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Simple algebraic groups with the same maximal tori, weakly commensurable Zariski-dense subgroups, and good reduction

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We provide a new condition for an absolutely almost simple algebraic group to have good reduction with respect to a discrete valuation of the base field which is formulated in terms of the existence of maximal tori with special properties. This characterization, in particular, shows that the Finiteness Conjecture for forms of an absolutely almost simple algebraic group over a finitely generated field that have good reduction at a divisorial set of places of the field (cf. [66]) would imply the finiteness of the genus of the group at hand. It also leads to a new phenomenon that we refer to as "killing the genus by a purely transcendental extension." Yet another application deals with the investigation of "eigenvalue rigidity" of Zariski-dense subgroups (cf. [64]), which in turn is related to the analysis of length-commensurable Riemann surfaces and general locally symmetric spaces. Finally, we

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analyze the Finiteness Conjecture and the genus problem for simple algebraic groups of type F_4.

1. Introduction

Let G be a reductive affine algebraic group over a field k. Given a discrete valuation v of k, we denote by k_v the corresponding completion, with valuation ring $\mathcal{O}_v$ and residue field $k^{(v)}$. We recall that G has *good reduction* at v if there exists a reductive group scheme $\mathcal{G}$ over $\mathcal{O}_v$ whose generic fiber $\mathcal{G} \times_{\mathcal{O}_v} k_v$ is isomorphic to $G \times_k k_v$; then the closed fiber $\mathcal{G} \times_{\mathcal{O}_v} k^{(v)}$ is called the *reduction* of G at v and will be denoted $\underline{G}^{(v)}$ (see §2 for more details, including the uniqueness of reduction, and variations). The focus of the recent work [14], [15], [16], [67], [68] was on the analysis of k-forms of G that have good reduction at all valuations in some natural set V of discrete valuations of k. We refer the reader to the survey [66] for a detailed discussion of this problem and some natural choices for k and V. One is particularly interested in the case where k is a finitely generated field and V is a *divisorial set* of valuations of k (which means that V consists of the discrete valuations that correspond to all prime divisors on a model $\mathfrak{X}$ of k, i.e. an irreducible separated normal scheme of finite type over $\mathbb{Z}$ with function field k — see [66, 5.3]). In this case, there is the following Finiteness Conjecture (cf. [66, Conjecture 5.7]): *the set of k-isomorphism classes of k-forms of G that have good reduction at all $v \in V$ is finite* (at least when the characteristic of k is "good"). This conjecture has been established in a number of cases, but the general case remains the focus of ongoing work. Its significance for the current effort to develop the arithmetic theory of algebraic groups over higher-dimensional fields is predicated on deep connections with other important problems. In particular, the validity of the conjecture for an absolutely almost simple simply connected k-group G and any divisorial set of places of k would imply the properness of the global-to-local map $H^1(k, \overline{G}) \to \prod_{v \in V} H^1(k_v, \overline{G})$ in Galois cohomology for the corresponding adjoint group $\overline{G}$ (cf. [66, §6]). In the present paper, we will focus on several other applications of the Finiteness Conjecture, particularly those related to the *genus problem* for absolutely almost simple algebraic groups. This includes a new phenomenon that we have termed "killing the genus by a purely transcendental extension," and the investigation of "eigenvalue rigidity" of Zariski-dense subgroups (cf. [64]) — the latter is related to the analysis of length-commensurable Riemann surfaces and general locally symmetric spaces in differential geometry (cf. [58], [60]). Finally, we develop new techniques for tackling the genus problem for some groups of type F_4 and obtain several finiteness results in this case.

To prepare for the discussion of the genus problem, we recall that two semisimple algebraic groups G_1 and G_2 defined over a field k are said to have the *same isomorphism classes of maximal k-tori* if every maximal k-torus T_1 of G_1 is k-isomorphic to some maximal k-torus T_2 of G_2, and vice versa. We then define the (k-)genus $\mathbf{gen}_k(G)$ (resp., the extended (k-)genus $\mathbf{gen}_k^+(G)$) of an absolutely almost simple k-group G as the set of k-isomorphism classes of *inner* k-forms of G (resp., *all* k-forms of G) that have the same isomorphism classes of maximal k-tori as G. (We note that we always have the inclusion $\mathbf{gen}_k(G) \subset \mathbf{gen}_k^+(G)$, which, in fact, is an equality whenever k is finitely generated — see Corollary 3.5.) The analysis of the genus is the subject of the *genus problem*. In particular, one expects to prove that the genus is always finite whenever the field k is finitely generated (of "good" characteristic) and is trivial in some special situations (see [66, §8]). One of our main results is the following theorem that relates the genus problem to good reduction.

Theorem 1.1. *Let G be an absolutely almost simple linear algebraic group over a field k and let v be a discrete valuation of k. Assume that the residue field $k^{(v)}$ is finitely generated and that* $\mathrm{char}\, k^{(v)} \neq 2$ *if G is of type B_ℓ $(\ell \geq 2)$. If G has good reduction at v, then any $G' \in \mathbf{gen}_k(G)$ also has good reduction at v. Moreover, the reduction $\underline{G'}^{(v)}$ lies in the genus $\mathbf{gen}_{k^{(v)}}(\underline{G}^{(v)})$ of the reduction $\underline{G}^{(v)}$.*

It should be pointed out that the proof of this theorem is based on an entirely new approach to good reduction of simple algebraic groups that shows that the existence of good reduction can be characterized in terms of the presence of maximal tori with certain specific properties — see Theorems 6.2 and 6.6 for precise statements. This approach enables us to extend to absolutely almost simple groups the techniques developed earlier in [13], [14], [17], and [65] to analyze the genus of a division algebra. In particular, just like the finiteness of the n-torsion of the unramified Brauer group ${}_n\mathrm{Br}(k)_V$ of a finitely generated field k with respect to a divisorial set of places V (provided that n is prime to $\mathrm{char}\, k$) implies the finiteness of the genus of any central simple algebra D of degree n over k (cf. [13], [14]), the above Finiteness Conjecture, in view of the following corollary of Theorem 1.1, would imply the finiteness of the genus of any absolutely almost simple algebraic k-group.

Corollary 1.2. *Let G be an absolutely almost simple algebraic group over an infinite finitely generated field k, and let V be a divisorial set of places of k. Assume that* $\mathrm{char}\, k \neq 2$ *if G is of type B_ℓ $(\ell \geq 2)$. Then there exists a finite subset $S \subset V$ such that every $G' \in \mathbf{gen}_k(G)$ has good reduction at all $v \in V \setminus S$.*

Next, applying Theorem 1.1, in conjunction with the theorem of Raghunathan-Ramanathan [63], we obtain the following statement concerning the effect of a purely transcendental base change on the genus.

Theorem 1.3. *Let G be an absolutely almost simple algebraic group over a finitely generated field k of characteristic $\neq 2$, and let $K = k(x)$ be the field of rational functions. Then any $H \in \mathbf{gen}_K(G \times_k K)$ is of the form $H = H_0 \times_k K$ for some $H_0 \in \mathbf{gen}_k(G)$.*

In view of [58, Theorem 7.5], the following is an immediate consequence of Theorem 1.3.

Corollary 1.4. *Let G be an absolutely almost simple simply connected algebraic group over a number field k, and let $K = k(x_1, \ldots, x_m)$ be the field of rational functions in $m \geqslant 1$ variables. Then the genus $\mathbf{gen}_K(G \times_k K)$ is finite, and in fact reduces to a single element if the type of G is different from A_ℓ $(\ell > 1)$, $\mathsf{D}_{2\ell+1}$ $(\ell > 1)$, and E_6.*

Theorem 1.3 prompts the question of whether for an absolutely almost simple algebraic group G over a field k and any $G' \in \mathbf{gen}_k(G)$, the group $G' \times_k K$ obtained by base change to the field of rational functions $K = k(x)$ lies in $\mathbf{gen}_K(G \times_k K)$. It turns out that not only is the answer to this question negative, but in fact one should expect an opposite phenomenon that we have termed "killing the genus by a purely transcendental extension." The nature of this phenomenon reveals itself in the following two statements.

Theorem 1.5. *Let A be a central simple algebra of degree n over a finitely generated field k, and let $G = \mathrm{SL}_{1,A}$. Assume that* char k *is prime to n, and let $K = k(x_1, \ldots, x_{n-1})$ be the field of rational functions in $(n-1)$ variables. Then $\mathbf{gen}_K(G \times_k K)$ consists of (the isomorphism classes of) groups of the form $H \times_k K$, where $H = \mathrm{SL}_{1,B}$ and B is a central simple algebra of degree n such that its class $[B]$ in the Brauer group* $\mathrm{Br}(k)$ *generates the same subgroup as the class $[A]$.*

The proof uses Amitsur's theorem on generic splitting fields [2], and a result of D. Saltman [73], [74] on function fields of Severi-Brauer varieties.

Theorem 1.6. *Let G be a group of type G_2 over a finitely generated field k of characteristic $\neq 2, 3$, and let $K = k(x_1, \ldots, x_6)$ be the field of rational functions in 6 variables. Then $\mathbf{gen}_K(G \times_k K)$ reduces to a single element.*

The proof relies on properties of Pfister forms (cf. [40]). These results prompt the following conjecture.

Conjecture 1.7. *Let G be an absolutely almost simple group over a finitely generated field k. Assume that* char k *is prime to the order of the Weyl group of G. Then there exists a purely transcendental extension $K = k(x_1, \ldots, x_m)$ of transcendence degree m depending only on the Cartan-Killing type of G such that every $H \in \mathbf{gen}_K(G \times_k K)$ is of the form $H_0 \times_k K$, where H_0 has the property that $H_0 \times_k F \in \mathbf{gen}(G \times_k F)$ for* any *field extension F/k.*

In §8.4, we relate this conjecture to the notion of the *motivic genus* that was proposed by A.S. Merkurjev.

Next, we will discuss applications of our results to the analysis of weakly commensurable Zariski-dense subgroups, which was initiated in [58] in connection with some problems in differential geometry. So, let G_1 and G_2 be absolutely almost simple algebraic groups over a field F of characteristic zero, and let $\Gamma_1 \subset G_1(F)$ and $\Gamma_2 \subset G_2(F)$ be two finitely generated Zariski-dense subgroups. We refer the reader to [58, §1] (see also §3 of the present paper) for the technical definition of the relation of *weak commensurability*; here, we only mention that it is a way of matching the eigenvalues of semisimple elements of Γ_1 and Γ_2. This relation is expected to lead to a new form of rigidity, called "eigenvalue rigidity," for arbitrary finitely generated Zariski-dense subgroups, where traditional forms of rigidity are inapplicable (cf. [64]). In this paper, we will show that one of the key issues in eigenvalue rigidity can be reduced to the Finiteness Conjecture. To provide more context, we recall that given a Zariski-dense subgroup $\Gamma \subset G(F)$, where G is an absolutely almost simple algebraic group defined over a field F, the *trace field* k_Γ is defined to be the subfield of F generated by the traces $\mathrm{tr}(\mathrm{Ad}\,\gamma)$ of elements $\gamma \in \Gamma$ in the adjoint representation on the Lie algebra $\mathfrak{g}$. According to a theorem of E.B. Vinberg [80], the field $k = k_\Gamma$ is the *minimal field of definition of* Γ. This means that k is the minimal subfield of F with the property that all transformations in $\mathrm{Ad}\,\Gamma$ can be simultaneously represented by matrices having all entries in k in a suitable basis of $\mathfrak{g}$. If such a basis is chosen, then the Zariski-closure of $\mathrm{Ad}\,\Gamma$ in $\mathrm{GL}(\mathfrak{g})$ is a simple algebraic k-group $\mathcal{G}$. It is an F/k-form of the adjoint group $\overline{G}$ called the *algebraic hull* of $\mathrm{Ad}\,\Gamma$. It should be mentioned that if Γ is *arithmetic*, the pair $(k, \mathcal{G})$ determines the commensurability class of Γ. While for general Zariski-dense subgroups this is no longer the case, the pair $(k, \mathcal{G})$ remains an important invariant of the commensurability class.

Now let $\Gamma_1 \subset G_1(F)$ and $\Gamma_2 \subset G_2(F)$ be finitely generated Zariski-dense subgroups of absolutely almost simple algebraic groups G_1 and G_2. Assume that Γ_1 and Γ_2 are weakly commensurable. Then $k_{\Gamma_1} = k_{\Gamma_2} =: k$ (cf. [58, Theorem 2]). Furthermore, G_1 and G_2 either have the same Cartan-Killing type, or one of them has type B_ℓ and the other type C_ℓ for some $\ell \geq 3$. So, apart from the ambiguity between types B and C, the corresponding algebraic hulls $\mathcal{G}_1$ and $\mathcal{G}_2$ are k-forms of one another. The remaining critical issue is the relationship between $\mathcal{G}_1$ and $\mathcal{G}_2$. More precisely, if we fix Γ_1, what can one say about the set of the forms $\mathcal{G}_2$ as $\Gamma_2 \subset G_2(F)$ runs through finitely generated Zariski-dense subgroups that are weakly commensurable to Γ_1? There is a conjecture (cf. [64, Conjecture 6.1]) that this set consists of finitely many k-isomorphism classes — see §10 for the precise formulation. If true, this would be a very strong statement[1] asserting that the eigenvalues of elements of a Zariski-dense subgroup (which could be, for example, just a free group on two generators) determine the ambient algebraic group up to finitely many possibilities. For example, if $G = \mathrm{SL}_{1,A}$, where A is a central simple algebra of degree n over a field k, and $\Gamma \subset G(k)$ is a finitely generated Zariski-dense subgroup with trace field k, then there would be only finitely many choices for a central simple k-algebra A' (necessarily of the same degree n) such that for $G' = \mathrm{SL}_{1,A'}$, the group $G'(k)$ contains a finitely generated Zariski-dense subgroup weakly commensurable to Γ. What we will see in §9 is that this conjecture again can be derived from the Finiteness

[1] Which, in particular, would be stronger than the finiteness of the genus.

Conjecture with the help of the following result (and in fact, the above statement for groups of type $\mathrm{SL}_{1,A}$ is already a theorem due to the fact that the Finiteness Conjecture has been confirmed in this case).

Theorem 1.8. *Let G be an absolutely almost simple algebraic group over a finitely generated field k of characteristic zero, and let V be a divisorial set of places of k. Given a finitely generated Zariski-dense subgroup $\Gamma \subset G(k)$ with trace field k, there exists a finite subset $S(\Gamma) \subset V$ such that every absolutely almost simple algebraic k-group G' with the property that there exists a finitely generated Zariski-dense subgroup $\Gamma' \subset G'(k)$ that is weakly commensurable to Γ has good reduction at all $v \in V \setminus S(\Gamma)$.*

The results on weakly commensurable arithmetic groups developed in [58] were used to settle some long-standing problems about isospectral and length-commensurable locally symmetric spaces. Here we will give only one application of the results on good reduction to not necessarily arithmetically defined Riemann surfaces. For a Riemannian manifold M, we denote by $L(M)$ the *(weak) length spectrum* of M, i.e. the collection of the lengths of all closed geodesics in M. We then call two Riemannian manifolds M_1 and M_2 *length-commensurable* if $\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2)$. Consider a Riemann surface M of the form $\mathbb{H}/\Gamma$, where $\mathbb{H}$ is the complex upper half-plane and $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ is a discrete subgroup with torsion-free image in $\mathrm{PSL}_2(\mathbb{R})$. We will assume that Γ is finitely generated and Zariski-dense in SL_2 (which is automatically true if M is, for example, compact). Then one can naturally associate to Γ a quaternion algebra A_Γ whose center is the trace field of Γ — see [42, 3.2] and §9. If Γ is arithmetic, then A_Γ is *the* quaternion algebra required for its description, and in the general case it is an invariant of the commensurability class of Γ. In §9, we will prove the following result that contains no arithmeticity assumptions.

Theorem 1.9. *Let $M_i = \mathbb{H}/\Gamma_i$ $(i \in I)$ be a family of length-commensurable Riemann surfaces, where $\Gamma_i \subset \mathrm{SL}_2(\mathbb{R})$ is a discrete finitely generated Zariski-dense subgroup with torsion-free image in $\mathrm{PSL}_2(\mathbb{R})$. Then the quaternion algebras A_{Γ_i} $(i \in I)$ belong to finitely many isomorphism classes over the common center (= trace field of all the Γ_i's).*

To the best of our knowledge, this statement is one of the first examples of applications of techniques from arithmetic geometry to nonarithmetic Riemann surfaces.

We conclude the paper with a series of results on forms with good reduction and the genus of simple algebraic groups of type F_4, which have never been previously analyzed over fields more general than number fields. The first three results treat those forms that split over a quadratic extension of the base field (see Appendix 2 for a characterization of such forms in terms of cohomological invariants). We recall that the $\mathbb{Q}$-forms of type F_4 that have good reduction at all primes were described in [29] and [21], and that for any simple group G of that type over a number field k, the genus $\mathbf{gen}_k(G)$ is trivial [58, Theorem 7.5]. We will prove the following version of the "Stability Theorem" that was established previously for groups of the form $\mathrm{SL}_{1,A}$, where A is a central simple algebra of exponent 2 (cf. [13]), and groups of type G_2 (cf. [16]).

Theorem 1.10. *Let k_0 be a number field, and set $k = k_0(x)$. Then for any absolutely simple algebraic k-group G of type F_4 that splits over a quadratic extension of k, the genus $\mathbf{gen}_k(G)$ is trivial.*

Next, following Kato [36], we recall that a 2-dimensional global field is defined to be the function field of either a curve over a number field or a surface over a finite field.

Theorem 1.11. *Let k be either a 2-dimensional global field of characteristic $\neq 2, 3$ or a purely transcendental extension $k = k_0(x, y)$ of transcendence degree 2 of a number field k_0. Then for any absolutely simple k-group G of type F_4 that splits over a quadratic extension of k, the genus $\mathbf{gen}_k(G)$ is finite.*

This is derived by combining Corollary 1.2 with the following theorem.

Theorem 1.12. *Let k be either a 2-dimensional global field of characteristic $\neq 2, 3$ or a purely transcendental extension $k = k_0(x, y)$ of transcendence degree 2 of a number field k_0, and let V be a divisorial set of discrete valuations of k. Then the set $\mathcal{I}$ of k-isomorphism classes of k-forms of type F_4 that split over a quadratic extension of k and have good reduction at all $v \in V$ is finite.*

We note that similar results for groups of type G_2 were obtained in [16] over 2-dimensional global field and in [67] over purely transcendental extensions of transcendence degree 2 of number fields.

Our final result applies to *all* forms of type F_4 and contributes to one of the main remaining problems in the theory of Jordan algebras. We refer the reader to subsection A2.1 of Appendix 2 for the definition of the map ϕ that describes forms of type F_4 in terms of the cohomological invariants f_3, f_5 and g_3. J.-P. Serre has raised the problem of whether ϕ is injective. We will show that, assuming the Finiteness Conjecture, we can at least confirm that ϕ is proper.

Theorem 1.13. *Let k be a finitely generated field of characteristic $\neq 2, 3$. Assume that the Finiteness Conjecture holds for k-groups of type F_4 with respect to any divisorial set V of discrete valuations of k. Then the map ϕ is proper, i.e. the preimage of a finite set is finite.*

Notations and conventions. We use standard notations associated with the Galois cohomology of algebraic groups (cf. [77]). In particular, given an algebraic group G defined over a field k and a Galois extension ℓ/k, we denote by $H^1(\ell/k, G)$ the set $H^1(\mathrm{Gal}(\ell/k), G(\ell))$ of noncommutative continuous Galois cohomology, and we write $H^1(k, G)$ for $H^1(k^{\mathrm{sep}}/k, G(k^{\mathrm{sep}}))$, where k^{sep} is a separable closure of k. Similar conventions are used for the set $Z^1(\ell/k, G)$ of noncommutative continuous 1-cocycles. We will slightly abuse notation and use lowercase Greek letters to denote both cocycles and cohomology classes whenever this does not lead to confusion. However, when we need to distinguish between the two, we will write ξ for a cocycle and $[\xi]$ for the corresponding cohomology class. We extend these notations also to étale (Čech) cocycles and the cohomology classes they define.

For an algebraic torus T, we let $X(T)$ and $X_*(T)$ denote the corresponding groups of characters and cocharacters, respectively. Furthermore, we denote by $\mathbb{G}_m$ the one-dimensional split torus.

Next, given a field k equipped with a discrete valuation v, we denote by k_v and $k^{(v)}$ the corresponding completion and residue field, respectively. Furthermore, we set $\mathcal{O}_v \subset k_v$ and $\mathcal{O}_{k,v} \subset k$ to be the associated valuation rings.

Finally, we recall some definitions and notations pertaining to commutative Galois cohomology and unramified cohomology, which will be needed mainly in §11 and in Appendix 2. For a $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-module M, we write $H^i(k, M)$ for the Galois cohomology group $H^i(\mathrm{Gal}(k^{\mathrm{sep}}/k), M)$. Now, if char $k^{(v)}$ is prime to n, then there exists a residue map

$$\rho_v^i \colon H^i(k, \mu_n^{\otimes j}) \to H^{i-1}(k^{(v)}, \mu_n^{\otimes (j-1)}).$$

We say that a class $x \in H^i(k, \mu_n^{\otimes j})$ is *unramified at* v if $x \in \ker \rho_v^i$ and that it is *ramified* otherwise. Furthermore, if V is a set of discrete valuations of k such that char $k^{(v)}$ is prime to n for all $v \in V$, then one defines the corresponding *unramified cohomology of degree* i to be

$$H^i(k, \mu_n^{\otimes j})_V = \bigcap_{v \in V} \ker \rho_v^i.$$

We refer the reader to [25, Ch. III and IV] for further details on these constructions.

2. Groups with good reduction

2.1. Good reduction: definition and examples

Even though the definition of good reduction for a reductive algebraic group at a discrete valuation of the base field has already been mentioned in §1, we begin by repeating it here for the convenience of further references.

Definition 2.1. Let G be a reductive algebraic group over a field k, and let v be a discrete valuation of k. We say that G has *good reduction* at v if there exists a reductive group scheme[2] $\mathcal{G}$ over the valuation ring $\mathcal{O}_v \subset k_v$ with generic fiber

$$\mathcal{G} \times_{\mathcal{O}_v} k_v \simeq G \times_k k_v.$$

[2] Let A be a commutative ring and $S = \mathrm{Spec}\, A$. A reductive A-group scheme is a smooth affine group scheme $\mathcal{G} \to S$ such that the geometric fibers $\mathcal{G}_{\bar{s}}$ are connected reductive groups for all geometric points $\bar{s}$ of S, cf. [22, Exp. XIX, Definition 2.7] or [21, Definition 3.1.1].

Then the $k^{(v)}$-group scheme $\mathcal{G} \times_{\mathcal{O}_v} k^{(v)}$ is called the *reduction* of G at v and will be denoted $\underline{G}^{(v)}$.

(We will see in subsection 2.2 below that the reduction $\underline{G}^{(v)}$ is well-defined.) We now consider a couple of examples of good reduction that are relevant for the present paper.

Example 2.2. Let $G = \mathrm{SL}_{1,A}$, where A is a central simple algebra of degree n over k, and let v be a discrete valuation of k. Assume that A is unramified at v, which means that there exists an Azumaya $\mathcal{O}_v$-algebra $\mathcal{A}$ such that $A \otimes_k k_v \simeq \mathcal{A} \otimes_{\mathcal{O}_v} k_v$ as k_v-algebras (cf. [69] and references therein). Let $\mathcal{G} = \mathrm{SL}_{1,\mathcal{A}}$ be the semisimple group scheme over $\mathcal{O}_v$ associated with $\mathcal{A}$ (cf. [9, 3.5.0.91]). Then

$$G \times_k k_v \simeq \mathcal{G} \times_{\mathcal{O}_v} k_v \tag{1}$$

as k_v-groups, hence G has good reduction at v.

Conversely, suppose $G = \mathrm{SL}_{1,A}$ has good reduction at v, and let $\mathcal{G}$ be the corresponding reductive scheme over $\mathcal{O}_v$. It is known that any inner form of the $\mathcal{O}_v$-group scheme SL_n is of the form $\mathrm{SL}_{1,\mathcal{A}}$ for some Azumaya $\mathcal{O}_v$-algebra $\mathcal{A}$ of degree n (cf. [9, 3.5.0.92]). So, if we write $\mathcal{G}$ as $\mathrm{SL}_{1,\mathcal{A}}$, the isomorphism (1) implies that

$$\text{either} \quad A \otimes_k k_v \simeq \mathcal{A} \otimes_{\mathcal{O}_v} k_v \quad \text{or} \quad A \otimes_k k_v \simeq \mathcal{A}^{\mathrm{op}} \otimes_{\mathcal{O}_v} k_v.$$

In either case, $A \otimes_k k_v$ comes from an Azumaya $\mathcal{O}_v$-algebra, and therefore A is unramified. Thus, $G = \mathrm{SL}_{1,A}$ has good reduction at v if and only if A is unramified at v.

Example 2.3. Let $G = \mathrm{Spin}_n(q)$, where q is a nondegenerate quadratic form of dimension $n \geq 2$ over a field k of characteristic $\neq 2$, and let v be a discrete valuation of k with residue characteristic $\mathrm{char}\, k^{(v)} \neq 2$. We will show that G has good reduction at v if and only if q is equivalent over k_v to a quadratic form of the shape

$$\lambda(u_1 x_1^2 + \cdots + u_n x_n^2), \quad \text{with} \quad \lambda \in k_v^{\times} \quad \text{and} \quad u_1, \ldots, u_n \in \mathcal{O}_v^{\times}. \tag{2}$$

First, let us assume that q is k_v-equivalent to such a form and set $q_0 = u_1 x_1^2 + \cdots + u_n x_n^2$. Then $G \times_k k_v = \mathrm{Spin}_n(q) = \mathrm{Spin}_n(q_0)$. On the other hand, since q_0 is a *regular* quadratic form on $\mathcal{O}_v^n$, there is a semisimple group scheme $\mathcal{G} = \mathcal{SPIN}_n(q_0)$ over $\mathcal{O}_v$ with generic fiber $G \times_k k_v$ (cf. [9, 4.5.2.6, 6.2.0.28, 8.2.0.59]). This means that G has good reduction at v.

Conversely, suppose $G = \mathrm{Spin}_n(q)$ has good reduction at v. When $n = 2$, the group G is a 1-dimensional torus whose splitting field is unramified at v, implying that q is equivalent to a form as in (2). Now suppose $n > 2$. Let q_0 be an n-dimensional split quadratic form, and let $\mathcal{G}_0 = \mathcal{SPIN}_n(q_0)$. Assume that $G = \mathrm{Spin}_n(q)$ has good reduction at v, i.e. there exists a reductive group $\mathcal{O}_v$-scheme $\mathcal{G}$ with generic fiber $G \times_k k_v$. Then $\mathcal{G}$ is obtained from $\mathcal{G}_0$ by twisting using an étale 1-cocycle $\xi \in Z^1(\mathcal{O}_v, \mathrm{Aut}(\mathcal{G}_0))$. If n

is odd, then $\mathrm{Aut}(\mathcal{G}_0) = \mathcal{SO}_n(q_0)$. Then ξ can be used to twist the quadratic form q_0 and obtain thereby a regular quadratic form q' over $\mathcal{O}_v$, in which case $\mathcal{G} = \mathcal{SPIN}_n(q')$. Passing to the generic fiber, we obtain that $\mathrm{Spin}_n(q) \simeq \mathrm{Spin}_n(q')$ over k_v and therefore q is k_v-equivalent to a scalar multiple of q'. On the other hand, since char $k^{(v)} \neq 2$, the form q', being regular, can be diagonalized over $\mathcal{O}_v$ as $u_1x_1^2 + \cdots + u_nx_n^2$ with $u_i \in \mathcal{O}_v^{\times}$, proving our claim.

The same argument works when n is even provided we can show that in this case, the cohomology class $[\xi]$ lies in the image of the map $\lambda\colon H^1(\mathcal{O}_v, \mathcal{O}_n(q_0)) \to H^1(\mathcal{O}_v, \mathrm{Aut}(\mathcal{G}_0))$ coming from the canonical morphism $\nu\colon \mathcal{O}_n(q_0) \to \mathrm{Aut}(\mathcal{G}_0)$. First, we observe that when $n = 8$, the group scheme $\mathcal{G}$ cannot be a triality form as otherwise the generic fiber G would also be a triality form, which is not the case. This means that in all cases, $[\xi]$ is represented by a cocycle having values in $B = \mathrm{Im}\ \nu$ (we note that B is represented by $\mathcal{PSO}_n(q_0) \rtimes \mathbb{Z}/2\mathbb{Z}$). The exact sequence

$$1 \to \mu_2 \longrightarrow \mathcal{O}_n(q_0) \xrightarrow{\nu} B \to 1$$

gives rise to the exact sequence

$$H^1(\mathcal{O}_v, \mathcal{O}_n(q_0)) \xrightarrow{\lambda} H^1(\mathcal{O}_v, B) \xrightarrow{\theta} H^2(\mathcal{O}_v, \mu_2) = {}_2\mathrm{Br}(\mathcal{O}_v). \tag{3}$$

We note that $\theta([\xi])$ is precisely the class of the Azumaya algebra involved in the description of $\mathcal{G}$. Since the generic fiber of $\mathcal{G}$ is the spinor group of a quadratic form, the image of $\theta([\xi])$ under the map $\mathrm{Br}(\mathcal{O}_v) \to \mathrm{Br}(k_v)$ is trivial, and then $\theta([\xi])$ is itself trivial since the latter map is well-known to be injective (cf. [47, Ch. IV, Corollary 2.6]). The exact sequence (3) then yields that $[\xi]$ lies in the image of λ, as required. □

2.2. The Grothendieck-Serre conjecture and its consequences

The Grothendieck-Serre conjecture predicts that for a reductive group scheme $\mathcal{G}$ over a regular local ring A with fraction field k, the map of nonabelian étale cohomology sets

$$H^1(A, \mathcal{G}) \to H^1(k, G) \quad (\text{where} \;\; G = \mathcal{G} \times_A k)$$

has trivial kernel. Very significant progress on the conjecture was achieved in [23], where it was proved under the assumption that R contains an infinite field; the case where A contains a finite field was treated in [52]. In the present paper, however, we will only need the case where A is a discrete valuation ring, which goes back to work of Y. Nisnevich [49], [50] (in fact, we will only need the case of a *complete* discrete valuation ring).

Theorem 2.4. *Let $\mathcal{G}$ be a reductive group scheme over a discrete valuation ring A. Then the map of nonabelian étale cohomology sets*

$$H^1(A, \mathcal{G}) \to H^1(k, G) \quad (\textit{where} \;\; G = \mathcal{G} \times_A k)$$

is injective.

This is [49, Ch. 2, Theorem 7.1] and [50, Theorem 4.2] in the case where the residue field of A is *perfect*. The general case is treated in [30, Theorem 1]. This result has the following important consequence.

Proposition 2.5. [30, §6, Corollary 3] *Let A be a discrete valuation ring, and k be its field of fractions. Then any reductive k-group has at most one reductive model over A.*

For semisimple groups and perfect residue fields, this statement appears as Theorem 5.1 in [50]; the general case is treated in [30, §6], where the argument combines Theorem 2.4 with a general statement concerning the uniqueness of reductive models over regular semilocal rings — see [30, §6, Proposition 14] for the details. In the context of Definition 2.1, Proposition 2.5 yields the uniqueness up to isomorphism of the $\mathcal{O}_v$-scheme of $\mathcal{G}$, implying, in particular, that the reduction $\underline{G}^{(v)}$ is well-defined.

2.3. A different approach to good reduction

The above definition of good reduction is most convenient for our purposes, in particular, for investigating connections with local-global principles. We note, however, that it is more traditional to define good reduction without passing to completions, i.e. by requiring the existence of a reductive group scheme $\mathcal{G}$ over the valuation ring $\mathcal{O}_{k,v} \subset k$ with generic fiber $\mathcal{G} \times_{\mathcal{O}_{k,v}} k$ isomorphic to G. Of course, our definition is less restrictive, so, a priori, we are considering a more general situation. For the sake of completeness, however, we will now briefly explain that for tori and absolutely almost simple groups, the two definitions are equivalent.

If a k-torus T has good reduction at v in the sense of Definition 2.1, then v is unramified in the minimal splitting field k_T (cf. [48]). Let $\widetilde{\mathcal{O}}$ denote the integral closure of $\mathcal{O}_{k,v}$ in k_T. Then $\widetilde{\mathcal{O}}/\mathcal{O}_{k,v}$ is a Galois extension of rings. Let $d = \dim T$, and let $\xi \in Z^1(k_T/k, \mathrm{GL}_d(\mathbb{Z}))$ be a cocycle such that the corresponding twist ${}_{\xi}(\mathbb{G}_m^d)$ of the d-dimensional k-split torus is k-isomorphic to T (here we identify the automorphism group $\mathrm{Aut}(\mathbb{G}_m^d)$ with $\mathrm{GL}_d(\mathbb{Z})$ through the action on the character group $X(\mathbb{G}_m^d) = \mathbb{Z}^d$). Then the Hopf k-algebra $k[T]$ is obtained by Galois descent from $k_T[T] = k_T[\mathbb{Z}^d]$ for the action of $\mathrm{Gal}(k_T/k)$ that coincides with the standard action on k_T and is given by ξ on $\mathbb{Z}^d$. Since $\widetilde{\mathcal{O}}/\mathcal{O}_{k,v}$ is a Galois extension of rings, we can likewise carry out Galois descent on $\widetilde{\mathcal{O}}[\mathbb{Z}^d]$ for the same action. This generates a Hopf $\mathcal{O}_{k,v}$-algebra that yields a torus $\mathcal{T}$ over $\mathcal{O}_{k,v}$ with generic fiber T, verifying thereby the traditional definition.

Next, let G be an absolutely almost simple simply connected algebraic k-group that has good reduction at v in the sense of Definition 2.1, and let ℓ be the minimal Galois extension of k over which G becomes an inner form of the split group G_0. The fact that G has good reduction at v implies that the extension ℓ/k is unramified at v. Let $\widetilde{\mathcal{O}}$ denote the integral closure of $\mathcal{O}_{k,v}$ in ℓ; then $\widetilde{\mathcal{O}}/\mathcal{O}_{k,v}$ is a Galois extension of rings.

Let G_1 be a quasi-split inner k-form of G. Then G_1 is isomorphic to the twist ${}_\xi G_0$ for some $\xi \in Z^1(\ell/k, \Sigma)$, where Σ is the group of symmetries of the Dynkin diagram of G_0, naturally considered as a subgroup of the automorphism group $\mathrm{Aut}(G_0)$. Recall that G_0 can be identified with the base change $\mathbf{G} \times_{\mathbb{Z}} k$ of the Chevalley group scheme $\mathbf{G}$ over $\mathbb{Z}$, and that the action of Σ comes from its action on $\mathbf{G}$. The Hopf algebra $k[G_1]$ is then obtained from $\ell[G_1] = \ell[G_0]$ by Galois descent for the action of $\mathrm{Gal}(\ell/k)$ on $\ell[G_0] = \ell \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbf{G}]$, which coincides with the standard action on ℓ and the action via the homomorphism $\xi\colon \mathrm{Gal}(\ell/k) \to \Sigma$ on $\mathbb{Z}[\mathbf{G}]$. This action leaves $\widetilde{\mathcal{O}}[G_0] := \widetilde{\mathcal{O}} \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbf{G}]$ invariant, and since $\widetilde{\mathcal{O}}/\mathcal{O}_{k,v}$ is a Galois extension of rings, we can carry out Galois descent in this situation. This yields a Hopf $\mathcal{O}_{k,v}$-algebra that corresponds to a reductive group scheme $\mathcal{G}_1$ over $\mathcal{O}_{k,v}$ with generic fiber G_1. This verifies that G_1 has good reduction in the traditional sense. To prove this fact for G, we need a result of Harder, whose proof ultimately depends on weak approximation.

To give the statement, we first need to introduce some notations that are different from the ones used elsewhere in this paper. So, let A be a Dedekind domain with fraction field k. For each maximal ideal $\mathfrak{p} \subset A$, denote by $\widehat{k}_{\mathfrak{p}}$ the corresponding completion of k with valuation ring $\widehat{A}_{\mathfrak{p}} \subset \widehat{k}_{\mathfrak{p}}$. Given a flat group scheme $\mathbb{G}$ over A, we let G denote its generic fiber $\mathbb{G} \times_A k$, and set $H^1_A(k, G)$ to be the image of the natural map $H^1_{fppf}(A, \mathbb{G}) \to H^1_{fppf}(k, G)$ of flat cohomology. Furthermore, for $\xi \in H^1_{fppf}(k, G)$, we denote by $\xi_{\mathfrak{p}}$ its image in $H^1_{fppf}(\widehat{k}_{\mathfrak{p}}, G)$ under the restriction map.

Proposition 2.6. ([31, Lemma 4.1.3]) *Let $\mathbb{G}$ be a flat group scheme of finite type over A whose generic fiber G is a reductive k-group. Then*

$$H^1_A(k, G) = \{\xi \in H^1(k, G) \mid \xi_{\mathfrak{p}} \in \mathrm{Im}(H^1_{fppf}(\widehat{A}_{\mathfrak{p}}, \mathbb{G}) \to H^1_{fppf}(\widehat{k}_{\mathfrak{p}}, G)) \text{ for all maximal ideals } \mathfrak{p} \subset A\}.$$

Next, let k be a field equipped with a discrete valuation v, and set A to be the corresponding valuation ring $\mathcal{O}_{k,v}$. Then it follows from the proposition that for any reductive group scheme $\mathcal{G}$ over $\mathcal{O}_{k,v}$ with generic fiber G, we have the following (note that since $\mathcal{G}$ is, by definition, smooth, its flat cohomology coincides with étale cohomology).

Corollary 2.7. *The natural diagram of pointed sets*

$$\begin{array}{ccc} H^1_{\text{ét}}(\mathcal{O}_{k,v}, \mathcal{G}) & \longrightarrow & H^1_{\text{ét}}(\mathcal{O}_v, \mathcal{G}) \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi_v} \\ H^1(k, G) & \xrightarrow{r_v} & H^1(k_v, G) \end{array}$$

is cartesian.

Now let $\overline{G}_1$ be the adjoint group for the quasi-split group G_1 considered above, and let $\overline{\mathcal{G}}_1$ be a reductive $\mathcal{O}_{k,v}$-scheme with generic fiber $\overline{G}_1$. By construction, G is an inner twist of G_1, so we can choose $\xi \in Z^1(k, \overline{G}_1)$ so that $G = {}_\xi G_1$. We then consider the cartesian diagram from Corollary 2.7 with $\mathcal{G}$ and G replaced by $\overline{\mathcal{G}}_1$ and $\overline{G}_1$, respectively, and keeping the notations for the maps. The fact that G has good reduction at v means that we may assume that $r_v([\xi]) \in \operatorname{im} \varphi_v$. We then conclude from the diagram that $[\xi] = \varphi([\zeta])$ for some $\zeta \in Z^1(\mathcal{O}_{k,v}, \overline{\mathcal{G}}_1)$. Then $\mathcal{G} := {}_\zeta\mathcal{G}_1$ is a reductive group scheme over $\mathcal{O}_{k,v}$ with generic fiber G, as required.

3. Generic tori, generic elements, and applications to weak commensurability

3.1. Generic tori

For an algebraic torus T defined over a field k, we denote by k_T the minimal splitting field of T. It is well-known that the Galois group $\mathcal{G}_T = \mathrm{Gal}(k_T/k)$ acts faithfully on the group of characters $X(T)$. Now, let G be a semisimple k-group, T a maximal k-torus of G, and $\Phi(G,T)$ the corresponding root system. Then the action of $\mathcal{G}_T$ on $X(T)$ permutes the roots, yielding a group homomorphism

$$\theta_T \colon \mathcal{G}_T \longrightarrow \mathrm{Aut}(\Phi(G,T)) \quad (\subset \mathrm{GL}(X(T) \otimes_{\mathbb{Z}} \mathbb{Q})).$$

Since $\Phi(G,T)$ generates a finite index subgroup of $X(T)$, this homomorphism is *injective*. We say that T is *generic* over k, or k-*generic*, if the image of θ_T contains the Weyl group $W(G,T)$. It is known that if k is an infinite finitely generated field, then every semisimple k-group G contains k-generic maximal k-tori. This can be established by first showing that G possesses a generic torus over a purely transcendental extension of k and then specializing the parameters in order to obtain a required generic torus defined over k — see Voskresenskiĭ [83, 4.2] and also [56]; we note that the specialization part is based on the fact that an infinite finitely generated field is Hilbertian — see [24, Theorem 13.4.2].

A different approach to the construction of generic tori was first developed in [57] over fields of characteristic zero and then extended to fields of arbitrary characteristic in [61]. Among other things, this approach demonstrates that to ensure the genericity of a maximal k-torus, it is enough to prescribe its local behavior at finitely many specially chosen valuations. More precisely, assuming that char $k = 0$, one can choose r distinct primes $p_1, \ldots, p_r$, where r is the number of conjugacy classes in the Weyl group of G, such that there exist embeddings $\iota_i \colon k \hookrightarrow \mathbb{Q}_{p_i}$ for $i = 1, \ldots, r$. Furthermore, letting v_i denote the pullback of the p_i-adic valuation under ι_i, for each $i = 1, \ldots, r$, one can specify a maximal k_{v_i}-torus T_i of G so that any maximal k-torus T of G that is conjugate to T_i by an element of $G(k_{v_i})$ for all $i = 1, \ldots, r$ is necessarily k-generic. For a very similar statement in the case of positive characteristic, we refer the reader to [61]. This construction of generic tori yields the following stronger form of the existence theorem.

Theorem 3.1. (cf. [59, Theorem 3.1], [61]) *Let G be a semisimple algebraic group over an infinite finitely generated field k. For any finitely generated extension ℓ of k, the group G contains a maximal k-torus that is generic over ℓ.*

In connection with this local-global construction, we would like to recall the following approximation statement for maximal tori and derive one consequence needed for our purposes.

Lemma 3.2. *Let G be a reductive algebraic group over a field k, and let V be a finite set of discrete valuations of k. Suppose that for each $v \in V$, we are given a maximal k_v-torus T_v of $G \times_k k_v$. Then there exists a maximal k-torus T of G that is conjugate to T_v by an element of $G(k_v)$ for all $v \in V$.*

This is Corollary 3 in [54, §7.2]; the proof uses the fact that the variety of maximal tori is rational over k.

Corollary 3.3. *Let G be a semisimple algebraic group over a field k, and let v be a discrete valuation of k. If $G' \in \mathbf{gen}_k(G)$, then $G' \times_k k_v \in \mathbf{gen}_{k_v}(G \times_k k_v)$.*

Indeed, the lemma implies that every maximal k_v-torus of $G \times_k k_v$ (resp., of $G' \times_k k_v$) is k_v-isomorphic to a maximal k-torus of G (resp., of G'), and our claim immediately follows from the definitions.

Proposition 3.4. *Let G_1 and G_2 be absolutely almost simple algebraic groups over a finitely generated field k, and let ℓ_i be the minimal Galois extension of k over which G_i becomes an inner form of the split group. Assume that G_1 and G_2 have the same isogeny classes of maximal k-tori. Then*

(i) *either G_1 and G_2 are of the same Killing-Cartan type, or one of them is of type B_ℓ and the other is of type C_ℓ for some $\ell \geqslant 3$;*
(ii) *$\ell_1 = \ell_2$, and consequently, if the groups G_1 and G_2 are of the same Killing-Cartan type and are both either simply connected or adjoint, then they are inner twists of one another.*

Proof. These statements were proved in the context of the analysis of weakly commensurable Zariski-dense subgroups in [58] and [60, §5] — see also Theorem 3.8 below. So, we will just briefly outline the argument in our present context of absolutely almost simple algebraic groups with the same tori. Set $\ell = \ell_1\ell_2$. Using Theorem 3.1, we can find a maximal k-torus T_1 of G_1 which is generic over ℓ. By our assumption, there exist a maximal k-torus T_2 of G_2 and a k-defined isogeny $\nu \colon T_1 \to T_2$. We then have the following commutative diagram

$$\begin{array}{ccc} & & \mathrm{GL}(X(T_1) \otimes_{\mathbb{Z}} \mathbb{Q}) \\ & \overset{\theta_{T_1}}{\nearrow} & \uparrow \tilde{\nu} \\ \mathrm{Gal}(k^{\mathrm{sep}}/k) & & \\ & \underset{\theta_{T_2}}{\searrow} & \\ & & \mathrm{GL}(X(T_2) \otimes_{\mathbb{Z}} \mathbb{Q}), \end{array}$$

where k^{sep} is a fixed separable closure of k, and $\tilde{\nu}$ is the isomorphism induced by ν. We note that for any field extension F of k contained in k^{sep}, the map $\tilde{\nu}$ gives an isomorphism between the images of $\mathrm{Gal}(k^{\mathrm{sep}}/F)$ under θ_{T_1} and θ_{T_2}, hence

$$|\theta_{T_1}(\mathrm{Gal}(k^{\mathrm{sep}}/F))| = |\theta_{T_2}(\mathrm{Gal}(k^{\mathrm{sep}}/F))|. \tag{4}$$

Since both G_1 and G_2 are inner forms over ℓ, by [58, Lemma 4.1] we have

$$\theta_{T_i}(\mathrm{Gal}(k^{\mathrm{sep}}/\ell)) \subset W(G_i, T_i) \quad \text{for} \quad i = 1, 2.$$

Combining this with the fact that T_1 was chosen to be generic over ℓ, we see that actually

$$\theta_{T_1}(\mathrm{Gal}(k^{\mathrm{sep}}/\ell)) = W(G_1, T_1). \tag{5}$$

Thus (4) with $F = \ell$ yields the inequality $|W(G_1, T_1)| \leqslant |W(G_2, T_2)|$. Starting now with a maximal k-torus T_2' of G_2 that is generic over ℓ and considering a maximal k-torus T_1' of G_1 that is k-isogenous to T_2', we similarly obtain the inequality $|W(G_2, T_2')| \leqslant |W(G_1, T_1')|$. Since $|W(G_i, T_i)| = |W(G_i, T_i')|$ for $i = 1, 2$, we conclude that

$$|W(G_1, T_1)| = |W(G_2, T_2)|. \tag{6}$$

This already yields assertion (i) as the type of a reduced irreducible root system is uniquely determined by the order of the corresponding Weyl group except for the ambiguity between types B_ℓ and C_ℓ for $\ell \geqslant 3$. In addition, (6) also implies that

$$\theta_{T_2}(\mathrm{Gal}(k^{\mathrm{sep}}/\ell)) = W(G_2, T_2). \tag{7}$$

Now assume that $\ell_2 \not\subset \ell_1$, i.e. $\ell_1 \subsetneqq \ell$. Since G_1 is an inner form already over ℓ_1, we conclude from (5) that

$$\theta_{T_1}(\mathrm{Gal}(k^{\mathrm{sep}}/\ell_1)) = W(G_1, T_1).$$

On the other hand, it follows from (7) that $\theta_{T_2}(\mathrm{Gal}(k^{\mathrm{sep}}/\ell_1))$ contains $W(G_2, T_2)$ but is *strictly bigger* as G_2 is *not* an inner form over ℓ_1. In view of (6), this contradicts (4) with $F = \ell_1$. Thus, $\ell_2 \subset \ell_1$, and by symmetry we conclude that $\ell_1 = \ell_2$, as required. It is well-known that for absolutely almost simple simply connected or adjoint groups, this fact implies that the groups are inner twists of one another. □

Corollary 3.5. *Let G be an absolutely almost simple algebraic group over a finitely generated field k. Then $\mathbf{gen}_k^+(G) = \mathbf{gen}_k(G)$.*

Proof. Let $G' \in \mathbf{gen}_k^+(G)$. Then according to Proposition 3.4, the group G' is an inner twist of G, i.e. $G' \in \mathbf{gen}_k(G)$. □

3.2. Generic elements

Let G be a (connected) absolutely almost simple algebraic group over a field k. A regular semisimple element $\gamma \in G(k)$ of infinite order is called *k-generic* if the k-torus $T = C_G(\gamma)^\circ$ (connected component of the centralizer) is k-generic. The following result yields the existence of generic elements in Zariski-dense subsemigroups under one natural assumption.

Theorem 3.6. ([61, Theorem 2]) *Let G be an absolutely almost simple algebraic group over a finitely generated field k, and let $\Gamma \subset G(k)$ be a Zariski-dense subsemigroup that contains an element of infinite order.*[3] *Then Γ contains a regular semisimple element $\gamma \in \Gamma$ of infinite order that is k-generic.*

In characteristic zero, the existence of generic elements of infinite order in an arbitrary Zariski-dense subgroup was established already in [57] for any semisimple G. The case of positive characteristic (particularly of characteristics 2 and 3) requires a more delicate argument, which was given in [61] assuming G to be absolutely almost simple. We will also need the following refined version of Theorem 3.6 over fields of characteristic zero.

Theorem 3.7. (cf. [59, Theorem 3.4]) *Let G be a connected absolutely almost simple algebraic group over a finitely generated field k of characteristic zero, v be a discrete valuation of k such that the completion k_v is locally compact, and $T(v)$ be a maximal k_v-torus of G. Given a finitely generated Zariski-dense subgroup $\Gamma \subset G(k)$ whose closure in $G(k_v)$ for the v-adic topology is open, there exists a regular semisimple element $\gamma \in \Gamma$ of infinite order such that the corresponding torus $T = C_G(\gamma)^\circ$ is generic over k and is conjugate to $T(v)$ by an element of $G(k_v)$.*

3.3. Weak commensurability

(Cf. [58]) Let $\gamma_1 \in \mathrm{GL}_{n_1}(F)$ and $\gamma_2 \in \mathrm{GL}_{n_2}(F)$ be two semisimple matrices over an infinite field F with respective eigenvalues

[3] We recall that any Zariski-dense subgroup $\Gamma \subset G(k)$, where G is a semisimple group over a field k of characteristic zero, automatically contains an element of infinite order.

$$\lambda_1, \ldots, \lambda_{n_1} \quad \text{and} \quad \mu_1, \ldots, \mu_{n_2}$$

(in an algebraic closure $\overline{F}$). We say that γ_1 and γ_2 are *weakly commensurable* if there exist integers $a_1, \ldots, a_{n_1}$ and $b_1, \ldots, b_{n_2}$ such that

$$\lambda_1^{a_1} \cdots \lambda_{n_1}^{a_{n_1}} = \mu_1^{b_1} \cdots \mu_{n_2}^{b_{n_2}} \neq 1.$$

Next, let $G_1 \subset \mathrm{GL}_{n_1}$ and $G_2 \subset \mathrm{GL}_{n_2}$ be two reductive algebraic F-groups, and let $\Gamma_1 \subset G_1(F)$ and $\Gamma_2 \subset G_2(F)$ be Zariski-dense subgroups that contain elements of infinite order. We say that Γ_1 and Γ_2 are *weakly commensurable* if every semisimple element $\gamma_1 \in \Gamma_1$ of infinite order is weakly commensurable to some semisimple element $\gamma_2 \in \Gamma_2$ of infinite order, and vice versa. It is easy to see that this relation does not depend on the choice of the matrix realizations of G_1 and G_2.

The following theorem summarizes the basic results about weakly commensurable subgroups.

Theorem 3.8. *Let G_1 and G_2 be absolutely almost simple algebraic groups over a finitely generated field k, and let ℓ_i be the minimal Galois extension of k over which G_i becomes an inner form of the split group. Furthermore, let $\Gamma_1 \subset G_1(k)$ and $\Gamma_2 \subset G_2(k)$ be Zariski-dense subgroups containing elements of infinite order. Assume that Γ_1 and Γ_2 are weakly commensurable. Then*

(1) *the groups G_1 and G_2 have the same order of the Weyl groups, or equivalently, they are either of the same type or one of them is of type B_ℓ and the other of type C_ℓ for some $\ell \geq 3$;*
(2) *if* char $k = 0$, *then the trace fields of Γ_1 and Γ_2 coincide: $k_{\Gamma_1} = k_{\Gamma_2}$;*
(3) *$\ell_1 = \ell_2$.*

In characteristic zero, part (1) is Theorem 1 in [58]. Its proof in positive characteristic remains exactly the same due to the existence of generic elements in *all* characteristics (Theorem 3.6). The result in part (2) as stated is specific to characteristic zero; in fact, it is false in positive characteristic. Technically, part (3) was proved in [58, Theorem 6.3(2)] only when k is a number field, so we will quickly sketch the general argument, which is similar to the proof of Proposition 3.4. We recall that a k-torus T is called *k-irreducible* if it does not contain any proper k-defined subtori; the irreducibility of T is equivalent to the fact that the Galois group $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ acts irreducibly on either of the $\mathbb{Q}$-vector spaces $X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ or $X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$, where $X(T)$ and $X_*(T)$ are, respectively, the groups of characters and cocharacters of T, hence the terminology. We will need the following result.

Lemma 3.9. ([58, Lemma 3.6]) *Let T be a k-irreducible torus. For any $t \in T(k)$ of infinite order and any nonzero character $\chi \in X(T)$, the Galois conjugates of $\lambda = \chi(t)$ generate the splitting field k_T.*

Proof of Theorem 3.8(3). Set $\ell = \ell_1\ell_2$. It is enough to prove the inclusion $\ell_1 \subset \ell_2$ as the opposite inclusion is obtained by a symmetric argument. Assume the contrary, i.e. $\ell_1 \not\subset \ell_2$. Using Theorem 3.6, we can find a regular semisimple element $\gamma_1 \in \Gamma_1$ of infinite order which is generic over ℓ. By our assumption, γ_1 is weakly commensurable to some semisimple element $\gamma_2 \in \Gamma_2$ of infinite order. Let T_i be a maximal k-torus of G_i containing γ_i. Since T_1 is ℓ-generic, we have the inclusion $\theta_{T_1}(\mathrm{Gal}(k^{\mathrm{sep}}/\ell)) \supset W(G_1, T_1)$. On the other hand, the fact that G_1 is an inner form of a split group over ℓ implies the opposite inclusion (see [58, Lemma 4.1]). Thus,

$$\theta_{T_1}(\mathrm{Gal}(k^{\mathrm{sep}}/\ell)) = W(G_1, T_1),$$

and in particular, $[\ell_{T_1} : \ell] = |W(G_1, T_1)|$. The condition that γ_1 and γ_2 are weakly commensurable means that there exist characters $\chi_i \in X(T_i)$ for $i = 1, 2$ such that

$$\lambda := \chi_1(\gamma_1) = \chi_2(\gamma_2) \neq 1.$$

It follows from Lemma 3.9 that the Galois conjugates of λ generate the splitting field k_{T_1}, yielding, in particular, the inclusion $k_{T_1} \subset k_{T_2}$, hence the inequality

$$[\ell_{T_2} : \ell] \geq [\ell_{T_1} : \ell] = |W(G_1, T_1)|. \tag{8}$$

At the same time, again by [58, Lemma 4.1], we have the inclusion $\theta_{T_2}(\mathrm{Gal}(k^{\mathrm{sep}}/\ell)) \subset W(G_2, T_2)$, so

$$[\ell_{T_2} : \ell] \leq |W(G_2, T_2)|. \tag{9}$$

However, by part (1) we have $|W(G_1, T_1)| = |W(G_2, T_2)|$, so comparing (8) and (9), we obtain that

$$\theta_{T_2}(\mathrm{Gal}(k^{\mathrm{sep}}/\ell)) = W(G_2, T_2).$$

By our assumption $\ell \neq \ell_2$, so the last equality implies that

$$|\theta_{T_2}(\mathrm{Gal}(k^{\mathrm{sep}}/\ell_2))| > |W(G_2, T_2)|.$$

This, however, contradicts the inclusion $\theta_{T_2}(\mathrm{Gal}(k^{\mathrm{sep}}/\ell_2)) \subset W(G_2, T_2)$, which again follows from [58, Lemma 4.1] as G_2 is an inner form over ℓ_2. □

We conclude this section with the following two statements.

Proposition 3.10. (cf. [58, Isogeny Theorem 4.2]) *Let G_1 and G_2 be two connected absolutely almost simple algebraic groups over an infinite field k, and for $i = 1, 2$, let ℓ_i be the minimal Galois extension of k over which G_i becomes an inner form of the split* *group. Assume that G_1 and G_2 have the same order of the Weyl groups and that $\ell_1 = \ell_2$. Furthermore, let T_i be a maximal k-torus of G_i, and let $\gamma_i \in T_i(k)$ be an element of infinite order. If T_1 is k-generic and the elements γ_1 and γ_2 are weakly commensurable, then there exists a k-isogeny $\pi \colon T_1 \to T_2$.*

Corollary 3.11. *Let G_1 and G_2 be absolutely almost simple algebraic groups over an infinite finitely generated field k, and let $\Gamma_1 \subset G_1(k)$ and $\Gamma_2 \subset G_2(k)$ be Zariski-dense subgroups containing elements of infinite order. Assume that Γ_1 and Γ_2 are weakly commensurable. If a k-generic element $\gamma_1 \in \Gamma_1$ of infinite order is weakly commensurable to a semisimple element $\gamma_2 \in \Gamma_2$ and T_i is a maximal k-torus of G_i containing γ_i, then there exists a k-defined isogeny $\pi \colon T_1 \to T_2$. In particular, the minimal splitting fields of T_1 and T_2 coincide: $k_{T_1} = k_{T_2}$, and hence the fact that T_1 is k-generic implies that T_2 is also k-generic.*

Proof. According to Theorem 3.8, the fact that Γ_1 and Γ_2 are weakly commensurable implies that G_1 and G_2 have the same order of the Weyl group and that $\ell_1 = \ell_2$. Now, our assertion follows immediately from Proposition 3.10. □

4. One consequence of a result of Klyachko

We refer to [5, Ch. VI] for the terminology and notations pertaining to root systems. In particular, for a reduced irreducible root system Φ in a $\mathbb{Q}$-vector space V, we let $\Phi^\vee$ denote the dual root system, $Q(\Phi)$ the sublattice of V generated by the roots (*root lattice*), and $P(\Phi)$ the dual lattice of $Q(\Phi^\vee)$ (*weight lattice*); recall that $Q(\Phi) \subset P(\Phi)$. Furthermore, we denote by $W(\Phi)$ the Weyl group of Φ, viewed as a subgroup of the automorphism group $\mathrm{Aut}(\Phi)$. The following result plays an important role in this paper.

Theorem 4.1. *Let Φ be a reduced irreducible root system. For any subgroup $\Gamma \subset \mathrm{Aut}(\Phi)$ containing $W(\Phi)$, we have $H^1(\Gamma, P(\Phi)) = 0$ if Φ is not of the type A_1 or C_ℓ, and $\mathbb{Z}/2\mathbb{Z}$ otherwise.*

This theorem is a particular case of the computations of $H^1(\Gamma, M)$ for any Γ as in the theorem and any Γ-invariant lattice $Q(\Phi) \subset M \subset P(\Phi)$ carried out by A. Klyachko [37]. Unfortunately, the result in [37] is *false* as stated; however, the argument given therein *does* work in the situation described in the theorem. Since [37] is not readily available, we will reproduce the argument in Appendix 1, where we will also present the original statement and explain the mistake that invalidates the argument in the general case.

Corollary 4.2. *Let $\overline{G}$ be a simple adjoint algebraic group over a field K, and $\overline{T}$ be a maximal K-torus of $\overline{G}$ which is K-generic. If the type of $\overline{G}$ is different from A_1 and B_ℓ ($\ell \geq 2$), then for the group of cocharacters $X_*(\overline{T})$, we have $H^1(\mathrm{Gal}(K_{\overline{T}}/K), X_*(\overline{T})) = 0$.*

Proof. Let $\Phi = \Phi(\overline{G}, \overline{T})$ be the root system of $\overline{G}$. Since $\overline{G}$ is adjoint, the character group $X(\overline{T})$ coincides with $Q(\Phi)$. So, the dual group $X_*(\overline{T})$ of cocharacters can be identified with $P(\Phi^\vee)$. By our assumption, the type of Φ is different from A_1 and B_ℓ, so the type of the dual system $\Phi^\vee$ is different from A_1 and C_ℓ. Furthermore, the fact that $\overline{T}$ is generic means that the Galois group $\mathrm{Gal}(K_{\overline{T}}/K)$ in its action on $X_*(\overline{T})$ contains the Weyl group $W(\Phi)$ — cf. §3.1. Our assertion now follows directly from Theorem 4.1. □

Since the proof of Theorem 4.1 is deferred to Appendix 1, we will now give an example that, on the one hand, shows a situation where the corollary can be checked by a direct computation, and on the other hand, demonstrates that the assertion can be false if the ambient group is not adjoint.

Example 4.3. Let L/K be a separable field extension of degree $n > 2$ such that the Galois group $\mathrm{Gal}(M/K)$ of the minimal Galois extension M of K that contains L is isomorphic to S_n. Set $\mathcal{G} = \mathrm{Gal}(M/K)$ and $\mathcal{H} = \mathrm{Gal}(M/L)$. Corresponding to the extension L/K, we have a maximal K-generic K-torus $\overline{T} = \mathrm{R}_{L/K}(\mathbb{G}_m)/\mathbb{G}_m$ of the adjoint group $G = \mathrm{PGL}_n$ of type A_{n-1}. The group of cocharacters $X_*(\overline{T})$ fits into the following exact sequence of $\mathcal{G}$-modules

$$0 \to \mathbb{Z} \longrightarrow \mathbb{Z}[\mathcal{G}/\mathcal{H}] \longrightarrow X_*(\overline{T}) \to 0,$$

leading to the exact sequence in cohomology

$$0 = H^1(\mathcal{G}, \mathbb{Z}[\mathcal{G}/\mathcal{H}]) \longrightarrow H^1(\mathcal{G}, X_*(\overline{T})) \longrightarrow H^2(\mathcal{G}, \mathbb{Z}) \xrightarrow{\alpha} H^2(\mathcal{G}, \mathbb{Z}[\mathcal{G}/\mathcal{H}]) = H^2(\mathcal{H}, \mathbb{Z}).$$

In terms of the natural identifications

$$H^2(\mathcal{G}, \mathbb{Z}) \simeq \mathrm{Hom}(\mathcal{G}, \mathbb{Q}/\mathbb{Z}) \text{ and } H^2(\mathcal{H}, \mathbb{Z}) \simeq \mathrm{Hom}(\mathcal{H}, \mathbb{Q}/\mathbb{Z}),$$

the map α corresponds to the restriction map

$$\mathrm{Hom}(\mathcal{G}, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Hom}(\mathcal{H}, \mathbb{Q}/\mathbb{Z}).$$

It easily follows that α is injective for $n > 2$, and we obtain $H^1(\mathcal{G}, X_*(\overline{T})) = 0$, in agreement with Corollary 4.2.

On the other hand, the norm torus $\widetilde{T} = \mathrm{R}^{(1)}_{L/K}(\mathbb{G}_m)$ is a maximal K-generic K-torus in the simply connected group $\widetilde{G} = \mathrm{SL}_n$. The co-character group $X_*(\widetilde{T})$ can be determined from the following exact sequence of $\mathcal{G}$-modules

$$0 \to X_*(\widetilde{T}) \longrightarrow \mathbb{Z}[\mathcal{G}/\mathcal{H}] \xrightarrow{\delta} \mathbb{Z} \to 0,$$

where δ is the augmentation map. This induces the exact sequence

$$\mathbb{Z}[\mathcal{G}/\mathcal{H}]^{\mathcal{G}} \xrightarrow{\delta} \mathbb{Z} \longrightarrow H^1(\mathcal{G}, X_*(\widetilde{T})) \longrightarrow H^1(\mathcal{G}, \mathbb{Z}[\mathcal{G}/\mathcal{H}]) = H^1(\mathcal{H}, \mathbb{Z}) = 0.$$

It follows that $H^1(\mathcal{G}, X_*(\widetilde{T})) \simeq \mathbb{Z}/n\mathbb{Z}$; in particular, it is nontrivial (including the case $n = 2$).

We will now discuss a consequence of Corollary 4.2 for unramified cohomology that will be needed in subsequent sections. Let $\mathcal{K}$ be a field complete with respect to a discrete valuation v. For any algebraic extension $\mathcal{L}/\mathcal{K}$, we let $\mathcal{O}_{\mathcal{L}}$ denote the valuation ring of the unique extension of v to $\mathcal{L}$. We also denote by $\mathcal{K}^{\mathrm{ur}}$ the maximal unramified extension of $\mathcal{K}$. Suppose now that T is a $\mathcal{K}$-torus whose minimal splitting field $\mathcal{L} = \mathcal{K}_T$ is unramified over $\mathcal{K}$. It follows from Hilbert's Theorem 90 and the inflation-restriction sequence that

$$H^1(\mathcal{K}, T) = H^1(\mathcal{K}^{\mathrm{ur}}/\mathcal{K}, T) = H^1(\mathcal{L}/\mathcal{K}, T),$$

and one also shows that

$$H^1(\mathcal{K}^{\mathrm{ur}}/\mathcal{K}, T(\mathcal{O}_{\mathcal{K}^{\mathrm{ur}}})) = H^1(\mathcal{L}/\mathcal{K}, T(\mathcal{O}_{\mathcal{L}})).$$

The subgroup of unramified cocycles $H^1(\mathcal{L}/\mathcal{K}, T)_{\{v\}} \subset H^1(\mathcal{L}/\mathcal{K}, T)$ is defined as the image of the natural homomorphism $H^1(\mathcal{L}/\mathcal{K}, T(\mathcal{O}_{\mathcal{L}})) \to H^1(\mathcal{L}/\mathcal{K}, T)$.

Proposition 4.4. *Let $\overline{T}$ be a maximal $\mathcal{K}$-torus of an absolutely simple adjoint algebraic $\mathcal{K}$-group $\overline{G}$ of type different from A_1 and B_ℓ. If $\overline{T}$ is $\mathcal{K}$-generic with unramified minimal splitting field $\mathcal{L} = \mathcal{K}_T$, then $H^1(\mathcal{L}/\mathcal{K}, \overline{T}) = H^1(\mathcal{L}/\mathcal{K}, \overline{T})_{\{v\}}$.*

Proof. We will view cocharacters of $\overline{T}$ as 1-parameter subgroups $\mathbb{G}_m \to \overline{T}$. Then the map

$$X_*(\overline{T}) \otimes_{\mathbb{Z}} \mathcal{L}^\times \to \overline{T}(\mathcal{L}), \quad \chi \otimes a \mapsto \chi(a),$$

is an isomorphism of $\mathrm{Gal}(\mathcal{L}/\mathcal{K})$-modules. Furthermore, if $\pi \in \mathcal{K}$ is a uniformizer, then since $\mathcal{L}/\mathcal{K}$ is unramified, π remains a uniformizer in $\mathcal{L}$, and therefore we have a decomposition of $\mathrm{Gal}(\mathcal{L}/\mathcal{K})$-modules

$$\mathcal{L}^\times = \langle \pi \rangle \times \mathcal{U}, \quad \text{where} \quad \mathcal{U} = (\mathcal{O}_{\mathcal{L}})^\times.$$

It follows that

$$\overline{T}(\mathcal{L}) \simeq X_*(\overline{T}) \times \overline{T}(\mathcal{O}_{\mathcal{L}})$$

as $X_*(\overline{T}) \otimes \mathcal{U} \simeq \overline{T}(\mathcal{O}_{\mathcal{L}})$. In view of our assumptions on $\overline{T}$, we have $H^1(\mathcal{L}/\mathcal{K}, X_*(\overline{T})) = 0$ by Corollary 4.2, and the required fact follows. □

Example 4.5. Let ℓ/k be a finite separable extension of degree n such that the minimal Galois extension m of k containing ℓ has Galois group S_n over k. Set $\mathcal{K} = k((x))$ and $\mathcal{L} = \ell((x))$. Then the norm torus $T = \mathrm{R}^{(1)}_{\mathcal{L}/\mathcal{K}}(\mathbb{G}_m)$ is a maximal $\mathcal{K}$-torus in the simply connected group $G = \mathrm{SL}_n$. Furthermore, this torus is $\mathcal{K}$-generic and its splitting field is unramified over $\mathcal{K}$ (with respect to the standard valuation v on the field of Laurent power series). We have

$$H^1(\mathcal{K}, T) = \mathcal{K}^\times / N_{\mathcal{L}/\mathcal{K}}(\mathcal{L}^\times) = k^\times / N_{\ell/k}(\ell^\times) \times \langle x \rangle / \langle x^n \rangle.$$

At the same time, the unramified part $H^1(\mathcal{K}, T)_v$ is easily seen to be $k^\times / N_{\ell/k}(\ell^\times)$. Thus, in this case, $H^1(\mathcal{K}, T) \neq H^1(\mathcal{K}, T)_{\{v\}}$. So, the assertion of Proposition 4.4 may fail if the ambient group is not adjoint.

5. Maximal tori with unramified splitting fields

Let $\mathcal{K}$ be a field that is complete with respect to a discrete valuation v, with valuation ring $\mathcal{O}$ and residue field k. We also fix a uniformizer $\pi \in \mathcal{K}$. The goal of this section is to establish the following result, which may be known to some experts, but which does not seem to have been recorded in the literature.

Theorem 5.1. *Let G be a reductive algebraic $\mathcal{K}$-group. Assume that G has* good reduction *at v, i.e. there exists a reductive group scheme $\mathcal{G}$ over $\mathcal{O}$ with generic fiber G. Then given a maximal $\mathcal{K}$-torus S of G whose splitting field $\mathcal{K}_S$ is unramified over $\mathcal{K}$, there exists a maximal torus $\mathcal{S}'$ of $\mathcal{G}$ such that for its generic fiber S', there exists $h \in G(\mathcal{K}^{\mathrm{ur}})$ satisfying $S' = hSh^{-1}$ and the isomorphism $\varphi \colon S \to S'$, $x \mapsto hxh^{-1}$, is defined over $\mathcal{K}$.*

We begin by recalling the well-known parametrization of the conjugacy classes of maximal tori in terms of Galois cohomology. So, let G be a (connected) reductive algebraic group over an arbitrary field K. Fix a maximal K-torus T of G, and let $N = N_G(T)$ denote its normalizer in G. Furthermore, let $W = N/T$ denote the Weyl group, $\theta \colon N \to W$ the corresponding quotient map, and $\theta^1 \colon H^1(K, N) \to H^1(K, W)$ the induced map on Galois cohomology. Given any other maximal K-torus S, we choose $g \in G(K^{\mathrm{sep}})$ so that $S = gTg^{-1}$. Then for any $\sigma \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$, the element $\xi(\sigma) := g^{-1} \cdot \sigma(g)$ belongs to $N(K^{\mathrm{sep}})$, and the correspondence $\sigma \mapsto \xi(\sigma)$ is a 1-cocycle with values in $N(K^{\mathrm{sep}})$ whose cohomology class $[\xi] \in H^1(K, N)$ is independent of the choice of the conjugating element g. Furthermore, the correspondence

$$S \ \mapsto \ [\xi]$$

sets up a bijection between the $G(K)$-conjugacy classes of maximal K-tori of G and the elements of $\ker(H^1(K, N) \to H^1(K, G))$. (More generally, if T splits over an extension L/K, then the above correspondence sets up a bijection between the $G(K)$-conjugacy classes of maximal K-tori of G that split over L and the elements of $\ker(H^1(L/K,N) \to H^1(L/K,G))$.) We will need the following version of this fact.

Lemma 5.2. *Let S_1 and S_2 be two maximal K-tori of G, and let $[\xi_1], [\xi_2] \in \ker(H^1(K,N) \to H^1(K,G))$ be the corresponding cohomology classes. Then $\theta^1([\xi_1]) = \theta^1([\xi_2])$ if and only if there exists $h \in G(K^{\mathrm{sep}})$ such that $S_2 = hS_1h^{-1}$ and the isomorphism $\varphi\colon S_1 \to S_2$, $x \mapsto hxh^{-1}$, is defined over K.*

Proof. Clearly, $\varphi^{-1} \circ \sigma(\varphi)\colon S_1 \to S_1$ is given by $x \mapsto (h^{-1}\sigma(h))x(h^{-1}\sigma(h))^{-1}$, and therefore φ is K-defined if and only if $s(\sigma) := h^{-1}\sigma(h) \in S_1$ for all $\sigma \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$.

$\Leftarrow$) Let $g_1 \in G(K^{\mathrm{sep}})$ be such that $S_1 = g_1Tg_1^{-1}$. Then $g_2 = hg_1$ satisfies $S_2 = g_2Tg_2^{-1}$, and the cocycles $\xi_i(\sigma) = g_i^{-1}\sigma(g_i)$, $i = 1,2$, corresponding to S_1 and S_2, are related by

$$\xi_2(\sigma) = g_1^{-1}s(\sigma)\sigma(g_1) = (g_1^{-1}s(\sigma)g_1)\xi_1(\sigma).$$

Since $g_1^{-1}s(\sigma)g_1 \in T$, we have $\theta(\xi_1(\sigma)) = \theta(\xi_2(\sigma))$ for all σ, and therefore $\theta^1([\xi_1]) = \theta^1([\xi_2])$.

$\Rightarrow$) Changing ξ_2 to an equivalent cocycle (which amounts to a different choice of g_2 for which $S_2 = g_2Tg_2^{-1}$), we may assume that the elements $\xi_i(\sigma) = g_i^{-1}\sigma(g_i)$, $i = 1,2$, satisfy $\theta(\xi_1(\sigma)) = \theta(\xi_2(\sigma))$, i.e.

$$\xi_2(\sigma) = \xi_1(\sigma)t(\sigma) \quad \text{with} \quad t(\sigma) \in T,$$

for all $\sigma \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$. Set $h = g_2g_1^{-1}$. It is enough to show that $h^{-1}\sigma(h) \in S_1$ for all σ. We have

$$\begin{aligned} h^{-1}\sigma(h) &= g_1\xi_2(\sigma)\sigma(g_1)^{-1} = g_1(\xi_1(\sigma)t(\sigma))\sigma(g_1)^{-1} \\ &= (g_1\xi_1(\sigma)\sigma(g_1)^{-1}) \cdot (\sigma(g_1)t(\sigma)\sigma(g_1)^{-1}) = \\ &= \sigma(g_1t(\sigma)g_1^{-1}) \in S_1(K^{\mathrm{sep}}), \end{aligned}$$

as required. □

Beginning the proof of Theorem 5.1, we pick a maximal torus $\mathcal{T}$ of $\mathcal{G}$ (cf. [22, Exp. IX, 7.3]); then its generic fiber T is a maximal $\mathcal{K}$-torus of G whose splitting field $\mathcal{K}_T$ is unramified over $\mathcal{K}$. Let $N = N_G(T)$ and $\mathcal{N} = N_{\mathcal{G}}(\mathcal{T})$ be the corresponding normalizers. We denote by $\mathcal{O}^{\mathrm{ur}}$ the valuation ring of the maximal unramified extension $\mathcal{K}^{\mathrm{ur}}$. Then the Weyl group $W = N/T$ can be identified with

$$N(\mathcal{K}^{\mathrm{ur}})/T(\mathcal{K}^{\mathrm{ur}}) = \mathcal{N}(\mathcal{O}^{\mathrm{ur}})/\mathcal{T}(\mathcal{O}^{\mathrm{ur}}). \tag{10}$$

Since by assumption the torus S splits over $\mathcal{K}^{\mathrm{ur}}$, it corresponds to some class $[\xi] \in \ker(H^1(\mathcal{K}^{\mathrm{ur}}/\mathcal{K}, N) \to H^1(\mathcal{K}^{\mathrm{ur}}/\mathcal{K}, G))$. Since the elements of $\ker(H^1(\mathcal{K}^{\mathrm{ur}}/\mathcal{K}, \mathcal{N}(\mathcal{O}^{\mathrm{ur}})) \to$

$H^1(\mathcal{K}^{\mathrm{ur}}/\mathcal{K}, \mathcal{G}(\mathcal{O}^{\mathrm{ur}}))$ correspond to the maximal tori of $\mathcal{G}$, it follows from Lemma 5.2 that it is enough to construct a class $[\xi']$ in this set that satisfies $\theta^1([\xi]) = \theta^1([\xi'])$.

Lemma 5.3. *There exists a cocycle* $\xi' \in Z^1(\mathcal{K}^{\mathrm{ur}}/\mathcal{K}, \mathcal{N}(\mathcal{O}^{\mathrm{ur}}))$ *such that*

(1) $\theta^1([\xi]) = \theta^1([\xi'])$;
(2) *there exists* $n \geq 1$ *such that for* $\mathcal{K}' = \mathcal{K}(\sqrt[n]{\pi})$, *the image of* $[\xi']$ *in* $H^1((\mathcal{K}')^{\mathrm{ur}}/\mathcal{K}', G)$ *is trivial.*

We will now assume the lemma and complete the proof of Theorem 5.1. In view of the validity of the Grothendieck-Serre conjecture over discrete valuation rings (cf. §2.2), the image of $[\xi']$ is trivial in $H^1((\mathcal{K}')^{\mathrm{ur}}/\mathcal{K}', \mathcal{G}(\mathcal{O}'^{\mathrm{ur}}))$, where $\mathcal{O}'^{\mathrm{ur}}$ is the valuation ring of $(\mathcal{K}')^{\mathrm{ur}}$. We note that $\mathcal{K}$ and $\mathcal{K}'$ have the same residue field k, and that the residue of ξ' is the trivial cocycle with values in $\underline{\mathcal{G}}(k^{\mathrm{sep}})$, where $\underline{\mathcal{G}}$ is the reduction of $\mathcal{G}$. Applying Hensel's Lemma, we conclude that the class $[\xi']$ is trivial in $H^1(\mathcal{K}^{\mathrm{ur}}/\mathcal{K}, \mathcal{G}(\mathcal{O}^{\mathrm{ur}}))$, as required.

Proof of Lemma 5.3. Using (10), for each $\sigma \in \mathrm{Gal}(\mathcal{K}^{\mathrm{ur}}/\mathcal{K})$ we can pick $n(\sigma) \in \mathcal{N}(\mathcal{O}^{\mathrm{ur}})$ so that $\theta(\xi(\sigma)) = \theta(n(\sigma))$, i.e.

$$\xi(\sigma) = n(\sigma)t(\sigma) \quad \text{with} \quad t(\sigma) \in T(\mathcal{K}^{\mathrm{ur}}). \tag{11}$$

As in the proof of Proposition 4.4, we have a canonical isomorphism of modules over $\Gamma = \mathrm{Gal}(\mathcal{K}^{\mathrm{ur}}/\mathcal{K})$:

$$X_*(T) \otimes_{\mathbb{Z}} (\mathcal{K}^{\mathrm{ur}})^\times \to T(\mathcal{K}^{\mathrm{ur}}), \quad \chi \otimes a \mapsto \chi(a).$$

Furthermore, we have the following direct product decomposition of Γ-modules:

$$(\mathcal{K}^{\mathrm{ur}})^\times = \mathcal{U} \times \langle \pi \rangle,$$

where $\mathcal{U} = (\mathcal{O}^{\mathrm{ur}})^\times$ is the group of units in $\mathcal{K}^{\mathrm{ur}}$. Now, set

$$A := X_*(T) \otimes_{\mathbb{Z}} \mathcal{U} \simeq T(\mathcal{O}^{\mathrm{ur}}) \subset T(\mathcal{K}^{\mathrm{ur}}) \quad \text{and} \quad B := X_*(T) \otimes_{\mathbb{Z}} \langle \pi \rangle \subset T(\mathcal{K}^{\mathrm{ur}}),$$

noting that A and B are invariant under the action of Γ as well as under conjugation by elements of $N(\mathcal{K}^{\mathrm{ur}})$ and that

$$T(\mathcal{K}^{\mathrm{ur}}) = A \times B \quad \text{as} \quad \Gamma\text{-modules}.$$

So, we can write $t(\sigma)$ in (11) as

$$t(\sigma) = a(\sigma)b(\sigma) \quad \text{with} \quad a(\sigma) \in A, \; b(\sigma) \in B.$$

Set $\xi'(\sigma) = n(\sigma)a(\sigma) \in \mathcal{N}(\mathcal{O}^{\mathrm{ur}})$. Using the cocycle condition for ξ in conjunction with the fact that $\xi(\sigma) = \xi'(\sigma)b(\sigma)$, we obtain the following relation:

$$(\xi'(\sigma) \cdot \sigma(\xi'(\tau)))^{-1} \cdot \xi'(\sigma\tau) = (\sigma(n(\tau))^{-1} \cdot b(\sigma) \cdot \sigma(n(\tau))) \cdot \sigma(b(\tau)) \cdot b(\sigma\tau)^{-1}.$$

Clearly, the left-hand side belongs to $\mathcal{N}(\mathcal{O}^{\mathrm{ur}})$, and the right-hand side to B. It follows that the left-hand side is actually in A, hence both sides are equal to 1. In other words, ξ' is a cocycle that satisfies $\theta^1([\xi]) = \theta^1([\xi'])$. Furthermore, we have

$$b(\sigma\tau) = ((\sigma(n(\tau)))^{-1} b(\sigma)\sigma(n(\tau))) \cdot \sigma(b(\tau)).$$

Conjugating this relation by $\xi(\sigma\tau)$ and using the fact that ξ is a cocycle and that $\xi(\sigma)t\xi(\sigma)^{-1} = n(\sigma)tn(\sigma)^{-1}$ for $t \in T(\mathcal{K}^{\mathrm{ur}})$, we see that $\nu(\sigma) = n(\sigma)b(\sigma)n(\sigma)^{-1}$ defines a Galois cocycle with values in ${}_\xi T(\mathcal{K}^{\mathrm{ur}})$, where ${}_\xi T$ denotes the twist of T by ξ. Let $\mathcal{L}$ be the minimal splitting field of ${}_\xi T$ (which by construction is unramified over $\mathcal{K}$), and let $n = [\mathcal{L} : \mathcal{K}]$. Set $K' = K(\sqrt[n]{\pi})$. We claim that the image of $[\nu] \in H^1(\mathcal{K}^{\mathrm{ur}}/\mathcal{K}, {}_\xi T)$ in $H^1(\mathcal{K}^{\mathrm{ur}}\mathcal{K}'/\mathcal{K}', {}_\xi T)$ is trivial. Indeed, it follows from Hilbert's Theorem 90 that every element in the latter group is annihilated by multiplication by n. Now, the cocycle ν has values in

$${}_\xi B := X_*({}_\xi T) \otimes_{\mathbb{Z}} \langle \pi \rangle \subset {}_\xi T(\mathcal{L}).$$

After base change from $\mathcal{K}$ to $\mathcal{K}'$, we can consider a similar subgroup

$${}_\xi B' = X_*({}_\xi T) \otimes_{\mathbb{Z}} \langle \pi^{1/n} \rangle \subset {}_\xi T(\mathcal{L}\mathcal{K}').$$

Since every element of ${}_\xi B$ can be uniquely divided by n in ${}_\xi B'$, there is a cocycle ν' with values in ${}_\xi B' \subset {}_\xi T(\mathcal{L}\mathcal{K}')$ such that $\nu = n \cdot \nu'$. But then it follows from the remark above that the image of the class $[\nu]$ in $H^1(\mathcal{K}^{\mathrm{ur}}\mathcal{K}'/\mathcal{K}', {}_\xi T)$ is trivial. This means that there exists $s \in {}_\xi T(\mathcal{L}\mathcal{K}')$ such that

$$n(\sigma)b(\sigma)n(\sigma)^{-1} = s^{-1} \cdot (n(\sigma)\sigma(s)n(\sigma)^{-1}).$$

Then $b(\sigma) = (n(\sigma)^{-1}s^{-1}n(\sigma)) \cdot \sigma(s)$, which implies that

$$\xi(\sigma) = s^{-1} \cdot \xi'(\sigma) \cdot \sigma(s),$$

for all $\sigma \in \mathrm{Gal}(K^{\mathrm{ur}}\mathcal{K}'/\mathcal{K}')$. This means that the classes $[\xi]$ and $[\xi']$ are mapped to the same element in $H^1(K^{\mathrm{ur}}\mathcal{K}'/\mathcal{K}', G)$, and therefore the image of $[\xi']$ is in fact trivial, as required. $\square$

6. Proof of Theorem 1.1

Theorem 1.1 is an easy consequence of the following result.

Theorem 6.1. *Let $\mathcal{K}$ be a field complete with respect to a discrete valuation v such that the residue field k is finitely generated, and let G be an absolutely almost simple $\mathcal{K}$-group that has good reduction at v. Assume that* $\mathrm{char}\, k \neq 2$ *if G is of type B_ℓ $(\ell \geq 2)$. Then any $G' \in \mathbf{gen}_{\mathcal{K}}(G)$ also has good reduction at v.*

For the proof, we will consider separately the two cases where the type of G is different from A_1 and B_ℓ $(\ell \geq 2)$ and where it is one of those types. In each case, we will characterize the existence of good reduction in terms of the presence of maximal tori with very specific properties — see Theorems 6.2 and 6.6. These characterizations will also be used in §9 for the analysis of weakly commensurable Zariski-dense subgroups. We begin with the following sufficient condition for good reduction for types different from A_1 and B_ℓ $(\ell \geq 2)$.

Theorem 6.2. *Let $\mathcal{K}$ be a field complete with respect to a discrete valuation v, and let G be an absolutely almost simple algebraic $\mathcal{K}$-group of type different from A_1 and B_ℓ $(\ell \geq 2)$. Assume that G contains a maximal $\mathcal{K}$-torus T which is $\mathcal{K}$-generic and whose minimal splitting field $\mathcal{K}_T$ is unramified over $\mathcal{K}$. Then G has good reduction at v.*

Proof. Let G_0 be the quasi-split inner form of G, and let $\mathcal{L}$ be the minimal Galois extension of $\mathcal{K}$ over which G_0 becomes split. Being a subextension of $\mathcal{K}_T/\mathcal{K}$, the extension $\mathcal{L}/\mathcal{K}$ is unramified, and therefore G_0 and the corresponding adjoint group $\overline{G}_0$ have good reduction (cf. [34, Corollary 7.9.4]). Let $\overline{\mathcal{G}}_0$ be the corresponding model for $\overline{G}_0$ over the valuation ring $\mathcal{O}$ of $\mathcal{K}$. Now, let $\rho\colon G \to \overline{G}$ be the isogeny onto the adjoint group, and $\overline{T} = \rho(T)$. It follows from Steinberg's Theorem (cf. [54, Proposition 6.19], [4, 8.6]) that there exist an embedding $\iota\colon \overline{T} \hookrightarrow \overline{G}_0$ and a 1-cocycle $\xi \in Z^1(\mathcal{K}, \overline{T})$ such that for the image $\bar{\xi}$ of ξ under the natural map $Z^1(\mathcal{K}, \overline{T}) \to Z^1(\mathcal{K}, \overline{G}_0)$, the twisted group ${}_{\bar{\xi}}G_0$ is $\mathcal{K}$-isomorphic to G. Since $\mathcal{K}_T/\mathcal{K}$ is unramified, according to Theorem 5.1, there exist a maximal torus $\overline{\mathcal{S}}$ of $\overline{\mathcal{G}}_0$ and an element $h \in \overline{G}_0(\mathcal{K}^{\mathrm{sep}})$ such that the generic fiber $\overline{S}$ satisfies $\overline{S} = h\overline{T}h^{-1}$ and the morphism $\varphi\colon \overline{T} \to \overline{S}$, $x \mapsto hxh^{-1}$, is defined over $\mathcal{K}$; recall that the latter is equivalent to the fact that $s(\sigma) := h^{-1} \cdot \sigma(h)$ lies in $\overline{T}(\mathcal{K}^{\mathrm{sep}})$ for all $\sigma \in \mathrm{Gal}(\mathcal{K}^{\mathrm{sep}}/\mathcal{K})$. Let ξ' be the image of ξ under the map $Z^1(\mathcal{K}, \overline{T}) \to Z^1(\mathcal{K}, \overline{S})$ induced by φ. We have

$$\xi(\sigma) = h^{-1}\xi'(\sigma)h = h^{-1}(\xi'(\sigma) \cdot (h\sigma(h)^{-1}))\sigma(h). \tag{12}$$

Next, the equation

$$\zeta(\sigma) := \xi'(\sigma) \cdot (h\sigma(h)^{-1}) = \xi'(\sigma) \cdot (hs(\sigma)^{-1}h^{-1}) = h(\xi(\sigma) \cdot s(\sigma)^{-1})h^{-1},$$

defines a cocycle $\zeta \in Z^1(\mathcal{K}, \overline{S})$, and we let $\bar{\zeta}$ denote its image under the map $Z^1(\mathcal{K}, \overline{S}) \to Z^1(\mathcal{K}, \overline{G}_0)$. It follows from (12) that $\bar{\xi} = \bar{\zeta}$, and hence $G \simeq {}_{\bar{\xi}}G_0 \simeq {}_{\bar{\zeta}}G_0$. So, it remains to show that ${}_{\bar{\zeta}}G_0$ has a reductive model over $\mathcal{O}$. Since φ is defined over $\mathcal{K}$, we have

$$\mathcal{K}_{\overline{S}} = \mathcal{K}_{\overline{T}} = \mathcal{K}_T,$$

which is unramified over $\mathcal{K}$. In addition, $\overline{S}$ is generic over $\mathcal{K}$, so by Proposition 4.4

$$H^1(\mathcal{K}, \overline{S}) = H^1(\mathcal{K}_{\overline{S}}/\mathcal{K}, \overline{S}) = H^1(\mathcal{K}_{\overline{S}}/\mathcal{K}, \mathcal{U}),$$

where $\mathcal{U} = \overline{S}(\mathcal{O}_{\mathcal{K}_{\overline{S}}})$ and $\mathcal{O}_{\mathcal{K}_{\overline{S}}}$ is the valuation ring of $\mathcal{K}_{\overline{S}}$. Thus, replacing ζ with an equivalent cocycle, we may assume that it has values in $\mathcal{U}$. Obviously, the inner automorphisms corresponding to the elements of $\mathcal{U}$ act on $\mathcal{G}_0 \times_{\mathcal{O}} \mathcal{O}_{\mathcal{K}_{\overline{S}}}$, and then the corresponding twisted $\mathcal{O}$-scheme $\mathcal{G} = {}_{\bar{\zeta}}\mathcal{G}_0$ is a required reductive model for ${}_{\bar{\zeta}}G_0 \simeq G$. □

Proposition 6.3. *Let $\mathcal{K}$ be a field complete with respect to a discrete valuation v and assume that the residue field $k = \mathcal{K}^{(v)}$ is infinite and finitely generated. If G is an absolutely almost simple algebraic $\mathcal{K}$-group that has good reduction at v, then G possesses a maximal $\mathcal{K}$-torus T that is $\mathcal{K}$-generic and whose minimal splitting field $\mathcal{K}_T$ is unramified over $\mathcal{K}$.*

Proof. Let $\mathcal{G}$ be a model of G over $\mathcal{O}$. Then the reduction $\underline{\mathcal{G}}$ is an absolutely almost simple algebraic k-group of the same type as G. Since k is infinite and finitely generated, one can find a maximal k-torus $\overline{\mathcal{T}}$ of $\underline{\mathcal{G}}$ that is generic over k (cf. Theorem 3.1). Let $\mathcal{T}$ be a lift of $\overline{\mathcal{T}}$ to $\mathcal{G}$ (cf. [20, Corollary B.3.5]). Then the generic fiber T is a maximal $\mathcal{K}$-torus of G that is $\mathcal{K}$-generic and whose splitting field $\mathcal{K}_T$ is unramified over $\mathcal{K}$. □

Corollary 6.4. *Let G be an absolutely almost simple algebraic $\mathcal{K}$-group of type different from A_1 and B_ℓ. Assume that the residue field k is finitely generated and G has good reduction at v. Then any $G' \in \mathbf{gen}_{\mathcal{K}}(G)$ has good reduction at v.*

Proof. We first consider the case where k is a *finite* field. In this case, it is well-known that the fact that G has good reduction implies that G is quasi-split over $\mathcal{K}$. Let $G' \in \mathbf{gen}_{\mathcal{K}}(G)$. Then clearly

$$\mathrm{rk}_{\mathcal{K}}\, G' = \mathrm{rk}_{\mathcal{K}}\, G. \tag{13}$$

On the other hand, according to Tits' classification [79], the group G' is quasi-split if and only if all vertices in the Tits index are distinguished. Since G' is an inner twist of G, this is equivalent to (13). Thus, G' is $\mathcal{K}$-quasi-split, hence $\mathcal{K}$-isomorphic to G. In particular, it has good reduction at v.

Next, suppose that k is *infinite* and finitely generated. In this case, our claim follows directly from the above results. Indeed, according to Proposition 6.3, the group G

contains a maximal $\mathcal{K}$-torus T that is generic over $\mathcal{K}$ and whose splitting field $\mathcal{K}_T$ is unramified over $\mathcal{K}$. Since $G' \in \mathbf{gen}_{\mathcal{K}}(G)$, it contains a maximal $\mathcal{K}$-torus T' isomorphic to T. Then of course $\mathcal{K}_{T'} = \mathcal{K}_T$ is unramified over $\mathcal{K}$. Furthermore, the assumption that G' is an inner twist of G implies that the subgroups $\theta_T(\mathrm{Gal}(\mathcal{K}_T/\mathcal{K})) \subset \mathrm{Aut}(\Phi(G,T))$ and $\theta_{T'}(\mathrm{Gal}(\mathcal{K}_{T'}/\mathcal{K})) \subset \mathrm{Aut}(\Phi(G,T'))$ have isomorphic images in the groups of symmetries of the corresponding Dynkin diagrams. Then the fact that T is $\mathcal{K}$-generic implies the same for T'. Hence, G' has good reduction by Theorem 6.2. $\square$

We now turn to the second case where the type of G is either A_1 or B_ℓ ($\ell \geq 2$). Let us first show that Theorem 6.2 may be false in this case.

Example 6.5. Let $\mathcal{K} = \mathbb{Q}((x))$, equipped with the standard valuation v.

(a) Let D be the quaternion algebra $\left(\dfrac{-1, x}{\mathcal{K}}\right)$ and $G = \mathrm{SL}_{1,D}$. Set $\mathcal{L} = \mathbb{Q}(\sqrt{-1})((x))$, and let $T = \mathrm{R}^{(1)}_{\mathcal{L}/\mathcal{K}}(\mathbb{G}_m)$ be the corresponding maximal $\mathcal{K}$-torus of G. Then T is $\mathcal{K}$-generic and splits over the unramified extension $\mathcal{L}/\mathcal{K}$, but the quaternion algebra D ramifies at v, hence G does not have good reduction according to Example 2.2.

(b) Let $\ell \geq 2$ and $\mathcal{K}$ be as above. Set

$$q_0 = x_1^2 + \cdots + x_{2\ell-1}^2 + 2x_{2\ell}^2 \quad \text{and} \quad q = q_0 + x x_{2\ell+1}^2,$$

and let

$$H = \mathrm{Spin}_{2\ell}(q_0) \ \subset \ \mathrm{Spin}_{2\ell+1}(q) = G.$$

Since H corresponds to a quadratic form defined over $\mathbb{Q}$, it has good reduction and contains a $\mathcal{K}$-generic maximal torus T that splits over an extension of the form $L\mathcal{K}$ for some finite extension L of $\mathbb{Q}$, which is obviously unramified. Now, T is also a maximal torus in G (because G and H have the same rank ℓ), and we claim that it remains $\mathcal{K}$-generic in G. Indeed, the group H is of type D_ℓ, the group G is of type B_ℓ, hence $|W(G,T)| = 2 \cdot |W(H,T)|$. Since T is generic in H, the image of $\mathrm{Gal}(\mathcal{K}_T/\mathcal{K})$ in $\mathrm{Aut}(\Phi(H,T))$ contains $W(H,T)$. Furthermore, the (signed) discriminant of q_0 is not a square, so the image is not entirely contained in $W(H,T)$ (cf. [58, Lemma 4.1]). It follows that the image is $W(G,T)$, making T generic in G. On the other hand, both the first and the second residues (cf. [40, Ch. VI]) of q are nontrivial, and therefore no scalar multiple of q can be equivalent to a diagonal quadratic form with all coefficients being units. So, G does not have good reduction at v by Example 2.3.

For types A_1 and B_ℓ we have the following modified condition for good reduction.

Theorem 6.6. *Let $\mathcal{K}$ be a field that is complete with respect to a discrete valuation v with residue field k. Suppose G is an absolutely almost simple algebraic $\mathcal{K}$-group of type either A_1 or B_ℓ ($\ell \geq 2$), and assume that* $\mathrm{char}\, k \neq 2$ *if G is of type B.*

(1) *If G contains two maximal $\mathcal{K}$-tori T_1 and T_2 that are $\mathcal{K}$-generic and whose splitting fields $\mathcal{K}_{T_i}$ are unramified over $\mathcal{K}$ and satisfy $\mathcal{K}_{T_1} \cap \mathcal{K}_{T_2} = \mathcal{K}$, then G has good reduction at v.*

(2) *Conversely, if k is infinite and finitely generated, and G has good reduction at v, then G contains two maximal $\mathcal{K}$-tori T_1 and T_2 that are $\mathcal{K}$-generic and whose splitting fields $\mathcal{K}_{T_1}$ and $\mathcal{K}_{T_2}$ are unramified over $\mathcal{K}$ and satisfy $\mathcal{K}_{T_1} \cap \mathcal{K}_{T_2} = \mathcal{K}$.*

Proof. Without loss of generality, we may assume that G is simply connected.

(1): The argument for type A_1 is rather simple. Indeed, here $G = \mathrm{SL}_{1,D}$, where D is a quaternion algebra over $\mathcal{K}$, and we need to show that D is unramified at v. Furthermore, we have $T_i = \mathrm{R}^{(1)}_{\mathcal{L}_i/\mathcal{K}}(G_m)$, where $\mathcal{L}_i = \mathcal{K}_{T_i}$ is a quadratic subfield of D, that, by assumption, is unramified. Now, if D were ramified at v, then the residue algebra $\overline{D}$ would be a quadratic extension of k. Then

$$\overline{D} = \overline{\mathcal{L}_1} = \overline{\mathcal{L}_2}.$$

This would imply that $\mathcal{L}_1 = \mathcal{L}_2$, contradicting the fact that $\mathcal{L}_1 \cap \mathcal{L}_2 = \mathcal{K}$. Thus, D is unramified, as required.

The argument for type B_ℓ is similar, but more technical. Here $G = \mathrm{Spin}_n(q)$, where $n = 2\ell + 1$ and q is a nondegenerate quadratic form on $\mathcal{K}^n$. We will use the standard action of G on the n-dimensional vector space. It is well-known that every maximal $\mathcal{K}$-torus T of G fixes an anisotropic vector $a \in \mathcal{K}^n$, and hence lies in the stabilizer $G(a)$, which can be identified with $H = \mathrm{Spin}_{n-1}(q')$, where q' is the restriction of q to the orthogonal complement $W = \langle a \rangle^\perp$; note that H is a group of type D_ℓ.

So, in our set-up, for each $i = 1, 2$, we can choose an anisotropic vector $a_i \in \mathcal{K}^n$ fixed by T_i, and let $H_i = \mathrm{Spin}_{n-1}(q_i)$, where q_i is the restriction of q to the orthogonal complement $W_i = \langle a_i \rangle^\perp$. Then T_i is a maximal $\mathcal{K}$-torus in H_i, which is $\mathcal{K}$-generic in H_i and has unramified splitting field. So, it follows from Theorem 6.2 for $\ell \geq 3$ and from Lemma 6.7 below for $\ell = 2$ (we note that the order of the Weyl group for type B_2 is 8) that H_i has good reduction at v. According to Example 2.3, this means that there exist an element $\lambda_i \in \mathcal{K}^\times$ and a basis $e^{(i)}_1, \ldots, e^{(i)}_{n-1}$ of W_i such that in this basis, the quadratic form $\lambda_i q_i$ has the following presentation

$$u^{(i)}_1 x_1^2 + \cdots + u^{(i)}_{n-1} x_{n-1}^2,$$

where $u^{(i)}_j$ are units in $\mathcal{K}$ for $i = 1, 2$ and $j = 1, \ldots, n-1$. Also, by scaling a_i, we may assume that the values of $v(\lambda_i q(a_i))$ are either 0 or 1. If it is 0 for at least one $i \in \{0, 1\}$, then G has good reduction at v (see Example 2.3). So, let us assume that the value is 1 for both $i = 1, 2$, i.e. $\lambda_i q(a_i) = \pi u^{(i)}_n$ where $u^{(i)}_n$ is a unit. Then

$$\lambda_i q = u^{(i)}_1 x_1^2 + \cdots + u^{(i)}_{n-1} x_{n-1}^2 + \pi u^{(i)}_n x_n^2.$$

We have $\lambda_1 q = (\lambda_1 \lambda_2^{-1})(\lambda_2 q)$. So, taking determinants and evaluating v, we obtain

$$1 \equiv nv(\lambda_1 \lambda_2^{-1}) + 1 (\mathrm{mod}\, 2),$$

which, in view of the fact that n is odd, implies that $v(\lambda_1 \lambda_2^{-1})$ is even. So, after scaling, we can actually assume that $\lambda_1 \lambda_2^{-1}$ is a unit. Consequently, setting $\lambda = \lambda_1$ and replacing q by λq, we may assume that

$$q = u_1^{(i)} x_1^2 + \cdots + u_{n-1}^{(i)} x_{n-1}^2 + \pi u_n^{(i)} x_n^2,$$

where $u_j^{(i)}$ are all units. Then the diagonal quadratic forms $\langle \overline{u_1^{(i)}}, \dots \overline{u_{n-1}^{(i)}} \rangle$ over the residue field k (where the bar denotes taking the residue in k) are both the so-called first residues of q (we note that the first and second residues were constructed by Springer assuming that the residue characteristic is $\neq 2$). Since the first-residue is well-defined (see [40, Ch. VI]), we conclude that these residues are equivalent, and then by Hensel's Lemma, the quadratic forms $q_i = u_1^{(i)} x_1^2 + \cdots + u_{n-1}^{(i)} x_{n-1}^2$ for $i = 1, 2$ themselves are equivalent.

Let d_i be the (signed) discriminant of q_i. Since T_i is generic in G, and the Weyl group of G contains the Weyl group of H_i (with respect to T_i) as a subgroup of index 2, we conclude that H_i is an outer form of a split group over $\mathcal{K}$. Furthermore, the minimal Galois extension of $\mathcal{K}$ over which it becomes an inner form is $\mathcal{K}(\sqrt{d_i})$. Since the forms q_1 and q_2 are equivalent, we conclude that

$$\mathcal{L} := \mathcal{K}(\sqrt{d_1}) = \mathcal{K}(\sqrt{d_2})$$

is a quadratic extension of $\mathcal{K}$. However, $\mathcal{L} \subset \mathcal{K}_{T_i}$ for both $i = 1, 2$, contradicting the assumption that $\mathcal{K}_{T_1}$ and $\mathcal{K}_{T_2}$ are disjoint over $\mathcal{K}$.

(2): Let $\mathcal{G}$ be the model of G over $\mathcal{O}$ with reduction $\underline{\mathcal{G}}$, which is an absolutely almost simple algebraic k-group of the same type as G. Since k is infinite and finitely generated, we can find a maximal k-torus $\overline{\mathcal{T}}_1$ of $\underline{\mathcal{G}}$ that is generic over k. Next, let $\overline{\mathcal{T}}_2$ be a maximal k-torus of $\underline{\mathcal{G}}$ that is generic over the splitting field $k_{\overline{\mathcal{T}}_1}$ of $\overline{\mathcal{T}}_1$. Since the Dynkin diagrams of the types A_1 and B_ℓ do not have nontrivial automorphisms, the degrees $[k_{\overline{\mathcal{T}}_i} : k]$ for $i = 1, 2$ are equal to the order w of the Weyl group. Besides, the degree $[k_{\overline{\mathcal{T}}_1} k_{\overline{\mathcal{T}}_2} : k_{\overline{\mathcal{T}}_1}]$ also equals w. This implies that

$$k_{\overline{\mathcal{T}}_1} \cap k_{\overline{\mathcal{T}}_2} = k. \tag{14}$$

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the lifts of $\overline{\mathcal{T}}_1$ and $\overline{\mathcal{T}}_2$ to $\mathcal{G}$, and let T_1 and T_2 be the corresponding generic fibers. Then T_1 and T_2 are maximal $\mathcal{K}$-tori of G that are generic over $\mathcal{K}$ and whose splitting fields K_{T_i} are unramified extensions of $\mathcal{K}$ with the residue fields $k_{\overline{\mathcal{T}}_i}$ for $i = 1, 2$. Then (14) implies that $\mathcal{K}_{T_1} \cap \mathcal{K}_{T_2} = \mathcal{K}$, as required. □

We will now prove the statement about good reduction of spinor groups of 4-dimensional quadratic forms that was used in the above argument. We recall that given a nondegenerate quadratic form q over $\mathcal{K}$ of dimension four, the spinor group $\mathrm{Spin}_4(q)$ is isomorphic to $G = \mathrm{R}_{\mathcal{L}/\mathcal{K}}(H)$, where $\mathcal{L}$ is a 2-dimensional étale $\mathcal{K}$-algebra and $H = \mathrm{SL}_{1,\mathcal{D}}$, where $\mathcal{D}$ is a central quaternion $\mathcal{K}$-algebra.

Lemma 6.7. *Let G be as above. If G possesses a maximal $\mathcal{K}$-torus such that $\mathcal{K}_T/\mathcal{K}$ is an unramified extension of degree 8, then G has good reduction.*

Proof. If $\mathcal{L} = \mathcal{K} \times \mathcal{K}$, then $[\mathcal{K}_T : \mathcal{K}] \leq 4$ for any $\mathcal{K}$-torus T of G. Thus, in our situation, $\mathcal{L}/\mathcal{K}$ is a quadratic field extension. It is enough to show that $\mathcal{L}/\mathcal{K}$ is unramified and $H_{\mathcal{L}} = H \times_{\mathcal{K}} \mathcal{L}$ has good reduction. Indeed, if $\mathcal{H}$ is a reductive $\mathcal{O}_{\mathcal{L}}$-model for $H_{\mathcal{L}}$, then $\mathcal{G} = \mathrm{R}_{\mathcal{O}_{\mathcal{L}}/\mathcal{O}_{\mathcal{K}}}(\mathcal{H})$ would be a reductive $\mathcal{O}_{\mathcal{K}}$-model for G. Since $\mathcal{L} \subset \mathcal{K}_T$, we immediately obtain that $\mathcal{L}/\mathcal{K}$ is unramified. Furthermore,

$$G \times_{\mathcal{K}} \mathcal{L} \simeq H_{\mathcal{L}} \times H_{\mathcal{L}}.$$

In terms of this $\mathcal{L}$-isomorphism, let $T \times_{\mathcal{K}} \mathcal{L} \simeq T_1 \times T_2$. We have $[\mathcal{L}_T : \mathcal{L}] = 4$, which means that T_1 and T_2 are nonisomorphic $\mathcal{L}$-tori of $H_{\mathcal{L}}$ having unramified splitting fields. So, the fact that $H_{\mathcal{L}}$ has good reduction follows from the first part of the proof of Theorem 6.6. □

Corollary 6.8. *Let G be an absolutely almost simple algebraic $\mathcal{K}$-group of type either A_1 or B_ℓ that has good reduction at v. Assume that k is finitely generated and* $\operatorname{char} k \neq 2$ *if G is of type B. Then any $G' \in \mathbf{gen}_{\mathcal{K}}(G)$ also has good reduction at v.*

Proof. As in the proof of Corollary 6.4, we consider two cases. First, if the residue field is finite, the fact that an absolutely almost $\mathcal{K}$-group G of type A_1 or B_ℓ has good reduction means that it actually splits over $\mathcal{K}$. Then G' also splits, and hence has good reduction. Next, suppose the residue field is infinite and finitely generated. Then by Theorem 6.6(2), the group G possesses two maximal $\mathcal{K}$-tori T_1 and T_2 that are generic over $\mathcal{K}$ and whose splitting fields are unramified over $\mathcal{K}$ and satisfy $\mathcal{K}_{T_1} \cap \mathcal{K}_{T_2} = \mathcal{K}$. On the other hand, G' contains maximal $\mathcal{K}$-tori T_1' and T_2' that are $\mathcal{K}$-isomorphic to T_1 and T_2, respectively. Clearly, T_1' and T_2' have properties analogous to those of T_1 and T_2, so G' has good reduction by Theorem 6.6(1). □

Now, Theorem 6.1 follows from Corollaries 6.4 and 6.8. Furthermore, to prove the first assertion of Theorem 1.1, one needs to use Corollary 3.3 and then apply Theorem 6.1. To prove the second assertion, we let $\overline{\mathcal{T}}$ be a maximal $k^{(v)}$-torus of the reduction $\underline{G}^{(v)}$. Let $\mathcal{G}$ be the model of $G \times_k k_v$ over $\mathcal{O}_v$ that yields the reduction $\underline{G}^{(v)}$. Then according to [20, Corollary B.3.5], the torus $\overline{\mathcal{T}}$ lifts to a maximal torus $\mathcal{T}$ of $\mathcal{G}$; let T be the corresponding generic fiber, which is a maximal k_v-torus of $G \times_k k_v$. As we already mentioned, $G' \in \mathbf{gen}_{k_v}(G)$, so there exists a maximal k_v-torus T' of G' that is isomorphic to T. We have already established that $G' \times_k k_v$ has a model $\mathcal{G}'$ over $\mathcal{O}_v$, and using Theorem 5.1 we may assume that T' is the generic fiber of a torus $\mathcal{T}'$ of $\mathcal{G}'$. According to Proposition 2.5, the fact that $T \simeq T'$ over k_v implies that $\mathcal{T} \simeq \mathcal{T}'$ over $\mathcal{O}_v$. Then the reduction $\overline{\mathcal{T}'}$ is a maximal

k_v-torus of the reduction $(\underline{G}')^{(v)}$ that is isomorphic to $\overline{T}$. A symmetric argument shows that every maximal k_v-torus of $(\underline{G}')^{(v)}$ is isomorphic to a maximal k_v-torus of $\underline{G}^{(v)}$. Thus, $(\underline{G}')^{(v)} \in \mathbf{gen}_{k_v}(\underline{G}^{(v)})$.

Corollary 6.9. *Let G be an absolutely almost simple algebraic group over an infinite finitely generated field k, and let V be a divisorial set of places of k. Assume that* $\operatorname{char} k \neq 2$ *if G is of type* B*. Then there exists a finite subset $S \subset V$ such that every $G' \in \mathbf{gen}_k(G)$ has good reduction at all $v \in V \setminus S$.*

Proof. First, we note that the residue field $k^{(v)}$ is finitely generated for all $v \in V$, so we can apply our previous results. Clearly, we can find a finite subset $S \subset V$ such that G has good reduction at all $v \in V \setminus S$. Besides, if G is of type B, then by our assumption $\operatorname{char} k \neq 2$ and we can include in S all $v \in V$ such that $v(2) \neq 0$. But then according to Theorem 1.1, every $G' \in \mathbf{gen}_k(G)$ also has good reduction at all $v \in V$. □

This corollary shows that the truth of the Finiteness Conjecture for forms G with good reduction and all divisorial sets of places of the given finitely generated field k would imply the finiteness of $\mathbf{gen}_k(G)$.

7. The behavior of the genus under a purely transcendental base change: proof of Theorem 1.3

In order to set the stage for the proof of Theorem 1.3, we would first like to present an analogous result for division algebras.

7.1. The genus of a division algebra

We recall that two finite-dimensional central division algebras D_1 and D_2 over a field K are said to have the *same maximal subfields* if they have the same degree n and satisfy the following property: a degree n extension P/K admits a K-embedding $P \hookrightarrow D_1$ if and only if it admits a K-embedding $P \hookrightarrow D_2$. Given a finite-dimensional central division algebra D over K, one defines its genus $\mathbf{gen}(D)$ as the set of classes $[D'] \in \operatorname{Br}(K)$ in the Brauer group corresponding to central division K-algebras D' that have the same maximal subfields as D. This concept has been analyzed in detail in [13], [14], [17], [65], and other publications. Our goal in the present subsection is to prove the following.

Proposition 7.1. *Let D be a central division algebra of degree n over a field k, and assume that n is prime to* $\operatorname{char} k$*. Set $K = k(x)$. Then every element of $\mathbf{gen}(D \otimes_k K)$ is represented by a division algebra of the form $D' \otimes_k K$, where D' is a central division algebra over k with $[D'] \in \mathbf{gen}(D)$.*

Proof. For any n that is prime to $\operatorname{char} k$, we have the following exact sequence that goes back to Faddeev (cf. [25, Example 9.2]):

$$0 \to {}_n\mathrm{Br}(k) \longrightarrow {}_n\mathrm{Br}(K) \xrightarrow{\rho} \bigoplus_p \mathrm{Hom}(\mathrm{Gal}((K^{(p)})^{\mathrm{sep}}/K^{(p)}), \mathbb{Z}/n\mathbb{Z}),$$

where p runs through all monic irreducible polynomials in $k[x]$, ρ is the direct sum of the corresponding residue maps

$$\rho^{(p)}\colon {}_n\mathrm{Br}(K) \longrightarrow \mathrm{Hom}(\mathrm{Gal}((K^{(p)})^{\mathrm{sep}}/K^{(p)}), \mathbb{Z}/n\mathbb{Z}),$$

and $K^{(p)} = k[x]/(p(x))$ is the residue field at p. Let $\Delta \in \mathbf{gen}(D \otimes_k K)$. Since the algebra $D \otimes_k K$ is unramified at all p, the latter implies that Δ is also unramified at all p (cf. [13], [65]), i.e. $\rho([\Delta]) = 0$. So, it follows from the above exact sequence that Δ is of the form $\Delta = D' \otimes_k K$ for some central division k-algebra D' of degree n. It remains to show that $D' \in \mathbf{gen}(D)$. For this, we let v denote the discrete valuation of K corresponding to the polynomial $p(x) = x$; then the completion K_v is $k((x))$ and the residue field $K^{(v)}$ is k. Since $D' \otimes_k K \in \mathbf{gen}(D \otimes_k K)$, it follows from [65, Lemma 2.1] that the degree n division algebras $\mathcal{D} = D \otimes_k K_v$ and $\mathcal{D}' = D' \otimes_k K_v$ have the same maximal subfields. Since $\mathcal{D}$ and $\mathcal{D}'$ are division algebras, the valuation v extends to valuations w and w' of these algebras, and we let $\overline{\mathcal{D}}$ and $\overline{\mathcal{D}'}$ denote the corresponding residue algebras. A standard argument shows that the fact that $\mathcal{D}$ and $\mathcal{D}'$ have the same maximal subfields implies that the residue algebras also have the same maximal subfields. Since $\overline{\mathcal{D}} \simeq D$ and $\overline{\mathcal{D}'} \simeq D'$, we see that $D' \in \mathbf{gen}(D)$, as required. $\square$

Using the proposition repeatedly, we obtain a similar statement for the field of rational functions $K = k(x_1, \ldots, x_m)$ in any number of variables. Our next goal is to prove Theorem 1.3 that extends the proposition to absolutely almost simple algebraic groups.

7.2. Proof of Theorem 1.3

The argument relies on the following fundamental fact.

Theorem 7.2. (Raghunathan, Ramanathan [63]) *Let G be a connected reductive algebraic group over a field k, and let $\mathbb{A}^1_k = \mathrm{Spec}\, k[x]$ be the affine line over k. Let $B \to \mathbb{A}^1_k$ be a principal G-bundle on $\mathbb{A}^1_k$ such that the bundle $B \times_{\mathbb{A}^1_k} \mathbb{A}^1_{k^{\mathrm{sep}}}$ on $\mathbb{A}^1_{k^{\mathrm{sep}}} = \mathrm{Spec}\, k^{\mathrm{sep}}[x]$, where k^{sep} is a separable closure of k, is trivial. Then B is constant, i.e. there exists a principal G-bundle $B_0 \to \mathrm{Spec}\, k$ such that $B = B_0 \times_{\mathrm{Spec}\, k} \mathbb{A}^1_k$.*

An alternative proof of this theorem was given in [26]. Later in [12], the theorem was extended to reductive, but not necessarily connected, groups; see also [1] for a new proof over fields of characteristic zero that uses buildings. In cohomological language, the theorem means that the natural map from $H^1(k, G)$ to $\mathrm{Ker}\left(H_{\text{ét}}(\mathbb{A}^1_k, G) \to H_{\text{ét}}(\mathbb{A}^1_{k^{\mathrm{sep}}}, G)\right)$ is a surjection (in fact, a bijection). We will use this interpretation to prove the following.

Proposition 7.3. *Let G be a semisimple algebraic group over a field k, let $K = k(x)$, and let V be the set of discrete valuations of K associated with the monic irreducible* *polynomials $p(x) \in k[x]$. If H is an inner K-form of $G \times_k K$ that has good reduction at all $v \in V$ and satisfies $H \times_K k^{\mathrm{sep}}(x) \simeq G \times_k k^{\mathrm{sep}}(x)$, then $H \simeq H_0 \times_k K$ for some k-form H_0 of G.*

Proof. The k-group G is an inner twist of a quasi-split k-group G_0. In terms of proving the proposition, we can replace G by G_0, and hence assume that G itself is quasi-split. Let $\overline{G}$ be the corresponding adjoint group. Since G is quasi-split, there exists a k-defined finite subgroup $\Sigma \subset \mathrm{Aut}(G)$ such that $\mathrm{Aut}(G) = \overline{G} \rtimes \Sigma$. Let $v \in V$, and denote by $\mathcal{O}_v$ the valuation ring of the completion K_v. Set $\mathcal{G} = G \times_k \mathcal{O}_v$ and $\overline{\mathcal{G}} = \overline{G} \times_k \mathcal{O}_v$. We say that a cohomology class in $H^1(K, \overline{G})$ is *unramified* at $v \in V$ if its image under the restriction map $H^1(K, \overline{G}) \to H^1(K_v, \overline{G})$ belongs to $\mathrm{Im}(H^1_{\text{ét}}(\mathcal{O}_v, \overline{\mathcal{G}}) \to H^1(K_v, \overline{G}))$.

Suppose now that H is an inner twist of $G \times_k K$ that has good reduction at all $v \in V$. Fix a cocycle $\xi \in Z^1(K, \overline{G})$ such that $H = {}_{\xi}(G \times_k K)$. We will first show that

$$[\xi] \in \mathrm{Im}\, \beta_k, \quad \text{where} \quad \beta_k \colon H^1_{\text{ét}}(\mathbb{A}^1_k, \overline{G}) \to H^1(K, \overline{G}) \tag{15}$$

is the map induced by passage to the generic point. According to Proposition 2.6, it is enough to show that $[\xi]$ is unramified at all $v \in V$. So, fix $v \in V$. By our assumption, there exists a reductive group $\mathcal{O}_v$-scheme $\mathcal{H}$ with generic fiber H. This scheme is necessarily an inner form of $\mathcal{G}$, so we can find a cocycle $\xi' \in Z^1_{\text{ét}}(\mathcal{O}_v, \overline{\mathcal{G}})$ such that $\mathcal{H} = {}_{\xi'}\mathcal{G}$. Passing to the generic point, we obtain $H \times_K K_v \simeq {}_{\xi'}(G \times_k K_v)$. This means that for the image ξ_v of ξ under the restriction map $Z^1(K, \overline{G}) \to Z^1(K_v, \overline{G})$, the cohomology classes $[\xi_v]$ and $[\xi']$ have the same image in $H^1(K_v, \mathrm{Aut}(G))$. Thus, there exists $g \in \mathrm{Aut}(G)(K_v^{\mathrm{sep}})$ such that $\xi_v(\sigma) = g\xi'(\sigma)\sigma(g)^{-1}$ for all $\sigma \in \mathrm{Gal}(K_v^{\mathrm{sep}}/K_v)$. We can write $g = hs$ with $h \in \overline{G}(K_v^{\mathrm{sep}})$ and $s \in \Sigma(K_v^{\mathrm{sep}}) = \Sigma(k^{\mathrm{sep}})$, and then define $\xi'' \in Z^1(K_v, \overline{G})$ by $\xi''(\sigma) := s\xi'(\sigma)\sigma(s)^{-1}$. Clearly, $[\xi''] = [\xi_v]$ in $H^1(K_v, \overline{G})$, and by construction $[\xi'']$ lies in the image of $H^1_{\text{ét}}(\mathcal{O}_v, \overline{\mathcal{G}}) \to H^1(K_v, \overline{G})$. Thus, $[\xi]$ is unramified at v.

Now, using (15), pick $[\zeta] \in H^1_{\text{ét}}(\mathbb{A}^1_k, \overline{G})$ such that $[\xi] = \beta_k([\zeta])$. To prove the proposition, it is enough to show that $[\xi]$ is the image of some $[\xi_0] \in H^1(k, \overline{G})$, as then one can take $H_0 = {}_{\xi_0}G$. As we discussed above, this would follow from Theorem 7.2 if we could show that $[\zeta] \in \mathrm{Ker}\left(H^1_{\text{ét}}(\mathbb{A}^1_k, \overline{G}) \to H^1_{\text{ét}}(\mathbb{A}^1_{k^{\mathrm{sep}}}, \overline{G})\right)$. We have the following commutative diagram

$$\begin{array}{ccc} H^1_{\text{ét}}(\mathbb{A}^1_k, \overline{G}) & \xrightarrow{\beta_k} & H^1(K, \overline{G}) \\ {\scriptstyle\gamma_1}\downarrow & & \downarrow{\scriptstyle\gamma_2} \\ H^1_{\text{ét}}(\mathbb{A}^1_{k^{\mathrm{sep}}}, \overline{G}) & \xrightarrow{\beta_{k^{\mathrm{sep}}}} & H^1(k^{\mathrm{sep}}(x), \overline{G}) \end{array}.$$

The fact that $H \times_K k^{\mathrm{sep}}(x) \simeq G \times_K k^{\mathrm{sep}}(x)$ means that $[\xi] \in \mathrm{Ker}(H^1(K, \overline{G}) \to H^1(k^{\mathrm{sep}}(x), \mathrm{Aut}(\overline{G}))$. Since $\overline{G}$ splits over k^{sep}, the map $H^1(k^{\mathrm{sep}}(x), \overline{G}) \to H^1(k^{\mathrm{sep}}(x), \mathrm{Aut}(\overline{G}))$ has trivial kernel, and therefore we conclude that actually $[\xi] \in \mathrm{Ker}\, \gamma_2$. So, it follows from the diagram that in order to show that $[\zeta] \in \mathrm{Ker}\, \gamma_1$, it suffices to prove that Ker $\beta_{k^{\mathrm{sep}}}$ is trivial. According to [50], there is a bijection between Ker $\beta_{k^{\mathrm{sep}}}$ and the double coset space

$$\mathrm{cl}(\overline{G}, k^{\mathrm{sep}}(x), V^s) := \overline{G}(\mathbf{A}^{\infty}(V^s))\backslash\overline{G}(\mathbf{A}(V^s))/\overline{G}(k^{\mathrm{sep}}(x))$$

where V^s is the set of discrete valuations of $k^{\mathrm{sep}}(x)$ associated with the closed points of $\mathbb{A}^1_{k^{\mathrm{sep}}}$, with $\overline{G}(\mathbf{A}(V^s))$ and $\overline{G}(\mathbf{A}^{\infty}(V^s))$ denoting the group of rational adèles of $\overline{G}$ associated with V^s and its subgroup of integral adèles (cf. [17, §4]). Fix a maximal k-defined torus T of $\overline{G}$. Then T splits over k^{sep} and a standard argument using strong approximation for the opposite maximal unipotent subgroups associated with T (cf. [17]) shows that every double coset in $\mathrm{cl}(\overline{G}, k^{\mathrm{sep}}(x), V^s)$ has a representative in $T(\mathbf{A}(V^s))$. On the other hand, for the multiplicative group $S = \mathbb{G}_m$, the double coset space $\mathrm{cl}(T, k^{\mathrm{sep}}(x), V^s)$ can be identified with the Picard group of $\mathbb{A}^1_{k^{\mathrm{sep}}}$, which is trivial. Since T is k^{sep}-split, we obtain that $\mathrm{cl}(T, k^{\mathrm{sep}}(x), V^s)$ reduces to a single element. Thus, $\mathrm{cl}(\overline{G}, k^{\mathrm{sep}}(x), V^s)$ also reduces to a single element, and the injectivity of $\beta_{k^{\mathrm{sep}}}$ follows. This completes the proof of the proposition. □

It is now easy to complete the proof of Theorem 1.3. Let $H \in \mathbf{gen}_K(G \times_k K)$, where $K = k(x)$. Since $G \times_k K$ has good reduction at all $v \in V$, we see from Theorem 1.1 that the same is true for H. Let T be any maximal k-torus of G. Then H has a maximal K-torus isomorphic to $T \times_k K$, which splits over $k^{\mathrm{sep}}(x)$. Thus, both $G \times_k K$ and H split over $k^{\mathrm{sep}}(x)$, hence $G \times_k k^{\mathrm{sep}}(x) \simeq H \times_K k^{\mathrm{sep}}(x)$. Since H is an inner twist of $G \times_k K$, we can apply Proposition 7.3 to conclude that $H = H_0 \times_k K$ for some inner k-form H_0 of G. Let v be the valuation of K associated with x. Then the reductions of $G \times_k K$ and H at v coincide with G and H_0, respectively. Consequently, Theorem 1.1 yields $H_0 \in \mathbf{gen}_k(G)$. (In fact, applying this argument to all valuations, we see that $H_0 \times_k \ell \in \mathbf{gen}_\ell(G \times_k \ell)$ for *every* finite simple extension ℓ/k.)

8. Killing the genus by a purely transcendental extension

As in the previous section, we will first explain the phenomenon of "killing the genus" in the case of division algebras.

8.1. Killing the genus of a division algebra

It turns out that Proposition 7.1 can be significantly strengthened as follows.

Theorem 8.1. *Let D be a central division algebra of degree n over a field k, and assume that n is prime to* char k. *Set $K = k(x_1, \ldots, x_{n-1})$. Then $\mathbf{gen}(D \otimes_k K)$ consists of (the Brauer classes of) central division K-algebras of the form $D' \otimes_k K$, where D' is a central division k-algebra of degree n such that the classes $[D]$ and $[D']$ generate the same subgroup of* $\mathrm{Br}(k)$.

We already know from Proposition 7.1 and the subsequent remark that every element of $\mathbf{gen}(D \otimes_k K)$ is represented by a division algebra of the form $D' \otimes_k K$ for some central division algebra D' over k of degree n. In order to show that the classes $[D]$ and $[D']$ generate the same subgroup of $\mathrm{Br}(k)$, we will eventually use Amitsur's Theorem [2]. However, its application requires some preparation. We refer the reader to [74, Ch. 13] for basic facts about Severi-Brauer varieties and their function fields.

Lemma 8.2. *Let D be a central division algebra of degree n over a field k, and let F_D be the function field of the corresponding Severi-Brauer variety $SB(D)$. Then there exist elements $x_1, \ldots, x_{n-1} \in F_D$ that are algebraically independent over k and such that $F_D/k(x_1, \ldots, x_{n-1})$ is an extension of degree n.*

Proof. Let $W \subset D$ be a k-subspace of dimension m. If we fix a k-basis $w_1, \ldots, w_m \in W$, then there exists a homogeneous polynomial $\nu_W \in k[t_1, \ldots, t_m]$ such that

$$\nu_W(\alpha_1, \ldots, \alpha_m) = \mathrm{Nrd}_{D/K}(\alpha_1 w_1 + \cdots + \alpha_m w_m) \quad \text{for all} \quad \alpha_1, \ldots, \alpha_m \in k.$$

Set Z_W to be the subvariety of the projective space $\mathbb{P}(W)$ defined by the equation $\nu_W = 0$. It was shown by E. Matzri [44] that for a Zariski-dense set of subspaces W in the Grassmannian $\mathrm{Gr}(n+1, D)$, the variety Z_W is absolutely irreducible and birationally k-isomorphic to $SB(D)$. For the purpose of proving our lemma, we pick one such $(n+1)$-dimensional subspace $W \subset D$ and fix a basis $w_1, \ldots, w_{n+1}$. Pick two distinct indices $i, j \in \{1, \ldots, n+1\}$, set

$$p(T) = \nu_W(t_1, \ldots, t_{i-1}, T, t_{i+1}, \ldots, t_{j-1}, 1, t_{j+1}, \ldots, t_{n+1}),$$

and then re-denote the variables $t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_{j-1}, t_{j+1}, \ldots, t_{n+1}$ as $x_1, \ldots, x_{n-1}$. Then F_D is isomorphic to the extension of $k(x_1, \ldots, x_{n-1})$ obtained by adjoining a root of $p(T)$. On the other hand, $p(T)$ is irreducible over $k(x_1, \ldots, x_{n-1})$ and its leading term is $\mathrm{Nrd}_{D/K}(w_i)T^n$, demonstrating that $\deg p = n$ and completing the argument. □

Proof of Theorem 8.1. As in the lemma, we denote by F_D the function field of the Severi-Brauer variety $SB(D)$, and pick algebraically independent elements $x_1, \ldots, x_{n-1} \in F_D$ so that F_D is a degree n extension of $K = k(x_1, \ldots, x_{n-1})$. According to Amitsur's theorem [2], the kernel of the base change map $\mathrm{Br}(k) \to \mathrm{Br}(F_D)$ coincides with the cyclic subgroup $\langle [D] \rangle \subset \mathrm{Br}(k)$. In particular,

$$D \otimes_k F_D \simeq M_n(F_D) \simeq (D \otimes_k K) \otimes_K F_D.$$

Since $[F_D : K] = n$, the latter means that F_D is K-isomorphic to a maximal subfield of $D \otimes_k K$. By our assumption, $D \otimes_k K$ and $D' \otimes_k K$ have the same maximal subfields, so F_D admits a K-embedding as a maximal subfield into $D' \otimes_k K$. It follows that

$$D' \otimes_k F_D \simeq (D' \otimes_k K) \otimes_K F_D \simeq M_n(F_D).$$

Then Amitsur's theorem yields the inclusion $[D'] \in \langle [D] \rangle$. A symmetric argument shows that $[D] \in \langle [D'] \rangle$, which completes the argument. □

Thus, no matter what the genus $\mathbf{gen}(D)$ is originally, after a suitable purely transcendental base change K/k, the genus $\mathbf{gen}(D \otimes_k K)$ becomes finite, and in fact minimal possible. We call this phenomenon "killing the genus by a purely transcendental extension." Later in this section, we will prove Theorems 1.5 and 1.6 that reveal a similar phenomenon for the norm one groups $\mathrm{SL}_{1,A}$ of central simple algebras A and groups of type G_2, after which we will discuss possible generalizations. But first, we would like to continue our discussion of this phenomenon in the context of division algebras. As an immediate consequence of Theorem 8.1, we have

Corollary 8.3. *Let D be a quaternion division algebra over a field k of characteristic $\neq 2$, and let $K = k(x)$. Then*

$$\mathbf{gen}(D \otimes_k K) = \{[D \otimes_k K]\}.$$

The above proof of Theorem 8.1 for quaternions yields the following statement:

(•) *Let D_1 and D_2 be two central quaternion division algebras over a field k of characteristic $\neq 2$, and let $K = k(x)$. If $D_1 \otimes_k K$ and $D_2 \otimes_k K$ are in the same genus, then $D_1 \simeq D_2$ over k.*

It turns out that (•) remains valid if the field of rational functions $K = k(x)$ is replaced by the function field of any absolutely irreducible curve over k having a k-rational point.

Proposition 8.4. *Let D_1 and D_2 be two central quaternion division algebras over a field k of characteristic $\neq 2$, and let C be a smooth geometrically integral curve over k with $C(k) \neq \emptyset$. If for the function field $K = k(C)$, the algebras $D_1 \otimes_k K$ and $D_2 \otimes_k K$ are in the same genus (as K-algebras), then $D_1 \simeq D_2$ over k.*

(We note that since $C(k) \neq \emptyset$, the algebras $D_1 \otimes_k K$ and $D_2 \otimes_k K$ are division algebras over K. Indeed, let $P \in C(k)$. Since P is nonsingular, we can consider the corresponding valuation v of K, and then the completion K_v can be identified with the field $k((t))$ of formal Laurent series. Then the algebras $D_1 \otimes_k K_v$ and $D_2 \otimes_k K_v$ are obviously division algebras, so the algebras $D_1 \otimes_k K$ and $D_2 \otimes_k K$ are also division algebras.)

Proof. Without loss of generality, we may assume that C is projective. Fix a rational point $P \in C(k)$, and consider the divisor $\Delta_n = nP$ on C for $n > 0$. It follows from the Riemann-Roch Theorem (see, for example, [76] for the statement and relevant notations) that the dimension $\ell(\Delta_n) = \dim_k L(\Delta_n)$ of the space $L(\Delta_n)$ associated with Δ_n for $n \gg 0$ is given by

$$\ell(\Delta_n) = n + 1 - g,$$

where g is the genus of C. Thus, we can find an *odd* $n \geq 1$ prime to char k such that there exists $f \in L(\Delta_n) \setminus L(\Delta_{n-1})$. Then the divisor of poles of the principal divisor (f) is Δ_n, hence has degree precisely n. Thinking of f as a morphism $C \to \mathbb{P}^1_k$, we conclude that the degree of this map is n, which means that K is a degree n extension of the field of rational functions $k(x)$. On the other hand, the function field F_{D_1} of the Severi-Brauer variety of D_1 can be viewed as a quadratic extension of $k(x)$. (More precisely, K and F_{D_1} can be embedded into an algebraic closure of the field $k(x)$ so that the images of these embeddings, for which we keep the same notations, have degrees n and 2 over $k(x)$, respectively.) We have

$$D_1 \otimes_k F_{D_1} \simeq M_2(F_{D_1}) \simeq (D_1 \otimes_k k(x)) \otimes_{k(x)} F_{D_1},$$

implying that F_{D_1} admits a $k(x)$-embedding into $D_1 \otimes_k k(x)$ as a maximal subfield (just as in the proof of Theorem 8.1). Then the composition $F_{D_1}K \simeq F_{D_1} \otimes_{k(x)} K$ admits a K-embedding into $(D_1 \otimes_k k(x)) \otimes_{k(x)} K \simeq D_1 \otimes_k K$ as a maximal subfield. By our assumption, the algebras $D_1 \otimes_k K$ and $D_2 \otimes_k K$ are in the same genus, so there is a K-embedding $F_{D_1}K \hookrightarrow D_2 \otimes_{k(x)} K$. It follows that

$$(D_2 \otimes_k F_{D_1}) \otimes_{F_{D_1}} F_{D_1}K \simeq D_2 \otimes_k F_{D_1}K \simeq (D_2 \otimes_k K) \otimes_K F_{D_1}K \simeq M_2(F_{D_1}K).$$

Thus, the degree n extension $F_{D_1}K/F_{D_1}$ splits the algebra $D_2 \otimes_k F_{D_1}$. Since n is odd, we conclude that $D_2 \otimes_k F_{D_1} \simeq M_2(F_{D_1})$. By Amitsur's theorem, this means that the quaternion division algebras D_1 and D_2 are isomorphic. $\square$

Remark 8.5. The assumption in Proposition 8.4 that C has a k-rational point cannot be omitted. Indeed, let D_1 and D_2 be two nonisomorphic quaternion division algebras having a common subfield (e.g., one can take $D_1 = (-1, 3)$ and $D_2 = (-1, 7)$ over $k = \mathbb{Q}$). Then $D_1 \otimes_K D_2 \simeq M_2(D)$ for the quaternion division algebra $D = (-1, 21)$. Let C be the Severi-Brauer variety for D (which is a conic without a rational point), and $K = k(C)$. Since K splits D, the K-algebras $D_1 \otimes_k K$ and $D_2 \otimes_k K$ are isomorphic, hence belong to the same genus. However, by construction, D_1 and D_2 are not isomorphic as k-algebras.

The result established in Corollary 8.3 prompts the following

Question 8.6. Does there exist a central quaternion division algebra D over the field of rational functions $K = k(x)$ over some field k having *nontrivial* genus $\mathbf{gen}(D)$?

8.2. Proof of Theorem 1.5

While the argument involves some of the same considerations as the proof of Theorem 8.1, it also contains several new elements. In order to apply Amitsur's theorem, we need to match certain maximal subfields of the algebras obtained by a purely transcendental base change, and not just the corresponding maximal tori in the associated norm one groups. It is well-known that given a central division algebra D of degree n over a field k, every maximal k-torus T of $G = \mathrm{SL}_{1,D}$ is the norm torus $\mathrm{R}^{(1)}_{F/k}(\mathbb{G}_m)$ for some maximal separable subfield $F \subset D$. The problem is that in general, given two separable degree n extensions F_1 and F_2 of K, the fact that the corresponding norm tori are K-isomorphic, may *not* imply that the extensions are isomorphic.[4] However, as the following lemma shows, this complication does not arise in the case of generic tori. We recall that a field extension F/k of degree n is called *generic* if it is separable and for its normal closure $\tilde{F}$, the Galois group $\mathrm{Gal}(\tilde{F}/k)$ is isomorphic to the symmetric group S_n.

Lemma 8.7. *Let F_1 and F_2 be two degree n extensions of a field k, and let $T_i = \mathrm{R}^{(1)}_{F_i/k}(\mathbb{G}_m)$ $(i = 1, 2)$ be the corresponding norm tori. If at least one of the extensions F_i is generic over k and $T_1 \simeq T_2$ as k-tori, then $F_1 \simeq F_2$ over k.*

Proof. It is well-known that the minimal splitting field of T_i is the normal closure of F_i over K, which we will denote by $\tilde{F}_i$. Since T_1 and T_2 are k-isomorphic, we have $\tilde{F}_1 = \tilde{F}_2 =: \tilde{F}$, and then by our assumption, the Galois group $G = \mathrm{Gal}(\tilde{F}/K)$ is isomorphic to S_n. Let $H_i = \mathrm{Gal}(\tilde{F}/F_i)$. To prove that $F_1 \simeq F_2$, it is enough to show that the subgroups H_1 and H_2 are conjugate in G. When $n \neq 6$, this follows from the elementary fact in group theory that, in this case, S_n has only one conjugacy class of subgroups of index n; in other words, every subgroup of index n is the stabilizer of some point — see [33, Kapitel II, Satz 5.5].

For $n = 6$, it is well-known that the group of outer automorphisms of G has order 2. Furthermore, if σ is an outer automorphism of G and $H \subset G$ is a fixed subgroup of index n, then it follows from [33, Kapitel II, Satz 5.5] that any subgroup of index n is conjugate to either H or $\sigma(H)$. To prove that H_1 and H_2 are conjugate in this case as well, we observe that a k-isomorphism between T_1 and T_2 yields an isomorphism of the character groups $X(T_1) \simeq X(T_2)$ as $\mathbb{Z}[G]$-modules, hence an isomorphism of the vector spaces $W_1 = X(T_1) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $W_2 = X(T_2) \otimes_{\mathbb{Z}} \mathbb{Q}$ as $\mathbb{Q}[G]$-modules. The representation of G afforded by W_i can be described as the permutation representation on $\mathbb{Q}[G/H_i]$ "minus" the trivial representation (we will refer to this representation as the *standard representation* associated with the subgroup H_i). So, to complete the argument, it remains to show that the standard representations $\rho\colon G \to \mathrm{GL}(W)$ and $\rho'\colon G \to$

[4] To construct such an example, it is enough to find a finite group G having two nonconjugate subgroups H_1 and H_2 such that the permutation lattices $\mathbb{Z}[G/H_1]$ and $\mathbb{Z}[G/H_2]$ are isomorphic as $\mathbb{Z}[G]$-modules. We learned from correspondence with R. Guralnick and D. Saltman that the following example of this situation was found by L. Scott [75]: $G = \mathrm{PSL}_2(F_{29})$ and H_1, H_2 are nonconjugate subgroups isomorphic to A_5.

$\mathrm{GL}(W')$ associated with H and $\sigma(H)$ are *not* equivalent. Since $\rho' = \rho \circ \sigma$, this follows from the explicit description of the character of the standard representation and the fact that σ switches the conjugacy classes of 3-cycles and the products of two disjoint 3-cycles. □

Next, we need the following strengthening of Lemma 8.2 that includes the genericity condition.

Proposition 8.8. *Let A be a central simple algebra of degree n over a finitely generated field k, and let F_A be the function field of the Severi-Brauer variety $SB(A)$. Then there exist elements $x_1, \ldots, x_{n-1} \in F_A$ that are algebraically independent over k and such that $F_A/k(x_1, \ldots, x_{n-1})$ is a* generic *field extension of degree n.*

Before giving the proof, we first discuss the following auxiliary construction. Let F/k be a separable field extension of degree n. Fix a basis $\omega_1 = 1, \omega_2, \ldots, \omega_n$ of F over k, and let $t_1, \ldots, t_n$ be variables. Set

$$\varphi_{F/k}(t_1, \ldots, t_n) = \prod_{\sigma} (\sigma(\omega_1)t_1 + \cdots + \sigma(\omega_n)t_n), \tag{16}$$

where the product is taken over all distinct embeddings $\sigma \colon F \hookrightarrow \overline{k}$. Clearly, $\varphi_{F/k}$ is a homogeneous polynomial of degree n in $t_1, \ldots, t_n$ with coefficients in k. When k is infinite, $\varphi_{F/k}$ is uniquely characterized by the condition

$$\varphi_{F/k}(\alpha_1, \ldots, \alpha_n) = \mathrm{N}_{F/k}(\alpha_1\omega_1 + \cdots + \alpha_n\omega_n) \quad \text{for all} \quad \alpha_1, \ldots, \alpha_n \in k. \tag{17}$$

Consider the polynomial $f_{F/k}(T) = \varphi_{F/k}(T, t_2, \ldots, t_{n-1}, 1)$ over $L := k(t_2, \ldots, t_{n-1})$.

Lemma 8.9. *Keeping the preceding notations, let $\tilde{F}$ be the normal closure of F over k. Then the splitting field E of $f_{F/k}(T)$ over L coincides with $\tilde{F}L = \tilde{F}(t_2, \ldots, t_{n-1})$, and therefore $\mathrm{Gal}(E/L) \simeq \mathrm{Gal}(\tilde{F}/k)$.*

Indeed, it follows from (16) that

$$f_{F/k}(T) = \prod_{\sigma} (T + (\sigma(\omega_2)t_2 + \cdots + \sigma(\omega_{n-1})t_{n-1} + \sigma(\omega_n))).$$

This shows that $E \subset \tilde{F}L$. On the other hand, if $\tau \in \mathrm{Gal}(\tilde{F}L/L) = \mathrm{Gal}(\tilde{F}/k)$ fixes the element $(\sigma(\omega_2)t_2 + \cdots + \sigma(\omega_{n-1})t_{n-1} + \sigma(\omega_n))$, then it fixes all the elements $\sigma(\omega_2), \ldots, \sigma(\omega_n)$. It follows that if τ fixes E then $\tau = \mathrm{id}$. So, $E = \tilde{F}L$, hence $\mathrm{Gal}(E/L) = \mathrm{Gal}(\tilde{F}/k)$.

Proof of Proposition 8.8. The argument is a refinement of the proof of Lemma 8.2, and we will freely use the notations introduced therein. Recall that the key point in realizing F_D as a degree n extension of the rational function field $F = k(x_1, \ldots, x_{n-1})$ was the fact that $SB(D)$ is birationally isomorphic to Z_W for a suitable choice of an $(n+1)$-dimensional subspace $W \subset D$. While this fact remains valid without any changes for any central simple algebra A, in order to ensure that the extension F_A/F is generic, we need to specialize the choice of W. First, since k is finitely generated, A contains a maximal subfield P that is a generic extension of k (this follows immediately, for example, from Theorem 3.1). It was shown by Saltman [73, 4.2(c)], [74, 13.28] that for a Zariski-dense set of $a \in A$, the space $W := P + ka$ is $(n+1)$-dimensional and the corresponding variety Z_W is birationally isomorphic to $SB(A)$. Fix one such a. Pick a basis $w_1 = 1, \dots, w_n$ of P/k; then $w_1, \dots, w_n, w_{n+1} := a$ is a basis of W. Take $i = 1$, $j = n$ as in the proof of Lemma 8.2 and consider the corresponding polynomial

$$p_W(T) = \nu_W(T, x_1, \dots, x_{n-2}, 1, x_{n-1}),$$

noting that p_W is monic and has coefficients in the ring $R := k[x_1, \dots, x_{n-1}]$. As in Lemma 8.2, the polynomial $p_W(T)$ is irreducible over $F = k(x_1, \dots, x_{n-1})$ and the extension F_A/F is obtained by adjoining a root of p_W. In order to prove that the extension is generic, we will use specialization. Let E be the splitting field of p_W and $G = \mathrm{Gal}(E/F)$ be the corresponding Galois group. Furthermore, let S be the integral closure of R in E, and let $\mathfrak{p}$ be the prime ideal of R generated by x_{n-1}. Since the restriction of the reduced norm map $\mathrm{Nrd}_{A/k}$ to P coincides with the usual norm map $\mathrm{N}_{P/k}$, we see from (17) that $\nu_W(t_1, \dots, t_n, t_{n+1}) (\mathrm{mod}\, t_{n+1})$ coincides with $\varphi_{P/k}(t_1, \dots, t_n)$, from which it follows that $p_W(T) (\mathrm{mod}\, \mathfrak{p})$ coincides with $f_{P/k}(T)$. In particular, since $f_{P/k}(T)$ is separable, so is $p_W(T)$. Fix a prime ideal $\mathfrak{P} \subset S$ lying above $\mathfrak{p}$, and let $G(\mathfrak{P})$ be its decomposition group. Then according to [41, Ch. VII, Proposition 2.5], there is a natural surjective homomorphism of $G(\mathfrak{P})$ to the automorphism group H of the field of fractions of $S/\mathfrak{P}$ over L (which is the field of fractions of $R/\mathfrak{p}$). On the other hand, it follows from our construction and Lemma 8.9 that the Galois group of the splitting field of $f_{P/k}(T)$ is the symmetric group S_n, so H admits a surjection onto S_n. Thus, a subgroup of G admits a surjection onto S_n, and therefore $|G| \geqslant n!$. However, G is the Galois group of the splitting field of a separable polynomial of degree n, hence must be isomorphic to a subgroup of S_n. Thus, $G \simeq S_n$, as required. □

It would be interesting to determine if the conclusion of the proposition remains valid without assuming that k is finitely generated.

We can now complete the proof of Theorem 1.5 by imitating the proof of Theorem 8.1. So, let $G = \mathrm{SL}_{1,A}$, where A is a central simple algebra of degree n over a finitely generated field k. Set $F = k(x_1, \dots, x_{n-1})$, and suppose that $G' \in \mathbf{gen}_F(G \times_k F)$. Using Theorem 1.3 repeatedly, we see that $G' = H \times_k F$ for some $H \in \mathbf{gen}_k(G)$. Since H is an inner twist of G, we have $H = \mathrm{SL}_{1,B}$ for some central simple algebra B of degree n over k. It remains to show that the classes $[A], [B] \in \mathrm{Br}(k)$ generate the same subgroup.

Using Proposition 8.8, we present the function field F_A of the Severi-Brauer variety $SB(A)$ as a degree n generic extension of F, and then arguing as in the proof of Theorem 8.1, we conclude that F_A is F-isomorphic to a maximal étale subalgebra of $A \otimes_k F$. Let $T = \mathrm{R}^{(1)}_{F_A/F}(\mathbb{G}_m)$ be the corresponding maximal F-torus of $G \times_k F$. By our assumption, $H \times_k F \in \mathbf{gen}_F(G \times_k F)$, so T is F-isomorphic to a maximal F-torus T' of G'; the latter is the norm torus $\mathrm{R}^{(1)}_{E/F}(\mathbb{G}_m)$ for some maximal étale subalgebra E of $B \otimes_k F$, which in fact is a field extension as T' is F-anisotropic. Since the field extension F_A/F is generic by construction, we can use Lemma 8.7 to conclude that F_A is F-isomorphic to E; in other words, F_A admits an F-embedding into $B \otimes_k F$. As in the proof of Theorem 8.1, we observe that then F_A splits B, so invoking Amitsur's Theorem, we see that $[B] \in \langle [A] \rangle$. The inclusion $[A] \in \langle [B] \rangle$ is established by a symmetric argument. $\square$

8.3. Proof of Theorem 1.6

We recall that an algebraic group G of type G_2 over a field k of characteristic $\neq 2$ can be realized as the automorphism group of an octonian algebra $\mathbb{O}(a,b,c)$ corresponding to a triple $(a,b,c) \in k^\times \times k^\times \times k^\times$. The norm form q of $\mathbb{O}(a,b,c)$ is the Pfister form $\ll a,b,c \gg$ in standard notations; we will write it as

$$q(x_0, x_1, \dots, x_7) = x_0^2 + q'(x_1, \dots, x_7) \quad \text{where} \quad q'(x_1, \dots, x_7) = -ax_1^2 - bx_2^2 + \cdots .$$

The following facts are well-known:

(1) for a field extension F/k, the group G is either split or anisotropic over F, cf. [79];
(2) Two K-groups G_1 and G_2 of type G_2 with associated norm forms q_1 and q_2 are F-isomorphic if and only if q_1 and q_2 are equivalent over F, cf. [38, Proposition 33.19];
(3) G is split over F if and only if q is hyperbolic (equivalently, isotropic) over F — this follows from (2).

It is enough to show that if G_1 and G_2 are two k-groups of type G_2 such that for $P := k(x_1, \dots, x_6)$ the groups $\mathcal{G}_1 := G_1 \times_k P$ and $\mathcal{G}_2 := G_2 \times_k P$ are in the same genus, then $G_1 \simeq G_2$ over k. We may assume that G_1 and G_2 are anisotropic over k, and let q_1 and q_2 be the corresponding norm forms. Set

$$L = k(x_1, \dots, x_6)\left(\sqrt{-q_1'(x_1, \dots, x_6, 1)}\right) = P\left(\sqrt{-q_1'(x_1, \dots, x_6, 1)}\right)$$

(the "homogeneous function field" of q_1 in the terminology of [40]). Then q_1 represents zero over L, so G_1 splits over L. A standard argument shows that $\mathcal{G}_1$ contains a maximal P-torus $\mathcal{T}$ of the form $\mathcal{T} = \mathrm{R}^{(1)}_{L/P}(\mathbb{G}_m) \times \mathrm{R}^{(1)}_{L/P}(\mathbb{G}_m)$ cf. ([54, Lemma 6.17]). By our assumption, $\mathcal{G}_1$ and $\mathcal{G}_2$ are in the same genus, and in particular, $\mathcal{T}$ is P-isomorphic to a maximal P-torus of $\mathcal{G}_2$, implying that $\mathcal{G}_2$ becomes split over L. Thus, the 3-Pfister form q_2 becomes split over the function field of the 3-Pfister form q_1, and therefore the forms

q_1 and q_2 are equivalent over k (cf. [40, Ch. X, Corollary 4.10]). So, the groups G_1 and G_2 are k-isomorphic, as required.

8.4. Motivic genus

The following variation of the notion of the genus, proposed by A.S. Merkurjev, provides a different perspective on the above results. He defined the *motivic genus* $\mathbf{gen}_m(G)$ of an absolutely almost simple algebraic k-group G to be the set of k-isomorphism classes of (inner) k-forms G' of G such that $G' \times_k F \in \mathbf{gen}_F(G \times_k F)$ for *all* field extensions F/k. Then Theorem 1.5 implies that for $G = \mathrm{SL}_{1,A}$, where A is a central simple algebra of degree n, the motivic genus is always finite of size $\leq n-1$, and reduces to a single element if A has exponent two. In addition, by Theorem 1.6, the motivic genus of a group of type G_2 also reduces to a single element. Furthermore, according to a result of Izhboldin [32], for given non-degenerate quadratic forms q and q' of *odd* dimension over a field k of characteristic $\neq 2$, the condition

(†) q and q' have the same Witt index over any extension F/k

implies that q and q' are scalar multiples of each other (this conclusion being false for even-dimensional forms). It follows that $|\mathbf{gen}_m(G)| = 1$ for $G = \mathrm{Spin}_n(q)$ with n odd. We note that condition (†) is equivalent to the fact that the motives of q and q' in the category of Chow motives are isomorphic (cf. Vishik [81], [82, Theorem 4.18], and also Karpenko [35]), which prompted the choice of terminology for this version of the genus. One can expect the motivic genus to be finite for all absolutely almost simple groups, of size bounded by a constant depending only on the type of the group (at least over fields of "good" characteristic, but not necessarily finitely generated). On the other hand, Conjecture 1.7 asserts that the genus gets reduced to the motivic genus (i.e., becomes as small as possible) after a suitable purely transcendental extension of the base field.

9. Weakly commensurable Zariski-dense subgroups

The goal of this section is to prove Theorem 1.8 that relates the presence of a finitely generated Zariski-dense subgroup weakly commensurable to a given one with good reduction. First, let us fix a model $\mathfrak{X} = \operatorname{Spec} A$ for k, i.e. an affine integral normal scheme of finite type over $\mathbb{Z}$ with function field k, and let V denote the set of discrete valuations of k associated with the prime divisors on $\mathfrak{X}$. We will consider separately the two cases where $\dim \mathfrak{X} = 1$ and $\dim \mathfrak{X} > 1$, respectively.

9.1. Proof of Theorem 1.8 in the case $\dim \mathfrak{X} = 1$

In this case, k is a number field (recall that $\operatorname{char} k = 0$), and V consists of almost all nonarchimedean valuations of k. The proof here is an adaptation of the argument developed in [60, §5] for a different, although related, purpose. By [54, Theorem 6.7], one can find a finite subset $S_1 \subset V$ such that G is quasi-split over the completion k_v for all $v \in V \setminus S_1$. Furthermore, let ℓ/k be the minimal Galois extension over which G becomes an inner form of the split group, and choose a finite subset $S_2 \subset V$ so that ℓ/k is unramified at all $v \in V \setminus S_2$. Finally, since k coincides with the trace field k_Γ, it follows from the Strong Approximation Theorem of Weisfeiler [85] that there exists a finite subset $S_3 \subset V$ such that the closure of Γ in $G(k_v)$ in the v-adic topology is open for all $v \in V \setminus S_3$. Set $S(\Gamma) = S_1 \cup S_2 \cup S_3$. The fact that this set is as required in Theorem 1.8 is an immediate consequence of the following.

Proposition 9.1. *Let G' be an absolutely almost simple k-group such that there exists a finitely generated Zariski-dense subgroup $\Gamma' \subset G'(k)$ that is weakly commensurable to Γ. Then G' is quasi-split over k_v, and consequently has good reduction, for all $v \in V \setminus S(\Gamma)$.*

Proof. Let ℓ' be the minimal Galois extension of k over which G' becomes an inner form of the split group. By Theorem 3.8, the existence of Γ' that is weakly commensurable to Γ implies that either G and G' have the same type or one of them is of type B_ℓ and the other of type C_ℓ, and also that $\ell = \ell'$. The latter means that when G and G' are of the same type, then the corresponding adjoint groups $\overline{G}'$ and $\overline{G}$ are inner twists of each other over k, hence over k_v. In order to prove that G' is quasi-split over k_v, it is enough to show that

$$\mathrm{rk}_{k_v}\, G' \geqslant \mathrm{rk}_{k_v}\, G. \tag{18}$$

Indeed, when one of the groups is of type B_ℓ and the other of type C_ℓ, we see that G is k_v-split, and then the inequality shows that G' is also k_v-split. Next, suppose that G and G' are of the same type. As we pointed out above, $\overline{G}'$ is an inner twist of $\overline{G}$, and therefore the $*$-actions of the absolute Galois group of k_v on the Tits indices of G and G' are identical (cf. [58, Lemma 4.1(a)]). It is well-known that the relative rank of a semisimple group equals the number of distinguished orbits under the $*$-action on its Tits index. By our construction, G is quasi-split over k_v, so all $*$-orbits are distinguished. Then (18) implies that all $*$-orbits in the Tits index of G' are also distinguished, so G' is k_v-quasi-split.

Now, to prove (18), we first use Theorem 3.7 to find a regular semisimple element γ so that the corresponding torus $T = C_G(\gamma)^\circ$ is generic over k and contains a maximal k_v-split torus of G over k_v. By our assumption, γ is weakly commensurable to some semisimple element $\gamma' \in \Gamma'$ of infinite order. Let T' be a maximal k-torus of G' containing γ'. Then it follows from Proposition 3.10 that there exists a k-defined isogeny $T \to T'$. Since $\mathrm{rk}_{k_v}\, T = \mathrm{rk}_{k_v}\, G$ by construction, the required inequality (18) follows. Thus, G' is quasi-split over k_v. Since $\ell' = \ell$, we obtain that ℓ' is unramified at v, and therefore G' has good reduction at v (cf. [34, Corollary 7.9.4]). □

9.2. Zariski-density of reductions when $\dim \mathfrak{X} > 1$

In this case, the residue fields $k^{(v)}$ of the valuations $v \in V$ are *infinite* finitely generated fields. The goal of this subsection is to establish a result about the Zariski-density of the reductions of Γ modulo almost all $v \in V$ that will play a crucial role in the proof of Theorem 1.8. Fix a faithful k-defined representation $G \hookrightarrow \mathrm{GL}_n$. By shrinking V, we may assume without loss of generality that for all $v \in V$, the reduction $\underline{G}^{(v)}$ associated with this realization is a connected absolutely almost simple algebraic group over the residue field $k^{(v)}$. We denote by $\rho_v \colon G(\mathcal{O}_{k,v}) \to \underline{G}^{(v)}(k^{(v)})$ the corresponding reduction map (where $\mathcal{O}_{k,v}$ is the valuation ring of v in k).

Proposition 9.2. *Let $\Gamma \subset G(k)$ be a finitely generated Zariski-dense subgroup. Then for almost all $v \in V$, we have the inclusion $\Gamma \subset G(\mathcal{O}_v)$ and the reduction $\Gamma^{(v)} := \rho_v(\Gamma)$ is Zariski-dense in $\underline{G}^{(v)}$.*

Proof. The first assertion is obvious. To prove the second one, we observe that since G is absolutely almost simple and k has characteristic zero, the adjoint representation $r \colon G \to \mathrm{GL}(\mathfrak{g})$ on the Lie algebra $\mathfrak{g} = L(G)$ is (absolutely) irreducible. Being Zariski-dense in G, the subgroup Γ also acts on $\mathfrak{g}$ absolutely irreducibly. Then by Burnside's Lemma (cf. [39, 7.3]), the image $r(\Gamma)$ spans $\mathrm{End}(\mathfrak{g})$. Fix a basis $a_1, \ldots, a_m$ ($m = \dim \mathfrak{g}$) of $\mathfrak{g}(k) \subset M_n(k)$ over k, and let e_{ij} ($i, j = 1, \ldots, m$) be the corresponding standard basis of $\mathrm{End}_k(\mathfrak{g}(k))$. Then we can find elements $\gamma_1, \ldots, \gamma_d \in \Gamma$ such that for all $i, j = 1, \ldots, m$ there are expressions

$$e_{ij} = \sum_{\ell=1}^{d} \alpha_{ij}^{\ell} r(\gamma_\ell) \quad \text{with} \quad \alpha_{ij}^{\ell} \in k. \tag{19}$$

Then for almost all $v \in V$, the following properties hold:

- the elements $a_1, \ldots, a_m$ belong to $\mathfrak{g}(\mathcal{O}_{k,v}) = \mathfrak{g}(k) \cap M_n(\mathcal{O}_{k,v})$ and their reductions $\bar{a}_1, \ldots, \bar{a}_m \in M_n(K^{(v)})$ form a $k^{(v)}$-basis of $\underline{\mathfrak{g}}^{(v)}(k^{(v)})$, where $\underline{\mathfrak{g}}^{(v)}$ is the Lie algebra of the reduction $\underline{G}^{(v)}$;
- all coefficients α_{ij}^{ℓ} belong to $\mathcal{O}_{k,v}$.

We note that for any such v, the endomorphisms e_{ij} leave $\mathfrak{g}(\mathcal{O}_{k,v})$ invariant, and their reductions $\bar{e}_{ij}$ form the standard basis of $\mathrm{End}_{k^{(v)}}(\underline{\mathfrak{g}}^{(v)}(k^{(v)}))$ associated with the basis $\bar{a}_1, \ldots, \bar{a}_m$. Reducing (19), we obtain the relations

$$\bar{e}_{ij} = \sum_{\ell=1}^{d} \bar{\alpha}_{ij}^{\ell} r(\rho_v(\gamma_\ell)),$$

where $\bar{\alpha}_{ij}^{\ell}$ denotes the image of α_{ij}^{ℓ} in $k^{(v)}$. These relations show that $\Gamma^{(v)}$ acts on $\underline{\mathfrak{g}}^{(v)}$ absolutely irreducibly. Letting H denote the Zariski-closure of $\Gamma^{(v)}$ in $\underline{G}^{(v)}$, we observe that the Lie algebra $L(H)$ is a $\Gamma^{(v)}$-invariant subspace of $\underline{\mathfrak{g}}^{(v)}$, and therefore there are only two possibilities: $L(H) = \underline{\mathfrak{g}}^{(v)}$ or $L(H) = \{0\}$. In the first case, $H = \underline{G}^{(v)}$, i.e. $\Gamma^{(v)}$ is Zariski-dense, as desired. In the second case, H, hence $\Gamma^{(v)}$, is finite. So, to complete the proof of the proposition we need to show that $\Gamma^{(v)}$ is *infinite* for almost all v. For this, we will consider two cases.

CASE 1: char $k^{(v)} = 0$. It follows from Jordan's Theorem (cf. [18]) that there exists an integer $j > 0$ (depending on n) such that every finite subgroup $\Phi \subset \mathrm{GL}_n(F)$, where F is any field of characteristic zero, contains an abelian normal subgroup of index dividing j, and then the commutator subgroup $[\Phi^{(j)}, \Phi^{(j)}]$ of the subgroup $\Phi^{(j)}$ generated by the jth powers is trivial. Now, since Γ is Zariski-dense in $G \subset \mathrm{GL}_n$, it follows that $\Delta := [\Gamma^{(j)}, \Gamma^{(j)}]$ is also Zariski-dense. In particular, we can find a *nontrivial* element $\delta \in \Delta$; then for almost all v the reduction $\rho_v(\delta)$ is nontrivial. Then the group $\Gamma^{(v)}$ must be *infinite*. Indeed, otherwise Jordan's Theorem would yield that

$$[(\Gamma^{(v)})^{(j)}, (\Gamma^{(v)})^{(j)}] = \rho_v(\Delta)$$

is trivial, which is not the case by our construction.

CASE 2: char $k^{(v)} > 0$. We will show that there exists $\gamma \in \Gamma$ such that $\rho_v(\gamma)$ has *infinite* order for almost all v at hand. For this, it is enough to make sure that the trace $\mathrm{tr}\,(r(\rho_v(\gamma)))$ is not a sum of roots of unity. Let k_0 be the algebraic closure of $\mathbb{Q}$ in k; we note that $[k_0 : \mathbb{Q}] < \infty$ as k is finitely generated, and $k \neq k_0$ since $\dim \mathfrak{X} > 1$. So, as k is the trace field of Γ, we can find $\gamma \in \Gamma$ so that $f := \mathrm{tr}\,(r(\gamma)) \notin k_0$. We will show that this γ is as required. First, we observe that for any v with char $k^{(v)} > 0$, the restriction v_0 of v to k_0 is nontrivial, and the residue field $k_0^{(v_0)}$ embeds into $k^{(v)}$. Furthermore, the residue $\bar{f}$ coincides with $\mathrm{tr}\,(r(\rho_v(\gamma)))$. Now, it is enough to show that $\bar{f}$ is not algebraic over $k_0^{(v_0)}$, which follows from the fact that the following two properties hold for almost all v:

(1) $k_0^{(v_0)}$ is algebraically closed in $k^{(v)}$;
(2) $\bar{f} \notin k_0^{(v_0)}$.

The proof is based on the following well-known fact: *Let X be an irreducible algebraic variety over a field K; then X is absolutely irreducible if and only if K is algebraically closed in the field of rational functions $K(X)$* (see, for example, [28, Proposition 5.50]). To apply this fact in our situation, we observe that since the model $\mathfrak{X} = \mathrm{Spec}\, A$ is normal, it can be viewed as a scheme over the ring of integers $\mathcal{O}_0$ of k_0. Besides, the k_0-variety $X = \mathfrak{X} \times_{\mathcal{O}_0} k_0$ is absolutely irreducible. Then it follows from the classical theorem of Bertini-Noether (cf. [24, Proposition 10.4.2]) that for almost all v, the reduction

$$\underline{\mathfrak{X}}^{(v_0)} = \mathfrak{X} \otimes_{\mathcal{O}_0} k^{(v_0)}$$

is an absolutely irreducible variety over $k_0^{(v_0)}$ whose field of rational functions coincides with $k^{(v)}$. Applying the statement mentioned above, we obtain property (1). Furthermore, since f is not constant on X, we can find two points $x_1, x_2 \in X(\overline{k_0})$ such that $f(x_1) \neq f(x_2)$. Then for almost all v, the points admit the reductions $\bar{x}_1$, $\bar{x}_2$ with respect to an extension of v_0, and for the reduction $\bar{f} \in k^{(v)}$, we have $\bar{f}(\bar{x}_1) \neq \bar{f}(\bar{x}_2)$. This means that $\bar{f} \notin k_0^{(v_0)}$, verifying property (2) and completing the argument. □

9.3. Proof of Theorem 1.8 in the case $\dim \mathfrak{X} > 1$

It follows from Proposition 9.2 that there exists a finite subset $S(\Gamma) \subset V$ such that for any $v \in V \setminus S(\Gamma)$ the following two conditions hold:

(a) G has good reduction at v, so that the $\underline{G}^{(v)}$ is a connected absolutely almost simple group;
(b) $\Gamma \subset G(\mathcal{O}_{k,v})$, and for the reduction map $\rho_v \colon G(\mathcal{O}_{k,v}) \to \underline{G}^{(v)}(k^{(v)})$, the image $\Gamma^{(v)} = \rho_v(\Gamma)$ is Zariski-dense in $\underline{G}^{(v)}$.

Suppose now that G' is an absolutely almost simple algebraic k-group such that there exists a finitely generated Zariski-dense subgroup $\Gamma' \subset G'(k)$ that is weakly commensurable to Γ. As before, we denote by ℓ (resp., ℓ') the minimal Galois extension of k over which G (resp., G') becomes an inner form of the split group. By Theorem 3.8, the Weyl groups of G and G' have the same order w and $\ell = \ell'$. Fix an extension u of v to ℓ; it follows from (a) that the extension ℓ/k is unramified at v.

Since $\Gamma^{(v)}$ is (finitely generated and) Zariski-dense in $\underline{G}^{(v)}$, by Theorem 3.6 there exists a regular semisimple element $\bar{\gamma} \in \Gamma^{(v)}$ that is generic over $\ell^{(u)}$. Write $\bar{\gamma} = \rho_v(\gamma)$ with $\gamma \in \Gamma$.

Lemma 9.3. (1) γ *is a regular semisimple element of infinite order.*
(2) *The maximal k-torus $T = C_G(\gamma)^\circ$ is generic over k_v and the extension k_T/k is unramified at v.*

Proof. (1): Let $c(t)$ be the characteristic polynomial of $\mathrm{Ad}\,\gamma$. Then its reduction $\bar{c}(t)$ is the characteristic polynomial of $\mathrm{Ad}\,\bar{\gamma}$. Since $\bar{\gamma}$ is regular semisimple, the multiplicity of 1 as a root of $\bar{c}(t)$ equals $r = \mathrm{rk}\,\underline{G}^{(v)} = \mathrm{rk}\,G$. So, the multiplicity of 1 as a root of $c(t)$ is $\leq r$, implying that it is in fact precisely r and hence the element γ is regular and semisimple. (To see the latter, let us consider the Jordan decomposition if $\gamma = \gamma_s\gamma_u$; then $c(t)$ is also the characteristic polynomial of $\mathrm{Ad}\,\gamma_s$. On the other hand, since $\mathrm{char}\, k = 0$, the assumption $\gamma_u \neq e$ would imply the existence of a nontrivial nilpotent element in the Lie algebra centralized by γ_s. But this would clearly make the multiplicity of 1 as a root of $c(t)$ greater than r, a contradiction.)

Furthermore, since $\bar{\gamma}$ has infinite order, so does γ.

(2): Let $E = (k_v)_T$ be the splitting field of T over k_v. Clearly, E contains ℓ_u, and we let $\tilde{u}$ denote the extension of u to E. Since ℓ/k is unramified at v, it follows from [58, Lemma 4.1] that it is enough to prove that

$$[E : \ell_u] =: w = [E^{(\tilde{u})} : \ell^{(u)}]. \tag{20}$$

Since E contains the splitting field of $c(t)$, the residue field $E^{(\tilde{u})}$ contains the splitting field of $\bar{c}(t)$. By our construction, the $k^{(v)}$-torus $\bar{T} := C_{\underline{G}^{(v)}}(\bar{\gamma})$ is generic over $\ell^{(u)}$, so $[(k^{(v)})_{\bar{T}} : \ell^{(u)}] = w$. On the other hand, the roots of $\bar{c}(t)$ include the values at $\bar{\gamma}$ of all roots $\alpha \in \Phi(\underline{G}^{(v)}, \bar{T})$. So, it follows from Lemma 3.9 that $(k^{(v)})_{\bar{T}} = E^{(\tilde{u})}$. Thus,

$$w = [E : \ell_u] \geq [E^{(\tilde{u})} : \ell^{(u)}] = [(k^{(v)})_{\bar{T}} : \ell^{(u)}] = w,$$

and (20) follows. $\square$

First, assume that the type of G is different from A_1 and B_ℓ, and let $v \in V \setminus S(\Gamma)$. It follows from Lemma 9.3 and the discussion preceding it that one can find a regular semisimple element $\gamma \in \Gamma$ of infinite order such that the k-torus $T = C_G(\gamma)^\circ$ is generic over ℓ_u and the splitting field k_T is unramified at v. By our assumption, γ is weakly commensurable to some semisimple element $\gamma' \in \Gamma'$ of infinite order. Let T' be a maximal k-torus of G' containing γ'. According to Corollary 3.11, the k-tori T and T' are isogenous over k. It follows that T' is generic over $\ell_u = \ell'_u$, hence over k_v. Then G' has good reduction at v by Theorem 6.2.

Next, let G be of one of the types A_1 or B_ℓ. We then first pick a regular semisimple element $\bar{\gamma}_1 \in \Gamma^{(v)}$ of infinite order that is generic over $k^{(v)}$ (note that here $\ell = k$), and let $\bar{T}_1 = C_{\underline{G}^{(v)}}(\bar{\gamma}_1)^\circ$ denote the corresponding $k^{(v)}$-torus. We then pick a regular semisimple element $\bar{\gamma}_2 \in \Gamma^{(v)}$ of infinite order that is generic over the splitting field $(k^{(v)})_{\bar{T}_1}$, and let $\bar{T}_2 = C_{\underline{G}^{(v)}}(\bar{\gamma}_2)^\circ$. Note that the Dynkin diagrams of the types at hand do not have nontrivial automorphisms, so

$$[(k^{(v)})_{\bar{T}_1}(k^{(v)})_{\bar{T}_2} : (k^{(v)})_{\bar{T}_1}] = w = [(k^{(v)})_{\bar{T}_2} : k^{(v)}],$$

which implies that

$$(k^{(v)})_{\bar{T}_1} \cap (k^{(v)})_{\bar{T}_2} = k^{(v)}. \tag{21}$$

Now, pick $\gamma_i \in \Gamma$ so that $\rho_v(\gamma_i) = \bar{\gamma}_i$, and let $T_i = C_G(\gamma_i)^\circ$, for $i = 1, 2$. Also, let $c_i(t)$ be the characteristic polynomial of $\mathrm{Ad}\gamma_i$. Then the reduction $\bar{c}_i(t)$ is the characteristic polynomial of $\mathrm{Ad}\,\bar{\gamma}_i$. Since T_i (resp., $\bar{T}_i$) is k_v- (resp., $k^{(v)}$-)generic, it follows from Lemma 3.9 that $(k_v)_{T_i}$ (resp., $(k^{(v)})_{\bar{T}_i}$) coincides with the splitting field of c_i (resp., $\bar{c}_i$), and therefore $(k^{(v)})_{\bar{T}_i}$ is precisely the residue field of $(k_v)_{T_i}$. We also know from Lemma 9.3 that the extension $(k_v)_{T_i}/k_v$ is unramified. Then (21) yields that $(k_v)_{T_1} \cap (k_v)_{T_2} = k_v$. By our assumption, the elements γ_1, γ_2 are weakly commensurable to semisimple elements

$\gamma'_1, \gamma'_2 \in \Gamma'$ of infinite order, respectively. Let T'_i be a maximal k-torus of G' containing γ'_i. Then by Corollary 3.11, the torus T_i is k-isogenous to the torus T'_i for $i = 1, 2$. It follows that T'_i is generic over k_v, the extension $(k_v)_{T'_i}/k_v$ is unramified for $i = 1, 2$, and

$$(k_v)_{T'_1} \cap (k_v)_{T'_2} = k_v.$$

So, G' has good reduction by Theorem 6.6.

9.4. Subgroups with the same exceptional set

A smaller Zariski-dense subgroup $\Delta \subset \Gamma$ may, a priori, require a larger exceptional set $S(\Delta)$ in Theorem 1.8. We will show in this subsection, however, that our construction of $S(\Gamma)$ produces an exceptional set that works for many subgroups $\Delta \subset \Gamma$ that are smaller than Γ. To describe precisely the possibilities for Δ, we will need the following definition.

Definition 9.4. Let Γ be an abstract group. The following subgroups will be called *principal standard subgroups* of Γ:

(1) the commutator subgroup $[\Gamma, \Gamma]$;
(2) the subgroups $\Gamma^{(n)}$ generated by the nth powers γ^n of elements $\gamma \in \Gamma$, for some $n \geqslant 1$.

Furthermore, a subgroup Δ of Γ is called *standard* if there exists a (finite) chain of subgroups

$$\Delta = \Gamma_m \subset \Gamma_{m-1} \subset \cdots \subset \Gamma_1 \subset \Gamma_0 = \Gamma$$

such that Γ_{i+1} contains a principal standard subgroup of Γ_i.

We note that all standard subgroups of a (finitely generated) Zariski-dense subgroup of a (connected) semisimple algebraic group are automatically Zariski-dense, although they may not be finitely generated. However, typically this does not create any additional problems in the analysis of weak commensurability since the statements dealing with the existence of generic elements having special properties remain valid for those Zariski-dense subgroups that are contained in a finitely generated subgroup of $G(k)$. Our goal in this subsection is to present the following strengthening of Theorem 1.8.

Theorem 9.5. *Let G be an absolutely almost simple algebraic group over a finitely generated field k of characteristic zero, and let V be a divisorial set of places of k. Given a Zariski-dense subgroup $\Gamma \subset G(k)$ with trace field k, there exists a finite subset $S(\Gamma) \subset V$ such that any absolutely almost simple algebraic K-group G' with the property that there exists a finitely generated Zariski-dense subgroup $\Gamma' \subset G'(K)$ that is weakly commensurable to some standard subgroup $\Delta \subset \Gamma$, has good reduction at all $v \in V \setminus S(\Gamma)$.*

Proof. It turns out that the set $V(\Gamma)$ we have constructed in the proof of Theorem 1.8 works in this more general setting. To see this, we will revisit our construction separately in the cases where $\dim \mathfrak{X} = 1$ and $\dim \mathfrak{X} > 1$. In the first case, the exceptional set $S(\Gamma)$ was constructed in subsection 9.1 as the union $S_1 \cup S_2 \cup S_3$ in the notations introduced therein. It follows from the definitions that the finite sets S_1 and S_2 are independent of Γ. Now, recall that the finite set S_3 is chosen so that the closure of Γ in $G(k_v)$ is open for all $v \in V \setminus S_3$. It is easy to see, however, that every principal standard subgroup, and hence any standard subgroup, of an open subgroup of $G(k_v)$ is itself open. This implies that if the closure of Γ in $G(k_v)$ is open, then so is the closure of any standard subgroup $\Delta \subset \Gamma$. In other words, if the set S_3 is chosen as in section 9.1 for Γ, then it also ensures the required property (i.e., the openness of the closure) for any standard subgroup $\Delta \subset \Gamma$. Thus, the set $S(\Gamma) = S_1 \cup S_2 \cup S_3$ will serve as an exceptional set for Δ as well.

Recall that the exceptional set in the case $\dim \mathfrak{X} > 1$ was actually constructed in subsection 9.3 as $V(\Gamma) = S_1 \cup S_2$, where S_1 consists of those $v \in V$ for which condition (a) fails, and S_2 consists of those $v \in V \setminus S_1$ for which condition (b) fails. Clearly, S_1 is independent of Γ. On the other hand, S_2 is chosen to be disjoint from S_1 so that for $v \in V \setminus (S_1 \cup S_2)$, the image $\rho_v(\Gamma)$ under the reduction map is a Zariski-dense subgroup of the connected absolutely almost simple algebraic $k^{(v)}$-group $\underline{G}^{(v)}$. Then $\rho_v(\Delta)$ is also Zariski-dense in $\underline{G}^{(v)}$ for any standard subgroup $\Delta \subset \Gamma$. This means that if we choose the finite set S_2 for Γ, then condition (b) will hold true for any standard subgroup $\Delta \subset \Gamma$ and for any $v \in V \setminus (S_1 \cup S_2)$. Thus $S(\Gamma) = S_1 \cup S_2$ can be taken for an exceptional set for Δ, completing the proof. □

10. Application to lattices and length-commensurable Riemann surfaces

As we already mentioned in the introduction, there is a conjecture that predicts the existence of only finitely many possibilities for the algebraic hull of a finitely generated Zariski-dense subgroup that is weakly commensurable to a given one ([64, Conjecture 6.1]); this conjecture is a crucial element in the so-called "eigenvalue rigidity."

Conjecture 10.1. *Let G_1 and G_2 be absolutely simple (adjoint) algebraic groups over a field F of characteristic zero, and let $\Gamma_1 \subset G_1(F)$ be a finitely generated Zariski-dense subgroup with trace field $k_{\Gamma_1} =: k$. Then there exists a finite collection $\mathcal{G}_1^{(2)}, \ldots, \mathcal{G}_r^{(2)}$ of F/k-forms of G_2 such that any finitely generated Zariski-dense subgroup $\Gamma_2 \subset G_2(F)$ that is weakly commensurable to Γ_1 is conjugate in $G_2(F)$ to a subgroup of one of the $\mathcal{G}_i^{(2)}(k)$ $(\subset G_2(F))$ for $i = 1, \ldots, r$.*

Due to Theorem 1.8, this conjecture would follow from the Finiteness Conjecture for forms with good reduction, and hence is valid in those cases where the Finiteness Conjecture has been established. For example, since the truth of the Finiteness Conjecture is known for inner forms of type A_n over all finitely generated fields of characteristic zero (cf. [14]), we see that given a central simple algebra A over a finitely generated field k with char $k = 0$ and a finitely generated Zariski-dense subgroup $\Gamma \subset G(k)$, where $G = \mathrm{SL}_{1,A}$, with trace field $k_\Gamma = k$, there are only finitely many isomorphism classes of central division k-algebras A' such that there exists a finitely generated Zariski-dense subgroup $\Gamma' \subset G'(k)$, where $G' = \mathrm{SL}_{1,A'}$, that is weakly commensurable to Γ. Other available results on the Finiteness Conjecture (cf. [66]) lead to a variety of cases where Conjecture 10.1 is known. We will not, however, provide a complete list of these cases here, but rather focus on the case of finitely generated Zariski-dense subgroups weakly commensurable to lattices (arithmetic or not), where our result (given in Theorem 10.2 below) may have applications to locally symmetric spaces.

10.1. Conjecture 10.1 for lattices

We refer the reader to [43], [62] for basic facts about lattices in semisimple Lie groups. In the case of simple Lie groups, we have the following finiteness result for weakly commensurable lattices (arithmetic or not).

Theorem 10.2. *Let G_1 be an absolutely simple (adjoint) real algebraic group, and let $\Gamma_1 \subset G_1(\mathbb{R})$ be a lattice with trace field $k = k_{\Gamma_1}$. Given an absolutely simple (adjoint) algebraic group G_2 over an extension F of k, there exists a finite collection $\mathcal{G}_1^{(2)}, \ldots, \mathcal{G}_r^{(2)}$ of F/k-forms of G_2 such that a finitely generated Zariski-dense subgroup $\Gamma_2 \subset G_2(F)$ that is weakly commensurable to Γ_1 is necessarily $G_2(F)$-conjugate to a subgroup of one of the $\mathcal{G}_i^{(2)}(k)$'s ($\subset G_2(F)$).*

(Here we assume that each F/k-form $\mathcal{G}_i^{(2)}$ comes with a *fixed* F-isomorphism $\phi_i \colon \mathcal{G}_i^{(2)} \times_k F \to G_2$, which then defines an embedding of groups $\mathcal{G}_i^{(2)}(k) \hookrightarrow \mathcal{G}_i^{(2)}(F) \hookrightarrow G_2(F)$.)

Proof. Since Γ_1 is finitely generated, the field k is also finitely generated. Let V be a divisorial set of places of k. Next, for the algebraic hull $\mathcal{G}^{(1)}$ of Γ_1, we have the inclusion $\Gamma_1 \subset \mathcal{G}^{(1)}(k)$. Now, let $\Gamma_2 \subset G_2(F)$ be an arbitrary finitely generated Zariski-dense subgroup weakly commensurable to Γ_1. Then the trace field k_{Γ_2} coincides with k (cf. Theorem 3.8(2)). Let $\mathcal{G}_{\Gamma_2}$ be the algebraic hull of Γ_2, so that $\Gamma_2 \subset \mathcal{G}_{\Gamma_2}(k)$. According to Theorem 1.8, we can find a finite subset $S(\Gamma_1) \subset V$ such that all such $\mathcal{G}_{\Gamma_2}$'s have good reduction at any $v \in V \setminus S(\Gamma_1)$. We now recall that unless $G_1 = \mathrm{PGL}_2$, the trace field k is a field of algebraic numbers (cf. [62, 7.67 and 7.68]), and that if G_1 is isomorphic to PGL_2, then so is G_2 (cf. Theorem 3.8(1)). Since the Finiteness Conjecture for forms with good reduction has already been established in the cases where either k is a number field (cf. [66, Proposition 5.2]) or the group is isogenous to PGL_2 (cf. [66, Theorem 7.6]), it follows that there exists a *finite* collection $\overline{\mathcal{G}}_1^{(2)}, \ldots, \overline{\mathcal{G}}_{\bar{r}}^{(2)}$ of F/k-forms of G_2 with the following property: Given a finitely generated Zariski-dense subgroup $\Gamma_2 \subset G_2(F)$, there exist an $i \in \{1, 2, \ldots, \bar{r}\}$ and a k-isomorphism $\varphi_{\Gamma_2,i} \colon \mathcal{G}_{\Gamma_2} \to \overline{\mathcal{G}}_i^{(2)}$, and then of course $\varphi_{\Gamma_2,i}(\Gamma_2) \subset \overline{\mathcal{G}}_i^{(2)}(k)$. On the other hand, we have F-isomorphisms

$$\iota_{\Gamma_2} \colon \mathcal{G}_{\Gamma_2} \times_k F \to G_2 \ \text{ and } \ \bar{\iota}_i \colon \overline{\mathcal{G}}_i^{(2)} \times_k F \to G_2.$$

Then $\sigma_{\Gamma_2,i} := \bar{\iota}_i \circ \varphi_{\Gamma_2,i} \circ \iota_{\Gamma_2}^{-1}$ is an F-automorphism of G_2, which, in terms of the embeddings of the groups of k-rational points given by ι_{Γ_2} and ι_i, has the property $\sigma_{\Gamma_2,i}(\Gamma_2) \subset \mathcal{G}_i^{(2)}(k)$. If $\sigma_{\Gamma_2,i}$ is inner, then it is conjugation by an element of $G_2(F)$ as G_2 is adjoint, giving the required fact. To handle the general case, we need to expand the collection of forms $\overline{\mathcal{G}}_i^{(2)}$ and (fixed) F-isomorphisms $\bar{\iota}_i \colon \overline{\mathcal{G}}_i \to G_2$, $i = 1, \ldots, \bar{r}$. Namely, let θ_j $(j = 1, \ldots, t)$ be a system of representatives of the cosets $\mathrm{Aut}(G_2)(F)/\mathrm{Int}(G_2)(F)$, where $\mathrm{Aut}(G_2)(F)$ is the group of F-defined automorphisms of the algebraic F-group G_2 and $\mathrm{Int}(G_2)(F)$ is the subgroup of inner automorphisms. Then for $i = 1, \ldots, \bar{r}$ and $j = 1, \ldots, t$, we set

$$\mathcal{G}_{i,j}^{(2)} = \overline{\mathcal{G}}_i \ \text{ and } \ \iota_{i,j} = \theta_j^{-1} \circ \iota_i.$$

We have already seen that given a finitely generated Zariski-dense subgroup $\Gamma_2 \subset G_2(F)$, we can find $i \in \{1, \ldots, \bar{r}\}$ such that $\sigma_{\Gamma_2,i}(\Gamma_2) \subset \overline{\mathcal{G}}_i(K)$. Furthermore, we can find $j \in \{1, \ldots, t\}$ so that $\sigma_{\Gamma_2,i} = \theta_j \circ \tau_{\Gamma_2,i,j}$, with $\tau_{\Gamma_2,i,j}$ inner. Then

$$\tau_{\Gamma_2,i,j}(\Gamma_2) \subset \iota_{i,j}(\mathcal{G}_{i,j}(k)),$$

as required. □

10.2. An application to length-commensurable Riemann surfaces

Let $\mathbb{H} = \{x + iy \mid y > 0\}$ be the complex upper half-plane equipped with the standard hyperbolic metric $ds^2 = \frac{1}{y^2}\left(dx^2 + dy^2\right)$. The action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ by fractional linear transformations is transitive and isometric. Furthermore, the stabilizer of $i \in \mathbb{H}$ is the special orthogonal group $\mathrm{SO}_2(\mathbb{R})$, allowing us to identify $\mathbb{H}$ with the symmetric space $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$. Let $\pi \colon \mathrm{SL}_2 \to \mathrm{PSL}_2$ be the canonical isogeny. Given a discrete subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ containing $\{\pm I\}$ and having torsion-free image $\pi(\Gamma) \subset \mathrm{PSL}_2(\mathbb{R})$, the quotient $M = \Gamma\backslash\mathbb{H}$ is a Riemann surface. It is well-known that every compact Riemann surface of genus > 1 is of this form. However, in this subsection, we will be interested in more general Riemann surfaces, where Γ is only assumed to be finitely generated and Zariski-dense. It was demonstrated in [42] that some properties of M can be understood in terms of the associated quaternion algebra A_Γ, which is constructed as follows.

Let $\Gamma^{(2)}$ denote the subgroup generated by the squares of all elements, and let A_Γ be the $\mathbb{Q}$-subalgebra of $M_2(\mathbb{R})$ generated by $\Gamma^{(2)}$. One shows that A_Γ is a quaternion algebra (although not necessarily a division algebra) with center $k_\Gamma = \mathbb{Q}(\mathrm{tr}\,\gamma \mid \gamma \in \Gamma^{(2)})$ (trace field) — cf. [42, Ch. 3]. If Γ_1 and Γ_2 are commensurable, then $A_{\Gamma_1} = A_{\Gamma_2}$; in other words,

A_Γ is an invariant of the commensurability class of Γ. Moreover, if Γ is an *arithmetic* Fuchsian group, then k_Γ is a number field and A_Γ is *the* quaternion algebra involved in the description of Γ (cf. [42, §8.2]). It follows that if Γ_1 and Γ_2 are arithmetic and the algebras A_{Γ_1} and A_{Γ_2} are isomorphic, then Γ_1 is commensurable with a conjugate of Γ_2, and hence the corresponding Riemann surfaces are commensurable, i.e. have a common finite-sheeted cover. The algebra A_Γ no longer determines the commensurability class of Γ if the latter is not arithmetic, but it nevertheless remains an important invariant of the commensurability class.

In differential geometry, one attaches to a Riemannian manifold M various spectra; in particular, the (weak) length spectrum $L(M)$ is defined as the set of the lengths of closed geodesics in M. Two Riemannian manifolds M_1 and M_2 are called *length-commensurable* if $\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2)$. For arithmetic Riemann surfaces, length-commensurability implies commensurability (cf. [71]). "Most" Riemann surfaces, however, are *not* arithmetic, and their investigation presents many challenges. In those cases where we are unable to characterize the commensurability class in terms of the length spectrum, we would like to understand at least the properties of the associated quaternion algebras. As we will see momentarily, the fact that $M_1 = \Gamma_1\backslash\mathbb{H}$ and $M_2 = \Gamma_2\backslash\mathbb{H}$ are length-commensurable implies that the trace fields are equal: $k_{\Gamma_1} = k_{\Gamma_2}$, i.e. the corresponding algebras A_{Γ_1} and A_{Γ_2} have a common center. We can now state the following finiteness result for families of length-commensurable surfaces.

Theorem 10.3. *Let $M_i = \Gamma_i\backslash\mathbb{H}$ $(i \in I)$ be a family of length-commensurable Riemann surfaces, with $\Gamma_i \subset \mathrm{SL}_2(\mathbb{R})$ Zariski-dense. Then the associated quaternion algebras A_{Γ_i} $(i \in I)$ belong to* finitely many *isomorphism classes (over the common center).*

Proof. We first recall that closed geodesics in $M = \Gamma\backslash\mathbb{H}$ correspond to hyperbolic elements in Γ different from $\pm I$, which are precisely the semisimple elements of Γ having infinite order. Furthermore, the length of the closed geodesic c_γ that corresponds to an element $\gamma \in \Gamma$ which is conjugate to $\begin{pmatrix} t_\gamma & 0 \\ 0 & t_\gamma^{-1} \end{pmatrix}$ is given by

$$\ell(c_\gamma) = \frac{2}{n_\gamma} \cdot |\log|t_\gamma||,$$

where n_γ is a certain integer ("winding number"). It follows that

$$\mathbb{Q} \cdot L(M) = \mathbb{Q} \cdot \{ \log|t_\gamma| \mid \gamma \in \Gamma \ \text{ semisimple of infinite order} \}. \tag{22}$$

Suppose now that two Riemann surfaces $M_i = \Gamma_i\backslash\mathbb{H}$ $(i = 1, 2)$ are length-commensurable. This means that for any semisimple element $\gamma_1 \in \Gamma_1$ of infinite order, there exists a semisimple element $\gamma_2 \in \Gamma_2$ of infinite order such that $\ell(c_{\gamma_1})/\ell(c_{\gamma_2}) \in \mathbb{Q}^\times$, and vice versa. This translates into the relation

$$t_{\gamma_1}^{n_1} = t_{\gamma_2}^{n_2} \neq 1 \quad \text{for some} \quad n_1, n_2 \in \mathbb{Z},$$

which implies that the subgroups Γ_1 and Γ_2 are weakly commensurable. Then applying Theorem 3.8(2), we conclude that their trace fields are the same: $k_{\Gamma_1} = k_{\Gamma_2}$ (we note that the definitions of the trace field given earlier and in the current subsection produce the same result). Fix one subgroup Γ_1 and set $k = k_{\Gamma_1}$; then $\Gamma_1^{(2)} \subset G_1(k)$, where $G_1 = \mathrm{SL}_{1,A_{\Gamma_1}}$. Since the group Γ_1 is finitely generated, the field k is finitely generated, and we let V denote a divisorial set of discrete valuations of k. Now let Γ_i $(i \in I)$ be any other subgroup in the family. Then $k_{\Gamma_i} = k$ and $\Gamma_i^{(2)} \subset G_i(k)$, where $G_i = \mathrm{SL}_{1,A_{\Gamma_i}}$. Since Γ_j for any $j \in I$ is finitely generated, we have $[\Gamma_j : \Gamma_j^{(2)}] < \infty$, so the weak commensurability of Γ_1 and Γ_i implies that of $\Gamma_1^{(2)}$ and $\Gamma_i^{(2)}$. Thus, it follows from Theorem 1.8 that there exists a finite subset $S(\Gamma_1) \subset V$ such that all G_i $(i \in I)$ have good reduction at every $v \in V \setminus S(\Gamma_1)$. Here the groups G_i are all of type A_1, and since the Finiteness Conjecture for forms with good reduction of this type over fields of characteristic $\neq 2$ has already been established (cf. [66, Theorem 7.6]), we conclude that they belong to finitely many isomorphism classes. Consequently, the quaternion algebras A_{Γ_i} also belong to finitely many isomorphism classes. □

Let us point out that this theorem is one of the first examples of the use in differential geometry of new techniques from arithmetic geometry that involve the notion of good reduction.

11. The genus problem and good reduction for groups of type F_4

In this section we will prove Theorems 1.10-1.13 that address some aspects of the genus problem and the Finiteness Conjecture for simple algebraic groups of type F_4. Our considerations will rely on properties of cohomological invariants, which we have assembled in Appendix 2. We therefore recommend that the reader consult Appendix 2 before continuing with this section. We note that while Theorems 1.10-1.12 deal only with forms that have trivial g_3-invariant, Theorem 1.13 shows that truth of the Finiteness Conjecture yields the properness of the map ϕ, which is expected to classify all forms of type F_4 (cf. Appendix 2).

Proof of Theorem 1.10. Let k_0 be a number field, set $k = k_0(x)$, and let V be the set of discrete valuations of k associated with the closed points of $\mathbb{A}^1_{k_0}$. Let G be a k-group of type F_4 that splits over a quadratic extension of k. We need to show that any $G' \in \mathbf{gen}_k(G)$ is k-isomorphic to G. According to Proposition A2.3, the group G possesses a maximal k-torus T that splits over a quadratic extension of k. Since G and G' lie in the same genus, T is k-isomorphic to a maximal k-torus of G', implying that G' splits over the same quadratic extension of k as G, and therefore the invariant $g_3(G')$ is trivial. Then it follows from Theorem A2.1 that in order to prove the isomorphism $G \simeq G'$ over k, it is enough to show that

$$f_3(G) = f_3(G') \text{ and } f_5(G) = f_5(G').$$

Recall that for any $i \geq 1$ and any $v \in V$, we have a residue map

$$\rho_v^i \colon H^i(k, \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^{i-1}(k^{(v)}, \mathbb{Z}/2\mathbb{Z}).$$

These maps enable us to construct, in each degree $i \geq 1$, the following exact sequence that in the case $i = 2$ goes back to Faddeev:

$$0 \to H^i(k_0, \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^i(k, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\oplus \rho_v^i} \bigoplus_{v \in V} H^{i-1}(k^{(v)}, \mathbb{Z}/2\mathbb{Z}) \to 0 \tag{23}$$

(cf. [25, Theorem 9.3]). In order to prove that $f_i(G) = f_i(G')$ for $i = 3, 5$, we will prove that

$$\rho_v^i(f_i(G)) = \rho_v^i(f_i(G')) \text{ for all } v \in V. \tag{24}$$

Assuming this, we obtain from (23) that

$$f_i(G') = f_i(G) + \alpha_i \text{ for some } \alpha_i \in H^i(k_0, \mathbb{Z}/2\mathbb{Z}).$$

To complete the proof, one shows by a specialization argument that $\alpha_i = 0$ for $i = 3, 5$. More precisely, the classes $f_i(G)$ and $f_i(G')$ are represented by symbols, and we can choose $x_0 \in k_0$ so that for the valuation v_0 of k corresponding to $x - x_0$, all factors of these symbols are units with respect to v_0. Then $k^{(v_0)} = k_0$, the groups G and G' have good reduction at v_0 (cf. Proposition A2.7 in Appendix 2), and the specializations of these symbols in $H^i(k_0, \mathbb{Z}/2\mathbb{Z})$ coincide with the invariants $f_i(\underline{G}^{(v_0)})$ and $f_i((\underline{G}')^{(v_0)})$ of the corresponding reductions (see Theorem A2.5 and the remark at the end of subsection A2.2). Since G and G' are in the same genus, their reductions are also in the same genus — see Theorem 1.1. But the genus of a group of type F_4 over a number field reduces to a single element by [58, Theorem 7.5], so $\underline{G}^{(v_0)} \simeq (\underline{G}')^{(v_0)}$ and therefore $f_i(\underline{G}^{(v_0)}) = f_i((\underline{G}')^{(v_0)})$. On the other hand,

$$f_i((\underline{G}')^{(v_0)}) = f_i(\underline{G}^{(v_0)}) + \alpha_i.$$

Thus, $\alpha_i = 0$ and $f_i(G) = f_i(G')$, as required.

In order to prove (24), we will use the following.

Lemma 11.1. *Suppose $\mathcal{K}$ is an infinite field of characteristic $\neq 2$ or 3. Let G be a $\mathcal{K}$-group of type F_4 that splits over a quadratic extension of $\mathcal{K}$, and let $G' \in \mathbf{gen}_{\mathcal{K}}(G)$. Then every extension $\mathcal{L}/\mathcal{K}$ of degree ≤ 2 that splits $f_3(G)$ (resp., $f_5(G)$) also splits $f_3(G')$ (resp., $f_5(G)$).*

Proof. Let σ be a generator of $\mathrm{Gal}(\mathcal{L}/\mathcal{K})$. Since $\mathcal{L}$ splits $f_3(G)$ (resp., $f_5(G)$), the group G is $\mathcal{L}$-split (resp., $\mathcal{L}$-isotropic), cf. section A2.1 of Appendix 2. Let B (resp., P) be an $\mathcal{L}$-defined Borel subgroup (resp., a proper $\mathcal{L}$-defined parabolic subgroup). Then the group $H = B \cap B^\sigma$ (resp., $H = P \cap P^\sigma$) is $\mathcal{K}$-defined. Let T be a maximal $\mathcal{K}$-torus of H that is also a maximal torus of G (cf. [3, 14.13]). Then T is $\mathcal{L}$-split (resp., $\mathcal{L}$-isotropic). Being in the same genus as G, the group G' contains a maximal $\mathcal{K}$-torus isomorphic to T. It follows that G' is also $\mathcal{L}$-split (resp., $\mathcal{L}$-isotropic), and therefore $\mathcal{L}$ splits $f_3(G')$ (resp., $f_5(G')$). □

We will now prove (24) for $i = 3$. According to (28) in subsection A2.1, we can write

$$f_3(G) = (a) \cup (b) \cup (c) \quad \text{and} \quad f_3(G') = (a') \cup (b') \cup (c'),$$

where $(t) \in H^1(k, \mathbb{Z}/2Z)$ denotes the cohomology class corresponding to $t{k^\times}^2$ under the canonical isomorphism $H^1(k, \mathbb{Z}/2\mathbb{Z}) \simeq k^\times/{k^\times}^2$ (other notations are explained in A2.1). If all values $v(a), \ldots, v(c')$ are even, then

$$\rho_v(f_3(G)) = 0 = \rho_v(f_3(G')). \tag{25}$$

Next, suppose that the values $v(a), v(b)$, and $v(c)$ are all even, but among the values $v(a'), v(b'), v(c')$, there is at least one that is odd; suppose, for example, that $v(c')$ is odd. We will apply Lemma 11.1 with $\mathcal{K} = k_v$ and $\mathcal{L} = \mathcal{K}(\sqrt{c'})$, noting that $G' \in \mathbf{gen}_{\mathcal{K}}(G)$ by Corollary 3.3. Thus, since $\mathcal{L}$ splits $f_3(G')$ (over $\mathcal{K}$) by Lemma 11.1, it also splits $f_3(G)$. Now, it follows from Hensel's Lemma that the unramified cohomology group $H^i(\mathcal{K}, \mathbb{Z}/2\mathbb{Z})_v$, defined as the kernel of the residue map $H^i(\mathcal{K}, \mathbb{Z}/2\mathbb{Z}) \to H^{i-1}(\mathcal{K}^{(v)}, \mathbb{Z}/2\mathbb{Z})$, is canonically isomorphic to $H^i(\mathcal{K}^{(v)}, \mathbb{Z}/2\mathbb{Z})$ (cf. [25, Proposition 7.7]). We may assume without loss of generality that $v(a) = v(b) = v(c) = 0$, so that the symbol $(a) \cup (b) \cup (c)$ is unramified. Since it splits over $\mathcal{L}$, the symbol $(\bar{a}) \cup (\bar{b}) \cup (\bar{c})$ is trivial in $H^3(\mathcal{L}^{(v)}, \mathbb{Z}/2\mathbb{Z}) = H^3(\mathcal{K}^{(v)}, \mathbb{Z}/2\mathbb{Z})$. It follows that $(a) \cup (b) \cup (c)$ is trivial in $H^3(\mathcal{K}, \mathbb{Z}/2\mathbb{Z})$, which implies that G is split. Then G' is also split, and we again obtain (25).

It remains to consider the case where each set $\{v(a), v(b), v(c)\}$ and $\{v(a'), v(b'), v(c')\}$ contains at least one odd value. Without loss of generality, we may assume that

$$v(a) = v(a') = v(b) = v(b') = 0 \quad \text{and} \quad v(c) = v(c') = 1.$$

Then

$$\rho_v(f_3(G)) = (\bar{a}) \cup (\bar{b}) \quad \text{and} \quad \rho_v(f_3(G')) = (\bar{a}') \cup (\bar{b}').$$

Identifying $H^2(k^{(v)}, \mathbb{Z}/2\mathbb{Z}) = H^2(k^{(v)}, \mu_2)$ with the 2-torsion subgroup ${}_2\mathrm{Br}(k^{(v)})$ of the Brauer group, we see that the residues are represented, respectively, by the classes of the quaternion algebras

$$\overline{D} = \left(\frac{\bar{a}, \bar{b}}{k^{(v)}}\right) \quad \text{and} \quad \overline{D}' = \left(\frac{\bar{a}', \bar{b}'}{k^{(v)}}\right).$$

Since the genus of a quaternion division algebra over a number field reduces to a single element (cf. [13], [65]), in order to prove that $\overline{D} \simeq \overline{D}'$, which would yield (24) in this case, one needs to prove that an extension $\ell/k^{(v)}$ of degree ≤ 2 splits $\overline{D}$ if and only if it splits $\overline{D}'$. Suppose that ℓ splits $\overline{D}$, and let $\mathcal{L}$ denote the unramified extension of $\mathcal{K} = k_v$ with residue field ℓ. It follows from Hensel's Lemma that $\mathcal{L}$ splits $D = \left(\frac{a, b}{\mathcal{K}}\right)$, and therefore also splits $f_3(G)$. By Lemma 11.1, the extension $\mathcal{L}$ also splits $f_3(G')$, and therefore its residue field ℓ splits $\rho_v^3(f_3(G')) = (\bar{a}') \cup (\bar{b}')$, i.e. splits $\overline{D}'$. By symmetry, every extension $\ell/k^{(v)}$ of degree ≤ 2 that splits $\overline{D}'$ also splits $\overline{D}$, completing the argument.

Next, we will prove (24) for $i = 5$. Write

$$f_5(G) = (a) \cup (b) \cup (c) \cup (d) \cup (e) \quad \text{and} \quad f_5(G') = (a') \cup (b') \cup (c') \cup (d') \cup (e').$$

As above, in the following two cases: 1) all values $v(a), \ldots, v(e')$ are even, and 2) all values $v(a), \ldots, v(e)$ are even and among the values $v(a'), \ldots, v(e')$ there is at least one that is odd, one proves that

$$\rho_v(f_5(G)) = 0 = \rho_v(f_5(G'))$$

by repeating basically the same argument. The remaining situations reduce to the case where

$$v(a) = \cdots = v(d) = v(a') = \cdots = v(d') = 0 \quad \text{and} \quad v(e) = v(e') = 1.$$

Then

$$\rho_v(f_5(G)) = (\bar{a}) \cup \cdots \cup (\bar{d}) \quad \text{and} \quad \rho_v(f_5(G')) = (\bar{a}') \cup \cdots \cup (\bar{d}').$$

Since $k^{(v)}$ is a number field, by the Poitou-Tate theorem (cf. [51, 8.6.13(ii)], [77, Ch. II, §6, Theorem B]), for any $j \geq 3$, we have an isomorphism

$$H^j(k^{(v)}, \mathbb{Z}/2\mathbb{Z}) \longrightarrow \prod_{w \in W} H^j((k^{(v)})_w, \mathbb{Z}/2\mathbb{Z}),$$

where W is the set of all archimedean places of $k^{(v)}$. Furthermore, the group $H^j((k^{(v)})_w, \mathbb{Z}/2\mathbb{Z})$ is trivial if w is complex, and has order 2 if w is real; in the latter case, any symbol $(a_1) \cup \cdots \cup (a_j)$ with all a_i's negative in $k^{(v)}$ gives the nontrivial element, while such a symbol in which at least one a_i is positive gives the trivial element. Thus, if (24) fails, then there exists a real place w of $k^{(v)}$ such that, say, $\rho_v(f_5(G))$ gives the trivial element and $\rho_v(f_5(G'))$ the nontrivial element of $H^4((k^{(v)})_w, \mathbb{Z}/2\mathbb{Z})$. This means that among $\bar{a}, \ldots, \bar{d}$ at least one element, say $\bar{d}$, is positive in $(k^{(v)})_w$, while all elements $\bar{a}', \ldots, \bar{d}'$ are negative. Consider the extension $\mathcal{L} = \mathcal{K}(\sqrt{d})$ of $\mathcal{K} = k$. Then $\mathcal{L}$ obviously splits $f_5(G)$. Let $\tilde{v}$ be an extension of v to $\mathcal{L}$. Then

$$\mathcal{L}^{(\tilde{v})} = k^{(v)}(\sqrt{\bar{d}}) \subset (k^{(v)})_w.$$

It follows that $(\mathcal{L}^{(\tilde{v})})_{\tilde{w}}$ for $\tilde{w}|w$ does not split the image of $\rho_v(f_5(G'))$ in $H^4((k^{(v)})_w, \mathbb{Z}/2\mathbb{Z})$, hence $\mathcal{L}^{(\tilde{v})}$ does not split $\rho_v(f_5(G'))$, and $\mathcal{L}$ does not split $f_5(G')$. This contradicts Lemma 11.1. $\square$

Proof of Theorem 1.12. Let G_0 be the k-split group of type F_4. According to Proposition A2.3, one can view $\mathcal{I}$ as a subset of the set $H^1(k, G_0)_{g_3=0}$ of cohomology classes having trivial g_3-invariant, and then by Theorem A2.1, the restriction to $\mathcal{I}$ of the map

$$\psi \colon H^1(k, G_0) \xrightarrow{(f_3, f_5)} H^3(k, \mathbb{Z}/2\mathbb{Z}) \times H^5(k, \mathbb{Z}/2\mathbb{Z})$$

is injective. On the other hand, it follows from Theorem A2.6 that the f_3- and f_5-invariants of the forms from $\mathcal{I}$ are V-unramified, i.e.

$$\psi(\mathcal{J}) \subset H^3(k, \mathbb{Z}/2\mathbb{Z})_V \times H^5(k, \mathbb{Z}/2\mathbb{Z})_V.$$

Furthermore, Proposition 4.2 and Corollary 6.2 in [16] yield the finiteness of the groups $H^3(k, \mathbb{Z}/2\mathbb{Z})_V$ and $H^5(k, \mathbb{Z}/2\mathbb{Z})_V$ in the case where k is a 2-dimensional global field, while Theorem 5.1(b) in [67] provides their finiteness over a purely transcendental extension $k = k_0(x, y)$ of transcendence degree 2 of a number field k_0. In both cases, we obtain the finiteness of $\mathcal{I}$. $\square$

Remark 11.2. Let k be a 2-dimensional global field with a divisorial set of places V. Fix a Killing-Cartan type $\tau \in \{\mathsf{A}_\ell\}_{\ell=1}^{\infty} \cup \cdots \cup \{\mathsf{G}_2\}$, and consider the set $\mathcal{Q}_V(\tau)$ of k-isomorphism classes of absolutely almost simple k-groups of type τ that split over a quadratic extension of k and have good reduction at all $v \in V$. The results of [16] imply that $\mathcal{Q}_V(\tau)$ is finite if τ is one of the types A_ℓ, B_ℓ, C_ℓ ($\ell \geq 1$) or G_2. Our Theorem 1.12 yields the finiteness of $\mathcal{Q}_V(\tau)$ for $\tau = \mathcal{F}_4$. The finiteness of $\mathcal{Q}_V(\tau)$ for $\tau = \mathsf{D}_\ell$ ($\ell \geq 4$) was recently established in [70]. So, over 2-dimensional global fields, it remains to investigate the finiteness of $\mathcal{Q}_V(\tau)$ for $\tau \in \{\mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8\}$. Of course, this question can be viewed in the context of the more general problem of classifying absolutely almost simple algebraic groups that split over a quadratic extension of the base field, considered by Weisfeiler [84], in terms of certain quadratic/Hermitian forms. It should also be mentioned that the finiteness of $\mathcal{Q}_V(\tau)$ (for all V) has immediate consequences for the finiteness of the genus of absolutely almost simple algebraic k-groups of type τ that split over a quadratic extension of k — see the derivation of Theorem 1.11 from Theorem 1.12 above.

Proof of Theorem 1.11. If G splits over a quadratic extension ℓ/k, then it has a maximal k-torus T that splits over ℓ. Then any $G' \in \mathbf{gen}_k(G)$ also splits over ℓ. On the other hand, according to Corollary 1.2, there exists a divisorial set V of discrete valuations of k such that every group $G' \in \mathbf{gen}_k(G)$ has good reduction at all $v \in V$. The finiteness of $\mathbf{gen}_k(G)$ now follows from Theorem 1.12. $\square$

The remainder of this section will be devoted to the proof of our last result concerning the properness of the map ϕ (cf. subsection A2.1 of Appendix 2 for the relevant definitions).

Proof of Theorem 1.13. We need to show that for any k-group G of type F_4, the fiber $\phi^{-1}(\phi(G))$ is finite. Choose a divisorial set of places V of k such that

- char $k^{(v)} \neq 2, 3$ for all $v \in V$;
- G has good reduction and the invariant $g_3(G)$ is unramified at all $v \in V$.

(We note that if G has good reduction at v and char $k^{(v)} \neq 3$, then $g_3(G)$ is *automatically* unramified — see Proposition A2.11, but this fact is not used in the argument.) Since we are assuming that the Finiteness Conjecture holds for all k-forms of type F_4 with respect to V, it is enough to show that every $G' \in \phi^{-1}(\phi(G))$ has good reduction at all $v \in V$. We will derive this fact from the following two propositions.

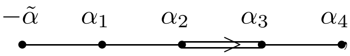
Proposition 11.3. *Assume that* char $k_v^{(v)} \neq 2$. *If a k-form G' of type F_4 does not have good reduction at $v \in V$, then $G' \times_k k_v$ either splits over an unramified Galois extension ℓ/k_v of degree 2^a $(a \geq 0)$ or contains a maximal k_v-torus that is anisotropic over k_v and splits over an unramified cubic Galois extension ℓ/k_v.*

Proposition 11.4. *Let v be a discrete valuation of k such that* char $k^{(v)} \neq 2, 3$, *and let G' be a simple k-group of type F_4 such that $G' \times_k k_v$ has a maximal k_v-torus that is anisotropic over k_v and splits over an unramified cubic Galois extension ℓ/k_v. If G' does not have good reduction at v, then the invariant $g_3(G')$ is ramified at v.*

Granting these facts, we will now complete the proof of Theorem 1.13. Assume that $G' \in \phi^{-1}(\phi(G))$ does not have good reduction at some $v \in V$. According to Proposition 11.3, the group $G' \times_k k_v$ either splits over an unramified Galois extension ℓ/k_v of degree 2^a $(a \geq 0)$ or contains a maximal k_v-torus that is anisotropic over k_v and splits over an unramified cubic Galois extension ℓ/k_v. In the first case, a standard restriction-corestriction argument yields $2^a \cdot g_3(G' \times_k k_v) = 0$, hence $g_3(G' \times_k k_v) = 0$ as we always have $3 \cdot g_3(G' \times_k k_v) = 0$. Thus, $G' \times_k k_v$ corresponds to a cohomology class in $H^1(k_v, G_0)_{g_3=0}$. But according to Theorem A2.1, the restriction of ϕ to $H^1(k_v, G_0)_{g_3=0}$ is injective. Thus $G' \times_k k_v \simeq G \times_k k_v$, contradicting the fact that G has good reduction at v and G' does not.

In the second case, it follows from Proposition 11.4 that the invariant $g_3(G' \times_k k_v)$ is ramified, while by construction the invariant $g_3(G \times_k k_v)$ is unramified. This contradicts the fact that $\phi(G) = \phi(G')$, hence $g_3(G \times_k k_v) = g_3(G' \times_k k_v)$. $\square$

The proof of Proposition 11.3 relies heavily on Bruhat-Tits theory, for which we refer the reader to [6], [7], and [8] (see also [34] for a modern exposition). As above, let G_0 be the k-split group of type F_4, T_0 be a maximal k_v-split torus of G_0, and $\Phi = \Phi(G_0, T_0)$ be the root system of G_0 with respect to T_0. Fix a system of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_4\}$ and let $\tilde{\alpha}$ denote the maximal root. We then have the following extended Dynkin diagram



whose set of vertices will be denoted $\tilde{\Pi}$. To each (non-empty) subset $\Omega \subset \tilde{\Pi}$, Bruhat-Tits theory associates a smooth group scheme $\mathcal{G}_\Omega$ over the valuation ring $\mathcal{O}_v$ of k_v with the following properties:

- the generic fiber $\mathcal{G}_\Omega \times_{\mathcal{O}_v} k_v$ is isomorphic to $G_0 \times_k k_v$;
- the closed fiber $\underline{\mathcal{G}}_\Omega^{(v)} = \mathcal{G}_\Omega \times_{\mathcal{O}_v} k_v^{(v)}$ is connected (because G_0 is simply connected);
- the unipotent radical U_Ω of $\underline{\mathcal{G}}_\Omega^{(v)}$ is defined and split over $k^{(v)}$, and $\underline{\mathcal{G}}_\Omega^{(v)}$ has a unique Levi subgroup L_Ω that contains the reduction $\underline{T}^{(v)}$, and hence is $k_v^{(v)}$-split.

We note that the Dynkin diagram of the semisimple part of L_Ω is obtained from the extended Dynkin diagram of G_0 by deleting the vertices belonging to Ω and the edges having at least one endpoint in Ω. Thus, L_Ω is a $k_v^{(v)}$-split reductive group with central torus of dimension $|\Omega| - 1$ and semisimple part (= commutator subgroup) H_Ω of rank $5 - |\Omega|$.

Since $\mathcal{G}_\Omega$ is smooth, the natural map

$$\lambda_\Omega \colon H^1(\mathcal{O}_v, \mathcal{G}_\Omega) \longrightarrow H^1(k_v^{(v)}, \underline{\mathcal{G}}_\Omega^{(v)}), \quad \xi \mapsto \bar{\xi},$$

given by reduction is bijective by Hensel's Lemma (cf. [8, Section 3.4, Lemme 2(2)]). Furthermore, since $\underline{\mathcal{G}}_\Omega^{(v)}$ has a Levi decomposition and its unipotent radical is split, we have

$$H^1(k_v^{(v)}, \underline{\mathcal{G}}_\Omega^{(v)}) = H^1(k_v^{(v)}, L_\Omega). \tag{26}$$

We say that a class $[\xi] \in H^1(k_v^{(v)}, L_\Omega)$ is *anisotropic* if the semisimple part of the twisted group ${}_\xi L_\Omega$ is $k_v^{(v)}$-anisotropic. The set of all anisotropic classes will be denoted $H^1(k_v^{(v)}, L_\Omega)_{\mathrm{an}}$, and its inverse image under λ_Ω will be denoted $H^1(\mathcal{O}_v, \mathcal{G}_\Omega)_{\mathrm{an}}$.

Theorem 11.5. (cf. [8, Theorem 3.12]) *The natural map*

$$\coprod_\Omega H^1(\mathcal{O}_v, \mathcal{G}_\Omega)_{\mathrm{an}} \to H^1(k_v, G_0)$$

is a bijection.

Remark 11.6. It should be noted that this result was established in [8] assuming that the residue field $k_v^{(v)}$ is *perfect*, which is not always the case in our situation. In this regard, we observe that since $\operatorname{char} k_v^{(v)} \neq 2, 3$, any simple algebraic k_v-group G of type F_4 splits over the maximal unramified extension k_v^{ur}. Indeed, in this case, the cohomological invariants f_3, f_5, and g_3 vanish, so the desired fact follows from the triviality of the kernel of ϕ (cf. Corollary A2.2). This implies that Bruhat-Tits theory in the sense of Prasad [55] is available for G over k_v. Another consequence is that

$$H^1(k_v, G_0) = H^1(k_v^{\mathrm{ur}}/k_v, G_0).$$

Then according to Theorem 3.8 in [27], the assertion of Theorem 11.5 remains valid for a k_v-split group G_0 of type F_4 whenever $\operatorname{char} k_v^{(v)} \neq 2, 3$.

Proof of Proposition 11.3. Let G' be a simple k_v-group of type F_4 that does not have good reduction at v. Write G' as a twist ${}_{\xi'}G_0$ for some cocycle $\xi' \in Z^1(k_v, G_0)$. According to Theorem 11.5 and subsequent remarks, there exists a subset $\Omega \subset \tilde{\Pi}$ such that the class $[\xi']$ is the image under the natural map $\nu_\Omega \colon H^1(\mathcal{O}_v, \mathcal{G}_\Omega) \to H^1(k_v, G_0)$ of some class from $H^1(\mathcal{O}_v, \mathcal{G}_\Omega)_{\mathrm{an}}$, and we still use ξ' to denote a cocycle representing this class. We note that for $\Omega = \{-\tilde{\alpha}\}$, the group $\mathcal{G}_\Omega$ coincides with the split $\mathcal{O}_v$-group scheme $\mathcal{G}_0$ of type F_4, and the classes in the image of ν_Ω correspond precisely to the k_v-forms of type F_4 that have good reduction. Since by assumption G' does *not* have good reduction, we conclude that $\Omega \neq \{-\tilde{\alpha}\}$. So, our task is to show that *in all other cases*, the group G' either splits over an unramified Galois extension ℓ/k_v of degree 2^a or contains a maximal k_v-torus that is anisotropic over k_v and splits over an unramified cubic Galois extension ℓ/k_v. Viewing ξ' as an element of $Z^1(\mathcal{O}_v, \mathcal{G}_\Omega)$, we can consider the twist $\mathcal{G}' = {}_{\xi'}\mathcal{G}_\Omega$, which is a smooth group scheme over $\mathcal{O}_v$ with generic fiber G' and closed fiber ${}_{\bar{\xi}'}\underline{\mathcal{G}}_\Omega^{(v)}$ in the above notations. We recall that if ℓ is an unramified extension k_v with residue field $\ell^{(v)}$, then every $\ell^{(v)}$-split torus of $\mathcal{G}' \times_{\mathcal{O}_v} \ell^{(v)}$ can be lifted to a split torus of $\mathcal{G} \times_{\mathcal{O}_v} \mathcal{O}(\ell)$, where $\mathcal{O}(\ell)$ is the valuation ring of ℓ (see [20, Corollary B.3.5]), implying that $\mathrm{rk}_\ell\, G' \geq \mathrm{rk}_{\ell^{(v)}}\, (\mathcal{G}' \times_{\mathcal{O}_v} \ell^{(v)})$.

In view of (26), we may assume that $\bar{\xi}'$ has values in L_Ω, so that we can consider the twisted groups $L'_\Omega := {}_{\bar{\xi}'}L_\Omega$ and $H'_\Omega := {}_{\bar{\xi}'}H_\Omega$. We observe that all absolutely simple components of H'_Ω are defined over $k_v^{(v)}$ and are inner forms. Since the central tori in L_Ω and L'_Ω are $k_v^{(v)}$-isomorphic, hence split, it is clear that if $|\Omega| \geq 3$, then $\mathrm{rk}_{k_v}\, G' \geq \mathrm{rk}_{k_v^{(v)}}\, \mathcal{G}' \geq 2$. In this case, it follows from the classification of forms of type F_4 (cf. [79]) that G' is k_v-split, hence has good reduction, a contradiction. Now, if $|\Omega| = 2$, then L'_Ω has a 1-dimensional central $k_v^{(v)}$-split torus. If H'_Ω is not absolutely almost simple then it has a component of type A_1 which splits over a separable quadratic extension $\bar{\ell}/k_v^{(v)}$. Letting ℓ denote the unramified extension of k_v with residue field $\ell^{(v)} = \bar{\ell}$, we find as above that $\mathrm{rk}_\ell\, G' \geq 2$, and hence G' splits over ℓ. Now, let H'_Ω be absolutely almost simple. Then it is of one of the following types: A_3, B_3, or C_3. Since every simple group of type B_r is isogenous to the special orthogonal group of a nondegenerate quadratic form (cf. [54, Proposition 2.20]), it obviously splits over a Galois extension of the form

$\bar{\ell} = k_v^{(v)}(\sqrt{a_1}, \ldots, \sqrt{a_t})$. Furthermore, every simple group of type C_r is isogenous to the special unitary group of a nondegenerate hermitian form over a central division algebra with a symplectic involution of the first kind (cf. [54, Proposition 2.19]). By Merkurjev's theorem [45], the algebra splits over a Galois extension of the same shape $\bar{\ell} = k_v^{(v)}(\sqrt{a_1}, \ldots, \sqrt{a_t})$, and this extension also splits the group. Picking such an extension if H'_Ω has type B_3 or C_3, and letting ℓ be the unramified extension of k_v with residue field $\ell^{(v)} = \bar{\ell}$ (which is automatically a Galois extension of k_v of degree 2^a) we will have that $\mathrm{rk}_\ell\, G' \geq 3$, implying that G' splits over ℓ.

In the remaining case, H'_Ω is an inner form of type A_3, i.e. a group of the form $\mathrm{SL}_{1,A}$ for some central simple $k_v^{(v)}$-algebra A of degree 4. Then it follows from the theorem of Merkurjev-Suslin [46] that this group splits over a Galois extension $\bar{\ell}/k_v^{(v)}$ of degree 2^a (note that one needs to adjoin $\sqrt{-1}$ to the base field before applying the Merkurjev-Suslin theorem), and then arguing as above we find that the unramified extension ℓ of k_v with residue field $\ell^{(v)} = \bar{\ell}$ is as required.

Finally, we consider the case $|\Omega| = 1$. Here the possible types of H_Ω (with the exception of F_4 itself are $\mathsf{A}_1 \times \mathsf{C}_3$, $\mathsf{A}_2 \times \mathsf{A}_2$, and $\mathsf{A}_1 \times \mathsf{B}_3$. The types $\mathsf{A}_1 \times \mathsf{C}_3$ and $\mathsf{A}_1 \times \mathsf{B}_3$ are handled just as above, so it remains to consider the type $\mathsf{A}_2 \times \mathsf{A}_2$. Let H_1 and H_2 be the almost simple components of H'_Ω; then H_i is isogenous to a group of the form SL_{1,D_i}, where D_i is a central simple $k_v^{(v)}$-algebra of degree 3 over $k_v^{(v)}$. Picking a separable maximal subfield $\bar{\ell}$ of D_1 and arguing as above, we see that G' splits over the unramified extension ℓ of k_v with residue field $\bar{\ell}$. We note that in this case, the group G' is automatically k_v-anisotropic. Indeed, it cannot be k_v-split because of bad reduction, so the only other k_v-isotropic possibility would have a simple group R of type C_3 as its anisotropic kernel. In the situation at hand, G' splits over an extension ℓ/k_v of degree 3, and then R must also split over ℓ. But for the groups of type C, this implies that R splits over k_v, a contradiction.

Next, by Wedderburn's theorem (cf. [38, Theorem 19.2]), D_1 contains a maximal subfield $\bar{\ell}$ that is a cubic Galois extension of the center. We note that, as follows from Hilbert's Theorem 90, the $k_v^{(v)}$-tori $\bar{\mathcal{S}} = \mathrm{R}^{(1)}_{\bar{\ell}/k_v^{(v)}}(\mathbb{G}_m)$ and $\bar{\mathcal{S}}/\mu_3 = \mathrm{R}_{\bar{\ell}/k^{(v)}}(\mathbb{G}_m)/\mathbb{G}_m$ are isomorphic. So, whether H_1 is simply connected or adjoint, it always contains a torus isomorphic to $\bar{\mathcal{S}}$ as a maximal torus, and we keep the notation $\bar{\mathcal{S}}$ for this torus. By [20, Corollary B.3.5], the torus $\bar{\mathcal{S}}$ lifts to a torus $\mathcal{S}$ of $\mathcal{G}_\Omega$. Let $S \subset G'$ be the generic fiber of $\mathcal{S}$. Then S splits over the unramified extension ℓ of k_v with residue field $\bar{\ell}$. It follows that $\mathrm{rk}_\ell\, G' \geq 2$, and therefore G' actually splits over ℓ. Now, we consider the centralizer $C = C_{G'}(S)$ and write it as an almost direct product SR, where R is a reductive group of rank 2. The complete list of possibilities for R is as follows:

- a 2-dimensional torus;
- an almost direct product of a 1-dimensional torus and an almost simple group of type A_1;
- a semisimple group of one of the following types: $\mathsf{A}_1 \times \mathsf{A}_1$, A_2, B_2, or C_2.

Since S and G' split over ℓ, so does R. Taking into account that G' is k_v-anisotropic and going through the above list, we find that R can only be either a 2-dimensional torus or an inner form of type A_2. In the first case, the almost direct product SR is a required torus that is anisotropic over k_v and splits over ℓ. In the second case, R is k_v-isogenous to a group of the form $\mathrm{SL}_{1,D}$, where D is a cubic division algebra (whose residue must coincide with D_2). Since ℓ splits R, it is isomorphic to a maximal subfield of D, implying that R contains a maximal k_v-torus S' of the form $\mathrm{R}^{(1)}_{\ell/k}(\mathbb{G}_m)$. Then SS' is a required maximal torus. □

Next, in preparation for the proof of Proposition 11.4, we will set up the necessary notations and state one result (Proposition 11.7) that will be proved in subsection A2.3 of Appendix 2. Let G_0 be a simple k-split group of type F_4, and let ℓ/k be a cubic Galois extension. We will now construct a special maximal k-torus T of G_0 whose cohomology classes yield all k-forms of G_0 that contain a maximal k-torus that is anisotropic over k and splits over ℓ. Fix a maximal k-split torus T_0 of G_0, and let $\Phi = \Phi(G_0, T_0)$ be the corresponding root system. We will continue using the above labeling of the roots of the extended Dynkin diagram. Then the subsets $\{-\tilde{\alpha}, \alpha_1\}$ and $\{\alpha_3, \alpha_4\}$ correspond to k-subgroups R_1 and R_2 of G_0 that are isomorphic to SL_3 and whose intersection is μ_3 (which is the center of both R_1 and R_2). Let $S = \mathrm{R}^{(1)}_{\ell/k}(\mathbb{G}_m)$. We consider the embeddings $\iota_i \colon S \hookrightarrow R_i$ $(i = 1, 2)$ afforded by the regular representation of ℓ over k, and let

$$T = \iota_1(S) \cdot \iota_2(S) \subset R_1 \cdot R_2.$$

We have an exact sequence

$$1 \to \mu_3 \xrightarrow{\alpha} S \times S \xrightarrow{\beta} T \to 1,$$

where $\alpha(s) = (s, s^{-1})$ and $\beta(s_1, s_2) = \iota_1(s_1)\iota_2(s_2)$. Set

$$\tilde{S}_1 = \{(s, s^{-1}) \mid s \in S\} \quad \text{and} \quad \tilde{S}_2 = \{(s, 1) \mid s \in S\},$$

and let $S_i = \beta(\tilde{S}_i)$ for $i = 1, 2$. Then $T = S_1 \times S_2$, so we can consider the homomorphisms $\gamma_i \colon H^1(k, T) \to H^1(k, S_i)$ $(i = 1, 2)$ given by the projections. Finally, let $\delta \colon H^1(k, S_1) \to H^2(k, \mu_3)$ be the coboundary map associated with the exact sequence

$$1 \to \mu_3 \longrightarrow \tilde{S}_1 \xrightarrow{\beta} S_1 \to 1.$$

Proposition 11.7. (1) *Every k-group G of type F_4 that contains a maximal k-torus that is k-anisotropic and splits over ℓ is of the form $G = {}_{\xi}G_0$ with $\xi = \nu(\zeta)$ for some $\zeta \in Z^1(k, T)$, where $\nu \colon Z^1(k, T) \to Z^1(k, G_0)$ is the natural map.*

(2) *In the above notations, $g_3(G) = \delta(\gamma_1([\zeta])) \cup \gamma_2([\zeta])$.*

The proof will be given in section A2.3.

Proof of Proposition 11.4. Applying Proposition 11.7 over k_v, we can write $G' \times_k k_v = {}_{\xi}(G_0 \times_k k_v)$, where $\xi = \nu(\zeta)$ and $\zeta \in Z^1(k_v, T)$. Suppose that G' does not have good reduction at v. We need to show that the g_3-invariant $g_3(G' \times_k k_v) \in H^3(k_v, \mathbb{Z}/3\mathbb{Z})$ is ramified. By Proposition 11.7, we have $g_3(G' \times_k k_v) = \delta(\gamma_1([\zeta])) \cup \gamma_2([\zeta])$. If $\delta(\gamma_1([\zeta]))$ is unramified, then by Lemma A2.10 the class of $\gamma_1([\zeta])$ in $H^1(k_v, S_1)$ lies in the image of the map[5] $H^1(\mathcal{O}_v, \mathcal{S}_1) \to H^1(k_v, S_1)$. If, in addition,

$$\gamma_2([\zeta]) \in H^1(k_v, S_2) = k_v^\times / N_{\ell_w/k_v}(\ell_w^\times)$$

is unramified, then it lies in the image of the map $H^1(\mathcal{O}_v, \mathcal{S}_2) \to H^1(k_v, S_2)$. This implies that there exists $\zeta_0 \in Z^1(\mathcal{O}_v, \mathcal{T})$ such that the image of $[\zeta_0]$ in $H^1(k_v, T)$ coincides with $[\zeta]$. Set $\xi_0 = \nu_0(\zeta_0)$ where $\nu_0 H^1(\mathcal{O}_v, \mathcal{T}) \to H^1(\mathcal{O}_v, \mathcal{G}_0)$ is the natural map. Then the twist ${}_{\xi_0}\mathcal{G}_0$ is a reductive $\mathcal{O}_v$-group scheme with generic fiber $G' \times_k k_v$, contradicting our assumptions that G' does not have good reduction at v.

Next, suppose that $\delta(\gamma_1([\zeta]))$ is unramified but $\gamma_2([\zeta])$ is ramified. Then the residue of the invariant $g_3(G' \times_k k_v)$ equals

$$s \cdot (\text{image of } \delta(\gamma_1([\zeta])) \text{ in } {}_3\mathrm{Br}(k^{(v)})), \quad \text{where} \quad s = 1, 2.$$

This element is trivial if and only if the element $\delta(\gamma_1([\zeta]))$ is trivial, in which case $g_3(G' \times_k k_v)$ is trivial. Since the f_3- and f_5-invariants of $G' \times_k k_v$ are also trivial as G' splits over a cubic extension, we conclude that $G' \times_k k_v$ splits, hence has good reduction, a contradiction.

Suppose now that $\delta(\gamma_1([\zeta]))$ is ramified. It obviously splits over ℓ, and therefore is represented by a cyclic cubic algebra of the form $(\ell/k_v, a)$ with $a \in k_v^\times$. Since ℓ/k_v is unramified, the valuation $v(a)$ is either 1 or 2. Then $g_3(G' \times_k k_v)$ can be written in the form $g_3(G' \times_k k_v) = \delta(\gamma_1([\xi])) \cup (c)$, where c is a unit, so its residue is $s \cdot \chi_{\ell^{(v)}} \cup (\bar{c})$, where $\chi_{\ell^{(v)}}$ is the character corresponding to the residue field $\ell^{(v)}$ of ℓ and $s = 1, 2$. If this element is trivial, then the element $\chi_\ell \cup (c)$ is trivial, and therefore $g_3(G' \times_k k_v)$ is trivial. So, again $G' \times_k k_v$ splits, hence G' has good reduction at v, a contradiction. □

Acknowledgments

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[5] We let $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{T}$ denote the $\mathcal{O}_v$-tori with generic fibers S_1, S_2, and T, respectively.

Appendix 1. On a result of A. Klyachko

A1.1. Proof of Theorem 4.1

We will freely use the notations introduced in §4. Let (,) be a $W(\Phi)$-invariant inner product on V. As usual, for $\alpha \in \Phi$, we define the *dual root* $\alpha^\vee = \dfrac{2\alpha}{(\alpha,\alpha)}$, and let $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$ denote the corresponding *dual root system*. Given a subgroup $\Gamma \subset \mathrm{Aut}(\Phi)$ containing $W(\Phi)$ and a Γ-invariant lattice M in V, we can write the inflation-restriction exact sequence

$$0 \to H^1(\Gamma/W(\Phi), M^{W(\Phi)}) \longrightarrow H^1(\Gamma, M) \longrightarrow H^1(W(\Phi), M)^{\Gamma/W(\Phi)} \\ \longrightarrow H^2(\Gamma/W(\Phi), M^{W(\Phi)}).$$

Since $M^{W(\Phi)} = 0$, we obtain an isomorphism

$$H^1(\Gamma, M) \simeq H^1(W(\Phi), M)^{\Gamma/W(\Phi)}.$$

It follows that it is enough to prove both assertions of the theorem for $\Gamma = W(\Phi)$, which we will assume to be the case throughout the rest of the argument.

Now, fix a system of simple roots $\Pi \subset \Phi$. It is well-known that Γ is a Coxeter group; more precisely, it is generated by reflections s_α for $\alpha \in \Pi$ (where $s_\alpha(v) = v - (\alpha^\vee, v)\alpha$ for $v \in V$) and is defined by the following relations

$$s_\alpha^2 = 1 \ , \quad (s_\alpha s_\beta)^{n_{\alpha,\beta}} = 1 \quad \text{for} \quad \alpha\,,\ \beta \in \Pi, \quad \alpha \neq \beta, \tag{A.1}$$

where $n_{\alpha,\beta}$ is the order of the product $s_\alpha s_\beta$ in Γ. For a Γ-invariant lattice $M \subset V$, we let $Z^1(\Gamma, M)$ and $B^1(\Gamma, M)$ denote the corresponding groups of cocycles and coboundaries, respectively, and for a function $f\colon \Gamma \to M$ set

$$\mu(f) = (f(s_\alpha)) \in \bigoplus_{\alpha\in\Pi} M.$$

Lemma A1.1. *The map μ sets up an isomorphism between $Z^1(\Gamma, M)$ and $\bigoplus_{\alpha\in\Pi}(\mathbb{Q}\alpha \bigcap M)$. Under this isomorphism, $B^1(\Gamma, M)$ corresponds to*

$$\{((\alpha^\vee, m)\alpha) \mid m \in M\}.$$

Proof. Any $f \in Z^1(\Gamma, M)$ is completely determined by its values on any generating set, making μ injective on $Z^1(\Gamma, M)$. We have

$$f(1) = 0 = f(s_\alpha) + s_\alpha f(s_\alpha),$$

implying that $f(s_\alpha) \in \mathbb{Q}\alpha$. So, $\mu(Z^1(\Gamma, M)) \subset \bigoplus_{\alpha\in\Pi}(\mathbb{Q}\alpha \bigcap M)$. To prove that this inclusion is actually an equality, take any (m_α) in the right-hand side, and let $\tilde{\Gamma}$ be the free group on s_α, $\alpha \in \Pi$. Recall that given a group Δ and a Δ-module T, a function $\varphi\colon \Delta \to T$ is a 1-cocycle iff the map

$$\varphi^+ \colon \Delta \to T \rtimes \Delta, \qquad x \mapsto (\varphi(x)\,,\, x)$$

is a group homomorphism. This allows us to define $\tilde{f} \in Z^1(\tilde{\Gamma}, M)$ by $\tilde{f}(s_\alpha) = m_\alpha$, and observe that $\tilde{f}$ descends to $f \in Z^1(\Gamma, M)$ satisfying $f(s_\alpha) = m_\alpha$ if and only if $\tilde{f}$ vanishes on the relations (A.1). The equation $\tilde{f}(s_\alpha^2) = 0$ immediately follows from the fact that $m_\alpha \in \mathbb{Q}\alpha$. Furthermore,

$$\tilde{f}((s_\alpha s_\beta)^{n_{\alpha,\beta}}) = (1 + (s_\alpha s_\beta) + \cdots + (s_\alpha s_\beta)^{n_{\alpha,\beta}-1})\tilde{f}(s_\alpha s_\beta).$$

Note that the right-hand side is fixed by $s_\alpha s_\beta$. On the other hand, since

$$\tilde{f}(s_\alpha s_\beta) = m_\alpha + s_\alpha m_\beta \in \mathbb{Q}\alpha + \mathbb{Q}\beta,$$

it belongs to $\mathbb{Q}\alpha + \mathbb{Q}\beta$. But $s_\alpha s_\beta$ is a nontrivial rotation of this 2-dimensional vector space, and therefore has no nonzero fixed vectors. So, $\tilde{f}((s_\alpha s_\beta)^{n_{\alpha,\beta}}) = 0$, completing the proof of the first assertion. The second assertion about $\mu(B^1(\Gamma, M))$ follows from the formula for s_α. □

It is now easy to complete the proof of Theorem 4.1. Set $M = P(\Phi)$, which, by definition, is the dual lattice of the lattice $Q(\Phi^\vee)$ generated by the dual root system. Thus, M has a basis consisting of *weights* ω_β ($\beta \in \Pi$) satisfying $(\alpha^\vee, \omega_\beta) = \delta_{\alpha\beta}$ (Kronecker delta). The crucial observation is that unless Φ is of type A_1 or C_ℓ ($\ell \geq 2$), for any $\alpha \in \Pi$ we have

$$\mathbb{Q}\alpha \bigcap M = \mathbb{Z}\alpha. \tag{A.2}$$

Indeed, this is obvious if $P(\Phi) = Q(\Phi)$ since Π is always a basis of $Q(\Phi)$. This proves (A.2) if Φ is one of the types E_8, F_4, or G_2. On the other hand, if Φ has rank > 1 with all the roots of the same length (equal to $\sqrt{2}$), then for a given $\alpha \in \Pi$, one can pick $\beta \in \Pi$ so that

$$(\alpha, \beta) = -1 = (\alpha, \beta^\vee).$$

If $\lambda\alpha \in M$, then $-\lambda = (\lambda\alpha, \beta^\vee) \in \mathbb{Z}$, proving (A.2) in this case. Apart from types A_1 and C_ℓ ($\ell \geq 2$), these two cases cover all types except B_ℓ with $\ell > 2$. For this remaining type, (A.2) follows immediately from the description of $P(\Phi)$ given in [5, Table II]. Thus, if Φ is not of type A_1 or C_ℓ then

$$\bigoplus_{\alpha\in\Pi}(\mathbb{Q}\alpha\bigcap M)=\bigoplus_{\alpha\in\Pi}\mathbb{Z}\alpha. \tag{A.3}$$

For types A_1 and C_ℓ, one easily finds, using [5, Tables I and II], that the left-hand side of (A.3) contains the right-hand side as a subgroup of index two. On the other hand, in all cases

$$\{((\alpha^\vee,m)\alpha)\mid m\in M\}=\bigoplus_{\alpha\in\Pi}\mathbb{Z}\alpha. \tag{A.4}$$

Indeed, the inclusion $\subset$ follows from the definition of $P(\Phi)$. On the other hand, for the weight $m=\omega_\beta$ with $\beta\in\Pi$, the element $((\alpha^\vee,m)\alpha)_{\alpha\in\Pi}$ has β in the β-slot and 0 in all other slots. Thus, the left-hand side of (A.4) contains all $\beta\in\Pi$, and (A.4) follows. Comparing these computations with Lemma A1.1, we obtain Theorem 4.1. □

Remark A1.2. Since $\mathsf{B}_2=\mathsf{C}_2$, the type B_2 in Theorem 4.1 should be treated as exceptional along with the types A_1 and C_ℓ ($\ell\geq 2$).

A1.2. On the mistake in [37]

In [37], Klyachko made a claim (p. 73, item c)) that for any subgroup $\Gamma\subset\mathrm{Aut}(\Phi)$ that contains $W(\Phi)$, and any Γ-invariant lattice $M\subset V$ satisfying $Q(\Phi)\subset M\subset P(\Phi)$, one has $H^1(\Gamma,M)=0$ *except* in the following three cases, where Γ coincides with $W(\Phi)=\mathrm{Aut}(\Phi)$: (1) $\Phi=\mathsf{A}_1$; (2) $\Phi=\mathsf{C}_\ell$ and $M=P(\Phi)$; and (3) $\Phi=\mathsf{B}_\ell$ and $M=Q(\Phi)$, where $H^1(\Gamma,M)=\mathbb{Z}/2\mathbb{Z}$. As we already mentioned, this result is *false* as stated. We will now indicate where the argument in [37] fails for $M=Q(\Phi)$, and then present a counter-example based on explicit computations.

For this M, we immediately have

$$\bigoplus_{\alpha\in\Pi}(\mathbb{Q}\alpha\bigcap M)\ =\ \bigoplus_{\alpha\in\Pi}\mathbb{Z}\alpha.$$

Then Klyachko observes that if Φ is of type different from $\mathsf{A}_1,\mathsf{B}_\ell$, then for each $\alpha\in\Pi$, the g.c.d. of the integers $(\alpha^\vee,\beta)$ as β varies in Π, equals 1, and concludes from this that

$$\{((\alpha^\vee,m)\alpha)\mid m\in M\}=\bigoplus_{\alpha\in\Pi}\mathbb{Z}\alpha; \tag{27}$$

in view of Lemma A1.1 this would prove that $H^1(\Gamma,M)=0$. The problem is that this "argument" proves only that for each $\alpha\in\Pi$, the projection of the left-hand side of (27) to $\mathbb{Z}\alpha$ is surjective, but does not fully justify (27). In fact, if we canonically identify the right-hand side of (27) with $\mathbb{Z}^\ell$, then the left-hand side gets identified with the submodule spanned by the columns of the Cartan matrix of Φ. It follows that $H^1(W(\Phi),Q(\Phi))$ is always isomorphic to the quotient $P(\Phi)/Q(\Phi)$, and in particular is nontrivial unless

$P(\Phi) = Q(\Phi)$ (in other words, Φ has one of the following types E_8, F_4, or G_2). We will now illustrate this by an explicit computation for the root system Φ of type A_ℓ.

Set $n = \ell + 1$, and consider the usual realization of Φ as the set of vectors

$$\varepsilon_i - \varepsilon_j, \quad i, j = 1, \ldots, n, \ i \neq j,$$

where $\varepsilon_1, \ldots, \varepsilon_n$ is the standard basis of $\mathbb{Q}^n$; then $\Gamma := W(\Phi)$ is identified with the symmetric group S_n acting by permutation of indices. Let

$$N = \bigoplus_{i=1}^{n} \mathbb{Z}\varepsilon_i.$$

Then for $M = Q(\Phi)$, we have the exact sequence of Γ-modules

$$0 \to M \longrightarrow N \xrightarrow{\delta} \mathbb{Z} \to 0,$$

with $\delta\left(\sum a_i e_i\right) = \sum a_i$, which yields the following exact sequence in cohomology

$$N^\Gamma \xrightarrow{\delta} \mathbb{Z} \longrightarrow H^1(\Gamma, M) \longrightarrow H^1(\Gamma, N). \tag{A.5}$$

We have an isomorphism of Γ-modules $N \simeq \mathbb{Z}[S_n/S_{n-1}]$, so by Shapiro's lemma

$$H^1(\Gamma, N) = H^1(S_{n-1}, \mathbb{Z}) = 0.$$

Then (A.5) implies that

$$H^1(\Gamma, M) = \mathbb{Z}/n\mathbb{Z},$$

which is consistent with our previous discussion. We note that this computation is another way to interpret the computation given in the second part of Example 4.3.

Appendix 2. On cohomological invariants and good reduction of groups of type F_4

A2.1. Cohomological invariants

Let k be an infinite field of characteristic $\neq 2, 3$, and let G_0 be the simple k-split group of type F_4 (which is both simply connected and adjoint). Then the k-isomorphism classes of simple k-groups of this type correspond bijectively to the elements of the Galois 1-cohomology set $H^1(k, G_0)$. We recall that J.-P. Serre constructed two cohomological invariants with coefficients in the group $\mathbb{Z}/2\mathbb{Z}$:

$$f_3 \colon H^1(k, G_0) \longrightarrow H^3(k, \mathbb{Z}/2\mathbb{Z}) \ \text{ and } \ f_5 \colon H^1(k, G_0) \longrightarrow H^5(k, \mathbb{Z}/2\mathbb{Z})$$

(see [25, Theorems 22.4 and 22.5]). Furthermore, M. Rost [72] defined a cohomological invariant with coefficients in $\mathbb{Z}/3\mathbb{Z}$:

$$g_3\colon H^1(k, G_0) \longrightarrow H^3(k, \mathbb{Z}/3\mathbb{Z}).$$

(We note that the maps f_3, f_5, and g_3 are natural in the base field k.) Given a cocycle $\xi \in Z^1(k, G_0)$ with corresponding twisted group $G = {}_\xi G_0$, we will often write $f_3(G)$, $f_5(G)$, and $g_3(G)$ instead of $f_3([\xi])$, $f_5([\xi])$, and $g_3([\xi])$. One assembles these three invariants into a map

$$\phi\colon H^1(k, G_0) \stackrel{(f_3, f_5, g_3)}{\longrightarrow} H^3(k, \mathbb{Z}/2\mathbb{Z}) \times H^5(k, \mathbb{Z}/2\mathbb{Z}) \times H^3(k, \mathbb{Z}/3\mathbb{Z}),$$

and one of the remaining fundamental open problems in the theory of Jordan algebras is to determine if ϕ is injective. The following theorem contains a partial result in this direction. We let $H^1(k, G_0)_{g_3=0}$ denote the subset of $H^1(k, G_0)$ consisting of cohomology classes/forms having trivial g_3-invariant.

Theorem A2.1. ([78]) *The map*

$$H^1(k, G_0)_{g_3=0} \stackrel{(f_3, f_5)}{\longrightarrow} H^3(k, \mathbb{Z}/2\mathbb{Z}) \times H^5(k, \mathbb{Z}/2\mathbb{Z})$$

is injective.

Corollary A2.2. *The map ϕ has trivial kernel.*

In Theorems 1.10-1.12, we deal with forms of type F_4 that have trivial g_3-invariant. It is well-known (cf. [38, 26.18], [53]) that to each k-group G of type F_4, one can associate a 27-dimensional simple exceptional Jordan k-algebra J known as the *Albert algebra.* Then the g_3-invariant of G vanishes if and only if J is *reduced*, i.e. has zero divisors. In this case, J admits a natural construction that involves an octonion algebra $\mathcal{O} = \mathcal{O}(a, b, c)$ corresponding to a triple $a, b, c \in k^\times$ and two additional parameters $d, e \in k^\times$. Then the cohomological invariants of G with coefficients in $\mathbb{Z}/2\mathbb{Z}$ are the following symbols

$$f_3(G) = (a) \cup (b) \cup (c) \quad \text{and} \quad f_5(G) = f_3(G) \cup (d) \cup (e), \tag{28}$$

where $(t) \in H^1(k, \mathbb{Z}/2Z)$ denotes the cohomology class corresponding to $tk^{\times 2}$ under the canonical isomorphism $H^1(k, \mathbb{Z}/2\mathbb{Z}) \simeq k^\times/k^{\times 2}$. A group G of type F_4 with trivial g_3-invariant is split over k if and only if the octonion algebra $\mathcal{O}$ is split, i.e. the invariant $f_3(G)$ vanishes, and G is k-isotropic if and only if the invariant $f_5(G)$ vanishes.

We will now establish an alternative characterization of groups of type F_4 having trivial g_3-invariant. Given a (separable) quadratic extension ℓ/k, we say that a k-torus T is *ℓ/k-admissible* if it is anisotropic over k and is split over ℓ, or, equivalently, if the nontrivial automorphism σ of ℓ/k acts on the character group $X(T)$ as multiplication by (-1).

Proposition A2.3. *Let G be a k-group of type F_4. Then $g_3(G) = 0$ if and only if G becomes split over some quadratic extension ℓ/k, in which case G possesses a maximal k-torus T that is ℓ/k-admissible.*

We say that a finite separable extension ℓ/k splits $x \in H^i(k, \mathbb{Z}/2\mathbb{Z})$ if the image of x under the restriction map $H^i(k, \mathbb{Z}/2\mathbb{Z}) \to H^i(\ell, \mathbb{Z}/2\mathbb{Z})$ is trivial. We have the following statement.

Lemma A2.4. *Let G be a k-group of type F_4 for which $g_3(G) = 0$. If ℓ/k is a quadratic extension that splits $f_3(G)$, then it also splits $f_5(G)$, and therefore splits G.*

Indeed, the first assertion immediately follows from the fact that $f_5(G) = f_3(G) \cup (d) \cup (e)$. Thus, over ℓ, all 3 invariants $f_3(G), f_5(G)$, and $g_3(G)$ become trivial. Then it follows from Corollary A2.2 (over ℓ) that the cohomology class in $H^1(k, G_0)$ that corresponds to G becomes trivial in $H^1(\ell, G_0)$, and therefore G is ℓ-isomorphic to G_0, hence is ℓ-split.

Proof of Proposition A2.3. Suppose $g_3(G) = 0$. Since the cohomology class $f_3(G)$ is a symbol, it splits over some quadratic extension ℓ/k. By Lemma A2.4, G splits over ℓ. The group G has an ℓ-defined Borel subgroup B such that $T := B \cap B^\sigma$ is a maximal k-torus of G (cf. [54, Lemma 6.17]). If Π is the system of positive roots in the root system $\Phi = \Phi(G, T)$ that corresponds to B, then for the action of σ on $X(T)$, we have $\sigma(\Pi) = -\Pi$. But the only element in $\mathrm{Aut}(\Phi)$ that has this property is multiplication by (-1). Thus, $\sigma = -1$, i.e. T is ℓ/k-admissible.

Conversely, if G becomes split over a quadratic extension ℓ/k, then ℓ splits the invariant $g_3(G)$. Using a standard restriction-corestriction argument, we see that $2 \cdot g_3(G) = 0$. But every element in $H^3(k, \mathbb{Z}/3\mathbb{Z})$ satisfies $3 \cdot g_3(G) = 0$. Thus, $g_3(G) = 0$. □

A2.2. An alternative description of invariants and good reduction

First, we recall some basic facts about absolutely almost simple k-groups G that possess a maximal k-torus T that is admissible with respect to a quadratic extension ℓ/k, assuming that char $k \neq 2$. These results were initially obtained in [84] and then systematically redeveloped in [10], [11] in the more general situation of groups over regular local rings (in particular, over discrete valuation rings). This generalization becomes particularly useful when we consider forms with good reduction. We will now review the theory over fields. Let $\mathfrak{g} = L(G)$ denote the Lie algebra of G, and let $\Phi = \Phi(G, T)$ be the root system of G with respect to the maximal torus T. Fix a Chevalley basis

$$\{H_{\alpha_1}, \ldots, H_{\alpha_r}\} \cup \{X_\alpha\}_{\alpha \in \Phi}$$

of $\mathfrak{g}(\ell)$ associated with T (where $r = \dim T$ is the rank of G and $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ is a system of simple roots). The key observation is that the action of the nontrivial automorphism σ of ℓ/k on the root elements of the Chevalley basis is described by equations of the form

$$\sigma(X_\alpha) = c_\alpha X_{-\alpha} \quad \text{with} \quad c_\alpha \in k^\times.$$

These constants c_α completely determine the k-isomorphism class of G (assuming that the latter is simply connected). One checks (cf. [11]) that $c_{-\alpha} = c_\alpha^{-1}$ and $c_{\alpha+\beta} = \pm c_\alpha c_\beta$ (with the sign depending only on α and β as elements of Φ). This means that the k-isomorphism class of G is determined by the quadratic extension ℓ/k and the constants c_α for only simple roots α.

In the rest of this subsection, G will denote a k-group of type F_4 with trivial g_3-invariant. According to Proposition A2.3, the group G has a maximal k-torus that is admissible over a quadratic extension $\ell = k(\sqrt{a})$. We will use the labeling of the simple roots $\alpha_1, \ldots, \alpha_4$ introduced in Bourbaki [5].

Theorem A2.5. ([11, Theorems 6.1 and 6.6]) *Let G be a simple algebraic k-group of type F_4 that has a maximal k-torus T that is admissible over a quadratic extension $\ell = k(\sqrt{a})$. Fix a system of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_4\}$ in the root system $\Phi(G,T)$, and let $c_{\alpha_1}, \ldots, c_{\alpha_4}$ be the constants defined above for some choice of root vectors in a Chevalley basis. Then*

$$f_3(G) = (a) \cup (c_{\alpha_1}) \cup (c_{\alpha_2}) \quad \text{and} \quad f_5(G) = (a) \cup (c_{\alpha_1}) \cup (c_{\alpha_2}) \cup (c_{\alpha_3}) \cup (c_{\alpha_4}).$$

We will now use this description of the invariants f_3 and f_5 to prove that they are unramified if the group G has good reduction at a discrete valuation v of k (see subsection A2.4 below for a more general result).

Theorem A2.6. *Let G be a k-group of type F_4 with trivial g_3-invariant that has good reduction at a discrete valuation v of k with* $\operatorname{char} k^{(v)} \neq 2$. *Then the invariants $f_3(G)$ and $f_5(G)$ are unramified at v.*

The proof will involve an application of the results from [11] over discrete valuation rings. So, suppose that our base field k is equipped with a discrete valuation v, and let $\mathcal{O}_{k,v} \subset k$ be the corresponding valuation ring. Let G be a simple algebraic k-group of type F_4 that splits over a quadratic extension $\ell = k(\sqrt{a})$. It follows from the description of $f_3(G)$ as a symbol (see (28)) that, without loss of generality, we can always assume that one of the elements is a unit, and we take this element for a. Then $\ell = k(\sqrt{a})$ is unramified over k at v, and therefore $\tilde{\mathcal{O}} := \mathcal{O}_{k,v}(\sqrt{a})$ is an étale extension of $\mathcal{O}_{k,v}$.

Proof of Theorem A2.6. Suppose that G has good reduction at v, i.e. there exists a reductive group scheme $\mathcal{G}$ over $\mathcal{O}_{k,v}$ with generic fiber G (see the discussion in §2.3). Then

$\mathcal{G} \times_{\mathcal{O}_{k,v}} \tilde{\mathcal{O}}$ is a reductive group scheme with generic fiber $G \times_k \ell$. Applying Proposition 2.5 in this situation and taking into account that $G \times_k \ell$ is split, we see that $\mathcal{G} \times_{\mathcal{O}_{k,v}} \tilde{\mathcal{O}}$ is split. Then one shows that $\mathcal{G}$ contains a maximal torus $\mathcal{T}$ whose generic fiber T is an ℓ/k-admissible torus. Furthermore, one verifies that the constants c_{α_i} $(i = 1, \dots, 4)$ belong to $\mathcal{O}_{k,v}^{\times}$ (see [11, Remark 44]). Combining this with the formulas for $f_3(G)$ and $f_5(G)$ given in Theorem A2.5 completes the proof of Theorem A2.6. □

We conclude this subsection with the following, which in some sense provides a converse to Theorem A2.6.

Proposition A2.7. *Let G be a simple algebraic k-group of type F_4 that has a maximal k-torus T that is ℓ/k-admissible for a quadratic extension ℓ/k. Assume that ℓ/k is unramified at v and the constants $c_{\alpha_1}, \dots, c_{\alpha_4}$ are v-units. Then G has good reduction at v.*

Proof. Let $\mathcal{G}_0$ (resp., $\mathfrak{g}_0$) be the split group scheme (resp., split Lie algebra) of type F_4 over $\mathcal{O}_{k,v}$, and let $\tilde{\mathcal{O}}$ be the integral closure of $\mathcal{O}_{k,v}$ in ℓ. It is enough to construct a Lie algebra $\mathfrak{g}$ over $\mathcal{O}_{k,v}$ such that

$$\mathfrak{g} \otimes_{\mathcal{O}_{k,v}} \tilde{\mathcal{O}} \simeq \mathfrak{g}_0 \otimes_{\mathcal{O}_{k,v}} \tilde{\mathcal{O}} \quad \text{and} \quad \mathfrak{g} \otimes_{\mathcal{O}_{k,v}} k \simeq L(G)_k,$$

where $L(G)$ is the Lie algebra of G. Indeed, it is well-known that the automorphism group of a split Lie algebra of type F_4 is a simple split algebraic group of type F_4, which is both adjoint and simply connected. So, $\mathfrak{g}$ can be obtained from $\mathfrak{g}_0$ by twisting using an ℓ/k-cocycle with values in $\mathcal{G}_0(\tilde{\mathcal{O}})$. Then twisting $\mathcal{G}_0$ by the same cocycle, we obtain the required reductive group $\mathcal{O}_{k,v}$-scheme $\mathcal{G}$ with generic fiber G (since its Lie algebra coincides with that of G by construction). On the other hand, the Lie algebra $\mathfrak{g}$ with the above properties is constructed from $\mathfrak{g}_0 \otimes_{\mathcal{O}_{k,v}} \tilde{\mathcal{O}}$ by Galois descent (which can be implemented due to the fact that ℓ/k is unramified at v) using the automorphism defined by

$$H_\alpha \mapsto -H_\alpha \quad \text{and} \quad X_\alpha \mapsto c_\alpha X_{-\alpha}$$

for all simple roots α (recall that $c_\alpha \in \mathcal{O}_{k,v}^{\times}$ by assumption). □

We note that the reduction $\underline{G}^{(v)}$ possesses a maximal $\ell^{(v)}/k^{(v)}$-admissible torus for which the corresponding constants are the reductions $\bar{c}_{\alpha_1}, \dots, \bar{c}_{\alpha_4}$.

A2.3. Two results about forms of type F_4 that split over a cubic extension

We will keep the notations introduced in §11 prior to the proof of Proposition 11.4. First, we will prove Proposition 11.7.

Proof of Proposition 11.7. (1): We begin with the following general fact.

Lemma A2.8. *Let G be a k-group of type F_4, and let ℓ/k be a cubic Galois extension. Given two maximal k-tori T_1 and T_2 of G that are anisotropic over k and split over ℓ, there exists $g \in G(\ell)$ such that the restriction of the inner automorphism* Int g *induces a k-defined isomorphism $T_1 \to T_2$.*

Proof. Let us return to the notations introduced immediately before the statement of Proposition 11.7. The Weyl group $W(R_i, \iota_i(S))$ is isomorphic to the symmetric group Σ_3, and we let $V_i \subset W(R_i, \iota_i(S))$ be its Sylow 3-subgroup. Then $V = V_1V_2 \subset W(G_0, T)$ has order 9, and therefore is a Sylow 3-subgroup of $W(G_0, T)$. It follows from this description that $W(G_0, T)$ has a unique conjugacy class of elements w of order 3 such that $X(T)^w = \{0\}$. Let $\theta^{(i)} \colon \mathrm{Gal}(\ell/k) \to W(G, T_i)$ for $i = 1, 2$ be the natural homomorphism (cf. subsection 3.1), and fix a generator $\sigma \in \mathrm{Gal}(\ell/k)$. Pick an arbitrary $g \in G(\ell)$ such that for the inner automorphism $\iota_g = \mathrm{Int}\, g$, we have $\iota_g(T_1) = T_2$, and let $\iota_g^* \colon X(T_2) \to X(T_1)$ be the corresponding comorphism. Considering $W(G, T_i)$ as a subgroup of $\mathrm{GL}(X(T_i))$, we can define an isomorphism $W(G, T_1) \to W(G, T_2)$ by $w \mapsto (\iota_g^*)^{-1} \circ w \circ \iota_g^*$. Since $w_i = \theta^{(i)}(\sigma) \in W(G, T_i)$ is an element of order 3 such that $X(T_i)^{w_i} = \{0\}$, it follows from the above remark that by replacing g with gn for an appropriate $n \in N_G(T_1)$, we can assume that

$$\theta^{(1)}(\sigma) \circ \iota_g^* = \iota_g^* \circ \theta^{(2)}(\sigma).$$

This means that the restriction $\iota_g|T_1$ is defined over k, as required. □

Corollary A2.9. *With notations and conventions as in Lemma A2.8, the maps $H^1(\ell/k, T_i) \to H^1(\ell/k, G)$ for $i = 1, 2$ have the same image.*

Proof. By the lemma, we can find $g \in G(\ell)$ such that the restriction of $\iota := \mathrm{Int}\, g$ induces a k-defined isomorphism $T_1 \to T_2$. Then for any $\sigma \in \mathrm{Gal}(\ell/k)$, we have $g\cdot\sigma(g)^{-1} \in T_2(\ell)$. It follows that an arbitrary cocycle $\xi(\sigma)$ on $\mathrm{Gal}(\ell/k)$ with values in $T_1(\ell)$ is equivalent in $H^1(\ell/k, G)$ to the cocycle

$$g\xi(\sigma)\sigma(g)^{-1} = (g\xi(\sigma)g^{-1}) \cdot (g \cdot \sigma(g)^{-1})$$

which has values in $T_2(\ell)$, and vice versa. □

It is now easy to conclude the proof of part (1) of Proposition 11.7. Let $\xi \in Z^1(k, G_0)$ be a cocycle such that $G = {}_\xi G_0$ contains a maximal k-torus T_1 that is anisotropic over k and splits over ℓ. It follows from Steinberg's theorem (cf. [54, Prop. 6.19]) that there exists a k-embedding $T_1 \hookrightarrow G_0$ such that $[\xi]$ lies in the image of the corresponding map $H^1(k, T_1) \to H^1(k, G_0)$. But according to Corollary A2.9, the image of this map coincides with the image of the map $H^1(k, T) \to H^1(k, G_0)$, and the required fact follows.

Turning now to part (2) of Proposition 11.7, we recall that for $S = \mathrm{R}^{(1)}_{\ell/k}(\mathbb{G}_m)$, we have $H^1(k, S) = k^\times/N_{\ell/k}(\ell^\times)$; in particular, we can write $\gamma_2([\zeta]) = bN_{\ell/k}(\ell^\times)$ for some $b \in k^\times$. On the other hand, $\delta(\gamma_1([\zeta]))$ corresponds to a Brauer class $[A] \in H^2(k, \mu_3) = {}_3\mathrm{Br}(k)$. The algebra A splits over ℓ, so that the corresponding division algebra has degree dividing 3. According to [25, 7.4], we have $g_3(G) = [A] \cup (b)$, as required. (We note that since ℓ splits A, the cup-product does not depend on the choice of b in the coset modulo the norm subgroup $N_{\ell/k}(\ell^\times)$.)

The second result of this subsection is the following.

Lemma A2.10.

(1) *δ is injective.*
(2) *Assume that k is complete with respect to a discrete valuation v with* $\operatorname{char} k^{(v)} \neq 3$, *and let $\mathcal{O}_v$ be the valuation ring in $k = k_v$. Furthermore, assume that the extension ℓ/k is unramified, so that there is an $\mathcal{O}_v$-torus $\mathcal{S}_1$ with generic fiber S_1. If $x \in H^1(k_v, S_1)$ is such that the image $\delta(x) \in H^2(k, \mu_3)$ is unramified, then x belongs to the image of the map $H^1(\mathcal{O}_v, \mathcal{S}_1) \to H^1(k_v, S_1)$.*

Proof. (1): We have the following long exact sequence

$$H^1(k, \mu_3) \xrightarrow{\alpha} H^1(k, \tilde{S}_1) \longrightarrow H^1(k, S_1) \xrightarrow{\delta} H^2(k, \mu_3).$$

Since α is surjective, δ is injective.

(2): Let $\tilde{\mathcal{S}}_1$ be an $\mathcal{O}_v$-torus with generic fiber $\tilde{S}_1$. We first show that the map

$$H^2(\mathcal{O}_v, \tilde{\mathcal{S}}_1) \xrightarrow{\varepsilon} H^2(k_v, \tilde{S}_1)$$

is injective. Consider the k_v-tori $T_0 = \mathbb{G}_m$ and $T = R_{\ell/k}(\mathbb{G}_m)$, and let $\mathcal{T}_0$ and $\mathcal{T}$ be the $\mathcal{O}_v$-tori with generic fibers T_0 and T. The exact sequence

$$1 \to \tilde{S}_1 \longrightarrow T \xrightarrow{N} T_0 \to 1,$$

where N is the norm map associated with the extension ℓ/k, induces the following commutative diagram with exact rows

$$\begin{array}{ccccc} H^1(\mathcal{O}_v, \mathcal{T}_0) & \longrightarrow & H^2(\mathcal{O}_v, \tilde{\mathcal{S}}_1) & \longrightarrow & H^2(\mathcal{O}_v, \mathcal{T}) \\ \rho_1 \downarrow & & \varepsilon \downarrow & & \rho_2 \downarrow \\ H^1(k_v, T_0) & \longrightarrow & H^2(k_v, \tilde{S}_1) & \longrightarrow & H^2(k_v, T) \end{array} .$$

But $H^1(k_v, T_0) = \{1\}$ by Hilbert's Theorem 90, and $H^1(\mathcal{O}_v, \mathcal{T}_0) = \operatorname{Pic} \mathcal{O}_v = \{1\}$. On the other hand, by Shapiro's Lemma, the homomorphism ρ_2 can be identified with the homomorphism

$$H^2(\mathcal{O}(\ell), \mathcal{T}_0 \times_{\mathcal{O}_v} \mathcal{O}(\ell)) \to H^2(\ell, \mathcal{T}_0 \times_k \ell),$$

where $\mathcal{O}(\ell)$ is the valuation ring of ℓ. So, the injectivity of ρ_2 immediately follows from the injectivity of the canonical map of the Brauer group of a discrete valuation ring to the Brauer group of its field of fractions (cf. [19, 3.6], [47, Ch. IV, §2]). Now, the injectivity of ε follows from the above commutative diagram.

Next, we have the following commutative diagram with exact rows

$$\begin{array}{ccccc} H^1(\mathcal{O}_v, \mathcal{S}_1) & \longrightarrow & H^2(\mathcal{O}_v, \mu_3) & \longrightarrow & H^2(\mathcal{O}_v, \tilde{\mathcal{S}}_1) \\ \downarrow & & \omega\downarrow & & \varepsilon\downarrow \\ H^1(k_v, S_1) & \xrightarrow{\delta} & H^2(k_v, \mu_3) & \longrightarrow & H^2(k_v, \tilde{S}_1) \end{array}.$$

It is well-known that $\mathrm{Im}\,\omega$ coincides with the subgroup of unramified cohomology classes. Then the required assertion follows from the injectivity of δ and ε by a diagram chase. □

A2.4. Cohomological invariant g_3 of forms with good reduction

The goal of this section is to prove the following.

Proposition A2.11. *Let k be a field with a discrete valuation v such that* $\mathrm{char}\, k^{(v)} \neq 3$. *If G is a k-form of type F_4 that has good reduction at v, then the invariant $g_3(G)$ is unramified at v.*

Proof. Without loss of generality, we may suppose that k is complete with respect to v, and let $\mathcal{O}_v$ be the valuation ring of k. By assumption, there exists a reductive group $\mathcal{O}_v$-scheme $\mathcal{G}$ with generic fiber G. Then the reduction $\underline{G}^{(v)} = \mathcal{G} \times_{\mathcal{O}_v} k^{(v)}$ is the automorphism group of a simple exceptional Jordan $k^{(v)}$-algebra $J^{(v)}$. It follows from [53, Theorem 58] that there exists a quadratic extension $\bar{\ell}/k^{(v)}$ such that the algebra $J^{(v)} \otimes_{k^{(v)}} \bar{\ell}$ is isomorphic to the Albert algebra $(A^{(v)}, \bar{\mu})$ obtained by Tits' first construction from a central cubic $k^{(v)}$-algebra $A^{(v)}$ and some $\bar{\mu} \in \bar{\ell}^{\times}$. Let ℓ be the unramified extension of k with residue field $\ell^{(v)} = \bar{\ell}$, and let $\mathcal{O}(\ell)$ be the valuation ring of ℓ. We now consider an Azumaya $\mathcal{O}(\ell)$-algebra $\mathcal{A}$ with residue algebra $A^{(v)}$, and let $\mu \in \mathcal{O}(\ell)^{\times}$ be an element with residue $\bar{\mu}$. Applying Tits' first construction with these $\mathcal{A}$ and μ, we obtain a Jordan $\mathcal{O}(\ell)$-algebra $\mathcal{J}$. It follows from Hensel's Lemma that the reductive group $\mathcal{O}(\ell)$-scheme $\tilde{\mathcal{G}}$ corresponding to $\mathcal{J}$ is isomorphic to $\mathcal{G} \times_{\mathcal{O}_v} \mathcal{O}(\ell)$. In particular, the generic fiber $\tilde{G}$ of $\tilde{\mathcal{G}}$ is isomorphic to $G \times_k \ell$ and corresponds to the Albert algebra (A, μ), where $A = \mathcal{A} \otimes_{\mathcal{O}(\ell)} \ell$. Let $\mathrm{res}_{\ell/k}$ denote the restriction map in cohomology. Then it follows from the definition of g_3 (see [72]) that

$$\mathrm{res}_{\ell/k}(g_3(G)) = g_3(G \times_k \ell) = [A] \cup (\mu) \in H^3(\ell, \mathbb{Z}/3\mathbb{Z}),$$

where $[A]$ denotes the class of A in ${}_3\mathrm{Br}(\ell) = H^2(\ell, \mu_3)$. Since by construction A comes from an Azumaya algebra and $\mu \in \mathcal{O}(\ell)^{\times}$, we conclude that $\mathrm{res}_{\ell/k}(g_3(G)) \in H^3(\ell, \mathbb{Z}/3\mathbb{Z})$

is unramified. On the other hand, since ℓ/k is unramified, we have the following commutative diagram

$$\begin{array}{ccc} H^3(k, \mathbb{Z}/3\mathbb{Z}) & \xrightarrow{\rho_k} & H^2(k^{(v)}, \mu_3) \\ {\scriptstyle \mathrm{res}_{\ell/k}} \downarrow & & \downarrow {\scriptstyle \mathrm{res}_{\bar{\ell}/k^{(v)}}} \\ H^3(\ell, \mathbb{Z}/3\mathbb{Z}) & \xrightarrow{\rho_\ell} & H^2(\bar{\ell}, \mu_3) \end{array}$$

where ρ_k and ρ_ℓ are the corresponding residue maps. We have

$$\rho_\ell(\mathrm{res}_{\ell/k}(g_3(G))) = 0 = \mathrm{res}_{\bar{\ell}/k^{(v)}}(\rho_k(g_3(G))).$$

But since $[\bar{\ell} : k^{(v)}] = 2$, a standard restriction-corestriction argument shows that $2 \cdot \rho_k(g_3(G)) = 0$. On the other hand, $3 \cdot H^2(k^{(v)}, \mu_3) = 0$, so $\rho_k(g_3(G)) = 0$, as required. □

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