y^*,Y}=\int_0^1\langle y^*,\mathbb{F}'(q_0)R_{q_0}^{q+\theta\underline{dq}}\,d\theta\underline{dq}\rangle_{Y^*,Y}$$

To prove convergence of a frozen regularised Newton method,

$$q^{k+1}=\mathrm{argmin}_{q\in X}\|\mathbb{F}(q^k)+\mathbb{F}'(q_0)(q^{k+1}-q^k)-g^\delta\|_Y^2+\gamma_k\|q-\tilde{q}_0\|_X^2,$$

or equivalently, with the Hilbert space adjoint $\mathbb{F}'(q_0)^*:Y\to X$ of $\mathbb{F}'(q_0)$,

$$q^{k+1}=q^k-(\mathbb{F}'(q_0)^*\mathbb{F}'(q_0)+\gamma_kI)^{-1}\Big(\mathbb{F}'(q_0)^*(\mathbb{F}(q^k)-g^\delta)+\gamma_k(q^k-\tilde{q}_0)\Big) \tag{67}$$

it suffices to guarantee (65) at some fixed $q=q_0$ where the derivative is evaluated, cf. [**?**]. In (67), the sequence of regularisation parameters γ_k is chosen to monotonically tend to zero as $k\to\infty$, e.g.

$$\gamma_k=\gamma_0\vartheta^k \text{ for some } \vartheta\in(0,1).$$

Indeed, with the exact solution denoted by $q^\dagger$, the error satisfies the recursion

$$\begin{aligned}q^{k+1}-q^\dagger=&(\mathbb{F}'(q_0)^*\mathbb{F}'(q_0)+\gamma_kI)^{-1}\Big(\mathbb{F}'(q_0)^*\mathbb{F}'(q_0)\int_0^1(I-R_{q_0}^{q^\dagger+\theta(q^k-q^\dagger)})\,d\theta\,(q^k-q^\dagger)\\ &+\mathbb{F}'(q_0)^*(g^\delta-y)+\gamma_k(\tilde{q}_0-q^\dagger)\Big).\end{aligned}$$

Using the estimates

$$\|(\mathbb{F}'(q_0)^*\mathbb{F}'(q_0)+\gamma_kI)^{-1}\mathbb{F}'(q_0)^*\mathbb{F}'(q_0)\|\leq 1,\quad \|(\mathbb{F}'(q_0)^*\mathbb{F}'(q_0)+\gamma_kI)^{-1}\mathbb{F}'(q_0)^*\|\leq\frac{1}{2\sqrt{\gamma_k}},$$

$$a_k:=\|(\mathbb{F}'(q_0)^*\mathbb{F}'(q_0)+\gamma_kI)^{-1}\gamma_k(\tilde{q}_0-q^\dagger)\|\to 0 \text{ as } k\to\infty$$

provided $\tilde{q}_0 - q^\dagger \in \mathcal{N}(\mathbb{F}'(q_0))^\perp$ (with the rate $a_k \leq \gamma_k^\nu$ under a source condition $\tilde{q}_0 - q^\dagger \in \mathscr{R}((\mathbb{F}'(q_0)^*\mathbb{F}'(q_0))^\nu)$), and assuming $\|q^k - q_0\|_X$, $\|q^\dagger - q_0\|_X \leq \rho$, we obtain

$$\|q^{k+1} - q^\dagger\|_X \leq C_R\rho\|q^k - q^\dagger\|_X + a_k + \frac{\delta}{2\sqrt{\gamma_k}}.$$

Thus with ρ small enough so that $C_R\rho < 1$, an induction proof yields $\|q^{k+1} - q^0\| \leq \rho$ and (by monotonicity of γ_k)

$$\|q^{k+1} - q^\dagger\|_X \leq (C_R\rho)^{k+1}\|q^0 - q^\dagger\|_X + \sum_{j=0}^{k}(C_R\rho)^{k-j}a_j + \frac{1}{2(1-C_R\rho)}\frac{\delta}{2\sqrt{\gamma_k}}$$

for all $k \leq k_* - 1$, where the stopping index $k_* = k_*(\delta)$ is defined a priori such that

$$k_*(\delta) \to \infty, \quad \frac{\delta}{\sqrt{\gamma_{k_*(\delta)}}} \to 0, \quad \text{as } \delta \to 0.$$

In particular, due to the fact that $\lim_{k\to\infty}\sum_{j=0}^{k}(C_R\rho)^{k-j}a_j = 0$, we have

$$\|q^{k_*(\delta)} - q^\dagger\|_X \to 0 \text{ as } \delta \to 0. \tag{68}$$

Theorem 5.2. *The regularised frozen Newton method* (67) *with* $\tilde{q}_0 - q^\dagger \in \mathcal{N}(\mathbb{F}'(q_0))^\perp$ *and starting value* q_0 *sufficiently close to* $q^\dagger$*, such that the multiplication operator with* $\frac{1}{\mathbb{S}(q_0)}$ *is bounded as an operator from* X *into itself*

$$\|\tfrac{1}{\mathbb{S}(q_0)} \cdot \|_{X\to X} := M < \infty \tag{69}$$

is well defined and converges as in equation (68)*.*

Without going into detail, we mention that under logarithmic source conditions (as natural in view of the exponential ill-posedness) logarithmic convergence rates can be shown.

An analogous proof can be carried out directly in the setting of Section 2, where we also have to allow for more variation in q to enable a range invariance condition to hold. More precisely, we consider recovery of $(q_\ell(x))_{\ell\in\mathbb{N}}$ in the system

$$-{}_0D^\alpha u_j(x) + \lambda_j u_j(x) + (A(\vec{u}(x))\vec{q}(x))_j = f_j(x), \quad x \in (0,1), \quad j \in \mathbb{N} \tag{70}$$

with boundary conditions (17), which results from testing (48) with φ_j and using the abbreviations $\vec{u}(x) = (u_i(x))_{i\in\mathbb{N}}$, $\vec{f}(x) = (f_i(x))_{i\in\mathbb{N}}$, $u_i(x) = \langle u(x,\cdot), \varphi_i\rangle_{\mathscr{H}}$, $f_i(x) = \langle f(x,\cdot), \varphi_i\rangle_{\mathscr{H}}$, $(A(\vec{w}))_{j\ell} = \langle \sum_{i=1}^{\infty} w_i\varphi_i\,\varphi_\ell, \varphi_j\rangle_{\mathscr{H}}$. The forward operator is defined as in Section 2 by $F(q) = (F_j(q))_{j\in\mathbb{N}}$, $F : \mathscr{D}(F) \to \ell^2$, $F_j(q) = u_j'(0)$, with $\mathscr{D}(F) \subseteq \hat{X}$ in the extended parameter space $\hat{X} = L^2(0,1;h^\sigma)$, where $h^\sigma = \{\vec{w} \in \ell^2 \,:\, (\lambda_i^{\sigma/2} w_i)_{i\in\mathbb{N}} \in \ell^2\}$. Likewise, we have a parameter-to-state map $S : \mathscr{D}(F) \to H_*^{\alpha/2}(0,1;h^\sigma) \cap L^2(0,1;h^{2+\sigma})$. Based on Theorem 5.1 and the transform

$u \mapsto (\langle u, \varphi_i \rangle_{\mathscr{H}})_{i\in\mathbb{N}}$ which is an isometric isomorphism from $L^2(0,1;H^\sigma(D))$ to $L^2(0,1;h^\sigma)$ for any $\sigma \in \mathbb{R}$, both operators F and S are well defined on $\mathscr{D}(F) = \{\vec{q} \in L^2(0,1;H^\sigma(D)) :$ $q(x,y) = \sum_{\ell=1}^\infty q_\ell(x)\varphi_\ell(y)$ satisfies (54)$\}$. Clearly, F' is given by $F'(\vec{q})\underline{\vec{dq}} = (S'(\vec{q})\underline{\vec{dq}})_x(x_0)$ where $\vec{v} = S'(\vec{q})\underline{dq}$ solves

$$-{}_0D^\alpha v_j(x) + \lambda_j v_j(x) + (A(\vec{v}(x))\vec{q}(x))_j = -(A(\vec{u}(x))\underline{\vec{dq}}(x))_j\,, \quad x \in (0,1)\,, \quad j \in \mathbb{N} \tag{71}$$

with homogeneous Dirichlet boundary conditions. Under condition (69), on $q_0(x,y) = \sum_{\ell=1}^\infty q_{0\ell}(x)\varphi_\ell(y)$ the infinite matrix $A(\vec{u}_0)$ is boundedly invertible at u_0:

$$\begin{aligned}\|A(\vec{u}_0)\vec{q}\|^2_{L^2(0,1;h^\sigma)} &= \int_0^1 \sum_{i=1}^\infty \lambda_i^\sigma \langle \Big(\sum_{j=1}^\infty u_{0,j}(x)\varphi_j\Big)\Big(\sum_{\ell=1}^\infty q_\ell(x)\varphi_\ell\Big), \varphi_i\rangle^2_{\mathscr{H}} \\ &= \|u_0\, q\|^2_X \geq \frac{1}{M^2}\|q\|^2_X = \frac{1}{M^2}\|\vec{q}\|^2_{\hat{X}}\end{aligned}$$

for $q(x,y) = \sum_{\ell=1}^\infty q_\ell(x)\varphi_\ell(y)$. Consequently, with

$$R^{\vec{\tilde{q}}}_{\vec{q}_0}\,\underline{\vec{dq}} = A(\vec{u}_0)^{-1}\Big((\vec{\tilde{q}} - \vec{q}_0)\,S'(\vec{\tilde{q}})\underline{\vec{dq}} + A(\vec{\tilde{u}})\underline{\vec{dq}}\Big) \tag{72}$$

and $\vec{u}_0 = S(\vec{q}_0)$, $\vec{\tilde{u}} = S(\vec{\tilde{q}})$, the range invariance condition is satisfied and an analogue of Theorem 5.2 follows.

Corollary 5.1. *The regularised frozen Newton method* (67) *with* $\mathbb{F}$ *replaced by* F, $\vec{q}_0 - \vec{q}^\dagger \in \mathscr{N}(F'(\vec{q}_0))^\perp$ *and starting value* $\vec{q}_0$ *sufficiently close to* $\vec{q}^\dagger$ *and such that* (69) *holds, is well defined and converges*

$$\|\vec{q}^{k_*(\delta)} - \vec{q}^\dagger\|_{\hat{X}} \to 0 \text{ as } \delta \to 0.$$

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